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Complex Variables.

★ Complex function:-

If $z = x + iy$ be a complex quantity then $\omega = f(z)$ which is again a complex quantity is called as complex function. (z & ω both are complex.)

As $z = x + iy$, one can write ω in the following form

$\omega = f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$

Here, u & v are functions of x & y.

Hence any complex function one can separate into real & img. parts.

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★ Analytic function:-
If a function $w=f(z)$ is defined & differentiable at each point of a domain D then it is called as analytic function or regular function or holomorphic function of z in the Domain D.

★ Singular point:-
If a function $f(z)$ is not analytic at a point in a domain D. then the point is called as singular point.
eg. $f(z)=\frac{1}{z-1}$ over $\mathbb{C}$

at $z=1$ $f(z)$ is not differentiable $\Rightarrow f(z)$ is not analytic at pt. $z=1 \Rightarrow z=1$ is singular point.

★ Cauchy-Riemann equations in Cartesian Co-ordinates:-
The necessary & sufficient conditions for a continuous function $w=f(z)=u(x,y)+iv(x,y)$ to be analytic in a region R are
i) u_x, u_y, v_x, v_y are continuous functions of x&y in region R.
ii) $u_x=v_y$ & $u_y=-v_x$ at each point of R

★ $\boxed{u_x=v_y \text{ \& } u_y=-v_x}$ these eq^ns are called as Cauchy-Riemann eq^ns or C-R. eq^ns.

★ When $f(z)$ is analytic, it's derivative is given by one of the following expression.

$\boxed{f'(z)=u_x+iv_x}$ or $f'(z)=v_y+iv_x$ or
$f'(z)=u_x-iu_y$ or $f'(z)=v_y-iu_y$

[Generally we'll use 1st expression i.e $f'(z)=u_x+iv_x$.]

★ Note:-

1) If $f(z)$ is analytic then it can be differentiated in usual manner eg. $f(z)=z^2 \Rightarrow f'(z)=2z$ as $f(z)=z^2$ is analytic function

2) C.R. eqⁿs are only necessary conditions for a function to be analytic. This means even if C.R. eqⁿs are satisfied the function need not be analytic at that point

3) If C-R eqⁿs are not satisfied then $f(z)$ is not analytic i.e $f'(z)$ does not exist.

★ Conjugate functions:-

If $w=f(z)=u+iv$ & $f(z)$ is analytic then the functions u & v are called as conjugate functions.

Problems:-

1) Show that the following functions are analytic & find their derivatives.

1) x^2-y^2+2ixy

$f(z)=x^2-y^2+i2xy \Rightarrow u=x^2-y^2$ & $v=2xy$

$u_x=2x,\ u_y=-2y,\ v_x=2y,\ v_y=2x$.

$\therefore u_x, u_y, v_x, v_y$ are all continuous.

Also $u_x=v_y$ & $u_y=-v_x$ i.e C-R eqⁿs are satisfied.

Hence $f(z)=x^2-y^2+i2xy$ is analytic.

$f'(z)=u_x+iv_x=2x+i2y=2(x+iy)=2z$

[Note:-
Here we are using above expression to find $f'(z)$ as given $f(z)$ is in the form of x & y.
otherwise if the function is analytic one can find the derivative in usual manner].

$\sin iy = i\sinh y$

2) $f(z) = \cos z$.

$f(z) = \cos z = \cos(x+iy) = \cos x \cos iy - \sin x \sin iy$
$= \cos x \cosh y - i \sin x \sinh y$

$\Rightarrow u = \cos x \cosh y \quad v = -\sin x \sinh y$

$u_x = -\sin x \cosh y$, $u_y = \cos x \sinh y$, $v_x = -\cos x \sinh y$,
$v_y = -\sin x \cosh y$

$\Rightarrow u_x, u_y, v_x, v_y$ are all continuous functions.

Also $u_x = v_y$ & $u_y = -v_x$ i.e. C-R eq^n's are satisfied

$\Rightarrow f(z) = \cos z$ is analytic.

$\therefore f'(z) = -\sin z$.

[Or $f'(z) = u_x + iv_x = -\sin x \cosh y - i \cos x \sinh y$.
$= -(\sin x \cos iy + \cos x \sin iy)$
$= -(\sin(x+iy)) = -\sin z$.]

