3) If $L[\sin\sqrt{t}] = \frac{\sqrt{\pi}}{s\sqrt{s}} e^{-1/4s}$ Find $L[\sin 2\sqrt{t}]$

Solⁿ Given, $L[\sin\sqrt{t}] = \frac{\sqrt{\pi}}{s\sqrt{s}} e^{-1/4s} = \phi(s)$

By Change of Scale property, $L[f(at)] = \frac{1}{a}\phi\left(\frac{s}{a}\right)$

Here $f(t) = \sin\sqrt{t} \Rightarrow f(at) = \sin\sqrt{at}$

$\therefore L[\sin 2\sqrt{t}] = L[\sin\sqrt{4t}] = \frac{1}{4}\phi\left(\frac{s}{4}\right)$ — here $a=4$

$$= \frac{1}{4}\frac{\sqrt{\pi}}{\frac{s}{4}\sqrt{\frac{s}{4}}} e^{-1/4 s/4}$$

$$= \frac{2\sqrt{\pi}}{s\sqrt{s}} e^{-1/s}$$

4) $L[\text{erf } 3\sqrt{t}]$

Solⁿ As $L[\text{erf}\sqrt{t}] = \frac{1}{s\sqrt{s+1}} = \phi(s)$

By Change of Scale property, $L[f(at)] = \frac{1}{a}\phi\left(\frac{s}{a}\right)$

Here, $f(t) = \text{erf}\sqrt{t} \Rightarrow f(at) = \text{erf}\sqrt{at}$

$\therefore L[\text{erf } 3\sqrt{t}] = L[\text{erf}\sqrt{9t}] = \frac{1}{9}\phi\left(\frac{s}{9}\right)$ — here $a=9$

$$= \frac{1}{9}\frac{1}{\frac{s}{9}\sqrt{\frac{s}{9}+1}}$$

$$= \frac{1}{s\sqrt{\frac{s+9}{9}}}$$

$$= \frac{3}{s\sqrt{s+9}}$$

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5) $L[e^{-t}\sqrt{t}]$

Sol^n $L[\sqrt{t}] = L[t^{1/2}] = \dfrac{\Gamma(\frac{1}{2}+1)}{s^{\frac{1}{2}+1}} = \dfrac{\Gamma(3/2)}{s^{3/2}}$

$= \dfrac{\frac{1}{2}\Gamma(\frac{1}{2})}{s^{3/2}}$ — using $\Gamma n = (n-1)\Gamma(n-1)$

$= \dfrac{\sqrt{\pi}}{2s^{3/2}} = \phi(s)$

By First shifting theorem $L[e^{-at}f(t)] = \phi(s+a)$

$\therefore L[e^{-t}\sqrt{t}] = \phi(s+1)$ — here $a=1$

$= \dfrac{\sqrt{\pi}}{2(s+1)^{3/2}}$

6) $L[e^{2t}\sin^2 t]$

Sol^n $L[\sin^2 t] = L\left[\dfrac{1-\cos 2t}{2}\right]$

$= \dfrac{1}{2}L[1-\cos 2t] = \dfrac{1}{2}\left(\dfrac{1}{s} - \dfrac{s}{s^2+4}\right) = \phi(s)$

$\therefore L[e^{2t}\sin^2 t] = \phi(s-2)$ — by First shifting theorem

$= \dfrac{1}{2}\left[\dfrac{1}{(s-2)} - \dfrac{(s-2)}{(s-2)^2+4}\right]$

7) $L[\cos 3t \cosh 4t]$

Sol^n $L[\cos 3t \cosh 4t] = L\left[\cos 3t\left(\dfrac{e^{4t}+e^{-4t}}{2}\right)\right]$

$= \dfrac{1}{2}L[e^{4t}\cos 3t + e^{-4t}\cos 3t]$ — (1)

