"Game Theory"

Strategic Game / Simultaneous Move Game.

It is a model of interacting players (decision makers). Each player has a set of possible action
The, so to say, 'Interaction between the players' allows each player's action to be affected by the action of all other players, but not only her own action.
'Action Profile' refers to the set of all possible actions for a player.

Strategic Game with ordinal Preferences :- consists of :-

1.) a set of players.
2.) for each player, a set of actions
3.) for each player, preferences over the set of action profiles.

Player's preferences are also frequently represented by giving the payoff functions, which have only ordinal significance.

Some basic aspects :-

1.) Each player chooses her action once & for all.
2.) Each of the players chooses their actions 'simultaneously' such that no player has the prior information about the action of other players, when he/she chooses her own action.

Prisoner's Dilemma :-

Players : The two suspects.

Actions :- Set of action plan is {Quiet, fink}.

Preference Ordering :- Player 01's preferences, from best to worst,

u_1 (fink, quiet) > u_1 (quiet, quiet) > u_1 (fink, fink) > u_1 (quiet, fink)

- u_1 (fink, quiet) ⇓ (P_1 (player 01) is freed)
- u_1 (quiet, quiet) ⇓ player 1 gets one year in prison
- u_1 (fink, fink) ⇓ Player 01 gets 3 years in prison.
- u_1 (quiet, fink) ⇓ Player 01 gets 04 years in prison

Similarly, for player 2,

u_2 (quiet, fink) > u_2 (quiet, quiet) > u_2 (fink, fink) > u_2 (fink, quiet)

- u_2 (quiet, fink) ⇓ Player 02 gets freed
- u_2 (quiet, quiet) ⇓ player 02 gets 1 year in prison.
- u_2 (fink, fink) ⇓ P_2 gets three years in prison.
- u_2 (fink, quiet) ⇓ P_2 gets 4 years in prison.

The Prisoner's dilemma.

		Suspect 02	
		Quiet	Fink
Suspect 01	Quiet	(1, 1)	(4,0)
	Fink	(0, 4)	(3,3)

This game actually represents the fact that Although there are gains from cooperation (both players choose quiet than they both choose fink) but each player has an incentive to "free ride" (Choose Fink) whatever the other player does.

1. Working on a joint project

		Player 02 Work hard	Goof off
Player 01	Work hard	(2,2)	(0,3)
	Goof off	(3,0)	(1,1)

Duopoly.

Two firms produce the same goods

Each firm ⟶ high price / ⟶ low price

⇓

wants to earn highest profits

Each firms care about their own profits, so, preferences can be represented by the profits it obtains.

		Firm 02 High	low
Firm 01	High	(1000, 1000)	(-200, 1200)
	low	(1200, -200)	(600, 600)

firm 01 would prefer (low, high) to (High, High) to (low, low) to (High, low).

Bach or Stravinsky / Battle of Sexes (BoS)

Player 01 prefers Bach whereas, player 02 prefers Stravinsky.

		Player 02 Bach	Stravinsky
Player 01	Bach	(2,1)	(0,0)
	Stravinsky	(0,0)	(1,2)

In this game of BoS, both players know that it is better to cooperate than not to cooperate. but they disagree about the best outcome.

Matching Pennies : This game is a conflictual game.

Two people choose, say, person 01 and person 02, to show the head or tail of the coin.

If both persons show the same side, then person 2 gives person 01 \$1. whereas, if person 01 and 02 show different sides, person 01 pays person 02 \$1.

Each person cares about the amount of money she receives and every person prefers more money than less.

Person 02.

		Head	Tail
Person 01	Head	(1, -1)	(-1, 1)
	Tail	(-1, 1)	(1, -1)

This game is 'strictly competitive' because person 01 wants to take the same action as what the other person chooses; but person 02 wants to choose the opposite action.

Stag Hunt Game:

Players: The hunters.

Actions: {Stag, Hare}

Player 02

		Stag	Hare.
Player 01	Stag	(2, 2)	(0, 1)
	Hare	(1, 0)	(1, 1)

Both players choose Stag → highest ranked profile.

For each player,

(Stag, Stag) ≻ (Hare, Stag) ≻

ash Equilibrium:-

In a game, the best action for any given player depends, in general, on the other player's actions. So, when choosing an action, a player must have in mind, the actions, that the other player will choose.

A Nash Equilibrium is an action profile a^* with the property that no player 'i' can do better by choosing an action different from a_i^*, given that every other player j adheres to a_j^*.

