We have heard /used statements like

- It will *mostly* rain today
- It is *highly unlikely* that India will qualify for Olympics
- I *may not* go to the college tomorrow etc.

We note that there is a *chance* attached to whether it rains today, India qualifies for the Olympics or whether I go to the college tomorrow.
In the above cases, we are actually saying that *probably* it will rain today or India *probably* will not qualify for Olympics or I *probably* will not go to the college tomorrow.

In this subject, we will study the theory of probability as applied to real life engineering problems where chance is associated with their occurrences.

Origins of the theory of Probability

Probability theory has its origins in gambling. A gambler Antoine Gombaud posed the problem of 'points' – 2 players roll dice and the player to **first** win a certain number of points is the winner; the problem was how to divide the money if the game stopped in the middle – to Mathematician Blaise Pascal (1623 – 1662). Pascal collaborated with Pierre de Fermat (1601–1665) and arrived at the 'laws of chance'.

- James Bernoulli (1654– 1705) wrote the (first) book ***–Theory of probability.***
- De Moivre (1667– 1754) published ***The doctrine of chances***.
- Karl Pearson (1857– 1936) worked on ***correlation analysis*** and introduced the ***chi-square test***.
- W.S. Gosset found the ***Student's t-distribution*** used in exact or small sampling.
- Ronald A. Fisher (1890–1962) is called the ***Father of Statistics***. He worked on ***Analysis of Variance*** and ***Design of Experiments***.

In probability theory, we deal with experiments that have different results known as **outcomes, (**unlike the experiments that one conducts in the laboratory that have fixed outcome for given inputs). Such experiments are known as **random experiments**.

Example:
Suppose we send signals as 0's and 1's through a binary channel. The receiver may not receive a sent 0 as a 0 or a sent 1 as a 1 always. Let's say

the receiver receives a sent 0 as a 0, 90% of the times and a sent 1 as a 1, 95% of the times. (This means that 10% of the times a sent 0 is received as a 1 and 5% of the times a sent 1 is received as a 0.) Then sending signals through this channel is a random experiment.

Definitions:
1. An experiment whose outcome is not known, but the set of all possible outcomes are known is called a **random experiment**.
2. A single performance of a random experiment is called a **trial**.
3. The set of all possible outcomes is known as the **sample space** of a given random experiment. It is denoted by **S** or Ω.
In the above example, $S = \{0,1\}$.
4. Subsets of the sample space S are called **events**.
5. Subsets of the sample space S having a single element are called **elementary events**.
6. Two events A and B are said to be **equal** if they consist of the same elements.
7. Events that cannot occur together are called **mutually exclusive events**. That is, the occurrence of one of the events *excludes* the occurrence of the others. Mutually exclusive events do not have elements in common.
8. If A is an event, then the *complement* of A is denoted by $\overline{A}$. We have $A \cup \overline{A} = S$

Let a random experiment be performed. Note that the sample space can be finite, countably infinite or uncountably infinite. The following examples will make it clear:

1. Finite:
(a) Random Experiment: Tossing a coin
S = {H, T}
(b) Random Experiment: Throwing a die:
S = {1, 2, 3, 4, 5, 6}
Let $A = \{1,2,3\}$ and $B = \{4,5\}$
Since *A* and *B* are subsets of the sample space S, they are events. Also *A* and *B* have no elements in common. That is, if one of them happens, the other does not happen. Thus *A* and *B* are mutually exclusive events.

2. Countably Infinite:
(a) Random Experiment: Registering the number of telephone calls received at a telephone exchange in two hours, say between 10 am and 12 noon
S = {0, 1, 2, . . .}
(b) Random Experiment: Counting the number of printing errors in a book of 150 pages
S = {0, 1, 2, . . .}

3. Uncountable:
(a) Random Experiment: A record of the height of II year engineering students in APSIT (in cm):
S = [0, 300]
(b) Random Experiment: Calculation of the total distance travelled by a II year Engineering student in APSIT from home to college (in km) everyday:
S = [0, 500]

Consider the following examples:
1. A box contains 100 capacitors of which 40 are 0.01μF with a 100-Voltage rating, 35 are 0. 1μF with a 50-Voltage rating and 25 are 1.0μF with a 10-Voltage rating. Determine the number of elements in the following sets:
(i) $A = \{capacitors\ with\ \text{capacitance} \geq 0.1\mu F\}$
(ii) $B = \{capacitors\ with\ \text{voltage rating} \geq 50V\}$
(iii) $C = \{capacitors\ with\ both\ \text{capacitance} \geq 0.1\mu F\ and\ voltage\ rating \geq 50V\}$
Solution: (i) $n(A) = 35 + 25 = 60$ (ii) $n(B) = 40 + 35 = 75$ (iii) $n(C) = 35$

2. An experiment consists of observing the sum of numbers showing up when 2 fair dice are thrown. Find the sample space and the probability of the following events;
(i) A: Sum is 7 (ii) B: Sum is >8 and ≤ 11 (iii) C: Sum is >10
Solution: Here the sample space

$$S = \begin{Bmatrix} (1,1),\ (1,2),\ (1,3),\ (1,4), (1,5),\ (1,6) \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6) \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6) \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6) \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6) \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \end{Bmatrix}$$

(Note that *S* has 36 elements)

(i) We have
$A = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$

$$\Rightarrow P(A) = \frac{6}{36} = \frac{1}{2}$$

(ii) We have
$B = \{(3,6),(4,5),(5,4),(6,3),(4,6),(5,5),(6,4),(5,6),(6,5)\}$

$$\Rightarrow P(B) = \frac{9}{36} = \frac{1}{4}$$

(iii) We have
$C = \{(5,6),(6,5),(6,6)\}$

$$\Rightarrow P(B) = \frac{3}{36} = \frac{1}{12}$$

Actually we have used the classical definition of probability to calculate the probabilities in the above example.

Classical definition of probability (*a-priori* definition):
Suppose a random experiment results in N equally likely (and mutually exclusive) outcomes. If N_A of these outcomes result in the event A, then the probability of the occurrence of the event A, is defined as $P(A) = \frac{N_A}{N}$.

Shortcomings: The classical definition cannot be used where the sample space is infinite. Also the concept of **equally likely** which is being used here, may not be applicable in all cases.

Frequency definition of probability (*a-posteriori* definition):
Suppose a random experiment is performed N times and an event A happens N_A times out of those N times. Then the probability of the occurrence of the event A is defined to be $P(A) = \lim_{N \to \infty} \frac{N_A}{N}$ if the limit exists.

Shortcomings: The frequency definition is not clear about how large N should be and also the limit in the definition may not exist.

Axiomatic definition of probability

Let S be a sample space associated with a random experiment and let $\mathcal{F}$ be a σ – algebra of subsets of S. All the subsets of $\mathcal{F}$ will be events. Then P is said to be a *probability measure* if it satisfies the following axioms:

Axiom 1: $P(A) \geq 0$ for all events A

Axiom 2: P(S) = 1

Axiom 3: If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$

The triplet (S, $\mathcal{F}, P$) is said to be a *probability space*.

<u>Some Theorems on Probability</u>

Theorem 1: The probability of the impossible event is zero. i.e $P(\varphi) = 0$.

Theorem 2: (complement rule): For any event A, we have $P(A^c) = 1 - P(A)$ where A^c is the complement of the event A.

Theorem 3: For any event A, we have $0 \leq P(A) \leq 1$.

Theorem 4: If $A \subseteq B$, then $P(A) \leq P(B)$.

Theorem 5: (Addition Rule) For any two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Prof. Nancy Sinollin