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Module No. - 3

Posets and Lattice.

* **Partially ordered relation** -
A relation R on a set A is called partial order relation if R is reflexive, antisymmetric and transitive. ~~poset~~.
(partial means not every pair is comparable).

* **Partially ordered set or Poset** -
The set A together with the partial order R is called a partially ordered set or poset.
It is denoted by (A, R).
Note:- (A, R) or (S, ≤) where $a \le b$, $a R b$.

ex. $A = \{1, 2, 3\}$
$R_1 = \{(1,1)\ (2,2)\ (3,3)\ (1,2)\ (2,3)\ (1,3)\}$
R_1 is reflexive as

$$\left.\begin{array}{l}(1,1) \in R_1 \\ (2,2) \in R_1 \\ (3,3) \in R_1\end{array}\right\} \quad \begin{array}{l}\forall x \in A \\ (x,x) \in R\end{array}$$

R_1 is transitive as

$(1,2) \in R_1$	$a R b$
$(2,3) \in R_1$	$b R c$
$(1,3) \in R_1$	$\therefore a R c$.

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R_1 is antisymmetric as
$(1,2) \in R_1$ but $(2,1) \notin R_1$
$(2,3) \in R_1$ $(3,2) \notin R_1$
$(1,3) \in R_1$ $(3,1) \notin R_1$
also $\left.\begin{array}{l}(1,1) \in R_1 \\ (2,2) \in R_1 \\ (3,3) \in R_1\end{array}\right\}$ for self loop allowed in antisymmetric.

$\therefore R_1$ is partial order relation

Note : Antisymmetric relation is defined as,
$(a,b) \in R$
if aRb with $a \neq b$ then
$bRa \notin R$
but if aRb and bRa then $a=b$.

ex. $A = \{1, 2, 3, 4\}$
$R = \{(1,1)(1,2)(2,2)(2,4)(1,3)(3,3),$
$(3,4)(1,4)(4,4)\}$

R is reflexive.
$(1,1) \in R$
$(2,2) \in R$
$(3,3) \in R$
$(4,4) \in R$

R is antisymmetric
$(1,2) \in R$ but $(2,1) \notin R$
$(2,4) \in R$ but $(4,2) \notin R$

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$(1,3) \in R$ but $(3,1) \notin R$
$(3,4) \in R$ but $(4,3) \notin R$
$(1,4) \in R$ but $(4,1) \in R$

~~R is transitive~~

also

$(1,1), (2,2), (3,3), (4,4) \in R$.

R is transitive

$(1,1)$ & $(1,2) \in R$, $(1,2) \in R$
$(1,1)$ & $(1,3) \in R$, $(1,3) \in R$
$(1,1)$ & $(1,4) \in R$, $(1,4) \in R$
$(1,2)$ & $(2,2) \in R$ $(1,2) \in R$
$(1,2)$ & $(2,4) \in R$ $(1,4) \in R$
$(2,4)$ & $(4,4) \in R$ $(2,4) \in R$
$(1,3)$ & $(3,4) \in R$ $(1,4) \in R$
$(1,3)$ & $(3,3) \in R$ $(1,3) \in R$
$(3,4)$ & $(4,4) \in R$ $(3,4) \in R$
$(1,4)$ & $(4,4) \in R$ $(1,4) \in R$

$\therefore$ R is partial order relation.

ex. Show that the relation $R = \{(a,b) \mid a \subseteq b\}$ defined on the power set of set $S = \{1,2,3\}$ is a partial order relation.

$\Rightarrow$ Relation R is said to be a partial ordering iff R is
- Reflexive
- Antisymmetric
- Transitive

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we have $S = \{1, 2, 3\}$

$P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

$R = \{(a, b) \mid a \subseteq b\}$

i) Reflexivity :

$a \subseteq a$ or aRa

ii) Antisymmetry :

$a \subseteq b$ and $b \subseteq a$ implies $a = b$.

iii) Transitivity :

$a \subseteq b$ and $b \subseteq c$ implies $a \subseteq c$.

Hence R is a partial order relation.

ex. show that the relation $R = \{(a,b) \mid a \text{ divides } b\}$ defined on the set $S = \{1, 2, 3, 4, 6\}$ is a partial order relation.

⇒ Relation R is said to be a partial ordering iff R is
- Reflexive
- Antisymmetric
- Transitive.

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$R = \{(a,b) \mid a \text{ divides } b\}$

$S = \{1, 2, 3, 4, 6\}$

i) Reflexive :

a divides a

$(1,1) \in R$

$(2,2) \in R$

$(3,3) \in R$

$(4,4) \in R$

$(6,6) \in R$

ii) Antisymmetric :

1. a divides b but b not divides a if $a \neq b$
2. a divides b and b divides a if $a = b$

$(1,2) \in R$ $(2,1) \notin R$

$(1,3) \in R$ $(3,1) \notin R$

$(1,4) \in R$ $(4,1) \notin R$

$(1,6) \in R$ $(6,1) \notin R$

$(2,4) \in R$ $(4,2) \notin R$

$(2,6) \in R$ $(6,2) \notin R$

$(3,6) \in R$ $(6,3) \notin R$

and $(1,1) \in R$, $(2,2) \in R$, $(3,3) \in R$

$(4,4) \in R$ & $(6,6) \in R$

iii) Transitive :

a divides b and b divides c implies a divides c.

like $(1,2) \in R$ and $(2,4) \in R$

$(1,4) \in R$.

R is partial order relation. Poset : (S, R) or $(S, \leq)$

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* Total order relations or chains and anti-chains -

Definition - If any two elements in a poset are comparable, then the partial order is called as a total order or linear order.

