Branch & Bound

- Travelling Salesperson Problem
- 15 puzzle problem.

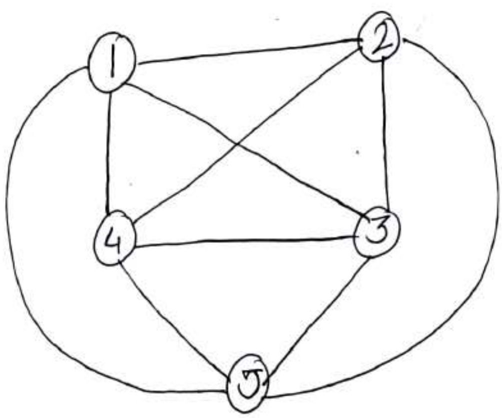
Branch & Bound

- Similar to backtracking
- State space tree
- Solves minimization problems can only be solved.

Traveling Salesman Branch-& Bound

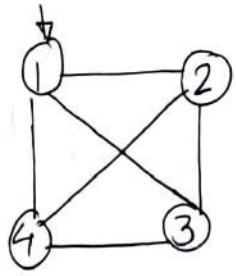
A weighted graph is given we need to identify shortest tour travelling through each vertex only once & returning back to home city i.e. starting vertex at the end of the tour.

Example 2



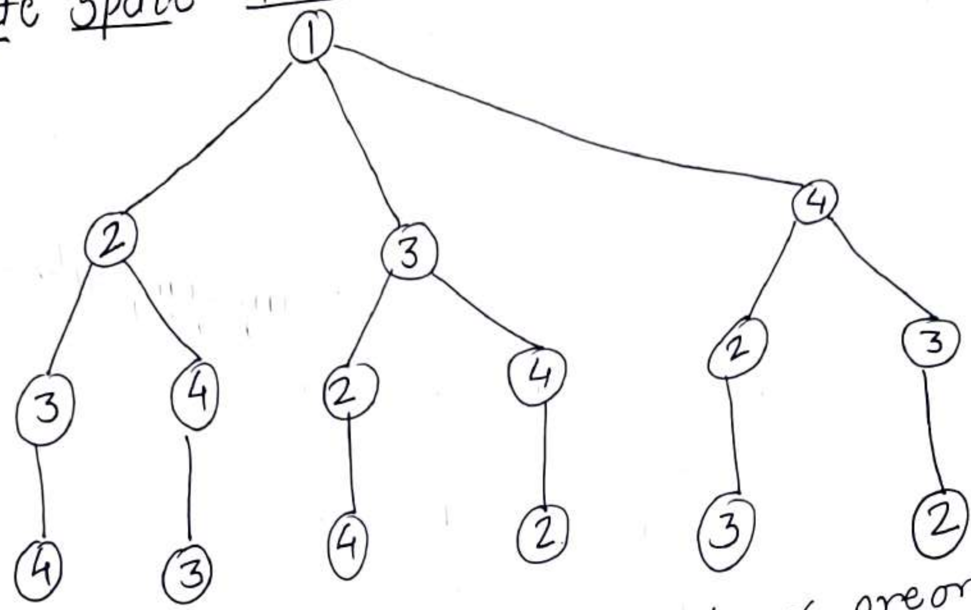
$$
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \infty & 20 & 30 & 10 & 11 \\
2 & 15 & \infty & 16 & 4 & 2 \\
3 & 3 & 5 & \infty & 2 & 4 \\
4 & 19 & 6 & 18 & \infty & 3 \\
5 & 16 & 4 & 7 & 16 & \infty
\end{array}
$$

First we will see simple example to understand the understand the basic difference betn branch & bound and backtracking.



