Cayley - Hamilton theorem

Every square matrix satisfies its characteristic equation.

① Verify Cayley-Hamilton theorem for the matrix A & hence find A^{-1}, A^{-2} & A^4 where

$$A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

Prove that $A^{-1} = A^2 - 5A + 9I$.

Soln:-

$$\lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0.$$

T.P $A^3 - 5A^2 + 9A - I = 0$

$$\begin{pmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{pmatrix} - 5\begin{pmatrix} -1 & +12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{pmatrix} + 9\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 0$$

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To find A^{-1} & A^{-2}

WKT $A^3 - 5A^2 + 9A - I = 0 \rightarrow (1)$

$\times$ by A^{-1}

$$A^2 - 5A + 9I - A^{-1} = 0$$

$$\Rightarrow A^{-1} = A^2 - 5A + 9I$$

$$= \begin{pmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{pmatrix} - 5\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix} + 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}$$

Multiply (1) by A^{-2}

$$A - 5I + 9A^{-1} - A^{-2} = 0$$

$$A^{-2} = A - 5I + 9A^{-1}.$$

$$= \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 9\begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 23 & 20 & 52 \\ 8 & 7 & 18 \\ 18 & -16 & 41 \end{pmatrix}$$

Multiply (1) by A

$$A^4 - 5A^3 + 9A^2 - A = 0$$

$$A^4 = 5A^3 - 9A^2 + A$$

$$= 5\begin{pmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{pmatrix} - 9\begin{pmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 32 & -40 & -23 \end{pmatrix}$$

② Verify cayley-Hamilton thm and hence find the matrix represented by $A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I$ where A is

$$\begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$$

Soln:- $\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$

T.P $A^3 - 7A^2 + 16A - 12I = 0$

$$= \begin{pmatrix} -8 & 15 & -10 \\ -52 & -157 & -118 \\ 92 & 270 & 208 \end{pmatrix} - 7\begin{pmatrix} 4 & 25 & 10 \\ -2 & -31 & -26 \\ 20 & 50 & 44 \end{pmatrix} + 16\begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix} - 12\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$= 0$.

$$
\begin{array}{r|l}
 & A^3 + A^2 \\
\hline
A^3 - 7A^2 + 16A - 12I & A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I \\
 & A^6 - 7A^5 + 16A^4 - 12A^3 \\
 & \quad - \quad + \quad - \quad + \\
\hline
 & A^5 - 7A^4 + 16A^3 - 12A^2 \\
 & A^5 - 7A^4 + 16A^3 - 12A^2 \\
 & \quad - \quad + \quad - \quad + \\
\hline
 & \qquad 2A - I
\end{array}
$$

$$A^6 - 6A^5 + 9A^4 + 4A^3 - 12A^2 + 2A - I =$$

$$(A^3 + A^2)(A^3 - 7A^2 + 16A - 12I) + 2A - I$$

$$= 2A - I$$

$$= 2\begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 20 & 10 \\ -4 & -7 & -8 \\ 6 & 10 & 13 \end{pmatrix}$$

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Exercise

(3) Find the characteristic equation of the matrix A given below and hence find the matrix represented by $A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$ where

$$A=\begin{pmatrix}2&1&1\\0&1&0\\1&1&2\end{pmatrix}$$

Soln:-

$$\lambda^3-5\lambda^2+7\lambda-3=0$$

T.P $A^3-5A^2+7A-3I=0$

$$A^5+A$$

$A^3-5A^2+7A-3I$) $A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$

$A^8-5A^7+7A^6-3A^5$

$-\quad+\quad-\quad+$

$A^4-5A^3+8A^2-2A$

$A^4-5A^3+7A^2-3A$

$-\quad+\quad-\quad+$

A^2+A+I

$$=(A^3-5A^2+7A-3I)(A^5+A)+A^2+A+I$$

$$=(0)+A^2+A+I=A^2+A+I$$

$$= \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{pmatrix}$$

(4) Verify that the matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ satisfies the characteristic eqn. Hence find A^{-2}.

$A^3 + A^2 - 5A - 5I = 0$ $\quad A^{-2} = \frac{1}{5}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(5) Verify Cayley-Hamilton theorem and find A^{-1} for $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$. Hence find $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ in terms of A.

$A + 5I$