CNF (Conjunctive Normal Form)

A formula which is equivalent to a given formula and which consists of a product of elementary sums is called a conjunctive normal form of given formula.

$(P\sim \vee Q) \wedge (Q \vee R) \wedge (\sim P \vee Q \vee \sim R)$

Steps to convert formula into CNF

- eliminate all the occurrences of $\oplus$ (XOR operator), $\longrightarrow$ (conditional), and $\leftrightarrow$ (biconditional)

    - $A \oplus B \equiv (A \vee B) \wedge \neg(A \wedge B)$
    - $A \rightarrow B \equiv \neg A \vee B$
    - $A \leftrightarrow B \equiv (\neg A \vee B) \wedge (A \vee \neg B)$
    - $A \leftrightarrow B \equiv (A \wedge B) \vee (\neg A \wedge \neg B)$

- move all the negations inwards(De Morgan's law)

Steps to convert formula into CNF

Some common equivalences that we use for the conversion are:

- **Commutativity for disjunction**: $A \vee B \equiv B \vee A$
- **Commutativity for conjunction**: $A \wedge B \equiv B \wedge A$
- **Associativity for disjunction**: $(A \vee B) \vee C \equiv A \vee (B \vee C)$
- **Associativity for conjunction**: $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$
- **Distribution over disjunction**: $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$
- **Distribution over conjunction**: $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$

Laws of Logical Equivalence

Identity Laws

1. $p \wedge T \equiv p$
2. $p \vee F \equiv p$

Domination Laws

3. $p \vee T \equiv T$
4. $p \wedge F \equiv F$

Idempotent Laws

5. $p \wedge p \equiv p$
6. $p \vee p \equiv p$

Double Negation Law

7. $\neg(\neg p) \equiv p$

Negation Laws

8. $p \vee \neg p \equiv T$
9. $p \wedge \neg p \equiv F$

Commutative Laws

10. $p \wedge q \equiv q \wedge p$
11. $p \vee q \equiv q \vee p$

Associative Laws

12. $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
13. $(p \vee q) \vee r \equiv p \vee (q \vee r)$

Distributive Laws

14. $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
15. $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

De Morgan's Laws

16. $\neg(p \wedge q) \equiv \neg p \vee \neg q$
17. $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Other operations

18. $p \rightarrow q \equiv \neg p \vee q$
19. $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
20. $p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$

Example 1

$$\neg((\neg A \rightarrow \neg B) \wedge \neg C)$$

$$\equiv \neg((\neg\neg A \vee \neg B) \wedge \neg C)$$

$$\equiv \neg((A \vee \neg B) \wedge \neg C)$$

$$\equiv \neg(A \vee \neg B) \vee \neg\neg C$$

$$\equiv \neg(A \vee \neg B) \vee C$$

$$\equiv (\neg A \wedge \neg\neg B) \vee C$$

$$\equiv (\neg A \wedge B) \vee C$$

$$\equiv (\neg A \vee C) \wedge (B \vee C)$$

Example 2

$$P \to \neg(R \vee \neg Q).$$

Step 1 produces: $\neg P \vee \neg(R \vee \neg Q)$.

Step 2 produces: $\neg P \vee (\neg R \wedge Q)$.

Step 3 produces: $(\neg P \vee \neg R) \wedge (\neg P \vee Q)$.

Examples

1. $(P \wedge Q) \to (P \to Q)$
2. $(P \wedge Q) \leftrightarrow (P \to Q)$
3. $(\neg P \vee Q) \to (P \to \neg Q)$
4. $(((P \to Q) \to P) \to Q)$
5. $(P \to (Q \to (P \to Q)))$
6. $((P \wedge \neg Q) \to \neg R) \leftrightarrow ((P \wedge R) \to Q)$
7. $(((P \vee Q) \vee R) \vee S) \leftrightarrow (P \vee (Q \vee (R \vee S)))$
8. $(((P \to Q) \to R) \to S) \leftrightarrow (P \to (Q \to (R \to S)))$
9. $(P \to (\neg R \to \neg S)) \vee ((S \to (P \vee \neg T)) \vee (\neg Q \to R))$