Simple example

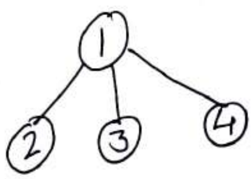
State Space Tree



In backtracking (Time consuming) ~~per~~ (useful for permitation problem) - DFS approach or preorder approach will give us all possible outcomes

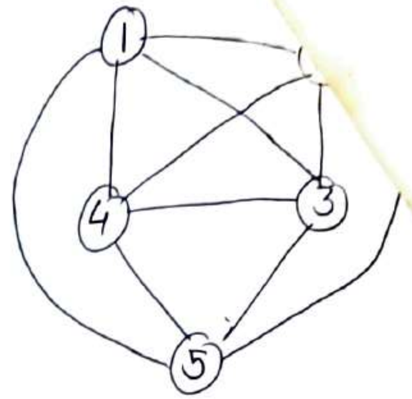
In branch & bound (useful for optimization problem) - we dont need all possible outcomes we need one path to provide tour with min^m cost

Branch and Bound - Level order is used & calculate cost



calculate cost for 2, 3 & 4 select the min^m & discard others (this is optimization)

Example 2



$$
M1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{bmatrix}
\infty & 20 & 30 & 10 & 11 \\
15 & \infty & 16 & 4 & 2 \\
3 & 5 & \infty & 2 & 4 \\
19 & 6 & 18 & \infty & 3 \\
16 & 4 & 7 & 16 & \infty
\end{bmatrix}
\end{array}
$$

Step 1 Reduce the adjacency matrix

minimum value from each row

$$
\begin{array}{c|ccccc|c}
 & 1 & 2 & 3 & 4 & 5 & \\
\hline
1 & \infty & 20 & 30 & 10 & 11 & 10 \\
2 & 15 & \infty & 16 & 4 & 2 & 2 \\
3 & 3 & 5 & \infty & 2 & 4 & 2 \\
4 & 19 & 6 & 18 & \infty & 3 & 3 \\
5 & 16 & 4 & 7 & 16 & \infty & 4
\end{array}
$$

Now substract this minimum value from the row

$$
\begin{array}{c|ccccc|c}
 & 1 & 2 & 3 & 4 & 5 & \\
\hline
1 & \infty & 10 & 20 & 0 & 1 & 10 \\
2 & 13 & \infty & 14 & 2 & 0 & 2 \\
3 & 1 & 3 & \infty & 0 & 2 & 2 \\
4 & 16 & 3 & 15 & \infty & 0 & 3 \\
5 & 12 & 0 & 3 & 12 & \infty & 4 \\
\hline
 & & & & & & 21
\end{array}
$$

21 cost of reducation of rows

Now take minimum value for each column. (3)

$$
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \infty & 10 & 20 & 0 & 1 \\
2 & 13 & \infty & 14 & 2 & 0 \\
3 & 1 & 3 & \infty & 0 & 2 \\
4 & 16 & 3 & 15 & \infty & 0 \\
5 & 12 & 0 & 3 & 12 & \infty \\
\hline
 & 1 & 0 & 3 & 0 & 0
\end{array}
$$

Reduce / substract the minimum value from the col^m

$$
M2 = \begin{array}{c|ccccc|c}
 & 1 & 2 & 3 & 4 & 5 & \\
\hline
1 & \infty & 10 & 17 & 0 & 1 & 10 \\
2 & 12 & \infty & 11 & 2 & 0 & 2 \\
3 & 0 & 3 & \infty & 0 & 2 & 2 \\
4 & 15 & 3 & 12 & \infty & 0 & 3 \\
5 & 11 & 0 & 0 & 12 & \infty & 4 \\
\hline
 & 1 & 0 & 3 & 0 & 0 & 21 \\
 & & & & & & +04 \\
 & & & & & & 25
\end{array}
$$

cost of reduction

By reducing the adjacency matrix we have found out the shortest distances from the matrix & substracted them from adj matrix. The cost of reduction may be the cost of tour. So the min^m cost of the tour may be 25.

Step2 Let's consider reduced matrix for furth[er] calculations.

$$
M2 = \begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \infty & 10 & 17 & 0 & 1 \\
2 & 12 & \infty & 11 & 2 & 0 \\
3 & 0 & 3 & \infty & 0 & 2 \\
4 & 15 & 3 & 12 & \infty & 0 \\
5 & 11 & 0 & 0 & 12 & \infty
\end{array}
$$

