Problem 1: Bayesian Game – Investment Decision under Uncertain Market Conditions

Problem: Two firms, A and B, decide whether to invest in a new technology. Each firm's profitability depends on the market conditions, which could be either good or bad. Firm A assigns a probability of 70% to good market conditions and 30% to bad conditions, while Firm B assigns a probability of 60% to good market conditions. Both firms must simultaneously decide whether to invest. Payoffs depend on the decision and the state of the market:

- If both invest, payoffs are 100 if the market is good and 10 if it is bad.
- If only one firm invests, the investing firm earns 120 if the market is good and -10 if it is bad, while the other firm earns 50.
- If neither invests, both firms earn 50 regardless of market conditions.

Task: Construct the Bayesian game model and find the Bayesian Nash Equilibrium.

Solution:

1. **Players**: Firm A and Firm B.
2. **Strategies**: Each firm can either **invest (I)** or **not invest (N)**.
3. **Payoffs**:
   - If both invest, the payoffs depend on market conditions: Good market (100, 100), Bad market (10, 10).
   - If one firm invests, the investing firm gets 120 in a good market, -10 in a bad market, and the non-investing firm gets 50.
   - If neither invests, both firms get 50.
4. **Market conditions**:
   - Firm A believes the probability of a good market is 70%.
   - Firm B believes the probability of a good market is 60%.

Bayesian Nash Equilibrium:

- Firm A's expected payoff from **Investing (I)**:

E[Payoff from I]=0.7×100+0.3×10=70+3=73

Firm A's expected payoff from **Not Investing (N)**: 50 (since market conditions don't matter).

Since 73 > 50, Firm A will invest.

- Firm B's expected payoff from **Investing (I)**:

E[Payoff from I]=0.6×100+0.4×10=60+4=64

Firm B's expected payoff from **Not Investing (N)**: 50.

Since 64 > 50, Firm B will also invest.

Thus, the **Bayesian Nash Equilibrium** is for both firms to **Invest**.

Problem 2: Bayesian Auction – Auction for a Painting

Problem: A rare painting is being auctioned, and two bidders, X and Y, value the painting differently. The private value of the painting to bidder X is drawn from a uniform distribution between 100 and 200, and for bidder Y, the value is uniformly distributed between 150 and 250. Each bidder submits a sealed bid. The highest bid wins, and the winner pays the amount of their bid.

Task: What is the optimal bidding strategy for each player under incomplete information? Illustrate how the Bayesian Nash Equilibrium is derived in this case.

Solution:

1. **Players**: Bidders X and Y.
2. **Private values**:
   - X's value $vX \sim U(100,200)$,
   - Y's value $vY \sim U(150,250)$.

Bayesian Nash Equilibrium:

- Each bidder's expected payoff is $vi - bi$ is the value and bi is the bid.
- The optimal bidding strategy for each player in a **first-price sealed-bid auction** (using Myerson's revenue equivalence theorem) is:

For bidder X:

$bX(vX)=Vx/2$

For bidder Y:

$bY(vY)=vY/2$

- This is the symmetric **Bayesian Nash Equilibrium**, where each bidder bids half of their private valuation.

Problem 3: Extensive Game with Imperfect Information – Card Game

Problem: Consider a simple card game with two players. Each player is dealt one card face down from a deck containing two red cards and two black cards. Player 1 can either bet or pass. If Player 1 bets, Player 2 can either fold or call. If Player 2 folds, Player 1 wins. If Player 2 calls, the player with the higher card wins.

Task: Model this game as an extensive game with imperfect information and find the sequential equilibrium of this game.

Solution:

1. **Players**: Player 1 and Player 2.
2. **Information**: Players cannot see each other's cards (imperfect information).
3. **Payoffs**:
   - Player 1 can **bet** or **pass**.
   - Player 2 can **fold** or **call** if Player 1 bets.
   - If Player 2 calls, the higher card wins.

Extensive Form Game:

- Model this game with two decision nodes for Player 1 (one for each possible card they hold) and Player 2's decision contingent on whether Player 1 bets.

Sequential Equilibrium:

- Using backward induction:
  - If Player 1 has a high card, they should bet.
  - If Player 2 sees a bet, they should call if they have a good card and fold otherwise.
  - Player 1 should only bet if they have a high card because Player 2's strategy will lead to a fold otherwise.

Thus, the **sequential equilibrium** is:

- Player 1 bets only with a high card, and Player 2 calls with a high card and folds with a low card.

Problem 4: Repeated Game – The Prisoner's Dilemma

Problem: Consider a repeated Prisoner's Dilemma game where the players interact over 5 rounds. Each round, they choose to either cooperate (C) or defect (D). The payoffs for each round are as follows:

- (C, C): Both players get 3 points.
- (D, D): Both players get 1 point.
- (C, D) or (D, C): The defector gets 5 points, and the cooperator gets 0 points.

Task: Analyze the possible subgame perfect Nash equilibria of this finitely repeated game.

Solution:

1. **Players**: Two players, P1 and P2.
2. **Strategies**: Cooperate (C) or Defect (D) in each round.
3. **Payoffs**:
   - (C, C): (3, 3)
   - (D, D): (1, 1)

- (C, D): (0, 5), (D, C): (5, 0).

Subgame Perfect Nash Equilibrium (SPNE):

- In a finitely repeated game, cooperation is not an equilibrium in the final round because defecting is dominant in a single-stage game.
- By backward induction, defection in the final round causes defection in the earlier rounds.

Thus, the **SPNE** is for both players to **defect in every round**.

Problem 5: Bargaining under Imperfect Information

Problem: Two players, A and B, are bargaining over how to divide a pie of size 100. A has private information about the actual size of the pie, which is either 100 or 50 with equal probability. B knows only the probabilities. A can propose an offer of x_A to keep for themselves, with B either accepting or rejecting the offer. If B rejects, both players get nothing.

Task: What is the optimal strategy for A and B under this imperfect information scenario? Analyze how the beliefs of player B influence the bargaining outcome and derive the sequential equilibrium.

Solution:

1. **Players**: A (informed) and B (uninformed).
2. **Private information**: A knows the size of the pie (either 50 or 100).
3. **Strategies**: A proposes a division, B accepts or rejects.

Beliefs:

- Player B believes the pie is large with 50% probability.

Sequential Equilibrium:

- A (when the pie is 100) should propose a division where B is indifferent between accepting and rejecting based on the expected size of the pie.
- If A proposes x_A leaves x_B, B will accept since their expected payoff $0.5 \times 50 + 0.5 \times 25 = 37.$, which is better than rejecting (getting 0).

Thus, the **sequential equilibrium** is that A offers 50 and keeps the rest, and B accepts.

Problem 6: Nash Equilibrium in Imperfect Information – Guessing Game

Problem: Consider a variation of the guessing game where two players choose a number between 1 and 100. The winner is the player whose number is closest to half of the average of the two numbers chosen. Each player has incomplete information about the other player's decision process.

Task: Model this as a game of imperfect information and determine the Nash equilibrium strategy for both players.

Solution:

1. **Players**: Two players.
2. **Objective**: Choose a number closest to half the average of the two numbers.
3. **Payoff**: Winner gets a fixed payoff.

Nash Equilibrium:

- The Nash equilibrium for this game, similar to the famous "beauty contest" game, converges on both players choosing the number 0.
- The logic is that if both players aim to pick half the average, the only consistent solution where both have no incentive to deviate is 0.

Thus, the **Nash equilibrium** is both players choosing 0.