* **Non-Linear Programming**:-

An optimisation problem in which either the objective function and /or some or all constraints are non-linear is called a non-linear programming problem (NLPP)

e.g. optimise $z = x_1^2 + x_2^2 - 5x_1x_2$

subject to $x_1^2 + x_1x_2 + 5x_3 = 70$

$x_1, x_2, x_3 \geq 0$

• **Quadratic Programming Problem**:-

The objective function is of the type:

$$z = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + \cdots + a_{1n}x_1x_n + a_{23}x_2x_3 + a_{24}x_2x_4 + \cdots + a_{2n}x_2x_n + \cdots + c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

• **Method to solve Quadratic Programming Problem**:-

- Assume that the first & second order partial derivatives i.e. $\frac{\partial f}{\partial x_i}$ & $\frac{\partial^2 f}{\partial x_i \partial x_j}$ exists $\forall i, j$, where $f(x_1, x_2 \dots x_n)$ is the objective function we want to maximise / minimise.
- Find the Hessian matrix, which is given by,

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

- we need to check whether this matrix is positive definite, negative definite or indefinite.

→ Given a matrix $A_{n \times n}$, the principal minors are,

$$A_1 = [a_{11}], \quad A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \dots, A_n = A_{n \times n}$$

Let the determinant of the matrices be

$D_1 = |A_1|, \quad D_2 = |A_2|, \dots, \quad D_n = |A_n|$

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→ If all the determinants,

- $D_1, D_2, \ldots, D_n$ are positive then A is <u>positive definite</u>
- $D_1, D_3, D_5, \ldots$ are negative & $D_2, D_4, \ldots$ are positive then A is <u>negative definite</u>.
- If a matrix is neither positive definite nor negative definite, then it is <u>indefinite</u>.

– Find the status of the Hession matrix H at a stationary point x_0.

- If H is positive definite at x_0, it has a <u>minima</u> at x_0
- If H is negative definite at x_0, it has a <u>maxima</u> at x_0
- If H is indefinite at x_0, it has <u>saddle point</u> at x_0

<u>Note</u>:- To find stationary points of a function,

Put $\frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} = 0, \ldots, \frac{\partial f}{\partial x_n} = 0$

On solving these equations, we get stationary point

$x_0 = (x_1, x_2, \ldots x_n)$

* <u>Examples</u>:-

1) Optimize $Z = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 8x_2 - 10x_3$

→ First we need to find the stationary points,

Let $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 8x_2 - 10x_3$

$\frac{\partial f}{\partial x_1} = 0 \Rightarrow 2x_1 - 6 = 0 \Rightarrow \underline{\underline{x_1 = 3}}$

$\frac{\partial f}{\partial x_2} = 0 \Rightarrow 2x_2 - 8 = 0 \Rightarrow \underline{\underline{x_2 = 4}}$

$\frac{\partial f}{\partial x_3} = 0 \Rightarrow 2x_3 - 10 = 0 \Rightarrow \underline{\underline{x_3 = 5}}$

$\therefore x_0 = (3, 4, 5)$ is the stationary point.

Now we find the second order partial derivatives,

$\frac{\partial^2 f}{\partial x_1^2} = 2, \quad \frac{\partial^2 f}{\partial x_2^2} = 2, \quad \frac{\partial^2 f}{\partial x_3^2} = 2$

Prof. Anushri Tambe

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_1 \partial x_3} = \frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{\partial^2 f}{\partial x_2 \partial x_3} = \frac{\partial^2 f}{\partial x_3 \partial x_1} = \frac{\partial^2 f}{\partial x_3 \partial x_2} = 0$$

$\therefore$ The Hessian matrix is $H = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Let

$A_1 = [2]$, $A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $A_3 = H$

$\therefore D_1 = 2$, $D_2 = 4$, $D_3 = 8$

all $D_1, D_2, D_3 > 0$ $\therefore$ H is positive definite

$\therefore f(x_1, x_2, x_3) = Z = x_1^2 + x_2^2 + x_3^2 - .. x_1 - 8x_2 - 10x_3$ has minimum

at $x_0 (3, 4, 5)$

$\therefore$ The minimum value of z is, $Z_{min} = 9 + 16 + 25 - 18 - 32 - 50$

$\therefore \boxed{Z_{min} = -50}$

2) Obtain the relative maximum or minimum (if any) of the function $Z = x_1 + 2x_3 + x_2 x_3 - x_1^2 - x_2^2 - x_3^2$

$\rightarrow$ Let $f(x_1, x_2, x_3) = x_1 + 2x_3 + x_2 x_3 - x_1^2 - x_2^2 - x_3^2$

The stationary points are given by,

$\frac{\partial f}{\partial x_1} = 0 \Rightarrow 1 - 2x_1 = 0 \Rightarrow \boxed{x_1 = 1/2}$