$L[\cos 3t] = \dfrac{s}{s^2+9} = \phi(s)$

From (1)

$$L[\cos 3t \cosh 4t] = \frac{1}{2} L\left[e^{4t}\cos 3t + e^{-4t}\cos 3t\right]$$

$$= \frac{1}{2}\left[\phi(s-4) + \phi(s+4)\right] \text{ — by first shifting theorem}$$

$$= \frac{1}{2}\left[\frac{(s-4)}{(s-4)^2+9} + \frac{(s+4)}{(s+4)^2+9}\right]$$

8) $L\left[e^{-3t}(1+t)^2\right]$

Soln $L\left[(1+t)^2\right] = L(1+2t+t^2)$

$$= \frac{1}{s} + 2\frac{\Gamma 2}{s^2} + \frac{\Gamma 3}{s^3}$$

$$= \frac{1}{s} + \frac{2(1!)}{s^2} + \frac{2!}{s^3} = \frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3} = \phi(s)$$

$$\therefore L\left[e^{-3t}(1+t)^2\right] = \phi(s+3) \text{ — by first shifting theorem}$$

$$= \frac{1}{(s+3)} + \frac{2}{(s+3)^2} + \frac{2}{(s+3)^3}$$

9) If $L[f(t)] = \dfrac{1}{s(s^2+1)}$ find $L\left[e^{-t} f(2t)\right]$

Soln Given, $L[f(t)] = \dfrac{1}{s(s^2+1)} = \phi(s)$

$$\therefore L[f(2t)] = \frac{1}{2}\phi\left(\frac{s}{2}\right) \text{ — by change of scale property (here } a=2)$$

$$= \frac{1}{2} \frac{1}{\frac{s}{2}\left(\left(\frac{s}{2}\right)^2+1\right)} = \frac{1}{s\left(\frac{s^2}{4}+1\right)} = \frac{4}{s(s^2+4)} = \phi_1(s)$$

$$\therefore L\left[e^{-t} f(2t)\right] = \phi_1(s+1) \text{ — by first shifting theorem}$$

$$= \frac{4}{(s+1)\left((s+1)^2+4\right)}$$

10) $L[\sin 2t \cos t \cosh 2t]$

Soln $L[\sin 2t \cos t \cosh 2t] = L\left[\sin 2t \cos t \left(\frac{e^{2t}+e^{-2t}}{2}\right)\right]$

$$= \frac{1}{2} L\left[e^{2t}\sin 2t \cos t + e^{-2t}\sin 2t \cos t\right] \quad \text{——— (1)}$$

$$L[\sin 2t \cos t] = L\left[\frac{\sin(2t+t)+\sin(2t-t)}{2}\right]$$

$$= \frac{1}{2} L[\sin 3t + \sin t]$$

$$= \frac{1}{2}\left[\frac{3}{s^2+9} + \frac{1}{s^2+1}\right] = \phi(s)$$

From (1)

$$L[\sin 2t \cos t \cosh 2t] = \frac{1}{2} L\left[e^{2t}\sin 2t \cos t + e^{-2t}\sin 2t \cos t\right]$$

$$= \frac{1}{2}\left[\phi(s-2) + \phi(s+2)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\left(\frac{3}{(s-2)^2+9} + \frac{1}{(s-2)^2+9}\right) + \frac{1}{2}\left(\frac{3}{(s+2)^2+9} + \frac{1}{(s+2)^2+9}\right)\right]$$

$$= \frac{1}{4}\left[\frac{3}{(s-2)^2+9} + \frac{1}{(s-2)^2+9} + \frac{3}{(s+2)^2+9} + \frac{1}{(s+2)^2+9}\right]$$

11) $L[e^{-2t} \cosh 2t \sin t]$

Soln $L[e^{-2t} \cosh 2t \sin t] = L\left[e^{-2t}\left(\frac{e^{2t}+e^{-2t}}{2}\right)\sin t\right]$

$$= \frac{1}{2}\left[(e^{0} + e^{-4t})\sin t\right]$$

$$= \frac{1}{2}\left[\sin t + e^{-4t}\sin t\right] \quad \text{——— (1)}$$

$$L[\sin t] = \frac{1}{s^2+1} = \phi(s)$$

$\therefore$ from ①

$$L[e^{-2t}\cosh 2t \sin t] = \frac{1}{2} L[\sin t + e^{-4t}\sin t]$$

$$= \frac{1}{2}\left[\frac{1}{s^2+1} + \phi(s+4)\right]$$ — by first shifting thm.

