Parshvanath Charitable Trust's
A. P. SHAH INSTITUTE OF TECHNOLOGY
(Approved by AICTE New Delhi & Govt. of Maharashtra, Affiliated to University of Mumbai)
(Religious Jain Minority)

Subject: Applied Mathematics III **SEM: III**

Evaluation of integration using Laplace transform.

i.e. Evaluation of $\int_0^\infty e^{-at} \cdot f(t)\,dt$

Steps : To evaluate $\int_0^\infty e^{-at} \cdot f(t)\,dt$

1] First calculate $L[f(t)] = \phi(s)$

2] Put the value of $L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\,dt = \phi(s)$

3] $\therefore \int_0^\infty e^{-st} \cdot f(t)\,dt = \phi(s)$

then put $s = a$

$$\therefore \int_0^\infty e^{-at} \cdot f(t)\,dt = \phi(a).$$

Examples :

1] Evaluate $\int_0^\infty e^{-t} \cdot \operatorname{erf}(3\sqrt{t})\,dt$.

Sol^n: let $f(t) = \operatorname{erf}(\sqrt{t})$

$$L[f(t)] = L[\operatorname{erf}\sqrt{t}] = \frac{1}{s\sqrt{s+1}} = \phi(s)$$

$$L[f(3t)] = L[\operatorname{erf}(3\sqrt{t})] = \frac{1}{9}\,\phi(s/9) = \frac{1}{9} \cdot \frac{1}{\frac{s}{9}\sqrt{\frac{s}{9}+1}}$$

$$= \frac{3}{s\sqrt{s+9}}$$

$$\therefore L[\operatorname{erf}(3\sqrt{t})] = L[\operatorname{erf}(\sqrt{9t})] = \frac{3}{s\sqrt{s+9}}$$

$$\therefore \int_0^\infty e^{-st} \cdot \operatorname{erf}(3\sqrt{t})\,dt = \frac{3}{s\sqrt{s+9}}.$$

put $s = 1$.

$$\therefore \int_0^\infty e^{-t} \cdot \operatorname{erf}(3\sqrt{t})\,dt = \frac{3}{\sqrt{10}}$$

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2] Evaluate $\int_0^\infty t^2 \cdot \frac{\sin 3t}{e^{2t}} \cdot dt$.

Sol^n: consider, $\int_0^\infty t^2 \cdot \frac{\sin 3t}{e^{2t}} \cdot dt = \int_0^\infty e^{-2t} \cdot t^2 \sin 3t \cdot dt$.

first we calculate $L[t^2 \sin 3t]$.

$\therefore L[\sin 3t] = \frac{3}{s^2+9}$

$$L[t^2 \sin 3t] = (-1)^2 \frac{d^2}{ds^2} \phi(s)$$

$$= (-1)^2 \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{3}{s^2+9}\right)\right]$$

$$= \frac{d}{ds}\left[\frac{-6s}{(s^2+9)^2}\right]$$

$$= -6\left[\frac{(s^2+9)^2 - s(2(s^2+9))(2s)}{(s^2+9)^4}\right]$$

$$= -6\left[\frac{9-3s^2}{(s^2+9)^3}\right]$$

$$L[t^2 \sin 3t] = 6\left[\frac{3s^2-9}{(s^2+9)^3}\right]$$

$$\therefore \int_0^\infty e^{-st} \cdot t^2 \sin 3t \cdot dt = 6\left[\frac{3s^2-9}{(s^2+9)^3}\right]$$

put, $s=2$.

$$\therefore \int_0^\infty e^{-2t} \cdot t^2 \sin 3t \cdot dt = 6\left[\frac{3(4)-9}{(4+9)^3}\right] = \frac{18}{(13)^3}.$$

3] Evaluate $\int_0^\infty \frac{\cos 6t - \cos 4t}{t} dt$.

