Bubbling of Kähler-Einstein metrics with cone singularities: examples in dimensions 1 and 2

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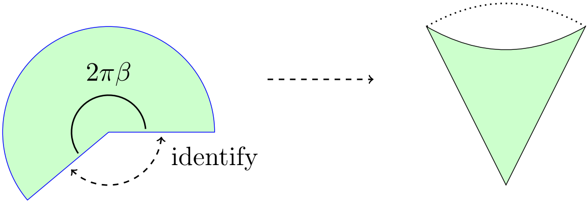
Outline

- $\dim_{\mathbb{C}} = 1$: Degeneration of polyhedral metrics on the 2-sphere
- $\dim_{\mathbb{C}} = 2$: Model Calabi-Yau metrics on $\mathbb{C}^2$ with cone singularities along a complex curve
- Conjectures

The 2-cone $C(2\pi\beta)$

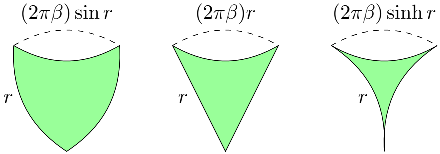
$0 < \beta < 1$

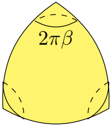


- In polar coordinates $g_\beta = dr^2 + \beta^2 r^2 d\theta^2$
- **Fact:** the induced complex structure on $\mathbb{R}^2 \setminus \{0\}$ is $\mathbb{C}^*$
- Proof: set $z = r^{1/\beta} e^{i\theta}$ then $g_\beta = |z|^{2\beta-2}|dz|^2$

Constant curvature surfaces with cone singularities

$$\begin{cases} dr^2 + \beta^2 \sin^2 r d\theta^2 \\ dr^2 + \beta^2 r^2 d\theta^2 \\ dr^2 + \beta^2 \sinh^2 r d\theta^2 \end{cases}$$



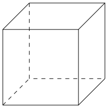


Double of spherical triangle. Constant curvature 1 metric on the 2-sphere with 3 cone points.

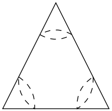
- Orbifolds: $\beta = 1/k$ for $k \geq 2$ integer
- Modular curve $\mathbb{H}/PSL(2,\mathbb{Z})$. Hyperbolic metric with two cone points $1/2, 1/3$

Flat metrics on the 2-sphere with cone points

- Surface of a polyhedron in $\mathbb{R}^3$
- Double of a polygon in $\mathbb{R}^2$



Surface of a cube. Flat metric on S^2 with 8 cone points of total angle $2\pi\beta = 3(\pi/2)$.



Double of a triangle with interior angles $\pi\beta_i$. Flat metric on S^2 with 3 cone points of total angle $2\pi\beta_i$

$$\boxed{\sum_{i=1}^{n}(1-\beta_i) = 2} \qquad \text{(Gauss-Bonnet)}$$

- Surface of a cube: $8 \cdot (1 - 3/4) = 2$
- Double of a triangle: $3 - (\beta_1 + \beta_2 + \beta_3) = 2$

Existence and uniqueness

- Fix $\vec{\beta} = (\beta_1, \dots, \beta_n)$ with $\sum_{i=1}^n (1 - \beta_i) = 2$
- $\mathrm{Met}(\vec{\beta})$ = flat metrics on S^2 with n cone points x_i of total angle $2\pi\beta_i$ modulo marked (orientation preserving) isometry and scale
- $\mathcal{M}_{0,n}$ = configuration of n distinct marked points in $\mathbb{CP}^1$ up to the action of Möbius transformations $PSL(2, \mathbb{C})$

Theorem (Troyanov)

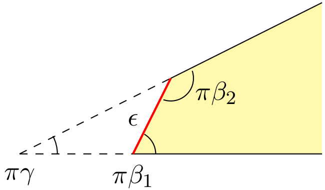
The forgetful map $F : \mathrm{Met}(\vec{\beta}) \to \mathcal{M}_{0,n}$ is a bijection

