Z-Transforms

Z-transform is used in discrete analysis. Its role in discrete analysis is same as that of Laplace & Fourier transforms in continuous system.

Z-transform is operate on sequence of the discrete integer valued arguments i.e. $k = 0, \pm 1, \pm 2, \dots$

Communication is one of the field whose development is based on discrete analysis.

Difference equations are also based on discrete system & their solutions & analysis are done by Z-transform. For every operational rule & application of Laplace transform there corresponds an operational rule & application of Z-transform.

e.g. Linearity property, convolution theorem.

* Sequence:-

Sequence $\{f(k)\}$ is an ordered list of real or complex numbers.

e.g. i) $\{5^0, 5^1, 5^2, \dots, 5^k, \dots\}$

For $k=0$, $f(k) = 5^0$, for $k=1$, $f(k) = 5^1$, ...

ii) $f(k) = \{15, 10, \underset{\uparrow}{7}, 4, 1, -1, 0\}$

arrow indicates the element corresponding to $k=0$

The elements on the left of the arrow correspond to $k = -1, -2, \dots$ & to the right correspond to $k = 1, 2, 3, \dots$

* Basic Operations on sequence:-

i) Let $\{f(k)\}$ & $\{g(k)\}$ be two sequences having same number of terms.

1) Addition :- $\{f(k)\} + \{g(k)\} = \{f(k) + g(k)\}$

e.g. $\{f(k)\} = \{1^3, 2^3, 3^3, 4^3, \ldots\}$

$\{g(k)\} = \{1^2, 2^2, 3^2, 4^2, \ldots\}$

$\therefore \{f(k)\} + \{g(k)\} = \{(1^3+1^2), (2^3+2^2), (3^3+3^2), \ldots\}$

& $\{f(k)\} - \{g(k)\} = \{(1^3-1^2), (2^3-2^2), \ldots\}$

2) Multiplication:- Let a be any scalar

$a\{f(k)\} = \{a f(k)\}$

e.g. $\{f(k)\} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$

$\therefore 3 \cdot \{f(k)\} = \{3, \frac{3}{2}, \frac{3}{3}, \ldots\}$

3) Linearity :- $a\{f(k)\} + b\{g(k)\} = \{a f(k) + b g(k)\}$

* Convergence & Divergence:-

Convergent sequence:- If $\{f(k)\}$ is a given sequence & if $f(k)$ tends to a (finite) real number L as k tends to ∞ then $\{f(k)\}$ is called convergent sequence.

A sequence which is not convergent i.e which does not tend to a (finite) real number is called Divergent sequence.

e.g. 1) $2, 2, 2, \ldots$ converges to 2

2) $1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{k}, \ldots$ converges to 0

3) $1, 2, 3, \ldots, k, \ldots$ diverges to ∞

4) $1, 3, 1, 3, \ldots$ oscillates between 1 & 3

* **Def^n: Z- transform:-**

The Z-transform of a sequence $\{f(K)\}$ is denoted by $Z\{f(K)\}$ & is defined as

$$Z\{f(K)\} = F(z) = \sum_{K=-\infty}^{\infty} f(K)\, z^{-K} = \sum_{K=-\infty}^{\infty} \frac{f(K)}{z^K}$$

where z is complex number & Z is an operator.

Ex: 1) If $f(K) = \{-6, -3, \underset{\uparrow}{0}, 2, 4\}$ then

$$Z\{f(K)\} = \sum_{K=-2}^{2} f(K)\, z^{-K}$$

$$= f(-2)\, z^{+2} + f(-1)\, z^{+1} + f(0)\cdot z^{0} + f(1)\cdot z^{-1} + f(2)\cdot z^{-2}$$

$$= (-6)\, z^2 + (-3)\cdot z + 0 + 2\left(\tfrac{1}{z}\right) + 4\left(\tfrac{1}{z^2}\right)$$

$$\therefore \boxed{Z\{f(K)\} = -6z^2 - 3z + \frac{2}{z} + \frac{4}{z^2}}$$

2) If $\{f(K)\} = \{3^0, 3^1, 3^2, 3^3, ---\}$ then

$$Z\{f(K)\} = \sum_{K=-\infty}^{\infty} f(K)\, z^{-K} = \sum_{K=0}^{\infty} f(K)\, z^{-K}$$

$$= 3^0\cdot z^0 + 3^1 z^{-1} + 3^2 z^{-2} + -----$$

$$= 1 + \frac{3}{z} + \frac{3^2}{z^3} + \frac{3^3}{z^3} + ---$$

$$= 1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + \left(\frac{3}{z}\right)^3 + ---$$

$$\left(\because 1 + ar + ar^2 + --- = \frac{a}{1-r}\ ; \text{if } |r|<1\right)$$

$$= \frac{1}{1-(3/z)} \quad \text{if } \left|\frac{3}{z}\right| < 1$$

$$\therefore \boxed{Z\{f(K)\} = \frac{z}{z-3}}$$

3) If $\{f(K)\} = \frac{1}{3^K}$

$$= \{ --- 27, 9, 3, 1, \tfrac{1}{3}, \tfrac{1}{9}, \tfrac{1}{27}, --- \}$$

$$\therefore Z\{f(K)\} = --- + 27z^3 + 9z^2 + 3z + 1 + \frac{1}{3z} + \frac{1}{9z^2} + \frac{1}{27z^3} + ---$$

