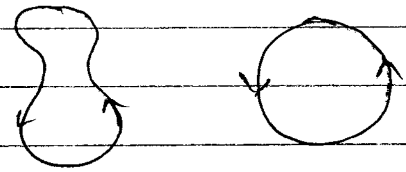
Defn :

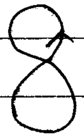
1) simple closed curve:-
If closed curve does not intersect itself then it is called as "simple closed curve" or "Jordan curve".

e.g.



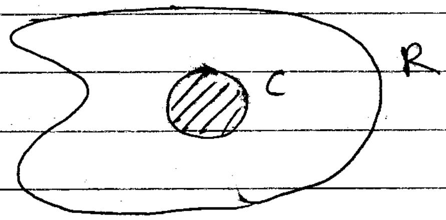
2) Multiple curve:-
If closed curve intersects itself then it is called as "multiple curve".

e.g.



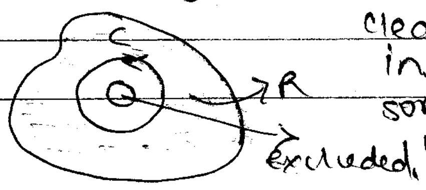
3) Simply Connected Region:-
The Region "R" is called as simply connected region if every closed curve in the region encloses points of region "R" only.

e.g.



4) Multiply Connected Region:-
A region which is not simply connected is called as "multiply connected region".

e.g.



Clearly C lies wholely inside arc but contain some points which are not in R.

* Cauchy's Integral Theorem:-

If $f(z)$ is analytic function & if its derivative $f'(z)$ is continuous at each point within and on a simple closed curve "c". then,

$$\oint_C f(z)\,dz = 0$$

① Verify Cauchy's Theorem for $f(z) = z^2$ along $|z| = 1$

→

$f(z) = z^2$

$z = (x+iy)^2$

$= x^2 + 2xiy - y^2$

$= x^2 - y^2 + i2xy$

$u = x^2 - y^2$ $\quad$ $v = 2xy$

$u_x = 2x$ $\quad$ $v_x = 2y$

$u_y = -2y$ $\quad$ $v_y = 2x$

$u_x = v_y$ & $u_y = -v_x$

The C-R eqns are satisfied

∴ The function $f(z)$ is analytic.

∴ By Cauchy's Theorem as $f(z)$ is analytic in an on closed curve $|z| = 1$

$$\oint_C f(z)\,dz$$

$$= \int_C z^2\,dz$$

$$= 0.$$

Given : $|z| = 1$

Put $z = e^{i\theta}$

$dz = e^{i\theta} \cdot i \cdot d\theta$

$\theta : 0 \to 2\pi$

$$\int_C z^2 dz$$

$$= \int_0^{2\pi} (e^{i\theta})^2 \, e^{i\theta} \cdot i \cdot d\theta$$

$$= i \int_0^{2\pi} e^{3i\theta} \cdot d\theta$$

$$= i \left[\frac{e^{3i\theta}}{3i} \right]_0^{2\pi}$$

$$= i \left[\frac{e^{3i2\pi}}{3i} - \frac{1}{3i} \right]$$

$$e^{i6\pi} = \cos 6\pi + i \sin 6\pi$$
$$= 1 + i(0)$$

$$= i \left[\frac{1}{3i} - \frac{1}{3i} \right]$$

$$= i(0)$$

$$= 0$$ //

2) $I = \int_C (z^2 - 2\bar{z} + 1)\, dz$

where c is $x^2+y^2=1$.

→

$I = \int_C (z^2+1-2\bar{z})\, dz$

$= \int_C (z^2+1)\, dz - \int_C 2\bar{z}\, dz$

$= \int_C (z^2+1)\, dz - 2\int_C \bar{z}\, dz$

$\int_C (z^2+1)\, dz$

$z^2+1 = (x+iy)^2+1$

$= x^2 + 2ixy - y^2 + 1$

$= x^2 - y^2 + 1 + i2xy$

$u = x^2-y^2+1 \qquad v = \cancel{i}2xy$

$u_x = 2x \qquad v_x = 2y$

$u_y = -2y \qquad v_y = \cancel{2y}\ \cancel{x}2x$

$u_x = \cancel{z} v_y$, $u_y = -\cancel{2y} v_x$

∴ The C-R eqns are satisfied.

