

Algorithm Analysis and Design

Complete Reference Guide

Comprehensive Coverage of All Topics

Md. Sadman Sakib

August 5, 2025

Contents

1	Complexity Analysis and Recurrence Relations	3
1.1	Asymptotic Notations	3
1.1.1	Big-O Notation (O)	3
1.1.2	Big-Omega Notation (Omega)	3
1.1.3	Big-Theta Notation (Theta)	3
1.1.4	Little-o Notation (o)	3
1.1.5	Little-omega Notation (omega)	3
1.2	Growth of Functions	3
1.3	Methods to Solve Recurrence Relations	3
1.3.1	Substitution Method	3
1.3.2	Recursion Tree Method	4
1.3.3	Master Method	4
2	Graph Traversal Algorithms	4
2.1	Breadth First Search (BFS)	4
2.2	Depth First Search (DFS)	6
2.3	Topological Sort	6
2.4	Strongly Connected Components (Kosaraju's Algorithm)	6
2.5	Euler Path and Circuit	6
2.6	Articulation Points (Cut Vertices)	7
2.7	Bridge Detection	7
2.8	Bi-connected Components	7
3	Shortest Path Algorithms	7
3.1	Dijkstra's Algorithm	7
3.2	Bellman-Ford Algorithm	7
3.3	Floyd-Warshall Algorithm	7
3.4	Shortest Path in DAG	9
4	Divide & Conquer Algorithms	9
4.1	Counting Inversions using Merge Sort	9
4.2	Closest Pair of Points	9
4.3	Computing A to the power k mod M using Divide and Conquer	9

4.4	Finding k-th Smallest Element (Quickselect)	11
5	Greedy Algorithms	11
5.1	Elements and Properties	11
5.2	Fractional Knapsack	11
5.3	Job Scheduling with Deadline	12
5.4	Minimum Spanning Tree	12
5.4.1	Prim's Algorithm	12
5.4.2	Kruskal's Algorithm	12
6	Dynamic Programming	12
6.1	Basic Principles	12
6.2	Memoization vs Tabulation	13
6.3	Coin Change Problems	13
6.3.1	Minimum Coins	13
6.3.2	Coin Change Ways	13
6.4	Longest Increasing Subsequence (LIS)	14
6.5	Longest Common Subsequence (LCS)	14
6.6	0/1 Knapsack	15
6.7	Matrix Chain Multiplication	15
6.8	Applications of Dynamic Programming	15
7	Network Flow	16
7.1	Flow Networks	16
7.2	Max-Flow Min-Cut Theorem	16
7.3	Ford-Fulkerson Method	16
7.4	Edmonds-Karp Algorithm	17
7.5	Maximum Bipartite Matching	17
7.6	Minimum Path Cover	18
7.7	Edge Cover	18
8	Advanced Topics and Complexity Results	19
8.1	NP-Completeness	19
8.2	Approximation Algorithms	19
8.3	Parameterized Complexity	20
9	Summary of Time Complexities	20
10	Important Recurrence Relations	20

1 Complexity Analysis and Recurrence Relations

1.1 Asymptotic Notations

Asymptotic notations describe the behavior of functions as input size approaches infinity.

1.1.1 Big-O Notation (O)

$f(n) = O(g(n))$ if $\exists c > 0, n_0 > 0$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$

Definition: Upper bound - worst case complexity.

1.1.2 Big-Omega Notation (Omega)

$f(n) = \Omega(g(n))$ if $\exists c > 0, n_0 > 0$ such that $f(n) \geq c \cdot g(n)$ for all $n \geq n_0$

Definition: Lower bound - best case complexity.

1.1.3 Big-Theta Notation (Theta)

$f(n) = \Theta(g(n))$ if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

Definition: Tight bound - average case complexity.