$$= \frac{1}{2}\left[\frac{1}{s^2+1} + \frac{1}{(s+4)^2+1}\right]$$

12) $L[t\, e^{2t} \sin 2t \cos t]$

Solⁿ: $L[t\, e^{2t}\sin 2t \cos t] = L[e^{2t}\, t \sin 2t \cos t]$

$$L[\sin 2t \cos t] = L\left[\frac{\sin 3t + \sin t}{2}\right]$$

$$= \frac{1}{2} L[\sin 3t + \sin t]$$

$$= \frac{1}{2}\left[\frac{3}{s^2+9} + \frac{1}{s^2+1}\right] = \phi(s)$$

$$L[t \sin 2t \cos t] = (-1)\frac{d}{ds}\phi(s)$$

$$= -\left[\frac{d}{ds}\frac{1}{2}\left(\frac{3}{s^2+9} + \frac{1}{s^2+1}\right)\right]$$

$$= -\frac{1}{2}\left[\frac{d}{ds}\left(\frac{3}{s^2+9} + \frac{1}{s^2+1}\right)\right]$$

$$= -\frac{1}{2}\left[\frac{(s^2+9)(0)-3(2s)}{(s^2+9)^2} + \frac{(s^2+1)(0)-(2s)}{(s^2+1)^2}\right]$$

$$= -\frac{1}{2}\left[\frac{(-6s)}{(s^2+9)^2} + \frac{(-2s)}{(s^2+1)^2}\right]$$ — by $\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$

$$= \frac{1}{2}\left[\frac{6s}{(s^2+9)^2} + \frac{2s}{(s^2+1)^2}\right] = \phi_1(s)$$

$\therefore L[e^{2t} \sin 2t \cos t] = \phi_1(s-2)$ —— by first shifting thm

$$= \frac{1}{2}\left[\frac{6(s-2)}{[(s-2)^2+9]^2} + \frac{2(s-2)}{[(s-2)^2+1]^2}\right]$$

$$= \frac{3(s-2)}{[(s-2)^2+9]^2} + \frac{(s-2)}{[(s-2)^2+1]^2}$$

13) $L[t^2 e^{-t} \sin 4t]$

Soln $L[t^2 e^{-t} \sin 4t] = L[e^{-t} t^2 \sin 4t]$

$L(\sin 4t) = \frac{4}{s^2+16} = \phi(s)$

$$\therefore L(t^2 \sin 4t) = (-1)^2 \frac{d^2}{ds^2} \phi(s)$$

$$= \frac{d}{ds}\left[\frac{d}{ds} \frac{4}{s^2+16}\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2+16)(0) - 4(2s)}{(s^2+16)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-8s}{(s^2+16)^2}\right] = -8\frac{d}{ds}\left[\frac{s}{(s^2+16)^2}\right]$$

$$= -8\left[\frac{(s^2+16)^2(1) - s\,2(s^2+16)(2s)}{(s^2+16)^4}\right]$$

$$= -8(s^2+16)\left[\frac{s^2+16-4s^2}{(s^2+16)^4}\right]$$

$$= -8\left[\frac{16-3s^2}{(s^2+16)^3}\right] = \phi_1(s)$$

$$L[e^{-t} t^2 \sin 4t] = \phi_1(s+1) = -8\left[\frac{16-3(s+1)^2}{((s+1)^2+16)^3}\right]$$

—— by first shifting theorem

14) $L[t e^{-3t} \cos 2t \cos 3t]$

Soln $L[t e^{-3t} \cos 2t \cos 3t] = L[e^{-3t}\, t \cos 2t \cos 3t]$

$$L[\cos 2t \cos 3t] = L\left[\frac{\cos(5t) + \cos(-t)}{2}\right]$$

$$= \frac{1}{2} L[\cos 5t + \cos t]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+25} + \frac{s}{s^2+1}\right] = \phi(s)$$

$$L[t \cos 2t \cos 3t] = (-1)\frac{d}{ds}\phi(s)$$

$$= -\frac{d}{ds}\left[\frac{1}{2}\left(\frac{s}{s^2+25} + \frac{s}{s^2+1}\right)\right]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{s}{s^2+25} + \frac{s}{s^2+1}\right]$$

