Bisection Method

The bisection method is a simple iterative method for successively reducing the uncertainty interval based on evaluation of the derivative.

Suppose the objective f^n, $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable unimodal f^n.

We have to find minimizer of f over an interval $[a, b]$.

Step 1: If the initial interval ~~contains~~ is not given, find it using ~~bracketing~~ the bracketing ~~steps to~~ method. ~~find bracket explained earlier~~ Call the bracket $[a, b]$.

Step 2: Find $f'(x)$

Step 3: Find approximate minimizer of f,

$$c = \frac{a+b}{2}.$$

Step 4: Find $f'(c)$.

Step 5: Case (i) : If $f'(c) > 0$, replace b by c and repeat from step 3.

Case (ii) If $f'(c) < 0$, replace a by c & repeat from step 3.

Step 6: Repeat above procedure till two consecutive values of c are same upto 4 decimal places or till $f'(c) = 0$.

This value of c is the required minimizer of f.

Ex.1. Find the minimizer of $f(x) = x^4 - 14x^3 + 60x^2$ by bisection method within a range of 0.3.

Sol^n: Step 1: To find initial interval, we check the values of f at 0, 1, 2.

$f(0) = 0, \quad f(1) = -23, \quad f(2) = 4$

~~i.e.~~ $\therefore f(1) < f(0)$ & $f(1) < f(2)$

$\therefore [0,2]$ is the bracket containing minimizer of f.

Step 2: $f'(x) = 4x^3 - \cancel{14x}\, 42x^2 + 120x - 70$

Iteration	a	b	$c = \frac{a+b}{2}$	$f'(c)$	Remark
1	0	2	1	12	Replace b by c
2	0	1	0.5	−20	Replace a by c
3	0.5	1	0.75	~~1.9375~~	~~Replace a by~~ c
~~4~~	~~0.75~~	~~1~~	~~0.875~~	~~5.5234~~	Replace b by c
~~5~~	~~0.75~~	~~0.875~~	~~0.8125~~		

$\because 0.75 - 0.5 < 0.3$,
~~$\because$ 0.875 − 0.8125 < 0.1~~, we stop here and the approximate minimizer is ~~0.8125~~ 0.75.

Find the minimizer of $f(x) = x^3 - x - 1$ in the interval $[0,1]$ correct upto 4 decimal places. using bisection method.

Soln: $f(x) = x^3 - x - 1 \quad \therefore f'(x) = 3x^2 - 1$

The Given initial interval is $[0,1]$ $\therefore a = 0,\ b = 1.$

Iteration	a	b	$c = \frac{a+b}{2}$	$f'(c)$	Remark
1	0	1	0.5	−0.25	Replace a by c
2	0.5	1	0.75	0.6875	Replace b by c
3	0.5	0.75	0.625	0.1719	Replace b by c
4	0.5	0.625	0.5625	−0.0508	Replace a by c
5	0.5625	0.625	0.5938	0.0578	Replace b by c
6	0.5625	0.5938	0.5782	0.0029	— " —
7	0.5625	0.5782	0.5704	−0.0239	Replace a by c
8	0.5704	0.5782	0.5743	−0.0105	— " —
9	0.5743	0.5782	0.5763	−0.0036	— " —
10	0.5763	0.5782	0.5773	−0.0002	— " —
11	0.5773	0.5782	0.5777	0.0012	Replace b by c
12	0.5773	0.5777	0.5775	0.0005	— " —
13	0.5773	0.5775	0.5774	0.0002	— " —
14	0.5773	0.5774	0.5774		

$\therefore$ 0.5774 is the minimizer of the given f^n $f(x)$.

Newton's Method

Consider optimisation problem

Minimise $f(x)$

subject to $x \in \mathbb{R}$

where $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable objective function.

<u>Steps</u>

1. To find initial value of minimizer of f, choose points $a < c < b$ such that $f(c) < f(a)$ & $f(c) < f(b)$.
   Select initial value $x_0 = c$.

