Math 481/542: (Introduction to)Stochastic Processes
Topic 1a): Discrete time Markov Chains

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1 Introduction to countably infinite Markov chains

A state space S is countably infinite if it can be put into one-to-one correspondence with the set of nonnegative integers $\{0, 1, 2, 3, \ldots\}$.

Lawler denotes the elements of any countably infinite state space S by x, y, z etc. instead of i, j, k etc. , as in the finite case.

The transition matrix P is now a matrix with countably many rows and as many columns.

Example 1.1 This is like example 1, page 38, in Lawler. Here,

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \ldots \end{array} \\ \begin{pmatrix} 1-p & p & 0 & 0 & 0 & \cdots \\ 1-p & 0 & p & 0 & 0 & \cdots \\ 0 & 1-p & 0 & p & 0 & \cdots \\ 0 & 0 & 1-p & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array},$$

where $p \in [0, 1]$. When $0 < p < 1$ then this model represents a random walk with elastic [partially reflecting] boundary. When $p = 1$ then this model represents a deterministic motion to the right on the positive integer lattice. When $p = 0$ then this model represents a deterministic motion to the left on the nonnegative integer lattice with 0 as the absorbing state.

Example 1.2 General random walk on the nonnegative integer lattice Here,

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \dots \end{array} \\ \begin{pmatrix} 1-p_0 & p_0 & 0 & 0 & 0 & \cdots \\ q_1 & r_1 & p_1 & 0 & 0 & \cdots \\ 0 & q_2 & r_2 & p_2 & 0 & \cdots \\ 0 & 0 & q_3 & r_3 & p_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}.$$

Example 1.1 is a special case of this Markov chain.

Example 1.3 General random walk on the integer lattice Here,

$$P = \begin{array}{c} \\ \vdots \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccccc} \cdots & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \dots \end{array} \\ \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & q_{-2} & r_{-2} & p_{-2} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & q_{-1} & r_{-1} & p_{-1} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & q_0 & r_0 & p_0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & q_1 & r_1 & p_1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & q_2 & r_2 & p_2 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & q_3 & r_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}.$$

If $r_i = 0$, $i = 0, \pm 1, \pm 2, \dots$ and $p_i = q_i = 1/2$, $i = 0, \pm 1, \pm 2, \dots$, then the random walk is called *symmetric*.

Example 1.4 Success Runs Here,

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \dots \end{array} \\ \begin{pmatrix} p_0 & q_0 & 0 & 0 & 0 & \cdots \\ p_1 & r_1 & q_1 & 0 & 0 & \cdots \\ p_2 & 0 & r_2 & q_2 & 0 & \cdots \\ p_3 & 0 & 0 & r_3 & q_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}.$$

2 Classification of states according to arithmetic properties of the n-step transition probabilities - countably infinite chains

Classification of states according the arithmetic properties of the n-step transition probabilities $p_n(i,j)$ [deja vu]:

$$S = \{essential\, states\} \cup \{inessential\, states\},$$

$$\{essential\, states\} = \bigcup_l essential\ communicating\ class_l,$$
$$\{inessential\, states\} = \bigcup_k inessential\ communicating\ class_k,$$
$$essential\ communicating\ class_l = \bigcup_{l_i} essential\ periodic\ class_{l_i},$$
$$inessential\ communicating\ class_l = \bigcup_{k_i} inessential\ periodic\ class_{k_i}.$$

Definition 2.1 State $x \in S$ is called *inessential* iff there exists $m \geq 0$ and $y \in S$ so that $p_m(x, y) > 0$ and $p_n(y, x) = 0$ for all $n \geq 0$. Otherwise, the state x is called essential.

Remark 2.1 Observe that if two states x, y are inessential then $\lim_{n\to\infty} p_n(x, y) = \lim_{n\to\infty} p_n(y, x) = 0$.

The above classification is the same here as in the finite case.

The concepts of irreducibility, reducibility, period, aperiodicity are the same here as in the finite case.

