Singlular Value Decomposition

step 1: $U = AA^T$

step 2: Calculate eigenvalue and eigenvector of U.

step 3: Normalize and get U.

step 4: $V = A^T A$

step 5: Calculate eignvalue and eignvector of V.

step 6: Normalize and get V and V^T.

step 7: Get Σ by square root of eignvalues of U.

Example: $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

$\rightarrow$

$$U = A \cdot A^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$$

Calculate eignvalue of above matrix.

$$\therefore |AA^T - \lambda I| = 0$$

$$\begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\begin{bmatrix} 17-\lambda & 8 \\ 8 & 17-\lambda \end{bmatrix} = 0$$

$$\therefore (17-\lambda)^2 - 64 = 0$$

$$\lambda^2 - 34\lambda + 225 = 0$$

$$\therefore (\lambda-25)(\lambda-9) = 0$$

$$\therefore \lambda_1 = 25 \quad \lambda_2 = 9$$

Get the eigenvectors of above eigenvalues.

$\lambda = 25$

$$\begin{bmatrix} 17-25 & 8 \\ 8 & 17-25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -8 & 8 \\ 8 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\therefore 8x_1 - 8x_2 = 0$$

$$\therefore x_1 = x_2$$

$$\therefore \text{vector} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = 9$

$$\begin{bmatrix} 17-9 & 8 \\ 8 & 17-9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$8x_1 + 8x_2 = 0$

$\therefore x_1 = -x_2$

$\therefore$ vector $= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore$ eigenvector $= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Normalize this vector by taking square root of every column.

$\therefore$ Normalized vector $= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

$(\because \sqrt{1^2+1^2} = \sqrt{2}$ &

$\sqrt{1^2+(-1)^2} = \sqrt{2})$

$$\therefore \boxed{U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}$$

Now calculate $A^T A$

$$\therefore A^T A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

$$\therefore \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

Here, S_1 = Trace of matrix

$= 13 + 13 + 8$

$= 34$

S_2 = addition of minor elements

$= [(13 \times 8) - (4)] + [(13 \times 8) - 4] + [(13 \times 13) - (12 \times 12)]$

$= (104 - 4) + (104 - 4) + (169 - 144)$

$= 225$

$|A| = 0$

$$\therefore \lambda^3 - 34\lambda^2 + 225\lambda = 0$$

From above equation, calculate eigenvalues.

$\lambda_1 = 25$, $\lambda_2 = 9$, $\lambda_3 = 0$

Calculate eigenvectors for each eigenvalue.

$\therefore$ For $\lambda = 25$

$$\begin{bmatrix} 13-25 & 12 & 2 \\ 12 & 13-25 & -2 \\ 2 & -2 & 8-25 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} -12 & 12 & 2 \\ 12 & -12 & -2 \\ 2 & -2 & -17 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$-12x + 12y + 2z = 0$

$12x - 12y - 2z = 0$

$2x - 2y - 17z = 0$

Using Cramer's Rule,

$$\frac{x}{\begin{vmatrix} -12 & -2 \\ -2 & -17 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 12 & -2 \\ 2 & -17 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 12 & -12 \\ 2 & -2 \end{vmatrix}}$$

$$\therefore \frac{x}{200} = \frac{-y}{-200} = \frac{z}{0}$$

$\therefore x = 1$, $y = 1$ $z = 0$

$\therefore$ vector is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda = 9$, calculate eigenvector.

$$\begin{bmatrix} 13-9 & 12 & 2 \\ 12 & 13-9 & -2 \\ 2 & -2 & 8-9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & 12 & 2 \\ 12 & 4 & -2 \\ 2 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\therefore \quad 4x + 12y + 2z = 0$$
$$12x + 4y - 2z = 0$$
$$2x - 2y - z = 0$$

Using Cramer's rule,

$$\frac{x}{\begin{vmatrix} 4 & -2 \\ -2 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 12 & -2 \\ 2 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 12 & 4 \\ 2 & -2 \end{vmatrix}}$$

$$\frac{-x}{8} = \frac{y}{8} = \frac{-z}{32}$$

$$\therefore \quad x = 1, \quad y = -1, \quad z = 1$$

$$\therefore \text{vector} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

For $\lambda = 0$, calculate eigenvalue.

$$\begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$13x + 12y + 2z = 0$$

$$12x + 13y - 2z = 0$$

$$2x - 2y + 8z = 0$$

Using, Cramer's Rule,

$$\frac{x}{\begin{vmatrix} 13 & -2 \\ -2 & 8 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 12 & -2 \\ 2 & 8 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 12 & 13 \\ 2 & -2 \end{vmatrix}}$$

$$\frac{x}{100} = \frac{-y}{100} = \frac{-z}{50}$$

$$x = 2, \quad y = -2, \quad z = -1$$

$$\therefore \text{vector} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$$

$$\therefore \text{Eigenvectors are} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & -2 \\ 0 & 4 & -1 \end{bmatrix}$$

After Normalization, above vectors will be

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \\ 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \end{bmatrix}$$

Transpose above vector like below

$$\boxed{V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & 4/\sqrt{18} \\ 2/3 & -2/3 & -1/3 \end{bmatrix}}$$

Next calculate Σ.

$$\therefore \boxed{\Sigma = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \Leftarrow \begin{bmatrix} \sqrt{25} & 0 & 0 \\ 0 & \sqrt{9} & 0 \\ 0 & 0 & \sqrt{0} \end{bmatrix}$$

$\therefore$ <u>SVD = $U \Sigma V^T$</u>

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{18} & -1/\sqrt{18} & 4/\sqrt{18} \\ 2/3 & -2/3 & -1/3 \end{bmatrix}$$