3) ze^{2z}.

$f(z) = ze^{2z} = (x+iy)e^{2(x+iy)}$
$= (x+iy)\, e^{2x+2iy} = (x+iy)\, e^{2x} \cdot e^{i2y}$
$= e^{2x}(x+iy)(\cos 2y + i\sin 2y)$
$= e^{2x}[x\cos 2y + ix\sin 2y + iy\cos 2y - y\sin 2y]$
$= e^{2x}[x\cos 2y - y\sin 2y] + i\, e^{2x}[x\sin 2y + y\cos 2y]$

$\therefore u = e^{2x}[x\cos 2y - y\sin 2y] \quad v = e^{2x}[x\sin 2y + y\cos 2y]$

$u_x = e^{2x}[\cos 2y] + [x\cos 2y - y\sin 2y]\,2e^{2x}$
$= e^{2x}[\cos 2y + 2x\cos 2y - 2y\sin 2y]$

$u_y = e^{2x}[\cancel{2x}\sin 2y - (2y\cos 2y + \sin 2y)]$
$= -e^{2x}[2x\sin 2y + 2y\cos 2y + \sin 2y]$

$v_x = e^{2x}[\sin 2y] + [x\sin 2y + y\cos 2y]\,2e^{2x}$
$= e^{2x}[\sin 2y + 2x\sin 2y + 2y\cos 2y]$

$v_y = e^{2x}[2x\cos 2y - 2y\sin 2y + \cos 2y]$

$\Rightarrow u_x, u_y, v_x, v_y$ are all continuous.
Also $u_x = v_y$ & $u_y = -v_x$ i.e C-R eq^{n}s are satisfied
$\Rightarrow f(z) = ze^{z}$ is analytic.
& $f'(z) = 2ze^{z} + e^{z}$

H.W. Show that following functions are analytic & also find their derivatives. e^z, $\sinh z$, $\sin z$.

1) e^z.

$f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$
$\therefore u = e^x \cos y \quad v = e^x \sin y$
$u_x = e^x \cos y, \quad u_y = -e^x \sin y, \quad v_x = e^x \sin y, \quad v_y = e^x \cos y$
$\Rightarrow u_x, u_y, v_x, v_y$ are all continuous function.
Also $u_x = v_y$ & $u_y = -v_x$ i.e C-R eq^{n}s are satisfied.
$\Rightarrow f(z) = e^z$ is analytic
$\Rightarrow f'(z) = e^z$.

2) $\sinh z$.

$f(z) = \sinh z = \sinh(x+iy) = \sinh x \cosh iy + \cosh x \sinh iy$
$= \sinh x \cos y + i \cosh x \sin y$.
$\therefore u = \sinh x \cos y, \quad v = \cosh x \sin y$
$u_x = \cosh x \cos y, \quad u_y = -\sinh x \sin y$
$v_x = \sinh x \sin y, \quad v_y = \cosh x \cos y$.
$\Rightarrow u_x, u_y, v_x, v_y$ are all continuous functions.
Also $u_x = v_y$ & $u_y = -v_x$ .i.e C-R eq^{n}s are satisfied.
$\Rightarrow f(z) = \sinh z$ is analytic.
$\therefore f'(z) = \cosh z$.

3) $\sin z$.

$f(z) = \sin z = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy$
$= \sin x \cosh y + i \cos x \sinh y$.
$u = \sin x \cosh y, \quad v = \cos x \sinh y$
$u_x = \cos x \cosh y, \quad u_y = \sin x \sinh y$

$v_x = -\sin x \sinh y$, $v_y = \cos x \cosh y$.
$\Rightarrow u_x, u_y, v_x, v_y$ are all continuous functions
Also $u_x = v_y$ & $u_y = -v_x$ i.e C-R eq^{n}s are satisfied.
$\Rightarrow f(z) = \sin z$ is analytic.
$\therefore f'(z) = \cos z$.

