- Convolution Theorem:

If $L^{-1}[\phi_1(s)] = f_1(t)$ and $L^{-1}[\phi_2(s)] = f_2(t)$, then,

$$L^{-1}[\phi(s)] = L^{-1}[\phi_1(s)\cdot\phi_2(s)] = \int_0^t f_1(u)\, f_2(t-u)\, du$$

Note:-

1) Always consider smaller function as $\phi_2(s)$ while choosing $\phi_1(s)$ & $\phi_2(s)$. so that $f_2(t-u)$ is smaller & then we can find integration easily

2) If $\phi(s)$ contains one of the term as $\frac{1}{s}$ then consider $\phi_2(s) = \frac{1}{s}$ while using convolution theorem.

Problems: Solve using Convolution Theorem.

1) $L^{-1}\left[\frac{1}{s(s+a)}\right]$

Sol^n Let $\phi_1(s) = \frac{1}{s+a}$ & $\phi_2(s) = \frac{1}{s}$

$$\therefore L^{-1}[\phi_1(s)] = L^{-1}\left[\frac{1}{s+a}\right] = e^{-at} = f_1(t)$$

$$L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{1}{s}\right] = 1 = f_2(t)$$

By Convolution theorem,

$$L^{-1}[\phi_1(s)\cdot\phi_2(s)] = \int_0^t f_1(u)\, f_2(t-u)\, du$$

$$L^{-1}\left[\frac{1}{(s+a)}\cdot\frac{1}{s}\right] = \int_0^t e^{-au}\, 1\, du$$

$$L^{-1}\left[\frac{1}{s(s+a)}\right] = \left[\frac{e^{-au}}{-a}\right]_0^t = \left[\frac{e^{-at}}{-a} - \frac{1}{-a}\right]$$

$$= \left[-\frac{e^{-at}}{a} + \frac{1}{a}\right]$$

2) $L^{-1}\left[\frac{1}{s(s+a)^2}\right]$

Sol^n Let $\phi_1(s) = \frac{1}{(s+a)^2}$ & $\phi_2(s) = \frac{1}{s}$

$\therefore L^{-1}[\phi_1(s)] = L^{-1}\left[\frac{1}{(s+a)^2}\right] = e^{-at} L^{-1}\left[\frac{1}{s^2}\right] = e^{-at}\frac{t}{\Gamma 2} = e^{-at}\frac{t}{1!}$

$= e^{-at}t = f_1(t)$

$L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{1}{s}\right] = 1 = f_2(t)$

$\therefore$ By Convolution theorem

$L^{-1}[\phi_1(s)\cdot\phi_2(s)] = \int_0^t f_1(u) f_2(t-u)\,du$

$L^{-1}\left[\frac{1}{(s+a)^2}\frac{1}{s}\right] = \int_0^t e^{-au} u \, 1 \, du$

$\therefore L^{-1}\left[\frac{1}{s(s+a)^2}\right] = \int_0^t u e^{-au} du$

$= \left[u\left(\frac{e^{-au}}{-a}\right) - (1)\left(\frac{e^{-au}}{(-a)^2}\right)\right]_0^t$

$= \frac{t e^{-at}}{-a} - \frac{e^{-at}}{a^2} - 0 + \frac{1}{a^2}$

$= -\frac{t e^{-at}}{a} - \frac{e^{-at}}{a^2} + \frac{1}{a^2}$

3) $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$

Sol^n Let $\phi_1(s) = \frac{1}{s^2+a^2}$ & $\phi_2(s) = \frac{1}{s}$

$\therefore L^{-1}[\phi_1(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at = f_1(t)$

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$$L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{1}{s}\right] = 1 = f_2(t)$$

By Convolution Theorem.

$$L^{-1}[\phi_1(s).\phi_2(s)] = \int_0^t f_1(u)\ f_2(t-u)\,du$$

$$L^{-1}\left[\frac{1}{(s^2+a^2)}\ \frac{1}{s}\right] = \int_0^t \frac{1}{a}\sin au\ 1\ \ du$$

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1}{a}\int_0^t \sin au\, du$$

$$= \frac{1}{a}\left[\frac{-\cos au}{a}\right]_0^t$$

$$= \frac{1}{a}\left[\frac{-\cos at}{a} + \frac{1}{a}\right]$$

$$= \frac{1}{a^2}(1-\cos at)$$

4) $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$

Sol^n Let $\phi_1(s) = \frac{s}{s^2+a^2}$ & $\phi_2(s) = \frac{s}{s^2+a^2}$

$$\therefore L^{-1}[\phi_1(s)] = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at = f_1(t)$$

$$\& L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at = f_2(t)$$

By Convolution Theorem.

