Bubbling of Kähler-Einstein metrics with cone singularities: examples in dimensions 1 and 2

Martin de Borbon (joint with Cristiano Spotti)
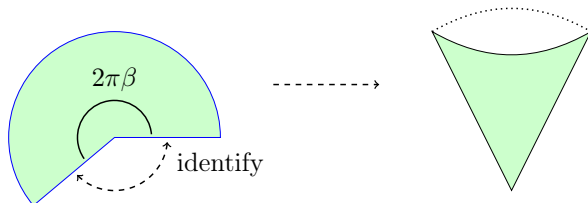
Seminar at Tsinghua University

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Outline

- $\dim_{\mathbb{C}} = 1$: Degeneration of polyhedral metrics on the 2-sphere

- $\dim_{\mathbb{C}} = 2$: Model Calabi-Yau metrics on $\mathbb{C}^2$ with cone singularities along a complex curve

- Higher dimensions, conjectures
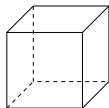
The 2-cone $C(2\pi\beta)$

$0 < \beta < 1$

$2\pi\beta$

identify

- In polar coordinates $g_\beta = dr^2 + \beta^2 r^2 d\theta^2$
- **Fact:** the induced complex structure on $\mathbb{R}^2 \setminus \{0\}$ is $\mathbb{C}^*$
- Proof: set $z = r^{1/\beta}e^{i\theta}$ then $g_\beta = |z|^{2\beta-2}|dz|^2$

Flat metrics on the 2-sphere with cone points

- Surface of a polyhedron in $\mathbb{R}^3$
- Double of a polygon in $\mathbb{R}^2$

Surface of a cube. Flat metric on S^2 with 8 cone points of total angle $2\pi\beta = 3(\pi/2)$.

Double of a triangle with interior angles $\pi\beta_i$. Flat metric on S^2 with 3 cone points of total angle $2\pi\beta_i$

$$\sum_{i=1}^{n}(1 - \beta_i) = 2$$

(Gauss-Bonnet)

- Surface of a cube: $8 \cdot (1 - 3/4) = 2$
- Double of a triangle: $3 - (\beta_1 + \beta_2 + \beta_3) = 2$

Existence and uniqueness

- Fix $\vec{\beta} = (\beta_1, \ldots, \beta_n)$ with $\sum_{i=1}^{n}(1 - \beta_i) = 2$
- $\mathrm{Met}(\vec{\beta})$ = flat metrics on S^2 with n cone points x_i of total angle $2\pi\beta_i$ modulo marked (orientation preserving) isometry and scale
- $\mathcal{M}_{0,n}$ = configuration of n distinct marked points in $\mathbb{CP}^1$ up to the action of Möbius transformations $PSL(2, \mathbb{C})$
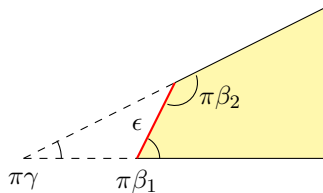
Theorem (Troyanov)

The forgetful map $F : \mathrm{Met}(\vec{\beta}) \to \mathcal{M}_{0,n}$ is a bijection

Proof: If $x_1, \ldots, x_n \in \mathbb{C}$ then

$$\left(\prod_{i=1}^{n} |z - x_i|^{2\beta_i - 2} \right) |dz|^2$$

extends smoothly over ∞ and defines a flat (Kähler) metric on $\mathbb{CP}^1$ with cone angles $2\pi\beta_i$ at the points x_i

Collision of two cone points
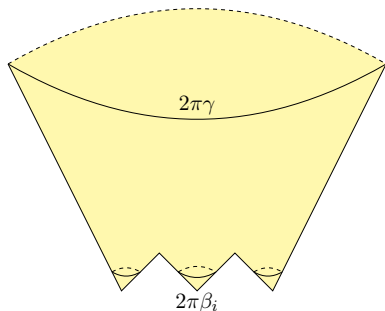

$$\gamma + (1 - \beta_1) + (1 - \beta_2) = 1$$

Model flat metrics on $\mathbb{C}$ with cone points

- $\displaystyle\sum_{i=1}^{p}(1-\beta_i) < 1$

$$g_F = \left(\prod_{i=1}^{p}|z - x_i|^{2\beta_i - 2}\right)|dz|^2$$

- flat Kähler metric on $\mathbb{C}$ with cone angles $2\pi\beta_i$ at x_i
- isometric to the 2-cone $C(2\pi\gamma)$ outside a compact set