Proof: If $x_1, \dots, x_n \in \mathbb{C}$ then

$$\left(\prod_{i=1}^n |z - x_i|^{2\beta_i - 2} \right) |dz|^2$$

extends smoothly over ∞ and defines a flat (Kähler) metric on $\mathbb{CP}^1$ with cone angles $2\pi\beta_i$ at the points x_i

Collision of two cone points



$$\gamma + (1 - \beta_1) + (1 - \beta_2) = 1$$

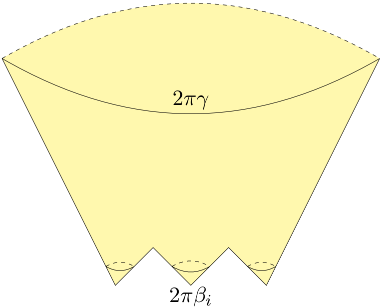
Model flat metrics on $\mathbb{C}$ with cone points

- $\displaystyle\sum_{i=1}^{p}(1-\beta_i) < 1$

$$g_F = \left(\prod_{i=1}^{p} |z - x_i|^{2\beta_i - 2}\right) |dz|^2$$

- flat Kähler metric on $\mathbb{C}$ with cone angles $2\pi\beta_i$ at x_i
- isometric to the 2-cone $C(2\pi\gamma)$ outside a compact set

$$1 - \gamma = \sum_{i=1}^{p}(1-\beta_i)$$



$$\lim_{\lambda\to 0} \lambda \cdot g_F = C(2\pi\gamma)$$

Bubble trees

Consider a family of flat metrics g_t for which a cluster of cone points $x_1(t), \ldots, x_p(t)$ collides to 0 as $t \to 0$

- For $f \in \mathcal{O}_{\mathbb{C},0}$ let $\nu(f) =$ order of vanishing of f at 0
- For $r \geq 0$ we have an equivalence relation $f \sim_r g$ if $\nu(f-g) \geq r$
- If $S = \{x_1(t), \ldots, x_p(t)\} \subset \mathcal{O}_{\mathbb{C},0}$ is a finite set then the equivalence classes of $\sim_r$ make a tree $\mathcal{T}$ with root S and leaves $\{x_i(t)\}$
- To every non-leaf vertex $\mathbf{v} \in \mathcal{T}$ we associate a model flat metric $B_{\mathbf{v}}$ on $\mathbb{C}$. The number of cone points of $B_{\mathbf{v}}$ is equal to the number of children of $\mathbf{v}$.
- The cone angles are $2\pi\gamma_{\mathbf{w}}$ with $1-\gamma_{\mathbf{w}} = \sum_{i | x_i \in \mathbf{w}} (1-\beta_i)$ and the position of the cone points are $x_{\mathbf{w}} = \lim_{t\to 0} t^{-k} x_i(t)$ where k is the smallest integer such that the elements of $\mathbf{v}$ are *not* $\sim_k$ equivalent.

Rescaled limits

- Let $s \in \mathcal{O}_{\mathbb{C},0}$ be a section
- For $\alpha > 0$ let h_α be the pointed Gromov-Hausdorff limit

$$h_\alpha = \lim_{t \to 0} (|t|^{-2\alpha} \cdot g_t, s(t))$$

- The section determines a path in the tree with vertices $\mathbf{v}_1, \dots, \mathbf{v}_\ell$
- $0 = \alpha_0 < \alpha_1 < \dots < \alpha_\ell$ such that $h_\alpha = C(2\pi\gamma_{\mathbf{v}_i})$ (with base point its vertex) if $\alpha_{i-1} < \alpha < \alpha_i$ and $h_\alpha = B_{\mathbf{v}_i}$ if $\alpha = \alpha_i$ (with base point $\lim_{t \to 0} t^{-k} s(t)$).

