Beliefs and Sequential Equilibrium in Imperfect Information Games

1. Formal Definitions

Sequential Equilibrium:

- A **Sequential Equilibrium** is a refinement of Nash Equilibrium applied to extensive-form games, particularly when players have imperfect information.
- It consists of:
    1. **A strategy profile**: A set of strategies, one for each player, specifying the actions to take at every possible point in the game.
    2. **A belief system**: A probability distribution over the nodes of the game tree, specifying the beliefs of each player about the game's state at each decision point.

Belief System:

- A **belief system** is a set of beliefs that each player holds at every information set, representing their understanding of where they are in the game.
- Beliefs are updated using **Bayes' rule** wherever possible, based on the observed actions of other players.

2. Mathematical Notation and Diagrams

Game Trees:

- **Game trees** are graphical representations of extensive-form games that illustrate the sequence of moves, players' choices, and information sets. They are essential for understanding Sequential Equilibrium.
- **Example:** In the Signaling Game, nodes represent different stages of the game (e.g., Candidate choosing education level, Employer deciding whether to hire). Arrows show possible actions, and branches represent different paths the game can take.

Notation:

- Let $\sigma=(\sigma 1,\sigma 2)$ represent a strategy profile for two players.
- Beliefs are denoted by $\mu(I)$, where I is an information set.
- Sequential rationality requires that each player's strategy maximizes their expected utility given their beliefs and the strategies of others.

3. Expanded Examples

Example 1: Signaling Game (with expanded steps)

- **Game Tree Setup:**

- Player 1 (Candidate) chooses $E \in \{EH, EL\}$ where EH is high education and EL is low education.
- Player 2 (Employer) observes E and decides whether to hire (H) or not (N).

Beliefs and Equilibrium:

- Suppose μ(H|EH) represents the employer's belief that the candidate has high ability given they chose high education.
- Sequential equilibrium requires that if Player 1 chooses EH, Player 2's belief μ(H|EH) should be high enough to make hiring optimal.
- If the employer's belief μ(H|EH) is sufficiently high, they will hire, supporting the candidate's decision to choose EH.

Example 2: Entry Deterrence Game (with expanded steps)

- **Game Tree Setup:**
  - Player 1 (Entrant) chooses whether to enter the market (In) or stay out (Out).
  - Player 2 (Incumbent) decides to fight (F) or accommodate (A) after observing the entry.

Beliefs and Equilibrium:

- If the entrant believes that the incumbent is more likely to accommodate (belief μ(A|In)is high), they will choose to enter.
- The incumbent, knowing this, might fight if they believe it will deter future entries, creating a self-fulfilling equilibrium.

4. Connection to Core Concepts

- **Nash Equilibrium:** Sequential Equilibrium refines Nash Equilibrium by ensuring that players' strategies are optimal given their beliefs at every point in the game.
- **Bayesian Nash Equilibrium:** In games with imperfect information, players' strategies are based on their beliefs, which are updated as the game progresses.
- **Subgame Perfect Equilibrium:** Sequential Equilibrium requires strategies to be optimal even in off-the-equilibrium-path subgames, further refining the concept of subgame perfection.

5. Real-World Applications

- **Economics:** Signaling games apply to markets where firms signal their quality through pricing or advertising.
- **Politics:** Entry deterrence games model situations where countries or firms deter rivals through threats of retaliation.
- **Negotiations:** Sequential equilibria are used to understand bargaining processes where parties form beliefs about each other's willingness to compromise.

6. Critical Analysis

- **Limitations of Sequential Equilibrium:**
  - **Rationality Assumptions:** Sequential Equilibrium assumes perfect rationality, which may not hold in real-world scenarios.
  - **Complexity:** Calculating beliefs and strategies can be complex, especially in games with large information sets.
  - **Dependence on Beliefs:** The equilibrium heavily depends on how beliefs are formed and updated, which can be challenging to model accurately.