RESIDUES :-

* Zero of an Analytic function :-

If an analytic function $f(z) = 0$ at point $z = z_0$ then z_0 is called as zero of an analytic function, $f(z)$ ~~f(z)~~

If $f(z_0) = 0$ but $f'(z_0) \neq 0$ then z_0 is simple zero or zero of order 1.

In general, if

$f(z_0) = 0 = f'(z_0) = f''(z_0) = \cdots = f^{(n-1)}(z_0)$

but $f^{(n)}(z_0) \neq 0$

then z_0 is zero of order "n".

Q. (1) Find zero of a function & also find its order.

1) $f(z) = (z-1)e^z$.

→ $f(z) = 0$

$(z-1)\, e^z = 0$

as $e^z \neq 0$

$z - 1 = 0$

$\Rightarrow z = 1$

$z = 1$ is zeros of function.

$f(z) = (z-1)e^z$

$f'(z) = (z-1)e^z + e^z$

$f'(1) = (1-1)e^1 + e^1$

$f'(1) = e \neq 0$

$\therefore$ $z = 1$ is zero of order 1.

Prof. Nancy Sinollin

$(z-3)^4 = 0$

~~$f^{4}(1) = e$~~ $\quad z-3 = \sqrt[4]{0}$

(2) $f(z) = (z-3)^4$

$z^2 = 1 \quad z = \pm 1$

$\rightarrow$ $f(z) = 0$

$(z-3)^4 = 0 \qquad z-3 = 0 \qquad \cancel{= \sqrt[4]{0}}$

$z = 3 \qquad z = 3$

$z = 3$ is zero of function.

$f(z) = (z-3)^4$

$f'(z) = 4(z-3)^3$

$f'(3) = 4(3-3)^3 = 0$

$f''(z) = 12(z-3)^2$

$f''(3) = 12(3-3)^2 = 0$

$f'''(z) = 24(z-3) \cancel{= 0}$

$\cancel{24(z-3)} = 24(3-3) = 0$

$f^{iv}(3) = 24 \neq 0$

$\therefore z = 3$ is zero of order 4

(3) $f(z) = \dfrac{(z+2)^2}{(z+3)^3}$

$\rightarrow$ $(z+2)^2 = 0$

$f(z) = 0$

~~z~~ $z = -2$ is zero of function.

$\therefore z = -2$ is zero of order 2.

(4) $f(z) = (z-1)^3 (z+4)^2$

→

$z = 1$ is a zero of function.
$z = 1$ is a zero of order 3

$z = -4$ is a zero of function.
$z = -4$ is a zero of order 2.

Definitions:-

(1) Singular Points (Problematic Points):-

~~The po~~ If $f(z)$ is not analytic at a point z_0 then z_0 is called as "singular point." or "singularity of $f(z)$"

eg:

$$f(z) = \frac{1}{z-1}$$

$z = 1$ is singularity of $f(z)$.

★ Types of singularities:

Type: I Isolated Singularity.

If there is no other singularity inside the neighbourhood (nbd) of z_0 other than z_0 then z_0 is called as "isolated singularity."

eg: $f(z) = \frac{1}{(z-1)(z-2)}$

$z = 1$ & $z = 2$ are isolated singularity.

Type II : Non isolated singularity

~~If there exist at least one singularity~~
If every nbd of z_0 consist at least one ~~st~~ more singular pt other than z_0 then it is called as "non-isolated singularity".

