Fourier Series

Fourier series in $(0, 2\pi)$

① Find a fourier series to represent $f(x) = x^2$ in $(0, 2\pi)$ and hence deduce that

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

② find the fourier series of the function $f(x) = e^{-x}$, $0 < x < 2\pi$ and $f(x+2\pi) = f(x)$. Hence deduce the value of

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+1}.$$

③ obtain the fourier expansion of $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the interval $0 \le x \le 2\pi$ and $f(x+2\pi) = f(x)$. Also deduce that

(i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

ii) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$

iv) $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$

(Use Parseval's Identity)

④ Expand $f(x) = x \sin x$ in the interval $0 \le x \le 2\pi$. Deduce that

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{3}{4}$$

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(5) Find the fourier expansion for $f(x) = \sqrt{1-\cos x}$ in $(0, 2\pi)$. Hence deduce that

$$\frac{1}{2} = \sum_{1}^{\infty} \frac{1}{4n^2-1}.$$

(6). Find the fourier series for $f(x) = \frac{1}{2}(\pi - x)$ in $(0, 2\pi)$. Hence deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

<u>Fourier series in $(-\pi, \pi)$</u> (<u>Neither even nor odd</u>)

(1) Find the fourier series of

$$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

Hence deduce that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

(2) Find the fourier series for $f(x) = \begin{cases} \cos x & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}$

(3) Find the fourier series for

$$f(x) = \begin{cases} x - \pi, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$$

Hence deduce that

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

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④ Find the fourier series for

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \sin x & 0 < x < \pi \end{cases}$$. Hence deduce that

i) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots = \frac{1}{2}$

ii) $\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots = \frac{1}{4}(\pi - 2)$.

Even and odd functions in $(-\pi, \pi)$

① Find the fourier expansion of $f(x) = x^2$ in $(-\pi, \pi)$. Hence deduce that

i) $\frac{\pi^2}{6} = \sum_{1}^{\infty} \frac{1}{n^2}$ ii) $\frac{\pi^2}{12} = \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

(iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

② Find the fourier expansion of $f(x) = |\cos x|$ in $(-\pi, \pi)$.

③ Find the fourier series for the function

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi} & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi} & 0 \le x \le \pi \end{cases}$$. Deduce that

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

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④ Find fourier series for

$$f(x) = \begin{cases} x + \pi/2 & -\pi < x < 0 \\ \pi/2 - x & 0 < x < \pi \end{cases}$$

. Hence deduce that

(i) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$ (ii) $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$

⑤ Find the fourier series of $x \cos x$ in $(-\pi, \pi)$.

⑥ Find the fourier series of $f(x) = |\sin x|$ in $(-\pi, \pi)$.

⑦ Find the fourier series for $f(x) = \sqrt{1 - \cos x}$ in $(-\pi, \pi)$. Hence deduce that $\frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$.

Fourier series in $(0, 2l)$

① Find the fourier expansion of $f(x) = 2x - x^2$, $0 \le x \le 3$ whose period is 3.

② Expand $f(x) = \begin{cases} \pi x & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$ period 2 into a fourier series.

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(3) If $f(x) = \begin{cases} \pi x & 0 \le x \le 1 \\ \pi(2-x) & 1 \le x \le 2 \end{cases}$ with period 2,

Show that $f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos(2n+1)\pi x$

Fourier series in $(-l, l)$

(1) Find the fourier expansion of

$f(x) = \begin{cases} 2 & -2 < x < 0 \\ x & 0 < x < 2 \end{cases}$

(2) Find the fourier series of $f(x) = |x|$, $-2 < x < 2$.

Hence deduce that $\sum \frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$.

(3) Find the fourier series for $f(x) = 1 - x^2$ in $(-1, 1)$

(4) Find the fourier series for $f(x) = x - x^2$, $-1 < x < 1$

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Half Range Series

(1) Find a cosine series of period 2π to represent $\sin x$ in $0 \le x \le \pi$.

(2) Find half range sine series for $f(x)$ when

$$f(x) = \begin{cases} x & 0 < x < \pi/2 \\ \pi - x & \pi/2 < x < \pi \end{cases}$$

. Hence find the sum of $\sum_{(2n-1)}^{\infty} \frac{1}{n^4}$.

(3) Find the half range cosine series for $f(x) = x$, $0 < x < 2$. Using Parseval's Identity deduce that

i) $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$ ii) $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$

(4) Find the half range sine series for $f(x) = x \sin x$ in $(0, \pi)$. Hence deduce that

$$\frac{\pi^2}{8\sqrt{2}} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots$$

(5) Expand $f(x) = \begin{cases} kx & 0 < x < l/2 \\ k(l-x) & l/2 < x < l \end{cases}$ into half

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range cosine series. Deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

(6) Obtain half range sine series in $(0, \pi)$ for $x(\pi - x)$. Hence find the value of

$$\Sigma \frac{(-1)^n}{(2n-1)^3}.$$

(7) Expand $f(x) = lx - x^2$, $0 < x < l$ in a half-range i) cosine series ii) sine series. Hence from sine series deduce that

i) $\frac{\pi^3}{32} = 1 - \frac{1}{1^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$ ii) $\frac{\pi^6}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \cdots$

iii) $\frac{\pi^6}{945} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \cdots$

(8) Find half range cosine series for $f(x) = e^x$, $0 < x < l$.

(9) Obtain half range sine series for $f(x) = x(2-x)$ in $0 < x < 2$ and hence deduce that

$$\Sigma \frac{1}{n^6} = \frac{\pi^6}{945}$$

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10) Find half range sine series of period $2l$ for $f(x) = \begin{cases} \frac{2x}{l} & 0 \le x \le l/2 \\ \frac{2}{l}(l-x) & l/2 \le x \le l \end{cases}$.

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