Module 3 - Two Dimensional Geometric Transformations

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Types of 2D Transformation

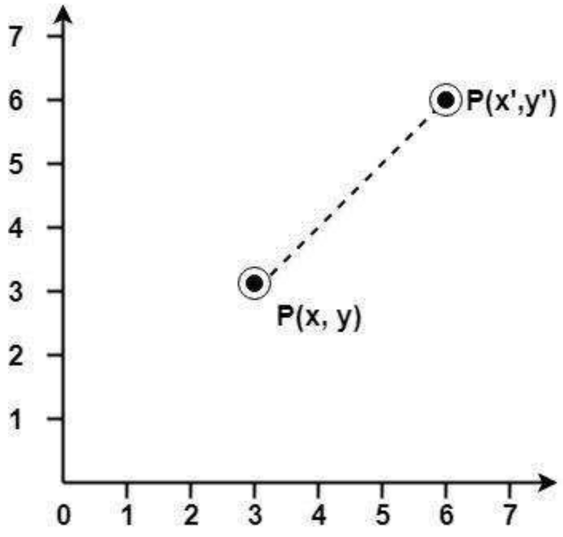
- **Translation**
  - A translation moves an object to a different position on the screen. you can translate a point in 2D by adding translation coordinate (tx,ty) to the original coordinate (x,y) to get the new coordinate (x',y').
- **Rotation**
  - In rotation, we rotate the object at particular angle θtheta from its origin. Normally the point P(x,y) is located at angle φfrom the horizontal x coordinate with distance r from the origin. We rotate it by angle θ,so that total rotation from the horizontal x axis is (φ+ θ)
- **Scaling**
  - To change the size of an object, scaling transformation is used. In the scaling process, you either expand or compress the dimensions of the object. Scaling can be achieved by multiplying the original coordinates of the object with the scaling factor to get the desired result.

Types of 2D Transformation

- **Shear**
  - A transformation that slants the shape of an object is called the shear transformation. There are two shear transformations X-Shear and Y-Shear. One shifts X coordinates values and other shifts Y coordinate values. However; in both the cases only one coordinate changes its coordinates and other preserves its values. **Shearing is also termed as Skewing.**
- **Reflection**
  - **Reflection** deals with obtaining a mirror image of the 2D object. In the Reflection process, the size of the object does not change.

Translation

- Translation Distance is t
  - tx - x direction
  - ty - y direction
- Therefore
  - x' = x + tx
  - y' = y + ty
- In Matrix Form, P’ = T + P



Translation

$$P' = \begin{bmatrix} X' \\ Y' \end{bmatrix} \qquad T = \begin{bmatrix} tx \\ ty \end{bmatrix} \qquad P = \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} X_{old} \\ Y_{old} \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \end{bmatrix}$$

Translation Examples

1. Given a circle C with radius 10 and center coordinates (3, 7). Apply the translation with distance 2 towards X axis and 4 towards Y axis. Obtain the new coordinates of C without changing its radius.

2. Given a square with coordinate points P(0, 4), Q(4, 4), R(4, 0), S(0, 0). Apply the translation with distance 4 towards X axis and 1 towards Y axis. Obtain the new coordinates of the square.

Translation Examples

Given a circle C with radius 10 and center coordinates (3, 7). Apply the translation with distance 2 towards X axis and 4 towards Y axis. Obtain the new coordinates of C without changing its radius.

Solution:

Given-

- Old center coordinates of C = $(X_{old}, Y_{old}) = (3, 7)$
- Translation vector = $(T_x, T_y) = (2, 4)$

Let the new center coordinates of C = (X_{new}, Y_{new}).

Applying the translation equations, we have-

- $X_{new} = X_{old} + T_x = 3 + 2 = 5$
- $Y_{new} = Y_{old} + T_y = 7 + 4 = 11$

Thus, New center coordinates of C = (5, 11).

Translation Examples

Given a square with coordinate points P(0, 4), Q(4, 4), R(4, 0), S(0, 0). Apply the translation with distance 4 towards X axis and 1 towards Y axis. Obtain the new coordinates of the square.

Solution:

For Coordinates P(0, 4) : Let the new coordinates of corner P = (X_{new}, Y_{new}).

- $X_{new} = X_{old} + T_x = 0 + 4 = 4$
- $Y_{new} = Y_{old} + T_y = 4 + 1 = 5$

Thus, New coordinates of corner P = (4, 5).

