Miyaoka-Yau inequality for hyperplane arrangements

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29/11/2024

Plan

- Main result: $Q \leq 0$ on C

- Proof: Bogomolov-Gieseker inequality for stable parabolic bundles

- Application: reflection arrangements

Basic definitions

- This is a paper about complex projective space

$$\mathbb{CP}^n = \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \, \mathbb{C}^* \, .$$

- Let $\mathcal{H}$ be a hyperplane arrangement in $\mathbb{CP}^n$. That is, $\mathcal{H}$ is a finite set of pairwise distinct complex hyperplanes

$$H \subset \mathbb{CP}^n \, .$$

- Let $L \subset \mathbb{CP}^n$ be a linear subspace obtained as intersection of hyperplanes $H \in \mathcal{H}$. The **multiplicity** of L is

$$m_L = \Big| \{ H \in \mathcal{H} \,|\, H \supset L \} \Big| \, .$$

Note: $m_L \geq \operatorname{codim} L$

Codimension 2 subspaces

Let $L \subset \mathbb{CP}^n$ be a codimension 2 intersection of hyperplanes in $\mathcal{H}$.
We say that

- L is **reducible** if $m_L = 2$
- L is **irreducible** if $m_L \geq 3$

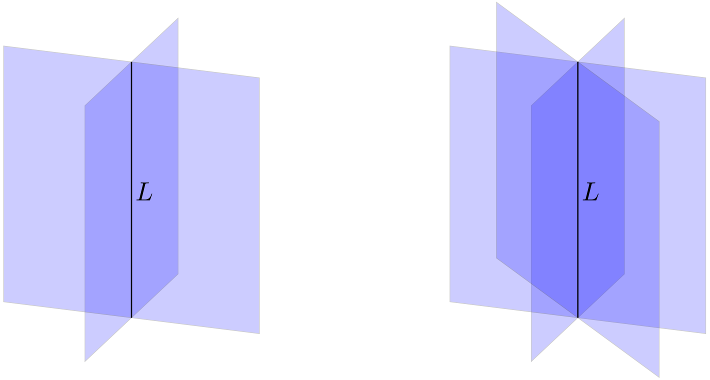


Figure 1: Reducible (left) and irreducible (right).

The Hirzebruch quadratic form

- $\mathcal{H} = \{H_1, \dots, H_N\}$ hyperplane arrangement in $\mathbb{CP}^n$
- σ_i = number of irreducible codimension 2 subspaces $L \subset H_i$
- The **Hirzebruch quadratic form** of $\mathcal{H}$ is the homogeneous degree 2 polynomial on $\mathbb{R}^N$ given by

$$Q(a_1, \dots, a_N) = \sum_{i,j=1}^{N} Q_{ij} a_i a_j$$

$$Q_{ij} = \begin{cases} -(n+1)\sigma_i + 2n & \text{if } i = j \\ -2 & \text{if } i \neq j \text{ and } L = H_i \cap H_j \text{ is reducible} \\ \ n-1 & \text{if } i \neq j \text{ and } L = H_i \cap H_j \text{ is irreducible} \end{cases}$$

Höfer's formula

Let σ_i be the number of points p with $r_p \geqslant 3$ lying on the i-th line of the given arrangement of k lines in the plane. We consider the $(k \times k)$-symmetric matrix A with

$$(3) \qquad A_{ij} = \begin{cases} 3\sigma_i - 4 & (i = j), \\ 2 & (i \neq j, \quad p \in L_i \cap L_j \text{ with } r_p = 2), \\ -1 & (i \neq j, \quad p \in L_i \cap L_j \text{ with } r_p \geqslant 3). \end{cases}$$

With the k lines we associate real variables x_i and let x be the column vector $(x_1, \ldots, x_k)$. With the s points p_j with $r_{p_j} \geqslant 3$ we associate real variables y_j. For each point p_j with $r_{p_j} \geqslant 3$ we consider the linear form

$$P_j(x, y) = 2y_j + \sum_{p_j \in L_i} x_i, \quad \text{where} \quad y = (y_1, \ldots, y_s).$$

Höfer's formula. *For the algebraic surface Y (a good covering of S of degree d with respect to $L_1, \ldots, L_k, E_1, \ldots, E_s$ and the given branching numbers $n_1, \ldots, n_k, m_1, \ldots, m_s$) we have*

$$(4) \qquad (3c_2(Y) - c_1^2(Y))/d = \frac{1}{4}\left(x^t A x + \sum_{j=1}^{s} P_j(x, y)^2\right),$$

where $x_i = 1 - \frac{1}{n_i}$ *and* $y_j = -1 - \frac{1}{m_j}$.