$$= \frac{-1}{2}\left[\frac{(s^2+25)(1) - s(2s)}{(s^2+25)^2} + \frac{(s^2+1)(1) - s(2s)}{(s^2+1)^2}\right]$$

$$= \frac{-1}{2}\left[\frac{s^2+25-2s^2}{(s^2+25)^2} + \frac{s^2+1-2s^2}{(s^2+1)^2}\right]$$

$$= \frac{-1}{2}\left[\frac{25-s^2}{(s^2+25)^2} + \frac{1-s^2}{(s^2+1)^2}\right] = \phi_1(s)$$

$$L[e^{-3t}\, t \cos 2t \cos 3t] = \phi_1(s+3)$$

$$= \frac{-1}{2}\left[\frac{25-(s+3)^2}{[(s+3)^2+25]^2} + \frac{1-(s+3)^2}{[(s+3)^2+1]^2}\right]$$

15) $L[t e^{-2t} \sinh 4t]$

Soln $L[t e^{-2t} \sinh 4t] = L\left[t e^{-2t}\left(\frac{e^{4t} - e^{-4t}}{2}\right)\right]$

$$= \frac{1}{2} L\left[t\left(e^{2t} - e^{-6t}\right)\right]$$

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$$= \frac{1}{2} L\left[e^{2t} t - e^{-6t} t\right] \text{ ——— } ①$$

$$L(t) = \frac{\Gamma 2}{s^2} = \frac{1!}{s^2} = \frac{1}{s^2} = \phi(s)$$

$$\therefore L\left[t e^{-2t} \sinh 4t\right] = \frac{1}{2} L\left[e^{2t} t - e^{-6t} t\right] \text{ ——— from } ①$$

$$= \frac{1}{2}\left[\phi(s-2) - \phi(s+6)\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s-2)^2} - \frac{1}{(s+6)^2}\right] \text{ — by first shifting theorem.}$$

16) $L\left[t e^{-3t} \operatorname{erf}(\sqrt{t})\right]$

Soln

$$L\left[t e^{-3t} \operatorname{erf}\sqrt{t}\right] = L\left[e^{-3t} t \operatorname{erf}\sqrt{t}\right]$$

$$L\left[\operatorname{erf}\sqrt{t}\right] = \frac{1}{s\sqrt{s+1}} = \phi(s)$$

$$L\left[t \operatorname{erf}\sqrt{t}\right] = (-1) \frac{d}{ds} \phi(s)$$

$$= -\frac{d}{ds}\left[\frac{1}{s\sqrt{s+1}}\right]$$

$$= -\frac{d}{ds}\left[s^{-1} (s+1)^{-1/2}\right]$$

$$= -\left[s^{-1}\left(\frac{-1}{2}\right)(s+1)^{-1/2-1} + (s+1)^{-1/2} (-1) s^{-2}\right]$$

$$= -\left[s^{-1}\left(\frac{-1}{2}\right)(s+1)^{-3/2} + (s+1)^{-1/2} (-1) s^{-2}\right]$$

$$= \left[\frac{1}{2s(s+1)^{3/2}} + \frac{1}{s^2(s+1)^{1/2}}\right] = \phi_1(s)$$

$$L\left[e^{-3t} t \operatorname{erf}\sqrt{t}\right] = \phi_1(s+3) = \left[\frac{1}{2(s+3)(s+3+1)^{3/2}} + \frac{1}{(s+3)^2 (s+3+1)^{1/2}}\right]$$

$$\therefore L\left[e^{-3t}\, t\, erf\sqrt{t}\right] = \frac{1}{2(s+3)(s+4)^{3/2}} + \frac{1}{(s+3)^2(s+4)^{1/2}}$$

