Chapter 1.1 - 1.4

Linear Equation: $c_1 x_1 + c_2 + x_2 + \ldots + c_v x_v = b$ 형태

Matrix Equation: $\begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_3 \end{bmatrix}$ 형태

$AX = [m \times n][m \times o] \to [m \times o]$ form

Example 1)
Linear Eq

$$x_1 + 2x_2 - x_3 = 4$$
$$-5x_2 + 3x_3 = 1$$

Vector Eq

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Matrix Eq

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

If $A = m \times n$ matrix, Everything below is equivalent.
(A is coefficient matrix, not augmented matrix)
[m equations; n unknowns; n columns; b in $\mathbb{R}^m$]

1. For each b in $\mathbb{R}^m$ $Ax = B$ has “a”(at least one) solution.
2. Each b in $\mathbb{R}^m$ is linear combination of columns of A
3. The columns of A span $\mathbb{R}^m$
4. Matrix A has pivot position in every row

선형성

$A(u + v) = A(u) + A(v)$
$A(cu) = c(Au)$

Chapter 1.5 - 1.9

Homogeneous Linear System

$Ax = b$ 꼴
$x = 0$ means trivial solution
if non-zero vector x exist, that is non-trivial solution.

For Homogeneous Equation(Ax=b),
if free variable exists, Ax=b has nontrivial solution.

- non-trivial solution has some zero entries, so long as not all of its entries are zero.

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix} x_3 \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix} = x_2 u + x_3 v$
Solution set is $span\{u, v\}$

- u, v are not scalar multiple of the other, the solution set is plane through the origin.

Parametric Vector Form

Parametric Equation: $\mathrm{x = su + tv \ (s, t \in \mathbb{R})}$
General Solution for Non-Homogeneous Equation $(\mathrm{Ax = b})$
$\mathrm{b = p + tv_h}$

- t as general parameter
- For homogeneous, $\mathrm{b = tv_h \ (t \in \mathbb{R})}$
- Non-Homogeneous => Solution of Homogenous Eq + vector p
- RREF 를 통하여 basic/free variables 를 구한 뒤, general solution 꼴로 표현
- $\mathrm{x = p + v_h}$에서 p 는 parameter 포함 X, $\mathrm{v_h}$는 parameter을 포함

Linear Dependency

Linearly dependent	Linearly Independent
Non-Trivial Solution	Trivial Solution
Det(A) == 0	Det(A) != 0
At least one is linear combination of others	Nothing is linear combination of others
p>n	Depend on
Zero vector in row vector	Depend on

- If $\mathrm{Ax = 0}$ has only trivial solution, columns of A is linearly independent.
- Linearly dependent: 한 평면에 존재
- Linearly independent: 한 평면에 존재 X

Linear Transformation

- $\mathbb{R}^n$ to $\mathbb{R}_m, \mathbb{R}^n$ is called Domain of $T, \mathbb{R}^m$ is codomain
- trivial(linearly independent) means one-to-one
- T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if columns of A span $\mathbb{R}^m$
- onto means Existence question, one-to-one means Uniqueness question.
- T is one-to-one $\mathrm{T(x) = b}$ has unique solution or none of all.

Matrix Operations

Matrix addtion: $\mathrm{A} + \mathrm{B}$

- They should have same size $m \times n$

$A = m \times n, B = n \times o \to AB = m \times o$

- $AB \neq BA$
- $\left(A^T\right)^T = A$
- $(A + B)^T = A^T + B^T$
- $(rA)^T = rA^T$
- $(AB)^T = B^T A^T$ (Remember reverse order)

Inverse of the Matrix

$A^{-1}A = I$

- Invertible(가역) : Singular matrix(특이 행렬)
- Non-Invertible(비가역): Non-Singular Matrix(정칙 행렬)
- If $(\mathrm{Det}\ A) \neq 0$, Matrix A is invertible.
- $\left(A^{-1}\right)^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$
- to find A^{-1}, use Identity Matrix. $[A \quad I]$ or $[I \quad A]$