$\frac{\partial f}{\partial x_2} = 0 \Rightarrow x_3 - 2x_2 = 0$ —— (1) & $\frac{\partial f}{\partial x_3} = 0 \Rightarrow 2 + x_2 - 2x_3 = 0$

$\Rightarrow x_2 - 2x_3 = -2$ —— (2)

Solving (1) & (2),

$x_3 = 2x_2$ & $x_2 = 2x_3 - 2 = 2(2x_2) - 2 = 4x_2 - 2 \Rightarrow \boxed{x_2 = 2/3}$

& $\boxed{x_3 = \frac{4}{3}}$ $\therefore$ The stationary point is $x_0 = \left(\frac{1}{2}, \frac{2}{3}, \frac{4}{3}\right)$

The second order partial derivatives are,

$\frac{\partial^2 f}{\partial x_1^2} = -2$, $\frac{\partial^2 f}{\partial x_2^2} = -2$, $\frac{\partial^2 f}{\partial x_3^2} = -2$, $\frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$, $\frac{\partial^2 f}{\partial x_2 \partial x_1} = 0$,

$\frac{\partial^2 f}{\partial x_1 \partial x_3} = 0$, $\frac{\partial^2 f}{\partial x_3 \partial x_1} = 0$, $\frac{\partial^2 f}{\partial x_2 \partial x_3} = 1$, $\frac{\partial^2 f}{\partial x_3 \partial x_2} = 1$

$\therefore$ The Hessian matrix is $H = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

$A_1 = [-2]$, $A_2 = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$, $A_3 = H$

$\therefore D_1 = -2$, $D_2 = 4$, $D_3 = -6$

Here $D_1, D_3 < 0$ & $D_2 > 0$ $\therefore$ H is negative definite

$\therefore f(x_1, x_2, x_3)$ has maximum at $x_0 = \left(\frac{1}{2}, \frac{2}{3}, \frac{4}{3}\right)$

$$\therefore Z_{max} = \frac{1}{2} + 2\left(\frac{4}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{4}{3}\right) - \frac{1}{4} - \frac{4}{9} - \frac{16}{9}$$

$$= \frac{1}{2} + \frac{8}{3} + \frac{8}{9} - \frac{1}{4} - \frac{20}{9} = \frac{18 + 96 + 32 - 9 - 80}{36} = \frac{57}{36}$$

$$\boxed{Z_{max} = \frac{19}{12}}$$

Practice Problems:-

Find the relative maximum or minimum of the function

(i) $Z = 2x_1 + 6x_3 + 9x_2x_3 - 4x_1^2 - 9x_2^2 - 9x_3^2$

(ii) $Z = x_1^2 + x_2^2 + x_3^2 - 6x_1 - 10x_2 - 14x_3 + 103$

* Optimisation with equality constraints:-

A NLPP in which the objective function is non-linear but the constraints are linear.

i.e. Optimise $z = f(x_1, x_2, ---, x_n)$

subject to $g_1(x_1, x_2, ---, x_n) = b_1$,

$g_2(x_1, x_2, ---, x_n) = b_2$,

$\vdots$

$g_n(x_1, x_2, ---, x_n) = b_m$

$x_1, x_2, ---, x_n \geq 0$

This type of problem is solved by forming Lagrangian Function with Lagrange's multiplier λ

(a) NLPP with n-variables & one equality constraint:-

optimise $z = f(x_1, x_2, \ldots, x_n)$

subject to $g(x_1, x_2, \ldots, x_n) = b$

$x_1, x_2, \ldots, x_n \geq 0$

- first express the constraints with RHS equal to zero

i.e. optimise $z = f(x_1, x_2, \ldots, x_n)$

subject to $h(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) - b$

$x_1, x_2, \ldots, x_n \geq 0$

- The Lagrangian function is (constructed as)

$$L(x_1, x_2, \ldots, x_n, \lambda) = f(x_1, \ldots, x_n) - \lambda h(x_1, \ldots, x_n) \quad \text{——— (1)}$$

where λ is called Lagrangian multiplier.

- The necessary condition for maxima or minima subject to the constraint $h(x_1, x_2, \ldots, x_n) = 0$ are

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \ldots\ldots, \frac{\partial L}{\partial x_n} = 0, \quad \frac{\partial L}{\partial \lambda} = 0 \quad \text{——— (2)}$$

$$(1) \Rightarrow \frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1}, \quad \frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2}, \ldots\ldots, \frac{\partial L}{\partial x_n} = \frac{\partial f}{\partial x_n} - \lambda \frac{\partial h}{\partial x_n}$$

$$\& \quad \frac{\partial L}{\partial \lambda} = -\lambda h \quad \text{——— (3)}$$

Using (2), we get from (3), the following (n+1) necessary cond^{n}s

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0, \ldots, \frac{\partial f}{\partial x_n} - \lambda \frac{\partial h}{\partial x_n} = 0$$

$$h(x_1, x_2, \ldots, x_n) = 0$$

Solving these equations, we get $x_1, x_2, \ldots, x_n, \lambda$.

i.e. we obtain the point of maxima or minima.