A Nash Equilibrium corresponds to a steady state. If, whenever the game is played, the action profile is the same Nash equilibrium a^*, then no player has a reason to choose any action different from her component of a^*; there is no pressure on the action profile to change.

2.7.1 Examples of Nash Equilibrium.

Prisoner's Dilemma

Player 02

Player 01	Quiet	Fink
Quiet	(2,2)	(0,3)
Fink	(3,0)	(1,1)

(Fink, Fink) is the unique Nash equilibrium of Prisoner's dilemma.

The only NASH equilibrium of the Prisoner's Dilemma is when both players choose Fink.
The incentive to free ride eliminates the possibility that the mutually desirable outcome (Quiet, Quiet) occurs

It is optimal for a player to choose Fink regardless the action what her opponent chooses.

Battle of Sexes (BoS)

		Player 02	
		Bach	Stravinsky
Player 01	Batch	(2,1)	(0,0)
	Stravinsky	(0,0)	(1,2)

This game has two NASH equilibria

(Bach, Bach) & (Stravinsky, Stravinsky)

(2,1) & (1,2)

If, in every counter, both players choose (Bach, Bach) then, no player has an incentive to deviate.

Matching Pennies

		Player 02	
		Head	Tail
Player 01	Head	(1, -1)	(-1, 1)
	Tail	(-1, 1)	(1, -1)

No unique NASH equilibria of this game.

Stag Hunt Game

		Player 02	
		Stag	Hunt.
Player 01	Stag	(2,2)	(0,1)
	Hunt	(1,0)	(1,1)

Two Nash Equilibrias exists : (Stag, Stag) (Hunt, Hunt)

Coordination Game:

Consider the game of Battle of Sexes (BoS), but let, both players prefer Bach. It is an example of a coordination game.

Player 02.

		Bach	Stravinsky
Player 01	Bach	(2,2)	(0,0)
	Stravinsky	(0,0)	(1,1)

A Coordinated Game.

Two NASH equilibria: (Bach, Bach) and (Stravinsky, Stravinsky)
If either of the action pair is reached, there is no reason to deviate from it from either of the two players.

Strict and Non Strict Equilibria

A deviation by a player leads to an outcome worse for that player than the equilibrium outcome.
The definition of 'NASH equilibria' requires only that the outcome of a deviation be no better for the deviant than the equilibrium outcome.
Some games have equilibria in which a player is indifferent between her equilibrium action and some other actions, given the other person's actions.

Player 02

		L	M	R
Player 01	T	(1,1)	(1,0)	(0,1)
	B	(1,0)	(0,1)	(1,0)

(Fig. 31.1) This game has a unique NASH equilibria, (T,L)

When player 02 chooses L, as she chose in this equilibrium player 1 is equally happy choosing T or B; if she deviates to 'B' then, she is no worse off than she is in the equilibrium.

Best Response functions :-

In BoS, 'Bach' is the best action for player 1 if player 2 chooses 'Bach'; 'Stravinsky' is the best action for player 1 if player 02 chooses 'Stravinsky'

In BoS, player 1 has a single best action for player 2's each best action.

In game of Fig. 31.1, both 'T' and 'B' are best actions for player 01 if player 02 chooses L; they both yield the payoff of 1; and player 1 has no actions that yields a higher payoff

Set of player i's best actions when the list of other player's actions is a_{-i} by $B_i(a_{-i})$.

In BoS,

$B_1(\text{bach}) = \{\text{bach}\}$ $\quad$ $B_1(\text{stravinsky}) = \{\text{Stravinsky}\}$

$B_1(L) = \{T, B\}$

$$B_i(a_{-i}) = \{a_i \text{ in } A_i;\ u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i})\ \forall\, a_i' \text{ in } A_i\}$$