6) Determine the fun^n $(x^3 - 3xy^2 + 3x) + i(3x^2y - y^3 + 3y)$ is analytic & if so find its derivative

$\rightarrow$ $f(z) = (x^3 - 3xy^2 + 3x) + i(3x^2y - y^3 + 3y)$
$u = x^3 - 3xy^2 + 3x$ $\quad v = 3x^2y - y^3 + 3y$.
$u_x = 3x^2 - 3y^2 + 3$, $u_y = -6xy$.
$v_x = 6xy$, $v_y = 3x^2 - 3y^2 + 3$
$\Rightarrow u_x, u_y, v_x, v_y$ all are continuous.
Also $u_x = v_y$ & $u_y = -v_x$ i.e C-R eq^{n}s are satisfied.
$\Rightarrow f(z) = (x^3 - 3xy^2 + 3x) + i(3x^2y - y^3 + 3y)$ is analytic
$f'(z) = u_x + i v_x = (3x^2 - 3y^2 + 3) + i\, 6xy$ —— 1>

[Note: Suppose we want to find derivative in terms of z, then we use Milne-Thompson method. (we'll see this later). Using this we put $x = z$ & $y = 0$ in 1>].

$\therefore$ By Milne-Thompson method put $x = z$ & $y = 0$.
$f'(z) = 3z^2 + 3$

7) Show that $w = \dfrac{x}{x^2+y^2} - \dfrac{iy}{x^2+y^2}$ is analytic & find $\dfrac{dw}{dz}$ in terms of z.

$\rightarrow$

$w = \dfrac{x}{x^2+y^2} - \dfrac{iy}{x^2+y^2} \Rightarrow u = \dfrac{x}{x^2+y^2}$, $v = -\dfrac{y}{x^2+y^2}$

$u_x = \dfrac{(x^2+y^2)\cdot 1 - x \cdot 2x}{(x^2+y^2)^2} = \dfrac{-x^2+y^2}{(x^2+y^2)^2}$

$u_y = \dfrac{-2xy}{(x^2+y^2)^2}$

$v_x = \dfrac{2xy}{(x^2+y^2)^2}$

$v_y = -\left[\dfrac{(x^2+y^2).1 - 2y^2}{(x^2+y^2)^2}\right] = -\left[\dfrac{x^2-y^2}{(x^2+y^2)^2}\right] = \dfrac{-x^2+y^2}{(x^2+y^2)^2}$

$\Rightarrow u_x, u_y, v_x, v_y$ are all continuous functions except at z

[i.e. $(x=0, y=0)$] Also $u_x = v_y$ & $u_y = -v_x$ i.e C-R eq's are satisfied.

$\therefore w$ is analytic function except at $z=0$

$\therefore \dfrac{dw}{dz} = u_x + i v_x = \dfrac{-x^2+y^2}{(x^2+y^2)^2} + i\dfrac{2xy}{(x^2+y^2)^2} = \dfrac{-x^2+y^2+i2xy}{(x^2+y^2)^2}$

By Milne-Thompson method put $x=z$ & $y=0$.

$\therefore \dfrac{dw}{dz} = \dfrac{-z^2+0}{(z^2)^2} = -\dfrac{1}{z^2}$

8) If $f(z)$ is equal to a) $z^2-\bar{z}$, b) $\bar{z}$ c) $2x+ixy^2$ show that $f'(z)$ does not exist.

$\rightarrow$ a) $z^2-\bar{z}$

$f(z) = z^2-\bar{z} = (x+iy)^2 - (x-iy)$

$= x^2 + i2xy - y^2 - x + iy$

$= (x^2-y^2-x) + i(2xy+y)$

$u = (x^2-y^2-x) \quad v = 2xy+y$

$u_x = 2x-1, \quad u_y = -2y, \quad v_x = 2y, \quad v_y = 2x+1$

$\Rightarrow u_x \neq v_y$

$\Rightarrow$ C-R eq^{n}s are not satisfied $\Rightarrow f'(z)$ does not exist

H.W b) $f(z) = \bar{z} = x-iy \Rightarrow u = x, v = -y$

$\therefore u_x = 1, u_y = 0, v_x = 0, v_y = -1$

$\Rightarrow u_x \neq v_y \Rightarrow$ C-R eq^{n}s are not satisfied

$\Rightarrow f'(z)$ does not exist.

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$w = u + iv$

H.W. c) $f(z) = 2x + ixy^2 \Rightarrow u = 2x,\ v = xy^2$

$u_x = 2,\ u_y = 0,\ v_x = y^2,\ v_y = 2xy$

$\Rightarrow u_x \neq v_y$ & $u_y \neq -v_x$.