$$1 - \gamma = \sum_{i=1}^{p}(1-\beta_i)$$

$2\pi\gamma$

$2\pi\beta_i$

$$\lim_{\lambda \to 0} \lambda \cdot g_F = C(2\pi\gamma)$$

Bubble trees

Consider a family of flat metrics g_t for which a cluster of cone points $x_1(t), \ldots, x_p(t)$ collides to 0 as $t \to 0$

- For $f \in \mathcal{O}_{\mathbb{C},0}$ let $\nu(f) = $ order of vanishing of f at 0
- For $r \geq 0$ we have an equivalence relation $f \sim_r g$ if $\nu(f - g) \geq r$
- If $S = \{x_1(t), \ldots, x_p(t)\} \subset \mathcal{O}_{\mathbb{C},0}$ is a finite set then the equivalence classes of $\sim_r$ make a tree $\mathcal{T}$ with root S and leaves $\{x_i(t)\}$
- To every non-leaf vertex $\mathbf{v} \in \mathcal{T}$ we associate a model flat metric $B_{\mathbf{v}}$ on $\mathbb{C}$. The number of cone points of $B_{\mathbf{v}}$ is equal to the number of children of $\mathbf{v}$.
- The cone angles are $2\pi\gamma_{\mathbf{w}}$ with $1 - \gamma_{\mathbf{w}} = \sum_{i | x_i \in \mathbf{w}} (1 - \beta_i)$ and the position of the cone points are $x_{\mathbf{w}} = \lim_{t \to 0} t^{-k} x_i(t)$ where k is the smallest integer such that the elements of $\mathbf{v}$ are *not* $\sim_k$ equivalent.

Rescaled limits

- Let $s \in \mathcal{O}_{\mathbb{C},0}$ be a section
- For $\alpha > 0$ let h_α be the pointed Gromov-Hausdorff limit

$$h_\alpha = \lim_{t \to 0}(|t|^{-2\alpha} \cdot g_t, s(t))$$

- The section determines a path in the tree with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_\ell$
- $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_\ell$ such that $h_\alpha = C(2\pi\gamma_{\mathbf{v}_i})$ (with base point its vertex) if $\alpha_{i-1} < \alpha < \alpha_i$ and $h_\alpha = B_{\mathbf{v}_i}$ if $\alpha = \alpha_i$ (with base point $\lim_{t \to 0} t^{-k} s(t)$).
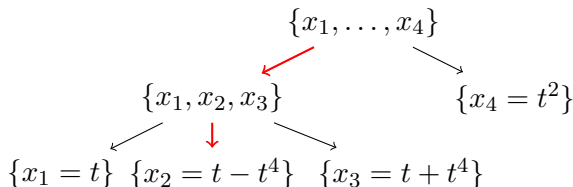
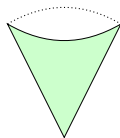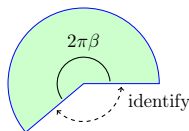$s(t) = t - t^4 + \text{(h.o.t.)}$ shown in red where $\mathcal{T}$
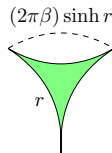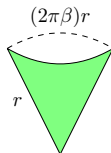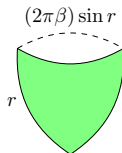
Cone singularities

Fix $\beta > 0$. **The 2-cone of total angle** $2\pi\beta$. In polar coordinates

$$dr^2 + \beta^2 r^2 d\theta^2$$

$$\begin{cases} dr^2 + \beta^2 \sin^2 r\, d\theta^2 \\ dr^2 + \beta^2 r^2 d\theta^2 \\ dr^2 + \beta^2 \sinh^2 r\, d\theta^2 \end{cases}$$
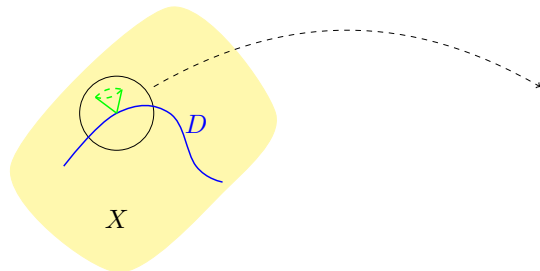
Kähler-Einstein metrics