For Coordinates Q(4, 4) : Let the new coordinates of corner Q = (X_{new}, Y_{new}).

- $X_{new} = X_{old} + T_x = 4 + 4 = 8$
- $Y_{new} = Y_{old} + T_y = 4 + 1 = 5$

Thus, New coordinates of corner Q = (8, 5).

For Coordinates R(4, 0) : Let the new coordinates of corner R = (X_{new}, Y_{new}).

- $X_{new} = X_{old} + T_x = 4 + 4 = 8$
- $Y_{new} = Y_{old} + T_y = 0 + 1 = 1$

Thus, New coordinates of corner R = (8, 1).

For Coordinates S(0, 0) : Let the new coordinates of corner S = (X_{new}, Y_{new}).

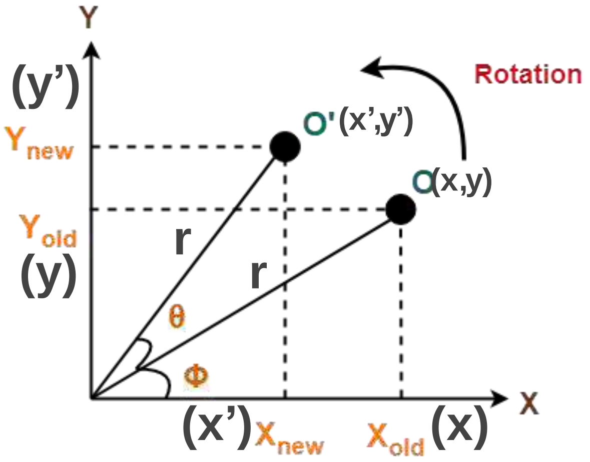
- $X_{new} = X_{old} + T_x = 0 + 4 = 4$
- $Y_{new} = Y_{old} + T_y = 0 + 1 = 1$

Thus, New coordinates of corner S = (4, 1).

$$R = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Rotation

- cos φ = base(x) / hypotenuse(r) , therefore x = r cos φ
- sin φ = height(y) / hypotenuse(r) , therefore y = r sin φ
- cos (A+B) = cos A cos B - sin A sin B
- sin (A+B) = sin A cos B + cos A sin B
- x' = r cos (φ + θ) and y' = r sin (φ + θ)
- x' = r cos φ cos θ - r sin φ sin θ
- y' = r sin φ cos θ + r cos φ sin θ
- As x = r cos φ and y = r sin φ
- x' = x cos θ - y sin θ
- y' = y cos θ + x sin θ
- P’ = R(θ) . P , where



$$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \end{bmatrix}$$

Rotation Values

θ	0^0 0^0	30^0 $\frac{\pi}{6}$	45^0 $\frac{\pi}{4}$	60^0 $\frac{\pi}{3}$	90^0 $\frac{\pi}{2}$	180^0 π	270^0 $\frac{3\pi}{2}$	360^0 2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$N.D.$	0	$N.D.$	0
$\operatorname{cosec}\theta$	$N.D.$	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$N.D.$	-1	$N.D$
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	$N.D.$	-1	$N.D.$	1
$\cot\theta$	$N.D.$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$N.D.$	0	$N.D$

Rotation Examples

1. Given a line segment with starting point as (0, 0) and ending point as (4, 4). Apply 30 degree rotation anticlockwise direction on the line segment and find out the new coordinates of the line.

2. Given a triangle with corner coordinates (0, 0), (1, 0) and (1, 1). Rotate the triangle by 90 degree anticlockwise direction and find out the new coordinates.

Rotation Examples

Given a line segment with starting point as (0, 0) and ending point as (4, 4). Apply 30 degree rotation anticlockwise direction on the line segment and find out the new coordinates of the line.

Solution:

Given-

- Old ending coordinates of the line = (X_{old}, Y_{old}) = (4, 4)
- Rotation angle = $\theta = 30^\circ$

Let new ending coordinates of the line after rotation = (X_{new}, Y_{new}).

Applying the rotation equations, we have-

- $X_{new} = X_{old} \cos\theta - Y_{old} \sin\theta = 4 \times \cos30^\circ - 4 \times \sin30^\circ = 4 \times (\sqrt{3} / 2) - 4 \times (1 / 2) = 2(\sqrt{3} - 1) = 2(1.73 - 1) = 1.46$
- $Y_{new} = X_{old} \sin\theta + Y_{old} \cos\theta = 4 \times \sin30^\circ + 4 \times \cos30^\circ = 4 \times (1 / 2) + 4 \times (\sqrt{3} / 2) = 2(1 + \sqrt{3}) = 2(1 + 1.73) = 5.46$

Thus, New coordinates = (1.46 , 5.46).