3 Return times, recurrent and transient states - countably infinite chains.

Denote *the first hitting time*:

$$T(x) = min\{n \geq 1 : X_n = x\}.$$

Next, define the *probability that the 1st visit to y after starting from x occurs at time* $n \leq 1$

$$f_{xy}^{(n)} = P(T(y) = n|X_0 = x) = P(X_n = y, X_m \neq y,\ m = 1, 2, \ldots, n-1|X_0 = x).$$

Note that $f_{xy}^{(1)} = p(x, y)$.

Now let

$$f_{xy} = \sum_{n=1}^{\infty} f_{xy}^{(n)} = P(T(y) < \infty|X_0 = x)$$
$$= Prob(\text{ever hitting } y \text{ after starting from } x).$$

Note that $f_{xx} = Prob(\text{ever returning to } x)$. Observe also that

$$f_{xx} \leq Prob(X_n = x \text{ for infinitely many } n).$$

Remark 3.1 What is the relationship between $\alpha(x, y)$ defined earlier and f_{xy}?

Definition 3.1 The state x is called recurrent if $f_{xx} = 1$. Otherwise the state x is called transient.

Definition 3.2 A Markov chain X_n is called recurrent if all its states are recurrent.

Remark 3.2 This definition agrees with Lawler's. Why?

Define the *number of returns to* x

$$R(x) = \sum_{n=0}^{\infty} I\{X_n = x\},$$

and

$$\bar{R}(x) = \sum_{n=1}^{\infty} I\{X_n = x\},$$

Note that $\bar{R}(x)$ does not count the possible initial visit. $R(x) = \bar{R}(x) + I\{X_0 = x\}$.

Fact If a state x is recurrent then $P(R(x) = \infty) = 1$.

We shall now derive criterion for recurrence. Consider first a transient state x. Thus $f_{xx} < 1$. Now, we have

$$P(\bar{R}(x) \geq 2 | X_0 = x) = P(\text{the chain returns to } x \text{ at least twice} | X_0 = x)$$

$$= P(\text{exist at least two times } m > n > 0 \text{ such that } X_m = X_n = x | X_0 = x)$$

$$= P(\text{exists at least one time } m > n > 0 \text{ such that } X_m = x |$$

$$\text{exists at least one time } n > 0 \text{ such that } X_n = x, \text{ and } X_0 = x) \times$$

$$P(\text{exists at least one time } n > 0 \text{ such that } X_n = x | X_0 = x)$$

$$= f_{xx} f_{xx} = f_{xx}^2.$$

Similarly,

$$P(\bar{R}(x) \geq k | X_0 = x) = P(\text{the chain returns to } x \text{ at least } k \text{ times} | X_0 = x) = f_{xx}^k, \quad \forall k \geq 1.$$

Thus, the conditional distribution of $\bar{R}(x)$ [conditional on $X_0 = x$] is geometric with the probability of failure f_{xx}. Consequently,

$$E(\bar{R}(x) | X_0 = x) = \frac{f_{xx}}{1 - f_{xx}}.$$

From the above we obtain the following fact,

Fact A state x is recurrent iff

$$\sum_{n=1}^{\infty} p_n(x,x) = \infty.$$

Equivalently, the state x is transient iff $\sum_{n=1}^{\infty} p_n(x,x) < \infty$.

Proof. $\Rightarrow$: Suppose first that x is transient. Thus,

$$\sum_{n=1}^{\infty} p_n(x,x) = \sum_{n=1}^{\infty} E(I\{X_n = x\}|X_0 = x)$$

$$= E(\bar{R}(x)|X_0 = x) = \frac{f_{xx}}{1 - f_{xx}} < \infty.$$

$\Leftarrow$: Suppose $\sum_{n=1}^{\infty} p_n(x,x) < \infty$. Thus $E(\bar{R}(x)|X_0 = x) < \infty$. So, $P(\bar{R}(x) < \infty|X_0 = x) = 1$. Consequently, with probability 1 the process starting from x returns to x only finitely many times. Thus, there is a positive probability that the process starting from x will never return to x, i.e. $1 - f_{xx} > 0$. This means that $f_{xx} < 1$ and thus the state x is transient. □

We also have the following fact

Fact Recurrence is the class property. That is, if $x \leftrightarrow y$ then x is recurrent iff y is recurrent.

