

Miyaoka-Yau inequality for hyperplane arrangements

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- Main result: $Q \leq 0$ on C
- Proof: Bogomolov-Gieseker inequality for stable parabolic bundles
- Application: complex reflection arrangements

Basic definitions

- This is a paper about complex projective space

$$\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{C}^*$$

- Let \mathcal{H} be a hyperplane arrangement in \mathbb{CP}^n
 \mathcal{H} is a finite set of pairwise distinct complex hyperplanes

$$H \subset \mathbb{CP}^n$$

- Let $L \subset \mathbb{CP}^n$ be a linear subspace obtained as intersection of hyperplanes in \mathcal{H} . The **multiplicity** of L is

$$m_L = |\{H \in \mathcal{H} \mid H \supset L\}|$$

Note: $m_L \geq \text{codim } L$

Codimension 2 subspaces

Let $L \subset \mathbb{CP}^n$ be a codimension 2 intersection of hyperplanes in \mathcal{H}

We say that

- L is **reducible** if its multiplicity is $m_L = 2$
- L is **irreducible** if its multiplicity $m_L \geq 3$

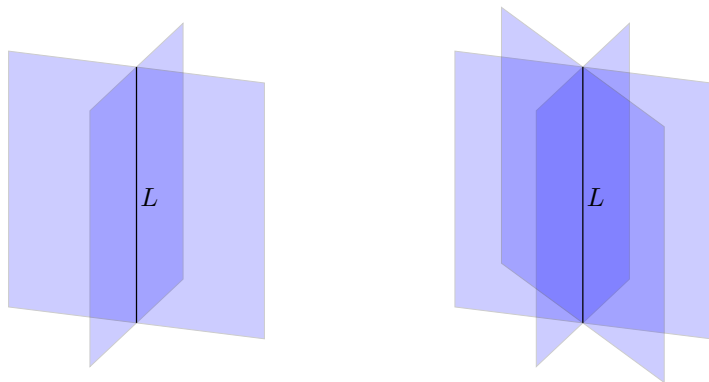


Figure 1: Reducible (left) and irreducible (right)

The Hirzebruch quadratic form

- $\mathcal{H} = \{H_1, \dots, H_N\}$ hyperplane arrangement in \mathbb{CP}^n
- σ_i = number of irreducible codimension 2 subspaces $L \subset H_i$
- The **Hirzebruch quadratic form** of \mathcal{H} is the homogeneous degree 2 polynomial on \mathbb{R}^N given by

$$Q(a_1, \dots, a_N) = \sum_{i,j=1}^N Q_{ij} a_i a_j$$

$$Q_{ij} = \begin{cases} -(n+1)\sigma_i + 2n & \text{if } i = j \\ -2 & \text{if } i \neq j \text{ and } L = H_i \cap H_j \text{ is reducible} \\ n-1 & \text{if } i \neq j \text{ and } L = H_i \cap H_j \text{ is irreducible} \end{cases}$$

Let σ_i be the number of points p with $r_p \geq 3$ lying on the i -th line of the given arrangement of k lines in the plane. We consider the $(k \times k)$ -symmetric matrix A with

$$(3) \quad A_{ij} = \begin{cases} 3\sigma_i - 4 & (i = j), \\ 2 & (i \neq j, \quad p \in L_i \cap L_j \text{ with } r_p = 2), \\ -1 & (i \neq j, \quad p \in L_i \cap L_j \text{ with } r_p \geq 3). \end{cases}$$

With the k lines we associate real variables x_i and let x be the column vector (x_1, \dots, x_k) . With the s points p_j with $r_{p_j} \geq 3$ we associate real variables y_j .

For each point p_j with $r_{p_j} \geq 3$ we consider the linear form

$$P_j(x, y) = 2y_j + \sum_{p_j \in L_i} x_i, \quad \text{where } y = (y_1, \dots, y_s).$$

Höfer's formula. For the algebraic surface Y (a good covering of S of degree d with respect to $L_1, \dots, L_k, E_1, \dots, E_s$ and the given branching numbers $n_1, \dots, n_k, m_1, \dots, m_s$) we have

$$(4) \quad (3c_2(Y) - c_1^2(Y))/d = \frac{1}{4} \left(x^t A x + \sum_{j=1}^s P_j(x, y)^2 \right),$$

where $x_i = 1 - \frac{1}{n_i}$ and $y_j = -1 - \frac{1}{m_j}$.