The matroid polytope

$\mathcal{H}$ is an *essential* and *irreducible* hyperplane arrangement in $\mathbb{CP}^n$

- A **basis** of $\mathcal{H} = \{H_1, \ldots, H_N\}$ is a subset $\mathcal{B} \subset \mathcal{H}$ consisting of $n+1$ linearly independent hyperplanes
- The indicator vector of $\mathcal{B}$ is the $1/0$ vector

$$\mathbf{e}_{\mathcal{B}} = \sum_{i \,|\, H_i \in \mathcal{B}} \mathbf{e}_i$$

  where $\mathbf{e}_1, \ldots, \mathbf{e}_N$ are the standard basis vectors of $\mathbb{R}^N$
- The **matroid polytope** is the convex hull of the vectors $\mathbf{e}_{\mathcal{B}}$

$$P = \mathrm{conv}\{\, \mathbf{e}_{\mathcal{B}} \mid \mathcal{B} \text{ is a basis of } \mathcal{H} \,\}.$$

Note: P is contained in the $(N-1)$-simplex $\Delta \subset \mathbb{R}^N$ with

$$\Delta = \left\{\, (a_1, \ldots, a_N) \in \mathbb{R}^N \,|\, a_i \geq 0, \ \sum_i a_i = n+1 \,\right\}$$

The semistable and stable cones

- The **semistable cone** is the cone over the matroid polytope

$$C = \mathbb{R}_{\geq 0} \cdot P = \text{cone}\,\{\,\mathbf{e}_{\mathcal{B}} \mid \mathcal{B} \text{ is a basis of } \mathcal{H}\,\}\,.$$

It is a convex polyhedral cone contained in the octant $\mathbb{R}^N_{\geq 0}$

- The **stable cone** is the interior of $C \subset \mathbb{R}^N$

$$C^\circ = \text{int}(C)\,.$$

- $\mathcal{H}$ is essential and irreducible $\iff \dim P = N - 1$
$\iff C^\circ$ is non empty

Defining linear inequalities of the stable cone

- Let $\mathcal{L}$ be the finite set of *non-empty* and *proper* linear subspaces $L \subset \mathbb{CP}^n$ obtained by intersecting members of $\mathcal{H}$
- $(a_1, \ldots, a_N) \in C^\circ \iff \forall i : a_i > 0$ and $\forall L \in \mathcal{L}$:

$$\boxed{\sum_{i \,|\, L \subset H_i} a_i < \frac{\operatorname{codim} L}{n+1} \cdot \sum_{i=1}^{N} a_i}$$

Relation to Geometric Invariant Theory:

- Standard embedding $\mathbb{CP}^n \subset \mathfrak{su}(n+1)^*$ as a coadjoint orbit. Let $p_i \in (\mathbb{CP}^n)^*$ be the annihilator of H_i
- $(a_1, \ldots, a_N) \in C^\circ \iff \exists\ F \in SL(n+1, \mathbb{C})$ such that the centre of mass of the points $F(p_i)$ with weights a_i is $0 \in \mathfrak{su}(n+1)^*$
- Familiar case $n = 1$, then $\mathbb{CP}^1 = S^2$ and $\mathfrak{su}(2)^* = \mathbb{R}^3$

Main Result

Theorem (Miyaoka-Yau inequality, dB-Panov 2024)