Type III : Pole

Let z_0 is singularity of $f(z)$ then Laurent's series expansion of $f(z)$ is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

If in Laurent's series expansion of $f(z)$ finite terms of ~~f(z)~~ negative power of $(z-z_0)$ are present i.e. if in the above series only M negative powers are present i.e. $b_{m+1} = b_{m+2} = \cdots = 0$ then $z = z_0$ is called as ~~whole~~ pole of order "~~M~~" "m"

Pole of ~~order~~ 1 ~~is~~ also called "simple pole"

eg. $f(z) = \dfrac{e^{3z}}{(z-1)^3}$

1. $z = 1$ is singularity
To find Laurent's series at $z = 1$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$f(z) = \frac{e^{3z}}{(z-1)^3}$$

$$= \frac{e^{3z-3+3}}{(z-1)^3}$$

$$= \frac{e^{3(z-1)}e^3}{(z-1)^3}$$

$$= \frac{e^3}{(z-1)^3}\left[1 + 3(z-1) + \frac{3^2(z-1)^2}{2!} + \frac{3^3(z-1)^3}{3!} + \cdots\right]$$

$$= e^3\left[\frac{1}{(z-1)^3} + \frac{3}{(z-1)^2} + \frac{3^2}{(z-1)2!} + \frac{3^3}{3!} + \cdots\right]$$

$z=1$ is pole of order 3.

(2) $f(z) = \frac{1}{(z-2)^3(z-1)}$

$\rightarrow$ $z=2$ is pole of order 3

$z=1$ is pole of order 1.

* Isolated Essential Singularity:-

If the Laurent's series expansion of $f(z)$ contains infinitely many negative powers of $(z-z_0)$ then z_0 is called as "isolated essential singularity"

eg. $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{z^2 2!} + \frac{1}{z^3 3!} + \cdots$

* Non-isolated Essential singularity:-

If we have sequence of poles of $f(z)$ $z_1, z_2, z_3, \ldots, z_n$ such that z_0 is limit point of this poles (as $n \to \infty$, $z_n \to z_0$). (Here $\lim\limits_{n\to\infty} z_n = z_0$) then z_0 is called as non-isolated essential singularity.

eg. $f(z) = \frac{1}{\sin\left(\frac{1}{z}\right)}$

we will get singularities when,

$$\sin\left(\frac{1}{z}\right) = 0$$

$$\frac{1}{z} = n\pi$$

$$z = \frac{1}{n\pi}, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\lim_{n\to\infty} z = \lim_{n\to\infty} \frac{1}{n\pi} = 0.$$

* Removable singularity :

If Laurent's series expansion of $f(z)$ does not contain negative power of $(z-z_0)$ then z_0 is called as "removable singularity."

eg. $f(z) = \frac{\sin z}{z}$

$z=0$ is singularity.

$$= \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)$$

$$= \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots\right)$$

$z=0$ is removable singularity.

Q. 1 Find singularities & explain its types.

1) $f(z) = \frac{\sin z}{z^3}$

→ $z=0$ is singularity.

$$= \frac{1}{z^3}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right]$$

$$= \left[\frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \dots\right]$$

$z=0$ is pole order of 2.

(2) $f(z) = \dfrac{\cot \pi z}{(z-a)^3}$.

$\longrightarrow$

$= \cancel{\sin \pi z}$

$= \dfrac{\cos \pi z}{\sin \pi z \, (z-a)^3}$

we'll get singularities

$\sin \pi z \, (z-a)^3 = 0$

$\sin \pi z = 0 \qquad\qquad (z-a)^3 = 0$

$\pi z = n\pi \qquad\qquad z = a.$

$n = 0, \pm 1, \pm 2, \ldots$

$z = n$

$\therefore$ singularities of $f(z)$ are,

$z = \cancel{a}n \;, \; n = 0, \pm 1, \pm 2, \ldots$

are pole of order 1

$z = a$ is a pole of order 3.

$z = n \;, \; n = 0, \pm, \pm 2, \ldots$

are pole of order 1. (isolated ~~essential~~ singularity).

(2) $f(z) = \dfrac{\cot \pi z}{(z-a)^3}$.

$\longrightarrow$

$= \cancel{\sin \pi z}$

$= \dfrac{\cos \pi z}{\sin \pi z \, (z-a)^3}$.

we'll get singularities

$\sin \pi z \, (z-a)^3 = 0$

$\sin \pi z = 0$ $\qquad$ $(z-a)^3 = 0$

$\pi z = n\pi$ $\qquad$ $z = a$.