$$\begin{tikzcd}
 & & \{x_1, \dots, x_4\} \arrow[dl] \arrow[dr] & \\
 & \{x_1, x_2, x_3\} \arrow[dl] \arrow[d] \arrow[dr] & & \{x_4 = t^2\} \\
\{x_1 = t\} & \{x_2 = t - t^4\} & \{x_3 = t + t^4\} &
\end{tikzcd}$$

$s(t) = t - t^4 + \text{(h.o.t.)}$ shown in red where $\mathcal{T}$

Moduli spaces

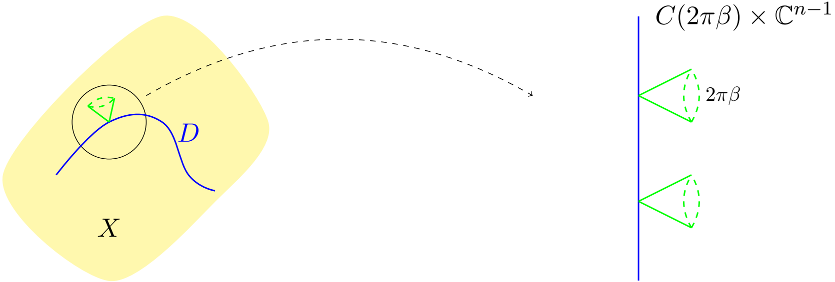
- Thurston: complex hyperbolic cone metric on $\overline{\mathrm{Met}}(\vec{\beta})$, divisors represent collision of pair of points

- Deligne-Mumford-Knudsen compactification $\overline{\mathcal{M}}_{0,n}$, divisors correspond to partitions of $\{1, \dots, n\}$ into two disjoint sets, its points represent nodal curves with 2 irreducible components

- Logarithmic resolution $\overline{\mathcal{M}}_{0,n} \to \overline{\mathrm{Met}}(\vec{\beta})$ (Koziarz-Nguyen), points in $\overline{\mathcal{M}}_{0,n}$ represent bubble trees

Kähler-Einstein metrics with cone singularities

$D \subset X$ *smooth* complex hypersurface and $0 < \beta < 1$ $\longleftrightarrow$ $(X, (1-\beta)D)$

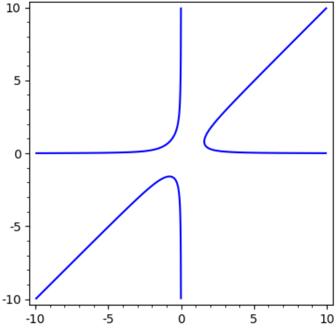


- $\mathrm{Ric}(g_{KE}) = \lambda \cdot g_{KE}$ on $X \setminus D$
- $g_{KE} \sim C(2\pi\beta) \times \mathbb{C}^{n-1}$ near D

Theory overview

- Donalson's interior Schauder estimate for the Laplace operator
- Existence and regularity theory, polyhomogeneous expansion (Jeffress-Mazzeo-Rubinstein)
- Chern-Weil formulas (Jian Song and Xiaowei Wang)
- Algebraic structure on non-collapsed Gromov-Hausdorff limits (Chen-Donaldson-Sun)

Model Ricci-flat solutions



- $C = \{P = 0\} \subset \mathbb{C}^2$ smooth with $\deg(P) \geq 2$
- *Different* asymptotic lines i.e. *no* parabola

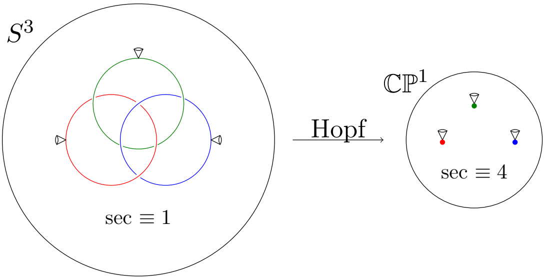
$$\frac{d-2}{d} < \beta < 1$$

Theorem (de Borbon, 2017)

1. $\omega_{RF}^2 = \Omega \wedge \bar{\Omega}$ *with* $\Omega = P^{\beta-1} dzdw$
2. ω_{RF} *is asymptotic to a polyhedral Kähler cone at infinity.*