$D \subset X$ *smooth* complex
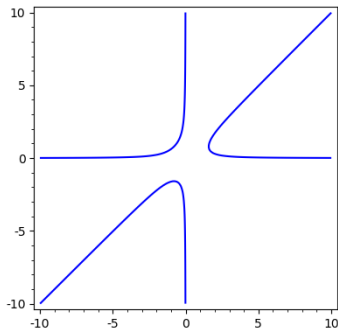hypersurface and $0 < \beta < 1$ $\longleftrightarrow$ $(X, (1-\beta)D)$

$\mathbb{C}_\beta \times \mathbb{C}^{n-1}$

$2\pi\beta$

$\operatorname{Ric}(g_{KE}) = \lambda g_{KE}$ on $X \setminus D$ and $g_{KE} \sim g_{\mathbb{C}_\beta \times \mathbb{C}^{n-1}}$ around D

- Donalson's Schauder estimate, existence and regularity theory, polyhomogeneous expansion, Chern-Weil, algebraic structure on non-collapsed GH limits, etc. Extensions to s.n.c. divisors.

Model Ricci-flat solutions

- $C = \{P = 0\} \subset \mathbb{C}^2$ smooth with $\deg(P) \geq 2$
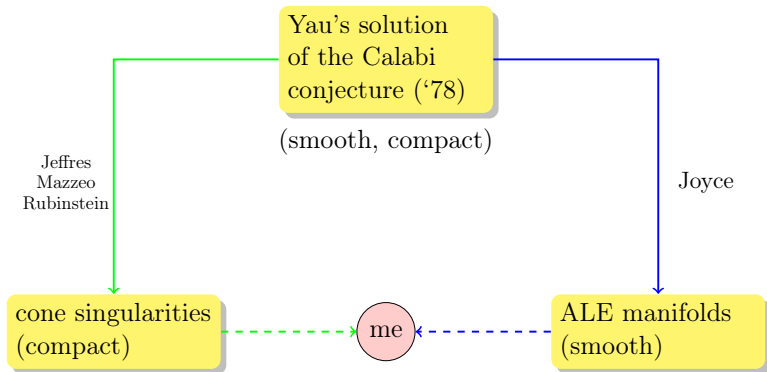- *Different* asymptotic lines i.e. *no* parabola

$$\frac{d-2}{d} < \beta < 1$$

Theorem (existence)

1. $\omega_{RF}^2 = \Omega \wedge \bar{\Omega}$ with $\Omega = P^{\beta-1} dz dw$
2. ω_{RF} *is asymptotic to a polyhedral Kähler cone at infinity.*

Monge-Ampère equation

continuity path: $(\omega_0 + i\partial\overline{\partial}\varphi_t)^2 = e^{tf}\omega_0^2$

Yau's solution
of the Calabi
conjecture ('78)

(smooth, compact)

Jeffres
Mazzeo
Rubinstein

Joyce

cone singularities
(compact)

me

ALE manifolds
(smooth)