1.1.4 Little-o Notation (o)

$f(n) = o(g(n))$ if $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$

1.1.5 Little-omega Notation (omega)

$f(n) = \omega(g(n))$ if $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty$

1.2 Growth of Functions

Common growth rates in ascending order:

$$O(1) < O(\log n) < O(n) < O(n \log n) < O(n^2) < O(n^3) < O(2^n) < O(n!)$$

1.3 Methods to Solve Recurrence Relations

1.3.1 Substitution Method

Steps:

1. Guess the form of solution
2. Verify by mathematical induction
3. Solve for constants

Example: $T(n) = 2T(n/2) + n$

Guess: $T(n) = O(n \log n)$, so $T(n) \leq c \cdot n \log n$

Proof by induction:

$$T(n) = 2T(n/2) + n \quad (1)$$

$$\leq 2c \cdot \frac{n}{2} \log\left(\frac{n}{2}\right) + n \quad (2)$$

$$= cn(\log n - \log 2) + n \quad (3)$$

$$= cn \log n - cn + n \quad (4)$$

$$\leq cn \log n \text{ (if } c \geq 1) \quad (5)$$

1.3.2 Recursion Tree Method

Steps:

1. Draw recursion tree
2. Calculate cost at each level
3. Sum costs across all levels

Example: $T(n) = 2T(n/2) + n$

$$\text{Level 0: } n \quad (6)$$

$$\text{Level 1: } 2 \cdot \frac{n}{2} = n \quad (7)$$

$$\text{Level 2: } 4 \cdot \frac{n}{4} = n \quad (8)$$

$$\vdots \quad (9)$$

$$\text{Level } \log n : n \cdot 1 = n \quad (10)$$

Total cost: $n \cdot (\log n + 1) = O(n \log n)$

1.3.3 Master Method

For recurrences of the form: $T(n) = aT(n/b) + f(n)$ where $a \geq 1, b > 1$

Let $c = \log_b a$

Case 1: If $f(n) = O(n^{c-\epsilon})$ for some $\epsilon > 0$, then $T(n) = \Theta(n^c)$

Case 2: If $f(n) = \Theta(n^c \log^k n)$ for some $k \geq 0$, then $T(n) = \Theta(n^c \log^{k+1} n)$

Case 3: If $f(n) = \Omega(n^{c+\epsilon})$ for some $\epsilon > 0$ and $af(n/b) \leq cf(n)$ for some $c < 1$, then $T(n) = \Theta(f(n))$

2 Graph Traversal Algorithms

2.1 Breadth First Search (BFS)

Time Complexity: $O(V + E)$ **Space Complexity:** $O(V)$

Algorithm 1 Breadth First Search

```
1:  $BFS(G, s)$ 
2: Initialize queue  $Q$  and visited array
3:  $visited[s] \leftarrow true$ 
4:  $Q.enqueue(s)$ 
5: while  $Q$  is not empty do
6:    $u \leftarrow Q.dequeue()$ 
7:   Process vertex  $u$ 
8:   for each vertex  $v$  adjacent to  $u$  do
9:     if  $visited[v] = false$  then
10:       $visited[v] \leftarrow true$ 
11:       $Q.enqueue(v)$ 
12:     end if
13:   end for
14: end while
```

Algorithm 2 Depth First Search

```
1:  $DFS(G, u)$ 
2:  $visited[u] \leftarrow true$ 
3: Process vertex  $u$ 
4: for each vertex  $v$  adjacent to  $u$  do
5:   if  $visited[v] = false$  then
6:      $DFS(G, v)$ 
7:   end if
8: end for
```

2.2 Depth First Search (DFS)

Time Complexity: $O(V + E)$ **Space Complexity:** $O(V)$

2.3 Topological Sort

Application: DAG (Directed Acyclic Graph) ordering **Time Complexity:** $O(V + E)$

Algorithm 3 Topological Sort using DFS

```

1: Initialize stack  $S$  and visited array
2: for each vertex  $u$  in  $G$  do
3:   if  $visited[u] = false$  then
4:      $TopologicalSortUtil(u, visited, S)$ 
5:   end if
6: end for
7: Print contents of stack  $S$ 

```

Algorithm 4 Topological Sort Utility

```

1:  $TopologicalSortUtil(u, visited, S)$ 
2:  $visited[u] \leftarrow true$ 
3: for each vertex  $v$  adjacent to  $u$  do
4:   if  $visited[v] = false$  then
5:      $TopologicalSortUtil(v, visited, S)$ 
6:   end if
7: end for
8:  $S.push(u)$ 