Proof. Since $x \leftrightarrow y$ then there are $m, n \geq 1$ so that $p_n(x,y) > 0$ and $p_m(y,x) > 0$. Now, since $p_{m+k+n}(x,x) \geq p_n(x,y)p_k(y,y)p_m(y,x)$ [see Lalwer, page 17], we have

$$\sum_{k=1}^{\infty} p_{m+k+n}(x,x) \geq \sum_{k=1}^{\infty} p_n(x,y)p_k(y,y)p_m(y,x)$$

$$\geq p_n(x,y)p_m(y,x) \sum_{k=1}^{\infty} p_k(y,y) = \infty.$$

□

Remark 3.3 a) In a finite Markov chain not all states are transient.

b) In a finite irreducible Markov chain all states are recurrent.

Example 3.1 (Random walk on integers) Consider random walk on the integer lattice with $r_i = 0$ and $p_i = p$, $q_i = q = 1 - p$, $p \in (0,1)$. We have that

$$p_{2n}(0,0) \sim \frac{(4pq)^n}{\sqrt{\pi n}}.$$

Thus, one dimensional random walk is recurrent iff $p = 1/2$. Intuitively: when $p > 1/2$ then the walk will eventually wander away to $+\infty$; when $p < 1/2$ then the walk will eventually wander away to $-\infty$.

Remark 3.4 On page 41 in Lawler it is shown that a symmetric random walk in Z^d is recurrent iff $d = 1$ or $d = 2$.

Remark 3.5 On pages 41-42 in Lawler another criterion for transience/recurrence is formulated. We shall perhaps discuss it when we talk about martingales.

Example 3.2 (Success runs) Consider the success runs process with $r_i = 0$. Assume that $p_i \in (0, 1)$. Then all the states form one communicating class, i.e. the chain is irreducible [all states are essential]. Thus the chain is recurrent iff the state $x = 0$ is recurrent. Now, we have

$$P(T(0) > k|X_0 = 0) = \prod_{i=0}^{k-1}(1 - p_i).$$

Since $P(T(0) = n|X_0 = 0) = f_{00}^{(n)}$, we have

$$P(T(0) > k|X_0 = 0) = 1 - \sum_{n=1}^{k} f_{00}^{(n)}.$$

So,

$$\sum_{n=1}^{k} f_{00}^{(n)} = 1 - \prod_{i=0}^{k-1}(1 - p_i).$$

Thus, the chain is recurrent iff $\prod_{i=0}^{\infty}(1 - p_i) = 0$. This is equivalent [proof omitted] to $\sum_{i=0}^{\infty} p_i = \infty$. In particular, if $p_i = p \in (0, 1)$, $i = 0, 1, 2, \ldots$ then the success runs chain is recurrent.

Example 3.3 (Random walk on nonnegative integers) Suppose in example 1.1 that $p_0 = 1$, $r_i = 0$ and $p_i = p \in [0, 1]$ for $i = 1, 2, 3, \ldots$. Then [we omit the proof] the chain is recurrent if $p < 1/2$.

4 Positive recurrence and null recurrence

Recall our notation,

$$m_x = E(T(x)|X_0 = x).$$

We then have the following theorem [which we state without proof]

Theorem 4.1 *Let $y \in S$. Then, for any $x \in S$ we have*

$$\lim_{n\to\infty} p_{nd(y)+s}(x, y) = \left[\sum_{r=0}^{\infty} f_{xy}^{(rd(y)+s)}\right] \frac{d(y)}{m_y}, \quad s = 0, 1, 2, \ldots, d(y) - 1.$$

Thus, in particular,

(i) If a state y is transient then $\lim_{n\to\infty} p_n(x,y) = 0, \quad \forall x \in S.$

(ii) If a state y is recurrent and $d(y) = 1$ then $\lim_{n\to\infty} p_n(x,y) = \frac{f_{xy}}{m_y}, \quad \forall x \in S.$

[Note: If y is recurrent and $x \leftrightarrow y$ then $f_{xy} = 1$, and so $\lim_{n\to\infty} p_n(x,y) = \frac{1}{m_y}$*; If y is recurrent and x, y do not communicate then $f_{xy} = \alpha(x,y)$.]*

(iii) If a state y is recurrent then $\lim_{n\to\infty} p_{nd(y)}(y,y) = \frac{d(y)}{m_y}$.

