

RESIDUES ::

* Zero of an Analytic function:-

If an analytic function $f(z)$ ~~is~~ $= 0$ at point $z = z_0$ then z_0 is called as zero of an analytic function, $f(z)$

~~$f(z)$~~

If $f(z_0) = 0$ but $f'(z_0) \neq 0$ then z_0 is simple zero or zero of order 1.

In general, if

$$f(z_0) = 0 = f'(z_0) = f''(z_0) = \dots = f^{(n-1)}(z_0)$$

$$\text{but } f^{(n)}(z_0) \neq 0$$

then z_0 is zero of order " n ".

Q. ① Find zero of a function & also find its order.

1) $f(z) = (z-1)e^z$



$$f(z) = 0$$

$$(z-1)e^z = 0$$

$$\text{as } e^z \neq 0$$

$$z-1 = 0$$

$$\Rightarrow z = 1$$

$z=1$ is zero of function.

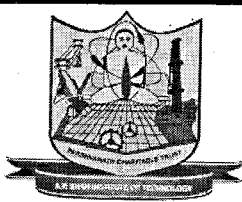
$$f(z) = (z-1)e^z$$

$$f'(z) = (z-1)e^z + e^z$$

$$f'(1) = (1-1)e^1 + e^1$$

$$f'(1) = e \neq 0$$

$\therefore z=1$ is ~~an~~ zero of order 1.



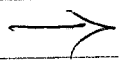
$$f'(1) = e.$$

$$(z-3)^4 = 0$$

$$z-3 = \sqrt[4]{0}$$

$$(2) \quad f(z) = (z-3)^4$$

$$z^2 = 1 \quad z = \pm 1$$



$$f(z) = 0$$

$$(z-3)^4 = 0$$

$$z = 3$$

$$z-3 = 0$$

$$z = 3$$

$z = 3$ is zero of function.

$$f(z) = (z-3)^4$$

$$f'(z) = 4(z-3)^3$$

$$f'(3) = 4(3-3)^3 = 0$$

$$f''(z) = 12(z-3)^2$$

$$f''(3) = 12(3-3)^2 = 0$$

$$f'''(z) = 24(z-3) = 0$$

$$f'''(3) = 24(3-3) = 0$$

$$f^{(4)}(3) = 24 \neq 0$$

$\therefore z = 3$ is zero of order 4

$$(3) \quad f(z) = \frac{(z+2)^2}{(z+3)^3}$$

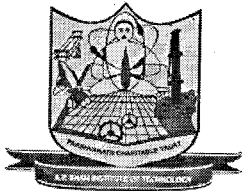


$$(z+2)^2 = 0$$

$$f(z) = 0$$

$z = -2$ is zero of function.

$\therefore z = -2$ is zero of order 2.



④ $f(z) = (z-1)^3 (z+4)^2$



$z = 1$ is a zero of $f(z)$ function.

$z = 1$ is a zero of order 3

$z = -4$ is a zero of $f(z)$ function.

$z = -4$ is a zero of order 2.

Definitions:-

① Singular Points (Problematic Points):-

~~The~~ If $f(z)$ is not analytic at a point z_0 then z_0 is called as "singular point" or "singularity of $f(z)$ "

eg.

$$f(z) = \frac{1}{z-1}$$

$z = 1$ is singularity of $f(z)$.

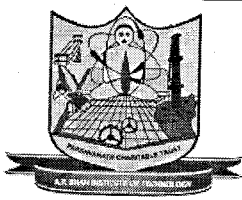
★ Types of Singularities:-

Type: I Isolated singularity.

If there is no other singularity inside the neighbourhood (nbd) of z_0 other than z_0 then z_0 is called as "isolated singularity".

eg' $f(z) = \frac{1}{(z-1)(z-2)}$

$z = 1$ & $z = 2$ are isolated singularity.



Type II: Non isolated singularity

If there exist ~~at~~ least one singularity
If every nbd of z_0 consist at
least one ~~at~~ more singular pt other than
 z_0 then it is called as "non-isolated
singularity".