Characterizations of Invertible Matrices

IMT (Invertible Matric Theorem) for $n \times n$ matrix

- A is invertible Matrix
- A is row equivalent to $n \times n$ identity Matrix I_n
- n has n pivot positions
- $\mathrm{Ax} = 0$ has only trivial solution.
- Columns of A form linearly independent set
- Linear transformation $\mathrm{x} \to \mathrm{Ax}$ is one-to-one
- $\mathrm{Ax} = \mathrm{b}$ has at least one solution for each b in $\mathbb{R}^n$
- Columns of A span $\mathbb{R}^n$
- Linear transformation $\mathrm{x} \to \mathrm{Ax}$ is onto
- There is $n \times n$ Matrix C/D such that $CA = I, AD = I$
- A^T is invertible matrix

LU Decomposition (factorization)

- Matrix multiplication involves synthesis of data
- Matrix decompostion involves analysis of data
- $U(m \times n)$ is echelon form of matrix A
- $L(m \times m)$ is unit lower triangular matrix
- $\mathrm{Ax} = \mathrm{b} \to \mathrm{L(Ux)} = \mathrm{b}$, $\text{solve y through L y} = \text{b, and get x through Ly} = \text{b}$
- U is upper triangular form (echelon form of A)
- L is unit lower triangular form (You can get L while finding U)

Subspace of $\mathbb{R}^n$

- Col A : All Linear combinations of Column of A
- Nul A : All Solutions of homogeneous equation $\mathrm{Ax} = 0$
- Basis of subspace H : Linearly independent set in H that spans H

Subspace of $\mathbb{R}^n$

- Zero vector is in H
- For each u and v in H, u+v is in H
- For each u and each scalar c, cu is in H
- {0} and $\mathbb{R}^n$ is subspace of $\mathbb{R}^n$

Col A

- Column space of matrix A is the set Col A of all linear combinations of columns of A
- If A = $\begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}$ with columns of $\mathbb{R}^m$, Col A is same as $\mathrm{span}\{a_1, ..., a_n\}$
- If $\mathrm{Ax} = \mathrm{b}$ is consistent, b is in Col A

Nul A

- Null space of matrix A is set of Nul A of all solutions of homogeneous equation $\mathrm{Ax} = 0$
- Zero vector is in Nul A ($A0 = 0$)
- $\mathrm{Au} = 0, \mathrm{Av} = 0,$ then $\mathrm{A(u + v)} = 0$

Basis for Subspace

- Basis for Subspace H of $\mathbb{R}^n$ is linearly independent set in H that spans H.
- Pivot Columns of matrix A forms basis of Column space of a

Dimension of Subspace

- Dimension of vector subspace H: dim H
- count of vector that consists Basis
- $H = \{0\} \to \dim H = 0$
- 직선: 1, 평면: 2, 공간: 3

Rank

- Rank A means dimension of Column space of matrix A
- Counts of Pivot Columns
- Rank A = Rank A|b -> Consistent
  - Rank A = Rank A|b = unknowns -> Unique
  - Rank A = Rank A|b $\neq$ unknowns -> Infinite Solutions
- Rank A $\neq$ Rank A|b -> No Solution

Theorem's about subspace

- n = rank A + dim(Nul A) [Rank theorem]
- Let H be a p-dimensional subspace of $\mathbb{R}^n$
  - Any linearly independent set of p elements in H is basis for H
  - any p elements of H that spans on H is basis for H

Extended IMT (Invertible Matric Theorem)

- The columns of A forms a basis of $\mathbb{R}^n$
- Col A = $\mathbb{R}^n$
- dim(Col A) = n
- rank A = n
- Nul A = {0}
- dim(Nul A) = 0

Determinant

Determinant

- 2×2 matrix: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \rightarrow |A| = a_{11}a_{22} - a_{12}a_{21}$
- 3×3 matrix: submatrix $A_{ij} \leftarrow$ exclude i-th row, j-th column

Example If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then $A_{13} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$, and $A_{22} = \begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$

- $\det A = \Sigma_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}$
- 일반화: $\det A = |A| = \Delta = \Sigma_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} = \Sigma_{j=1}^{n} a_{ij} C_{ij}$
- i 의 값이 달라도 det A의 값은 똑같음.
- $n \times n$ matrix,$_{\text{submatrix}}$ $A_{ij}(n-1 \times n-1)$
- $\det A \neq 0 \rightarrow$ Invertible matrix
- Cofactor: $C_{ij} = (-1)^{i+j} \det A_{ij}$
- If A is $n \times n$ matrix, $\det A =$ product of entries of main diagonals of A
- $-np \leq \det A \leq np[n \times n \text{ matrix}, p = \text{abs(n largest elements in matrix } A)]$