$n = 0, \pm 1, \pm 2, \ldots$

$z = n$

$\therefore$ singularities of $f(z)$ are,

$z = n$, $n = 0, \pm 1, \pm 2, \ldots$

are pole of order 1

$z = a$ is a pole of order 3.

$z = n$, $n = 0, \pm, \pm 2, \ldots$

are pole of order 1. (isolated ~~essential~~ singularity).

Residues:-

The coefficient of $(z-z_0)^{-1}$ or $\frac{1}{(z-z_0)}$ in the L's Expansion of $f(z)$ at $z=z_0$ is called as "residues" of $f(z)$ at $z=z_0$.

Procedure to find out Residues:-

case i) If $z=z_0$ is pole of order 1 i.e. simple pole then,

$$\text{Residue of } f(z) \text{ at } z=z_0 = \lim_{z\to z_0}\left[(z-z_0)\,f(z)\right]$$

case ii) If $z=z_0$ is a pole of order n then.

$$\text{Residue of } f(z) \text{ at } z=z_0 = \lim_{z\to z_0} \frac{1}{(n-1)!}\,\frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n f(z)\right]$$

Q. (1) Determine the poles & find residues at each pole.

1) $f(z) = \dfrac{z^2}{(z+2)(z-1)^2}$

→

$z=-2$ is pole of order 1

&

$z=1$ is pole of order 2.

$z = -2$

Resi of $f(z)$ at $z = -2 = \lim_{z \to -2} \left[(z+2)\, f(z) \right]$

$$= \lim_{z \to -2} \left[(z+2) \cdot \frac{z^2}{(z+2)(z-1)^2} \right]$$

$$= \lim_{z \to -2} \left[\frac{z^2}{(z-1)^2} \right]$$

$$= \cancel{\lim_{z \to -2}} \left[\frac{(2)^2}{(-2-1)^2} \right]$$

$$= \frac{4}{9}$$

Residue of $f(z)$ at $z = 1$

$$= \lim_{z \to 1} \frac{d}{dz} \left[(z-1)^2 \cdot \frac{z^2}{(z+2)(z-1)^2} \right]$$

$$= \lim_{z \to 1} \frac{d}{dz} \left[\frac{z^2}{(z+2)} \right]$$

$$= \lim_{z \to 1} \frac{(z+2)(2z) - z^2(1)}{(z+2)^2}$$

$$= \cancel{\lim} \frac{(1+2)(2(1)) - (1)^2(1)}{(1+2)^2}$$

$$= \frac{(3)(2) - (1)}{9} = \frac{5}{9}$$

$$f(z) = \frac{e^z}{(z-1)^3}$$

→ $z=1$ is a pole of order 3.

$z=1$

Residue of $f(z)$ at $z=1$

$$= \lim_{z\to 1} \frac{1}{(3-1)!} \frac{d^2}{dz^2}\left[(z-1)^3 \cdot \frac{e^z}{(z-1)^3}\right]$$

$$= \lim_{z\to 1} \frac{1}{2!} \frac{d^2}{dz^2}\left[e^z\right]$$

$$= \lim_{z\to 1} \frac{1}{2} \frac{d^2}{dz^2} e^z$$

$$= \lim_{z\to 1} \frac{1}{2} e^z$$

$$= \cancel{\lim} \ \frac{1}{2} e.$$

$$= \frac{e}{2}$$

(3) $f(z) = \dfrac{1}{z^3+z^5}$

→ $f(z) = \dfrac{1}{z^3(1+z^2)}$

$z^3(1+z^2) = 0$

$z^3 = 0 \quad , \quad (1+z^2) = 0$

$z = 0 \qquad z^2 = -1$

$z = \pm\sqrt{-1}$

$z = \pm i$

$z = 0$ is a pole of order 3.