Type III: Pole

Let z_0 is singularity of $f(z)$ then
Laurent's series expansion of $f(z)$ is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

If in Laurent's series expansion of $f(z)$
finite terms of ~~the~~ negative power of
 $(z-z_0)$ are present i.e. if in the above
series only M negative powers are present
i.e. $b_{m+1} = b_{m+2} = \dots = 0$
then $z=z_0$ is called as ~~whole~~ ^{pole} of order
"~~the~~" m

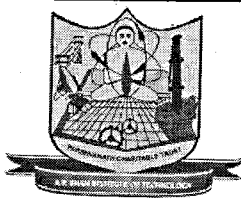
Pole of order 1 is also called "simple
pole"

$$\text{eg. } f(z) = \frac{e^{3z}}{(z-1)^3}$$

①

$z=1$ is singularity.
To find Laurent's series at $z=1$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$



$$f(z) = \frac{e^{3z}}{(z-1)^3}$$

$$= \frac{e^{3z-3+3}}{(z-1)^3}$$

$$= \frac{e^{3(z-1)} e^3}{(z-1)^3}$$

$$= \frac{e^3}{(z-1)^3} \left[1 + 3(z-1) + \frac{3^2(z-1)^2}{2!} + \frac{3^3(z-1)^3}{3!} + \dots \right]$$

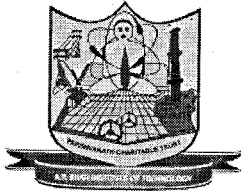
$$= e^3 \left[\frac{1}{(z-1)^3} + \frac{3}{(z-1)^2} + \frac{3^2}{(z-1)2!} + \frac{3^3}{3!} + \dots \right]$$

$z=1$ is pole of order 3.

② $f(z) = \frac{1}{(z-2)^3(z-1)}$

→ $z=2$ is pole of order 3

$z=1$ is pole of order 1.



* Isolated Essential Singularity:-

If the Laurent's series expansion of $f(z)$ contains infinitely many negative powers of $(z - z_0)$ then z_0 is called as "isolated essential singularity".

eg. $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{z^2 2!} + \frac{1}{z^3 3!} + \dots$

* Non-isolated Essential Singularity:-

If we have sequence of poles of $f(z)$ $z_1, z_2, z_3, \dots, z_n$ such that z_0 is limit point of this poles (as $n \rightarrow \infty, z_n \rightarrow z_0$).
(Here $\lim_{n \rightarrow \infty} z_n = z_0$) then z_0 is called as non-

isolated essential singularity.

eg. $f(z) = \frac{1}{\sin(\frac{1}{z})}$

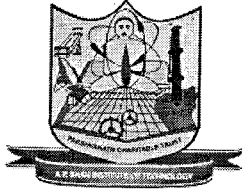
we will get singularities when,

$$\sin\left(\frac{1}{z}\right) = 0$$

$$\frac{1}{z} = n\pi$$

$$z = \frac{1}{n\pi}, \quad n = 0, \pm 1, \pm 2, \dots$$

$$\lim_{n \rightarrow \infty} z = \lim_{n \rightarrow \infty} \frac{1}{n\pi} = 0.$$



* removable singularity:-

If Laurent's series expansion of $f(z)$ does not contain negative power of $(z - z_0)$ then z_0 is called a "removable singularity."

eg. $f(z) = \frac{\sin z}{z}$

$z = 0$ is singularity.

$$= \frac{1}{z} \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right)$$

$$= \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots \right)$$

$z = 0$ is removable singularity.

Q. ① Find singularities & explain its types.

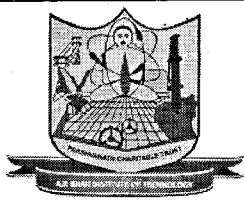
↳ $f(z) = \frac{\sin z}{z^3}$

→ $z = 0$ is singularity.

$$= \frac{1}{z^3} \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right]$$

$$= \left[\frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \dots \right]$$

$z = 0$ is pole order of 2.



$$(2) f(z) = \frac{\cos \pi z}{(z-a)^3}$$

→

$$= \frac{\sin \pi z}{(z-a)^3}$$

$$= \frac{\cos \pi z}{\sin \pi z (z-a)^3}$$

we'll get singularities

$$\sin \pi z (z-a)^3 = 0$$

$$\sin \pi z = 0$$

$$(z-a)^3 = 0$$

$$\pi z = n\pi$$

$$z = a$$

$$n = 0, \pm 1, \pm 2, \dots$$

$$z = n$$

∴ Singularities of $f(z)$ are

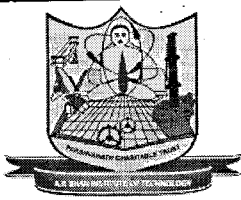
$$z = n, n = 0, \pm 1, \pm 2, \dots$$

are pole of order 1

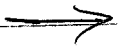
$z = a$ is a pole of order 3.

$$z = n, n = 0, \pm 1, \pm 2, \dots$$

are pole of order 1. (isolated ~~essential~~ singularity).