Note:-

1) $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$ if $|x| < 1$

2) $1 - x + x^2 - x^3 + \cdots = \frac{1}{1+x}$ if $|x| < 1$

3) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$

4) $1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots = (1+x)^n$

The region for which $\Sigma f(K) z^{-K}$ is convergent is called the <u>region</u> <u>of</u> <u>convergence</u> & is denoted by R.O.C

<u>Ex</u>: 1) Find the Z-transform & region of convergence of

$$f(K) = 5^K, \quad K < 0$$
$$= 3^K, \quad K \geq 0$$

$\rightarrow$

$$Z\{f(K)\} = \sum_{K=-\infty}^{\infty} f(k) z^{-K}$$

$$= \sum_{K=-\infty}^{-1} 5^K . z^{-K} + \sum_{K=0}^{\infty} 3^K . z^{-K}$$

$$= (\cdots + 5^{-3}z^3 + 5^{-2}z^2 + 5^{-1}z) + (1 + 3.z^{-1} + 3^2.z^{-2} + \cdots)$$

$$= (\cdots + (5^{-1}z)^3 + (5^{-1}z)^2 + 5^{-1}z) + \left(1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + \cdots\right)$$

$$= \frac{5^{-1}z}{1 - 5^{-1}z} + \frac{1}{1 - 3/z}$$

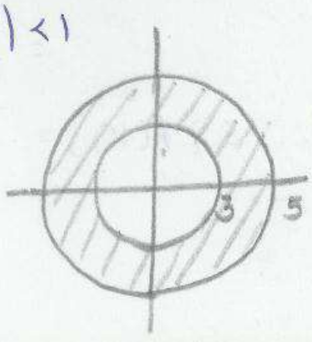
$$= \frac{z}{5 - z} + \frac{z}{z - 3}$$

$$= \frac{2z}{(5-z)(z-3)}$$

which is convergent if $\left|\frac{z}{5}\right| < 1$, $\left|\frac{3}{z}\right| < 1$

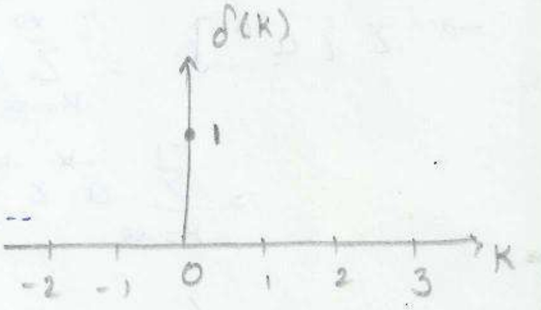
i.e. $|z| < 5$, $|z| > 3$

$\therefore$ ROC is $3 < |z| < 5$



2) find z-transform of unit Impulse function

$$\delta(K) = 1 \;,\; K=0$$
$$= 0 \;,\; \text{otherwise}$$



$\rightarrow$ $$Z\{\delta(K)\} = \sum_{K=-\infty}^{\infty} \delta(K)\, z^{-K}$$

$$= \cdots\, 0 + 0 + 1\cdot z^{0} + 0 + 0 + \cdots$$

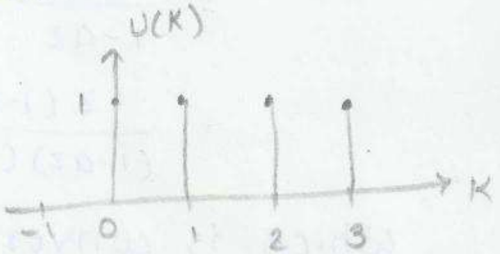
$$= 1$$

which is convergent $\forall z$

$\therefore$ ROC is whole of z-plane.

3) find z-transform of discrete unit step function

$$U(K) = 1 \;,\; K \geq 0$$
$$= 0 \;,\; K < 0$$



$\rightarrow$ $$Z\{U(K)\} = \sum_{K=-\infty}^{\infty} U(K)\cdot z^{-K}$$

$$= \sum_{K=-\infty}^{-1} 0\cdot z^{-K} + \sum_{K=0}^{\infty} 1\cdot z^{-K} = \sum_{K=0}^{\infty} z^{-K}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots = \frac{1}{1 - 1/z} = \frac{z}{z-1}$$

which is convergent if $\left|\frac{1}{z}\right| < 1$ i.e. $|z| > 1$

4) $$f(K) = \frac{a^K}{K!} \;,\; K \geq 0$$

$\rightarrow$ $$Z\{f(K)\} = \sum_{K=0}^{\infty} \frac{a^K}{K!}\, z^{-K}$$

$$= \sum_{K=0}^{\infty} \frac{a^K}{z^K}\cdot\frac{1}{K!} = \sum_{K=0}^{\infty} \left(\frac{a}{z}\right)^K \frac{1}{K!}$$

$$= 1 + \frac{a}{z} + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \frac{1}{3!}\left(\frac{a}{z}\right)^3 + \cdots$$

$$= e^{a/z}$$

ROC is whole of z-plane except at $z=0$

Ⓒ Linearity

1) Find $Z\{a^{|K|}\}$

$\rightarrow$ $Z\{a^{|K|}\} = \sum_{K=-\infty}^{\infty} a^{|K|} z^{-K}$

$= \sum_{K=-\infty}^{-1} a^{-K} z^{-K} + \sum_{K=0}^{\infty} a^{K} z^{-K}$

$= (\cdots\cdots + a^3 z^3 + a^2 z^2 + az) + (1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \cdots\cdots)$