We can write, $\int_0^\infty \frac{\cos 6t - \cos 4t}{t} dt = \int_0^\infty e^{-0t} \cdot \frac{\cos 6t - \cos 4t}{t} \cdot dt$

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first to find $L\left[\frac{\cos 6t - \cos 4t}{t}\right]$.

$$L[\cos 6t - \cos 4t] = L[\cos 6t] - L[\cos 4t]$$

$$= \frac{s}{s^2+36} - \frac{s}{s^2+16}$$

$$L\left[\frac{\cos 6t - \cos 4t}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+36} - \frac{s}{s^2+16}\right) ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{2s}{s^2+36} - \frac{2s}{s^2+16}\right) ds$$

$$= \frac{1}{2}\left[\log(s^2+36) - \log(s^2+16)\right]_s^\infty$$

$$= \frac{1}{2}\left[\infty - \infty - \left(\log(s^2+36) - \log(s^2+16)\right)\right]$$

$$= \frac{-1}{2}\cdot\log\left(\frac{s^2+36}{s^2+16}\right).$$

$$L\left[\frac{\cos 6t - \cos 4t}{t}\right] = \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right).$$

$$\int_0^\infty e^{-st}\,\frac{\cos 6t - \cos 4t}{t}\,dt = \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right).$$

put $s = 0$

$$\therefore \int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt = \frac{1}{2}\log\left(\frac{16}{36}\right) = \log\left(\frac{4}{6}\right) = \log\left(\frac{2}{3}\right)$$

4] Evaluate $\int_0^\infty e^{-t}\int_0^t \frac{\sin u}{u}\,du\cdot dt$.

$\underline{\text{sol}^n}$: first to calculate, $L\left[\int_0^t \frac{\sin u}{u}\,du\right]$.

$$\therefore L[\sin u] = \frac{1}{s^2+1}.$$

$$L\left[\frac{\sin u}{u}\right] = \int_s^\infty \frac{1}{s^2+1}\,ds$$
$$= \left[\tan^{-1}(s)\right]_s^\infty$$
$$= \tan^{-1}(\infty) - \tan^{-1}s.$$
$$= \pi/2 - \tan^{-1}s$$
$$= \cot^{-1}s.$$

$$\therefore L\left[\int_0^t \frac{\sin u}{u}\,du\right] = \frac{1}{s}\cdot\cot^{-1}s.$$

$$\therefore \int_0^\infty e^{-st}\int_0^t \frac{\sin u}{u}\cdot du\cdot dt = \frac{1}{s}\cdot\cot^{-1}s.$$

put $s = 1$. we get,

$$\int_0^\infty e^{-t}\int_0^t \frac{\sin u}{u}\,du\cdot dt = \cot^{-1}(1) = \frac{\pi}{4}.$$

5] IF $\int_0^\infty e^{-2t}\cdot\sin(t+\alpha)\cdot\cos(t-\alpha)\,dt = \frac{1}{4}$. find α.

Sol^n: We will find $L\left[\sin(t+\alpha)\cdot\cos(t-\alpha)\right]$.

$$\therefore L\left[\sin(t+\alpha)\cdot\cos(t-\alpha)\right] = L\left[\frac{\sin(t+\alpha+t-\alpha)+\sin(t+\alpha-(t-\alpha))}{2}\right]$$
$$= L\left[\frac{\sin 2t + \sin 2\alpha}{2}\right]$$

$$L\left[\sin(t+\alpha)\cdot\cos(t-\alpha)\right] = \frac{1}{2}\left[\frac{2}{s^2+4} + \sin 2\alpha\cdot\left(\frac{1}{s}\right)\right]$$

$$\therefore \int_0^\infty e^{-st}\sin(t+\alpha)\cdot\cos(t-\alpha)\,dt = \frac{1}{2}\left[\frac{2}{s^2+4} + \frac{\sin 2\alpha}{s}\right].$$

put $s = 2$.

$$\therefore \int_0^\infty e^{-2t}\cdot\sin(t+\alpha)\cdot\cos(t-\alpha)\,dt = \frac{1}{2}\left[\frac{2}{4+4} + \frac{\sin 2\alpha}{2}\right].$$

$$\therefore \int_0^\infty e^{-2t} \sin(t+\alpha).\cos(t-\alpha)\, dt = \frac{1}{2}\left[\frac{2}{8} + \frac{\sin 2\alpha}{2}\right]$$

$$\therefore \quad \frac{1}{4} = \frac{1}{2}\left[\frac{1}{4} + \frac{\sin 2\alpha}{2}\right]$$

$$\therefore \quad \frac{1}{4} = \frac{1}{4}\left[\frac{1}{2} + \sin 2\alpha\right]$$

$$\sin 2\alpha = \frac{1}{2}$$

$$2\alpha = \sin^{-1}(1/2) = \pi/6$$

$$\boxed{\alpha = \pi/12}$$