[Other cases are more complicated to formulate as they require the use of periodic subclasses.]

□

We may now introduce the following definition

Definition 4.1 Let x be a recurrent state. Then,

(i) x is *positive recurrent* if $m_x < \infty$,

(ii) x is *null recurrent* if $m_x = \infty$.

Note: if x is transient then $m_x = \infty$.

Remark 4.1 Positive and null recurrence are class properties.

Remark 4.2 An irreducible finite Markov chain is positive recurrent [i.e. all its states are positive recurrent].

Suppose [in order to simplify the discussion below] that P is irreducible [not necessarily aperiodic]. From the above theorem we conclude that

$$\lim_{n\to\infty} (1/n) \sum_{k=0}^{n-1} P^k = \Pi = \begin{pmatrix} \pi \\ \pi \\ \pi \\ \vdots \end{pmatrix},$$

where

$$\pi = [\pi(x)]_{x\in S},$$

and

$$\pi(x) = \frac{1}{m_x}, \quad \forall x \in S.$$

Remark 4.3 If x is a null recurrent state, or a transient state then $\pi(x) = 0$. Thus, for a null recurrent chain, or for a transient chain we have $\pi = \mathbf{0}$.

Remark 4.4 Similarly as in the finite case it can be shown that the vector π satisfies

$$\pi P = \pi, \quad \pi \geq \mathbf{0}.$$

If the chain is positive recurrent then, additionally, we have $\pi > \mathbf{0}$ and $\pi\, \mathbf{1}^T = 1$. In this case the vector π is the invariant probability vector, and is the unique solution to the above system.

Remark 4.5 If the chain is aperiodic, then the above Cesaro limit may be replaced with an ordinary limit. In this case, if additionally the chain is positive recurrent, the vector π represents the limiting probability distribution for the chain.

Remark 4.6 A positive recurrent and aperiodic chain is called *ergodic*.

Example 3.2 continued

Case 1: $0 < p_i = p < 1, \quad i = 0, 1, 2, 3, \ldots$. Then, all states are positive recurrent and $\pi(k) = p(1-p)^k > 0, \quad k = 0, 1, 2, 3, \ldots$.

Case 2: $p_i = p = 0, \quad i = 0, 1, 2, 3, \ldots$. Then, all states are transient.

Case 3: $p_i = p = 1, \quad i = 0, 1, 2, 3, \ldots$. Then the state 0 is positive recurrent, all other states are transient.

Case 4: $p_i = \frac{1}{i+2}, \quad i = 0, 1, 2, 3, \ldots$. Then, all states are null recurrent. $\square$

Example 3.3 continued All assumptions made there are in force. Let $q = 1 - p$. Then,

(a) If $p < q$ [i.e. $p < 1/2$] then all states are positive recurrent, and we have

$$\pi(0) = (1/2)(1 - p/q), \quad \pi(k) = (1/2p)(1 - p/q)(p/q)^k,\ k = 1, 2, 3 \ldots.$$

(b) If $p \geq q$ then all states are transient **[but essential]**. Thus, $\pi = \mathbf{0}$.

Now, study example on page 44 in Lawler.

HW

1) A device services customers one at a time such that if it is handling a customer at time n, it has probability p of finishing his/her service before time $n + 1$. In this latter case, it begins servicing the next waiting customer at time $n+1$. Between time n and $n + 1$, Z_n customers arrive, where

$$P(Z_n = 1) = \gamma, P(Z_n = 0) = 1 - \gamma, 0 < \gamma < 1.$$

Let X_n be the number of customers both awaiting service and being serviced at time n.

a. Find the transition probabilities of $(X_n, n \geq 0)$.

b. Find the condition for the existence of the stationary distribution and find the stationary distribution under this condition. Interpret this distribution. What is the limiting behavior of the chain if this condition is not satisfied?

2) Lawler, Ex. 2.3

3) Lawler, Ex. 2.4