2. Find $f'(x)$ & $f''(x)$

3. Approximate minimizer $x_{n+1} = x_n - \dfrac{f'(x_n)}{f''(x_n)}$

4. Repeat above procedure till two consecutive values are same upto 4 decimal places.

<u>Note:</u>

Newton's method works well if $f''(x) > 0$ everywhere. However, if $f''(x) < 0$ for some x, Newton's method may fail to converge to the minimizer.

Ex. 1 Using Newton's method find minimiz

of $f(x) = \frac{x^2}{2} - \sin x$

Sol^n $f(x) = \frac{x^2}{2} - \sin x$

To find initial value,

$f(0) = 0$, $f(1) = -0.3415$, $f(2) = 1.0907$

$0 < 1 < 2$ s.t. $f(1) < f(0)$ & $f(1) < f(2) \Rightarrow$ Bracket is $[0, 2]$.

$\therefore x_0 = 1$

$f'(x) = x - \cos x$, $f''(x) = 1 + \sin x$

Iteration	$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}$ $x_{n+1} = x_n - \frac{x_n - \cos x_n}{1 + \sin x_n}$
1	$x_1 = 0.7504$
2	$x_2 = 0.7391$
3	$x_3 = 0.7391$

Required minimizer of f is 0.7391.

since $x_2 = x_3$

Ex. 1 Using Newton's method find minimiz[er] of $f(x) = \frac{x^2}{2} - \sin x$

Sol^n:- $f(x) = \frac{x^2}{2} - \sin x$

To find initial value,

$f(0) = 0$, $f(1) = -0.3415$, $f(2) = 1.0907$

$0<1<2$ s.t. $f(1) < f(0)$ & $f(1) < f(2) \Rightarrow$ Bracket is $[0, 2]$.

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1	$x_1 = 0.7504$
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Required minimizer of f is 0.7391.

since $x_2 = x_3$

Find the minimizer of $f(x) = \frac{x^5}{5} - \frac{x^2}{2} - 9x$ using Newton's method.

Soln: To find bracket $[a,b]$:

$f(0) = 0, \quad f(1) = -9.3, \quad f(2) = -13.6, \quad f(3) = 17.1$

$\therefore f(2) < f(1)$ & $f(2) < f(3)$

$\Rightarrow$ the required bracket is $[1, 3]$.

Take $x_0 = 2$.

$f(x) = \frac{x^5}{5} - \frac{x^2}{2} - 9x \qquad \therefore f'(x) = x^4 - x - 9$

$f''(x) = 4x^3 - 1$

By Newton's method,

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)} = x_n - \frac{(x_n^4 - x_n - 9)}{4x_n^3 - 1}$$

$$= \frac{4x_n^4 - x_n^4 + x_n + 9}{4x_n^3 - 1}$$

$$\therefore x_{n+1} = \frac{3x_n^4 + 9}{4x_n^3 - 1}$$

Iteration (n)	$x_{n+1} = \frac{3x_n^4 + 9}{4x_n^3 - 1}$
0	$x_1 = \frac{3x_0^4 + 9}{4x_0^3 - 1} = \frac{3 \times 2^4 + 9}{4 \times 2^3 - 1} = 2.3551$
1	$x_2 = 1.9764$
2	$x_3 = 1.8331$
3	$x_4 = 1.8137$
4	$x_5 = 1.8134$
5	$x_6 = 1.8134$

$\because x_5 = x_6$, the approx. minimizer of the given f^n is 1.8134

False Position method

Suppose the objective function, $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function

- Steps to find minimizer of $f(x)$ over an interval $[a, b]$ using False Position method:

<u>Step 1</u>: If the initial interval is not given, find it using the bracketing method. Call the bracket as $[a, b]$.

<u>Step 2</u>: Find $f'(x)$.

<u>Step 3</u>: Find approximate minimizer of f,

$$c = \frac{a f'(b) - b f'(a)}{f'(b) - f'(a)}$$

<u>Step 4</u>: Find $f'(c)$.

<u>Step 5</u>: Case (i) If $f'(c) > 0$, replace b by c and repeat from step 3

Case (ii) If $f'(c) < 0$, replace a by c & repeat from step 3.