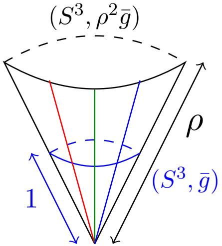
Polyhedral Kähler cones

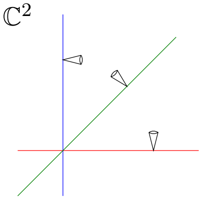
$\bar{g}$ = constant curvature 1 metric on the 3-sphere



Cone:

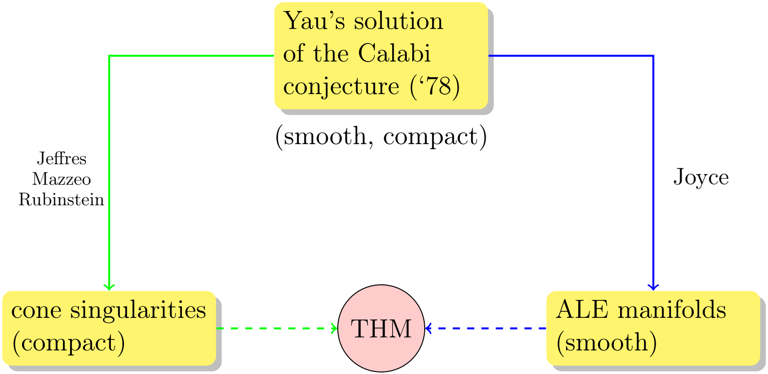
$$d\rho^2 + \rho^2 \bar{g}$$





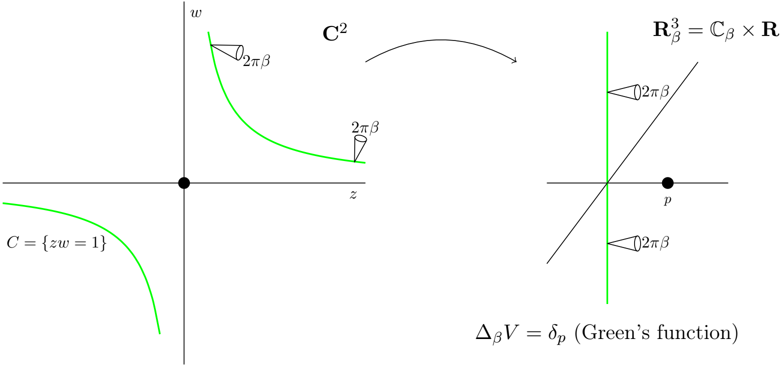
Proof of Theorem

continuity path: $(\omega_0 + i\partial\overline{\partial}\varphi_t)^2 = e^{tf}\omega_0^2$



$d = 2 \to S^1$-symmetry

Gibbons-Hawking ansatz



$\Delta_\beta V = \delta_p$ (Green's function)

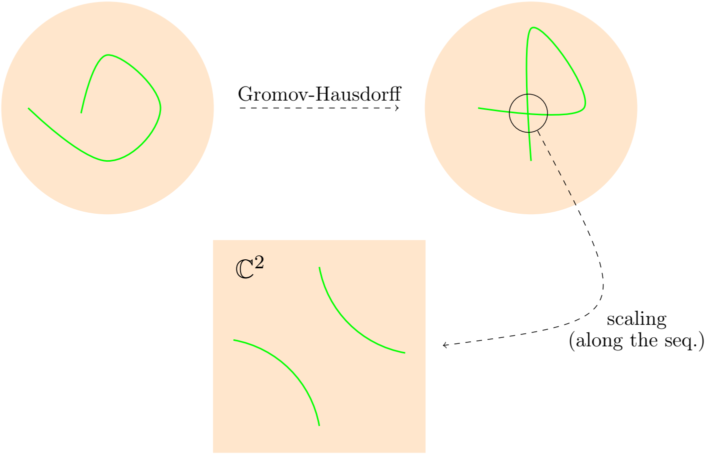
$$g_{RF} = V g_{\mathbf{R}^3_\beta} + (1/V)\alpha^2, \quad d\alpha = - \star_\beta dV$$