$= \frac{az}{1-az} + \frac{1}{1-a/z}$

$= \frac{az}{1-az} + \frac{z}{z-a} = \frac{az(z-a) + z(1-az)}{(1-az)(z-a)}$

$= \frac{z(1-a^2)}{(1-az)(z-a)}$

which is convergent if $|az| < 1$ & $|\frac{a}{z}| < 1$ i.e. $|a| < |z|$

i.e. $|z| < \frac{1}{a}$ & $|z| > a$

$\therefore$ ROC is $a < |z| < \frac{1}{a}$

2) $Z\{(\frac{1}{2})^{|K|}\}$

$= \sum_{K=-\infty}^{\infty} (\frac{1}{2})^{|K|} z^{-K} = \sum_{K=-\infty}^{-1} (\frac{1}{2})^{-K} z^{-K} + \sum_{K=0}^{\infty} (\frac{1}{2})^{K} z^{-K}$

$= \{\cdots\cdots + (\frac{1}{2})^3 z^3 + (\frac{1}{2})^2 z^2 + (\frac{1}{2}) z\} + \{1 + (\frac{1}{2}) z^{-1} + (\frac{1}{2})^2 z^{-2} + \cdots\}$

$= (\frac{z}{2} + (\frac{z}{2})^2 + (\frac{z}{2})^3 + \cdots\cdots) + (1 + \frac{1}{2z} + \frac{1}{(2z)^2} + \frac{1}{(2z)^3} + \cdots\cdots)$

$= \frac{z}{2}[1 + \frac{z}{2} + (\frac{z}{2})^2 + \cdots\cdots] + \frac{1}{1 - 1/2z}$

$= \frac{z}{2} \cdot \frac{1}{1 - z/2} + \frac{1}{1 - 1/2z}$, if $|\frac{z}{2}| < 1$, $|\frac{1}{2z}| < 1$

$= \frac{z}{2-z} + \frac{2z}{2z-1}$, if $|z| < 2$ & $\frac{1}{2} < |z|$

$= \frac{3z}{(2-z)(2z-1)}$, $\frac{1}{2} < |z| < 2$

* Find Z-transform of

1) $f(K) = \sin \alpha K$, $K \geq 0$, α is real

$\rightarrow$ $Z\{f(K)\} = \sum_{K=-\infty}^{\infty} f(K) z^{-K} = \sum_{K=0}^{\infty} f(K) z^{-K}$

$= \sum_{K=0}^{\infty} \sin\alpha K \cdot z^{-K} = \sum_{K=0}^{\infty} \left(\frac{e^{i\alpha K} - e^{-i\alpha K}}{2i} \right) z^{-K}$

$= \frac{1}{2i} \sum_{K=0}^{\infty} (e^{i\alpha K} \cdot z^{-K} - e^{-i\alpha K} \cdot z^{-K})$

$= \frac{1}{2i} \Big[(1 + e^{i\alpha} \cdot z^{-1} + e^{2i\alpha} \cdot z^{-2} + \dots) -$

$(1 + e^{-i\alpha} \cdot z^{-1} + e^{-2i\alpha} \cdot z^{-2} + \dots) \Big]$

$= \frac{1}{2i} \left[\left(1 + \frac{e^{i\alpha}}{z} + \frac{e^{2i\alpha}}{z^2} + \dots\right) - \left(1 + \frac{1}{e^{i\alpha} z} + \frac{1}{e^{2i\alpha} z^2} + \dots\right) \right]$

$= \frac{1}{2i} \left[\frac{1}{1 - \frac{e^{i\alpha}}{z}} - \frac{1}{1 - \frac{e^{-i\alpha}}{z}} \right]$, $\left|\frac{e^{i\alpha}}{z}\right| < 1$ & $\left|\frac{e^{-i\alpha}}{z}\right| < 1$

$= \frac{1}{2i} \left[\frac{z}{z - e^{i\alpha}} - \frac{z}{z - e^{-i\alpha}} \right]$, $|e^{i\alpha}| < |z|$ & $|z| > |e^{-i\alpha}|$

$= \frac{z}{2i} \left[\frac{e^{i\alpha} - e^{-i\alpha}}{(z - e^{i\alpha})(z - e^{-i\alpha})} \right]$, $|z| > 1$ $(\because |e^{i\alpha}| = 1)$

$= \frac{z}{2i} \left[\frac{2i \cdot \sin\alpha}{(z - e^{i\alpha})(z - e^{-i\alpha})} \right]$

$$\therefore \boxed{Z\{\sin\alpha K\} = \frac{z \cdot \sin\alpha}{z^2 - 2z\cos\alpha + 1}} , \quad |z| > 1$$

Similarly,

2) $Z\{\cos\alpha K\} = \frac{z(z - \cos\alpha)}{z^2 - 2z\cos\alpha + 1}$, $|z| > 1$

3) $Z\{\sinh\alpha K\} = \frac{z \cdot \sinh\alpha}{z^2 - 2z\cosh\alpha + 1}$, $|z| > \max\{|e^{\alpha}|, |e^{-\alpha}|\}$

4) $Z\{\cosh\alpha K\} = \frac{z(z - \cosh\alpha)}{z^2 - 2z\cosh\alpha + 1}$, $|z| > \max\{|e^{\alpha}|, |e^{-\alpha}|\}$