& $z = \pm i$ is a pole of order 1.

~~$z = 0$~~

~~Residues of $f(z)$ at $z=0$~~

$$\cancel{= \lim_{z\to 0} \frac{1}{(3-1)!} \frac{d^2}{dz^2}\left[(z)^3 \frac{1}{z^3(1+z^2)} \right]}$$

$$f(z) = \frac{1}{z^3+z^5} = \frac{1}{z^3(1+z^2)}$$

$$= \frac{1}{z^3(z-i)(z+i)}$$

$z = 0$

Residues of $f(z)$ at $z=0$

$$= \lim_{z\to 0} \frac{1}{(3-1)!} \frac{d^2}{dz^2}\left[z^3 \frac{1}{z^3(1+z^2)} \right]$$

$$= \lim_{z\to 0} \frac{1}{2} \frac{d^2}{dz^2} \frac{1}{(1+z^2)}$$

$$= \lim_{z\to 0} \frac{1}{2} \frac{d}{dz}\left[-1(1+z^2)^{-2}(2z)\right]$$

$$= \lim_{z\to 0} \frac{1}{2}(-2) \frac{d}{dz}\left[(1+z^2)^{-2}(z)\right]$$

$$= \lim_{z\to 0} (-1)\left[(1+z^2)^{-2}(1) + (z)(-2)(1+z^2)^{-3}(2z)\right]$$

$$= \lim_{z\to 0} \frac{1}{2} \frac{d^2}{dz^2}\left(\frac{1}{1+z^2}\right)$$

$$= \lim_{z\to 0} \frac{1}{2} \frac{d}{dz}\left(\frac{-1}{1+z^2}(2z)\right)$$

$$= \lim_{z\to 0} \frac{1}{2} \frac{d}{dz}\left[\frac{-2z}{1+z^2}\right]$$

$$= -\lim_{z\to 0} \frac{1}{2} \frac{d}{dz}\left[\frac{+2z}{(1+z^2)}\right]$$

$$= -\lim_{z\to 0} \frac{1}{2}\left[\frac{(1+z^2)(2)-(2z)(0+2z)}{(1+z^2)^2}\right]$$

$$= -\frac{1}{2}\left[\frac{(1+0)(2)-(0)}{(1+0)^2}\right]$$

$$= -\frac{1}{2}\left[\frac{2}{1}\right]$$

$$= -1$$

Prof. Nancy Sinollin

$z = i$ is a pole of order 1.

Residues of $f(z)$ at $z = i$

$$= \lim_{z \to i} \cancel{(z-i)} \cdot \frac{1}{z^3 \cancel{(z-i)}(z+i)}$$

$$= \lim_{z \to i} \frac{1}{z^3(z+i)}$$

$$= \frac{1}{i^3(i+i)}$$

$$= \frac{1}{i^3(2i)}$$

$$= \frac{1}{2}$$

$z = -i$ is a pole of order 1

Residues of $f(z)$ at $z = -i$

$$= \lim_{z \to -i} \cancel{(z+i)} \cdot \frac{1}{z^3(z-i)\cancel{(z+i)}}$$

$$= \lim_{z \to -i} \frac{1}{z^3(z-i)}$$

$$= \frac{1}{(-i)^3(-i-i)} = \frac{1}{(-i)^3(-2i)}$$

$$= \frac{1}{(-i)^3(-2i)} \qquad = \frac{1}{\cancel{(+i)}\cancel{(-2i)}}$$

$$= \frac{1}{\cancel{(-i)}\cancel{(-2i)}} = \frac{1}{\cancel{(-i)}\cancel{(-2i)}} = \frac{1}{}$$

$$= \frac{1}{\cancel{2i^2}} = \frac{1}{\cancel{2i^2}}$$

$$= \cancel{\frac{1}{2}} \; \cancel{4}$$

$$= \frac{1}{\cancel{(-1)}\cancel{(i3)}\cancel{(-2i)}}$$

$$= \frac{1}{(+1)(2i4)}$$

$$= \frac{1}{2} \; 4$$

(4) $f(z) = \dfrac{\sin^2 z}{z^3}$

$\rightarrow$ $z=0$ is ~~polar~~ singularity.