The matroid polytope

\mathcal{H} is an *essential* and *irreducible* hyperplane arrangement in \mathbb{CP}^n

- A **basis** of $\mathcal{H} = \{H_1, \dots, H_N\}$ is a subset $\mathcal{B} \subset \mathcal{H}$ consisting of $n + 1$ linearly independent hyperplanes
- The indicator vector of \mathcal{B} is the 1/0 vector

$$\mathbf{e}_{\mathcal{B}} = \sum_{i | H_i \in \mathcal{B}} \mathbf{e}_i$$

where $\mathbf{e}_1, \dots, \mathbf{e}_N$ are the standard basis vectors of \mathbb{R}^N

- The **matroid polytope** is the convex hull of the vectors $\mathbf{e}_{\mathcal{B}}$

$$P = \text{conv}\{\mathbf{e}_{\mathcal{B}} \mid \mathcal{B} \text{ is a basis of } \mathcal{H}\}$$

Note: P is contained in the $(N - 1)$ -simplex $\Delta \subset \mathbb{R}^N$ with

$$\Delta = \left\{ (a_1, \dots, a_N) \in \mathbb{R}^N \mid a_i \geq 0, \sum_i a_i = n + 1 \right\}$$

The semistable and stable cones

- The **semistable cone** is the cone over the matroid polytope

$$C = \mathbb{R}_{\geq 0} \cdot P = \text{cone} \{ \mathbf{e}_{\mathcal{B}} \mid \mathcal{B} \text{ is a basis of } \mathcal{H} \}$$

It is a convex polyhedral cone contained in the octant $(\mathbb{R}_{\geq 0})^N$

- The **stable cone** is the interior of $C \subset \mathbb{R}^N$

$$C^\circ = \text{int}(C)$$

- \mathcal{H} is essential and irreducible $\iff \dim P = N - 1$
 $\iff C^\circ$ is non empty

Defining linear inequalities of the stable cone

- Let \mathcal{L} be the finite set of *non-empty* and *proper* linear subspaces $L \subset \mathbb{CP}^n$ obtained by intersecting members of \mathcal{H}
- $(a_1, \dots, a_N) \in C^\circ \iff \forall i : a_i > 0 \text{ and } \forall L \in \mathcal{L} :$

$$\sum_{i \mid L \subset H_i} a_i < \frac{\text{codim } L}{n+1} \cdot \sum_{i=1}^N a_i$$

Relation to Geometric Invariant Theory:

- Standard embedding $\mathbb{CP}^n \subset \mathfrak{su}(n+1)^*$ as a coadjoint orbit
Let $p_i \in (\mathbb{CP}^n)^*$ be the annihilator of H_i
- $(a_1, \dots, a_N) \in C^\circ \iff \exists F \in SL(n+1, \mathbb{C})$ such that the centre of mass of the points $F(p_i)$ with weights a_i is $0 \in \mathfrak{su}(n+1)^*$
- If $n = 1$ then $\mathbb{CP}^1 = S^2$ and $\mathfrak{su}(2)^* = \mathbb{R}^3$

Main Result

Theorem (Miyaoka-Yau inequality, dB-Panov 2024)

The Hirzebruch quadratic form is non-positive on the semistable cone:

$$C \subset \{Q \leq 0\}$$

- If $n = 1$ then $Q \equiv 0$
- If $n = 2$ this follows from Panov's *Polyhedral Kähler Manifolds*, Geometry & Topology, 2009
- **Conjecture:** if $\mathbf{a} = (a_1, \dots, a_N) \in C^\circ$ is such that $Q(\mathbf{a}) = 0$ and $a_i \in (0, 1)$. Then there is a Kähler metric on \mathbb{CP}^n of constant holomorphic sectional curvature with cone angles $2\pi\alpha_i$ in transverse directions to the hyperplanes $H_i \in \mathcal{H}$, with $\alpha_i = 1 - a_i$
 - If \mathcal{H} is a complex reflection arrangement then the metrics have been constructed by Couwenberg-Heckman-Looijenga (IHÉS, 2005)

klt and CY arrangements

A weighted arrangement is a pair $(\mathcal{H}, \mathbf{a})$ consisting of:

- a hyperplane arrangement \mathcal{H} in \mathbb{CP}^n
- a weight vector $\mathbf{a} \in \mathbb{R}^{\mathcal{H}}$ with components $a_H > 0$

The weighted arrangement $(\mathcal{H}, \mathbf{a})$ is

- klt if

$$\forall L \in \mathcal{L} : \sum_{H \supset L} a_H < \text{codim } L$$

where \mathcal{L} is the set of non-empty and proper subspaces $L \subset \mathbb{CP}^n$ obtained by intersecting hyperplanes in \mathcal{H} . In particular,

$$0 < a_H < 1$$

- Calabi-Yau (CY) if

$$\sum_{H \in \mathcal{H}} a_H = n + 1$$

Restatement of the main theorem

Theorem (Miyaoka-Yau inequality, dB-Panov 2024)

Suppose that the weighted arrangement $(\mathcal{H}, \mathbf{a})$ is klt and CY. Then

$$Q(\mathbf{a}) = \sum_{L \in \mathcal{L}_{\text{irr}}^{n-2}} a_L^2 - \frac{1}{2} \sum_{H \in \mathcal{H}} B_H \cdot a_H^2 - \frac{n+1}{2} \leq 0$$

- $\mathcal{L}_{\text{irr}}^{n-2}$ is the set of irreducible codimension 2 subspaces $L \subset \mathbb{CP}^n$
- The weight a_L at $L \in \mathcal{L}_{\text{irr}}^{n-2}$ is given by

$$a_L = \frac{1}{2} \cdot \sum_{H \supset L} a_H$$

- $B_H + 1$ is the number of $L \in \mathcal{L}_{\text{irr}}^{n-2}$ with $L \subset H$

Sketch proof: the resolution

- Logarithmic resolution

$$X \xrightarrow{\pi} \mathbb{CP}^n$$

with $D = \pi^{-1}(\mathcal{H})$ a simple normal crossing divisor

- X is the *minimal De Concini-Procesi wonderful model* of \mathcal{H}
- The irreducible components of D are in bijective correspondence with non-empty and proper irreducible subspaces $L \in \mathcal{L}_{\text{irr}}$

$$D = \bigcup_{L \in \mathcal{L}_{\text{irr}}} D_L$$

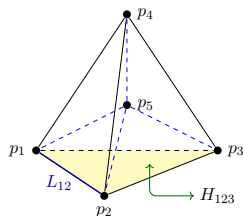
where D_L is the unique irreducible component of D such that

$$\pi(D_L) = L$$

Sketch proof: the resolution (example)

Let $p_1, \dots, p_5 \in \mathbb{CP}^3$ be five points in general linear position

$$\mathcal{H} = 10 \text{ planes } \overline{p_i p_j p_k}$$



- **Step 1:** $X_1 \xrightarrow{\sigma_1} \mathbb{CP}^3$ is the blowup at the five points p_i
- **Step 2:** $X \xrightarrow{\sigma_2} X_1$ is the blowup at the 10 (disjoint) proper transforms of the lines L_{ij}

$$\pi^{-1}(\mathcal{H}) = \underbrace{\left(\bigcup_H D_H \right)}_{10 \text{ divisors } \cong \text{Bl}_4 \mathbb{P}^2} \cup \underbrace{\left(\bigcup_L D_L \right)}_{10 \text{ divisors } \cong \mathbb{P}^1 \times \mathbb{P}^1} \cup \underbrace{\left(\bigcup_p D_p \right)}_{5 \text{ divisors } \cong \text{Bl}_4 \mathbb{P}^2}$$

Remark: $X = \overline{M_{0,6}}$

Sketch proof: the parabolic bundle

Parabolic bundle \mathcal{E}_* on (X, D) defined by:

- vector bundle $\mathcal{E} = \pi^*(T\mathbb{CP}^n)$
- weights a_L for $L \in \mathcal{L}_{\text{irr}}$ given by

$$a_L = (\text{codim } L)^{-1} \sum_{H \supset L} a_H$$

- increasing filtrations of $\mathcal{E}|_{D_L}$ by vector subbundles

$$F_a^L = \begin{cases} \pi^*(TL) & \text{if } a < a_L \\ \mathcal{E}|_{D_L} & \text{if } a \geq a_L \end{cases}$$