Ex 1) Find $Z\{\sin(ak+b)\}$, $k \geq 0$

$\rightarrow$ $\sin(ak+b) = \sin ak \cdot \cos b + \cos ak \cdot \sin b$

$$\therefore Z\{\sin(ak+b)\} = \cos b \; Z\{\sin ak\} + \sin b \; Z\{\cos ak\} \quad \text{... linearity}$$

$$= \cos b \cdot \frac{z \cdot \sin a}{z^2 - 2z\cos a + 1} + \sin b \cdot \frac{z(z-\cos a)}{z^2 - 2z\cos a + 1}$$

$$= \frac{z[\sin a \cdot \cos b + z\sin b - \cos a \cdot \sin b]}{z^2 - 2z \cdot \cos a + 1}$$

$$= \frac{z[\sin(a-b) + z\sin b]}{z^2 - 2z\cos a + 1}$$

2) Find $Z\{\cos(\frac{k\pi}{8} + \alpha)\}$, $k \geq 0$

$$\rightarrow Z\{\cos(\tfrac{k\pi}{8} + \alpha)\} = \cos\alpha \cdot Z\{\cos \tfrac{k\pi}{8}\} + \sin\alpha \cdot Z\{\sin \tfrac{k\pi}{8}\}$$

$$= \cos\alpha \cdot \frac{z^2 - z\cos(\pi/8)}{z^2 - 2z\cos(\pi/8) + 1} - \sin\alpha \cdot \frac{z \cdot \sin(\pi/8)}{z^2 - 2z\cos(\pi/8) + 1}$$

$$= \frac{z[z\cos\alpha - \cos(\pi/8) \cdot \cos\alpha - \sin\alpha \cdot \sin(\pi/8)]}{z^2 - 2z\cos(\pi/8) + 1}$$

$$= \frac{z[z\cos\alpha - \cos(\pi/8 - \alpha)]}{z^2 - 2z\cos(\pi/8) + 1}$$

* **Change of Scale :-**

If $Z\{f(k)\} = F(z)$ then $Z\{a^k \cdot f(k)\} = F(\frac{z}{a}) \cdot f$

if R.O.C of $Z\{f(k)\}$ is $R_1 < |z| < R_2$ then ROC of $Z\{a^k \cdot f(k)\}$ is $|a| \cdot R_1 < |z| < |a| \cdot R_2$

1) Find $Z\{c^k \cdot \sin\alpha k\}$

$\rightarrow$ Let $Z\{\sin\alpha k\} = \dfrac{z \cdot \sin\alpha}{z^2 - 2z\cos\alpha + 1}$

$\therefore$ by change of scale property,

$$Z\{c^k \cdot \sin\alpha k\} = \frac{(z/c)\cdot \sin\alpha}{(z/c)^2 + 2(z/c)\cdot\cos\alpha + 1}$$

$$= \frac{cz\sin\alpha}{z^2 - 2cz\cos\alpha + c^2}$$

2) Find $Z\{2^k \cdot \cos(3k+2)\}$, $k \geq 0$

$\rightarrow$ $\cos(3k+2) = \cos 3k \cdot \cos 2 - \sin 3k \cdot \sin 2$

$\therefore$ $Z\{\cos(3k+2)\} = \cos 2 \cdot Z\{\cos 3k\} - \sin 2 \cdot Z\{\sin 3k\}$ --- Linearity

$$= \cos 2 \cdot \frac{z(z-\cos 3)}{z^2 - 2z\cos 3 + 1} - \sin 2 \, \frac{z\sin 3}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z[z\cos 2 - \cos 3\cdot\cos 2 - \sin 3\cdot\sin 2]}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z[z\cos 2 - \cos 1]}{z^2 - 2z\cos 3 + 1}$$

by change of scale property,

$$Z\{2^k \cdot \cos(3k+2)\} = \frac{(z/2)\,[(z/2)\cos 2 - \cos 1]}{(z/2)^2 - 2(z/2)\cos 3 + 1}$$

$$= \frac{z[z\cos 2 - 2\cos 1]}{z^2 - 4z\cos 3 + 4}$$

HW 3) Find $Z\{3^k \cdot \cosh\alpha k\}$, $k \geq 0$

* **Shifting Property** :-

If $Z\{f(k)\} = F(z)$ then $Z\{f(k+n)\} = z^n F(z)$

& $Z\{f(k-n)\} = z^{-n} F(z)$

1) Find $Z\{\frac{1}{k+1}\}$, $k \geq 1$ & ROC

→ Consider $Z\{\frac{1}{k}\} = \sum_{k=1}^{\infty} \frac{1}{k} \cdot z^{-k}$

$$= \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \cdots$$

$$= -\log\left(1 - \frac{1}{z}\right), \quad |z| > 1$$

By shifting property,

$$Z\left\{\frac{1}{k+1}\right\} = z\left\{-\log\left(1-\frac{1}{z}\right)\right\}$$

$$= -z \cdot \log\left(1 - \frac{1}{z}\right), \quad |z| > 1$$

2) Find $Z\{a^k \cdot \delta(k-n)\}$, $k \geq 0$

→ let $Z\{\delta(k)\} = 1$

∴ By shifting property,

$$Z\{\delta(k-n)\} = z^{-n} \cdot (1) = z^{-n}$$

∴ By change of scale property,

$$Z\{a^k \cdot \delta(k-n)\} = \left(\frac{z}{a}\right)^{-n}$$

* **Multiplication by k** :-

If $Z\{f(k)\} = F(z)$ then

$$Z\{k \cdot f(k)\} = -z \frac{d}{dz} F(z)$$

In general, $Z\{k^n \cdot f(k)\} = \left(-z \frac{d}{dz}\right)^n F(z)$

1) Find $Z\{(k+1)a^k\}$, $k \geq 0$

$\rightarrow$ $Z\{(k+1)a^k\} = Z\{k \cdot a^k\} + Z\{a^k\}$

$$Z\{a^k\} = \sum_{k=0}^{\infty} a^k z^{-k} = 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots$$