Rotation Examples

Given a triangle with corner coordinates (0, 0), (1, 0) and (1, 1). Rotate the triangle by 90 degree anticlockwise direction and find out the new coordinates.

Solution:

For Coordinates A(0, 0) : Let the new coordinates of corner A = (X_{new}, Y_{new}).

- $X_{new} = X_{old} \cos\theta - Y_{old} \sin\theta = 0 \times \cos 90^o - 0 \times \sin 90^o = 0$
- $Y_{new} = X_{old} \sin\theta + Y_{old} \cos\theta = 0 \times \sin 90^o + 0 \times \cos 90^o = 0$ Thus,

New coordinates of corner A = (0, 0).

For Coordinates B(1, 0) : Let the new coordinates of corner B = (X_{new}, Y_{new}).

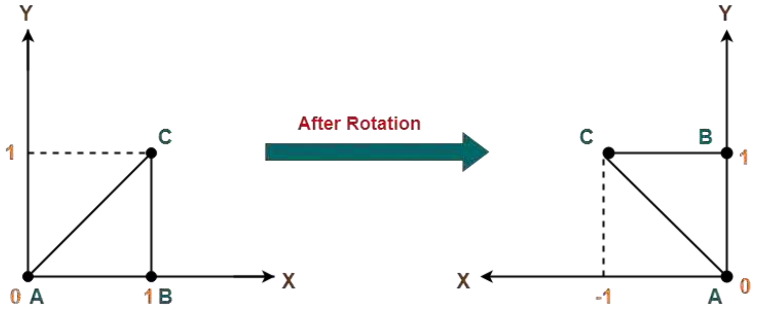
- $X_{new} = X_{old} \cos\theta - Y_{old} \sin\theta = 1 \times \cos 90^o - 0 \times \sin 90^o = 0$
- $Y_{new} = X_{old} \sin\theta + Y_{old} \cos\theta = 1 \times \sin 90^o + 0 \times \cos 90^o = 1 + 0 = 1$ Thus,

New coordinates of corner B = (0, 1).

For Coordinates C(1, 1) : Let the new coordinates of corner C = (X_{new}, Y_{new}).

- $X_{new} = X_{old} \cos\theta - Y_{old} \sin\theta = 1 \times \cos 90^o - 1 \times \sin 90^o = 0 - 1 = -1$
- $Y_{new} = X_{old} \sin\theta + Y_{old} \cos\theta = 1 \times \sin 90^o + 1 \times \cos 90^o = 1 + 0 = 1$ Thus,

New coordinates of corner C = (-1, 1).



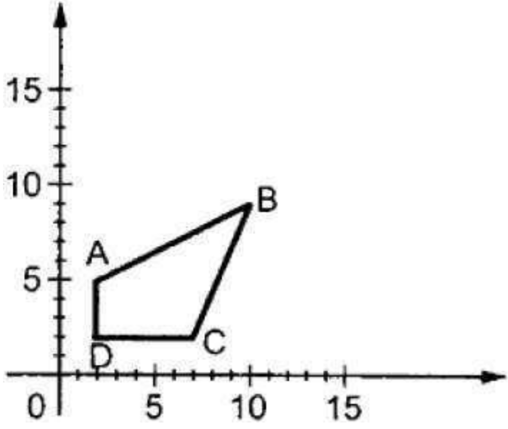
Scaling

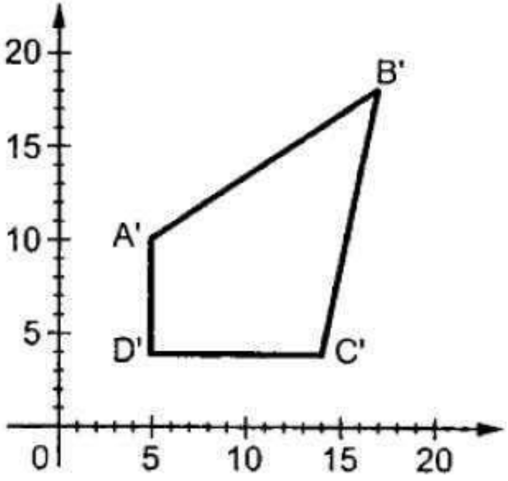
- The scaling factor S_X, S_Y scales the object in X and Y direction respectively. The above equations can also be represented in matrix form as below