Note:-

Comparable elements -

If A is given set and R is a partial order relation on A then the elements a, b of A are said to be comparable if aRb or bRa. This means if $a \not R b$ or $b \not R a$ then a and b are not comparable.

e.g.

$A = \{1, 2, 3, 4, 6, 12\}$

$R = \{(a, b) \mid a \text{ divides } b\}$

$R = \{(1,1)(1,2)(1,3)(1,4)(1,6)(1,12)$
$(2,2)(2,4)(2,6)(2,12)(3,3)$
$(3,6)(3,12)(4,4)(4,12)(6,6)$
$(6,12)(12,12)\}$

here we can say that,

$(2,4) \Rightarrow 2R4$ and $4R2$

but $\cancel{4 \not R 2} \in 4R2 \notin R$

but 2 and 4 are comparable.

but for $(2,11) \Rightarrow 2 \not R 11$ & $11 \not R 2$

hence 2 & 11 not comparable.

here (a,b) is a relation where a divides b.
so R is a relation called as total order relation or chain.

but $(2, 11) \Rightarrow 2 \not R 11$ & $11 \not R 2$
hence it is not comparable
so we can say it is anti-chain.

Chain -

Hence such total order relation or linear order relation or simple ordering relation with set A is called as total ordered set or simply ordered set or a chain.

Anti-chain -

If on the other hand, in a poset, if no two elements of the set are comparable, then poset is called as anti-chain.

Hasse Diagram -

The diagraph of a poset can be considerably as follows.
for instance, since in a poset, the relation is reflexive, we drop the loops around the vertices, Since in a poset R is

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transitive i.e. if a R b and b R c then a R c, we can drop edge from a to c. Thus we drop all edges implied by transitivity.

Finally we arrange the whole diagram such that all arrows point upwards and then drop the arrow heads. The resulting diagram is Hasse diagram.

we consider the example.

$A = \{1, 2, 3, 4, 12\}$

$R = \{(a, b) \mid a \text{ divides } b)$

show that (A, R) is a poset. Also construct the digraph of the poset and its Hasse diagram.

⇒ we have,

$R = \{(1,1)(2,2)(3,3)(4,4)(12,12)$
$(1,2)(1,3)(1,4)(1,12)(2,4)(2,12)$
$(3,12)(4,12)\}$

Let a, b, c be any three elements of A.

i) aRa, R is reflexive.

$(1,1) \in R \quad (2,2) \in R$

$(3,3) \in R \quad (4,4) \in R \quad (12,12) \in R.$

ii) a R b ~~but~~ and b R a if $a = b$

or a R b but b $\not R$ a

$(a, b) \in R$ $\quad$ $(b, a) \notin R$.

so R is antisymmetric

$(1,2) \in R$	$(2,1) \notin R$
$(1,3) \in R$	$(3,1) \notin R$
$(1,4) \in R$	$(4,1) \notin R$
$(1,12) \in R$	$(12,1) \notin R$
$(2,4) \in R$	$(4,2) \notin R$
$(2,12) \in R$	$(12,2) \notin R$
$(3,12) \in R$	$(12,3) \notin R$
$(4,12) \in R$	$(12,4) \notin R$.

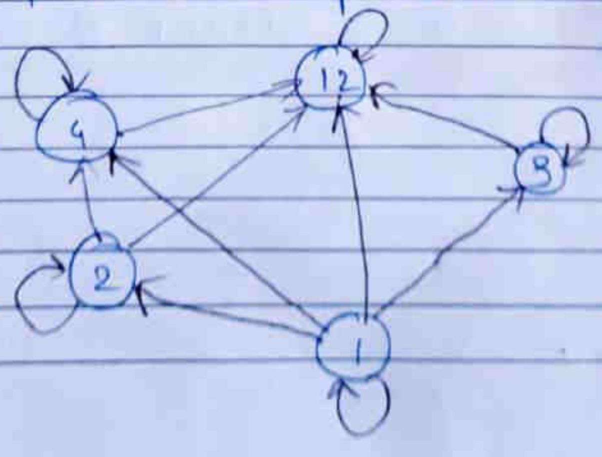
iii) R is transitive.

$\therefore$ $(1,2) \in R$ $\quad$ $(2,4) \in R$

$(1,4) \in R$. and so on.

R is partial order relation on A

(A, R) is a poset



construct Hasse dia.

1) Delete loop at each vertex.

2) Delete edges implied by transitivity

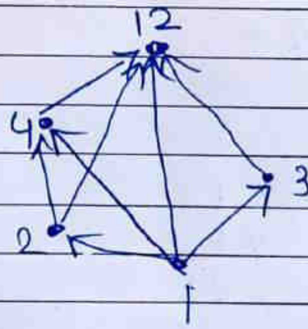


fig (a)

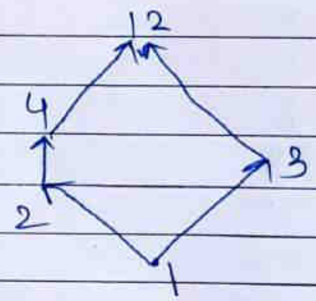


fig (b)

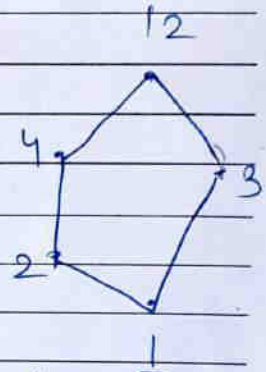


fig (c)

3) Rearrange the digraph, such that all edges go upward.
& finally we need to drop only arrow heads.
fig (c) is a resultant Hasse dia.