What Propositional Logic Cannot Do

- We saw that some declarative sentences are not statements without specifying the value of `indeterminates'

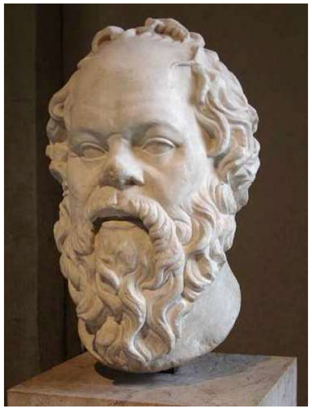
  ``x + 2 is an even number''

  ``If x + 1 > 0, then x > 0''

  ``A man has a brother''

- Some valid arguments cannot be expressed with all our machinery of tautologies, equivalences, and rules of inference

  Every man is mortal.
  Socrates is a man.
  ___
  ∴ Socrates is mortal



Open Statements or Predicates

- Sentences like \`x is greater than 3' or \`person x has a brother' are not true or false unless the variable is assigned some particular value.
- Sentence \`x is greater than 3' consists of 2 parts.
  - The first part, x, is called the variable or the subject of the sentence.
  - The second part – the predicate, \`is greater than 3' – refers to a property the subject can have.
- Sentences that have such structure are called open statements or predicates
- We write P(x) to denote a predicate with variable x

Unary, Binary, and so on

`x is greater than 3'
`x is my brother'
`x is a human being'

contain only 1 variable, unary predicates
P(x)

`x is greater than y'
`x is the mother of y'
`car x has colour y'

contain 2 variables, binary predicates
Q(x,y)

`x divides y + z'
`x sits between y and z'
`x is a son of y and z'

contain 3 variables, ternary predicates
R(x,y,z)

Assigning a Value

- When a variable is assigned a value, the predicates turns into a statement, whose truth value can be evaluated.

P(x) = `x is greater than 3' $\xrightarrow{x=2}$ P(2) = `2 is greater than 3' false

P(x) = `x is greater than 3' $\xrightarrow{x=4}$ P(4) = `4 is greater than 3' true

Q(x,y) = `car x has colour y' $\xrightarrow{x=\text{my car},\ y=\text{red}}$ Q(my car,red) = `my car is red' true

Q(x,y) = `car x has colour y' $\xrightarrow{x=\text{my car},\ y=\text{grey}}$ Q(my car,grey) = `my car is grey' false

Universe

- We cannot assign a variable of a predicate ANY value. We need to obtain a meaningful statement!

`x is greater than 3'

23 0 a cat a big red truck

- Every variable of a predicate is associated with a **universe** or **universe of discourse**, and its values are taken from this universe

`x is greater than 3'	x is a number
`x is my brother'	x is a human
`x is an animal'	x is a ???
`car x has colour y'	x is a car y is a colour

Relational Databases

- A relational database is a collection of tables like

No.	Name	Student ID	Supervisor	Thesis title
1.	Bradley Coleman	30101234	Petra Berenbrink	Algebraic graph theory
…	…	…	…	…

A table consists of a schema and an instance. A schema is a collection of attributes, where each attribute has an associated universe of possible values. An instance is a collection of rows, where each row is a mapping that associates with each attribute of the schema a value in its universe.

- Every table is a predicate that is true on the rows of the instance and false otherwise.

Quantifiers

- One way to obtain a statement from a predicate is to assign all its variables some values

- Another way to do that is to use expressions like

  `For every …’

  `There is … such that …’

  `A … can be found …’

  `Any … is …’

`Every man is mortal’

`There is x such that x is greater than 3’

`There is a person who is my father’

`For any x, $x^2 \geq 0$ '

quantification

Universal Quantifiers

- Abbreviates constructions like

  For all …
  For any …
  Every …
  Each …

- Asserts that a predicate is true for all values from the universe

  `Every man is mortal’
  `All lions are fierce’
  `For any x, $x^2 \geq 0$ '

- Notation: $\forall$
- $\forall x\ P(x)$ means that for every value a from the universe P(a) is true

Universal Quantifiers (cntd)

`For any x, $x^2 \geq 0$ ' true!

`Every car is grey' false! my car is not grey

- $\forall$x P(x) is false if and only if there is at least one value a from the universe such that P(a) is false
- Such a value a is called a counterexample

Existential Quantifiers

- Abbreviates constructions like

  For some …
  For at least one …
  There is …
  There exists …

- Asserts that a predicate is true for at least one value from the universe

  `There is a living king'
  `Some people are fierce'
  `There is x such that $x^2 \geq 10$'

- Notation: $\exists$

- $\exists x\ P(x)$ means that there is a value a from the universe such that P(a) is true

Existential Quantifiers (cntd)

`There is a grey car' true! my friend's car is red

`For some x, $x^2 < 0$' false!

- $\exists x$ P(x) is false if and only if for all a from the universe P(a) is false
- Disproving an existential statement is difficult!

Quantifiers and Negations

- Summarizing

	true	false
$\forall x\, P(x)$	For every value a from the universe $P(a)$ is true	There is a counterexample – a value a from the universe such that $P(a)$ is false
$\exists x\, P(x)$	There is a value a from the universe such that $P(a)$ is true	For all values a from the universe $P(a)$ is false

- Observe that

  $\forall x\, P(x)$ is false if and only if $\exists x\, \neg P(x)$ is true

  $\exists x\, P(x)$ is false if and only if $\forall x\, \neg P(x)$ is true

Example

- What is the negation of each of the following statements?

Statement		Negation
All lions are fierce	$\forall x\, P(x)$	There is a peaceful lion
Everyone has two legs	$\forall x\, P(x)$	There is a person having more than two legs, one leg, or no legs at all
Some people like coffee	$\exists x\, P(x)$	All people hate coffee
There is a lady in one of these rooms (Some rooms contain a lady)	$\exists x\, P(x)$	There is a tiger in every room

Multiple quantifiers

- Often predicates have more than one variable. In this case we need more than one quantifier.

P(x,y) = ``car x has colour y"

$\forall x \forall y\ P(x,y)$	``every car is painted all colours
$\exists x \exists y\ P(x,y)$	``there is a car that is painted some colour
$\forall x \exists y\ P(x,y)$	``every car is painted some colour
$\exists x \forall y\ P(x,y)$	``there is a car that is painted all colours

Open and Bound Variables

- In the statement

  ``car x has some colour'' $\exists y\, P(x,y)$

  variables x and y play completely different roles.

- Variable y is **bound** by the existential quantifier. Effectively it disappeared from the statement.

- Variable x is not bound, it is **free**.

- Another example:

  ``x is the least number''  $Q(x,y)$ = ``x is less than y''

  $\forall y\ Q(x,y)$ or $\forall y\ (x \le y)$

  ``x is the greatest number'' $\forall y\ Q(y,x)$ or $\forall y\ (y \le x)$