- To check the point obtained above is maxima/minima, consider the following determinant

$$\Delta_{n+1} = \begin{vmatrix} 0 & \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_n} \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} - \lambda \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} - \lambda \frac{\partial^2 h}{\partial x_1 \partial x_n} \\ \frac{\partial h}{\partial x_2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} - \lambda \frac{\partial^2 h}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} - \lambda \frac{\partial^2 h}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial h}{\partial x_n} & \frac{\partial^2 f}{\partial x_n \partial x_1} - \lambda \frac{\partial^2 h}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} - \lambda \frac{\partial^2 h}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} - \lambda \frac{\partial^2 h}{\partial x_n^2} \end{vmatrix}$$

- If the principal minors $\Delta_3, \Delta_4, \Delta_5, \ldots$ are alternately positive & negative i.e. $\Delta_3 > 0$, $\Delta_4 < 0$, $\Delta_5 > 0, \ldots$ then pt. x_0 is <u>maxima</u>
- If all $\Delta_3, \Delta_4, \Delta_5, \ldots$ are negative, i.e. $\Delta_3 < 0$, $\Delta_4 < 0, \ldots$ then x_0 is <u>minima</u>

<u>Note:</u>- (i) If z is a function of two variables only, then we get only Δ_3.

If Δ_3 is <u>positive</u> then x_0 is <u>maxima</u>

If Δ_3 is <u>negative</u> then x_0 is <u>minima</u>

(ii) If z is a function of three variables then we get Δ_3 & Δ_4

If <u>both</u> Δ_3 & Δ_4 are <u>negative</u> then x_0 is <u>minima</u>

If <u>$\Delta_3 > 0$</u> & <u>$\Delta_4 < 0$</u> then x_0 is <u>maxima</u>.

<u>Examples:-</u>

1) Using Lagrange's Multipliers, solve the following NLPP

(i) Optimise $z = 4x_1 + 8x_2 - x_1^2 - x_2^2$

subject to $x_1 + x_2 = 2$

$x_1, x_2 \geq 0$

$\rightarrow$ NLPP is optimise $z = 4x_1 + 8x_2 - x_1^2 - x_2^2$

subject to $x_1 + x_2 - 2 = 0$

$x_1, x_2 \geq 0$

The Lagrangian function is

$$L(x_1, x_2, \lambda) = (4x_1 + 8x_2 - x_1^2 - x_2^2) - \lambda(x_1 + x_2 - 2) \quad \text{—— (1)}$$

$\therefore$ the partial derivatives are

$$\frac{\partial L}{\partial x_1} = 4 - 2x_1 - \lambda \quad , \quad \frac{\partial L}{\partial x_2} = 8 - 2x_2 - \lambda \quad , \quad \frac{\partial L}{\partial \lambda} = -(x_1 + x_2 - 2)$$

Solving the equations,

$$\frac{\partial L}{\partial x_1} = 0 \quad , \quad \frac{\partial L}{\partial x_2} = 0 \quad , \quad \frac{\partial L}{\partial \lambda} = 0 \quad \text{we get}$$

$\therefore\ 4 - 2x_1 - \lambda = 0$ —— (2)

$8 - 2x_2 - \lambda = 0$ —— (3)

$-(x_1 + x_2 - 2) = 0$ —— (4)

$(2) + (3) \Rightarrow 12 - 2(x_1 + x_2) - 2\lambda = 0$

$\Rightarrow 12 - 2(2) = 2\lambda$ from (3)

$\Rightarrow \boxed{\lambda = 4}$

$\therefore (1) \Rightarrow 2x_1 = 4 - 4 = 0 \Rightarrow \boxed{x_1 = 0}$

$(2) \Rightarrow 2x_2 = 8 - 4 = 4 \Rightarrow \boxed{x_2 = 2}$ $\quad \therefore \boxed{X_0 = (0, 2)}$

Now, $h(x_1, x_2) = x_1 + x_2 - 2 = 0$

$\therefore \dfrac{\partial h}{\partial x_1} = 1 \ , \ \dfrac{\partial h}{\partial x_2} = 1$ & all other partial derivatives are zero.