$$= \frac{1}{1 - a/z} = \frac{z}{z-a}$$

$\therefore$ by multiplication by k property,

$$Z\{k \cdot a^k\} = -z \frac{d}{dz}\left(\frac{z}{z-a}\right)$$

$$= -z\left[\frac{(z-a) - z(1)}{(z-a)^2}\right] = \frac{-az}{(z-a)^2}$$

$$\therefore Z\{(k+1)a^k\} = \frac{-az}{(z-a)^2} + \frac{z}{z-a} = \frac{z^2}{(z-a)^2}$$

2) Find $Z\{k^2 a^{k-1}\}$, $k \geq 0$

$\rightarrow$ Let $Z\{a^k\} = \frac{z}{z-a}$

$\therefore$ by shifting property, $Z\{f(k-n)\} = z^{-n}F(z)$

$$\therefore Z\{a^{k-1}\} = z^{-1}\left(\frac{z}{z-a}\right) = \frac{1}{z-a}$$

$\therefore$ by multiplication by k,

$$Z\{k \cdot a^{k-1}\} = -z \frac{d}{dz}\left(\frac{1}{z-a}\right) = \frac{z}{(z-a)^2}$$

$$\therefore Z\{k^2 a^{k-1}\} = -z \frac{d}{dz}\left(\frac{z}{(z-a)^2}\right)$$

$$= \frac{-z\left[(z-a)^2(1) - z\,2(z-a)\right]}{(z-a)^4}$$

$$= -z\left[\frac{(z-a) - 2z}{(z-a)^3}\right]$$

$$= \frac{z(z+a)}{(z-a)^3}, \quad |z| > |a|$$

* Initial Value :-

If $Z\{f(K)\} = F(z)$, $K \geq 0$ then $f(0) = \lim_{z \to \infty} F(z)$

* Final value

$$\lim_{K \to 0} f(K) = \lim_{z \to 1} (z-1) F(z)$$

* Convolution Theorem :-

If $Z\{f(K)\} = F(z)$ & $Z\{g(K)\} = G(z)$ then

$Z\{f(K) * g(K)\} = F(z) \cdot G(z)$

where $f(K) * g(K) = \sum_{n=-\infty}^{\infty} f(m) \cdot g(k-m)$

$$= \sum_{n=-\infty}^{\infty} g(m) \cdot f(K-m).$$

Ex: 1) If $f(K) = \frac{1}{2^K} * \frac{1}{3^K}$ find $Z\{f(K)\}$, $K \geq 0$

$\rightarrow$ $Z\left\{\frac{1}{2^K}\right\} = \sum_{K=0}^{\infty} \frac{1}{2^K} z^{-K} = 1 + \frac{1}{2z} + \left(\frac{1}{2z}\right)^2 + \cdots$

$$= \frac{1}{1 - 1/2z} = \frac{2z}{2z-1} , \quad |1/2z| < 1 \Rightarrow |z| > \frac{1}{2}$$

& $Z\left\{\frac{1}{3^K}\right\} = \sum_{K=0}^{\infty} \frac{1}{3^K} z^{-K} = 1 + \frac{1}{3z} + \frac{1}{(3z)^2} + \cdots$

$$= \frac{1}{1 - 1/3z} = \frac{3z}{3z-1} , \quad |1/3z| < 1 \Rightarrow |z| > \frac{1}{3}$$

$\therefore$ By convolution th^m,

$$Z\{f(K)\} = Z\left\{\frac{1}{2^K}\right\} \cdot Z\left\{\frac{1}{3^K}\right\}$$

$$= \left(\frac{2z}{2z-1}\right) \cdot \left(\frac{3z}{3z-1}\right) , \quad |z| > 1/2$$

2) If $f(K) = 4^K \cdot U(K)$ & $g(K) = 5^K \cdot U(K)$. find $Z\{f(K) * g(K)\}$, $K \geq 0$

$\rightarrow$ $f(K) = 4^K \cdot U(K) = \{4^0, 4^1, 4^2, \cdots\}$

& $g(K) = 5^K \cdot U(K) = \{5^0, 5^1, 5^2, \cdots\}$

$$\therefore Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} = 1 + \frac{4}{z} + \left(\frac{4}{z}\right)^2 + \cdots$$

$$= \frac{1}{1-4/z} = \frac{z}{z-4} \quad , \quad |4/z| < 1 \Rightarrow |z| > 4$$

$$\& \quad Z\{g(k)\} = \sum_{k=0}^{\infty} g(k) \cdot z^{-k} = 1 + \frac{5}{z} + \left(\frac{5}{z}\right)^2 + \cdots$$

$$= \frac{1}{1-5/z} = \frac{z}{z-5} \quad , \quad |5/z| < 1 \Rightarrow |z| > 5$$

$\therefore$ by convolution thm,

$$Z\{f(k) * g(k)\} = \frac{z}{z-4} \cdot \frac{z}{z-5} = \frac{z^2}{(z-4)(z-5)}$$