$$(2) f(z) = \frac{\cos \pi z}{(z-a)^3}$$



$$= \frac{\sin \pi z}{(z-a)^3}$$

$$= \frac{\cos \pi z}{\sin \pi z (z-a)^3}$$

we'll get singularities

$$\sin \pi z (z-a)^3 = 0$$

$$\sin \pi z = 0$$

$$(z-a)^3 = 0$$

$$\pi z = n\pi$$

$$z = a$$

$$n = 0, \pm 1, \pm 2, \dots$$

$$z = n$$

\therefore Singularities of $f(z)$ are

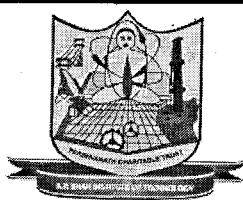
$$z = n, n = 0, \pm 1, \pm 2, \dots$$

are pole of order 1

$z = a$ is a pole of order 3.

$$z = n, n = 0, \pm 1, \pm 2, \dots$$

are pole of order 1. (isolated ~~essential~~ singularity).



Residues:

The coefficient of $(z-z_0)^{-1}$ or $\frac{1}{(z-z_0)}$ in the L.S. Expansion of $f(z)$ at $z=z_0$ is called as "residues" of $f(z)$ at $z=z_0$.

Procedure to find out Residues:-

case i) If $z=z_0$ is pole of order 1 i.e. simple pole then,

$$\text{Residue of } f(z) \text{ at } z=z_0 = \lim_{z \rightarrow z_0} [(z-z_0) f(z)]$$

case ii) If $z=z_0$ is a pole of order n then,

$$\text{Residue of } f(z) \text{ at } z=z_0 = \lim_{z \rightarrow z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} [(z-z_0)^n f(z)]$$

Q. ① Determine the poles & find residues at each pole.

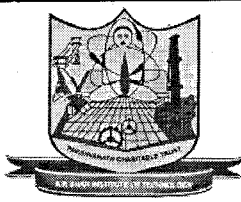
$$1) f(z) = \frac{z^2}{(z+2)(z-1)^2}$$

→

$z = -2$ is pole of order 1

+

$z = 1$ is pole of order 2.



$$z = -2$$

$$\text{Resi of } f(z) \text{ at } z = -2 = \lim_{z \rightarrow -2} \left[(z+2) f(z) \right]$$

$$= \lim_{z \rightarrow -2} \left[(z+2) \cdot \frac{z^2}{(z+2)(z-1)^2} \right]$$

$$= \lim_{z \rightarrow -2} \left[\frac{z^2}{(z-1)^2} \right]$$

$$= \lim_{z \rightarrow -2} \left[\frac{(-2)^2}{(-2-1)^2} \right]$$

$$= \frac{4}{9}$$

$$\text{Residue of } f(z) \text{ at } z = 1$$

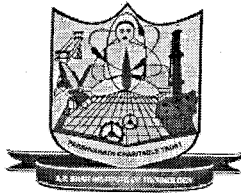
$$= \lim_{z \rightarrow 1} \frac{d}{dz} \left[(z-1)^2 \cdot \frac{z^2}{(z+2)(z-1)^2} \right]$$

$$= \lim_{z \rightarrow 1} \frac{d}{dz} \left[\frac{z^2}{(z+2)} \right]$$

$$= \lim_{z \rightarrow 1} \frac{(z+2)(2z) - z^2(1)}{(z+2)^2}$$

$$= \frac{(1+2)(2(1)) - (1)^2(1)}{(1+2)^2}$$

$$= \frac{(3)(2) - (1)}{9} = \frac{5}{9}$$



$$(z-1)^3$$

→ $z=1$ is a pole of order 3.

$$z=1$$

Residue of $f(z)$ at $z=1$

$$= \lim_{z \rightarrow 1} \frac{1}{(3-1)!} \frac{d^2}{dz^2} \left[(z-1)^3 \cdot \frac{e^z}{(z-1)^3} \right]$$

$$= \lim_{z \rightarrow 1} \frac{1}{2!} \frac{d^2}{dz^2} [e^z]$$

$$= \lim_{z \rightarrow 1} \frac{1}{2} \frac{d^2}{dz^2} e^z$$

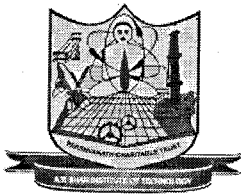
$$= \lim_{z \rightarrow 1} \frac{1}{2} e^z$$

$$= \frac{1}{2} e$$

$$= \frac{e}{2}$$

③ ~~$f(z) = \frac{1}{z^3+z^5}$~~

→ $f(z) = \frac{1}{z^3(1+z^2)}$



$$z^3(1+z^2) = 0$$

$$z^3 = 0, \quad (1+z^2) = 0$$

$$z = 0$$

$$z^2 = -1$$

$$z = \pm \sqrt{-1}$$

$$z = \pm i$$

$z = 0$ is a pole of order 3.

