Miyaoka-Yau inequality for hyperplane arrangements

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Plan

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- Main result:

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- Main result: $Q \leq 0$ on C

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- Proof:

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- Application: complex reflection arrangements

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$$H \subset \mathbb{CP}^n$$

- Let $L \subset \mathbb{CP}^n$ be a linear subspace obtained as intersection of hyperplanes in $\mathcal{H}$. The **multiplicity** of L is

$$m_L = \big|\{H \in \mathcal{H} \,|\, H \supset L\}\big|$$

Note: $m_L \geq \operatorname{codim} L$

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We say that

- L is **reducible** if its multiplicity is $m_L = 2$
- L is **irreducible** if its multiplicity is $m_L \geq 3$

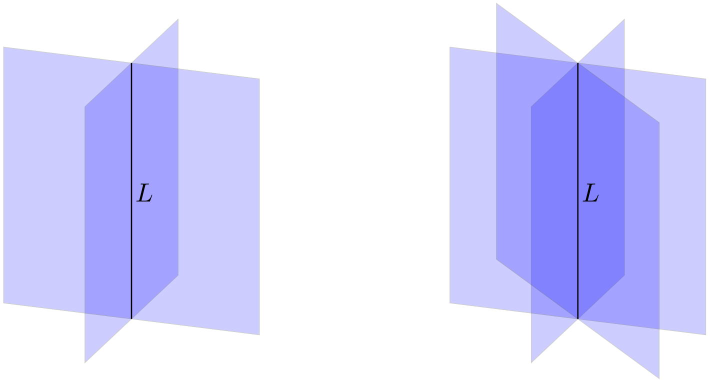

Figure 1: Reducible (left) and irreducible (right)

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- $\mathcal{H} = \{H_1, \ldots, H_N\}$ hyperplane arrangement in $\mathbb{CP}^n$
- $\sigma_i =$ number of irreducible codimension 2 subspaces $L \subset H_i$
- The **Hirzebruch quadratic form** of $\mathcal{H}$ is the homogeneous degree 2 polynomial on $\mathbb{R}^N$ given by

$$Q(a_1, \ldots, a_N) = \sum_{i,j=1}^{N} Q_{ij} a_i a_j$$

$$Q_{ij} = \begin{cases} -(n+1)\sigma_i + 2n & \text{if } i = j \\ -2 & \text{if } i \neq j \text{ and } L = H_i \cap H_j \text{ is reducible} \\ \ n-1 & \text{if } i \neq j \text{ and } L = H_i \cap H_j \text{ is irreducible} \end{cases}$$

Höfer's formula

Let σ_i be the number of points p with $r_p \geqslant 3$ lying on the i-th line of the given arrangement of k lines in the plane. We consider the $(k \times k)$-symmetric matrix A with

$$(3) \qquad A_{ij} = \begin{cases} 3\sigma_i - 4 & (i = j), \\ 2 & (i \neq j, \quad p \in L_i \cap L_j \text{ with } r_p = 2), \\ -1 & (i \neq j, \quad p \in L_i \cap L_j \text{ with } r_p \geqslant 3). \end{cases}$$

With the k lines we associate real variables x_i and let x be the column vector $(x_1, \ldots, x_k)$. With the s points p_j with $r_{p_j} \geqslant 3$ we associate real variables y_j. For each point p_j with $r_{p_j} \geqslant 3$ we consider the linear form

$$P_j(x, y) = 2y_j + \sum_{p_j \in L_i} x_i, \quad \text{where} \quad y = (y_1, \ldots, y_s).$$

Höfer's formula. *For the algebraic surface Y (a good covering of S of degree d with respect to $L_1, \ldots, L_k, E_1, \ldots, E_s$ and the given branching numbers $n_1, \ldots, n_k, m_1, \ldots, m_s$) we have*

$$(4) \qquad (3c_2(Y) - c_1^2(Y))/d = \frac{1}{4}\left(x^t A x + \sum_{j=1}^{s} P_j(x, y)^2\right),$$

where $x_i = 1 - \frac{1}{n_i}$ *and* $y_j = -1 - \frac{1}{m_j}$.

