- Evaluation of integration using Laplace Transform i.e. evaluation of $\int_0^\infty e^{-at} f(t)\,dt$

To evaluate $\int_0^\infty e^{-at} f(t)\,dt$.

- First consider $\int_0^\infty e^{-st} f(t)\,dt = L[f(t)] = \phi(s)$ —① (just by replacing a by s in problem)
- then put $s=a$ in ① we get

$$\int_0^\infty e^{-at} f(t)\,dt = \phi(a)$$

Problems

1) Evaluate $\int_0^\infty e^{-t} \operatorname{erf}(3\sqrt{t})\,dt$.

Sol^n Consider $\int_0^\infty e^{-st} \operatorname{erf}(3\sqrt{t})\,dt = L[\operatorname{erf}(3\sqrt{t})]$ —— ①

$$L[\operatorname{erf}\sqrt{t}] = \frac{1}{s\sqrt{s+1}} = \phi(s)$$

$f(t) = \operatorname{erf}\sqrt{t} \Rightarrow f(at) = \operatorname{erf}\sqrt{at}$.

we know by change of scale property

$$L[f(at)] = \frac{1}{a}\phi\left(\frac{s}{a}\right)$$

—— where $L[f(t)] = \phi(s)$

$$\therefore L[\operatorname{erf}(3\sqrt{t})] = L[\operatorname{erf}\sqrt{9t}]$$

$$= \frac{1}{9}\phi\left(\frac{s}{9}\right) \quad \text{—— here } a=9$$

$$= \frac{1}{9}\frac{1}{\frac{s}{9}\sqrt{\frac{s}{9}+1}}$$

$$= \frac{1}{s\sqrt{\frac{s+9}{9}}} = \frac{3}{s\sqrt{s+9}}$$

$\therefore$ from (1)

$$\int_0^\infty e^{-st}\, erf(3\sqrt{t})\, dt = L\left[erf(3\sqrt{t})\right]$$

$$= \frac{3}{s\sqrt{s+9}}$$

put $s = 1$, we get

$$\int_0^\infty e^{-t}\, erf(3\sqrt{t})\, dt = \frac{3}{1\sqrt{1+9}}$$

$$= \frac{3}{\sqrt{10}}$$

2) Evaluate $\int_0^\infty t^2 \frac{\sin 3t}{e^{2t}}\, dt$

Sol^n $\int_0^\infty t^2 \frac{\sin 3t}{e^{2t}}\, dt = \int_0^\infty e^{-2t} t^2 \sin 3t\, dt$.

Consider,

$$\int_0^\infty e^{-st} t^2 \sin 3t\, dt = L\left[t^2 \sin 3t\right] \quad \text{------ (1)}$$

$$L[\sin 3t] = \frac{3}{s^2+9} = \phi(s)$$

$$L[t^2 \sin 3t] = (-1)^2 \frac{d^2}{ds^2}\phi(s)$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\frac{3}{s^2+9}\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2+9)(0) - 3(2s)}{(s^2+9)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-6s}{(s^2+9)^2}\right]$$

$$= -6 \frac{d}{ds}\left[\frac{s}{(s^2+9)^2}\right]$$

$$= -6\left[\frac{(s^2+9)^2 - s\,2(s^2+9)(2s)}{((s^2+9)^2)^2}\right]$$

$$= -6\left[\frac{(s^2+9)(s^2+9-4s^2)}{(s^2+9)^4}\right]$$

$$= -6\left[\frac{9-3s^2}{(s^2+9)^3}\right]$$

$$= 6\left[\frac{3s^2-9}{(s^2+9)^3}\right]$$

$\therefore$ from (1)

$$\int_0^\infty e^{-st} t^2 \sin 3t\,dt = L\left[t^2 \sin 3t\right] = 6\left[\frac{3s^2-9}{(s^2+9)^3}\right]$$

put $s=2$, we get

$$\int_0^\infty e^{-2t} t^2 \sin 3t\,dt = 6\left[\frac{12-9}{(4+9)^3}\right] = 6\left[\frac{3}{(13)^3}\right] = \frac{18}{(13)^3}$$

3) Evaluate $\int_0^\infty \left(\frac{\cos 6t - \cos 4t}{t}\right) dt$

Soln

$$\int_0^\infty \left(\frac{\cos 6t - \cos 4t}{t}\right) dt = \int_0^\infty e^{-0t}\left(\frac{\cos 6t - \cos 4t}{t}\right) dt$$

Consider,

$$\int_0^\infty e^{-st}\left(\frac{\cos 6t - \cos 4t}{t}\right) dt = L\left[\frac{\cos 6t - \cos 4t}{t}\right] \quad \text{— (1)}$$

$$L[\cos 6t - \cos 4t] = \frac{s}{s^2+36} - \frac{s}{s^2+16} = \phi(s)$$

$$L\left[\frac{\cos 6t - \cos 4t}{t}\right] = \int_s^\infty \phi(s)\,ds$$

$$= \int_s^{\infty}\left(\frac{s}{s^2+36} - \frac{s}{s^2+16}\right) ds$$

$$= \frac{1}{2}\int_s^{\infty}\left(\frac{2s}{s^2+36} - \frac{2s}{s^2+16}\right) ds$$

$$= \frac{1}{2}\left[\log(s^2+36) - \log(s^2+16)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+36}{s^2+16}\right)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\log\left(\frac{\cancel{s^2}(1+36/s^2)}{\cancel{s^2}(1+16/s^2)}\right)\right]_s^{\infty}$$

$$= \frac{1}{2}\left[\log(1) - \log\left(\frac{1+36/s^2}{1+16/s^2}\right)\right]$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{s^2+36}{s^2+16}\right)\right]$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+36}{s^2+16}\right)^{-1}\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right)$$