17) $L\left[t\sqrt{1+\sin 2t}\right]$

Sol^n

$$L\left[\sqrt{1+\sin 2t}\right] = L\left[\sqrt{\cos^2 t+\sin^2 t+2\sin t\cos t}\right]$$

$$= L\left[\sqrt{(\cos t+\sin t)^2}\right]$$

$$= L[\cos t+\sin t]$$

$$= \frac{s}{s^2+1} + \frac{1}{s^2+1} = \frac{s+1}{s^2+1} = \phi(s)$$

$$L\left[t\sqrt{1+\sin 2t}\right] = (-1)\frac{d}{ds}\phi(s)$$

$$= -\frac{d}{ds}\left[\frac{s+1}{s^2+1}\right]$$

$$= -\left[\frac{(s^2+1)(1)-(s+1)(2s)}{(s^2+1)^2}\right]$$

$$= -\left[\frac{s^2+1-2s^2-2s}{(s^2+1)^2}\right]$$

$$= -\left[\frac{1-s^2-2s}{(s^2+1)^2}\right]$$

18) $L\left[\frac{1}{t}e^{2t}\sin t\right]$

Sol^n $L\left[e^{2t}\frac{1}{t}\sin t\right]$

$$L[\sin t] = \frac{1}{s^2+1} = \phi(s)$$

$$L\left[\frac{1}{t}\sin t\right] = \int_s^\infty \phi(s)\,ds = \int_s^\infty \frac{1}{s^2+1}\,ds$$

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$$L\left[\frac{1}{t}\sin t\right] = \left[\tan^{-1}s\right]_s^{\infty}$$

$$= \tan^{-1}\infty - \tan^{-1}s$$

$$= \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s = \phi_1(s)$$

$$L\left[e^{2t}\frac{1}{t}\sin t\right] = \phi_1(s-2) = \cot^{-1}(s-2)$$

— by first shifting theorem

19) $L\left[\frac{2\sin t \sin 2t}{t}\right]$

Sol^n $L[2\sin t \sin 2t] = L[\cos(-t) - \cos(3t)]$

$$= L[\cos t - \cos 3t]$$

$$= \frac{s}{s^2+1} - \frac{s}{s^2+9} = \phi(s)$$

$$L\left[\frac{2\sin t \sin 2t}{t}\right] = \int_s^{\infty} \phi(s)\, ds$$

$$= \int_s^{\infty}\left(\frac{s}{s^2+1} - \frac{s}{s^2+9}\right) ds$$

$$= \frac{1}{2}\int_s^{\infty}\left(\frac{2s}{s^2+1} - \frac{2s}{s^2+9}\right) ds$$

$$= \frac{1}{2}\left[\log(s^2+1) - \log(s^2+9)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+1}{s^2+9}\right)\right]_s^{\infty}$$

$$= \frac{1}{2}\left\{\log\left[\frac{s^2(1+1/s^2)}{s^2(1+9/s^2)}\right]\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\log(1) - \log\left(\frac{1+1/s^2}{1+9/s^2}\right)\right]$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{s^2+1}{s^2+9}\right)\right]$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+1}{s^2+9}\right)^{-1}\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2+9}{s^2+1}\right)$$

20) $L\left[\frac{e^{-2t}\cosh t \sin 2t}{t}\right]$

Soln: $L\left[\frac{e^{-2t}\cosh t \sin 2t}{t}\right] = L\left[\frac{e^{-2t}}{t}\left(\frac{e^{t}+e^{-t}}{2}\right)\sin 2t\right]$

$$= \frac{1}{2}L\left[\frac{1}{t}(e^{-t}+e^{-3t})\sin 2t\right]$$

$$= \frac{1}{2}L\left[e^{-t}\frac{\sin 2t}{t} + e^{-3t}\frac{\sin 2t}{t}\right] \quad \text{—— (1)}$$

$$L(\sin 2t) = \frac{2}{s^2+4} = \phi(s)$$

$$L\left[\frac{\sin 2t}{t}\right] = \int_s^\infty \phi(s)\,ds = \int_s^\infty \frac{2}{s^2+4}\,ds$$

$$= 2\int_s^\infty \frac{1}{s^2+2^2}\,ds$$

$$= \frac{\cancel{2}}{\cancel{2}}\left[\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}\left(\frac{s}{2}\right)$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{2}\right)$$

$$= \cot^{-1}\left(\frac{s}{2}\right) = \phi_1(s)$$

From (1)

$$L\left[\frac{e^{-2t}\cosh t \sin 2t}{t}\right] = \frac{1}{2}L\left[e^{-t}\frac{\sin 2t}{t} + e^{-3t}\frac{\sin 2t}{t}\right]$$