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$$\mathbf{e}_{\mathcal{B}} = \sum_{i \,|\, H_i \in \mathcal{B}} \mathbf{e}_i$$

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Note: P is contained in the $(N-1)$-simplex $\Delta \subset \mathbb{R}^N$ with

$$\Delta = \left\{\, (a_1, \ldots, a_N) \in \mathbb{R}^N \,|\, a_i \geq 0, \ \sum_i a_i = n+1 \,\right\}$$

The semistable and stable cones

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- The **semistable cone** is the cone over the matroid polytope

$$C = \mathbb{R}_{\geq 0} \cdot P = \text{cone}\,\{\, \mathbf{e}_{\mathcal{B}} \mid \mathcal{B} \text{ is a basis of } \mathcal{H} \,\}$$

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- $\mathcal{H}$ is essential and irreducible $\iff \dim P = N - 1$
$\iff C^\circ$ is non empty

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- If $n = 1$ then $\mathbb{CP}^1 = S^2$ and $\mathfrak{su}(2)^* = \mathbb{R}^3$

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    - If $\mathcal{H}$ is a complex reflection arrangement then the metrics have been constructed by Couwenberg-Heckman-Looijenga (IHÉS, 2005)

klt and CY arrangements

A weighted arrangement is a pair $(\mathcal{H}, \mathbf{a})$ consisting of:

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The weighted arrangement $(\mathcal{H}, \mathbf{a})$ is

- klt if

$$\forall L \in \mathcal{L} : \sum_{H \supset L} a_H < \operatorname{codim} L$$

where $\mathcal{L}$ is the set of non-empty and proper subspaces $L \subset \mathbb{CP}^n$ obtained by intersecting hyperplanes in $\mathcal{H}$. In particular,

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- Calabi-Yau (CY) if

$$\sum_{H \in \mathcal{H}} a_H = n + 1$$

Restatement of the main theorem

Theorem (Miyaoka-Yau inequality, dB-Panov 2024)

Suppose that the weighted arrangement $(\mathcal{H}, \mathbf{a})$ is klt and CY. Then

$$Q(\mathbf{a}) = \sum_{L \in \mathcal{L}^{n-2}_{\mathrm{irr}}} a_L^2 - \frac{1}{2} \sum_{H \in \mathcal{H}} B_H \cdot a_H^2 - \frac{n+1}{2} \leq 0$$

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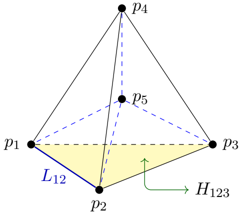
with $D = \pi^{-1}(\mathcal{H})$ a simple normal crossing divisor
- X is the *minimal De Concini-Procesi wonderful model* of $\mathcal{H}$
- The irreducible components of D are in bijective correspondence with non-empty and proper irreducible subspaces $L \in \mathcal{L}_{\mathrm{irr}}$
$$D = \bigcup_{L \in \mathcal{L}_{\mathrm{irr}}} D_L$$
where D_L is the unique irreducible component of D such that
$$\pi(D_L) = L$$

Sketch proof: the resolution (example)

Let $p_1, \ldots, p_5 \in \mathbb{CP}^3$ be five points in general linear position

$$\mathcal{H} \ = 10 \text{ planes } \overline{p_i p_j p_k}$$



- **Step 1:** $X_1 \xrightarrow{\sigma_1} \mathbb{CP}^3$ is the blowup at the five points p_i
- **Step 2:** $X \xrightarrow{\sigma_2} X_1$ is the blowup at the 10 (disjoint) proper transforms of the lines L_{ij}

$$\pi^{-1}(\mathcal{H}) = \underbrace{\left(\bigcup_H D_H\right)}_{10 \text{ divisors } \cong \mathrm{Bl}_4\, \mathbb{P}^2} \ \bigcup\ \underbrace{\left(\bigcup_L D_L\right)}_{10 \text{ divisors } \cong \mathbb{P}^1 \times \mathbb{P}^1} \ \bigcup\ \underbrace{\left(\bigcup_p D_p\right)}_{5 \text{ divisors } \cong \mathrm{Bl}_4\, \mathbb{P}^2}$$