$\& z = \pm i$ is a pole of order 1.

~~$z = 0$~~

~~Residues of $f(z)$ at $z = 0$~~

~~$$= \lim_{z \rightarrow 0} \frac{1}{(3-1)!} \frac{d^2}{dz^2} \left[\frac{(-z)^3 \cdot 1}{z^3(1+z^2)} \right]$$~~

$$f(z) = \frac{1}{z^3 + z^5} = \frac{1}{z^3(1+z^2)}$$

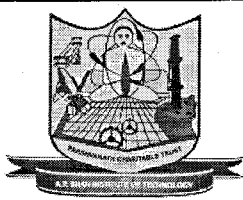
$$= \frac{1}{z^3(z-i)(z+i)}$$

$$z = 0$$

Residues of $f(z)$ at $z = 0$

$$= \lim_{z \rightarrow 0} \frac{1}{(3-1)!} \frac{d^2}{dz^2} \left[\frac{z^3 \cdot 1}{z^3(1+z^2)} \right]$$

$$= \lim_{z \rightarrow 0} \frac{1}{2} \frac{d^2}{dz^2} \frac{1}{(1+z^2)}$$



$$= \lim_{z \rightarrow 0} \frac{1}{2} \frac{d}{dz} \left[-1 (1+z^2)^{-2} (2z) \right]$$

$$= \lim_{z \rightarrow 0} \frac{1}{2} (-2) \frac{d}{dz} \left[(1+z^2)^{-2} (z) \right]$$

$$= \lim_{z \rightarrow 0} (-1) \left[(1+z^2)^{-2} (1) + (z) (-2) (1+z^2)^{-3} (2z) \right]$$

$$= \lim_{z \rightarrow 0} \frac{1}{2} \frac{d^2}{dz^2} \left(\frac{1}{1+z^2} \right)$$

$$= \lim_{z \rightarrow 0} \frac{1}{2} \frac{d}{dz} \left(\frac{-1}{1+z^2} (2z) \right)$$

$$= \lim_{z \rightarrow 0} \frac{1}{2} \frac{d}{dz} \left(\frac{-2z}{1+z^2} \right)$$

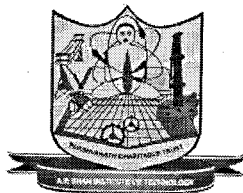
$$= - \lim_{z \rightarrow 0} \frac{1}{2} \frac{d}{dz} \left(\frac{+2z}{(1+z^2)} \right)$$

$$= - \lim_{z \rightarrow 0} \frac{1}{2} \left[\frac{(1+z^2)(2) - (2z)(0+2z)}{(1+z^2)^2} \right]$$

$$= - \frac{1}{2} \left[\frac{(1+0)(2) - (0)}{(1+0)^2} \right]$$

$$= - \frac{1}{2} \left[\frac{2}{1} \right]$$

$$= -1 //$$



$z = i$ is a pole of order 1.

Residues of $f(z)$ at $z = i$

$$= \lim_{z \rightarrow i} (z - i) \cdot \frac{1}{z^3(z-i)(z+i)}$$

$$= \lim_{z \rightarrow i} \frac{1}{z^3(z+i)}$$

$$= \frac{1}{i^3(i+i)}$$

$$= \frac{1}{i^3(2i)}$$

$$= \frac{1}{2}$$

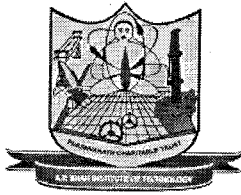
$z = -i$ is a pole of order 1

Residues of $f(z)$ at $z = -i$

$$= \lim_{z \rightarrow -i} (z + i) \cdot \frac{1}{z^3(z-i)(z+i)}$$

$$= \lim_{z \rightarrow -i} \frac{1}{z^3(z-i)}$$

$$= \frac{1}{(-i)^3(-i-i)} = \frac{1}{(-i)^3(-2i)}$$



$$= \frac{1}{(-i)^3 (-2i)}$$

$$= \frac{1}{(\cancel{-i})^3 (-2i)}$$

$$= \frac{1}{(\cancel{-i})(-2i)} = \frac{1}{(\cancel{-i})(-2i)} = \frac{1}{2i^2}$$

$$= \frac{1}{2i^2} = \frac{1}{2i^2}$$

$$= \frac{1}{2} \cdot \frac{1}{i^2}$$

$$= \frac{1}{(-1)(i^3)(-2i)}$$

$$= \frac{1}{(+1)(2i^4)}$$

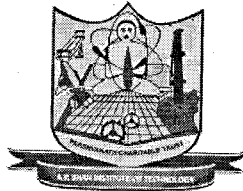
$$= \frac{1}{2}$$

(4) $f(z) = \frac{\sin^2 z}{z^3}$

→ $z=0$ is ~~poles~~ singularity.