$$= \frac{1}{2}\left[\phi_1(s+1) + \phi_1(s+3)\right]$$

$$= \frac{1}{2}\left[\cot^{-1}(s+1) + \cot^{-1}(s+3)\right]$$

21) $L\left[\frac{1-\cos t}{t^2}\right]$

Soln $L[1-\cos t] = \frac{1}{s} - \frac{s}{s^2+1} = \phi(s)$

$$L\left[\frac{1}{t}(1-\cos t)\right] = \int_s^\infty \phi(s)\,ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds$$

$$= \int_s^\infty \left(\frac{1}{s} - \frac{1}{2}\frac{2s}{s^2+1}\right) ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[2\log s - \log(s^2+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log s^2 - \log(s^2+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2}{s^2+1}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2}{s^2(1+1/s^2)}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log(1) - \log\left(\frac{1}{1+1/s^2}\right)\right]$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{s^2}{s^2+1}\right)\right]$$

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$$= \frac{1}{2} \log\left(\frac{s^2}{s^2+1}\right)^{-1}$$

$$= \frac{1}{2} \log\left(\frac{s^2+1}{s^2}\right) = \phi_1(s)$$

$$L\left[\frac{1}{t}\left(\frac{1}{t}(1-\cos t)\right)\right] = \int_s^{\infty} \phi_1(s)\, ds$$

$$= \frac{1}{2}\int_s^{\infty} \log\left(\frac{s^2+1}{s^2}\right) 1 \, ds$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+1}{s^2}\right)\int 1\, ds - \int\left[\int 1\, ds\right]\frac{d}{ds}\log\left(\frac{s^2+1}{s^2}\right) ds\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+1}{s^2}\right) s - \int s \frac{1}{\left(\frac{s^2+1}{s^2}\right)} \frac{d}{ds}\left(\frac{s^2+1}{s^2}\right) ds\right]_s^{\infty}$$

$$= \frac{1}{2}\left[s\log\left(\frac{s^2+1}{s^2}\right) - \int \frac{s(s^2)}{s^2+1}\left(\frac{s^2(2s)-(s^2+1)(2s)}{s^4}\right) ds\right]_s^{\infty}$$

$$= \frac{1}{2}\left[s\log\left(\frac{s^2+1}{s^2}\right) - \int \frac{s^3\cancel{s}(2s^2-2s^2-2)}{(s^2+1)\, s^4} ds\right]$$

$$= \frac{1}{2}\left[s\log\left(\frac{s^2+1}{s^2}\right) + 2\int \frac{1}{s^2+1} ds\right]_s^{\infty}$$

$$= \frac{1}{2}\left[s\log\left(\frac{s^2+1}{s^2}\right) + 2\tan^{-1} s\right]_s^{\infty}$$

$$= \frac{1}{2}\left[0 + 2\tan^{-1}\infty - s\log\left(\frac{s^2+1}{s^2}\right) - 2\tan^{-1} s\right]$$

$$= \frac{1}{2}\left[\frac{2\pi}{2} - s\log\left(\frac{s^2+1}{s^2}\right) - 2\tan^{-1} s\right]$$

$$= \frac{1}{2}\left[\pi - s\log\left(\frac{s^2+1}{s^2}\right) - 2\tan^{-1} s\right]$$

22) Find $L[f'(t)]$ where $f(t) = t \quad 0 \le t \le 3$

$\qquad\qquad\qquad\qquad\qquad\qquad = 6 \; ; \quad t > 3$

Sol^n $L[f'(t)] = -f(0) + sL[f(t)]$ ——— (1)

as $f(t) = t \quad 0 \le t \le 3$

$f(0) = 0$ ——— (2)