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$

- $x' = x \cdot s_x$ and $y' = y \cdot s_y$

$$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \end{bmatrix}$$

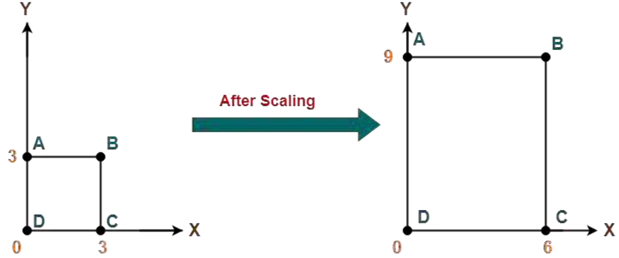




Cases of Scaling

1. $(S_x = S_y) = 1$ ---- No Scaling
2. $(S_x = S_y) \,!= 1$ ---- Uniform Scaling
3. $S_x \,!= S_y$ ---- Non-Uniform Scaling
4. $(S_x = S_y) < 1$ ---- Uniform Compression
5. $(S_x = S_y) > 1$ ---- Uniform expansion

Scaling Examples



Given a square object with coordinate points A(0, 3), B(3, 3), C(3, 0), D(0, 0). Apply the scaling parameter 2 towards X axis and 3 towards Y axis and obtain the new coordinates of the object.

Solution:

Given-

- Old corner coordinates of the square = A (0, 3), B(3, 3), C(3, 0), D(0, 0)
- Scaling factor along X axis = 2
- Scaling factor along Y axis = 3

For Coordinates A(0, 3) : Let the new coordinates of corner A = (X_{new}, Y_{new}).

- $X_{new} = X_{old} \times S_x = 0 \times 2 = 0$
- $Y_{new} = Y_{old} \times S_y = 3 \times 3 = 9$

Thus, New coordinates of corner A = (0, 9).

For Coordinates B(3, 3) : Let the new coordinates of corner B = (X_{new}, Y_{new}).

- $X_{new} = X_{old} \times S_x = 3 \times 2 = 6$
- $Y_{new} = Y_{old} \times S_y = 3 \times 3 = 9$

Thus, New coordinates of corner B = (6, 9).

For Coordinates C(3, 0) : Let the new coordinates of corner C = (X_{new}, Y_{new}).

- $X_{new} = X_{old} \times S_x = 3 \times 2 = 6$
- $Y_{new} = Y_{old} \times S_y = 0 \times 3 = 0$

Thus, New coordinates of corner C = (6, 0).

For Coordinates D(0, 0) : Let the new coordinates of corner D = (X_{new}, Y_{new}).

- $X_{new} = X_{old} \times S_x = 0 \times 2 = 0$
- $Y_{new} = Y_{old} \times S_y = 0 \times 3 = 0$

Thus, New coordinates of corner D = (0,0).

Shear

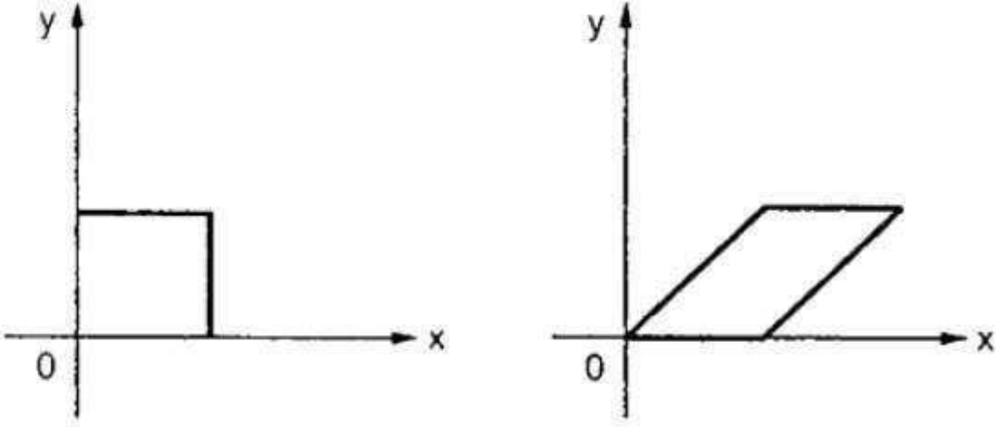
A transformation that slants the shape of an object is called the shear transformation. There are two shear transformations X-Shear and Y-Shear. One shifts X coordinates values and other shifts Y coordinate values. However; in both the cases only one coordinate changes its coordinates and other preserves its values. Shearing is also termed as Skewing.