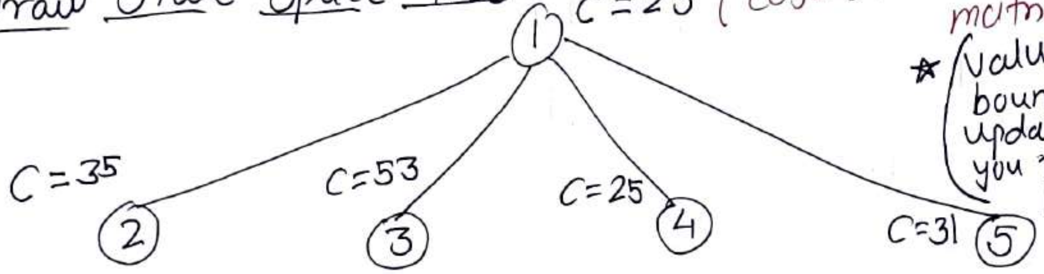
reduced cost = 25

Draw State space tree

upper bound = ∞

C = 25 (cost of reduced matrix)

★ (Value of upper bound will update once you reach to leaf node)



t(1, 2) →

Make 1^{st} row & 2^{nd} column as infinity.

$$M3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \\ \end{array} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \left[\begin{array}{c} \infty \\ \overset{\infty}{\cancel{12}} \\ 0 \\ 15 \\ 11 \end{array}\right. & \begin{array}{c} \infty \\ \infty \\ \infty \\ \infty \\ \infty \end{array} & \begin{array}{c} \infty \\ 11 \\ \infty \\ 12 \\ 0 \end{array} & \begin{array}{c} \infty \\ 2 \\ 0 \\ \infty \\ 12 \end{array} & \left.\begin{array}{c} \infty \\ 0 \\ 2 \\ 0 \\ \infty \end{array}\right] \\ 0 & 0 & 0 & 0 & 0 \end{array} \begin{array}{c} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \\ \end{array}$$

once we travel from vertex 1 to vertex 2 we need not travel back from vertex 2 to vertex 1. so (2,1) will be ∞

All rows & columns are having 0 that means all rows & columns are reduced.

$$\text{cost}(1,2) = \underset{\text{value from matrix}}{c(1,2)} + \underset{\text{reducation cost}}{r} + \underset{\text{new reduction cost}}{\hat{r}}$$

$$= 10 + 25 + 0$$

$$= 35$$

* $\boxed{\text{New reduction cost}}$
  ↳ It the cost/value substracted from rows/columns to get atleast one zero '0'.

cost (1, 3)

1st row & 3rd column as infinity

$$
\begin{array}{c|c}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\left[\begin{array}{ccccc}
\infty & \infty & \infty & \infty & \infty \\
12 & \infty & \infty & 2 & 0 \\
\infty & 3 & \infty & 0 & 2 \\
15 & 3 & \infty & \infty & 0 \\
11 & 0 & \infty & 12 & \infty
\end{array}\right]
\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \\
\hline
 & \begin{array}{ccccc} 11 & 0 & 0 & 0 & 0 \end{array}
\end{array}
$$

$(3, 1) = \infty$

First column is not reduced so we need to substruct 11 from column 1

$$
M4 = \begin{array}{c|c}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\left[\begin{array}{ccccc}
\infty & \infty & \infty & \infty & \infty \\
1 & \infty & \infty & 2 & 0 \\
\infty & 3 & \infty & 0 & 2 \\
4 & 3 & \infty & \infty & 0 \\
0 & 0 & \infty & 12 & 0
\end{array}\right]
\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \\
\hline
 & \begin{array}{cccccc} 11 & 0 & 0 & 0 & 0 & 0 \end{array}
\end{array}
$$

$+ 11$

$\overline{11}$

new reduction cost

$$
\begin{aligned}
\text{cost}(1,3) &= c(1,3) + r + \hat{r} \\
&= 17 + 25 + 11 \\
&= 53
\end{aligned}
$$

$$\begin{array}{c|c}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\left[\begin{array}{ccccc}
\infty & \infty & \infty & \infty & \infty \\
10 & \infty & 9 & 0 & \infty \\
0 & 3 & \infty & 0 & \infty \\
12 & 0 & 9 & \infty & \infty \\
\infty & 0 & 0 & 12 &
\end{array}\right]
\end{array}$$