<u>Step 6</u> Repeat above procedure till two consecutive values of c are same upto 4 decimal places or till $f'(c) = 0$.

This value of c is the required minimizer of f.

Using False Position method, find minimizer of $f(x) = \frac{x^4}{4} - \frac{x^2}{2} - 4x$, $x \in \mathbb{R}$

→ To find a bracket,

$f(0) = 0$

$f(0.5) = -2.1094$

$f(1) = -4.25$

$f(1.5) = -5.8594$

$f(2) = -6$

$f(2.5) = -3.3594$

$f'(x) = x^3 - x - 4$

∴ We get $1.5 < 2 < 2.5$ such that $f(2) < f(1.5)$ & $f(2) < f(2.5)$

∴ $[1.5, 2.5]$ is the inyeeval in which minimizer of f lies.

Initially $a = 1.5$, $b = 2.5$, $c = \frac{a f'(b) - b f'(a)}{f'(b) - f'(a)}$

Iteration	a	$f'(a)$	b	$f'(b)$	c	$f'(c)$	Remark
1	1.5	-2.125	2.5	9.125	1.6889	-0.8715	$a \leftrightarrow c$
2	1.6889	-0.875	2.5	9.125	1.7596	-0.3115	$a \leftrightarrow c$
3	1.7596	-0.3115	2.5	9.125	1.784	-0.1061	$a \leftrightarrow c$
4	1.7840	-0.1061	2.5	9.125	1.7922	-0.0357	$a \leftrightarrow c$
5	1.7922	-0.0357	2.5	9.125	1.795	-0.0115	$a \leftrightarrow c$
6	1.795	-0.0115	2.5	9.125	1.7959	-0.0037	$a \leftrightarrow c$
7	1.7959	-0.0037	2.5	9.125	1.7962	-0.0011	$a \leftrightarrow c$
8	1.7962	-0.0011	2.5	9.125	1.7963	-0.0002	$a \leftrightarrow c$
9	1.7963	-0.0002	2.5	9.125	1.7963		

$\therefore$ Minimizer of f in the interval $[1.5, 2.5]$
is 1.7963

Q.2 Using False Position method, find minimizer of $f(x) = x e^x - \sin x$ in the interval $[-3, -2.5]$.

Soln: Given $f(x) = x e^x - e^x + \cos x$

$$\therefore f'(x) = x e^x + e^x - e^x - \sin x$$

$$= x e^x - \sin x$$

Given interval is $[-3, -2.5]$

$\therefore$ Initially $a = -3$, $b = -2.5$,

$$c = \frac{a f'(b) - b f'(a)}{f'(b) - f'(a)}$$

Iteration Number	a	$f'(a)$	b	$f'(b)$	c	$f'(c)$
1	-3	-0.0082	-2.5	0.3933	-2.9898	$0.00083 > 0$
2	-3	-0.0082	-2.9898	0.00083	-2.9907	$0.00083 > 0$
3	-3	-0.0082	-2.9907	0.00003	-2.9907	

$\therefore$ Minimizer of f in the interval $[-3, -2.5]$
is -2.9907.

$\therefore$ Minimizer of f in the interval $[1.5, 2.5]$
is 1.7963

Q.2 Using False Position method, find minimizer of $f(x) = xe^x - \sin x$ in the interval $[-3, -2.5]$.

Sol^n: Given $f(x) = xe^x - e^x + \cos x$

$\therefore f'(x) = xe^x + e^x - e^x - \sin x$

$= xe^x - \sin x$

Given interval is $[-3, -2.5]$

$\therefore$ ally $a = -3$, $b = -2.5$,

$$c = \frac{a f'(b) - b f'(a)}{f'(b) - f'(a)}$$

Iteration Number	a	f'(a)	b	f'(b)	c	f'(c)
1	-3	-0.0082	-2.5	0.3933	-2.9898	0.0008370
2	-3	-0.0082	-2.9898	0.00083	-2.9907	0.0008370
3	-3	-0.0082	-2.9907	0.00003	-2.9907	

$\therefore$ Minimizer of f in the interval $[-3, -2.5]$
is -2.9907.