```

2.4 Strongly Connected Components (Kosaraju's Algorithm)

Time Complexity: $O(V + E)$

Algorithm 5 Kosaraju's Algorithm

```

1: Perform DFS on original graph  $G$  and store vertices in stack by finish time
2: Create transpose graph  $G^T$ 
3: Pop vertices from stack and perform DFS on  $G^T$ 
4: Each DFS tree in step 3 is a strongly connected component

```

2.5 Euler Path and Circuit

Euler Path: Visits every edge exactly once **Euler Circuit:** Euler path that starts and ends at same vertex

Conditions:

- **Euler Circuit:** All vertices have even degree
- **Euler Path:** Exactly 0 or 2 vertices have odd degree

2.6 Articulation Points (Cut Vertices)

Definition: Vertex whose removal increases number of connected components

Algorithm 6 Tarjan's Algorithm for Articulation Points

```

1: Initialize  $disc[], low[], parent[], visited[]$ 
2:  $time \leftarrow 0$ 
3: for each vertex  $u$  do
4:   if  $visited[u] = false$  then
5:      $APUtil(u)$ 
6:   end if
7: end for

```

Key Formula: $low[u] = \min(low[u], low[v])$ for tree edges, $low[u] = \min(low[u], disc[v])$ for back edges

2.7 Bridge Detection

Definition: Edge whose removal increases number of connected components

Condition: Edge (u, v) is bridge if $low[v] > disc[u]$

2.8 Bi-connected Components

Definition: Maximal bi-connected subgraph

Algorithm 7 Bi-connected Components

```

1: Use modified Tarjan's algorithm
2: Maintain stack of edges
3: When articulation point found, pop edges until current edge
4: Each set of popped edges forms a bi-connected component

```

3 Shortest Path Algorithms

3.1 Dijkstra's Algorithm

Application: Single-source shortest path with non-negative weights **Time Complexity:** $O((V + E) \log V)$ with binary heap

3.2 Bellman-Ford Algorithm

Application: Single-source shortest path with negative weights **Time Complexity:** $O(VE)$

3.3 Floyd-Warshall Algorithm

Application: All-pairs shortest path **Time Complexity:** $O(V^3)$

Recurrence Relation:

$$dist^{(k)}[i][j] = \min(dist^{(k-1)}[i][j], dist^{(k-1)}[i][k] + dist^{(k-1)}[k][j])$$

Algorithm 8 Dijkstra's Algorithm

```

1: Initialize  $dist[s] = 0$ ,  $dist[v] = \infty$  for all  $v \neq s$ 
2: Create min-heap  $Q$  with all vertices
3: while  $Q$  is not empty do
4:    $u \leftarrow ExtractMin(Q)$ 
5:   for each vertex  $v$  adjacent to  $u$  do
6:     if  $dist[u] + weight(u, v) < dist[v]$  then
7:        $dist[v] \leftarrow dist[u] + weight(u, v)$ 
8:        $DecreaseKey(Q, v, dist[v])$ 
9:     end if
10:  end for
11: end while

```

Algorithm 9 Bellman-Ford Algorithm

```

1: Initialize  $dist[s] = 0$ ,  $dist[v] = \infty$  for all  $v \neq s$ 
2: for  $i = 1$  to  $|V| - 1$  do
3:   for each edge  $(u, v)$  in  $E$  do
4:     if  $dist[u] + weight(u, v) < dist[v]$  then
5:        $dist[v] \leftarrow dist[u] + weight(u, v)$ 
6:     end if
7:   end for
8: end for
9: // Check for negative cycles
10: for each edge  $(u, v)$  in  $E$  do
11:   if  $dist[u] + weight(u, v) < dist[v]$  then
12:     return "Negative cycle detected"
13:   end if
14: end for

```

Algorithm 10 Floyd-Warshall Algorithm

```

1: Initialize  $dist[i][j] = weight(i, j)$  if edge exists,  $\infty$  otherwise
2:  $dist[i][i] = 0$  for all  $i$ 
3: for  $k = 1$  to  $n$  do
4:   for  $i = 1$  to  $n$  do
5:     for  $j = 1$  to  $n$  do
6:       if  $dist[i][k] + dist[k][j] < dist[i][j]$  then
7:          $dist[i][j] \leftarrow dist[i][k] + dist[k][j]$ 
8:       end if
9:     end for
10:  end for
11: end for