$\boxed{M6}$

$\boxed{cost(1,5)}$ $\rightarrow$ $C = 31$

$$\begin{array}{c|c}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\left[\begin{array}{ccccc}
\infty & \infty & \infty & \infty & \infty \\
12 & \infty & 11 & \infty & 0 \\
0 & 3 & \infty & \infty & 2 \\
\infty & 3 & 12 & \infty & 0 \\
11 & 0 & 0 & \infty & \infty
\end{array}\right]
\end{array}$$

$\boxed{M5}$

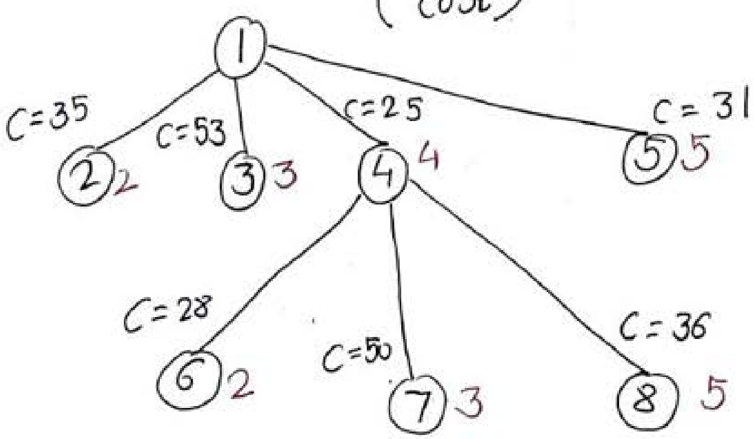
$\boxed{cost(1,4)}$ $\rightarrow$ $C = 25$

$c(1,2) = 35$

$c(1,3) = 53$

$c(1,4) = \boxed{25}$ ← we expand tree for vertex 4 & as it has minimum value (L.C. Branch & Bound) (Least cost)

$c(1,5) = 31$



For calculation C(4,6), C(4,7) & C(4,8)
matrix M5 will be used.

$$M5 = \begin{array}{c|c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 12 & \infty & 11 & \infty & 0 \\ 0 & 3 & \infty & \infty & 2 \\ \infty & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & \infty & \infty \end{bmatrix} \end{array}$$

$\left.\begin{array}{l} \text{cost } (4,2) \neq \\ C(4,6) \neq \end{array}\right\} =$ so make 4th row & 2nd column ∞

$$M6 = \begin{array}{c|cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} & \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \boxed{\cancel{12}}\ \infty & \infty & 11 & \infty & 0 \\ 0 & \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & \infty & 0 & \infty & \infty \end{bmatrix} & \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \\ & \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \end{array} & \end{array}$$

reduced matrix

~~No~~ Reduction cost = 0

~~C(4,2)~~ = ∞

cost (4,6) $= C(4,2) + r(4) + \hat{r}$

$= 3 + 25 + 0$

$= 28$

$$M7 = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & \infty & \infty & 0 \\ \infty & 1 & \infty & \infty & 0 \\ \infty & \infty & \infty & \infty & \infty \\ 0 & 0 & \infty & \infty & \infty \end{bmatrix} \end{array}$$

$$M8 = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & 0 & \infty & \infty \\ 0 & 3 & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & 0 & 0 & \infty & \infty \end{bmatrix} \end{array}$$