& $f(x_1, x_2) = 4x_1 + 8x_2 - x_1^2 - x_2^2$

$$\therefore \frac{\partial f}{\partial x_1} = 4 - 2x_1 \quad , \quad \frac{\partial^2 f}{\partial x_2 \partial x_1} = 0 \quad , \quad \frac{\partial^2 f}{\partial x_1^2} = -2$$

$$\frac{\partial f}{\partial x_2} = 8 - 2x_2 \quad , \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0 \quad , \quad \frac{\partial^2 f}{\partial x_2^2} = -2$$

$$\therefore \Delta_3 = \begin{vmatrix} 0 & \dfrac{\partial h}{\partial x_1} & \dfrac{\partial h}{\partial x_2} \\ \dfrac{\partial h}{\partial x_1} & \dfrac{\partial^2 f}{\partial x_1^2} - \lambda \dfrac{\partial^2 h}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \dfrac{\partial^2 h}{\partial x_1 \partial x_2} \\ \dfrac{\partial h}{\partial x_2} & \dfrac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \dfrac{\partial^2 h}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} - \lambda \dfrac{\partial^2 h}{\partial x_2^2} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix}$$

$$= 0 - 1(-2) + 1(2) = 4$$

Here $\Delta_3 = 4 > 0 \quad \therefore X_0$ is <u>maxima</u>

$\therefore \boxed{x_1 = 0}$

$\boxed{x_2 = 2}$

& $Z_{max} = 4(0) + 8(2) - 0 - (2)^2 = 16 - 4 = 12$

$\therefore \boxed{Z_{max} = 12}$

Prof. Anushri Tambe

(ii) Optimise $Z = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23$

subject to $x_1 + x_2 + x_3 = 10$

$x_1, x_2, x_3 \geq 0$

→ The Lagrangian function is

$$L(x_1, x_2, x_3, \lambda) = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23 - \lambda(x_1 + x_2 + x_3 - 10)$$

$\therefore \dfrac{\partial L}{\partial x_1} = 12 - 2x_1 - \lambda$, $\dfrac{\partial L}{\partial x_2} = 8 - 2x_2 - \lambda$, $\dfrac{\partial L}{\partial x_3} = 6 - 2x_3 - \lambda$

$\dfrac{\partial L}{\partial \lambda} = -(x_1 + x_2 + x_3 - 10)$

Solving , $\dfrac{\partial L}{\partial x_1} = 0$, $\dfrac{\partial L}{\partial x_2} = 0$, $\dfrac{\partial L}{\partial x_3} = 0$, $\dfrac{\partial L}{\partial \lambda} = 0$

$\therefore 12 - 2x_1 - \lambda = 0$ —— (1) , $8 - 2x_2 - \lambda = 0$ —— (2),

$6 - 2x_3 - \lambda = 0$ —— (3) , $x_1 + x_2 + x_3 - 10 = 0$ —— (4)

$(1) + (2) + (3) \rightarrow 26 - 2(x_1 + x_2 + x_3) - 3\lambda = 0$

$\Rightarrow 26 - 2(10) - 3\lambda = 0$ from (4)

$\Rightarrow 6 = 3\lambda \Rightarrow \boxed{\lambda = 2}$

$\therefore (1) \Rightarrow 12 - 2x_1 - 2 = 0 \Rightarrow 2x_1 = 10 \Rightarrow \boxed{x_1 = 5}$

$(2) \Rightarrow 8 - 2x_2 - 2 = 0 \Rightarrow 2x_2 = 6 \Rightarrow \boxed{x_2 = 3}$

$(3) \Rightarrow 6 - 2x_3 - 2 = 0 \Rightarrow 2x_3 = 4 \Rightarrow \boxed{x_3 = 2}$ $\therefore \boxed{X_0 = (5, 3, 2)}$

Now,

$$\Delta_4 = \begin{vmatrix} 0 & \dfrac{\partial h}{\partial x_1} & \dfrac{\partial h}{\partial x_2} & \dfrac{\partial h}{\partial x_3} \\ \dfrac{\partial h}{\partial x_1} & \dfrac{\partial^2 f}{\partial x_1^2} - \lambda \dfrac{\partial^2 h}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} - \lambda \dfrac{\partial^2 h}{\partial x_1 \partial x_2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_3} - \lambda \dfrac{\partial^2 h}{\partial x_1 \partial x_3} \\ \dfrac{\partial h}{\partial x_2} & \dfrac{\partial^2 f}{\partial x_2 \partial x_1} - \lambda \dfrac{\partial^2 h}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} - \lambda \dfrac{\partial^2 h}{\partial x_2^2} & \dfrac{\partial^2 f}{\partial x_2 \partial x_3} - \lambda \dfrac{\partial^2 h}{\partial x_2 \partial x_3} \\ \dfrac{\partial h}{\partial x_3} & \dfrac{\partial^2 f}{\partial x_3 \partial x_1} - \lambda \dfrac{\partial^2 h}{\partial x_3 \partial x_1} & \dfrac{\partial^2 f}{\partial x_3 \partial x_2} - \lambda \dfrac{\partial^2 h}{\partial x_3 \partial x_2} & \dfrac{\partial^2 f}{\partial x_3^2} - \lambda \dfrac{\partial^2 h}{\partial x_3^2} \end{vmatrix}$$

where $f(x_1, x_2, x_3) = 12x_1 + 8x_2 + 6x_3 - x_1^2 - x_2^2 - x_3^2 - 23$.