$$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} 1 & Sh_x \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \end{bmatrix}$$

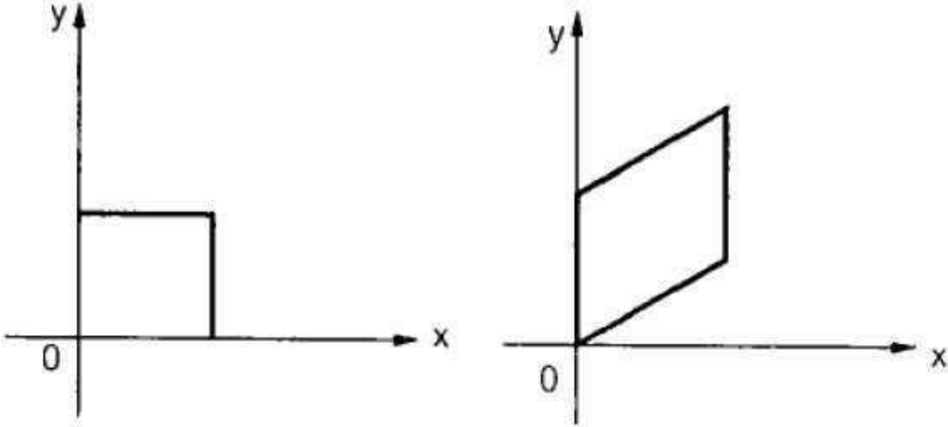
$X' = X + Sh_x \cdot Y$

$$\begin{bmatrix} X_{new} \\ Y_{new} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Sh_y & 1 \end{bmatrix} \times \begin{bmatrix} X_{old} \\ Y_{old} \end{bmatrix}$$

$Y' = Y + Sh_y \cdot X$



(a) Original object (b) Object after x shear



(a) Original object (b) Object after y shear

Shearing in X-axis Example

Given a triangle with points (1, 1), (0, 0) and (1, 0). i) Apply shear parameter 2 on X axis and find out the new coordinates of the object. ii) Apply shear parameter 2 on Y axis and find out the new coordinates of the object.

Solution:

Given-

- Old corner coordinates of the triangle = A (1, 1), B(0, 0), C(1, 0)
- Shearing parameter towards X direction (Sh_x) = 2
- Shearing parameter towards Y direction (Sh_y) = 2

For Coordinates A(1, 1) : Let the new coordinates of corner A = (X_{new}, Y_{new}).

- $X_{new} = X_{old} + Sh_x \times Y_{old} = 1 + 2 \times 1 = 3$
- $Y_{new} = Y_{old} = 1$

Thus, New coordinates of corner A = (3, 1).

For Coordinates B(0, 0) : Let the new coordinates of corner B = (X_{new}, Y_{new}).

- $X_{new} = X_{old} + Sh_x \times Y_{old} = 0 + 2 \times 0 = 0$
- $Y_{new} = Y_{old} = 0$

Thus, New coordinates of corner B = (0, 0).

For Coordinates C(1, 0) : Let the new coordinates of corner C = (X_{new}, Y_{new}).

- $X_{new} = X_{old} + Sh_x \times Y_{old} = 1 + 2 \times 0 = 1$
- $Y_{new} = Y_{old} = 0$

Thus, New coordinates of corner C = (1, 0).

Shearing in Y-axis Example

Given a triangle with points (1, 1), (0, 0) and (1, 0). i) Apply shear parameter 2 on X axis and find out the new coordinates of the object. ii) Apply shear parameter 2 on Y axis and find out the new coordinates of the object.

Solution:

Given-

- Old corner coordinates of the triangle = A (1, 1), B(0, 0), C(1, 0)
- Shearing parameter towards X direction (Sh_x) = 2
- Shearing parameter towards Y direction (Sh_y) = 2

For Coordinates A(1, 1) : Let the new coordinates of corner A = (X_{new}, Y_{new}).

- $X_{new} = X_{old} = 1$
- $Y_{new} = Y_{old} + Sh_y \times X_{old} = 1 + 2 \times 1 = 3$

Thus, New coordinates of corner A = (1, 3).