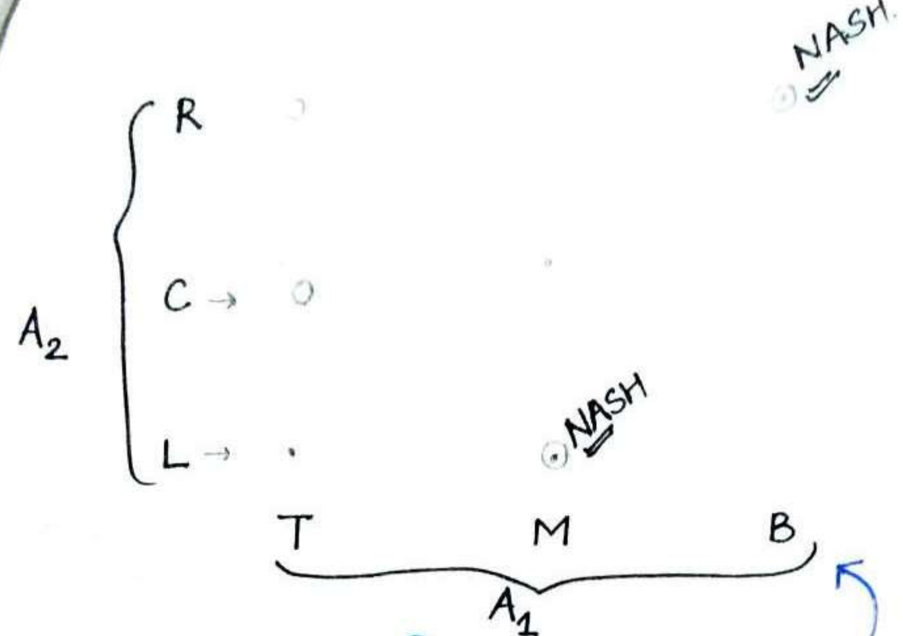
↓ Best Response function of player 'i' to a_{-i}.

Player 2

Player 1	L	C	R
T	(1, 2*)	(2*, 1)	(1*, 0)
M	(2*, 1*)	(0, 1*)	(0, 0)
B	(0, 1)	(0, 0)	(1*, 2*)

(M, L) and (B, R) are two NASH equilibria.

{ Using best response functions to find NASH equilibria in a two player game in which each player has 3 actions }



$BR_2(T) = L \to \cdot$

$BR_2(M) = \{L, C\} \to \cdot$

$BR_2(B) = R \to \cdot$

$BR_1(L) = M \to \circ$

$BR_1(C) = T \to \circ$

$BR_1(R) = \{T, B\} \to \circ$

Player 2

Player 1

	L	C	R
T	1,2	2,1	1,0
M	2,1	0,1	0,0
B	0,1	0,0	1,2

Player 1's best responses are indicated by O, player 2's best response are indicated by ·(dot).

(T,C) marked by a circle just represents that T is player 01's best response to player 02's choice of C.

Circles at (T,R) and (B,R) reflects the fact that T and B are best responses of player 1, given Player 2's choice of R.

Any action profile marked by both a circle and a dot is a NASH equilibria.

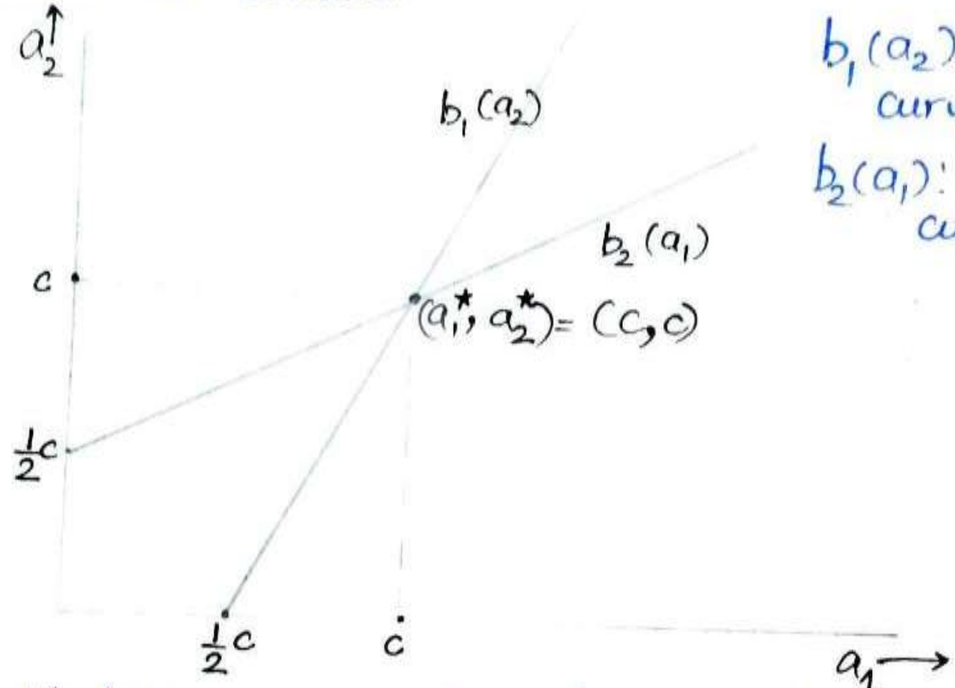
☐ A Synergistic Relationship

Two individuals are involved in a synergistic relationship. If both individuals devote more efforts to the relationship, then both are better off.

