MATRICES

Eigen Values and Eigen Vectors

An eigenvalue is a scalar λ which satisfies the equation $A_{nxn}X_{nx1} = \lambda X_{nx1}$ and a non-zero X_{nx1} is called the corresponding eigenvector.

Problem

01) Verify that $X = [2 \;\; 3 \;\; -2 \;\; -3]^T$ is an eigenvector corresponding to the

eigen value $\lambda=2$ of the matrix $A=\begin{bmatrix} 1 & -4 & -1 & -4 \\ 2 & 0 & 5 & -4 \\ -1 & 1 & -2 & 3 \\ -1 & 4 & -1 & 6 \end{bmatrix}$

Homework

02) If X is an eigenvector of A corresponding to an eigenvalue λ then prove that

kX is also an eigenvector corresponding to the same eigenvalue.

03) Prove that distinct eigenvalues of a matrix have distinct eigenvectors.

Eigen Values and Eigen Vectors

Let A be n x n matrix, λ a scalar and, I the unit matrix of same order as A.
$|A - \lambda I| = 0$ is called the **characteristic equation** of the matrix A.

For $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ the characteristic equation is given by

$$\lambda^3 - (a_{11} + a_{22} + a_{33})\lambda^2 + \left\{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right\}\lambda - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

The roots λ of this equation are called **characteristic roots** or **eigen values** of A .

If there exists a non-zero vector X such that $|A-\lambda I|=0$ then X said to be a **Characteristic/eigen vector** of a matrix A, corresponding to the **eigen value** λ.

Problems

4) Show that the following matrices have the same characteristic equation

$$\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\begin{bmatrix} b & c & a \\ c & a & b \\ a & b & c \end{bmatrix}\begin{bmatrix} c & a & b \\ a & b & c \\ b & c & a \end{bmatrix}$$

Homework

5) Find the characteristic equation of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Note

Sum of the eigenvalues of A = sum of the diagonal elements of A(trace of A)
Product of the eigenvalues of A = determinant of A

Problem

6) Find the sum and product of the eigenvalues of A= $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Homework

7) Find the sum & product of the eigenvalues of A= $\begin{bmatrix} -2 & -9 & 5 \\ -5 & -10 & 7 \\ -9 & -21 & 14 \end{bmatrix}$

Properties

The eigenvalues of a triangular/diagonal matrix are its diagonal elements.
The eigenvalues of an orthogonal matrix are +1 or -1
Eigenvalues of a Hermitian/symmetric matrix are all real
0 is an eigenvalue of a matrix A if and only if A is singular.
λ^n is an eigenvalue of A^n if λ is an eigenvalue of A.

λ^{-1} is an eigenvalue of A^{-1} if λ is an eigenvalue of A.

$\frac{|A|}{\lambda}$ is an eigenvalue of $adj\,A$ if λ is an eigenvalue of non-singular A.

(D-07)
Eigen vectors corresponding to distinct eigenvalues are linearly independent.

Problems

8) Find the eigen values and the eigen vectors of

(a) $A=\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ **(M-12)**(b) $A=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 3 & 2 & 3 \end{bmatrix}$ **(M-11)(c)** $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$

9) Find eigen values of $A^2 -3A+4I$ and eigen vectors of adjA where A= $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$

10) Find the sum and product of the eigenvalues of $\begin{bmatrix} 4 & 0 & 0 & 1 \\ 2 & -1 & 3 & 0 \\ 1 & 1 & 1 & 0 \\ -4 & 2 & -6 & 1 \end{bmatrix}$ **(M-10)**

Homework

11) Find eigen values of adj A and eigen vectors of A^{-1} where A=

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

12) Find eigenvalues & eigenvectors of adj A & A^3 where $A=\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ **(M-10,D-08)**

13) Find eigen values & eigen vectors of of A^3+I where $A=\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ **(M-15,D-11, M-09)**

14) Find the eigen values of the adjoint of the matrix $A=\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ **(D-15)**

Cayley-Hamilton Theorem

Every square matrix A satisfies its own characteristic equation that is $\lvert A-\lambda I\rvert=0$

Problems

15) Verify Cayley-Hamilton theorem for the following matrices

a) A= $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and hence find A^{-1} and $2A^5-3A^4+A^2-4I$

b) $A=\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and hence find A^{-1} and $A^5-2A^4+3A^3+A$ **(D-08)**

c) $A=\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and hence find A^{-2} **(M-12)**

Homework

16) Verify Cayley-Hamilton theorem for the following matrix and hence find A^{-1} & A^4

(a) $A=\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ find $A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$

(b) A= $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$ **(D-11, M-09)** **(c) A=** $\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 0 & 2 & 1 \end{bmatrix}$ **(M-11)**

(d) Find $A^5-4A^4-7A^3+11A^2-A-10I$ interms of A where $A=\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ **(M-14,D-14)**

(e) Find the characteristic equation of the matrix $A=\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ and hence find the matrix represented by $A^6-6A^5+9A^4+4A^3-12A^2+2A-I$ **(D-15)**

Minimal polynomial & Derogatory/Non-derogatory matrix

The monic polynomial of lowest degree which annihilates the matrix A is called the minimal polynomial of the matrix A.
Minimal polynomial is a divisor of the characteristic polynomial.
If the minimal polynomial is of degree lesser than order of the matrix then the matrix is said to be a derogatory matrix; otherwise it is called a non-derogatory matrix.
A matrix with distinct eigenvalues is non-derogatory.