For Coordinates B(0, 0) : Let the new coordinates of corner B = (X_{new}, Y_{new}).

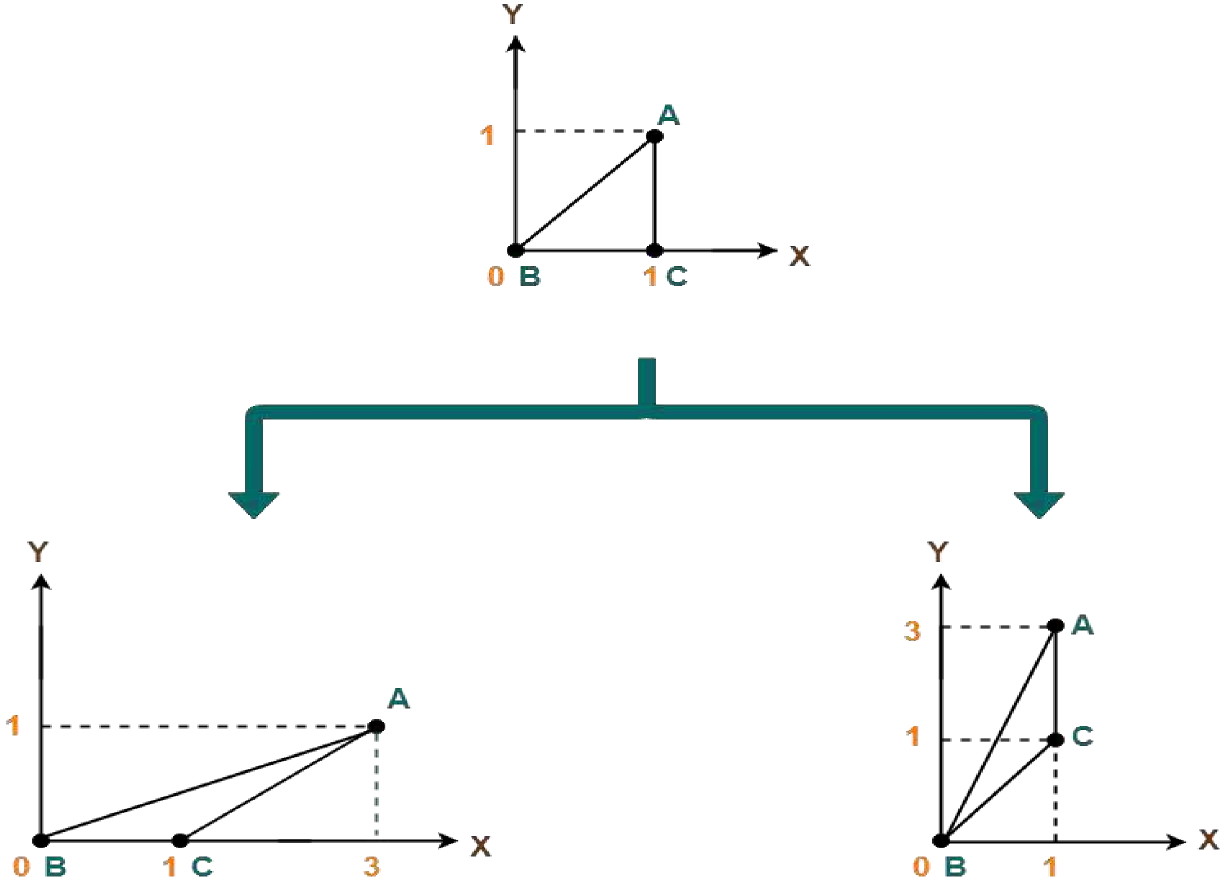
- $X_{new} = X_{old} = 0$
- $Y_{new} = Y_{old} + Sh_y \times X_{old} = 0 + 2 \times 0 = 0$

Thus, New coordinates of corner B = (0, 0).

For Coordinates C(1, 0) : Let the new coordinates of corner C = (X_{new}, Y_{new}).

- $X_{new} = X_{old} = 1$
- $Y_{new} = Y_{old} + Sh_y \times X_{old} = 0 + 2 \times 1 = 2$

Thus, New coordinates of corner C = (1, 2).



