4) Find Fourier series of $f(x) = x^2$ in $(-\pi, \pi)$

Sol^n As given interval is $(-\pi, \pi)$.

(We will check $f(x)$ is even or odd)

$f(-x) = (-x)^2 = x^2 = f(x)$

$\Rightarrow f(x)$ is even

$\Rightarrow b_n = 0$

Also, $l = \pi$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{———} \quad (1)$$

$$a_0 = \frac{2}{l} \int_0^l f(x)\, dx$$

$$= \frac{2}{\pi} \int_0^\pi x^2\, dx$$

$$= \frac{2}{\pi} \left[\frac{x^3}{3}\right]_0^\pi$$

$$= \frac{2}{\pi} \cdot \frac{\pi^3}{3}$$

$$= \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{\pi} \int_0^\pi x^2 \cos nx\, dx$$

$$= \frac{2}{\pi} \left[(x^2)\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^\pi$$

$$= \frac{2}{\pi} \left[0 + \frac{2\pi \cos n\pi}{n^2} + 0 - 0 + 0 - 0\right]$$

$$a_n = \frac{2}{\pi} \frac{2\pi (-1)^n}{n^2}$$

$$= \frac{4(-1)^n}{n^2}$$

$\therefore$ From (1)

$$f(x) = \frac{2\pi^2}{3\times 2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

5) Find Fourier Series of $f(x) = x$ in $(-\pi, \pi)$

soln As interval $(-\pi, \pi)$ is given

we can check $f(x)$ is either odd or even

$f(-x) = -x = -f(x)$

$\Rightarrow f(x)$ is odd $\Rightarrow a_0 = 0,\ a_n = 0$ Here, $l = \pi$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{(1)}$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi\left(\frac{-\cos n\pi}{n}\right) + 0 - 0 + 0\right]$$

$$= \frac{-2}{\pi} \times \pi \frac{\cos n\pi}{n} = -\frac{2(-1)^n}{n}$$

$\therefore$ from (1)

$$f(x) = \sum_{n=1}^{\infty} \frac{(-2)(-1)^n}{n} \sin nx.$$

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6) Find Fourier Series of $f(x) = x + x^2$ in $(-\pi, \pi)$

Hence find 1) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots$

2) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \cdots$

Sol^n $f(x) = x + x^2 = f_1(x) + f_2(x)$

(as $f(x)$ is neither even nor odd function)

Hence, $f_1(x) = x$ & $f_2(x) = x^2$

From 4) & 5) we have

$$f_2(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

$$f_1(x) = x = \sum_{n=1}^{\infty} \frac{(-2)(-1)^n}{n} \sin nx$$

$$\therefore f(x) = x^2 + x = f_2(x) + f_1(x)$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2} + \sum_{n=1}^{\infty} \frac{(-2)(-1)^n}{n} \sin nx$$

To prove remaining part we'll just use Fourier series of $f_2(x) = x^2$

$$f_2(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$$

$$= \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \cdots\right] \quad \text{--- (1)}$$

put $x = \pi$ in (1)

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{-(-1)}{1^2} + \frac{1}{2^2} - \frac{(-1)}{3^2} + \cdots\right]$$

$$\pi^2 - \frac{\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{2\pi^2}{3\times 4} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

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$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \quad \text{---} (2)$$

put $x=0$ in (1)

$$0 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right]$$

$$-\frac{\pi^2}{3} = -4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \quad \text{---} (3)$$

(2) + (3) gives us

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = \left(\frac{1}{1^2} + \cancel{\frac{1}{2^2}} + \frac{1}{3^2} + \cdots\right) + \left(\frac{1}{1^2} - \cancel{\frac{1}{2^2}} + \frac{1}{3^2} - \cdots\right)$$

$$\frac{\cancel{3}\,\pi^2}{\cancel{12}\,4} = \frac{2}{1^2} + \frac{2}{3^2} + \cdots$$

$$\frac{\pi^2}{4} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

7) Find Fourier series of $f(x) = \begin{cases} x + \frac{\pi}{2} & -\pi < x < 0 \\ \frac{\pi}{2} - x & 0 < x < \pi \end{cases}$

Sol^n $f(-x) = \begin{cases} -x + \frac{\pi}{2} & -\pi < -x < 0 \\ \frac{\pi}{2} + x & 0 < -x < \pi \end{cases}$

$$= \begin{cases} -x + \frac{\pi}{2} & +\pi > x > 0 \\ \frac{\pi}{2} + x & 0 > x > -\pi \end{cases}$$

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$$= \begin{cases} \frac{\pi}{2} + x & -\pi < x < 0 \\ \frac{\pi}{2} - x & 0 < x < \pi \end{cases}$$

$= f(x)$

$\therefore$ $f(x)$ is even function.