& $h(x_1, x_2, x_3) = x_1 + x_2 + x_3 - 10$

Prof. Anushri Tambe

$$\therefore \Delta_4 = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{vmatrix}$$

$$= 0\begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & -2 \end{vmatrix} - 1\begin{vmatrix} 1 & -2 & 0 \\ 1 & 0 & -2 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= 0 - 1\{1(4)\} + 1\{1(0) + 2(-2) + 0\} - 1\{1(0) + 2(2) + 0\}$$

$$= -4 - 4 - 4 = -12$$

$$\& \ \Delta_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = 0 - 1(-2) + 1(2) = 2 + 2 = 4$$

$\therefore \Delta_4 = -12 < 0 \ \& \ \Delta_3 = 4 > 0 \quad \therefore X_0$ is maxima

$\therefore \boxed{x_1 = 5} \quad \boxed{x_2 = 3} \quad \boxed{x_3 = 2}$

$\& \ Z_{max} = 12(5) + 8(3) + 6(2) - 25 - 9 - 4 - 23 = 35$

$\therefore \boxed{Z_{max} = 35}$

* Practice Problems:-

Using Lagrange's multipliers, solve the following NLPP

(i) optimise $z = 6x_1^2 + 5x_2^2$
subject to $x_1 + 5x_2 = 3$
$x_1, x_2 \geq 0$

(ii) Optimise $z = 2x_1 + 6x_2 - x_1^2 - x_2^2 + 14$
subject to $x_1 + x_2 = 4$
$x_1, x_2 \geq 0$

(iii) optimise $z = 2x_1^2 + x_2^2 + 3x_3^2 + 10x_1 + 8x_2 + 6x_3 - 100$
subject to $x_1 + x_2 + x_3 = 20$
$x_1, x_2, x_3 \geq 0$

(iv) optimise $z = 3x_1^2 + x_2^2 + x_3^2$
subject to $x_1 + x_2 + x_3 = 2$
$x_1, x_2, x_3 \geq 0$

Prof. Anushri Tambe

(b) NLPP with n variables & more than one (m) equality constraints (m<n) :-

Optimise $Z = f(x_1, x_2, \ldots, x_n)$

subject to $h_1(x_1, \ldots, x_n) = 0$

$h_2(x_1, \ldots, x_n) = 0$

$\vdots$

$h_m(x_1, \ldots, x_n) = 0$

$x_1, \ldots, x_n \geq 0$

The Lagrangian function with m multipliers $\lambda_1, \ldots, \lambda_m$ is

$$L(x_1, \ldots, x_n, \lambda) = f(x_1, \ldots, x_n) - \lambda_1 h_1(x_1, \ldots, x_n) - \ldots - \lambda_m h_m(x_1, \ldots, x_n)$$

$=0$

The necessary conditions for maxima / minima is,

$$\frac{\partial L}{\partial x_i} = 0, \quad i = 1, 2, \ldots, n \quad , \quad \frac{\partial L}{\partial \lambda_j} = 0, \quad j = 1, \ldots, m$$

Solving these equations, we get stationary point.

To decide whether the point is maxima / minima, consider bordered Hessian matrix denoted by H^B

$$H^B = \left[\begin{array}{c|c} O & P \\ \hline P' & Q \end{array}\right]_{(m+n)\times(m+n)}$$

where

$$O = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{m \times n} \quad \& \quad P = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial h_m}{\partial x_1} & \frac{\partial h_m}{\partial x_2} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}_{m \times n}$$

$$Q = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \cdots & \frac{\partial^2 L}{\partial x_2 \partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 L}{\partial x_n \partial x_1} & \frac{\partial^2 L}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 L}{\partial x_n^2} \end{bmatrix}$$

& P' is transpose of P

Prof. Anushri Tambe

- Condition for maxima & minima:

The nature of the function at x_0 is determined by the signs of $(n-m)$ principal minors of the matrix H^B

(i) start with the principal minor of order $(2m+1)$ & check the signs of $(n-m)$ principal minors. If these signs are alternately positive & negative, starting with $(-1)^{m+n}$ then x_0 is maxima.

(ii) If the signs of these minors are $(-1)^m$, then x_0 is minima

Note: (1) If there are two unknowns & two linear constraints then bordered Hessian matrix H^B is not useful. So find the signs of two principal minors of
Find determinant of both principal minors
say A_1 & A_2

$$\begin{bmatrix} \frac{\partial^2 z}{\partial x_1^2} & \frac{\partial^2 z}{\partial x_1 \partial x_2} \\ \frac{\partial^2 z}{\partial x_2 \partial x_1} & \frac{\partial^2 z}{\partial x_2^2} \end{bmatrix}$$

(i) If both A_1 & A_2 are positive then x_0 is minima.
(ii) If A_1 is negative & A_2 is positive then x_0 is maxima

(2) Geometrically, the constraints are two straight lines & they intersect in only one point at the most.

