MATHEMATICAL EXPECTATION:

Using the pdf or the distribution function, we can obtain the average value of the r.v. X.

The average value, called the **expectation** or **the expected value** or the **mean** gives an idea of where the values of the r.v. X are concentrated.

Mean is a first order quantity.

To know how the individual values are scattered around the mean, we consider the **variance** and its positive square root, the **standard deviation**.

Variance is a second order quantity.

Definition [Expectation] : Suppose X is a **discrete** r.v. with probability mass function $p_X = P(X = x)$, the **expectation** or the **expected value** of X is defined as:

$$E(X) = \sum_X x p_X.$$

If X is a **continuous** r.v. with p.d.f. $f(x)$ then the expectation or the expected value of X is defined as:

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

(Here $-\infty$ denotes the lower limit and ∞ denotes the upper limit)

REMARKS:

(i) If Y = g(X) is a r.v. (i.e. Y is a function of X), then

$$E(Y) = E(g(X)) = \begin{cases} \sum_X g(x) p_X, & \text{if } X \text{ is discrete} \\ \int_X g(x) f(x) dx, & \text{if } X \text{ is continuous} \end{cases}$$

This implies that

$$E(X^2) = \begin{cases} \sum_X x^2 p_X, & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x^2 f(x) dx, & \text{if } X \text{ is continuous} \end{cases}$$

(ii) Expectation of a constant is that constant itself i.e. E(a) = a, if a is a constant

Examples

1. A r.v. X has the following distribution. Find k and the mean.

X	-1	0	1	2	3
P(X=x)	0.2	0.1	k	2k	0.1

Solution: Since $p(x)$ is a probability mass function, we have

$$\sum_X p_X = 1$$
$$\Rightarrow 0.2 + 0.1 + k + 2k + 0.1 = 1$$
$$\Rightarrow 3k = 1 - 0.4$$
$$\Rightarrow k = 0.2$$

Therefore, the probability distribution of X is:

X	-1	0	1	2	3
P(X=x)	0.2	0.1	0.2	0.4	0.1

Therefore, the mean = E(X) is given by,

$$E(X) = \sum_X x p_X$$
$$= (-1)(0.2) + (0)(0.1) + (1)(0.2) + (2)(0.4) + (3)(0.1)$$
$$= 1.1$$

2. If the mean of the following distribution is 16, find m and n.

X	8	12	16	20	24
P(X=x)	1/8	m	n	1/4	1/12

Solution: Since $p(x)$ is a probability mass function, we have

$$\sum_X p_X = 1$$
$$\Rightarrow \frac{1}{8} + m + n + \frac{1}{4} + \frac{1}{12} = 1$$
$$\Rightarrow m + n = \frac{13}{24} ......(1)$$

Also Mean = E(X)=16

$$\Rightarrow \sum_X x p_X = 16$$
$$\Rightarrow 8 * \frac{1}{8} + 12m + 16n + 20 * \frac{1}{4} + 24 * \frac{1}{12} = 16$$
$$\Rightarrow 12m + 16n = 8.....(2)$$

Solving (1) and (2), we get, $n = \dfrac{9}{24}; m = \dfrac{4}{24}$.

3. A fair coin is tossed until a head appears. What is the expectation of the number of tosses required?

Solution: Let the random variable X denote the number of tosses required till a head appears. Then X is a discrete r.v taking values 1,2,3,… Therefore the probability distribution of X is given by

X	1	2	3	4	5	…	
$P(X=x)$	1/2	$(1/2)^2$	$(1/2)^3$	$(1/2)^4$	$(1/2)^5$	…	

$$Then\ E(X)=\sum_{x} xp_x$$

$$=1\cdot\frac{1}{2}+2.\left(\frac{1}{2}\right)^2+3.\left(\frac{1}{2}\right)^3+4.\left(\frac{1}{2}\right)^4+5.\left(\frac{1}{2}\right)^5+\ldots$$

$$=\frac{1}{2}\left(1+2.\left(\frac{1}{2}\right)+3.\left(\frac{1}{2}\right)^2+4.\left(\frac{1}{2}\right)^3+5.\left(\frac{1}{2}\right)^4+\ldots\right)$$

$$=\frac{1}{2}\left(\frac{1}{(1-1/2)^2}\right)$$

$$=2$$