$\Rightarrow b_n = 0$ & $l = \pi$

Hence,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{———} \quad (1)$$

$$a_0 = \frac{2}{l} \int_0^l f(x)\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \left(\frac{\pi}{2} - x\right) dx$$

$$= \frac{2}{\pi} \left[\frac{\pi}{2} x - \frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi^2}{2} - \frac{\pi^2}{2} - 0\right]$$

$$= 0$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \left(\frac{\pi}{2} - x\right) \cos nx\, dx$$

$$= \frac{2}{\pi} \left[\left(\frac{\pi}{2} - x\right)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[(0) - \frac{\cos n\pi}{n^2} - 0 + \frac{1}{n^2}\right] = \frac{2}{\pi} \left[\frac{-(-1)^n}{n^2} + \frac{1}{n^2}\right]$$

$$a_n = \frac{2}{\pi}\left[\frac{1-(-1)^n}{n^2}\right]$$

$\therefore$ from (1)

$$f(x) = 0 + \sum_{n=1}^{\infty} \frac{2}{\pi}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx.$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx.$$

$$= \frac{2}{\pi}\left[\frac{2}{1^2}\cos x + 0 + \frac{2}{3^2}\cos 3x + 0 \cdots\right]$$

$$f(x) = \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \cdots\right] \quad \text{---} (2)$$

put $x=0$ in (2), we get

$$f(0) = \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right] \quad \text{---} (3)$$

$$f(0) = \frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right]$$

$$= \frac{1}{2}\left[\lim_{x\to 0^-}\left(\frac{\pi}{2}+x\right) + \lim_{x\to 0^+}\left(\frac{\pi}{2}-x\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2}+\frac{\pi}{2}\right] = \frac{\pi}{2}$$

$$\therefore \frac{\pi}{2} = \frac{4}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\cdots\right] \quad \text{--- from (3)}$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\cdots$$

To prove second series we will use Parseval's identity.

$$\frac{2}{l}\int_0^l [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{2}{\pi}\int_0^{\pi}\left[\frac{\pi}{2}-x\right]^2 dx = 0 + \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\left(\frac{1-(-1)^n}{n^2}\right)\right]^2$$

$$\Rightarrow \frac{2}{\pi}\left[\frac{\left[\frac{\pi}{2}-x\right]^3}{3\times(-1)}\right]_0^{\pi} = \sum_{n=1}^{\infty} \frac{4}{\pi^2}\frac{(1-(-1)^n)^2}{n^4}$$

$$\Rightarrow \frac{2}{\pi}\left[\frac{\left(\frac{-\pi}{2}\right)^3}{-3} - \frac{\left(\frac{\pi}{2}\right)^3}{-3}\right] = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)^2}{n^4}$$

$$\Rightarrow \frac{2}{\pi}\left[+\frac{\pi^3}{24}+\frac{\pi^3}{24}\right] = \frac{4}{\pi^2}\left[\frac{2^2}{1^4}+0+\frac{2^2}{3^4}+\cdots\right]$$

$$\Rightarrow \frac{2}{\cancel{\pi}}\times\frac{\cancel{2}\pi^3}{24}\times\frac{\pi^{\cancel{2}}}{\cancel{4}} = 4\left[\frac{1}{1^4}+\frac{1}{3^4}+\cdots\right]$$

$$\Rightarrow \frac{\pi^4}{24\times 4} = \frac{1}{1^4}+\frac{1}{3^4}+\cdots$$

$$\Rightarrow \frac{\pi^4}{96} = \frac{1}{1^4}+\frac{1}{3^4}+\cdots$$

8) Find fourier Series of

$$f(x) = \begin{cases} 0 & -2<x<-1 \\ 1+x & -1<x<0 \\ 1-x & 0<x<1 \\ 0 & 1<x<2 \end{cases}$$

Sol^n As interval is $(-2,2)$ we'll check $f(x)$ is even or odd.

$$f(-x) = \begin{cases} 0 & -2<-x<-1 \\ 1-x & -1<-x<0 \\ 1+x & 0<-x<1 \\ 0 & 1<-x<2 \end{cases}$$

$$= \begin{cases} 0 & 2>x>1 \\ 1-x & 1>x>0 \\ 1+x & 0>x>-1 \\ 0 & -1>-x>-2 \end{cases}$$

$$= \begin{cases} 0 & 1 < x < 2 \\ 1-x & 0 < x < 1 \\ 1+x & -1 < x < 0 \\ 0 & -2 < x < -1 \end{cases}$$

$$= \begin{cases} 0 & -2 < x < -1 \\ 1+x & -1 < x < 0 \\ 1-x & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