Problems

17) S.T. the following matrices are derogatory & find the minimal polynomial of the matrices

(a) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ $(b)\begin{bmatrix} 1 & -2 & 0 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$

18) Show that the following matrices are non-derogatory

$(a)\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ $(b)\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

Homework

19) Determine if the following matrices are derogatory or non-derogatory

(a) $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$

Functions of a square matrix

If A is a matrix of order of **2** then a function of the matrix A is given by $\lambda_1 \text{and} \lambda_2$ $f(A) = a_0 A + a_1 I$ $f(A)=a_0 A + a_1 I$ where $a_0 \text{and} a_1$ are obtained by solving simultaneously the equation $f(\lambda) = a_0\lambda + a_1$ for the two eigenvalues $\lambda_1 \text{and} \lambda_2$

(If the eigenvalue repeats the second equation is given by $f'(\lambda) = a_0$)

If A is a matrix of order of **3** then a function of the matrix A is given by $f(A)=a_0 A^2 + a_1 A + a_2 I$ where $a_0, a_1 \text{and} a_2$ are obtained by solving simultaneously the equation $f(\lambda) = a_0\lambda^2 + a_1\lambda + a_2$ for the three eigenvalues $\lambda_1, \lambda_2, \text{and} \lambda_3$ (If the eigenvalue repeat the other equations are given by taking derivatives of the equation $f(\lambda) = a_0\lambda^2 + a_1\lambda + a_2$)

Problems

20) If $A=\begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}$ find $3A^{57}+2A^{18}$

21) If $A=\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ find sin A

22)If $A=\begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$ find A^{50} **(D-14)**

Homework

23) If $A=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ find A^{50} 23) If $A=\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ find A^{50} (M-14)

24) If $A=\begin{bmatrix} \frac{\pi}{2} & \frac{3\pi}{2} \\ \pi & \pi \end{bmatrix}$ find cosA

25) If $A=\begin{bmatrix} -1 & 4 \\ 2 & 1 \end{bmatrix}$ then prove that $3\tan A = A\tan 3$ **(M-14)**

26) Find e^A and 4^A if A= $A=\begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$ **(D-15)**

Algebraic multiplicity and Geometric multiplicity

Algebraic multiplicity is the number of times an eigenvalue occurs and geometric multiplicity is the number of corresponding linearly independent eigenvectors.

Algebraic multiplicity ≥ Geometric multiplicity

Similarity

A & B are said to be similar if there exists a non-singular matrix P such that $P^{-1}AP=B$

Problems

27) Prove that similar matrices have the same eigen values.

28) Show that A= $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}$ are not similar matrices

Homework

29) Prove that AB and BA have the same eigen values, if A or B is non-singular.

30) Show that $A=\begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ are similar matrices

Diagonalization

If a matrix A is similar to a diagonal matrix D then it is said to be diagonalizable.

Condition that a matrix A is diagonalizable is that for each eigenvalue of A

Algebraic multiplicity = Geometric multiplicity.

Then $P^{-1}AP=D$ where diagonal elements of D are the eigenvalues of A and the corresponding eigenvectors are the columns of P.

A matrix with distinct eigenvalues is always diagonalizable.

Problems

31) Show that the following matrices A are diagonalizable and find the diagonalizing (or modal)matrix P and the diagonal(or spectral) matrix D in each case

(a) $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$

(d) $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ (e) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ **(M-15)**

32) Show that the following matrices A are not diagonalizable

(a) $\begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$ $(b)\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ $(c)\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$

$(d)\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $(e)\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

Homework

33) Find a matrix P which diagonalizes A= $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ and verify that $P^{-1}AP=D$

34) Determine if the following matrices are diagonalizable

(a) $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$ $(b)\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ $(c)\begin{bmatrix} -17 & 18 & -6 \\ -18 & 19 & -6 \\ -9 & 9 & 2 \end{bmatrix}$

35) If A= $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 0 \\ \frac{1}{2} & 2 \end{bmatrix}$ then show that both A and B are not diagonalizable

but AB is diagonalizable

36) Diagonalize the Hermitian matrix $\begin{bmatrix} -3 & 2+2i \\ 2-2i & 4 \end{bmatrix}$

Orthogonal similarity and Orthogonal Reduction

A and B are said to be orthogonally similar if there exist an orthogonal matrix P such that B= $P^{-1}AP = P^TAP$

A symmetric matrix A is always orthogonally similar to a diagonal matrix D such that D= $P^{-1}AP = P^TAP$ where the diagonal elements of D are the eigenvalues of A and the corresponding normalized eigenvectors are the columns of P.

This is known as orthogonal reduction of symmetric matrix A to diagonal matrix D.

Problems

37)Find the orthogonal matrix P that will diagonalize the following symmetric matrix A and also find the diagonal matrix D

(a) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ **(M-14,D-14)** (b) $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

38) If A is symmetric matrix of order 3 and the eigenvalues are $\lambda_1=0, \lambda_2=3, \lambda_3=15$ and the corresponding eigenvectors are $X_1=[1,2,2]^T$ for $\lambda_1=0$, $X_2=[-2,-1,2]^T$ for $\lambda_2=3$ and X_3 for $\lambda_3=15$ .Find X_3 $AX_1, AX_2, A^{10}X_3$.

Homework

39) Find the orthogonal matrix P that will diagonalize the following symmetric matrix A and also find the diagonal matrix D

$(a)\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

40) If A is symmetric matrix of order 3 and the eigenvalues are $\lambda_1=8, \lambda_2=2, \lambda_3=2$ and the corresponding eigenvectors are $X_1=[2,-1,1]^T$ for $\lambda_1=8$, $X_2=[-1/2,0,1]^T$ for $\lambda_2=2$ and X_3 for $\lambda_3=2$ .Find X_3 $AX_1, AX_2, A^{10}X_3$.

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