The Hirzebruch quadratic form is non-positive on the semistable cone:

$$C \subset \{Q \leq 0\}$$

- If $n = 1$ then $Q \equiv 0$
- If $n = 2$ this follows from Panov's *Polyhedral Kähler Manifolds*, Geometry & Topology, 2009
- **Conjecture:** if $\mathbf{a} = (a_1, \ldots, a_N) \in C^\circ$ is such that $Q(\mathbf{a}) = 0$ and $a_i \in (0, 1)$. Then there is a Kähler metric on $\mathbb{CP}^n$ of constant holomorphic sectional curvature with cone angles $2\pi\alpha_i$ in transverse directions to the hyperplanes $H_i \in \mathcal{H}$, with $\alpha_i = 1 - a_i$.
  - If $\mathcal{H}$ is a complex reflection arrangement then the metrics have been constructed by Couwenberg-Heckman-Looijenga (IHÉS, 2005)

klt and CY arrangements

A weighted arrangement is a pair $(\mathcal{H}, \mathbf{a})$ consisting of:

- a hyperplane arrangement $\mathcal{H}$ in $\mathbb{CP}^n$;
- a weight vector $\mathbf{a} \in \mathbb{R}^{\mathcal{H}}$ with components $a_H > 0$.

The weighted arrangement $(\mathcal{H}, \mathbf{a})$ is

- klt if

  $$\forall L \in \mathcal{L} : \sum_{H \supset L} a_H < \operatorname{codim} L$$

  where $\mathcal{L}$ is the set of non-empty and proper subspaces $L \subset \mathbb{CP}^n$ obtained by intersecting hyperplanes in $\mathcal{H}$. In particular,

  $$0 < a_H < 1$$

- Calabi-Yau (CY) if

  $$\sum_{H \in \mathcal{H}} a_H = n + 1$$

Restatement of the main theorem

Theorem (Miyaoka-Yau inequality, dB-Panov 2024)

Suppose that the weighted arrangement $(\mathcal{H}, \mathbf{a})$ *is klt and CY. Then*

$$Q(\mathbf{a}) = \sum_{L \in \mathcal{L}^{n-2}_{\mathrm{irr}}} a_L^2 - \frac{1}{2} \sum_{H \in \mathcal{H}} B_H \cdot a_H^2 - \frac{n+1}{2} \leq 0$$

- $\mathcal{L}^{n-2}_{\mathrm{irr}}$ is the set of irreducible codimension 2 subspaces $L \subset \mathbb{CP}^n$
- The weight a_L at $L \in \mathcal{L}^{n-2}_{\mathrm{irr}}$ is given by

$$a_L = \frac{1}{2} \cdot \sum_{H \supset L} a_H$$

- $B_H + 1$ is the number of $L \in \mathcal{L}^{n-2}_{\mathrm{irr}}$ with $L \subset H$

Sketch proof: the resolution

- Logarithmic resolution
$$X \xrightarrow{\pi} \mathbb{CP}^n$$
with $D = \pi^{-1}(\mathcal{H})$ a simple normal crossing divisor.
- X is the *minimal De Concini-Procesi wonderful model* of $\mathcal{H}$.
- The irreducible components of D are in bijective correspondence with non-empty and proper irreducible subspaces $L \in \mathcal{L}_{\mathrm{irr}}$
$$D = \bigcup_{L \in \mathcal{L}_{\mathrm{irr}}} D_L$$
where D_L is the unique irreducible component of D such that
$$\pi(D_L) = L\,.$$

Sketch proof: the parabolic bundle

Parabolic bundle $\mathcal{E}_*$ on (X, D) defined by:

- vector bundle $\mathcal{E} = \pi^*(T\mathbb{CP}^n)$
- weights a_L for $L \in \mathcal{L}_{\text{irr}}$ given by

$$a_L = (\operatorname{codim} L)^{-1} \sum_{H \supset L} a_H$$

- increasing filtrations of $\mathcal{E}|_{D_L}$ by vector subbundles

$$F_a^L = \begin{cases} \pi^*(TL) & \text{if } a < a_L, \\ \mathcal{E}|_{D_L} & \text{if } a \geq a_L, \end{cases}$$

Remark: klt implies $a_L \in (0, 1)$ and CY implies $\text{par-c}_1(\mathcal{E}_*) = 0$