Thm:- If $Z\{f(k)\} = F(z)$ then $Z\{e^{-ak} \cdot f(k)\} = F(e^a z)$

$$\rightarrow \quad Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) \cdot z^{-k} = F(z)$$

$$\therefore Z\{e^{-ak} \cdot f(k)\} = \sum_{k=-\infty}^{\infty} e^{-ak} \cdot f(k) z^{-k} = \sum_{k=-\infty}^{\infty} f(k) (e^a z)^{-k}$$

$$= F(e^a z)$$

* **Inverse Transform:-**

If $Z\{f(k)\} = F(z)$ then $\{f(k)\}$ is called an inverse Z transform of $F(z)$ & written as $f(k) = Z^{-1}\{F(z)\}$

- To find inverse z-transform we should know its region of convergence i.e. ROC
- Three methods to find inverse z-transform.

1) Direct Division
2) Binomial expansion
3) Partial fraction

1) **Direct Division:-**

Here we divide the numerator by the denominator & obtain a power series.

Ex: 1) $\dfrac{z}{z-a}$ i) $|z| > a$ ii) $|z| < a$

$\rightarrow$ i) $|z| > a \Rightarrow \left|\frac{a}{z}\right| < 1$

$$
\begin{array}{r|l}
 & 1 + a/z + a^2/z^2 + a^3/z^3 \\
\hline
z-a & z \\
 & -\,z - a \\
 & \quad a \\
 & -\,a - a^2/z \\
 & \quad a^2/z \\
 & -\,a^2/z - a^3/z^2 \\
 & \quad a^3/z^2 \\
 & -\,a^3/z^2 - a^4/z^3 \\
 & \quad a^4/z^3
\end{array}
$$

$$\therefore \frac{z}{z-a} = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots$$

$$= 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots = \sum_{k=0}^{\infty}\left(a^k z^{-k}\right), \quad k \geq 0$$

$$= Z(a^k), \quad k \geq 0$$

$$\therefore Z^{-1}\left\{\frac{z}{z-a}\right\} = a^k, \quad k \geq 0$$

ii) $|z| < a \Rightarrow |z/a| < 1$. consider $F(z) = \dfrac{z}{-a+z}$

$$
\begin{array}{r|l}
 & -z/a - z^2/a^2 - z^3/a^3 \\
\hline
-a+z & z \\
 & -\,z - z^2/a \\
 & \quad z^2/a \\
 & -\,z^2/a - z^3/a^2 \\
 & \quad z^3/a^2 \\
 & -\,z^3/a^2 - z^4/a^3 \\
 & \quad z^4/a^3
\end{array}
$$

$$\therefore \frac{z}{-a+z} = -\frac{z}{a} - \frac{z^2}{a^2} - \frac{z^3}{a^3} - \cdots = -\sum_{k=1}^{\infty} a^{-k} z^{k} = -\sum_{k=-\infty}^{-1} a^k z^{-k}$$

$$= Z(-a^k), \quad k < 0$$

$$\therefore Z^{-1}\left\{\frac{z}{-a+z}\right\} = -a^k, \quad k < 0$$

2) $\dfrac{z^2+z}{z^3-3z^2+3z-1}$, $|z|>1$

$\longrightarrow$ $|z|>1 \Rightarrow \left|\frac{1}{z}\right|<1$

$$\begin{array}{r|l} & \frac{1}{z}+\frac{4}{z^2}+\frac{9}{z^2} \\ z^3-3z^2+3z-1 & z^2+z \\ & -\left(z^2-3z+3-\frac{1}{z}\right) \\ \hline & 4z-3+\frac{1}{z} \\ & -\left(4z-12+\frac{12}{z}-\frac{4}{z^2}\right) \\ \hline & 9-\frac{11}{z}-\frac{4}{z^2} \\ & -\left(9-\frac{27}{z}+\frac{27}{z^2}-\frac{9}{z^3}\right) \\ \hline & \frac{16}{z}-\frac{23}{z^2}+\frac{9}{z^3} \end{array}$$

$$\therefore F(z) = \frac{z^2+z}{z^3-3z^2+3z-1} = \frac{1}{z}+\frac{4}{z^2}+\frac{9}{z^3}+\cdots$$

$$= 1\cdot z^{-1}+4z^{-2}+9z^{-3}+\cdots$$

$$= \sum_{K=0}^{\infty} K^2 z^{-K} = Z(K^2),\ K\geq 0$$

- For $|z|<a$, first rearrange the polynomials in ascending powers of z & then divide.

 For $|z|>a$, rearrange the polynomials in descending powers of z & then divide.

* 2) Binomial Expansion :-

In this method we take suitable factor common depending upon ROC from denominator, so that denominator is of the form $1-r$ where $|r|<1$ & use binomial theorem.