∴ The $f(z)$ is analytic in & on curve C

By Cauchy's Thm

∴ $\int_C (z^2+1)\, dz$

$= 0$.

$= 2 \int_C \bar{z}\, dz$

$z = x + iy$
$|z - z_0| = r$
$(x-a)^2 + (y-b)^2 = r^2$
$|z| = r$
$\sqrt{x^2+y^2} = r$
$x^2+y^2 = r^2$
$|z|^2 = 1$
$\therefore \underline{|z| = 1}$

Put $z = e^{i\theta}$

$dz = e^{i\theta} \cdot i \cdot d\theta$

$\bar{z} = e^{-i\theta}$

$\theta : 0 \to 2\pi$

$= 2 \int_0^{2\pi} e^{-i\theta} \cdot e^{i\theta} \cdot i \cdot d\theta$

$= 2 \int_0^{2\pi} i \cdot d\theta$

$= 2i \int_0^{2\pi} d\theta$

$= 2i \left[\theta\right]_0^{2\pi}$

$= 2i\left[2\pi - 0\right]$

$= 4i\pi.$

from ①,

$\therefore \quad I = 0 - 4i\pi$

$\therefore \quad \boxed{I = -4i\pi}$ //

H.W.

2) $I = \int_C (z^2 - 2\bar{z} + 1)\, dz.$
$x^2+y^2 = 2$

without using cauchy's thm.

$x^2+y^2 = 2$
$|z|^2 = 2$
$|z| = \sqrt{2}$

Put. $z = \sqrt{2}\, e^{i\theta}$.

~~Put~~.

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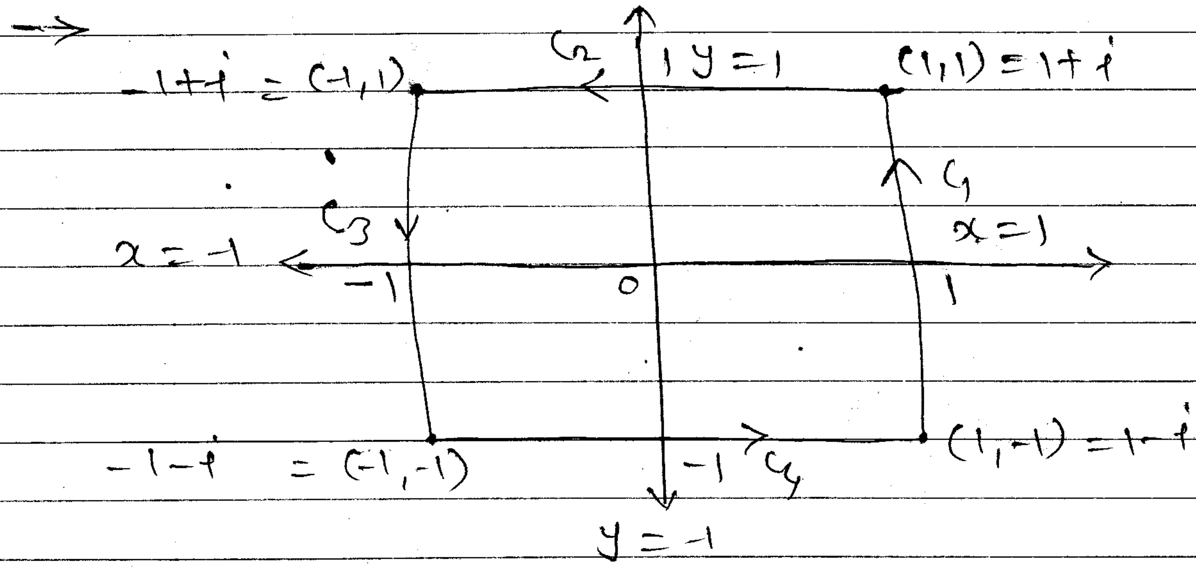
H.w. (3) Verify Cauchy's Theorem for,

$f(z) = 3z^2 + iz - 4$

if C is perimeter of square with vertices are $1 \pm i$ & $-1 \pm i$

$\rightarrow$



$$\int_C f(z)\, dz.$$

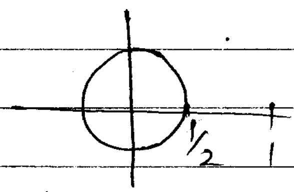
(4) Evaluate:

$$\int_C \frac{e^{2z}}{z-1}\, dz$$

where C is $|z| = \frac{1}{2}$.