```

3.4 Shortest Path in DAG

Time Complexity: $O(V + E)$

Algorithm 11 Shortest Path in DAG

```

1: Topologically sort the vertices
2: Initialize  $dist[s] = 0$ ,  $dist[v] = \infty$  for all  $v \neq s$ 
3: for each vertex  $u$  in topological order do
4:   for each vertex  $v$  adjacent to  $u$  do
5:     if  $dist[u] + weight(u, v) < dist[v]$  then
6:        $dist[v] \leftarrow dist[u] + weight(u, v)$ 
7:     end if
8:   end for
9: end for

```

4 Divide & Conquer Algorithms

4.1 Counting Inversions using Merge Sort

Inversion: Pair (i, j) where $i < j$ but $arr[i] > arr[j]$ **Time Complexity:** $O(n \log n)$

Algorithm 12 Count Inversions

```

1: MergeAndCount(arr, temp, left, mid, right)
2:  $i \leftarrow left, j \leftarrow mid + 1, k \leftarrow left, inv\_count \leftarrow 0$ 
3: while  $i \leq mid$  and  $j \leq right$  do
4:   if  $arr[i] \leq arr[j]$  then
5:      $temp[k++] \leftarrow arr[i++]$ 
6:   else
7:      $temp[k++] \leftarrow arr[j++]$ 
8:      $inv\_count \leftarrow inv\_count + (mid - i + 1)$ 
9:   end if
10: end while
11: Copy remaining elements
12: return  $inv\_count$ 

```

4.2 Closest Pair of Points

Time Complexity: $O(n \log n)$

4.3 Computing A to the power k mod M using Divide and Conquer

Time Complexity: $O(\log k)$

Formula:

$$A^k = \begin{cases} 1 & \text{if } k = 0 \\ (A^{k/2})^2 & \text{if } k \text{ is even} \\ A \times A^{k-1} & \text{if } k \text{ is odd} \end{cases}$$

Algorithm 13 Closest Pair of Points

```

1: Sort points by x-coordinate
2: ClosestPairRec( $P_x, P_y, n$ )
3: if  $n \leq 3$  then
4:   return brute force solution
5: end if
6: Divide points into left and right halves
7:  $d_l \leftarrow \text{ClosestPairRec}(P_{x\_left}, P_{y\_left}, n/2)$ 
8:  $d_r \leftarrow \text{ClosestPairRec}(P_{x\_right}, P_{y\_right}, n/2)$ 
9:  $d \leftarrow \min(d_l, d_r)$ 
10: Create strip of points within distance  $d$  from dividing line
11: Check distances in strip and update  $d$  if necessary
12: return  $d$ 

```

Algorithm 14 Fast Modular Exponentiation

```

1: FastPower( $A, k, M$ )
2: if  $k = 0$  then
3:   return 1
4: end if
5: if  $k$  is even then
6:    $temp \leftarrow \text{FastPower}(A, k/2, M)$ 
7:   return  $(temp \times temp) \bmod M$ 
8: else
9:   return  $(A \times \text{FastPower}(A, k - 1, M)) \bmod M$ 
10: end if

```

4.4 Finding k-th Smallest Element (Quickselect)

Average Time Complexity: $O(n)$ **Worst Case:** $O(n^2)$

Algorithm 15 Randomized Quickselect

```

1: QuickSelect(arr, left, right, k)
2: if left = right then
3:   return arr[left]
4: end if
5: pivotIndex  $\leftarrow$  RandomizedPartition(arr, left, right)
6: if k = pivotIndex then
7:   return arr[k]
8: else if k < pivotIndex then
9:   return QuickSelect(arr, left, pivotIndex - 1, k)
10: else
11:   return QuickSelect(arr, pivotIndex + 1, right, k)
12: end if

```

5 Greedy Algorithms

5.1 Elements and Properties

Greedy Choice Property: Locally optimal choices lead to globally optimal solution