L.C. Branch & Bound

~~C(4,6)~~ = $C(1,2) = 35$

~~C(4,~~ $C(1,3) = 53$

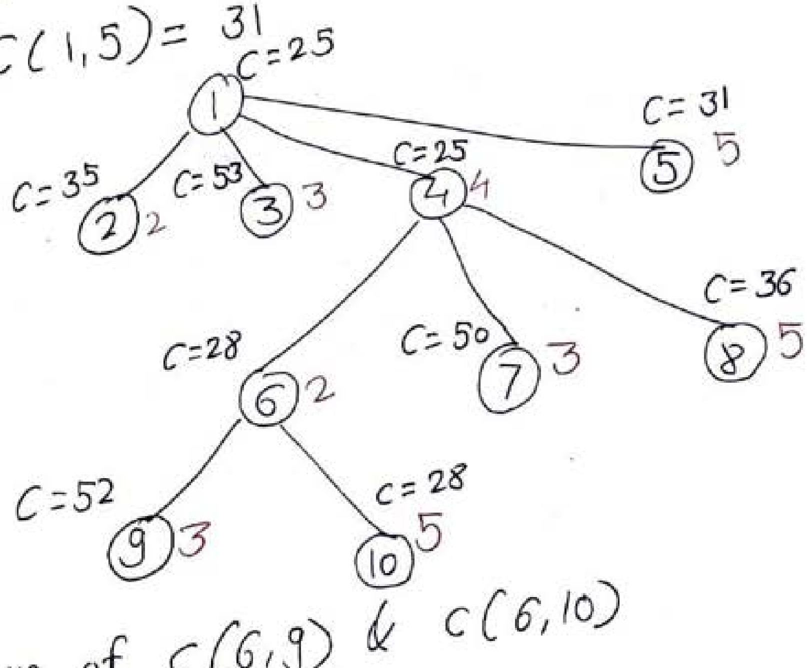
$C(1,4) = 25$

$C(1,5) = 31$

$C(4,6) = \boxed{28}$ ← L.C.

$C(4,7) = 50$

$C(4,8) = 36$



For calculation of $C(6,9)$ & $C(6,10)$ matrix M6 will be used

$$M6 = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & \infty & \infty & \infty & \infty & \infty \\ 2 & \infty & \infty & 11 & \infty & 0 \\ 3 & 0 & \infty & \infty & \infty & 2 \\ 4 & \infty & \infty & \infty & \infty & \infty \\ 5 & 11 & \infty & 0 & \infty & \infty \end{array}$$

$C(6,9)$ vertex 2 to vertex 3 $\rightarrow$ 2^{nd} row & 3^{rd} column ∞
& $c(3,1) = \infty$

$$\cancel{M7} = \begin{array}{c|ccccc|c} & 1 & 2 & 3 & 4 & 5 & \\ \hline 1 & \infty & \infty & \infty & \infty & \infty & 0 \\ 2 & \infty & \infty & \infty & \infty & \infty & 0 \\ 3 & \boxed{\cancel{0}}\ \infty & \infty & \infty & \infty & 2 & 2 \\ 4 & \infty & \infty & \infty & \infty & \infty & 0 \\ 5 & 11 & \infty & \infty & \infty & \infty & 11 \\ \hline & 0 & 0 & 0 & 0 & 0 & \end{array}$$

$13 + 0 = \boxed{13}$ cost of reduction

$$M7 = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & \infty & \infty & \infty & \infty & \infty \\ 2 & \infty & \infty & \infty & \infty & \infty \\ 3 & \infty & \infty & \infty & \infty & 0 \\ 4 & \infty & \infty & \infty & \infty & \infty \\ 5 & 0 & \infty & \infty & \infty & \infty \end{array}$$

$C(\cancel{6,10})$

$cost(6,9) = c(2,3) + c(6) + 13$

$= 11 + 28 + 13$

$= 52$

$cost(6,10) = 28$

$$M8 = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & \infty & \infty & \infty & \infty & \infty \\ 2 & \infty & \infty & \infty & \infty & \infty \\ 3 & 0 & \infty & \infty & \infty & \infty \\ 4 & \infty & \infty & \infty & \infty & \infty \\ 5 & \infty & \infty & \infty & \infty & \infty \end{array}$$