For any given efforts of individual j, the returns to individual i's efforts first increases, then decreases.

$$\boxed{b_i(a_j) = \frac{1}{2}(c + a_j)}$$

To find the NASH equilibria of the game, we can construct and analyze the player's best response functions. Given a_j, individual i's payoff is a quadratic function of a_i which is zero when $a_i = 0$ and when $a_i = c + a_j$, and reaches a maximum in between



$b_1(a_2)$: Best response curve for player 1
$b_2(a_1)$: Best response curve for player 2

Player 1's best response function associates an action for player 1 with every possible action for player 2.
Player 2's best response function associates an action for player 2 with every action of player 1.

<u>Best response functions :-</u>

$$\boxed{a_1 = \frac{1}{2}(c + a_2)} \rightarrow ① \qquad \boxed{a_2 = \frac{1}{2}(c + a_1)} \rightarrow ②$$

Eq. ① represents $BR_1(a_2)$ and eq. ② represents $BR_2(a_1)$

Solving the NASH equilibrium via the intersection of both the Best response functions :-

$$a_1 = \frac{1}{2}\left(c + \frac{1}{2}(c + a_1)\right)$$

$$\Rightarrow a_1 = \frac{1}{2}\left(\frac{2c + (c + a_1)}{2}\right) \Rightarrow a_1 = \frac{1}{2 \times 2}(3c + a_1)$$

$$a_1 = \frac{1}{4}(3c + a_1)$$

$$\Rightarrow 4a_1 = 3c + a_1$$

$$\Rightarrow 4a_1 - a_1 = 3c$$

$$\Rightarrow \cancel{3}a_1 = \cancel{3}c \Rightarrow \boxed{a_1^* = c}$$

Substituting $a_1^* = c$ in eq. (2), we have

$$a_2 = \frac{1}{2}(c + c) \quad (\because a_1 = c)$$

$$a_2 = \frac{1}{\cancel{2}} \times \cancel{2}c = c$$

$$\Rightarrow \boxed{a_2^* = c}$$

$$\boxed{a_1^* = a_2^* = c}$$

$\therefore$ Unique NASH equilibria : $(a_1, a_2) = (c, c)$.

2.9.1 Strict Domination.

By the term '<u>strictly dominant strategy</u>', we mean to say that

In any game, a player's action "Strictly dominates" another action if it is superior, no matter what the other players do.

<u>Example</u> :- Prisoner's Dilemma.

We can eliminate 'Quiet' for each player, leaving 'Fink, Fink' as the only outcome for a 'NASH'.

Player 2

Player 1		Quiet	Fink
	Quiet	(2,2)	(0,3)
	Fink	(3,0)	(1,1)

The action profile 'Fink' strictly dominates the action 'Quiet'. A player prefers the action profile when she chooses Fink to the outcome when she chooses 'Quiet'.

Battle of Sexes

		Person 2: Bach	Person 2: Stravinsky
Person 1	Bach	(2,1)	(0,0)
	Stravinsky	(0,0)	(1,2)

Neither of the action dominates the other's actions.

'Strictly Dominant Strategy'

If an action strictly dominates the action a_i, we say that a_i is strictly dominated. Since a player's Nash Equilibrium action is a best response to the other player's Nash Equilibrium actions, a strictly dominated action is not used in any NASH equilibrium.

Player 1 \ Player 2	L	R
T	1	0
M	2	1
B	1	3

Action profile 'M' strictly dominates 'T' but 'B' is better than 'M' if player 2 chooses 'R'. Since 'T' is strictly dominated by M, so the game ~~may~~ has no unique NASH equilibria in which player 1 uses 'T', the game may also not have any NASH equilibria in which player 1 uses 'M'.

Player 1 \ Player 2	L	R
T	1	0
M	2	1
B	3	2

'M' strictly dominates 'T', but 'B' strictly dominates 'M', so, in this → game, in any NASH equilibria, player 1's action is 'B'.