(3) If there are three unknowns & two constraints, then simplify the bordered matrix H^B
If its sign is negative, then x_0 is maxima
& it its sign is positive, then x_0 is minima

(4) Geometrically the constraints are the two planes & they intersect in a line.

Examples: Using Lagrangian multipliers, solve the following NLPP

1) Optimise $z = 4x_1 + 9x_2 - x_1^2 - x_2^2$

subject to $4x_1 + 3x_2 = 15$

$3x_1 + 5x_2 = 14$

$x_1, x_2 \geq 0$

$\longrightarrow$ Let $f(x_1, x_2) = 4x_1 + 9x_2 - x_1^2 - x_2^2$

$h_1(x_1, x_2) = 4x_1 + 3x_2 - 15$

$h_2(x_1, x_2) = 3x_1 + 5x_2 - 14$

The Lagrangian function is

$$L(x_1, x_2, \lambda_1, \lambda_2) = f(x_1, x_2) - \lambda_1 h_1(x_1, x_2) - \lambda_2 h_2(x_1, x_2)$$

$$\therefore L(x_1, x_2, \lambda_1, \lambda_2) = 4x_1 + 9x_2 - x_1^2 - x_2^2 - \lambda_1(4x_1 + 3x_2 - 15) - \lambda_2(3x_1 + 5x_2 - 14)$$

$$\therefore \frac{\partial L}{\partial x_1} = 4 - 2x_1 - 4\lambda_1 - 3\lambda_2 \quad , \quad \frac{\partial L}{\partial x_2} = 9 - 2x_2 - 3\lambda_1 - 5\lambda_2$$

$$\frac{\partial L}{\partial \lambda_1} = -(4x_1 + 3x_2 - 15) \quad , \quad \frac{\partial L}{\partial \lambda_2} = -(3x_1 + 5x_2 - 14)$$

Solving, $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0$, $\frac{\partial L}{\partial \lambda_1} = 0$, $\frac{\partial L}{\partial \lambda_2} = 0$

$\therefore$ $4 - 2x_1 - 4\lambda_1 - 3\lambda_2 = 0$ —— (1) $\qquad 9 - 2x_2 - 3\lambda_1 - 5\lambda_2 = 0$ —— (2)

$4x_1 + 3x_2 - 15 = 0$ $\longrightarrow$ (3) $\qquad 3x_1 + 5x_2 - 14 = 0$ —— (4)

$\therefore 4 \times (1) + 3 \times (2) \Rightarrow$

(we want eqn in the form of (3))

$$\begin{aligned} & 16 - 8x_1 - 16\lambda_1 - 12\lambda_2 = 0 \\ + \; & 27 - 6x_2 - 9\lambda_1 - 15\lambda_2 = 0 \\ \hline & 43 - 8x_1 - 6x_2 - 25\lambda_1 - 27\lambda_2 = 0 \end{aligned}$$

$\Rightarrow 43 - 2(4x_1 + 3x_2) - 25\lambda_1 - 27\lambda_2 = 0$

$\Rightarrow 43 - 2(15) - 25\lambda_1 - 27\lambda_2 = 0$ --- from (3)

$\Rightarrow 25\lambda_1 + 27\lambda_2 = 13$ —— (5)

Now $3 \times (1) + 5 \times (2) \Rightarrow$

(in the form of (4))

$$\begin{aligned} & 12 - 6x_1 - 12\lambda_1 - 9\lambda_2 = 0 \\ + \; & 45 - 10x_2 - 15\lambda_1 - 25\lambda_2 = 0 \\ \hline & 57 - 6x_1 - 10x_2 - 27\lambda_1 - 34\lambda_2 = 0 \end{aligned}$$

$\Rightarrow 57 - 2(3x_1 + 5x_2) - 27\lambda_1 - 34\lambda_2 = 0$

$\Rightarrow 57 - 2(14) - 27\lambda_1 - 34\lambda_2 = 0$

$\Rightarrow 27\lambda_1 + 34\lambda_2 = 29$ —— (6)

$27\times(5) - 25\times(6) \Rightarrow \quad -121\lambda_2 = -374 \quad \rightarrow \boxed{\lambda_2 = \frac{374}{121}}$

$\& \; 25\lambda_1 = 13 - 27\left(\frac{374}{121}\right) = \frac{-8521}{121} \quad \Rightarrow \boxed{\lambda_1 = -\frac{341}{121}}$

from (1), $2x_1 = 4 - 4\lambda_1 - 3\lambda_2 = 4 - 4\left(\frac{-341}{121}\right) - 3\left(\frac{374}{121}\right) = \frac{726}{121} = 6$

$\therefore \boxed{x_1 = 3}$

from (2), $2x_2 = 9 - 3\lambda_1 - 5\lambda_2 = 9 - 3\left(\frac{-341}{121}\right) - 5\left(\frac{374}{121}\right) = \frac{242}{121} = 2$

$\therefore \boxed{x_2 = 1}$ (or solve eq^n (3) & (4) simultaneously)

$\therefore X_0 = (3, 1)$ is the stationary point.