Optimal Substructure: Optimal solution contains optimal solutions to subproblems

5.2 Fractional Knapsack

Time Complexity: $O(n \log n)$

Algorithm 16 Fractional Knapsack

```

1: Calculate value-to-weight ratio for each item
2: Sort items by ratio in descending order
3: totalValue  $\leftarrow$  0, currentWeight  $\leftarrow$  0
4: for each item i in sorted order do
5:   if currentWeight + weight[i]  $\leq$  capacity then
6:     currentWeight  $\leftarrow$  currentWeight + weight[i]
7:     totalValue  $\leftarrow$  totalValue + value[i]
8:   else
9:     remainingCapacity  $\leftarrow$  capacity - currentWeight
10:    totalValue  $\leftarrow$  totalValue + value[i]  $\times$   $\frac{\text{remainingCapacity}}{\text{weight}[i]}$ 
11:    BREAK
12:   end if
13: end for
14: return totalValue

```

5.3 Job Scheduling with Deadline

Time Complexity: $O(n^2)$

Algorithm 17 Job Scheduling with Deadline

```

1: Sort jobs by profit in descending order
2: Initialize result[] array and slot[] boolean array
3: for each job i do
4:   for  $j = \min(n, \text{job}[i].\text{deadline}) - 1$  down to 0 do
5:     if slot[j] = false then
6:       result[j]  $\leftarrow i$ 
7:       slot[j]  $\leftarrow \text{true}$ 
8:       BREAK
9:     end if
10:  end for
11: end for

```

5.4 Minimum Spanning Tree

5.4.1 Prim's Algorithm

Time Complexity: $O(E \log V)$ with binary heap

Algorithm 18 Prim's Algorithm

```

1: Initialize key[v] =  $\infty$  for all vertices, key[0] = 0
2: Create min-heap Q with all vertices
3: Initialize parent[] array
4: while Q is not empty do
5:    $u \leftarrow \text{ExtractMin}(Q)$ 
6:   for each vertex v adjacent to u do
7:     if  $v \in Q$  and  $\text{weight}(u, v) < \text{key}[v]$  then
8:       parent[v]  $\leftarrow u$ 
9:       key[v]  $\leftarrow \text{weight}(u, v)$ 
10:      DecreaseKey(Q, v, key[v])
11:    end if
12:  end for
13: end while

```

5.4.2 Kruskal's Algorithm

Time Complexity: $O(E \log E)$

6 Dynamic Programming

6.1 Basic Principles

Optimal Substructure: Optimal solution contains optimal solutions to subproblems

Overlapping Subproblems: Same subproblems are solved multiple times

Algorithm 19 Kruskal's Algorithm

```

1: Sort all edges by weight in ascending order
2: Initialize Union-Find data structure
3:  $MST \leftarrow \emptyset$ 
4: for each edge  $(u, v)$  in sorted order do
5:   if  $Find(u) \neq Find(v)$  then
6:      $MST \leftarrow MST \cup \{(u, v)\}$ 
7:      $Union(u, v)$ 
8:     if  $|MST| = V - 1$  then
9:       BREAK
10:    end if
11:  end if
12: end for
13: return  $MST$ 

```

Comparison with Other Paradigms:

- **vs Divide & Conquer:** DP has overlapping subproblems, D&C has independent subproblems
- **vs Greedy:** DP considers all possibilities, Greedy makes locally optimal choices

6.2 Memoization vs Tabulation

Memoization: Top-down approach, store results of subproblems **Tabulation:** Bottom-up approach, build solution iteratively

6.3 Coin Change Problems**6.3.1 Minimum Coins**

Time Complexity: $O(n \times amount)$

Algorithm 20 Minimum Coins

```

1:  $CoinChange(coins[], amount)$ 
2: Initialize  $dp[0...amount]$  with  $\infty$ ,  $dp[0] = 0$ 
3: for  $i = 1$  to  $amount$  do
4:   for each coin  $c$  in  $coins$  do
5:     if  $i \geq c$  then
6:        $dp[i] = \min(dp[i], dp[i - c] + 1)$ 
7:     end if
8:   end for
9: end for
10: return  $dp[amount]$ 

```

Recurrence: $dp[i] = \min(dp[i], dp[i - coin] + 1)$ for all valid coins

6.3.2 Coin Change Ways

Algorithm 21 Number of Ways to Make Change

```

1: Initialize  $dp[0...amount]$  with 0,  $dp[0] = 1$ 
2: for each coin  $c$  in  $coins$  do
3:   for  $i = c$  to  $amount$  do
4:      $dp[i] = dp[i] + dp[i - c]$ 
5:   end for
6: end for
7: return  $dp[amount]$ 