$$(1-a)^n = 1-na+\frac{n(n-1)}{2!}a^2-\frac{n(n-1)(n-2)}{3!}a^3+\cdots$$

$$(1+a)^n = 1+na+\frac{n(n-1)}{2!}a^2+\frac{n(n-1)(n-2)}{3!}a^3+\cdots$$

Ex: *) Find inverse z-transform of the following

1) $F(z) = \frac{2z}{z-a}$ i) $|z| > |a|$ ii) $|z| < |a|$

→ i) $|z| > |a| \Rightarrow \left|\frac{a}{z}\right| < 1$ ∴ we take z outside

$$\therefore F(z) = \frac{2z}{z-a} = \frac{2z}{z(1-a/z)}$$

$$= 2\left[1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \cdots\right]$$

$$= 2\sum_{k=0}^{\infty} a^k z^{-k}$$

$$\therefore \frac{2z}{z-a} = Z\{2a^k\}, \quad k \geq 0$$

$$\therefore \boxed{Z^{-1}\left\{\frac{2z}{z-a}\right\} = 2a^k, \quad k \geq 0}$$

ii) $|z| < |a| \Rightarrow \left|\frac{z}{a}\right| < 1$ ∴ we take 'a' outside

$$\therefore F(z) = \frac{2z}{z-a} = \frac{2z}{-a(1-z/a)}$$

$$= -\frac{2z}{a}\left[1 + \left(\frac{z}{a}\right) + \left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)^3 + \cdots\right]$$

$$= -2\left[\frac{z}{a} + \left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)^3 + \cdots\right]$$

$$= -2\sum_{k=1}^{\infty} z^k a^{-k}$$

$$= -2\sum_{k=-\infty}^{-1} z^{-k} a^k$$

$$\therefore \frac{2z}{z-a} = Z\{-2a^k\}, \quad k < 0$$

$$\therefore \boxed{Z^{-1}\left\{\frac{2z}{z-a}\right\} = -2a^k, \quad k < 0}$$

2) $F(z) = \frac{1}{(z-a)^2}$ i) $|z| < a$, ii) $|z| > a$

$\rightarrow$ i) $|z| < a \Rightarrow \left|\frac{z}{a}\right| < 1$

$$\therefore F(z) = \frac{1}{(z-a)^2} = \frac{1}{(-a)^2(1 - z/a)^2}$$

$$= \frac{1}{a^2}\left(1 - \frac{z}{a}\right)^{-2}$$

$$= \frac{1}{a^2}\left[1 + 2\frac{z}{a} + 3\left(\frac{z}{a}\right)^2 + 4\left(\frac{z}{a}\right)^3 + \cdots\right]$$

$$= \frac{1}{a^2} + \frac{2z}{a^3} + \frac{3z^2}{a^4} + \frac{4z^3}{a^5} + \cdots$$

$\therefore$ coefficient of $z^n = \frac{n+1}{a^{n+2}}$, $n \geq 0$

$\Rightarrow$ coefficient of $z^{-k} = \frac{-k+1}{a^{-k+2}}$, $-k \geq 0$ i.e. $k \leq 0$

$$\therefore \boxed{z^{-1}\left\{\frac{1}{(z-a)^2}\right\} = \frac{-k+1}{a^{-k+2}}, \quad k \leq 0}$$

ii) $|z| > a \Rightarrow \left|\frac{a}{z}\right| < 1$

$$\therefore F(z) = \frac{1}{(z-a)^2} = \frac{1}{z^2(1 - a/z)^2}$$

$$= \frac{1}{z^2}\left[1 + 2\left(\frac{a}{z}\right) + 3\left(\frac{a}{z}\right)^2 + 4\left(\frac{a}{z}\right)^3 + \cdots\right]$$

$$= \frac{1}{z^2} + \frac{2a}{z^3} + \frac{3a^2}{z^3} + \frac{4a^3}{z^4} + \cdots$$

$\therefore$ coefficient of $z^{-n} = (n-1)a^{n-2}$, $n \geq 2$

$\Rightarrow$ coefficient of $z^{-k} = (k-1)a^{k-2}$, $k \geq 2$

$$\therefore \boxed{z^{-1}\left\{\frac{1}{(z-a)^2}\right\} = (k-1)a^{k-2}, \quad k \geq 2}$$

3) Method of Partial Fraction:-

Ex: Find inverse z transform of

1) $\dfrac{z}{(z-1)(z-2)}$, $|z| > 2$

$\rightarrow$ $\dfrac{z}{(z-1)(z-2)} = \dfrac{A}{z-1} + \dfrac{B}{z-2}$

$\Rightarrow z = A(z-2) + B(z-1)$

Put $z=2 \Rightarrow 2 = B(1) \Rightarrow B =$

Put $z=1 \Rightarrow 1 = A(-1) \Rightarrow \boxed{A=-1}$

$\therefore \dfrac{z}{(z-1)(z-2)} = \dfrac{2}{z-2} - \dfrac{1}{z-1}$

Now $|z| > 2$ then $|z| > 1$

$\Rightarrow |2/z| < 1$ & $|1/z| < 1$ $\therefore$ take z common

$\therefore F(z) = \dfrac{2}{z-2} - \dfrac{1}{z-1}$

$= \dfrac{2}{z(1-2/z)} - \dfrac{1}{z(1-1/z)}$

$= \dfrac{2}{z}\left[1 + \dfrac{2}{z} + \left(\dfrac{2}{z}\right)^2 + \cdots\right] - \dfrac{1}{z}\left[1 + \dfrac{1}{z} + \left(\dfrac{1}{z}\right)^2 + \cdots\right]$

$= \left[\dfrac{2}{z} + \left(\dfrac{2}{z}\right)^2 + \left(\dfrac{2}{z}\right)^3 + \cdots\right] - \left[\dfrac{1}{z} + \left(\dfrac{1}{z}\right)^2 + \left(\dfrac{1}{z}\right)^3 + \cdots\right]$