Now $z = 4x_1 + 9x_2 - x_1^2 - x_2^2$

$\therefore \frac{\partial z}{\partial x_1} = 4 - 2x_1, \quad \frac{\partial^2 z}{\partial x_1^2} = -2, \quad \frac{\partial^2 z}{\partial x_1 \partial x_2} = 0$

$\frac{\partial z}{\partial x_2} = 9 - 2x_2, \quad \frac{\partial^2 z}{\partial x_2^2} = -2, \quad \frac{\partial^2 z}{\partial x_2 \partial x_1} = 0$

$\therefore$ The Hessian matrix $H = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

$\therefore A_1 = [-2], \quad A_2 = H$

$\therefore D_1 = -2, \quad D_2 = 4 \quad$ i.e. $D_1 < 0, \; D_2 > 0 \quad \therefore X_0$ is maxima

$\& \; z = 4(3) + 9(1) - (3)^2 - (1)^2 = 11 \quad \Rightarrow \boxed{z_{max} = 11}$

2) optimise $z = x_1^2 + x_2^2 + x_3^2$

subject to $x_1 + x_2 + x_3 = 13$

$3x_1 + x_2 + x_3 = 27$

$x_1, x_2, x_3 \geq 0$

$\rightarrow$ Let $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$

$h_1(x_1, x_2, x_3) = x_1 + x_2 + x_3 - 13$

$h_2(x_1, x_2, x_3) = 3x_1 + x_2 + x_3 - 27$

The Lagrangian function is

$L(x_1, x_2, x_3, \lambda_1, \lambda_2) = f(x_1, x_2, x_3) - \lambda_1 h_1(x_1, x_2, x_3) - \lambda_2 h_2(x_1, x_2, x_3)$

$\therefore L(x_1, x_2, x_3, \lambda_1, \lambda_2) = x_1^2 + x_2^2 + x_3^2 - \lambda_1(x_1 + x_2 + x_3 - 13) - \lambda_2(3x_1 + x_2 + x_3 - 27)$

$\therefore \dfrac{\partial L}{\partial x_1} = 2x_1 - \lambda_1 - 3\lambda_2$, $\dfrac{\partial L}{\partial x_2} = 2x_2 - \lambda_1 - \lambda_2$,

$\dfrac{\partial L}{\partial x_3} = 2x_3 - \lambda_1 - \lambda_2$, $\dfrac{\partial L}{\partial \lambda_1} = -(x_1 + x_2 + x_3 - 13)$, $\dfrac{\partial L}{\partial \lambda_2} = -(3x_1 + x_2 + x_3 - 27)$

Solving, $\dfrac{\partial L}{\partial x_1} = 0$, $\dfrac{\partial L}{\partial x_2} = 0$, $\dfrac{\partial L}{\partial x_3} = 0$, $\dfrac{\partial L}{\partial \lambda_1} = 0$, $\dfrac{\partial L}{\partial \lambda_2} = 0$

$\therefore$ $2x_1 - \lambda_1 - 3\lambda_2 = 0$ —— (1) $2x_2 - \lambda_1 - \lambda_2 = 0$ —— (2)

$2x_3 - \lambda_1 - \lambda_2 = 0$ —— (3) $x_1 + x_2 + x_3 = 13$ —— (4) $3x_1 + x_2 + x_3 = 27$ —— (5)

(Now to find values of $x_1, x_2, x_3, \lambda_1, \lambda_2$)

$(1) + (2) + (3) \Rightarrow 2x_1 + 2x_2 + 2x_3 - \lambda_1 - \lambda_1 - \lambda_1 - 3\lambda_2 - \lambda_2 - \lambda_2 = 0$

$\Rightarrow 2(x_1 + x_2 + x_3) - 3\lambda_1 - 5\lambda_2 = 0$

$\Rightarrow 2(13) - 3\lambda_1 - 5\lambda_2 = 0$ --- from (4)

$\Rightarrow 3\lambda_1 + 5\lambda_2 = 26$ —— (6)

$3 \times (1) + (2) + (3) \Rightarrow 6x_1 + 2x_2 + 2x_3 - 3\lambda_1 - \lambda_1 - \lambda_1 - 9\lambda_2 - \lambda_2 - \lambda_2 = 0$

$\Rightarrow 2(3x_1 + x_2 + x_3) - 5\lambda_1 - 11\lambda_2 = 0$

$\Rightarrow 2(27) - 5\lambda_1 - 11\lambda_2 = 0$

$\Rightarrow 5\lambda_1 + 11\lambda_2 = 54$ —— (7)

$5 \times (6) - 3 \times (7) \Rightarrow$

$$\begin{array}{r} 15\lambda_1 + 25\lambda_2 = 130 \\ -\;\; 15\lambda_1 + 33\lambda_2 = 162 \\ \hline -8\lambda_2 = -32 \end{array} \Rightarrow \boxed{\lambda_2 = 4}$$