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^3 e^{-st} f(t)\,dt + \int_3^\infty e^{-st} f(t)\,dt$$

$$= \int_0^3 e^{-st}\, t\,dt + \int_3^\infty e^{-st}\, 6\,dt$$

$$= \int_0^3 t\, e^{-st}\,dt + 6\int_3^\infty e^{-st}\,dt$$

$$= \left[t\left(\frac{e^{-st}}{(-s)}\right) - (1)\left(\frac{e^{-st}}{(-s)^2}\right)\right]_0^3 + 6\left[\frac{e^{-st}}{-s}\right]_3^\infty$$

$$= \left[3\frac{e^{-3s}}{-s} - \frac{e^{-3s}}{s^2} - 0 + \frac{1}{s^2}\right] + 6\left[0 - \frac{e^{-3s}}{-s}\right]$$

$$= \left[\frac{-3e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2} + \frac{6e^{-3s}}{s}\right]$$

$$= \left[\frac{3e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2}\right] \quad \text{——— (3)}$$

Substituting (2) & (3) in (1) we get,

$$L[f'(t)] = -(0) + s\left[\frac{3e^{-3s}}{s} - \frac{e^{-3s}}{s^2} + \frac{1}{s^2}\right]$$

$$= 3e^{-3s} - \frac{e^{-3s}}{s} + \frac{1}{s}$$

23) $L\left[\frac{d}{dt}\left(\frac{\sin 2t}{t}\right)\right]$

Sol^n We know $L\left[\frac{d}{dt} f(t)\right] = L[f'(t)] = -f(0) + s\,L[f(t)]$ —— ①

Here, $f(t) = \frac{\sin 2t}{t}$

As $f(0) = \frac{\sin 0}{0} = \frac{0}{0}$. Hence, $f(0) = \lim_{t\to 0} \frac{\sin 2t}{t} = \frac{0}{0}$

By L'Hospital Rule

$= \lim_{t\to 0} \frac{(\sin 2t)'}{(t)'}$

$= \lim_{t\to 0} \frac{2\cos 2t}{(1)}$

$f(0) = 2\cos(0) = 2$ —— ②

$L[f(t)] = L\left[\frac{\sin 2t}{t}\right]$

$L[\sin 2t] = \frac{2}{s^2+4} = \phi(s)$

$\therefore L\left[\frac{\sin 2t}{t}\right] = \int_s^\infty \phi(s)\,ds = \int_s^\infty \frac{2}{s^2+4}\,ds$

$= 2\int_s^\infty \frac{1}{s^2+2^2}\,ds$

$= \frac{2}{2}\left[\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$

$= \tan^{-1}\infty - \tan^{-1}\left(\frac{s}{2}\right)$

$= \pi/2 - \tan^{-1}\left(\frac{s}{2}\right) = \cot^{-1}\left(\frac{s}{2}\right)$ —— ③

Substituting ② & ③ in ① we get

$L\left[\frac{d}{dt}\left(\frac{\sin 2t}{t}\right)\right] = -2 + s\cot^{-1}\left(\frac{s}{2}\right)$

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24) $L\left[\int_0^t e^{-3u} \sin 4u \, du\right]$

Sol^n $L[\sin 4u] = \dfrac{4}{s^2+16} = \phi(s)$

$L[e^{-3u} \sin 4u] = \phi(s+3) = \dfrac{4}{(s+3)^2+16} = \phi_1(s)$

$L\left[\int_0^t e^{-3u} \sin 4u \, du\right] = \dfrac{1}{s}\phi_1(s) = \dfrac{1}{s}\left[\dfrac{4}{(s+3)^2+16}\right]$

25) $L\left[e^{-2t} \int_0^t u \sin 3u \, du\right]$

Sol^n $L[\sin 3u] = \dfrac{3}{s^2+9} = \phi(s)$

$L[u \sin 3u] = (-1)\dfrac{d}{ds}\phi(s) = -\left[\dfrac{d}{ds}\left(\dfrac{3}{s^2+9}\right)\right]$

$= -\left[\dfrac{(s^2+9)(0) - 3(2s)}{(s^2+9)^2}\right]$

$= -\left[\dfrac{-6s}{(s^2+9)^2}\right]$

$= \dfrac{6s}{(s^2+9)^2} = \phi_1(s)$

$L\left[\int_0^t u \sin 3u \, du\right] = \dfrac{1}{s}\phi_1(s) = \dfrac{1}{\cancel{s}} \dfrac{6\cancel{s}}{(s^2+9)^2} = \dfrac{6}{(s^2+9)^2} = \phi_2(s)$