$\therefore$ coefficient of $z^{-k} = 2^k - 1$, $k \geq 1$

$\therefore \boxed{z^{-1}\{F(z)\} = 2^k - 1 \ , \ k \geq 1}$

2) $F(z) = \dfrac{1}{(z-3)(z-2)}$

i) $|z| < 2$

ii) $2 < |z| < 3$

iii) $|z| > 3$

$\rightarrow$ By partial fraction,

$$\frac{1}{(z-3)(z-2)} = \frac{1}{z-3} - \frac{1}{z-2}$$

i) $|z| < 2$ then $|z| < 3$

$\Rightarrow \left|\frac{z}{2}\right| < 1$, $\left|\frac{z}{3}\right| < 1$

$$\therefore \frac{1}{(z-3)(z-2)} = \frac{1}{z-3} - \frac{1}{z-2} = \frac{1}{-3(1-z/3)} + \frac{1}{2(1-z/2)}$$

$$= \frac{-1}{3}\left[1+\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2+\cdots\right] + \frac{1}{2}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\cdots\right]$$

$$= \left[\frac{-1}{3} - \frac{z}{3^2} - \frac{z^2}{3^3} - \cdots\right] + \left[\frac{1}{2} + \frac{z}{2^2} + \frac{z^2}{2^3} + \cdots\right]$$

$\therefore$ coeff. of $z^k = -3^{-k-1} + 2^{-k+1}$, $k \geq 0$

$\therefore$ coeff of $z^{-k} = -3^{k-1} + 2^{k+1}$, $-k \geq 0$ i.e. $k \leq 0$

$$\therefore \boxed{Z^{-1}\{F(z)\} = -3^{k-1} + 2^{k-1}, \quad k \leq 0}$$

ii) $2 < |z| < 3 \Rightarrow |z| > 2$ & $|z| < 3$.

$\therefore \left|\frac{2}{z}\right| < 1$ & $\left|\frac{z}{3}\right| < 1$

$$\therefore \frac{1}{(z-3)(z-2)} = \frac{1}{z-3} - \frac{1}{z-2} = \frac{1}{-3(1-z/3)} - \frac{1}{z(1-2/z)}$$

$$= \frac{-1}{3}\left[1+\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2+\cdots\right] - \frac{1}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\cdots\right]$$

$$= \left[\frac{-1}{3} - \frac{z}{3^2} - \frac{z^2}{3^3} - \cdots\right] - \left[\frac{1}{z} + \frac{2}{z^2} + \frac{2^2}{z^3} + \cdots\right]$$

$\therefore$ coeff of z^k from first series $= -3^{-k-1}$, $k \geq 0$

$\therefore$ coeff of z^{-k} from first series $= -3^{k-1}$, $k \leq 0$

& coeff of z^{-k} from second series $= -2^{k-1}$, $k \geq 1$

$$\therefore \boxed{Z^{-1}\{F(z)\} = \begin{cases} -3^{k-1}, & k \leq 0 \\ -2^{k-1}, & k \geq 1 \end{cases}}$$

iii) $|z| > 3 \Rightarrow \left|\frac{3}{z}\right| < 1$ & $|z| > 2 \Rightarrow \left|\frac{2}{z}\right| < 1$.

$$\therefore \frac{1}{(z-3)(z-2)} = \frac{1}{z-3} - \frac{1}{z-2}$$

$$= \frac{1}{z(1-3/z)} - \frac{1}{z(1-2/z)}$$

$$= \frac{1}{z}\left[1 + \frac{3}{z} + \left(\frac{3}{z}\right)^2 + \cdots\right] - \frac{1}{z}\left[1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \cdots\right]$$

$$= \left[\frac{1}{z} + \frac{3}{z^2} + \frac{3^2}{z^3} + \cdots\right] - \left[\frac{1}{z} + \frac{2}{z^2} + \frac{2^2}{z^3} + \cdots\right]$$

$\therefore$ coeff. of $z^{-k} = 3^{k-1} - 2^{k-1}, \ k \geq 1$

$$\therefore \boxed{z^{-1}\{F(z)\} = 3^{k-1} - 2^{k-1}, \quad k \geq 1}$$

3) $F(z) = \dfrac{z+2}{z^2 - 2z + 1}, \quad |z| > 1$

$$\therefore \frac{z+2}{z^2-2z+1} = \frac{z+2}{(z-1)^2} = \frac{A}{z-1} + \frac{B}{(z-1)^2}$$

$$\Rightarrow z+2 = A(z-1) + B$$

Put $z = 1 \Rightarrow \boxed{3 = B}$

comparing coeff of z, $\boxed{A = 1}$

Now, $|z| > 1 \Rightarrow \left|\frac{1}{z}\right| < 1$

$$\therefore \frac{z+2}{(z-1)^2} = \frac{1}{z-1} + \frac{3}{(z-1)^2}$$

$$= \frac{1}{z(1-1/z)} + \frac{3}{z^2(1-1/z)^2}$$

$$= \frac{1}{z}\left[1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \cdots\right] + \frac{3}{z^2}\left[1 + \frac{2}{z} + 3\frac{1}{z^2} + \frac{4}{z^3} + \cdots\right]$$

$$= \left[\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right] + \left[\frac{3}{z^2} + \frac{2\cdot3}{z^3} + \frac{3\cdot3}{z^4} + \cdots\right]$$

$\therefore$ coeff of $z^{-k} = 1 + 3(k-1) = 3k-2, \quad k \geq 1$

$$\therefore \boxed{z^{-1}\{F(z)\} = 3k-2, \quad k \geq 1}$$