Cumulative distribution function (c.d.f) or Distribution function

If X is a r.v, discrete or continuous, then the function $F(x) = P(X \le x)$ is called the cumulative distribution function (c.d.f) or distribution function of X.

If X is discrete, then

$$F(x) = P(X \le x) = \sum_{x_i \le x} p_i \quad \text{where } p_i = P(X = x_i) \text{ is the probability mass function of } X.$$

If X is continuous, then

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)dx \quad \text{where } f(x) \text{ is the probability density function of } X.$$

Examples:

1. Suppose X is a discrete r.v with probability distribution

X	1	2	3
P(X=x)	1/4	1/2	1/4

Obtain the distribution function of X.

Solution: We have the distribution function of X to be given by

$$F(x) = P(X \le x) = \sum_{x_i \le x} p_i \text{ if X is discrete. Therefore,}$$

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{4}, & 1 \le x < 2 \\ \frac{3}{4}, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

2. Consider the experiment of tossing a coin. Obtain the distribution function F of the random variable which takes on the value 1 for the outcome tail and the value 0 for the outcome head. Obtain F (.07) and F (1.99).

Solution: Let P(Head) = p and P(Tail) = q $(0 \le p, q \le 1;\ p+q=1)$

Then we have the probability distribution of X to be

Outcome	Head	Tail
X	0	1
P(X=x)	p	q

Then the distribution function of X is given as

$$F(x) = P(X \le x)$$
$$= \begin{cases} 0 & x<0 \\ p & 0 \le x < 1 \\ 1(= p+q) & x \ge 1 \end{cases}$$

3. Suppose X is a continuous r.v. with $f(x) = \frac{1}{2}, 0 \le x \le 2$. Obtain the distribution function of X.

Solution: We have the distribution function of X to be given by

$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)dx$ if X is continuous. Therefore,

$$F(x) = \int_0^x \frac{1}{2}dx$$
$$= \frac{1}{2}x\Big|_0^x$$
$$= \frac{1}{2}x$$
$$\therefore F(x) = \begin{cases} \frac{1}{2}x, & 0 \le x \le 2 \\ 1, & x \ge 2 \end{cases}$$

4. A random variable has the following exponential pdf: $P(x) = ke^{-|x|}$. Determine the value of k and the corresponding cumulative distribution function and sketch the pdf and cdf as functions of x.

Solution: Since $P(x) = ke^{-|x|}$ is a pdf, we should have

$$\int_{-\infty}^{\infty} P(x)dx = 1$$
$$\Rightarrow \int_{-\infty}^{\infty} ke^{-|x|}dx = 1$$
$$\Rightarrow \int_{-\infty}^{0} ke^{x}dx + \int_{0}^{\infty} ke^{-x}dx = 1 \quad \left(\because |x| = \begin{cases} -x, & x<0 \\ x, & x \ge 0 \end{cases}\right)$$
$$\Rightarrow k\left\{ e^{x}\Big]_{-\infty}^{0} + \frac{e^{-x}}{-1}\Big]_{0}^{\infty} \right. = 1$$
$$\Rightarrow k = \frac{1}{2}$$
$$\therefore P(x) = \frac{1}{2}e^{-|x|}, \; -\infty < x < \infty$$

Now the distribution function $F(x)$ is given by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} P(x)dx$$

$Case 1: x < 0$

$$F(x) = \int_{-\infty}^{x} \frac{1}{2} e^{-|x|} dx$$

$$= \int_{-\infty}^{x} \frac{1}{2} e^{x} dx$$

$$= \frac{1}{2} e^{x}$$

$Case 2: x \ge 0$

$$F(x) = \int_{-\infty}^{0} \frac{1}{2} e^{-|x|} dx + \int_{0}^{x} \frac{1}{2} e^{-|x|} dx$$

$$= \int_{-\infty}^{0} \frac{1}{2} e^{x} dx + \int_{0}^{x} \frac{1}{2} e^{-x} dx$$

$$= \frac{1}{2} e^{x} \Big]_{-\infty}^{0} - \frac{1}{2} e^{-x} \Big]_{0}^{x}$$

$$= \frac{1}{2} - \left(\frac{1}{2} e^{-x} - \frac{1}{2}\right)$$

$$= 1 - \frac{1}{2} e^{-x}$$

$$\therefore F(x) = \begin{cases} \frac{1}{2} e^{x}, & x < 0 \\ 1 - \frac{1}{2} e^{-x}, & x \ge 0 \end{cases}$$

Properties of the distribution function:

1. *F(x)* is a non-decreasing function of x. That is, if $x_1 < x_2$, then $F(x_1) \le F(x_2)$

   Proof: Let A be the event, $\{X \le x_1\}$ and B be the event, $\{X \le x_2\}$

   Since $x_1 < x_2$, we have $A \subset B$.

   $This\ implies\ P(A) \le P(B)$

   $i.e. P(X \le x_1) \le P(X \le x_2)$

   $i.e. F(x_1) \le F(x_2)$

2. $F(-\infty) = 0;\ F(+\infty) = 1$

Proof: We have $F(-\infty) = P(X \le -\infty) = 0$ and

$F(+\infty) = P(X \le +\infty) = P(S) = 1$

3. If $x_1 < x_2$, then $P(x_1 < X \le x_2) = F(x_2) - F(x_1)$

   Proof: We have, $\{X \le x_2\} = \{X \le x_1\} \cup \{x_1 < X \le x_2\}$ ....(1)

   Since the events in the R.H.S are mutually exclusive, by Axiom 3, we get from (1),

   $P(X \le x_2) = P(X \le x_1) + P(x_1 < X \le x_2)$

   $\Rightarrow F(x_2) = F(x_1) + P(x_1 < X \le x_2)$

   $\Rightarrow F(x_2) - F(x_1) = P(x_1 < X \le x_2)$

4. $P(X > x) = 1 - F(x)$

   Proof: We have

   $L.H.S = P(X > x) = 1 - P(X \le x)$

   $= 1 - F(x) = R.H.S$

5. If $x_1 < x_2$, then $P(x_1 \le X \le x_2) = F(x_2) - F(x_1^-)$ where

   $x_1^- = \lim_{h \to 0} x_1 - h$

6. $P(X = x) = F(x) - F(x^-)$