$(6) \Rightarrow 3\lambda_1 = 26 - 5(4) = 6 \Rightarrow \boxed{\lambda_1 = 2}$

$(1) \Rightarrow 2x_1 = \lambda_1 + 3\lambda_2 = 2 + 12 = 14 \Rightarrow \boxed{x_1 = 7}$

$(2) \Rightarrow 2x_2 = \lambda_1 + \lambda_2 = 2 + 4 = 6 \Rightarrow \boxed{x_2 = 3}$

$(3) \Rightarrow 2x_3 = \lambda_1 + \lambda_2 = 2 + 4 = 6 \Rightarrow \boxed{x_3 = 3}$

Prof. Anushri Tambe

Now, $\frac{\partial h_1}{\partial x_1} = 1$, $\frac{\partial h_1}{\partial x_2} = 1$, $\frac{\partial h_1}{\partial x_3} = 1$

$\frac{\partial h_2}{\partial x_1} = 3$, $\frac{\partial h_2}{\partial x_2} = 1$, $\frac{\partial h_2}{\partial x_3} = 1$

$\frac{\partial^2 L}{\partial x_1^2} = 2$, $\frac{\partial^2 L}{\partial x_1 \partial x_2} = 0$, $\frac{\partial^2 L}{\partial x_1 \partial x_3} = 0$

$\frac{\partial^2 L}{\partial x_2^2} = 2$, $\frac{\partial^2 L}{\partial x_2 \partial x_1} = 0$, $\frac{\partial^2 L}{\partial x_2 \partial x_3} = 0$

$\frac{\partial^2 L}{\partial x_3^2} = 2$, $\frac{\partial^2 L}{\partial x_3 \partial x_1} = 0$, $\frac{\partial^2 L}{\partial x_3 \partial x_2} = 0$

$$\therefore P = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \frac{\partial h_1}{\partial x_3} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \frac{\partial h_2}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} , \quad P' = \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{\partial^2 L}{\partial x_1^2} & \frac{\partial^2 L}{\partial x_1 \partial x_2} & \frac{\partial^2 L}{\partial x_1 \partial x_3} \\ \frac{\partial^2 L}{\partial x_2 \partial x_1} & \frac{\partial^2 L}{\partial x_2^2} & \frac{\partial^2 L}{\partial x_2 \partial x_3} \\ \frac{\partial^2 L}{\partial x_3 \partial x_1} & \frac{\partial^2 L}{\partial x_3 \partial x_2} & \frac{\partial^2 L}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore H_0^B = \left[\begin{array}{c|c} 0 & P \\ \hline P' & Q \end{array}\right] = \left[\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 1 & 1 \\ \hline 1 & 3 & 2 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & 2 \end{array}\right]$$

Here $n = 3$ (number of variables)

$m = 2$ (number of constraints)

$\therefore n - m = 1$ & $2m+1 = 5$

Use <u>Laplace</u> <u>method</u> to find determinant of H_0^B

Prof. Anushri Tambe

$$\therefore H_0^B = (-1)^{3+4+1} \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \begin{vmatrix} 1 & 3 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} + (-1)^{3+5+1} \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} \begin{vmatrix} 1 & 3 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 0 \end{vmatrix}$$

$$+ (-1)^{4+5+1} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} 1 & 3 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= (1-3)\left[1(2)-3(2)\right] - (1-3)\left[1(-2)-3(-2)\right] + (1-1)\left[+2(0)\right]$$

$$= (-2)(-4) - (-2)(4) + 0$$

$$\boxed{H_0^B = 16}$$

as H_0^B is positive, $X_0 = (7, 3, 3)$ is a minima.

& $Z_{min} = (7)^2 + (3)^2 + (3)^2 = 67$ $\quad \therefore \boxed{Z_{min} = 67}$

Practice Problems:-

Using Lagrangian multiplier solve the following NLPP.

1) Maximise $z = 6x_1 + 8x_2 - x_1^2 - x_2^2$

subject to $4x_1 + 3x_2 = 16$

$3x_1 + 5x_2 = 15$

$x_1, x_2 \geq 0$

2) optimise $z = 2x_1^2 + 3x_2^2 + x_3^2$

subject to $x_1 + x_2 + 2x_3 = 13$

$2x_1 + x_2 + x_3 = 10$

$x_1, x_2, x_3 \geq 0$

(Laplace Method:- consider H_0^B from eg. (2). Multiply the determinant $\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}$ by the determinant obtained by deleting rows & col^{n}s in which these elements lie. i.e. by $\begin{vmatrix} 1 & 3 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$.
The sign is determined by $(-1)^{3+4+1}$ as the elements of $\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}$ lie in the third & forth column. Similarly take the product of all other determinants with proper sign]

Prof. Anushri Tambe