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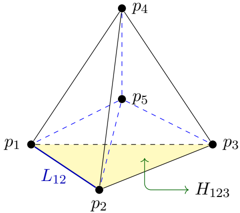

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Remark: $X = \overline{M_{0,6}}$

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- increasing filtrations of $\mathcal{E}|_{D_L}$ by vector subbundles

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Remark: klt implies $a_L \in (0,1)$ and CY implies

$$\boxed{\text{par-c}_1(\mathcal{E}_*) = 0} \in H^2(X, \mathbb{R})$$

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- **Stability Theorem.** If $\mathcal{V} \subset \mathcal{E}$ is a non-zero and proper *saturated subsheaf*. Then
$$\text{par-c}_1(\mathcal{V}_*) \cdot c_1(P_k)^{n-1} < 0$$
where $\mathcal{V}_*$ is the naturally induced parabolic structure on $\mathcal{V}$

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- Since $p(k) < 0$ for $k \gg 1$, we must have $Q(\mathbf{a}) \leq 0$ $\square$

Stability Theorem and distributions on $\mathbb{CP}^n$

Key estimate

Suppose that $(\mathcal{H}, \mathbf{a})$ is a weighted arrangement that is klt and CY. Then there is $\delta > 0$ such that, for any distribution $\mathcal{V} \subset T\mathbb{CP}^n$ with index $\imath = c_1(\mathcal{V}) \geq 0$, we have

$$\sum_{H \mid H \pitchfork \mathcal{V}} a_H \geq \imath + \delta \,,$$

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Example. Let $M \subset \mathbb{CP}^n$ be a linear subspace with $\dim M = r - 1$ for some $1 \leq r \leq n-1$. The collection of all r-dimensional subspaces that contain M defines a distribution $\mathcal{V} \cong \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r}$ of index $\imath = r$.
A hyperplane is tangent to $\mathcal{V}$ if and only if $H \supset M$.

Application: lower bound on total sum of multiplicities

Theorem (dB-Panov, 2024)

Suppose that for all $L \in \mathcal{L}$ we have

$$m_L < \operatorname{codim} L \cdot \frac{N}{n+1} \quad \left(i.e.\ \mathbf{1} \in C^\circ \right)$$

Then

$$\sum_{L \in \mathcal{L}^{n-2}} m_L \geq \left(1 - \frac{2}{n+1}\right) N^2 + N \quad \left(i.e.\ Q(\mathbf{1}) \leq 0 \right)$$

Equality holds if and only if every $H \in \mathcal{H}$ intersects $\mathcal{H} \setminus \{H\}$ along

$$\left(1 - \frac{2}{n+1}\right) N + 1 \quad \left(i.e.\ \mathbf{1} \in \ker Q \right)$$

codimension 2 subspaces.

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Examples: (i) The $n+1$ coordinate hyperplanes in $\mathbb{CP}^n$.
(ii) If $\mathcal{H} \subset \mathbb{CP}^1$ and $|\mathcal{H}| \geq 2$ then $\mathcal{H}$ is Hirzebruch.
However, if $n \geq 2$ then the Hirzebruch condition is more rigid.

Complex reflection groups

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Example: Take in $\mathbb{C}P^{n+1}$ the plane $\mathbb{C}P^n = \{x_1 + \ldots + x_{n+2} = 0\}$. The *braid arrangement* $A_{n+1} \subset \mathbb{C}P^n$ is given by planes $x_i - x_j = 0$ for $i \neq j \in \{1, \ldots, n+2\}$. A_{n+1} consists of $(n+1)(n+2)/2$ hyperplanes and induces A_n on each of them.

Hirzebruch line arrangements/Geometrization

Hirzebruch's problem 1985. Consider an arrangement of N lines in $\mathbb{C}P^2$ such that each line intersects others in exactly $N/3+1$ points. Does such an arrangement consist of mirrors of a finite reflection group?