$$L^{-1}[\phi_1(s).\phi_2(s)] = \int_0^t f_1(u)\ f_2(t-u)\,du$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)}\ \frac{s}{(s^2+a^2)}\right] = \int_0^t \cos au\ \ \cos a(t-u)\,du$$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \int_0^t \cos au\ \ \cos(at-au)\,du$$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \int_0^t \frac{1}{2}\left[\cos(au+at-au)+\cos(au-(at-au))\right]du$$

$$= \frac{1}{2}\int_0^t \left[\cos(at)+\cos(au-at+au)\right] du$$

$$= \frac{1}{2}\int_0^t \left[\cos at+\cos(2au-at)\right] du$$

$$= \frac{1}{2}\left[u\cos at+\frac{\sin(2au-at)}{2a}\right]_0^t$$

$$= \frac{1}{2}\left[t\cos at+\frac{\sin(at)}{2a}-0-\frac{\sin(-at)}{2a}\right]$$

$$= \frac{1}{2}\left[t\cos at+\frac{\sin at}{2a}+\frac{\sin at}{2a}\right] \quad \because \sin(-at) = -\sin(at)$$

$$= \frac{1}{2}\left[t\cos at+\frac{\cancel{2}\sin at}{\cancel{2}a}\right]$$

$$= \frac{1}{2}\left[t\cos at+\frac{\sin at}{a}\right]$$

5) Ex. Using Convolution Theorem find $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$

6) $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

Sol^n Let, $\phi_1(s) = \frac{1}{s^2+a^2}$ & $\phi_2(s) = \frac{s}{s^2+a^2}$

$$\therefore L^{-1}[\phi_1(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at = f_1(t)$$

$$L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at = f_2(t)$$

$\therefore$ By, Convolution Theorem

$$L^{-1}[\phi_1(s)\phi_2(s)] = \int_0^t f_1(u)\, f_2(t-u)\, du$$

$$L^{-1}\left[\frac{1}{(s^2+a^2)} \frac{s}{(s^2+a^2)}\right] = \int_0^t \frac{1}{a}\sin au\, \cos a(t-u)\, du$$

$$= \frac{1}{a}\int_0^t \sin au\, \cos(at-au)\, du$$

$$= \frac{1}{a}\int_0^t \frac{1}{2}\left[\sin(\cancel{au}+at-\cancel{au}) + \sin(au-(at-au))\right] du$$

$$= \frac{1}{2a}\int_0^t \left[\sin at + \sin(au-at+au)\right] du$$

$$= \frac{1}{2a}\int_0^t \left[\sin at + \sin(2au-at)\right] du$$

$$= \frac{1}{2a}\left[u\sin at - \frac{\cos(2au-at)}{2a}\right]_0^t$$

$$= \frac{1}{2a}\left[t\sin at - \frac{\cos at}{2a} - 0 + \frac{\cos(-at)}{2a}\right]$$

$$= \frac{1}{2a}\left[t\sin at - \cancel{\frac{\cos at}{2a}} + \cancel{\frac{\cos at}{2a}}\right]$$

$$= \frac{1}{2a}\, t\sin at$$

7) $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$

Soln Let, $\phi_1(s) = \frac{s}{s^2+a^2}$ & $\phi_2(s) = \frac{s}{s^2+b^2}$

$$\therefore L^{-1}[\phi_1(s)] = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at = f_1(t)$$

$$L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{s}{s^2+b^2}\right] = \cos bt = f_2(t)$$

$\therefore$ By Convolution theorem

$$L^{-1}[\phi_1(s)\cdot \phi_2(s)] = \int_0^t f_1(u)\, f_2(t-u)\, du$$

$$L^{-1}\left[\frac{s}{s^2+a^2}\ \frac{s}{s^2+b^2}\right] = \int_0^t \cos au \ \cos b(t-u)\, du$$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \int_0^t \cos au \ \cos(bt-bu)\, du$$

$$= \frac{1}{2}\int_0^t \left[\cos(au+bt-bu) + \cos(au-(bt-bu))\right] du$$

$$= \frac{1}{2}\int_0^t \left[\cos[(a-b)u+bt] + \cos[au-bt+bu]\right] du$$

$$= \frac{1}{2}\int_0^t \left[\cos[(a-b)u+bt] + \cos[(a+b)u-bt]\right] du$$

$$= \frac{1}{2}\left[\frac{\sin[(a-b)u+bt]}{(a-b)} + \frac{\sin[(a+b)u-bt]}{(a+b)}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin((a-b)t+bt)}{a-b} + \frac{\sin((a+b)t-bt)}{a+b} - \frac{\sin bt}{a-b} - \frac{\sin(-bt)}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin(at-\cancel{bt}+\cancel{bt})}{a-b} + \frac{\sin(at+\cancel{bt}-\cancel{bt})}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

8) Ex $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$ solve using convolution theorem.

9) Ex. $L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right]$ solve using Convolution theorem.

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