$$f(z) = \frac{(\sin z)^2}{z^3}$$

$$= \frac{1}{z^3} \left[\frac{1}{z} - \frac{1}{z^3} \right]$$



$$f(z) = \frac{\sin^2 z}{z^3}$$

$$= \frac{1}{z^3} \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right]^2$$

$$= \frac{1}{z^3} \left[z^2 - \frac{2z^6}{(3!)^2} + z^3 \right]$$

$z=0$ is a pole of order "3"

$$\text{Resid of } f(z) \text{ at } z=0 = \lim_{z \rightarrow 0} \frac{1}{2!} \frac{d^2}{dz^2} \left[\frac{z^3 \sin^2 z}{z^3} \right]$$

$$= \lim_{z \rightarrow 0} \frac{1}{2} \frac{d^2}{dz^2} [\sin^2 z]$$

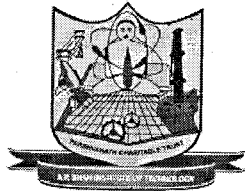
$$= \lim_{z \rightarrow 0} \frac{1}{2} \frac{d}{dz} [2 \sin z \cos z]$$

$$= \lim_{z \rightarrow 0} \frac{1}{2} \frac{d}{dz} [\sin 2z]$$

$$= \lim_{z \rightarrow 0} \frac{1}{2} [2 \cos 2z]$$

$$= \cos 2(0)$$

$$= 1 //$$



$$f(z) = \frac{\sin^4 z}{z^3}$$

$$= \frac{(\sin z)^2}{z^3}$$

$$= \frac{1}{z^3} \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right)^2$$

$$= \frac{1}{z^3} \left[z^2 - \frac{2zz^3}{3!} + \frac{(-3z^5)^2}{(3!)^2} - \dots \right]$$

$$= \left(\frac{1}{z} - \frac{2z}{3!} + \frac{z^3}{(3!)^2} - \dots \right)$$

$\therefore z=0$ is pole of order 1

\therefore Residue of $f(z)$ at $z=0$

$$= \text{coefficient of } \frac{1}{(z-0)}$$

$$= 1 //$$

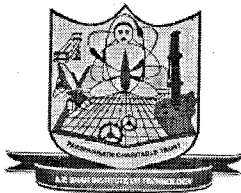
⑤ $f(z) = z^2 \sin\left(\frac{1}{z}\right)$

$\rightarrow z=0$ is singularity

$$f(z) = z^2 \left[\frac{1}{z} - \frac{1}{z^3 3!} + \frac{1}{z^5 5!} - \dots \right]$$

$$= \left[z - \frac{1}{z 3!} + \frac{1}{z^3 5!} - \dots \right]$$

$\therefore (z=0 \text{ is isolated essential singularity})$



∴ Residue of $f(z)$ at $z=0$

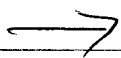
$$= \text{coefficient of } \left(\frac{1}{z-0} \right)$$

$$= \frac{-1}{36} = -\frac{1}{36} //$$

V. Imp 6) Find sum of residues of $f(z)$

$$f(z) = \frac{\tan z}{z} \text{ of its poles}$$

inside $|z|=2$



$$f(z) = \frac{\sin z}{z \cdot \cos z} \quad |z|=2$$

we'll get singularities,

$$z \cos z = 0$$

$$z=0 \quad \text{or} \quad \cos z = 0$$

$$z = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$$

$$z = \pm (2n+1) \frac{\pi}{2}$$

$z = 0, \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots$ are poles
of order 1.

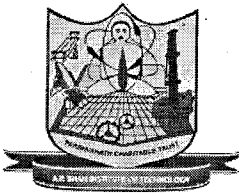
Given $|z|=2$

$$z=0 \quad |z|=|0|=0 < 2$$

$z=0$ is inside $|z|=2$

$$z = \pm \frac{\pi}{2} \quad |z| = \left| \pm \frac{\pi}{2} \right| = \frac{\pi}{2} < 2$$

$z = \pm \frac{\pi}{2}$ are inside $|z|=2$.



$$z = \pm \frac{3\pi}{2} \quad \left| \pm \frac{3\pi}{2} \right| = \frac{3\pi}{2} > 2$$

$$\therefore z = \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2} \dots \text{etc. all}$$

outside $|z| = 2$.