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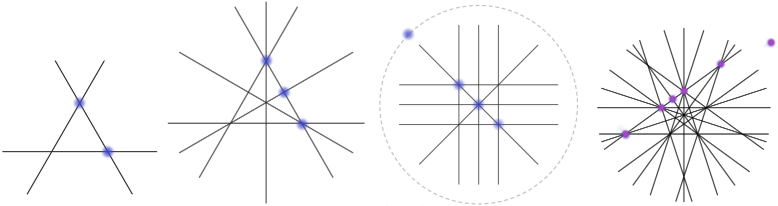

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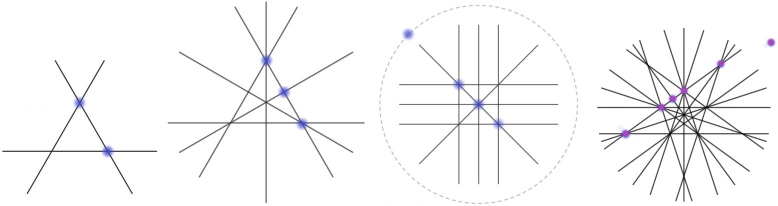

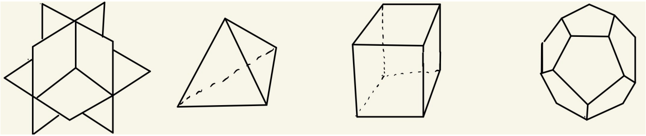

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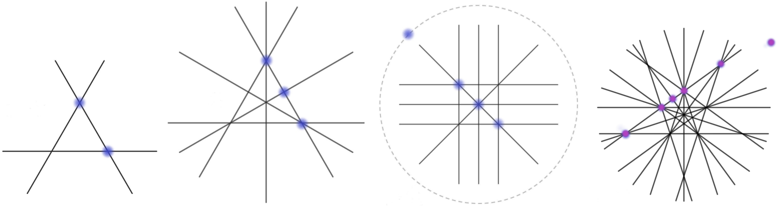
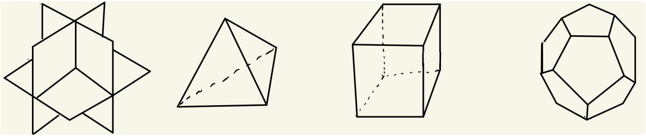
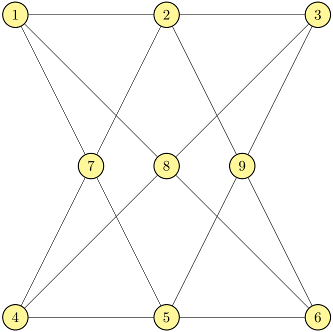




Panov (Geometry & Topology, 2018): there exist exactly four Hirzebruch line arrangements in $\mathbb{RP}^2$

Example: Non-Pappus matroid



$$\begin{pmatrix}
-5 & 1 & 1 & -2 & 1 & 1 & 1 & 1 & -2 \\
1 & -5 & 1 & 1 & -2 & 1 & 1 & -2 & 1 \\
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-2 & 1 & 1 & -5 & 1 & 1 & 1 & 1 & -2 \\
1 & -2 & 1 & 1 & -5 & 1 & 1 & -2 & 1 \\
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The Hirzebruch quadratic form is ≤ 0 on the positive octant of $\mathbb{R}^9$.

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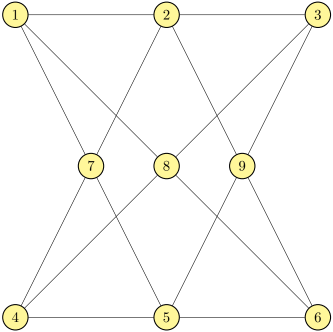

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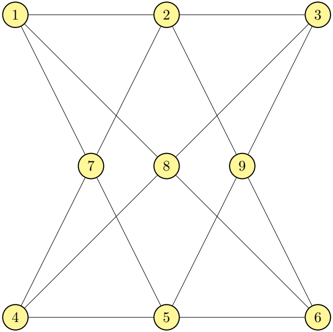

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THANK YOU!