Shearing in X Axis

Shearing in Y Axis

Reflection

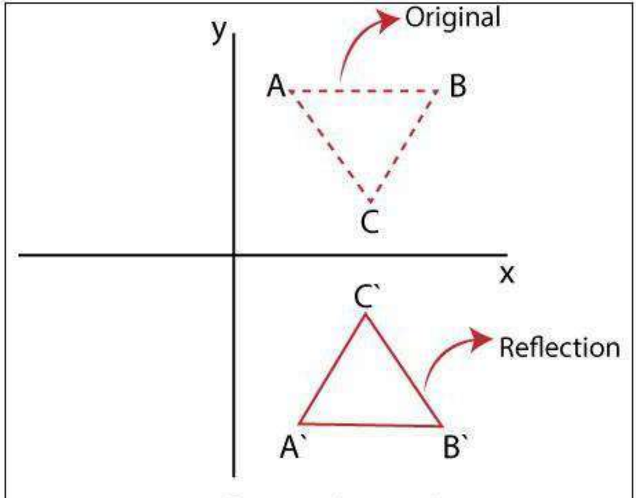
- Reflection is a kind of rotation where the angle of rotation is 180 degree.
- The reflected object is always formed on the other side of mirror.
- The size of reflected object is same as the size of original object.

Along X - axis :

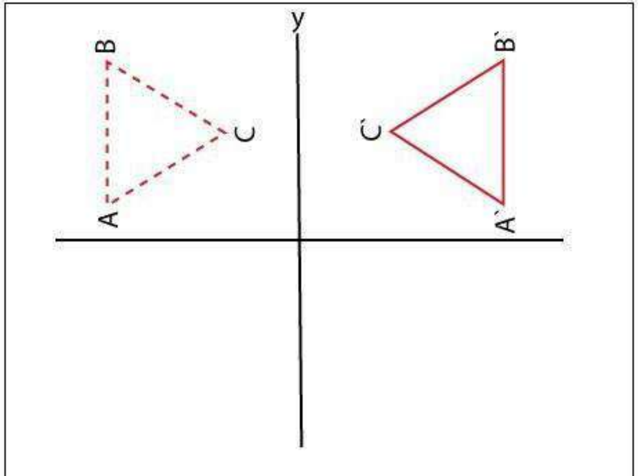
- $X_{new} = X_{old}$
- $Y_{new} = -Y_{old}$

Along Y - axis :

- $X_{new} = -X_{old}$
- $Y_{new} = Y_{old}$



Reflection along with x-axis



Reflection along with y-axis

Reflection on X-axis Example

Given a triangle with coordinate points A(3, 4), B(6, 4), C(5, 6). Apply the reflection on the X axis and obtain the new coordinates of the object.

Solution:

Given-

- Old corner coordinates of the triangle = A (3, 4), B(6, 4), C(5, 6)
- Reflection has to be taken on the X axis

For Coordinates A(3, 4) : Let the new coordinates of corner A = (X_{new}, Y_{new}).

- $X_{new} = X_{old} = 3$
- $Y_{new} = -Y_{old} = -4$

Thus, New coordinates of corner A = (3, -4).

For Coordinates B(6, 4) : Let the new coordinates of corner B = (X_{new}, Y_{new}).

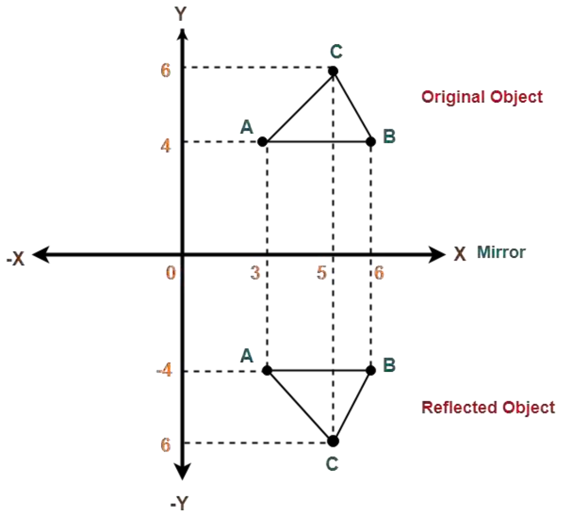
- $X_{new} = X_{old} = 6$
- $Y_{new} = -Y_{old} = -4$

Thus, New coordinates of corner B = (6, -4).

For Coordinates C(5, 6) : Let the new coordinates of corner C = (X_{new}, Y_{new}).