Residue at $f(z)$ at $z=0$

$$\phi = \lim_{z \rightarrow 0} (z) f(z)$$

$$= \lim_{z \rightarrow 0} z \cdot \frac{\sin z}{z \cos z}$$

$$= \lim_{z \rightarrow 0} \frac{\sin z}{\cos z}$$

$$= 0$$

Residue at $f(z)$ at $z = \frac{\pi}{2}$

$$= \lim_{z \rightarrow \frac{\pi}{2}} z \cdot \frac{\sin z}{z \cos z}$$

$$= \frac{\sin \frac{\pi}{2}}{\cos \frac{\pi}{2}} \lim_{z \rightarrow \frac{\pi}{2}} (z - \frac{\pi}{2}) \frac{\sin z}{z \cos z}$$

$$= \lim_{z \rightarrow \frac{\pi}{2}} (z - \frac{\pi}{2}) \cdot \frac{\sin z}{z \cos z}$$

By L-Hospital rule,

$$= \frac{0}{0}$$



$$= \lim_{z \rightarrow \frac{\pi}{2}} \left[\frac{(z - \frac{\pi}{2}) \sin z}{z \cos z} \right]$$

$$= \lim_{z \rightarrow \frac{\pi}{2}} \left[\frac{(z - \frac{\pi}{2}) \sin \frac{\pi}{2}}{\frac{\pi}{2} \cos z} \right]$$

$$= \frac{2}{\pi} \lim_{z \rightarrow \frac{\pi}{2}} \left[\frac{(z - \frac{\pi}{2})}{\cos z} \right]$$

$$= \frac{2}{\pi} \lim_{z \rightarrow \frac{\pi}{2}} \left[\frac{1}{-\sin z} \right]$$

$$= \frac{2}{\pi} \left[\frac{-1}{\sin \frac{\pi}{2}} \right]$$

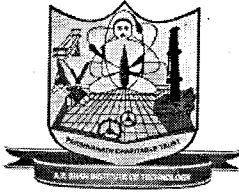
$$= \frac{2}{\pi} \left[\frac{-1}{1} \right]$$

$$= -\frac{2}{\pi}$$

$$\cancel{z = \pi} \quad z = +\frac{\pi}{2}$$

$$= \lim_{z \rightarrow \frac{\pi}{2}} \frac{(z + \frac{\pi}{2}) \sin z}{z \cos z}$$

$$= \left(\frac{-2}{\pi} \right) \lim_{z \rightarrow \frac{\pi}{2}} \left[\frac{z + \frac{\pi}{2}}{\cos z} \right]$$



$$= \frac{+2}{\pi} \lim_{z \rightarrow \frac{\pi}{2}} \left[\frac{1}{-\sin z} \right]$$

$$= \frac{+2}{\pi} \left[\frac{1}{-\sin(\pi - \frac{\pi}{2})} \right]$$

$$= \frac{+2}{\pi} \left[\frac{1}{-(-1)} \right]$$

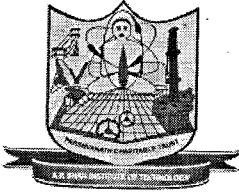
$$= \frac{+2}{\pi} //$$

$$= \frac{2}{\pi} //$$

$$\text{Sum of residue} = 0 - \frac{2}{\pi} + \frac{2}{\pi}$$

$$= 0 //$$

Prof. Nancy Sinollin



① Find residue of $z^2 \sec \pi z$

$$f(z) = z^2 \sec \pi z$$

$$= \frac{z^2}{\cos \pi z}$$

we'll get singularities when

$$\cos \pi z = 0$$

$$\Rightarrow \pi z = \left(2n + \frac{1}{2}\right) \frac{\pi}{2} \quad \dots n = 0, \pm 1, \pm 2, \dots$$

$$z = \frac{2n+1}{2} \quad \dots n = 0, \pm 1, \pm 2, \dots$$

~~$z = \frac{2n+1}{2}$~~ which are poles of order 1

Residue of $f(z)$ at $z = z_0$

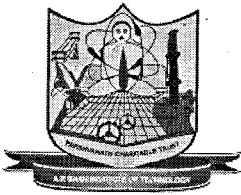
$$= \lim_{z \rightarrow z_0} (z - z_0) f(z)$$