$d = 2 \rightarrow S^1$-symmetry
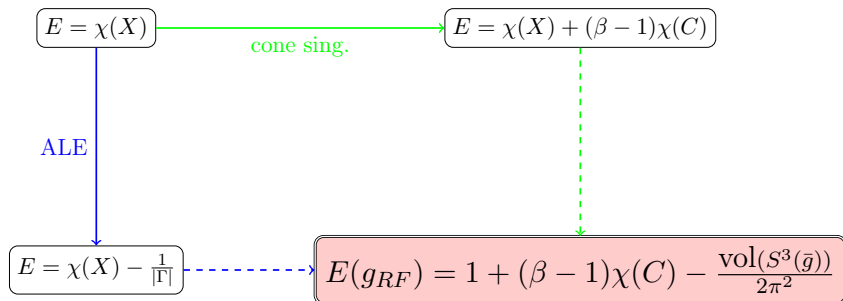
Gibbons-Hawking ansatz
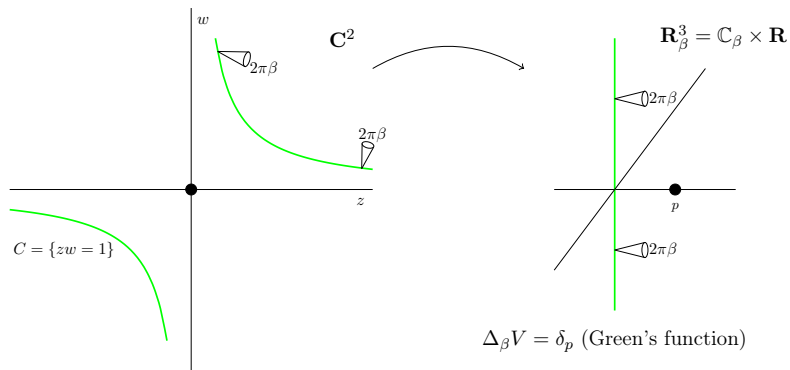
$$g_{RF} = V g_{\mathbf{R}_\beta^3} + (1/V)\alpha^2, \quad d\alpha = - \star_\beta \, dV$$

L^2-norm of the curvature

Chern-Weil: the energy of a KE metric depends only on topology.

$$E(g) := \frac{1}{8\pi^2} \int |\mathrm{Riem}(g)|^2$$

$E = \chi(X)$ $\xrightarrow{\text{cone sing.}}$ $E = \chi(X) + (\beta - 1)\chi(C)$

ALE

$E = \chi(X) - \frac{1}{|\Gamma|}$ $\dashrightarrow$ $E(g_{RF}) = 1 + (\beta - 1)\chi(C) - \frac{\mathrm{vol}(S^3(\bar{g}))}{2\pi^2}$

Blow-up limits

Non-collapsed limit space of a KE sequence

Gromov-Hausdorff

$\mathbb{C}^2$

scaling
(along the seq.)

KLT pairs
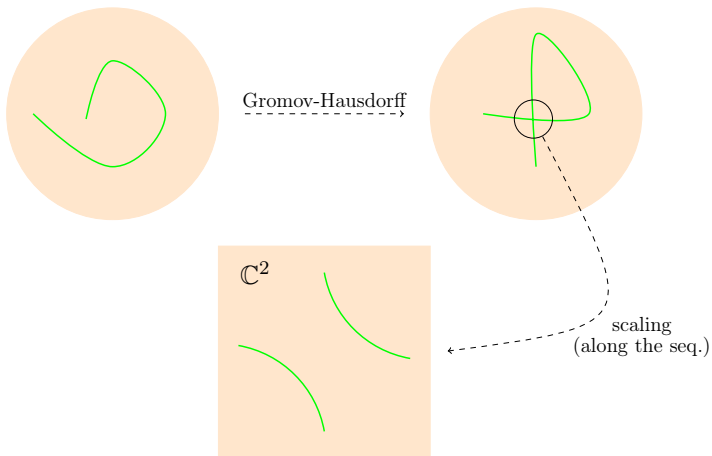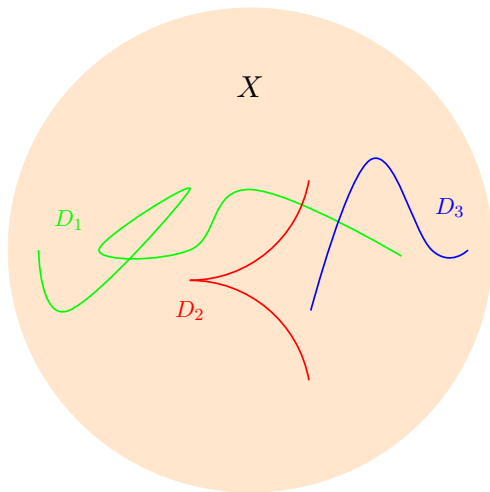
$$\left(X, \sum_i (1 - \beta_i) D_i\right)$$

$$D_i = \{s_i = 0\}$$

$$\prod_i |s_i|^{2\beta_i - 2} \in L^1_{loc}$$

Tangent cones (overview)

Non-collapsed polarized KE limit space (X, d_{KE})

$$C_p X := \lim_{\lambda \to 0}(X, \lambda^{-1}d_{KE}, p)$$

- Uniqueness, $C_p X$ is an affine algebraic variety (Donaldson-Sun).
  Order of vanishing w.r.t. $d_{KE} \implies$ two steps degeneration.
- $C_p X$ depends only on $\mathcal{O}_{X,p}$ (the germ of X at p).
  Chi Li's normalized volumes of valuations.