'B' is an action profile that is not strictly dominated.

- **Weak Domination** :- In a strategic game with the ordinal preferences, player i's action a_i'' weakly dominates her action a_i' if

a_i (
and

$u_i(a_i'', a_{-i}) \geq u_i(a_i', a_{-i})$ $\forall$ list a_{-i} of the other player's actions.

and

$u_i(a_i'', a_i) > u_i(a_i', a_i)$ $\forall$ list a_i of the other player's actions.

where u_i is a payoff function that represents player i's preferences.

	L	R
T	1	0
M	2	0
B	2	1

'M' weekly dominates 'T', $(2,0) \succsim (1,0)$

'B' strictly dominates 'T', $(2,1) \succ (1,0)$.

'B' weekly dominates 'M', $(2,1) \succsim (2,0)$

- Two strategic games with a unique NASH equilibrium in which both player's actions are weakly dominated.

	B	C
B	1,1	0,0
C	0,0	0,0

'B' weakly dominates 'C'.

	B	C
B	(1,1)	(2,0)
C	(0,2)	(2,2)

('C, C') is the NASH EQUILIBRIUM.

Another example :-

	L	C	R
T	(0,0)	(1,0)	(1,1)
M	(1,1)	(1,1)	(3,0)
B	(1,1)	(2,1)	(2,2)

Two player symmetric game.

A two player **strategic** game with ordinal pref is symmetric if the player's set of actions are the sa and player's preferences are represented by payoffs functions u_1 and u_2 for which $u_1(a_1, a_2) = u_2(a_2, a_1)$ $\forall$ action pair (a_1, a_2).

A two player symmetric game.

Player 02

Player 1	A	B
A	(ω, ω)	(x, y)
B	(y, x)	(z, z)

In this game, both player 1 and 2 have same set of action

Two symmetric Games

Prisoner's Dilemma

	Quiet	Fink
Quiet	(2, 2)	(0, 3)
Fink	(3, 0)	(1, 1)

Both players have the action profile Quiet or fink. i.e They have the same set of action profile.

Stag Hunt Game

	Stag	Hunt
Stag	(2, 2)	(0, 1)
Hunt	(1, 0)	(1, 1)

Both players have same set of actions i.e Stag or Hunt.

Symmetric Nash Equilibrium : An action profile a^* in a strategic game with ordinal preferences in which each player has the same set of actions is a Symmetric Nash Equilibrium if it is a Nash Equilibrium and a_i^* is the same for every player 'i'.

ample :-

Approaching Pedestrians. → This game (symmetric game) have two Symmetric NASH equilibrium.

		Player 2	
		Left	Right
Player 1	left	(1,1)	(0,0)
	Right	(0,0)	(1,1)

This game has two Symmetric Nash Equilibria i.e (left, left) and (Right, Right). However, we must note that both players are better off when they step in the same direction.

Example :- A symmetric game may have no symmetric Nash Equilibrium.

		Player 2	
		X	Y
Player 1	X	(0,0)	(1,1)
	Y	(1,1)	(0,0)

(1,1) and (1,1) are two Nash equilibrium of this game but (X,Y) (Y,X) neither of them is symmetric.

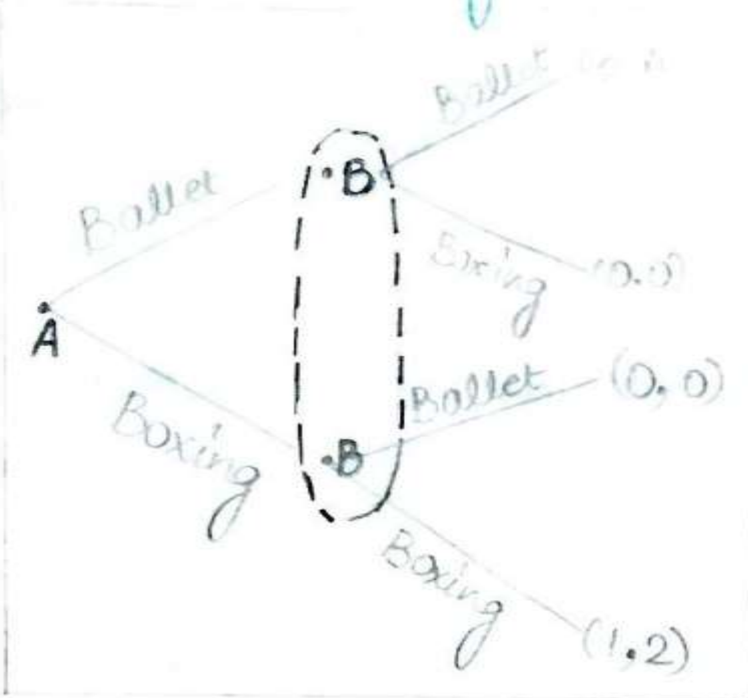
Sub Game Perfect NASH Equilibrium (SPNE)

A 'Subgame' is a part of an extensive form beginning with a decision nod & including everything that branches out to the right of it.