Chern-Weil: the energy of a KE metric depends only on topology.

$$E(g) := \frac{1}{8\pi^2}\int |\mathrm{Riem}(g)|^2$$

$$\begin{tikzcd}
\boxed{E = \chi(X)} \arrow[rr, "\text{cone sing.}"'] \arrow[d, "\text{ALE}"'] & & \boxed{E = \chi(X) + (\beta - 1)\chi(C)} \arrow[d, dashed] \\
\boxed{E = \chi(X) - \frac{1}{|\Gamma|}} \arrow[rr, dashed] & & \boxed{E(g_{RF}) = 1 + (\beta - 1)\chi(C) - \frac{\mathrm{vol}(S^3(\bar{g}))}{2\pi^2}}
\end{tikzcd}$$

Blow-up limits



$C_\epsilon = \{P_d(z,w) + (\mathrm{h.o.t.}) = \epsilon\}$ as $\epsilon \to 0$. If we rescale coordinates by $z = \epsilon^{1/d}\tilde{z}$, $w = \epsilon^{1/d}\tilde{w}$ then C_ϵ converges to $C = \{P_d(\tilde{z},\tilde{w}) = 1\}$ as $\epsilon \to 0$.

Multiple asymptotic lines

Conjecture:

- For every $1/2 < \beta < 1$ there is a Calabi-Yau metric on $\mathbb{C}^2$ with cone angle $2\pi\beta$ along the parabola $\{w = z^2\}$
- The tangent cone at infinity equal to $\mathbb{C}_\gamma \times \mathbb{C}$, where $\gamma = 2\beta - 1$
- The energy of the metric is finite and given by $E = 1 - \beta$

Moduli spaces of pairs

- Ascher-DeVleming-Liu: K-moduli space of pairs $(\mathbb{CP}^2, (1-\beta)C)$ with $\deg C = 4$ is the GIT quotient for $5/8 < \beta < 1$
- If $\gamma = 2\beta - 1$ then $1/4 < \gamma < 1$. By Li-Sun there is a KE metric with cone angle $2\pi\gamma$ along $C_0 = \{Q = 0\}$ where $Q = X_0^2 + X_1^2 + X_2^2$.
- Let P be a generic homogeneous degree 4 and $C_\epsilon = \{\epsilon P + Q^2 = 0\}$. There are eight branch points $\{P = 0\} \cap \{Q = 0\}$
- Family of KE metrics g_ϵ with cone angle $2\pi\beta$ along C_ϵ and a limiting KE metric g_0 with cone angle $2\pi\gamma$ along C_0
- The convergence $g_\epsilon \to g_0$ is modelled in directions transverse to C_0 by two cone points of angle $2\pi\beta$ which collide into a single cone point of angle $2\pi\gamma$
- It is expected that the parabola CY metric is a re-scaled limit of the sequence g_ϵ at each of the eight branch points
- The Song-Wang energy formula $E = \chi(X) + (\beta - 1)\chi(D)$ gives $E(g_\epsilon) - E(g_0) = 8(1-\beta)$

A general theory

Non-collapsed polarized KE limit space (X, d_{KE})

$$C_p X := \lim_{\lambda \to 0} (X, \lambda^{-1} d_{KE}, p)$$

- Uniqueness, $C_p X$ is an affine algebraic variety (Donaldson-Sun). Order of vanishing w.r.t. $d_{KE} \implies$ two steps degeneration.
- $C_p X$ depends only on $\mathcal{O}_{X,p}$ (the germ of X at p). Chi Li's normalized volumes of valuations.

Sasaki geometry precedents: Martelli-Sparks-Yau (volume minimization), Collins-Székelyhidi (K-stability) .

Question

Describe all possible rescaled limits of a family.

Song Sun's *minimal bubbles*.

THANK YOU!