$\therefore L\left[e^{-2t} \int_0^t u \sin 3u \, du\right] = \phi_2(s+2)$

$= \dfrac{6}{[(s+2)^2+9]^2}$

26) $L\left[\cosh t \int_0^t \cosh u \, e^u \, du\right]$

Sol^n $L\left[\cosh t \int_0^t \cosh u \, e^u \, du\right]$

$= L\left[\left(\frac{e^t+e^{-t}}{2}\right)\int_0^t \left(\frac{e^u+e^{-u}}{2}\right)e^u \, du\right]$

$= \frac{1}{4} L\left[(e^t+e^{-t})\int_0^t (e^u \cdot e^u + e^{-u} \cdot e^u)\,du\right]$

$= \frac{1}{4} L\left[e^t \int_0^t (e^{2u}+1)\,du + e^{-t}\int_0^t (e^{2u}+1)\,du\right]$ —— (1)

$L\left[e^{2u}+1\right] = \frac{1}{s-2} + \frac{1}{s} = \phi(s)$

$L\left[\int_0^t (e^{2u}+1)\,du\right] = \frac{1}{s}\phi(s) = \frac{1}{s}\left[\frac{1}{(s-2)} + \frac{1}{s}\right] = \phi_1(s)$

$\therefore$ From (1)

$L\left[\cosh t \int_0^t \cosh u \, e^u \, du\right]$

$= \frac{1}{4} L\left[e^t \int_0^t (e^{2u}+1)\,du + e^{-t}\int_0^t (e^{2u}+1)\,du\right]$

$= \frac{1}{4}\left[\phi_1(s-1) + \phi_1(s+1)\right]$

$= \frac{1}{4}\left[\frac{1}{(s-1)}\left(\frac{1}{s-1-2} + \frac{1}{s-1}\right) + \frac{1}{(s+1)}\left(\frac{1}{s+1-2} + \frac{1}{s+1}\right)\right]$

$= \frac{1}{4}\left[\frac{1}{(s-1)}\left(\frac{1}{s-3} + \frac{1}{s-1}\right) + \frac{1}{(s+1)}\left(\frac{1}{s-1} + \frac{1}{s+1}\right)\right]$

27) Prove that $L[\operatorname{erf}\sqrt{t}] = \frac{1}{s\sqrt{s+1}}$

Sol^n $\operatorname{erf}\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx$

put $x^2 = u \Rightarrow x = u^{1/2}$

$\Rightarrow dx = \frac{1}{2}u^{-1/2}\,du$

as $x: 0 \longrightarrow \sqrt{t}$ $\quad u: 0 \longrightarrow t$

$$\operatorname{erf}\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^t e^{-u}\frac{1}{2}u^{-1/2}\,du$$

$$= \frac{2}{\sqrt{\pi}}\frac{1}{2}\int_0^t e^{-u}u^{-1/2}\,du$$

$$= \frac{1}{\sqrt{\pi}}\int_0^t e^{-u}u^{-1/2}\,du$$

$$L[\operatorname{erf}\sqrt{t}] = \frac{1}{\sqrt{\pi}}L\left[\int_0^t e^{-u}u^{-1/2}\,du\right]$$

$$L[u^{-1/2}] = \frac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}} = \frac{\Gamma\frac{1}{2}}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}} = \phi(s)$$

$$L[e^{-u}u^{-1/2}] = \phi(s+1) = \frac{\sqrt{\pi}}{\sqrt{s+1}} = \phi_1(s)$$

$$L\left[\int_0^t e^{-u}u^{-1/2}\,du\right] = \frac{1}{s}\phi_1(s) = \frac{1}{s}\left(\frac{\sqrt{\pi}}{\sqrt{s+1}}\right)$$

$$\therefore L[\operatorname{erf}\sqrt{t}] = \frac{1}{\sqrt{\pi}}L\left[\int_0^t e^{-u}u^{-1/2}\,du\right]$$

$$= \frac{1}{\cancel{\sqrt{\pi}}}\frac{1}{s}\frac{\cancel{\sqrt{\pi}}}{\sqrt{s+1}}$$

$$= \frac{1}{s\sqrt{s+1}}$$