```

6.4 Longest Increasing Subsequence (LIS)**Time Complexity:** $O(n^2)$ basic DP, $O(n \log n)$ optimized**Algorithm 22** LIS using DP

```

1: Initialize  $lis[0...n - 1]$  with 1
2: for  $i = 1$  to  $n - 1$  do
3:   for  $j = 0$  to  $i - 1$  do
4:     if  $arr[i] > arr[j]$  and  $lis[i] < lis[j] + 1$  then
5:        $lis[i] = lis[j] + 1$ 
6:     end if
7:   end for
8: end for
9: return  $\max(lis[0...n - 1])$ 

```

Recurrence: $LIS[i] = \max(LIS[j] + 1)$ for all $j < i$ where $arr[j] < arr[i]$ **6.5 Longest Common Subsequence (LCS)****Time Complexity:** $O(mn)$ **Algorithm 23** LCS using DP

```

1: Initialize  $dp[0...m][0...n]$  with 0
2: for  $i = 1$  to  $m$  do
3:   for  $j = 1$  to  $n$  do
4:     if  $X[i - 1] = Y[j - 1]$  then
5:        $dp[i][j] = dp[i - 1][j - 1] + 1$ 
6:     else
7:        $dp[i][j] = \max(dp[i - 1][j], dp[i][j - 1])$ 
8:     end if
9:   end for
10: end for
11: return  $dp[m][n]$ 

```

Recurrence:

$$LCS[i][j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ LCS[i - 1][j - 1] + 1 & \text{if } X[i] = Y[j] \\ \max(LCS[i - 1][j], LCS[i][j - 1]) & \text{otherwise} \end{cases}$$

6.6 0/1 Knapsack

Time Complexity: $O(nW)$

Algorithm 24 0/1 Knapsack

```

1: Initialize  $dp[0...n][0...W]$  with 0
2: for  $i = 1$  to  $n$  do
3:   for  $w = 1$  to  $W$  do
4:     if  $weight[i - 1] \leq w$  then
5:        $dp[i][w] = \max(dp[i - 1][w], dp[i - 1][w - weight[i - 1]] + value[i - 1])$ 
6:     else
7:        $dp[i][w] = dp[i - 1][w]$ 
8:     end if
9:   end for
10: end for
11: return  $dp[n][W]$ 

```

Recurrence:

$$dp[i][w] = \begin{cases} 0 & \text{if } i = 0 \text{ or } w = 0 \\ dp[i - 1][w] & \text{if } weight[i - 1] > w \\ \max(dp[i - 1][w], dp[i - 1][w - weight[i - 1]] + value[i - 1]) & \text{otherwise} \end{cases}$$

6.7 Matrix Chain Multiplication

Time Complexity: $O(n^3)$

Algorithm 25 Matrix Chain Multiplication

```

1: Initialize  $dp[1...n][1...n]$  with 0 on diagonal
2: for length  $l = 2$  to  $n$  do
3:   for  $i = 1$  to  $n - l + 1$  do
4:      $j = i + l - 1$ 
5:      $dp[i][j] = \infty$ 
6:     for  $k = i$  to  $j - 1$  do
7:        $cost = dp[i][k] + dp[k + 1][j] + p[i - 1] \times p[k] \times p[j]$ 
8:       if  $cost < dp[i][j]$  then
9:          $dp[i][j] = cost$ 
10:      end if
11:    end for
12:  end for
13: end for
14: return  $dp[1][n]$ 

```

Recurrence: $dp[i][j] = \min_{i \leq k < j} (dp[i][k] + dp[k + 1][j] + p_{i-1} \times p_k \times p_j)$

6.8 Applications of Dynamic Programming

- Edit Distance (Levenshtein Distance)

- Subset Sum Problem
- Palindrome Partitioning
- Maximum Sum Subarray (Kadane's Algorithm)
- Optimal Binary Search Tree
- Travelling Salesman Problem (TSP)

7 Network Flow

7.1 Flow Networks

Flow Network: Directed graph $G = (V, E)$ with:

- Source vertex s (no incoming edges)
- Sink vertex t (no outgoing edges)
- Capacity function $c : E \rightarrow \mathbb{R}^+$

Flow Properties:

1. **Capacity Constraint:** $0 \leq f(u, v) \leq c(u, v)$ for all $(u, v) \in E$
2. **Flow Conservation:** $\sum_{v \in V} f(u, v) = 0$ for all $u \in V \setminus \{s, t\}$

Value of Flow: $|f| = \sum_{v \in V} f(s, v)$

7.2 Max-Flow Min-Cut Theorem

Cut: Partition of vertices (S, T) where $s \in S$ and $t \in T$

Capacity of Cut: $c(S, T) = \sum_{u \in S, v \in T} c(u, v)$

Theorem: Maximum flow value equals minimum cut capacity $\max |f| = \min c(S, T)$

7.3 Ford-Fulkerson Method

Time Complexity: $O(E \cdot |f^*|)$ where $|f^*|$ is maximum flow

Algorithm 26 Ford-Fulkerson Method

- 1: Initialize flow $f(u, v) = 0$ for all edges (u, v)
 - 2: **while** there exists augmenting path P from s to t in residual graph **do**
 - 3: Find bottleneck capacity $c_f(P) = \min\{c_f(u, v) : (u, v) \in P\}$
 - 4: **for** each edge (u, v) in P **do**
 - 5: $f(u, v) = f(u, v) + c_f(P)$
 - 6: $f(v, u) = f(v, u) - c_f(P)$
 - 7: **end for**
 - 8: **end while**
 - 9: **return** flow f
-

Residual Graph: $G_f = (V, E_f)$ where $E_f = \{(u, v) : c_f(u, v) > 0\}$

$$\text{Residual Capacity: } c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E \\ f(v, u) & \text{if } (v, u) \in E \\ 0 & \text{otherwise} \end{cases}$$

Limitation: Can be slow with irrational capacities (infinite runtime possible)

7.4 Edmonds-Karp Algorithm

Improvement: Use BFS to find shortest augmenting path **Time Complexity:** $O(VE^2)$

Algorithm 27 Edmonds-Karp Algorithm

```

1: Initialize flow  $f(u, v) = 0$  for all edges
2: while BFS finds augmenting path from  $s$  to  $t$  do
3:    $path\_flow = \infty$ 
4:   Trace path and find minimum residual capacity
5:   Update flow along the path
6:    $max\_flow = max\_flow + path\_flow$ 
7: end while
8: return  $max\_flow$ 

```

Algorithm 28 BFS for Augmenting Path

```

1: Initialize  $parent[]$  array and visited array
2:  $queue.push(s)$ ,  $visited[s] = true$ 
3: while queue is not empty do
4:    $u = queue.pop()$ 
5:   for each vertex  $v$  adjacent to  $u$  do
6:     if  $visited[v] = false$  and  $c_f(u, v) > 0$  then
7:        $parent[v] = u$ 
8:        $visited[v] = true$ 
9:        $queue.push(v)$ 
10:      if  $v = t$  then
11:        return true
12:      end if
13:    end if
14:  end for
15: end while
16: return false

```

7.5 Maximum Bipartite Matching

Bipartite Graph: $G = (U \cup V, E)$ where no edges within U or V

Reduction to Max Flow:

1. Create source s connected to all vertices in U with capacity 1
2. Create sink t connected from all vertices in V with capacity 1
3. Set capacity 1 for all original edges

Time Complexity: $O(VE)$

Algorithm 29 Maximum Bipartite Matching using DFS

```

1: Initialize match[] array with  $-1$ 
2: result = 0
3: for each vertex u in left set do
4:   Initialize visited[] array with false
5:   if dfs(u, visited, match) then
6:     result = result + 1
7:   end if
8: end for
9: return result

```

Algorithm 30 DFS for Augmenting Path in Bipartite Matching

```

1: dfs(u, visited, match)
2: for each vertex v adjacent to u do
3:   if visited[v] = true then
4:     CONTINUE
5:   end if
6:   visited[v] = true
7:   if match[v] =  $-1$  OR dfs(match[v], visited, match) then
8:     match[v] = u
9:     return true
10:  end if
11: end for
12: return false