$$f(z) = \frac{(\sin z)^2}{z^3}$$

$$= \cancel{\frac{1}{z^3}} \left[\frac{1}{z} \quad \frac{1}{z^3} \right.$$

$$f(z) = \frac{\sin^2 z}{z^3}$$

$$= \frac{1}{z^3}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]^2$$

$$= \frac{1}{z^3}\left[z^2 - \frac{2z^6}{(3!)^2} + z^3\right]$$

$z = 0$ is a pole of order "3"

$$\text{Res of } f(z) \text{ at } z=0 = \lim_{z\to 0} \frac{1}{2!}\frac{d^2}{dz^2}\left[\frac{z^3 \sin^2 z}{z^3}\right]$$

$$= \lim_{z\to 0} \frac{1}{2}\frac{d^2}{dz^2}\left[\sin^2 z\right]$$

$$= \lim_{z\to 0} \frac{1}{2}\frac{d}{dz}\left[2\sin z \cos z\right]$$

$$= \lim_{z\to 0} \frac{1}{2}\frac{d}{dz}\left[\sin 2z\right]$$

$$= \lim_{z\to 0} \frac{1}{2}\left[+2\cos 2z\right]$$

$$= \cos 2(0)$$

$$= 1$$

$$f(z) = \frac{\sin^2 z}{z^3}$$

$$= \frac{(\sin z)^2}{z^3}$$

$$= \frac{1}{z^3}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)^2$$

$$= \frac{1}{z^3}\left[z^2 - \frac{2zz^3}{3!} + \frac{(z^3)^2}{(3!)^2} \cdots\right]$$

$$= \left(\frac{1}{z} - \frac{2z}{3!} + \frac{z^3}{(3!)^2} - \cdots\right)$$

$\therefore$ $z=0$ is pole of order 1

$\therefore$ Residue of $f(z)$ at $z=0$

$= \text{coefficient of } \frac{1}{(z-0)}$

$= 1$

(5) $f(z) = z^2 \sin\left(\frac{1}{z}\right)$

$\rightarrow$ $z=0$ is singularity

$$f(z) = z^2\left[\frac{1}{z} - \frac{1}{z^3 3!} + \frac{1}{z^5 5!} - \cdots\right]$$

$$= \left[z - \frac{1}{z\,3!} + \frac{1}{z^3\,5!} - \cdots\right]$$

$\therefore$ ($z=0$ is isolated essential singularity)

∴ Residue of $f(z)$ at $z=\infty$

$= \text{coefficient of } \left(\frac{1}{z-0}\right)$

$= \frac{-1}{36} = \frac{-1}{6}$ //

V. Imp 6> Find sum of residues of $f(z) = \frac{\tan z}{z}$ at its poles inside $|z|=2$

→

$f(z) = \frac{\sin z}{z \cdot \cos z}$ $\quad |z|=2$

we'll get singularities,

$z\cos z = 0$

$z=0$ or $\cos z = 0$

$z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

$z = \pm(2n+1)\frac{\pi}{2}$

$z=0, \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$ are all poles of order 1.

Given $|z|=2$

$z=0$ $\quad |z| = |0| = 0 < 2$

$z=0$ is inside $|z|=2$

$z = \pm\frac{\pi}{2}$ $\quad |z| = \left|\pm\frac{\pi}{2}\right| = \frac{\pi}{2} < 2$

$z = \pm\frac{\pi}{2}$ are inside $|z|=2$.

$z = \pm \frac{3\pi}{2} \quad \left|\pm \frac{3\pi}{2}\right| = \frac{3\pi}{2} > 2$

$\therefore z = \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2} \ldots$ ~~the~~ all outside $|z| = 2$.