$\rightarrow$

clearly $f(z) = \dfrac{e^{2z}}{z-1}$ is



analytic everywhere except $z=1$

Given C is $|z| = \frac{1}{2}$.

for $|z| = 1$, $|z| = |1| = 1 > \frac{1}{2}$.

$\Rightarrow$ $z=1$ lies outside curve C $|z| = \frac{1}{2}$.

Hence
∵ function is analytic in and on closed curve C.

∴ By Cauchy's thm.,

$$\int_C \frac{e^{2z}}{z-1}\,dz = 0.$$

★ Cauchy's Integral OR Fundamental formula ::

If $f(z)$ is analytic inside and on closed curve "C" of simply connected region and if z_0 is any point within C then,

$$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0).$$

Also,

$$\int_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0)$$

① Evaluate :

i) $\int_C \frac{e^{2z}}{z-1}\,dz$ ii) $\int_C \frac{e^{2z}}{(z-1)^3}\,dz.$

where C is $|z| = 2$

→ ~~f(z)~~ ~~f(z)~~ = $\frac{e^{2z}}{z-1}$ is analytic every where except $z = 1$

Given C is $|z| = 2$

for $z = 1$ $|z| = |1| = 1 < 2.$

⇒ $z = 1$ lies inside curve C $|z| = 2.$

Clearly, $f(z) = e^{2z}$ is analytic inside and on curve C.

By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$$

$$\Rightarrow \int_C \frac{e^{2z}}{z-1} dz = 2\pi i f(1) = 2\pi i e^2.$$

2) $\int_C \frac{e^{2z}}{(z-1)^3} dz$

By Cauchy's formula,

$$\int_C \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i f^{(n-1)}(z_0)}{(n-1)!}$$

$$\int_C \frac{e^{2z}}{(z-1)^3} dz = \frac{2\pi i f^{(2)}(1)}{(3-1)!}$$

$f(z) = e^{2z}$

$f'(z) = 2e^{2z}$

$f''(z) = 4e^{2z}$

$\Rightarrow f''(1) = 4e^2.$

$$\therefore \int_C \frac{e^{2z}}{(z-1)^3} dz = \frac{2\pi i}{2} 4e^2$$

$$\therefore \int_C \frac{e^{2z}}{(z-1)^3} dz = 4\pi i e^2 \text{ //}$$

Procedure :-

Type - I :

If $\int \frac{f(z)}{(z-z_0)} dz$ or $\int \frac{f(z)}{(z-z_0)^n} dz$

case i) : If point z_0 is outside curve "C" then by cauchy's thm,

$$\int_C \frac{f(z)}{z-z_0} dz = 0.$$

or $$\int_C \frac{f(z)}{(z-z_0)^n} dz = 0.$$

case ii) : If point z_0 is inside curve "c" then by Cauchy's formula,

$$\int_C \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$$

& $$\int_C \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(z_0).$$

Type - II :

If $I = \int_C \frac{f(z)}{(z-a)(z-b)} dz.$

case i) : If points $z = a, b$ both are outside curve "c"

By Cauchy's thm,

$$\int_C \frac{f(z)}{(z-a)(z-b)} dz = 0.$$

case ii) : If points $z=a$ & $z=b$ both are inside curve C

By Partial fraction,

$$\frac{1}{(z-a)(z-b)} = \frac{A}{(z-a)} + \frac{B}{(z-b)}$$

$$\int_C \frac{f(z)}{(z-a)(z-b)} dz$$

$$= \int_C f(z)\left[\frac{1}{(z-a)(z-b)}\right] dz$$

$$= \int_C f(z)\left[\frac{A}{(z-a)} + \frac{B}{(z-b)}\right] dz$$

$$= \int_C \left[\frac{A f(z)}{(z-a)} + \frac{B f(z)}{(z-b)}\right] dz$$

$$= A \int_C \frac{f(z)}{(z-a)} dz + B \int_C \frac{f(z)}{(z-b)} dz.$$

By Cauchy's formula,

$$= A\, 2\pi i f(a) + B\, 2\pi i f(b).$$

case iii) : If points $z=a$ is inside & $z=b$ is outside the curve C

$\therefore$ $F(z) = \dfrac{f(z)}{(z-b)}$ is analytic in & on C

$$\int_C \frac{f(z)}{(z-a)(z-b)}\,dz$$

$$= \int_C \frac{\left[\frac{f(z)}{(z-b)}\right]}{(z-a)}\,dz$$

$$= \int_C \frac{F(z)}{(z-a)}\,dz$$

By Cauchy's Integral formula,

$$\therefore \int_C \frac{f(z)}{(z-a)(z-b)}\,dz = 2\pi i\, F(a).$$

① $\displaystyle\int_C \frac{z+3}{z^2-2z+5}\,dz$

where C is $|z-1| = 1$.