$\Rightarrow$ C-R eqns are not satisfied. $\Rightarrow f'(z)$ does not exist.

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14> Find a, b, c, d, e if

1> $f(z) = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3y - exy^3 + 4xy)$ is analytic

→ Here, $u = ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2$, $v = 4x^3y - exy^3 + 4xy$

$u_x = 4ax^3 + 2bxy^2 + 2dx$, $u_y = 2bx^2y + 4cy^3 - 4y$

$v_x = 12x^2y - ey^3 + 4y$ $v_y = 4x^3 - 3exy^2 + 4x$.

As $f(z)$ is analytic f(z) satisfies C-R eqns $u_x = v_y$ & $u_y = -v_x$

as $u_x = v_y \Rightarrow 4ax^3 + 2bxy^2 + 2dx = 4x^3 - 3exy^2 + 4x$

Comparing the coefficients we get,

$4a = 4 \Rightarrow \boxed{a = 1}$

$2b = -3e$ —— 1>

$2d = 4 \Rightarrow \boxed{d = 2}$

also $u_y = -v_x \Rightarrow 2bx^2y + 4cy^3 - 4y = -12x^2y + ey^3 - 4y$

Comparing the coefficients we get,

$2b = -12 \Rightarrow \boxed{b = -6}$ $\therefore$ from 1> $-12 = -3e \Rightarrow \boxed{e = 4}$

$4c = e$ —— 2>

$\Rightarrow 4c = 4 \Rightarrow \boxed{c = 1}$

$\therefore a = 1, \ b = -6, \ c = 1, \ d = 2, \ e = 4$.

H.W 2> $f(z) = (ax^3 + bxy^2 + 3x^2 + cy^2 + x) + i(dx^2y - 2y^3 + exy + y)$ is analytic.

→ Here, $u = ax^3 + bxy^2 + 3x^2 + cy^2 + x$ $v = dx^2y - 2y^3 + exy + y$

Given $f(z)$ is analytic $\Rightarrow f(z)$ satisfies C-R eqn

$\Rightarrow u_x = v_y$ & $u_y = -v_x$.

$u_x = 3ax^2 + by^2 + 6x + 1$ $u_y = 2bxy + 2cy$

$v_x = 2dxy + ey$ $v_y = dx^2 - 6y^2 + ex + 1$

As $u_x = v_y \Rightarrow 3ax^2 + by^2 + 6x + \cancel{1} = dx^2 - 6y^2 + ex + \cancel{1}$

Comparing the coefficients we get,

$3a = d$ —— 1> , $\boxed{b = -6}$, $\boxed{e = 6}$

Also, $u_y = -v_x \Rightarrow 2bxy + 2cy = -2dxy - ey$

Comparing the coefficients we get,

$2b = -2d \Rightarrow b = -d$. as $b = -6 \Rightarrow -d = -6 \Rightarrow \boxed{d = 6}$

$2c = -e$, as $e = 6 \Rightarrow 2c = -6 \Rightarrow \boxed{c = -3}$

from 1> $3a = d$ as $d = 6 \Rightarrow 3a = 6 \Rightarrow \boxed{a = 2}$

$\therefore a = 2, \ b = -6, \ c = -3, \ d = 6, \ e = 6$.

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HW15) Find a, b, c, d if $f(z) = x^2 + 2axy + by^2 + i(cx^2 + 2dxy + y^2)$ is analytic

Here, $u = x^2 + 2axy + by^2$; $v = cx^2 + 2dxy + y^2$

$\therefore$ As $f(z)$ is analytic it satisfies C-R eqns.

i.e $u_x = v_y$ & $u_y = -v_x$

$\therefore u_x = 2x + 2ay$, $u_y = 2ax + 2by$.

$v_x = 2cx + 2dy$, $v_y = 2dx + 2y$.

as $u_x = v_y \Rightarrow 2x + 2ay = 2dx + 2y$

Comparing the coefficients we get,

$2d = 2 \Rightarrow \boxed{d = 1}$ & $2a = 2 \Rightarrow \boxed{a = 1}$

Also, $u_y = -v_x \Rightarrow 2ax + 2by = -2cx - 2dy$.