$= f(x)$

$\Rightarrow f(x)$ is even function

$\Rightarrow b_n = 0$ & $l = 2$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right) \quad \text{——— (1)}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{2}\int_0^2 f(x)\,dx$$

$$= \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$= \int_0^1 (1-x)\,dx + 0$$

$$= \left[x - \frac{x^2}{2}\right]_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$a_n = \int_0^1 f(x)\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^1 (1-x)\cos\left(\frac{n\pi x}{2}\right)dx + 0$$

$$= \left[\frac{(1-x)\left(\sin\left(\frac{n\pi x}{2}\right)\right)}{(n\pi/2)} - \frac{(-1)\left(-\cos\left(\frac{n\pi x}{2}\right)\right)}{(n\pi/2)^2}\right]_0^1$$

$$= \left[0 - \cos(n\pi/2)\frac{4}{n^2\pi^2} - 0 + (1)\frac{4}{n^2\pi^2}\right]$$

$$= \frac{4}{n^2\pi^2}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right]$$

Hence from ①

$$f(x) = \frac{1}{2\times 2} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left[1-\cos\left(\frac{n\pi}{2}\right)\right]\cos\left(\frac{n\pi x}{2}\right)$$

$$= \frac{1}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-\cos\left(\frac{n\pi}{2}\right)\right]\cos\left(\frac{n\pi x}{2}\right)$$

9) Find Fourier Series of

$$f(x) = \pi x \qquad 0 \le x \le 1$$
$$= \pi(2-x) \qquad 1 \le x \le 2$$

Sol^n Given interval is $(0,2)$

$\therefore (c, c+2l) = (0,2) \Rightarrow c = 0$ & $c+2l = 2 \Rightarrow 2l = 2$

$\Rightarrow \boxed{l = 1}$

as, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right)$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\pi x) + \sum_{n=1}^{\infty} b_n\sin(n\pi x) \quad \text{— ①}$$

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$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$= \frac{1}{1}\int_0^2 f(x)\,dx$$

$$= \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$= \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$= \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \pi\left[\frac{1}{2}\right] + \pi\left[4 - \frac{4}{2} - 2 + \frac{1}{2}\right]$$

$$= \pi\left[\frac{1}{2} + \cancel{2} - \cancel{2} + \frac{1}{2}\right]$$

$$= \pi$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{1}\int_0^2 f(x)\cos\left(\frac{n\pi x}{1}\right)dx$$

$$= \int_0^1 \pi x\cos(n\pi x)\,dx + \int_1^2 \pi(2-x)\cos(n\pi x)\,dx$$

$$= \pi\left[\int_0^1 x\cos(n\pi x)\,dx + \int_1^2 (2-x)\cos(n\pi x)\,dx\right]$$

$$= \pi\left[\left(x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right)_0^1 + \left((2-x)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right)_1^2\right]$$

$$= \pi\left[0 + \frac{\cos n\pi}{n^2\pi^2} - 0 + \frac{-1}{n^2\pi^2} + 0 - \frac{\cos 2n\pi}{n^2\pi^2} - 0 + \frac{\cos n\pi}{n^2\pi^2}\right]$$

$$a_n = \pi\left[\frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} + \frac{(-1)^n}{n^2\pi^2}\right]$$

$$= \pi\left[\frac{2(-1)^n}{n^2\pi^2} - \frac{2}{n^2\pi^2}\right]$$

$$= \frac{2\pi}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= 1\int_0^2 f(x)\sin(n\pi x)\,dx$$

$$= \int_0^1 \pi x\sin(n\pi x)\,dx + \int_1^2 \pi(2-x)\sin(n\pi x)\,dx$$

$$= \pi\left[\int_0^1 x\sin(n\pi x)\,dx + \int_1^2 (2-x)\sin(n\pi x)\,dx\right]$$

$$= \pi\left[\left(x\left(\frac{-\cos(n\pi x)}{n\pi}\right) - (1)\left(-\frac{\sin(n\pi x)}{n^2\pi^2}\right)\right)_0^1 + \left((2-x)\left(\frac{-\cos(n\pi x)}{n\pi}\right) - (-1)\left(-\frac{\sin(n\pi x)}{n^2\pi^2}\right)\right)_1^2\right]$$

$$= \pi\left[(1)\left(\frac{-\cos n\pi}{n\pi}\right) - 0 - 0 + 0 + 0 - 0 - (1)\left(-\frac{\cos n\pi}{n\pi}\right)\right]$$

$$= \pi\left[\frac{-\cos n\pi}{n\pi} + \frac{\cos n\pi}{n\pi}\right]$$

$$= \pi(0)$$

$$b_n = 0$$

$\therefore$ from (1)

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left[(-1)^n - 1\right]\cos(n\pi x) + 0$$

10) Find Fourier series for $f(x) = \sqrt{1-\cos x}$ in $(0, 2\pi)$ also prove that $\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}$.