Remark: klt implies $a_L \in (0, 1)$ and CY implies $\text{par-c}_1(\mathcal{E}_*) = 0$

Sketch proof: the stability theorem

- Fix positive integers b_L for L in $\mathcal{L}_{\text{irr}}^\circ = \mathcal{L}_{\text{irr}} \setminus \mathcal{H}$ such that

$$P_k = k \cdot \pi^*(\mathcal{O}_{\mathbb{P}^n}(1)) - \sum_{L \in \mathcal{L}_{\text{irr}}^\circ} b_L \cdot D_L$$

is an ample line bundle on X for all $k \gg 1$

- **Stability Theorem.** If $\mathcal{V} \subset \mathcal{E}$ is a non-zero and proper *saturated subsheaf*. Then

$$\text{par-c}_1(\mathcal{V}_*) \cdot c_1(P_k)^{n-1} < 0$$

where \mathcal{V}_* is the naturally induced parabolic structure on \mathcal{V}

Sketch proof: the Bogomolov-Gieseker inequality

- The Bogomolov-Gieseker inequality for stable parabolic bundles (proved by Takuro Mochizuki in 2006, *Astérisque*) asserts that

$$\text{par-ch}_2(\mathcal{E}_*) \cdot c_1(P_k)^{n-2} \leq 0$$

- The expression $\text{par-ch}_2(\mathcal{E}_*) \cdot c_1(P_k)^{n-2}$ defines a polynomial of degree $n - 2$ in k that we write as $p(k)$
- Calculation of $\text{par-ch}_2(\mathcal{E}_*)$ and certain cup products in $H^*(X)$ show that

$$p(k) = Q(\mathbf{a}) \cdot k^{n-2} + O(k^{n-3})$$

- Since $p(k) < 0$ for $k \gg 1$, we must have $Q(\mathbf{a}) \leq 0$ \square

Stability Theorem and distributions on \mathbb{CP}^n

Key estimate

Suppose that $(\mathcal{H}, \mathbf{a})$ is a weighted arrangement that is klt and CY. Then there is $\delta > 0$ such that, for any distribution $\mathcal{V} \subset T\mathbb{CP}^n$ with index $\iota = c_1(\mathcal{V}) \geq 0$, we have

$$\sum_{H|H \pitchfork \mathcal{V}} a_H \geq \iota + \delta,$$

where the sum is over all $H \in \mathcal{H}$ that are transverse to \mathcal{V} .

Example. Let $M \subset \mathbb{CP}^n$ be a linear subspace with $\dim M = r - 1$ for some $1 \leq r \leq n - 1$. The collection of all r -dimensional subspaces that contain M defines a distribution $\mathcal{V} \cong \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r}$ of index $\iota = r$. A hyperplane is tangent to \mathcal{V} if and only if $H \supset M$.

Application: lower bound on total sum of multiplicities

Theorem (dB-Panov, 2024)

Suppose that for all $L \in \mathcal{L}$ we have

$$m_L < \operatorname{codim} L \cdot \frac{N}{n+1} \quad (\text{i.e. } \mathbf{1} \in C^\circ)$$

Then

$$\sum_{L \in \mathcal{L}^{n-2}} m_L \geq \left(1 - \frac{2}{n+1}\right) N^2 + N \quad (\text{i.e. } Q(\mathbf{1}) \leq 0)$$

Equality holds if and only if every $H \in \mathcal{H}$ intersects $\mathcal{H} \setminus \{H\}$ along

$$\left(1 - \frac{2}{n+1}\right) N + 1 \quad (\text{i.e. } \mathbf{1} \in \ker Q)$$

codimension 2 subspaces.

Hirzebruch arrangements

A hyperplane arrangement \mathcal{H} is Hirzebruch if:

- for every subspace $L \in \mathcal{L}$ we have

$$\frac{m_L}{\operatorname{codim} L} \leq \frac{N}{n+1}$$

- every $H \in \mathcal{H}$ intersects the others along

$$\left(1 - \frac{2}{n+1}\right) N + 1$$

codimension 2 subspaces.

Examples: (i) The $n+1$ coordinate hyperplanes in \mathbb{CP}^n .

