# Matrices

## Eigen values & Eigen vectors

### Basic definitions

**Matrix**

A matrix is a system of $mn$ numbers arranged in $m$ rows & $n$ columns. It is called an $m \times n$ matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & & \cdots & a_{mn} \end{bmatrix}$$

**Row & Column matrix**

A matrix having only one row is called a row matrix & a matrix having only one column is called a column matrix.

Ex:- $\begin{bmatrix} 3 & 2 \end{bmatrix} \rightarrow$ row matrix $\quad \begin{bmatrix} 3 \\ 2 \end{bmatrix} \rightarrow$ column matrix

**Square matrix :-**

If the number of rows of a matrix is equal to the number of columns then the matrix is called a square matrix.

Diagonal matrix:-

A square matrix whose all non-diagonal elements are zero is called a diagonal matrix

Ex:-

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow \begin{array}{c} \text{diagonal matrix} \\ \text{square matrix} \end{array}$$

Trace of a matrix

The sum of all diagonal elements of a square matrix is called the trace of a matrix.

Singular matrix

A ~~ma~~ square matrix whose determinant is zero is called a singular matrix.

Non-singular matrix

A square matrix whose determinant is not zero is called a non-singular matrix.

Unit Matrix
A diagonal matrix whose all diagonal elements are equal to one is called a unit matrix.

Ex:- $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Transpose of a matrix
A matrix obtained from a given matrix A by interchanging rows & columns is called the transpose of a given matrix & is denoted by $A'$ or $A^T$. i.e) $a_{ij} = a_{ji}$

Ex:- $A = \begin{bmatrix} a & b & c \\ d & e & f \\ x & y & z \end{bmatrix}$ $A'$ (or) $A^T = \begin{bmatrix} a & d & x \\ b & e & y \\ c & f & z \end{bmatrix}$

Symmetric & skew-symmetric matrix
A square matrix A is said to be symmetric if $A = A^T$

A square matrix A is said to be skew-symmetric if $A = -A^T$. i.e) $a_{ij} = -a_{ji}$

**Hermitian matrix**

A square matrix $A = (a_{ij})$ is said to be Hermitian if $a_{ij} = \bar{a}_{ji} \ \forall \ i, j$.

**Ex:-**

$$A = \begin{pmatrix} 1 & 1+2i \\ 1-2i & 2 \end{pmatrix}$$

**Skew-Hermitian matrix**

A square matrix $A = (a_{ij})$ is said to be Skew-Hermitian if $a_{ij} = -\bar{a}_{ji} \ \forall \ i, j$.

**Orthogonal matrices**

A real square matrix A is called orthogonal if $AA' = A'A = I$

**Unitary Matrix**

A square matrix A is said to be unitary if the product of A & its transpose of conjugate complex is $\bar{A}' = A^{\theta}$ is a unit matrix.

$$A^{\theta}A = AA^{\theta} = I$$

Eigen values

Let $A$ be any square matrix, $\lambda$ a scalar & $I$ the unit matrix of the same order. Then $A-\lambda I$ is called the characteristic matrix.

$|A-\lambda I|$ is called the characteristic polynomial.

$|A-\lambda I|=0$ is called the characteristic equation of the matrix $A$.

The roots of the characteristic equation are called the characteristic roots or latent roots or characteristic values or eigen values of $A$.

Ex:- $A=\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}$

$$A-\lambda I=\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}=\begin{bmatrix}1-\lambda & 2\\ 4 & 3-\lambda\end{bmatrix}$$

$|A-\lambda I|=0$ is the characteristic equation of A.

If $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

$\therefore$ the characteristic equation is

$$\lambda^2 - (\text{trace of } A)\lambda + |A| = 0$$

$$\lambda^2 - (1+3)\lambda + (3-8) = 0$$

$$\lambda^2 - 4\lambda - 5 = 0.$$

The roots of characteristic equation are eigen values $\lambda^2 - 4\lambda - 5 = 0$

$\Rightarrow \lambda = -1, 5$ are the eigen values.

(2) Find the eigen values of

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

For 3x3 matrix

ex: $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

The characteristic equation is

$$\lambda^3 - (\text{trace of } A)\lambda^2 + ((ae+ei+ai) - (db+gc+hf))\lambda - |A| = 0.$$

If $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

The characteristic equation is

$$\lambda^3 - [2+2+2]\lambda^2 + [(2\times2 + 2\times2 + 2\times2) - ((1\times-1)+(1\times1)+(-1\times-1))]\lambda - [2(4-1)+1(2+1)+1(-1-2)] = 0$$

$$\lambda^3 - 6\lambda^2 + [12-1]\lambda - [6] = 0$$

$$\therefore \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

The roots of the characteristic equation are called eigen values

$\lambda = 1, 2, 3$ are the eigen values of A.