Sketch proof: the stability theorem

- Fix positive integers b_L for L in $\mathcal{L}_{\mathrm{irr}}^{\circ} = \mathcal{L}_{\mathrm{irr}} \setminus \mathcal{H}$ such that

$$P_k = k \cdot \pi^*\big(\mathcal{O}_{\mathbb{P}^n}(1)\big) - \sum_{L \in \mathcal{L}_{\mathrm{irr}}^{\circ}} b_L \cdot D_L$$

  is an ample line bundle on X for all $k \gg 1$

- **Stability Theorem.** If $\mathcal{V} \subset \mathcal{E}$ is a non-zero and proper *saturated subsheaf*. Then

$$\text{par-c}_1(\mathcal{V}_*) \cdot c_1(P_k)^{n-1} < 0$$

  where $\mathcal{V}_*$ is the naturally induced parabolic structure on $\mathcal{V}$

Sketch proof: the Bogomolov-Gieseker inequality

- The Bogomolov-Gieseker inequality for stable parabolic bundles (proved by Takuro Mochizuki in 2006, *Astérisque*) asserts that

$$\text{par-ch}_2(\mathcal{E}_*) \cdot c_1(P_k)^{n-2} \leq 0$$

- The expression $\text{par-ch}_2(\mathcal{E}_*) \cdot c_1(P_k)^{n-2}$ defines a polynomial of degree $n-2$ in k that we write as $p(k)$
- Calculation of $\text{par-ch}_2(\mathcal{E}_*)$ and certain cup products in $H^*(X)$ show that

$$p(k) = Q(\mathbf{a}) \cdot k^{n-2} + O(k^{n-3})$$

- Since $p(k) < 0$ for $k \gg 1$, we must have $Q(\mathbf{a}) \leq 0$ $\square$

Stability Theorem and distributions on $\mathbb{CP}^n$

Key estimate

Suppose that $(\mathcal{H}, \mathbf{a})$ is a weighted arrangement that is klt and CY. Then there is $\delta > 0$ such that, for any distribution $\mathcal{V} \subset T\mathbb{CP}^n$ with index $\imath = c_1(\mathcal{V}) \geq 0$, we have

$$\sum_{H \,|\, H \pitchfork \mathcal{V}} a_H \geq \imath + \delta \,,$$

where the sum is over all $H \in \mathcal{H}$ that are transverse to $\mathcal{V}$.

Example. Let $M \subset \mathbb{CP}^n$ be a linear subspace with $\dim M = r - 1$ for some $1 \leq r \leq n-1$. The collection of all r-dimensional subspaces that contain M defines a distribution $\mathcal{V} \cong \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r}$ of index $\imath = r$.
A hyperplane is tangent to $\mathcal{V}$ if and only if $H \supset M$.

Application: lower bound on total sum of multiplicities

Theorem (dB-Panov, 2024)

Suppose that for all $L \in \mathcal{L}$ we have

$$m_L < \operatorname{codim} L \cdot \frac{N}{n+1} \quad \left(i.e.\ \mathbf{1} \in C^\circ \right)$$

Then

$$\sum_{L \in \mathcal{L}^{n-2}} m_L \geq \left(1 - \frac{2}{n+1}\right) N^2 + N \quad \left(i.e.\ Q(\mathbf{1}) \leq 0 \right)$$

Equality holds if and only if every $H \in \mathcal{H}$ intersects $\mathcal{H} \setminus \{H\}$ along

$$\left(1 - \frac{2}{n+1}\right) N + 1 \quad \left(i.e.\ \mathbf{1} \in \ker Q \right)$$

codimension 2 subspaces.

Hirzebruch arrangements

A hyperplane arrangement $\mathcal{H}$ is Hirzebruch if:

- for every subspace $L \in \mathcal{L}$ we have

$$\frac{m_L}{\operatorname{codim} L} \leq \frac{N}{n+1}$$

- every $H \in \mathcal{H}$ intersects the others along

$$\left(1 - \frac{2}{n+1}\right) N + 1$$

  codimension 2 subspaces.

Examples: (i) The $n+1$ coordinate hyperplanes in $\mathbb{CP}^n$.
(ii) If $\mathcal{H} \subset \mathbb{CP}^1$ and $|\mathcal{H}| \geq 2$ then $\mathcal{H}$ is Hirzebruch.
However, if $n \geq 2$ then the Hirzebruch condition is more rigid.