Soln $(c, c+2l) = (0, 2\pi)$

$c = 0$ & $c + 2l = 2\pi \Rightarrow 2l = 2\pi \Rightarrow l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{——} (1)$$

$$f(x) = \sqrt{1-\cos x}$$

$$= \sqrt{2\sin^2\left(\frac{x}{2}\right)}$$

$$= \sqrt{2}\sin\left(\frac{x}{2}\right)$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\left(\frac{x}{2}\right)dx$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{-\cos(x/2)}{(1/2)}\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{-\cos\pi}{1/2} + \frac{1}{1/2}\right]$$

$$= \frac{2\sqrt{2}}{\pi}\left[-(-1)+1\right] = \frac{2\sqrt{2}}{\pi}(2)$$

$$= \frac{4\sqrt{2}}{\pi}$$

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$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\left(\frac{x}{2}\right)\cos nx \, dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{2\pi} \frac{1}{2}\left[\sin\left(\frac{x}{2}+nx\right)+\sin\left(\frac{x}{2}-nx\right)\right]dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\sin\left(\frac{1+2n}{2}\right)x+\sin\left(\frac{1-2n}{2}\right)x\right]dx$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{-\cos\left(\frac{1+2n}{2}\right)x}{\left(\frac{1+2n}{2}\right)} - \frac{\cos\left(\frac{1-2n}{2}\right)x}{\left(\frac{1-2n}{2}\right)}\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{-2\cos(1+2n)\pi}{(1+2n)} - \frac{2\cos(1-2n)\pi}{(1-2n)} + \frac{2(1)}{1+2n} + \frac{2(1)}{1-2n}\right]$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{-2(-1)}{1+2n} - \frac{2(-1)}{1-2n} + \frac{2}{1+2n} + \frac{2}{1-2n}\right]$$ — $(1-2n)$ & $(1+2n)$ are odd no.

$$= \frac{\sqrt{2}}{\cancel{2}\pi}\times\cancel{2}\left[\frac{1}{1+2n}+\frac{1}{1-2n}+\frac{1}{1+2n}+\frac{1}{1-2n}\right]$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{2}{1+2n}+\frac{2}{1-2n}\right]$$

$$= \frac{2\sqrt{2}}{\pi}\left[\frac{1-\cancel{2n}+1+\cancel{2n}}{1-(2n)^2}\right]$$

$$= \frac{2\sqrt{2}}{\pi}\left[\frac{2}{1-4n^2}\right]$$

$$= \frac{4\sqrt{2}}{\pi(1-4n^2)}$$

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$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\ \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin\left(\frac{x}{2}\right)\ \sin nx\ dx.$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{2\pi} \frac{1}{2}\left[\cos\left(\frac{x}{2}-nx\right) - \cos\left(\frac{x}{2}+nx\right)\right]dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\cos\left(\frac{1-2n}{2}\right)x - \cos\left(\frac{1+2n}{2}\right)x\right]dx$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{\sin\left(\frac{1-2n}{2}\right)x}{\left(\frac{1-2n}{2}\right)} - \frac{\sin\left(\frac{1+2n}{2}\right)x}{\left(\frac{1+2n}{2}\right)}\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{2\sin(1-2n)\pi}{(1-2n)} - 2\,\frac{\sin(1+2n)\pi}{(1+2n)} - 0 + 0\right]$$

$$= \frac{2\sqrt{2}}{\cancel{2}\pi}\left[0 - 0\right]$$

$$= 0$$

$\therefore$ from (1)

$$f(x) = \frac{4\sqrt{2}}{2\times\pi} + \sum_{n=1}^{\infty}\frac{4\sqrt{2}}{\pi(1-4n^2)}\cos nx\ . + 0$$

$$\sqrt{2}\sin\left(\frac{x}{2}\right) = \frac{2\sqrt{2}}{\pi} + \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{-(4n^2-1)}$$

put $x = 0$

$$\sqrt{2}(0) = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{\cos(0)}{4n^2-1}$$

$$0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\frac{4\cancel{\sqrt{2}}}{\cancel{\pi}}\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{2\cancel{\sqrt{2}}}{\cancel{\pi}}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}$$

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