Residue ~~at~~ $f(z)$ at $z=0$

$= \lim_{z \to 0} (z) f(z)$

$= \lim_{z \to 0} z \cdot \frac{\sin z}{z \cos z}$

$= \lim_{z \to 0} \cdot \frac{\sin z}{\cos z}$

$= 0.$

Residue at $f(z)$ at $z = \frac{\pi}{2}$

$= \lim_{z \to \frac{\pi}{2}} \cdot \cancel{\frac{z \cdot \sin z}{z \cos z}}$

$= \cancel{\frac{\sin \frac{\pi}{2}}{\cos \frac{\pi}{2}}} \lim_{z \to \frac{\pi}{2}} \left(z - \frac{\pi}{2}\right) \frac{\sin z}{z \cos z}$

$= \lim_{z \to \frac{\pi}{2}} \left(z - \frac{\pi}{2}\right) \cdot \frac{\sin z}{z \cos z}$

By L-Hospital rule.

$= \cancel{\ell}$

$$= \lim_{z \to \frac{\pi}{2}} \left[\frac{(z - \frac{\pi}{2}) \sin z}{z \cos z} \right]$$

$$= \lim_{z \to \frac{\pi}{2}} \left[\frac{(z - \frac{\pi}{2}) \sin \frac{\pi}{2}}{\frac{\pi}{2} \cos z} \right]$$

$$= \frac{2}{\pi} \lim_{z \to \frac{\pi}{2}} \left[\frac{(z - \frac{\pi}{2})}{\cos z} \right]$$

$$= \frac{2}{\pi} \lim_{z \to \frac{\pi}{2}} \left[\frac{1}{-\sin z} \right]$$

$$= \frac{2}{\pi} \left[\frac{-1}{\sin \frac{\pi}{2}} \right]$$

$$= \frac{2}{\pi} \left[\frac{-1}{1} \right]$$

$$= \frac{-2}{\pi}$$

~~$z = \pi$~~ $z = -\frac{\pi}{2}$

$$= \lim_{z \to -\frac{\pi}{2}} \left(z + \frac{\pi}{2}\right) \frac{\sin z}{z \cos z}$$

$$= \left(\frac{-2}{\pi}\right)(-1) \lim_{z \to \frac{\pi}{2}} \left[\frac{z + \frac{\pi}{2}}{\cos z} \right]$$

$$= \frac{+2}{\pi} \lim_{z \to \frac{\pi}{2}} \left[\frac{1}{-\sin z} \right]$$

$$= \frac{+2}{\pi} \left[\frac{1}{-\sin(-\frac{\pi}{2})} \right]$$

$$= \frac{+2}{\pi} \left[\frac{1}{-(-1)} \right]$$

$$= \frac{+2}{\pi}$$

$$= \frac{2}{\pi}$$

Sum of residue $= 0 - \frac{2}{\pi} + \frac{2}{\pi}$

$= 0$

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① find residue of $z^2 \sec \pi z$

→

$$f(z) = z^2 \sec \pi z$$

$$= \frac{z^2}{\cos \pi z}$$

we'll get singularities when
$\cos \pi z = 0$

$$\Rightarrow \pi z = \left(2n + 1\right)\frac{\pi}{2} \quad \ldots n = 0, \pm 1, \pm 2, \ldots$$

$$z = \frac{2n+1}{2} \quad \ldots n = 0, \pm 1, \pm 2, \ldots$$

$z = \frac{2n+1}{2}$ which are poles of order 1

Residue of $f(z)$ at $z = z_0$

$$= \lim_{z \to 0} (z - z_0) f(z)$$

Residue of $f(z)$ at $z = \left(\frac{2n+1}{2}\right)$

$$= \lim_{z \to \left(\frac{2n+1}{2}\right)} \left(z - \frac{2n+1}{2}\right) \frac{z^2}{\cos \pi z} = \frac{0}{0}$$

$$= \lim_{z \to \left(\frac{2n+1}{2}\right)} \frac{\left(z - \frac{2n+1}{2}\right)\left(\frac{2n+1}{2}\right)^2}{\cos \pi z}$$

$$= \left(\frac{2n+1}{2}\right)^2 \lim_{z \to \left(\frac{2n+1}{2}\right)} \frac{\left(z - \frac{2n+1}{2}\right)}{\cos \pi z} = \frac{0}{0}$$