$$\text{Residue of } f(z) \text{ at } z = \left(\frac{2n+1}{2}\right)$$

$$= \lim_{z \rightarrow \left(\frac{2n+1}{2}\right)} \left(z - \frac{2n+1}{2}\right) \frac{z^2}{\cos \pi z} = \frac{0}{0}$$

$$= \lim_{z \rightarrow \left(\frac{2n+1}{2}\right)} \frac{\left(z - \frac{2n+1}{2}\right) \left(\frac{2n+1}{2}\right)^2}{\cos \pi z}$$

$$= \left(\frac{2n+1}{2}\right)^2 \lim_{z \rightarrow \left(\frac{2n+1}{2}\right)} \frac{\left(z - \frac{2n+1}{2}\right)}{\cos \pi z} = \frac{0}{0}$$



By L'Hospital Rule.

$$= \left(\frac{2n+1}{2} \right)^2 \lim_{z \rightarrow \left(\frac{n+1}{2} \right)} \frac{(1)}{(-\sin \pi z) \pi}$$

$$= \left(\frac{2n+1}{2} \right)^2 \left[-\frac{1}{\left[\sin \pi \left(\frac{2n+1}{2} \right) \right] \pi} \right]$$

② Find poles of $f(z) = \frac{\sec^2 z}{z^2}$ which lies inside $|z|=2$. & also find residues

→ $f(z) = \frac{\sec^2 z}{z^2}$

$$f(z) = \frac{1}{z^2 \cos^2 z}$$

We'll get singularities when

$$z^2 \cos^2 z = 0$$

$$f(z) = \frac{1}{z^2 \cos^2 z}$$

$f(z) \neq$

We'll get singularities when

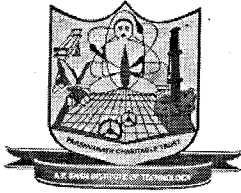
$$z^2 \cos z = 0$$

$$z^2 = 0 \text{ or } \cos z = 0$$

$$z = 0 \text{ or } z = \pm \left(\frac{2n+1}{2} \right)$$

~~$n \neq 0$~~

$$z = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \dots$$



$\Rightarrow z=0$ is a pole of order 2.

$$z = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots \text{ pole of order 1}$$

given $|z| = 2$

$$z = 0, |z| = |0| = 0 < 2 \rightarrow \text{inside.}$$

$$z = \pm \frac{\pi}{2} \quad |z| = \left| \pm \frac{\pi}{2} \right| = \frac{\pi}{2} < 2 \rightarrow \text{inside}$$

$$z = \pm \frac{3\pi}{2} \quad |z| = \left| \pm \frac{3\pi}{2} \right| = \frac{3\pi}{2} > 2 \rightarrow \text{outside.}$$

$$z = z_0.$$

Residue of $f(z)$ at $z = z_0$

$$= \lim_{z \rightarrow z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z)$$

$z = 0$ is pole of order 2 ($n = 2$)

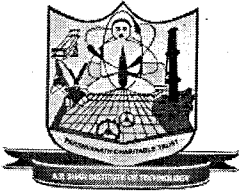
Residue of $f(z)$ at $z = 0$

$$= \lim_{z \rightarrow 0} \frac{1}{1!} \frac{d}{dz} \left[\frac{z^2}{z^2 \cos z} \right]$$

$$= \lim_{z \rightarrow 0} \frac{d}{dz} \left(\frac{1}{\cos z} \right)$$

$$= \lim_{z \rightarrow 0} \left[\frac{\cos z(0) - (-\sin z)}{(\cos z)^2} \right]$$

$$= \lim_{z \rightarrow 0} \left[\frac{\sin z}{(\cos z)^2} \right] = 0$$



$$\text{Residue of } f(z) \text{ at } (z=0) = \lim_{z \rightarrow 0} \left[(z-0) f(z) \right]$$

$$= \lim$$

$$\text{Residue of } f(z) \text{ at } (z = \frac{\pi}{2})$$

$$= \lim_{z \rightarrow \frac{\pi}{2}} \left[\left(z - \frac{\pi}{2} \right) \frac{1}{z^2 \cos z} \right] = \frac{0}{0}$$

$$= \lim_{z \rightarrow \frac{\pi}{2}} \left[\left(z - \frac{\pi}{2} \right) \frac{1}{\left(\frac{\pi}{2} \right)^2 \cos z} \right]$$

$$= \frac{4}{\pi^2} \lim_{z \rightarrow \frac{\pi}{2}} \left[\frac{\left(z - \frac{\pi}{2} \right)}{\cos z} \right]$$

$$= \frac{4}{\pi^2} \lim_{z \rightarrow \frac{\pi}{2}} \left[\frac{1}{-\sin z} \right]$$