Complex reflection groups

- $G \subset U(n+1)$ irreducible complex reflection group
- $\mathcal{H}$ = reflecting hyperplanes

Theorem (dB-Panov, 2024)

$\mathcal{H}$ is Hirzebruch and its quadratic form Q is negative semidefinite

Proof: show that $\mathbf{1} \in C^\circ$ and $Q(\mathbf{1}) = 0$. Key identity:

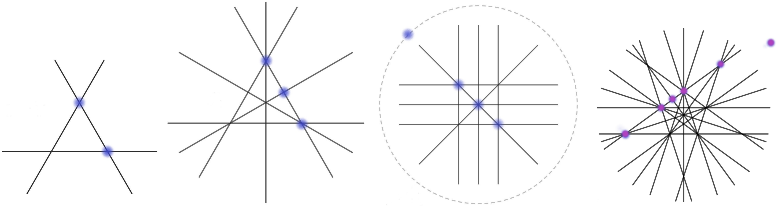
$$\sum_{H \in \mathcal{H}} |\langle v, n_H \rangle|^2 = \frac{N}{n+1} \cdot \|v\|^2$$

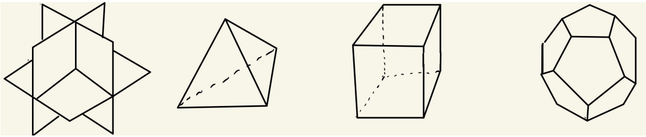
which follows from Schur's lemma. $\square$

Example: Take in $\mathbb{C}P^{n+1}$ the plane $\mathbb{C}P^n = \{x_1 + \ldots + x_{n+2} = 0\}$. The *braid arrangement* $A_{n+1} \subset \mathbb{C}P^n$ is given by planes $x_i - x_j = 0$ for $i \neq j \in \{1, \ldots, n+2\}$. A_{n+1} consists of $(n+1)(n+2)/2$ hyperplanes and induces A_n on each of them.

Hirzebruch line arrangements/Geometrization

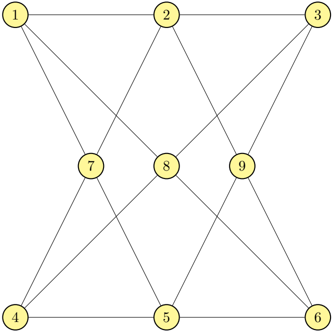
Hirzebruch's problem 1985. Consider an arrangement of N lines in $\mathbb{C}P^2$ such that each line intersects others in exactly $N/3+1$ points. Does such an arrangement consist of mirrors of a finite reflection group?





Panov (Geometry & Topology, 2018): there exist exactly four Hirzebruch line arrangements in $\mathbb{RP}^2$

Example: Non-Pappus matroid



$$\begin{pmatrix}
-5 & 1 & 1 & -2 & 1 & 1 & 1 & 1 & -2 \\
1 & -5 & 1 & 1 & -2 & 1 & 1 & -2 & 1 \\
1 & 1 & -5 & 1 & 1 & -2 & -2 & 1 & 1 \\
-2 & 1 & 1 & -5 & 1 & 1 & 1 & 1 & -2 \\
1 & -2 & 1 & 1 & -5 & 1 & 1 & -2 & 1 \\
1 & 1 & -2 & 1 & 1 & -5 & -2 & 1 & 1 \\
1 & 1 & -2 & 1 & 1 & -2 & -2 & -2 & -2 \\
1 & -2 & 1 & 1 & -2 & 1 & -2 & -2 & -2 \\
-2 & 1 & 1 & -2 & 1 & 1 & -2 & -2 & -2
\end{pmatrix}$$

The Hirzebruch quadratic form is ≤ 0 on the positive octant of $\mathbb{R}^9$.

Question: does Miyaoka-Yau inequality holds for oriented matroids?

Open problem: do symplectic 4-manifolds of ‘general type’ satisfy $c_1^2 \leq 3c_2$?

THANK YOU!