By L'Hospital Rule

$$= \left(\frac{2n+1}{2}\right)^2 \lim_{z \to \left(\frac{2n+1}{2}\right)} \frac{(1)}{(-\sin \pi z)\,\pi}$$

$$= \left(\frac{2n+1}{2}\right)^2 \left[\frac{1}{-\left[\sin \pi \left(\frac{2n+1}{2}\right)\right]\pi}\right]$$

② Find poles of $f(z) = \frac{\sec z}{z^2}$ which lies inside $|z| = 2$ & also find residues

→ $f(z) = \cancel{\frac{\sec z}{z^2}} \quad \frac{\sec z}{z^2}$

$\cancel{f(z)} = \frac{1}{\cancel{z^2 \cos \pi z}}$

~~We'll get singularities when~~

$\cancel{z = 0} \quad \cancel{z^2 \cos \pi z = 0}$

$$f(z) = \frac{1}{z^2 \cos z}$$

We'll get singularities when

$z^2 \cos z = 0.$

$z^2 = 0$ or $\cos z = 0$

$z = 0$ or $z = \pm\left(\frac{2n+1}{2}\right).$

$\cancel{n, 0},$

$z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2} \ldots$

$\Rightarrow z=0$ is a pole of order 2.

$z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$ pole of order 1

given $|z|=2$

$z=0$, $|z| = |0| = 0 < 2 \rightarrow$ inside.

$z = \pm\frac{\pi}{2}$ $\quad |z| = \left|\pm\frac{\pi}{2}\right| = \frac{\pi}{2} < 2$
$\rightarrow$ inside

$z = \pm\frac{3\pi}{2}$ $\quad |z| = \left|\pm\frac{3\pi}{2}\right| = \frac{3\pi}{2} > 2$
outside.

$z=0$.

Residue of $f(z)$ at $z=z_0$

$$= \lim_{z\to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z)$$

$z=0$ is pole of order 2 $(n=2)$

Residue of $f(z)$ at $z=0$

$$= \lim_{z\to 0} \frac{1}{1!} \frac{d}{dz}\left[z^2 \cdot \frac{1}{z^2 \cdot \cos z}\right]$$

$$= \lim_{z\to 0} \frac{d}{dz}\left(\frac{1}{\cos z}\right)$$

$$= \lim_{z\to 0}\left[\frac{\cos z\,(0) - (-\sin z)}{(\cos z)^2}\right]$$

$$= \lim_{z\to 0}\left[\frac{\sin z}{(\cos z)^2}\right] = 0$$

Residue of $f(z)$ at $(z=0)$ $= \lim_{z \to z_0} [(z-z_0) f(z)]$

$= \cancel{\lim}$

Residue of $f(z)$ at $\left(z = \frac{\pi}{2}\right)$

$$= \lim_{z \to \frac{\pi}{2}} \left[\left(z - \frac{\pi}{2}\right) \frac{1}{z^2 \cos z} \right] = \frac{0}{0}$$

$$= \lim_{z \to \frac{\pi}{2}} \left[\left(z - \frac{\pi}{2}\right) \frac{1}{\left(\frac{\pi}{2}\right)^2 \cos z} \right]$$

$$= \frac{4}{\pi^2} \lim_{z \to \frac{\pi}{2}} \left[\frac{\left(z - \frac{\pi}{2}\right)}{\cos z} \right]$$