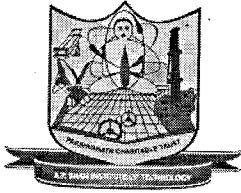
$$= \frac{4}{\pi^2} \frac{1}{-\sin\left(\frac{\pi}{2}\right)}$$

$$= -\frac{4}{\pi^2}$$

$$\text{Residue of } f(z) \text{ at } (z = -\frac{\pi}{2})$$

$$= \lim_{z \rightarrow -\frac{\pi}{2}} \left[\left(z + \frac{\pi}{2} \right) f(z) \right]$$

$$= \lim_{z \rightarrow -\frac{\pi}{2}} \left[\left(z + \frac{\pi}{2} \right) \frac{1}{z^2 \cos z} \right]$$



$$= \lim_{z \rightarrow -\frac{\pi}{2}} \left[\left(z + \frac{\pi}{2} \right) \cdot \frac{1}{z^2 \cos z} \right]$$

$$= \lim_{z \rightarrow -\frac{\pi}{2}} \left[\left(z + \frac{\pi}{2} \right) \frac{1}{\left(-\frac{\pi}{2} \right)^2 \cos z} \right]$$

$$= \lim_{z \rightarrow -\frac{\pi}{2}} \left[\left(z + \frac{\pi}{2} \right) \cdot \frac{1}{\frac{\pi^2}{4} \cos z} \right]$$

$$= \frac{4}{\pi^2} \lim_{z \rightarrow -\frac{\pi}{2}} \left[\frac{1}{-\sin z} \right]$$

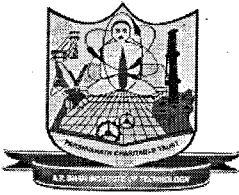
$$= \frac{4}{\pi^2} \cdot \frac{1}{-\sin \left(-\frac{\pi}{2} \right)}$$

$$= \frac{4}{\pi^2} //$$

*** Cauchy's Residue Theorem:-**

If $f(z)$ is analytic in & on simple closed curve C except at finite no. of isolated singular points: $z_1, z_2, z_3, \dots, z_n$ inside C then ~~is~~

$$\oint_C f(z) dz = 2\pi i \left(\text{sum of residues at } z_1, z_2, \dots, z_n \right)$$



①

$$\int_C \frac{z^2}{(z-2)(z-1)^2} dz \text{ where } C \text{ is } |z| = 2.5$$

$$\rightarrow f(z) = \frac{z^2}{(z-2)(z-1)^2}$$

clearly, $z=2, 1$ are singular pts.

$z=2$ is pole of order 1

$z=1$ is pole of order 2.

Given $|z| = 2.5$

$z=2$ $|2| = 2 < 2.5 \rightarrow$ inside C

$z=1$ $|1| = 1 < 2.5 \rightarrow$ inside C

~~pole of order 4~~

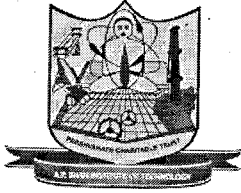
Residue of $f(z)$ at $z=2$

$$= \lim_{z \rightarrow 2} (z-2) \frac{z^2}{(z-1)^2 (z-2)}$$

$$= \lim_{z \rightarrow 2} \frac{z^2}{(z-1)^2}$$

$$= \frac{(2)^2}{(2-1)^2}$$

$$= 4$$



Residue of $f(z)$ at $z = 1$

$$= \lim_{z \rightarrow 1} \frac{1}{(1-1)!} \frac{d}{dz} \left[\frac{(z-1)^2 \cdot z^2}{(z-2)(z-1)^2} \right]$$

$$= \lim_{z \rightarrow 1} \frac{d}{dz} \left[\frac{z^2}{(z-2)} \right]$$

$$= \lim_{z \rightarrow 1} \left[\frac{(z-2)(2z) - z^2(1)}{(z-2)^2} \right]$$

$$= \left[\frac{(1-2)(2) - (1)^2}{(1-2)^2} \right]$$

$$= \left[\frac{(-1)(2) - 1}{(-1)^2} \right]$$

$$= \frac{-2-1}{+1}$$

$$= -3$$

By Cauchy's Residue Theorem,

~~residue~~ $\oint f(z) dz = 2\pi i$ (sum of residues)

$$\oint \frac{z^2}{(z-2)(z-1)^2} dz = 2\pi i (4-3)$$

$$= 2\pi i //$$