Comparing the coefficients we get,

$2a = -2c \Rightarrow a = -c \Rightarrow 1 = -c \Rightarrow \boxed{c = -1}$

$2b = -2d \Rightarrow b = -d \Rightarrow \boxed{b = -1}$

$\therefore a = 1,\ b = -1,\ c = -1,\ d = 1$.

HW16) Find k such that $\frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{kx}{y}$ is analytic.

Let, $f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{kx}{y}$

$\therefore u = \frac{1}{2}\log(x^2+y^2)$ $v = \tan^{-1}\left(\frac{kx}{y}\right)$

as $f(z)$ is analytic, it satisfies C-R. eqns.

$\therefore u_x = v_y$ & $u_y = -v_x$

$u_x = \frac{\cancel{2}x}{\cancel{2}(x^2+y^2)} = \frac{x}{x^2+y^2}$, $u_y = \frac{\cancel{2}y}{\cancel{2}(x^2+y^2)} = \frac{y}{x^2+y^2}$

$v_x = \frac{1}{1+\left(\frac{kx}{y}\right)^2}\left(\frac{k}{y}\right) = \frac{y^{\cancel{2}}}{y^2+k^2x^2}\frac{k}{\cancel{y}} = \frac{ky}{y^2+k^2x^2}$

$v_y = \frac{1}{1+\left(\frac{kx}{y}\right)^2}\left(\frac{-kx}{y^2}\right) = \frac{\cancel{y^2}}{y^2+k^2x^2}\frac{(-kx)}{\cancel{y^2}} = \frac{-kx}{y^2+k^2x^2}$

as $u_x = v_y$ & $u_y = -v_x$

$\Rightarrow \frac{x}{x^2+y^2} = \frac{-kx}{x^2+y^2}$ & $\frac{y}{x^2+y^2} = -\frac{ky}{y^2+k^2x^2}$.

which will be true when $\boxed{k = -1}$.

★ Cauchy - Riemann Eq^ns in Polar Co-ordinates.

* Prove the C-R eq^ns in polar form.

Let $f(z)$ be analytic function.

Consider the polar coordinates i.e. $x = r\cos\theta$ & $y = r\sin\theta$.

as $z = x+iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$

As $f(z) = u+iv$ i.e. $f(re^{i\theta}) = u+iv$. —— 1>.

Diff. 1> partially w.r.t. r. we get,

$$f'(re^{i\theta})\,e^{i\theta} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \quad \text{—— 2>}$$

Diff 1> partially w.r.t. θ we get,

$$f'(re^{i\theta})\,re^{i\theta}\cdot i = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}.$$

$$f'(re^{i\theta})\,e^{i\theta}\,r\cdot i = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

∴ using 2> we get, $\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) r\cdot i = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$.

$$\Rightarrow ir\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}.$$

$$\Rightarrow -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

Comparing we get $-r\frac{\partial v}{\partial r} = \frac{\partial u}{\partial \theta}$ & $r\frac{\partial u}{\partial r} = \frac{\partial v}{\partial \theta}$.

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

i.e. $\boxed{u_r = \frac{1}{r}v_\theta \quad \& \quad u_\theta = -r\,v_r}$

which are C-R. eq^ns in polar form.

★ From 2> we get. $f'(re^{i\theta}) = e^{-i\theta}(u_r + iv_r)$

i.e. $f'(z) = e^{-i\theta}(u_r + iv_r)$.

or $f'(z) =$

1) Determine whether ω is analytic & find $\frac{d\omega}{dz}$ for follow.
1) $\omega = \log z$ 2) $\omega = z^n$.