$$= \frac{4}{\pi^2} \lim_{z \to \frac{\pi}{2}} \left[\frac{1}{-\sin z} \right]$$

$$= \frac{4}{\pi^2} \frac{1}{-\sin\left(\frac{\pi}{2}\right)}$$

$$= -\frac{4}{\pi^2}$$

Residue of $f(z)$ at $\left(z = -\frac{\pi}{2}\right)$

$$= \lim_{z \to -\frac{\pi}{2}} \left[\left(z + \frac{\pi}{2}\right) f(z) \right]$$

$$= \lim_{z \to -\frac{\pi}{2}} \left[\left(z + \frac{\pi}{2}\right) \frac{1}{z^2 \cos z} \right]$$

$$= \lim_{z \to -\frac{\pi}{2}} \left[\left(z + \frac{\pi}{2}\right) \cdot \frac{1}{z^2 \cos z} \right]$$

$$= \lim_{z \to -\frac{\pi}{2}} \left[\left(z + \frac{\pi}{2}\right) \frac{1}{\left(-\frac{\pi}{2}\right)^2 \cos z} \right]$$

$$= \lim_{z \to -\frac{\pi}{2}} \left[\left(z + \frac{\pi}{2}\right) \frac{1}{\frac{\pi^2}{4} \cos z} \right]$$

$$= \frac{4}{\pi^2} \lim_{z \to -\frac{\pi}{2}} \left[\frac{1}{-\sin z} \right]$$

$$= \frac{4}{\pi^2} \frac{1}{-\sin\left(-\frac{\pi}{2}\right)}$$

$$= \frac{4}{\pi^2} \quad //$$

★ Cauchy's Residue Theorem :-

If $f(z)$ is analytic in & on simple closed curve c except at finite no. of isolated singular points $z_1, z_2, z_3, \ldots z_n$ inside c then

$$\oint_c f(z)\,dz = 2\pi i \left(\begin{array}{c}\text{sum of residues} \\ \text{at } z_1, z_2, \ldots z_n\end{array}\right)$$

(1) $\int_c \frac{z^2}{(z-2)(z-1)^2} dz$ where c is $|z| = 2.5$

$\rightarrow f(z) = \frac{z^2}{(z-2)(z-1)^2}$

clearly $z = 2, 1$ are singular pts.

$z = 2$ is pole of order 1
$z = 1$ is pole of order 2.
given $|z| = 2.5$

$z = 2$ $\quad |2| = 2 < 2.5 \rightarrow$ inside c

$z = 1$ $\quad |1| = 1 < 2.5 \rightarrow$ inside. c

~~pole of order 1~~

Residue of $f(z)$ at $z = 2$

$$= \lim_{z \to 2} (z-2) \frac{z^2}{(z-1)^2 (z-2)}$$

$$= \lim_{z \to 2} \frac{z^2}{(z-1)^2}$$

$$= \frac{(2)^2}{(2-1)^2}$$

$$= 4$$

Residue of f(z) at z = 1

$$= \lim_{z\to 1} \frac{1}{(1-1)!} \frac{d}{dz}\left[(z-1)^2 \frac{z^2}{(z-2)(z-1)^2}\right]$$

$$= \lim_{z\to 1} \frac{d}{dz}\left[\frac{z^2}{(z-2)}\right]$$

$$= \lim_{z\to 1}\left[\frac{(z-2)(2z) - z^2(1)}{(z-2)^2}\right]$$

$$= \left[\frac{(1-2)(2) - (1)^2}{(1-2)^2}\right]$$

$$= \left[\frac{(-1)(2) - 1}{(-1)^2}\right]$$

$$= \frac{-2-1}{+1}$$

$$= -3$$

By Cauchy's Residue Theorem,

~~Resid~~ $\oint f(z)\,dz = 2\pi i$ (sum of residues)

$$\int_C \frac{z^2}{(z-2)(z-1)^2}\,dz = 2\pi i\,(4-3)$$

$$= 2\pi i$$