(ii) If $\mathcal{H} \subset \mathbb{CP}^1$ and $|\mathcal{H}| \geq 2$ then \mathcal{H} is Hirzebruch.

However, if $n \geq 2$ then the Hirzebruch condition is more rigid.

Complex reflection groups

- $G \subset U(n+1)$ irreducible complex reflection group
- \mathcal{H} = reflecting hyperplanes

Theorem (dB-Panov, 2024)

\mathcal{H} is Hirzebruch and its quadratic form Q is negative semidefinite

Proof: show that $\mathbf{1} \in C^\circ$ and $Q(\mathbf{1}) = 0$. Key identity:

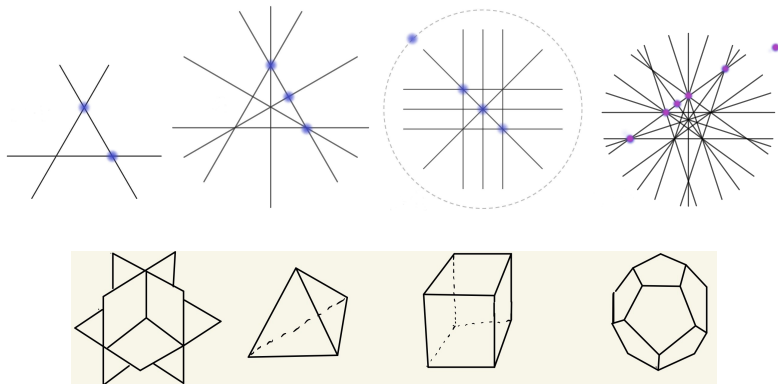
$$\sum_{H \in \mathcal{H}} |\langle v, n_H \rangle|^2 = \frac{N}{n+1} \cdot \|v\|^2$$

which follows from Schur's lemma. \square

Example: Take in $\mathbb{C}P^{n+1}$ the plane $\mathbb{C}P^n = \{x_1 + \dots + x_{n+2} = 0\}$. The *braid arrangement* $A_{n+1} \subset \mathbb{C}P^n$ is given by planes $x_i - x_j = 0$ for $i \neq j \in \{1, \dots, n+2\}$. A_{n+1} consists of $(n+1)(n+2)/2$ hyperplanes and induces A_n on each of them.

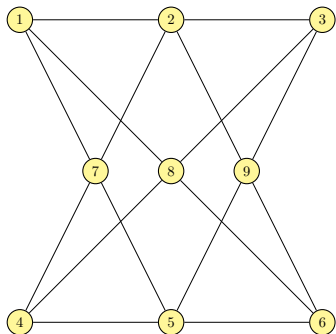
Hirzebruch line arrangements/Geometrization

Hirzebruch's problem 1985. Consider an arrangement of N lines in \mathbb{CP}^2 such that each line intersects others in exactly $N/3 + 1$ points. Does such an arrangement consist of mirrors of a finite reflection group?



Panov (Geometry & Topology, 2018): there exist exactly four Hirzebruch line arrangements in \mathbb{RP}^2

Example: Non-Pappus matroid



$$\begin{pmatrix} -5 & 1 & 1 & -2 & 1 & 1 & 1 & 1 & -2 \\ 1 & -5 & 1 & 1 & -2 & 1 & 1 & -2 & 1 \\ 1 & 1 & -5 & 1 & 1 & -2 & -2 & 1 & 1 \\ -2 & 1 & 1 & -5 & 1 & 1 & 1 & 1 & -2 \\ 1 & -2 & 1 & 1 & -5 & 1 & 1 & -2 & 1 \\ 1 & 1 & -2 & 1 & 1 & -5 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 & 1 & -2 & -2 & -2 & -2 \\ 1 & -2 & 1 & 1 & -2 & 1 & -2 & -2 & -2 \\ -2 & 1 & 1 & -2 & 1 & 1 & -2 & -2 & -2 \end{pmatrix}$$

The Hirzebruch quadratic form is ≤ 0 on the positive octant of \mathbb{R}^9 .

Question: does Miyaoka-Yau inequality holds for oriented matroids?

Open problem: do symplectic 4-manifolds of ‘general type’ satisfy $c_1^2 \leq 3c_2$?

THANK YOU!