Sasaki geometry precedents: Martelli-Sparks-Yau (volume minimization), Collins-Székelyhidi (K-stability) .

Question
Identify tangent cones of weak KE metrics on KLT (surface) pairs.

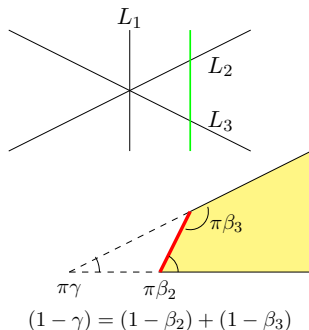# Stable case (joint with C. Spotti)
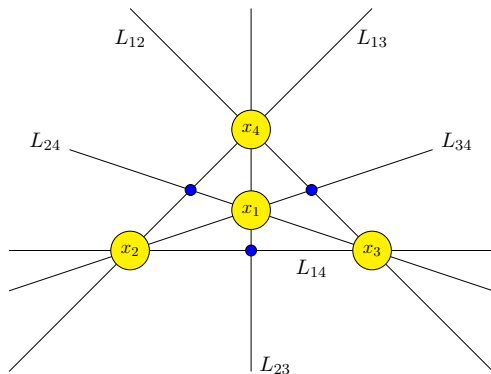
$(\mathbf{CP}^2, \sum_k \mu_k L_k)$ KLT line arrangement, here $\mu_k = 1 - \beta_k$. Assume that

- Calabi-Yau condition: $\sum_k \mu_k = 3$;
- Troyanov holds at points with multiplicity ≥ 3.

$\implies$ tangent cones at multiple points are the obvious PK cones.

Strictly unstable case (joint with G. Edwards)

L_1

L_2

L_3

$\pi\beta_3$

$\pi\gamma$ $\pi\beta_2$

$(1-\gamma) = (1-\beta_2) + (1-\beta_3)$

Let $L_i = \{\ell_i = 0\}$ and $0 < \beta_1 < \beta_2 \le \beta_3 < 1$. Assume that

$$(1-\beta_2) + (1-\beta_3) < (1-\beta_1)$$

Then

- $\omega_{RF}^2 = \Omega \wedge \bar{\Omega}$
- $C_0 \omega_{RF} = \mathbb{C}_{\beta_1} \times \mathbb{C}_\gamma$

where $\Omega = \ell_1^{\beta_1-1} \ell_2^{\beta_2-1} \ell_3^{\beta_3-1} dz dw$

My papers

1. *Calabi-Yau metrics with cone singularities along intersecting complex lines* (with G. Edwards); forthcoming.

2. *Toric Sasaki-Einstein metrics with conical singularities* (with E. Legendre); arXiv, 2020.

3. *Schauder estimates on products of cones* (with G. Edwards); arXiv, 2020.

4. *ALE Calabi-Yau metrics with conical singularities along compact divisors* (with C. Spotti); International Mathematics Research Notices, 2019.

5. *Local models for Kähler-Einstein metrics* (with C. Spotti); Proceedings of the AMS, 2018.

6. *Calabi-Yau metrics with cone singularities along line arrangements* (with C. Spotti); arXiv, 2017.

7. *Asymptotically conical Ricci-flat Kähler metrics on $\mathbb{C}^2$ with cone singularities along a complex curve*; Journal of the London Mathematical Society, 2017.

8. *The Gibbons-Hawking ansatz over a wedge*; Journal of Geometry and Physics, 2017.

9. *Kähler metrics with cone singularities along a divisor of bounded Ricci curvature*; Annals of Global Analysis and Geometry, 2017.

10. *Singularities of plane complex curves and limits of Kähler metrics with cone singularities. I: Tangent cones*; Complex Manifolds, 2017.