1) $\omega = \log z$.
→ Consider the polar co-ordinates $x = r\cos\theta$, $y = r\sin\theta$
as $z = x+iy \Rightarrow \omega = \log z = \log(x+iy)$
$= \log(r\cos\theta + ir\sin\theta)$
$\Rightarrow \omega = \log(r(\cos\theta + i\sin\theta)) = \log(re^{i\theta}) = \log r + \log e^{i\theta}$
$= \log r + i\theta \log e = \log r + i\theta$.
$\Rightarrow u = \log r \quad v = \theta$.
$\therefore u_r = \frac{1}{r} \quad u_\theta = 0, \quad v_r = 0 \quad v_\theta = 1$

$\therefore u_r = \frac{1}{r} v_\theta$ & $u_\theta = 0 = -r v_r \Rightarrow u_\theta = -r v_r$
$\Rightarrow$ C-R eq^{n}s are satisfied by ω.
also $u_r, u_\theta, v_r, v_\theta$ are continuous.
Hence, ω is analytic.
& $\frac{d\omega}{dz} = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(\frac{1}{r}\right) = \frac{1}{re^{i\theta}} = \frac{1}{z}$

HW 2) $\omega = z^n$.
→ Consider the polar co-ordinates $x = r\cos\theta$, $y = r\sin\theta$.
as $z = x+iy$
$\omega = z^n = (x+iy)^n = (r\cos\theta + ir\sin\theta)^n = r^n(\cos\theta + i\sin\theta)^n$
$\Rightarrow \omega = r^n(\cos n\theta + i\sin n\theta)$. — By De Moivre's thm.
$\therefore u = r^n \cos n\theta \quad v = r^n \sin n\theta$.
$\therefore u_r = nr^{n-1}\cos n\theta, \quad u_\theta = -r^n n \sin n\theta$
$v_r = nr^{n-1}\sin n\theta \quad v_\theta = nr^n \cos n\theta$.
$\therefore \frac{1}{r} v_\theta = nr^{n-1}\cos n\theta = u_r \Rightarrow u_r = \frac{1}{r} v_\theta$.
Also $-r v_r = -nr^n \sin n\theta = u_\theta \Rightarrow u_\theta = -r v_r$
$\Rightarrow$ C-R eq^{n}s are satisfied by ω
also, $u_r, u_\theta, v_r, v_\theta$ are continuous.
Hence, ω is analytic. $\Rightarrow \frac{d\omega}{dz} = nz^{n-1}$

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2) Find β if $f(z) = r^2\cos 2\theta + i r^2 \sin \beta\theta$ is analytic.

→ Given, $f(z) = r^2\cos 2\theta + i r^2 \sin\beta\theta$

$u = r^2 \cos 2\theta$ & $v = r^2 \sin \beta\theta$.

As $f(z)$ is analytic it satisfies C-R eq^{ns}. (in polar form)

i.e $u_r = \frac{1}{r} v_\theta$ & $u_\theta = -r v_r$.

Here, $u_r = 2r\cos 2\theta$, $u_\theta = -2r^2 \sin 2\theta$

$v_r = 2r \sin\beta\theta$, $v_\theta = \beta r^2 \cos\beta\theta$.

As $u_r = \frac{1}{r} v_\theta \Rightarrow 2r\cos 2\theta = \frac{1}{r}\beta r^2 \cos\beta\theta \Rightarrow 2r\cos 2\theta = \beta r\cos\beta\theta$

$\Rightarrow \beta = 2$.

Also $u_\theta = -r v_r \Rightarrow -2r^2 \sin 2\theta = -2r^2 \sin\beta\theta \Rightarrow \sin 2\theta = \sin\beta\theta$.

$\Rightarrow \beta = 2$. Hence, we get $\beta = 2$ Here.

3) Is $f(z) = z/\bar{z}$ is analytic?

→ Let $z = x + iy$

By polar co-ordinates $x = r\cos\theta$, $y = r\sin\theta$.

$\therefore z = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

$\bar{z} = x - iy = r(\cos\theta - i\sin\theta) = re^{-i\theta}$.

$\therefore f(z) = \frac{z}{\bar{z}} = \frac{\cancel{r}e^{i\theta}}{\cancel{r}e^{-i\theta}} = e^{i2\theta} = \cos 2\theta + i\sin 2\theta = u + iv$.

$\Rightarrow u = \cos 2\theta$, $v = \sin 2\theta$.

Here $u_r = 0$, $u_\theta = -2\sin 2\theta$.

$v_r = 0$, $v_\theta = 2\cos 2\theta$.

$\therefore u_r \neq \frac{1}{r} v_\theta$ & $u_\theta \neq -r v_r$

$\Rightarrow$ C-R eq^{ns} are not satisfied.

$\Rightarrow \therefore f(z)$ is not analytic.