~~C(6,9)~~ =

~~C(6,10)~~ =

L.C. Brand & Bound

$C(1,2) = 35$

$C(1,3) = 53$

$C(1,4) = 25$

$C(1,5) = 31$

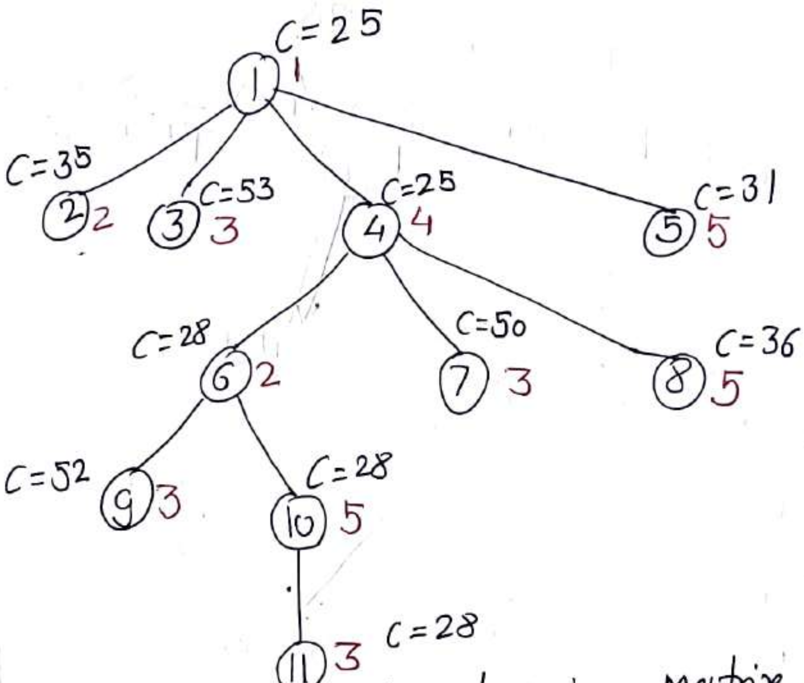
$C(4,6) = 28$

$C(4,7) = 50$

$C(4,8) = 36$

$C(6,9) = 52$

$C(6,10) = \boxed{28}$ ← L.C.



C(10,11) will be calculated using Matrix M8

$$M8 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ 0 & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 0 & \infty & \infty \end{bmatrix} \end{array}$$

5^{th} row & 3^{rd} column set to ∞

$C(10, 11) = 28$

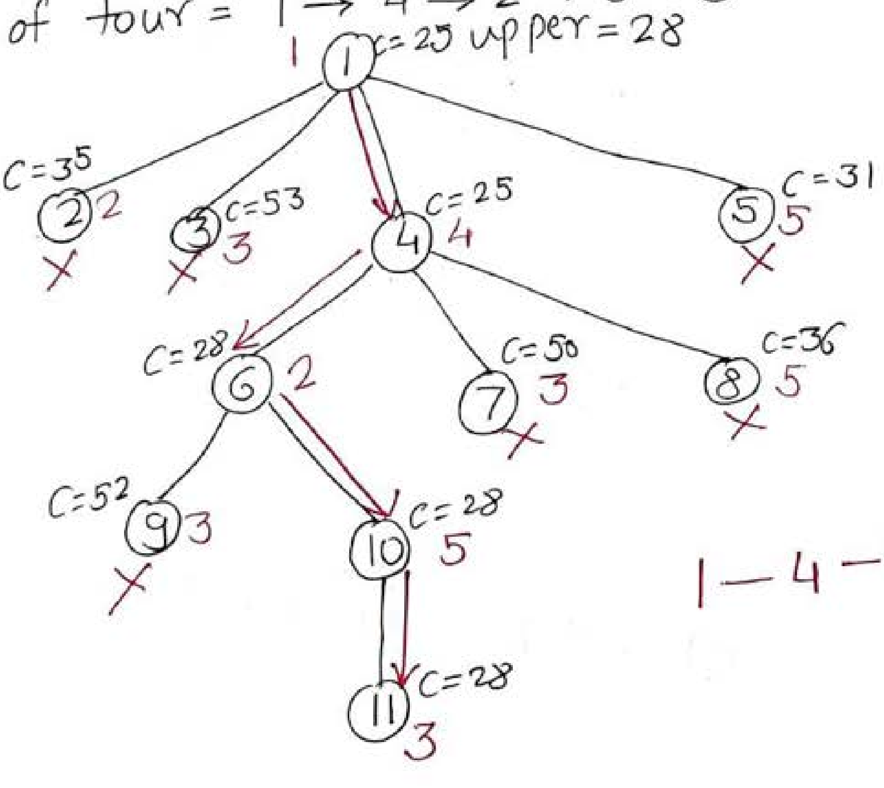
$\boxed{\text{Now update upper} = 28}$ as we have reached to leaf node

Now kill all the nodes having value greater than 28.

So the remaining node is 11.

cost of tour = 28

Path of tour = $1 \to 4 \to 2 \to 5 \to 3 \to 1$



Kill nodes having cost greater than 28

1 — 4 — 2 — 5 — 3 - 1