$\therefore$ from ①

$$\int_0^{\infty} e^{-st}\left(\frac{\cos 6t - \cos 4t}{t}\right) dt = L\left[\frac{\cos 6t - \cos 4t}{t}\right] = \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right)$$

put $s=0$ we get,

$$\int_0^{\infty} e^{-0t}\left(\frac{\cos 6t - \cos 4t}{t}\right) dt = \frac{1}{2}\log\left(\frac{16}{36}\right)$$

$$\Rightarrow \int_0^{\infty}\left(\frac{\cos 6t - \cos 4t}{t}\right) dt = \log\left(\frac{16}{36}\right)^{1/2}$$

$$= \log\sqrt{\frac{16}{36}} = \log\left(\frac{4}{6}\right)$$

$$= \log\left(\frac{2}{3}\right)$$

4) Evaluate $\int_0^\infty e^{-t} \int_0^t \frac{\sin u}{u}\, du\, dt$

Soln Consider,

$$\int_0^\infty e^{-st} \int_0^t \frac{\sin u}{u}\, du\, dt = L\left[\int_0^t \frac{\sin u}{u}\, du\right] \quad \text{--- (1)}$$

$$L(\sin u) = \frac{1}{s^2+1} = \phi(s)$$

$$L\left(\frac{\sin u}{u}\right) = \int_s^\infty \phi(s)\, ds$$

$$= \int_s^\infty \frac{1}{s^2+1}\, ds$$

$$= \left[\tan^{-1}(s)\right]_s^\infty = \tan^{-1}\infty - \tan^{-1}s$$

$$= \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s = \phi_1(s)$$

$$\therefore L\left[\int_0^t \frac{\sin u}{u}\, du\right] = \frac{1}{s}\phi_1(s) = \frac{1}{s}\cot^{-1}s$$

From (1)

$$\int_0^\infty e^{-st} \int_0^t \frac{\sin u}{u}\, du\, dt = L\left[\int_0^t \frac{\sin u}{u}\, du\right] = \frac{1}{s}\cot^{-1}s$$

put $s=1$ we get,

$$\int_0^\infty e^{-t} \int_0^t \frac{\sin u}{u}\, du\, dt = \frac{1}{1}\cot^{-1}(1)$$

$$= \frac{\pi}{4}$$

5) If $\int_0^{\infty} e^{-2t} \sin(t+\alpha) \cos(t-\alpha)\, dt = \frac{1}{4}$ find α.

Sol^n Consider,

$$\int_0^{\infty} e^{-st} \sin(t+\alpha) \cos(t-\alpha)\, dt = L[\sin(t+\alpha)\cos(t-\alpha)] \quad \text{—— (1)}$$

$$L[\sin(t+\alpha)\cos(t-\alpha)] = L\left[\frac{\sin(t+\cancel{\alpha}+t-\cancel{\alpha})+\sin(t+\alpha-(t-\alpha))}{2}\right]$$

$$= \frac{1}{2}L[\sin(2t) + \sin(\cancel{t}+\alpha-\cancel{t}+\alpha)]$$

$$= \frac{1}{2}L[\sin 2t + \sin 2\alpha]$$

$$= \frac{1}{2}\left[\frac{2}{s^2+4} + \sin 2\alpha\left(\frac{1}{s}\right)\right]$$

from (1)

$$\int_0^{\infty} e^{-st} \sin(t+\alpha) \cos(t-\alpha)\, dt = L[\sin(t+\alpha)\cos(t-\alpha)]$$

$$= \frac{1}{2}\left[\frac{2}{s^2+4} + \frac{\sin 2\alpha}{s}\right]$$

put $s = 2$ we get

$$\int_0^{\infty} e^{-2t} \sin(t+\alpha) \cos(t-\alpha)\, dt = \frac{1}{2}\left[\frac{2}{4+4} + \frac{\sin 2\alpha}{2}\right]$$

Also $\int_0^{\infty} e^{-2t} \sin(t+\alpha) \cos(t-\alpha)\, dt = \frac{1}{4}$ (given)

$$\Rightarrow \frac{1}{2}\left[\frac{\cancel{2}1}{\cancel{8}4} + \frac{\sin 2\alpha}{2}\right] = \frac{1}{4}$$

$$\Rightarrow \frac{1}{4} + \frac{\sin 2\alpha}{2} = \frac{2}{4} \Rightarrow \frac{\sin 2\alpha}{2} = \frac{2}{4} - \frac{1}{4} = \frac{1}{4}$$

$$\Rightarrow \sin 2\alpha = \frac{\cancel{2}1}{\cancel{4}2} \Rightarrow \sin 2\alpha = \frac{1}{2} \Rightarrow 2\alpha = \frac{\pi}{6} \Rightarrow \alpha = \frac{\pi}{12}$$