Diagonalisation of Matrices

Modal matrix

Theorem:- A square non singular matrix A whose eigen values are all distinct can be diagonalised by a similarity transformation $D = M^{-1}AM$ where M is the matrix whose columns are the eigen vectors of A and D is the diagonal matrix whose diagonal elements are the eigen values of A.

Note:-
If all the eigen values of A are distinct then A is diagonalisable.

Algebraic & Geometric multiplicity

* If an eigen value λ_1 of matrix A is repeated t times then t is called the algebraic multiplicity of λ_1.

* If corresponding to an eigen value λ_1, there are s linearly independent eigen vectors then s is called the geometric multiplicity of λ_1.

① Show that the matrix $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ is diagonalisable. Find the transforming matrix and the diagonal matrix.

Soln:- In previous example the eigen values and eigen vectors are calculated.

$$\lambda = 0, 3, 15$$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \quad X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Since all the eigen values are distinct, the matrix is diagonalisable.

[The eigen vectors correspond to distinct eigen values of real symmetric matrix are orthogonal].

Since A is symmetric matrix M can be calculated by normalising the eigen vectors.

Norm of x_1, x_2, x_3 is $\sqrt{1+4+4} = \sqrt{9} = 3$.

$$\therefore M = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Since $M^{-1}AM = D$, the given matrix can be diagonalised to diagonal matrix $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

by transforming matrix $M = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

② Show that the matrix $A = \begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix}$ is diagonalisable. Find the diagonal form D & the diagonalising matrix M.

Soln:- The characteristic eqn is

$$\lambda^2 - (-9+3+7)\lambda + [(-27+21-63)-(-32-64+24)]\lambda - |A| = 0.$$

$$\lambda^2 - \lambda + \overset{-5}{\cancel{3}}\lambda - 3 = 0.$$

$$\lambda = -1, -1, 3.$$

$\lambda = -1$

$$\begin{pmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

By using Row-transformations

$R_2 - R_1$
$$\begin{pmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ -16 & 8 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$R_3 - 2R_1$
$$\begin{pmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

$$-8x_1 + 4x_2 + 4x_3 = 0.$$

$-2x_1 + x_2 + x_3 = 0$

Rank = 1

No. of unknown variables = 3

No. of linearly independent solutions = $n - r$

$= 3 - 2$

$= 1$

Let $x_2 = s$

$x_3\ t$

$-2x_1 + s + t = 0$

$-2x_1 = -s - t$

$x_1 = s/2 + t/2$

$$\therefore X = \begin{pmatrix} s/2 + t/2 \\ s + 0 \\ 0 + t \end{pmatrix}$$

$$= s\begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix}$$

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Algebraic multiplicity = 2 $[\lambda = -1, -1]$

Geometric multiplicity = 2. $\left[X = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix}\right]$

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$\lambda = 3$

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

By cramer's rule,

$$\frac{x_1}{16-0} = \frac{-x_2}{-48+32} = \frac{x_3}{0+32} = t$$

$$\frac{x_1}{16} = \frac{-x_2}{-16} = \frac{x_3}{32} = t$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{2} = t$$

$x_1 = t$, $x_2 = t$, $x_3 = 2t$

$$\therefore X = \begin{pmatrix} t \\ t \\ 2t \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

Algebraic multiplicity $= 1$

Geometric multiplicity $= 1$.

$\therefore$ The matrix A can be diagonalised to a diagonal matrix $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ using the transforming matrix $M = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

③ Show that the matrix $A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$ is not similar to a diagonal matrix.

Soln:- The characteristic eqn is

$$\lambda^3 - 5\lambda^2 + ((4+2+2) - (0+0+0))\lambda - |A| = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\lambda = 1, 2, 2$$

$\lambda = 2$

$$\begin{pmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

By cramer's rule

$$\frac{x_1}{-3} = \frac{-x_2}{0} = \frac{x_3}{0}$$ (By considering first 2 rows)

$$\therefore X = \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix}$$

Algebraic multiplicity = 2 (2 eigen values)
Geometric multiplicity = 1 (1 eigen vector)

Since algebraic multiplicity is not equal to geometric multiplicity, the matrix A is not diagonalisable.

Remark :-
The necessary and sufficient condition of a square matrix to be similar to a diagonal matrix is that the geometric multiplicity of each of its eigen values coincides with the algebraic multiplicity.

Exercise (i) $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ ii) $A = \begin{pmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{pmatrix}$

iii) $A = \begin{pmatrix} -17 & 18 & -6 \\ -18 & 19 & -6 \\ -9 & 9 & 2 \end{pmatrix}$

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Note :-

* A matrix A is orthogonal iff the columns [rows] of A form an orthonormal set of vectors

* Every symmetric matrix is orthogonally diagonalisable.

Exercise

Show that the following matrices are similar to diagonal matrices. Find the diagonal form & the diagonalising matrix

(i) $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ (ii) $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

(1) S.T $A = \begin{pmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{pmatrix}$ is not diagonalisable