In a simultaneous game, there is only one proper subgame i.e game itself.

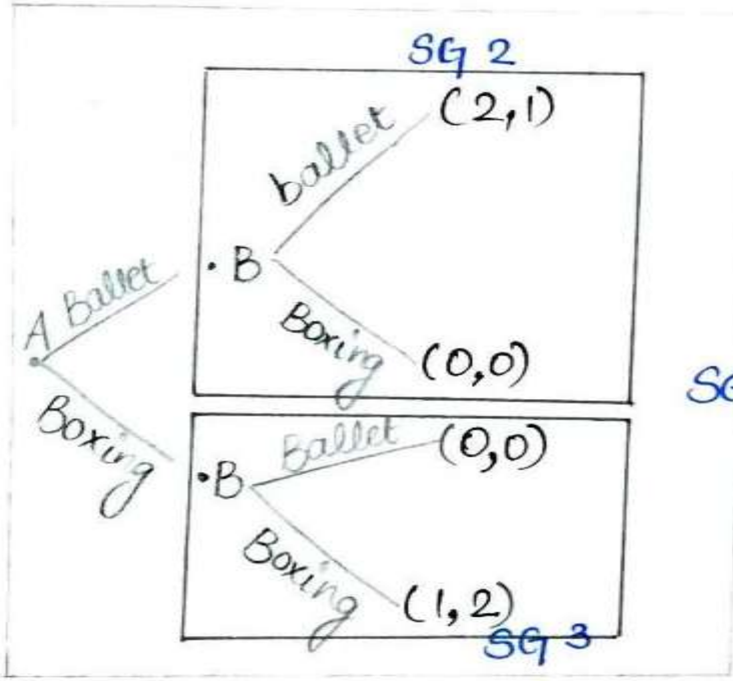
However, in a sequential game, there will 3 proper subgames, i.e game itself & 2 subgames starting with decision nod where player B takes decision.

Proper Subgames :- A proper subgame is a subgame that starts at a decision nod not connected another in an information set.



Simultaneous Game

→ denotes that the action profile that player B would choose is not known to 'A'.