Ans:- $z^2 - 2z + 5 = 0$

$z = 1 \pm 2i$

$\dfrac{z+3}{z^2-2z+5}$ is analytic except $z = 1 \pm 2i$

Given C is $|z-1| = 1$

$z = 1+2i$ $\quad = |1+2i-1| \neq 1$

$= |2i| = \sqrt{4} = 2 > 1$

$\Rightarrow z = 1+2i$ is outside curve C.

$z = 1-2i$ $\quad |z-1| = |1-2i-1|$

$= |-2i|$

$= |2i|$

$= 2 > 1$

$\Rightarrow z = 1-2i$ is outside curve C.

$\therefore \dfrac{z+3}{z^2-2z+5}$ is analytic in and on curve c

$\therefore$ By Cauchy's Integral th^m,

$$\int_c \frac{z+3}{z^2-2z+5}\,dz = 0$$

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H.W.

1) $I = \int_c (z^2 - 2\bar{z} + 1)\,dz$

where c is $x^2+y^2=2$
without using Cauchy's th^m.

Ans:-

$x^2+y^2=2$

$|z|^2 = 2$

$|z| = \sqrt{2}$

Put $z = \sqrt{2}\,e^{i\theta}$

$dz = \sqrt{2}\,e^{i\theta}\cdot i\cdot d\theta$

θ : 0 to 2π

$$\therefore I = \int_c (z^2 - 2\bar{z} + 1)\,dz$$

$$= \int_0^{2\pi} \left[(\sqrt{2}\,e^{i\theta})^2 - 2(\sqrt{2}\,e^{-i\theta}) + 1\right]\sqrt{2}\,e^{i\theta}\cdot i\cdot d\theta$$

$$= \int_0^{2\pi} \left[2e^{2i\theta} - 2\sqrt{2}\,e^{-i\theta} + 1\right]\sqrt{2}\,e^{i\theta}\cdot i\cdot d\theta$$

$$= \sqrt{2}\,i \int_0^{2\pi} \left[2e^{2i\theta} - 2\sqrt{2}\,e^{-i\theta} + 1\right] e^{i\theta}\,d\theta$$

$$\therefore I = \sqrt{2}i \int_0^{2\pi} \left[2.e^{2i\theta}.e^{i\theta} - 2\sqrt{2}\, e^{-i\theta}.e^{i\theta} + e^{i\theta}\right] d\theta$$

$$= \sqrt{2}i \int_0^{2\pi} \left[2.e^{3i\theta} - 2\sqrt{2} + e^{i\theta}\right] d\theta$$

$$= \sqrt{2}i \left[\frac{2.e^{3i\theta}}{3i} - 2\sqrt{2}\theta + \frac{e^{i\theta}}{i}\right]_0^{2\pi}$$

$$= \sqrt{2}i \left[\frac{2e^{3i2\pi}}{3i} - 2\sqrt{2}(2\pi) + \frac{e^{i2\pi}}{i} - 2.\frac{1}{3i} + 0 - \frac{1}{i}\right]$$

$$I = \sqrt{2}i \left[\frac{2e^{6i\pi}}{3i} - 4\sqrt{2}\pi + e^{i2\pi} - \frac{2}{3i} - \frac{1}{i}\right]$$

$$e^{6i\pi} = \cos 6\pi + i\sin 6\pi$$
$$= 1 + i(0)$$
$$= 1$$

$$e^{i2\pi} = \cos 2\pi + i\sin 2\pi$$
$$= 1 + i(0)$$
$$= 1$$

$$I = \sqrt{2}i \left[\frac{\cancel{2}}{\cancel{3}i} - 4\sqrt{2}\pi + 1 - \frac{\cancel{2}}{\cancel{3}i} - \frac{1}{i}\right]$$

$$I = \sqrt{2}i \left[1 - 4\sqrt{2}\pi - \frac{1}{i}\right]$$

$$\boxed{I = \sqrt{2}i - 8\pi i - \sqrt{2}}$$ //