```

7.6 Minimum Path Cover

Definition: Minimum number of vertex-disjoint paths that cover all vertices in DAG

Theorem: For DAG with n vertices, minimum path cover = $n - \text{maximum matching}$ in corresponding bipartite graph

Construction:

1. Create bipartite graph with left copy and right copy of vertices
2. Add edge (u_L, v_R) if (u, v) exists in original DAG
3. Find maximum bipartite matching

Formula: Minimum Path Cover = $|V| - |M|$ where M is maximum matching

7.7 Edge Cover

Definition: Set of edges such that every vertex is incident to at least one edge

Minimum Edge Cover:

- For connected graph: $|V| - |M|$ where M is maximum matching
- General formula: $|V| - c$ where c is number of connected components

Algorithm 31 Minimum Edge Cover

```

1: Find maximum matching  $M$  in graph  $G$ 
2:  $cover = M$ 
3: for each unmatched vertex  $v$  do
4:   Add any edge incident to  $v$  to  $cover$ 
5: end for
6: return  $cover$ 

```

8 Advanced Topics and Complexity Results

8.1 NP-Completeness

Decision Problems:

- **P:** Problems solvable in polynomial time
- **NP:** Problems verifiable in polynomial time
- **NP-Complete:** Hardest problems in NP
- **NP-Hard:** At least as hard as NP-complete problems

Famous NP-Complete Problems:

- 3-SAT (Boolean Satisfiability)
- Hamiltonian Path/Cycle
- Travelling Salesman Problem (Decision version)
- Knapsack Problem (Decision version)
- Graph Coloring
- Clique Problem
- Vertex Cover

8.2 Approximation Algorithms

Approximation Ratio: For minimization problem, algorithm A has ratio ρ if: $\frac{A(I)}{OPT(I)} \leq \rho$

Examples:

- **Vertex Cover:** 2-approximation using maximal matching
- **TSP:** 2-approximation using MST (metric TSP)
- **Set Cover:** H_n -approximation where $H_n = \sum_{i=1}^n \frac{1}{i}$

8.3 Parameterized Complexity

Fixed Parameter Tractable (FPT): Algorithm runs in $O(f(k) \cdot n^c)$ time where k is parameter

Examples:

- **Vertex Cover:** $O(2^k \cdot n)$ where k is size of vertex cover
- **Graph Coloring:** $O(2^k \cdot n)$ where k is number of colors

9 Summary of Time Complexities

Algorithm	Time Complexity	Space Complexity
BFS/DFS	$O(V + E)$	$O(V)$
Dijkstra	$O((V + E) \log V)$	$O(V)$
Bellman-Ford	$O(VE)$	$O(V)$
Floyd-Warshall	$O(V^3)$	$O(V^2)$
Kruskal	$O(E \log E)$	$O(V)$
Prim	$O((V + E) \log V)$	$O(V)$
Ford-Fulkerson	$O(E \cdot f^*)$	$O(V + E)$
Edmonds-Karp	$O(VE^2)$	$O(V + E)$
Merge Sort	$O(n \log n)$	$O(n)$
Quick Sort (avg)	$O(n \log n)$	$O(\log n)$
Heap Sort	$O(n \log n)$	$O(1)$
Counting Sort	$O(n + k)$	$O(k)$
Radix Sort	$O(d(n + k))$	$O(n + k)$

10 Important Recurrence Relations

$$T(n) = 2T(n/2) + O(n) \Rightarrow O(n \log n) \text{ (Merge Sort)} \quad (11)$$

$$T(n) = 2T(n/2) + O(1) \Rightarrow O(n) \text{ (Binary Search)} \quad (12)$$

$$T(n) = T(n-1) + O(n) \Rightarrow O(n^2) \text{ (Selection Sort)} \quad (13)$$

$$T(n) = T(n-1) + O(1) \Rightarrow O(n) \text{ (Linear Search)} \quad (14)$$

$$T(n) = 2T(n-1) + O(1) \Rightarrow O(2^n) \text{ (Fibonacci)} \quad (15)$$

$$T(n) = 7T(n/2) + O(n^2) \Rightarrow O(n^{\log_2 7}) \text{ (Strassen)} \quad (16)$$