Properties of Determinants

Theorem 3

- $A \sim U \sim I_n$ (If A is invertible)
- Replacement: $A \to B : \det A = \det B$
- Interchange: $A \leftrightarrow B : \det B = -\det A$
- Scaling: $\det B = k \det A$
- zero column/row $\to \det A = 0$
- $\text{Col } 1 = k \text{Col b} \to \det A = 0$
- $\det(kA) = k^n \det A$
- $\det A = (-1)^{\text{count of interchange}} \det U$
- $\det U = U_{11} + U_{22} + ... + U_{nn}$
- if A = invertible $\to \det U \neq 0$
- $\det A = 0 \to$ rows/columns of A is linearly dependent
-

Column Operations

- $\det A^T = \det A$
- $\det AB = \det A \det B$
- $\det(A + B) \neq \det A + \det B$

Cramer's rule

- $A_i(b) = \begin{bmatrix} a_1 & a_2 & \dots & b & \dots a_n \end{bmatrix}$
- $x_i = \frac{\det A_i(b)}{\det A} (i = 1, 2, 3, \dots, n)$
- $AI_i(x) = A[e_1 \dots x \dots e_n] = [a_1 \dots b \dots a_n] = A_i(b)$
- $(\det A)(\det I_i(x)) = \det(A_i(b))$

A Fomula for A^{-1}

- $A^{-1} = \frac{1}{\det A} \text{ adj } A$
- adjugate matrix is transpose of matrix of cofactors

Questions

1 선형 방정식, 벡터 방정식, 행렬 방정식의 차이점을 설명하고, 각각의 예시를 들어주세요.

2 선형 시스템이 적어도 하나의 해를 가지기 위한 충분조건을 설명해주세요. 이때, 계수 행렬의 특성에 대해 언급해주세요.

3 선형 독립과 선형 종속의 개념을 설명하고, 이들의 특징을 비교해주세요. 또한, 행렬식(determinant)과의 관계에 대해 설명해주세요.

4 선형 변환(linear transformation)이란 무엇인가요? 선형 변환이 일대일 대응(one-to-one)이거나 전사함수(onto)가 되기 위한 조건을 설명해주세요.

5 행렬의 곱셈에 대해 설명하고, 행렬 곱셈의 성질 중 3가지를 예시와 함께 설명해주세요.

6 가역행렬(invertible matrix)이란 무엇인가요? 가역행렬이 되기 위한 필요충분조건을 3가지 이상 나열해주세요.

7 LU 분해(LU decomposition)에 대해 설명하고, LU 분해를 이용하여 선형 시스템을 푸는 과정을 간단히 설명해주세요.

8 $\mathbb{R}^n$ 공간의 부분공간(subspace)에 대해 설명하고, Col A와 Nul A의 의미를 각각 설명해주세요.

선형대수학 문제

다음 선형 시스템을 행렬 형태로 나타내고, 가우스 소거법을 사용하여 해를 구하세요.

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 5 \\ x_1 - 2x_2 + 4x_3 &= 3 \\ 3x_1 + x_2 - 2x_3 &= 4 \end{aligned}$$

다음 벡터들이 선형 독립인지 판별하세요. $v_1 = (1, 2, -1)$ $v_2 = (2, 1, 3)$ $v_3 = (3, 5, 1)$ 다음 행렬 A의 LU 분해를 구하세요.

$$A = \begin{pmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{pmatrix}$$

다음 행렬 A가 가역행렬인지 판별하고, 가역행렬이라면 역행렬을 구하세요.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 2 & 3 & 4 \end{pmatrix}$$

다음 선형 변환 T가 일대일 대응이면서 전사함수인지 판별하세요.

$$T : \mathbb{R}^3 \to \mathbb{R}^3, \quad T(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3, x_1 + x_3)$$

다음 행렬 A의 열공간($ColA$)과 영공간($NulA$)의 기저를 구하세요.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 2 & 3 \\ 2 & -4 & -3 \end{pmatrix}$$