



Existence of social functions

The **existence of social welfare functions** is a fundamental concept in social choice theory, which examines how individual preferences can be aggregated into a collective decision or social choice. A **social welfare function (SWF)** is a mechanism or rule that takes individual preferences and combines them to produce a ranking or selection that represents the "best" collective choice for society.

1. Arrow's Impossibility Theorem

One of the key results related to the existence of social welfare functions is **Arrow's Impossibility Theorem** (also known as **Arrow's Paradox**), formulated by economist Kenneth Arrow in 1951. Arrow's theorem is a cornerstone of social choice theory and describes the limitations of social welfare functions under certain fairness criteria.

Conditions for a Fair Social Welfare Function

Arrow identified five desirable conditions that a fair social welfare function should meet:

1. **Unrestricted Domain (or Universality)**: The social welfare function should be able to produce a collective decision for any set of individual preferences. That is, it should work no matter how voters rank the alternatives.
2. **Pareto Efficiency (or Pareto Optimality)**: If every individual prefers one option (say A) over another (say B), then the social welfare function should rank A higher than B in the collective decision.
3. **Non-Dictatorship**: No single individual (a "dictator") should have the power to determine the social ranking of alternatives based solely on their own preferences, regardless of the preferences of others.
4. **Independence of Irrelevant Alternatives (IIA)**: The collective ranking of any two alternatives (say A and B) should depend only on the individuals' preferences between A and B, not on their preferences regarding other alternatives (e.g., C).
5. **Transitivity**: The social welfare function should produce a ranking that is logically consistent. If the function ranks alternative A higher than B and B higher than C, then it should also rank A higher than C (i.e., the collective preferences should be transitive).

Arrow's Impossibility Result

Arrow's theorem states that **no social welfare function** can satisfy all five of these conditions simultaneously when there are three or more alternatives and at least two voters.

- **Interpretation**: This means that any attempt to aggregate individual preferences into a collective decision will either violate one or more of the fairness conditions or be dictated by a single individual's preferences.
- **Implications**: The theorem shows that designing a voting system or a collective decision-making process that is perfectly fair and rational is impossible in practice. This does not mean that collective decision-making is impossible, but it highlights the trade-offs involved in choosing a social welfare function.

2. Social Welfare Function vs. Social Choice Function



It is important to distinguish between a **social welfare function** and a **social choice function**:

- **Social Welfare Function (SWF)**: This function assigns a complete ordering or ranking of all possible alternatives based on the preferences of individuals in society. It provides a way to rank every possible outcome.
- **Social Choice Function (SCF)**: This function selects one or more alternatives from a set of possible choices based on individual preferences. In contrast to an SWF, an SCF does not need to rank all alternatives but simply chooses one or more as the "best" choices.

Example of SWF

Suppose there are three alternatives, A, B, and C, and three individuals with the following preferences:

- Individual 1: $A > B > C$
- Individual 2: $B > C > A$
- Individual 3: $C > A > B$

A social welfare function would take these individual rankings and produce a societal ranking, for example: $A > B > C$, or some other ordering, depending on the function used.

3. Other Approaches in Social Choice

Given the limitations imposed by Arrow's Theorem, alternative methods have been explored to aggregate preferences that relax one or more of the conditions Arrow set forth. Some of these include:

3.1 Relaxing the Independence of Irrelevant Alternatives (IIA)

In practice, many voting systems, like the **Borda count** and **approval voting**, violate the IIA condition. These systems allow rankings or choices to be influenced by irrelevant alternatives, yet they are still widely used because they provide a reasonable aggregation of preferences.

3.2 Relaxing Unrestricted Domain

One approach is to restrict the set of allowable individual preferences. For example, if individual preferences are required to be "single-peaked" (i.e., there is a clear consensus on what is worse or better), certain aggregation methods can meet Arrow's conditions without contradiction.

3.3 Condorcet Criterion

A **Condorcet winner** is an alternative that can defeat every other alternative in pairwise comparison. Methods like the **Condorcet method** focus on finding the Condorcet winner, if one exists, although this approach may still face the issue of cyclical preferences (Condorcet paradox).

4. Gibbard-Satterthwaite Theorem



The **Gibbard-Satterthwaite Theorem** is another important result in social choice theory, extending the impossibility results of Arrow. It states that for any non-trivial voting system that chooses between three or more options, there is always some incentive for voters to vote strategically (i.e., insincerely) rather than according to their true preferences. This means that strategic manipulation is unavoidable in certain voting systems.