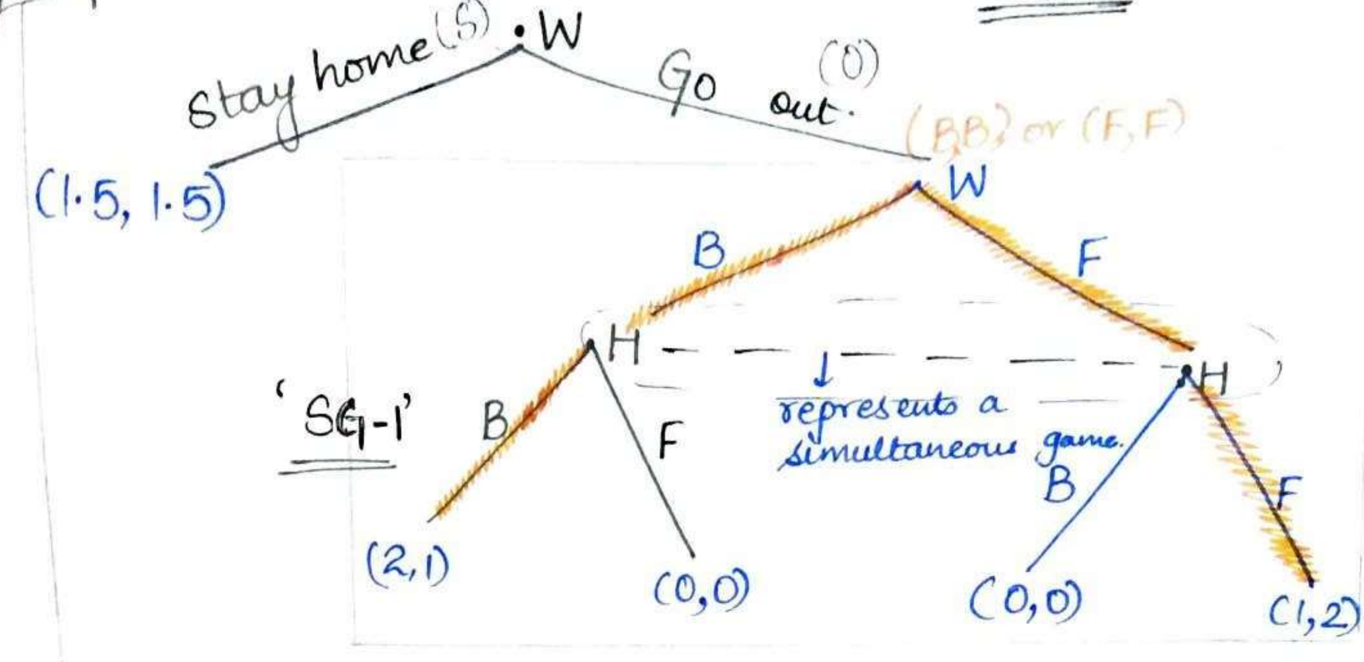


'Sequential Game'

SG 1

In a sequential game, there will be 3 proper subgames i.e 1 the game itself & 2 subgames starting with decision nod where player B takes action.

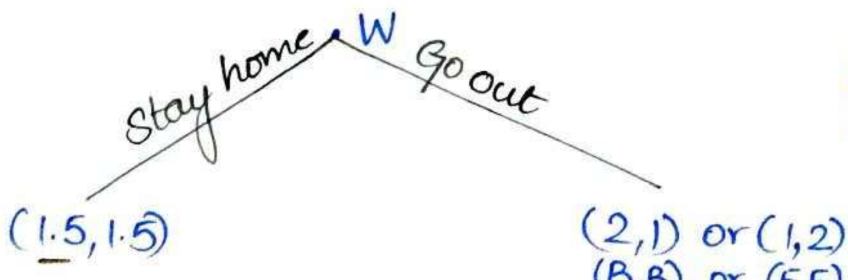
ample :-



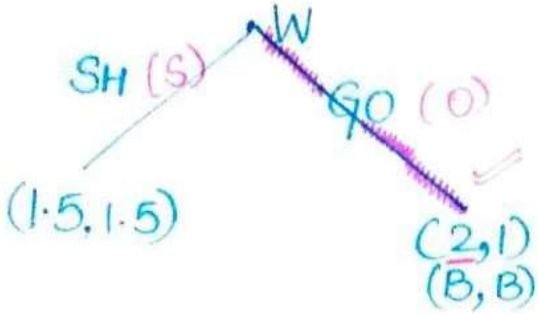
Above game has 2 subgames i·e SG–1 and SG–2
So, we would first solve the lowest & smallest game i·e SG–1. SG–1 is of imperfect information (simultaneous Game)
If we solve this game by making normal form, we get two NASH equilibrias : (B, B) and (F, F).

W \ H	B	F
B	(2,1)	(0,0)
F	(0,0)	(1,2)

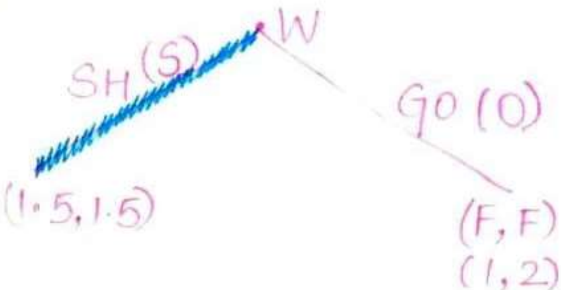
The next step is to solve SG – 2. Now, imagine :



{ This is the game left }



WH W
BB + O
$\Rightarrow \{OB, B\}$
wife strategy, husband strategy



WH W
FF + S
$\Rightarrow \{SF, F\}$

Strategy set

$S_W = \{SB, SF, OB, OF\}$

$S_H = \{B, F\}$.

If Subgame 1 ends at (B,B), then wife prefers to get go out and if SG-1 ends at (F,F), then wife prefers to stay home. Therefore, SPNE is {OB, B} and {SF, F}

$\therefore \boxed{SPNE = \{OB, B\}, \{SF, F\}}$