- $X_{new} = X_{old} = 5$
- $Y_{new} = -Y_{old} = -6$

Thus, New coordinates of corner C = (5, -6).



Reflection on Y-axis Example

Given a triangle with coordinate points A(3, 4), B(6, 4), C(5, 6). Apply the reflection on the Y axis and obtain the new coordinates of the object.

Solution:

Given-

- Old corner coordinates of the triangle = A (3, 4), B(6, 4), C(5, 6)
- Reflection has to be taken on the Y axis

For Coordinates A(3, 4) : Let the new coordinates of corner A = (X_{new}, Y_{new}).

- $X_{new} = -X_{old} = -3$
- $Y_{new} = Y_{old} = 4$

Thus, New coordinates of corner A = (-3, 4).

For Coordinates B(6, 4) : Let the new coordinates of corner B = (X_{new}, Y_{new}).

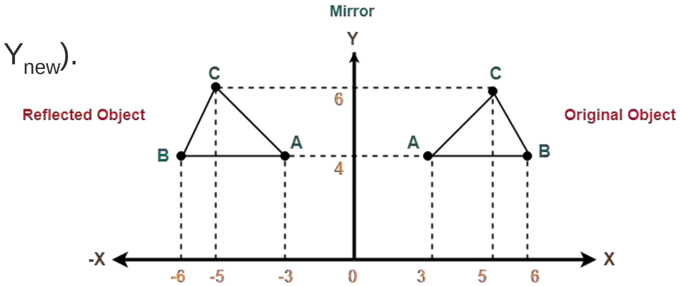
- $X_{new} = -X_{old} = -6$
- $Y_{new} = Y_{old} = 4$

Thus, New coordinates of corner B = (-6, 4).

For Coordinates C(5, 6) : Let the new coordinates of corner C = (X_{new}, Y_{new}).

- $X_{new} = -X_{old} = -5$
- $Y_{new} = Y_{old} = 6$

Thus, New coordinates of corner C = (-5, 6).



Matrix representation and Homogeneous Coordinates

Matrix Representation

- **Matrix representation** is a method used by a **computer** language to store **matrices** of more than one dimension in memory.

- Any transformation can be written using general equation i.e.

$$P' = M_1P + M_2$$

Where, $P' \rightarrow$ New Coordinates

$P \rightarrow$ Old Coordinates

$M_1 \rightarrow$ Multiplication Matrix

$M_2 \rightarrow$ Additive Matrix

1. Translation

$P' = T + P$

$P' = M_1P + M_2$

M_1 = Identity Matrix

M_2 = Translation Matrix $\begin{matrix} T_x \\ T_y \end{matrix}$

1. Scaling

$P' = SP$

$P' = M_1P + M_2$

M_1 = Scaling Matrix $\begin{bmatrix} S_X & 0 \\ 0 & S_Y \end{bmatrix}$

M_2 = Pivot point/Fixed point

1. Rotation

$P' = R(\theta)P$

$P' = M_1P + M_2$

M_1 = Rotation Matrix $\begin{bmatrix} \cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix}$

M_2 = Pivot point/Fixed point

Homogeneous Coordinate

- Homogeneous coordinates provide a method to perform certain standard operations on points in by means of matrix multiplications.
- For two-dimensional geometric transformation, we can choose homogeneous parameter h to any non-zero value.
- Each two-dimensional position is then represented with homogeneous coordinates (x, y, 1).
- It has to be added to the 2D coordinate to expand it to 3D coordinate

2D $\rightarrow$ 3D

$(x,y) \rightarrow (x_h, y_h, h)$

$$\mathbf{x_h = x.h}$$
$$\mathbf{y_h = y.h}$$

3D $\rightarrow$ 2D

$(x_h, y_h, h) \rightarrow (x,y)$

$$\mathbf{x = x_h/h}$$
$$\mathbf{y = y_h/h}$$

1. Translation Matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} T_x \\ T_y \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

P’ = T(t_x, t_y) . P

1. Scaling Matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} S_X & 0 \\ 0 & S_Y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_X & 0 & 0 \\ 0 & S_Y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

P’ = S(S_x, S_y) . P

1. Rotation Matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & sin\,\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

P’ = R(θ) . P

Composite Transformations

Composite Transformation

- A number of **transformations** or sequence of **transformations** can be combined into single one called as composition. The resulting matrix is called as **composite** matrix.
- The process of combining is called as concatenation. The ordering sequence of these numbers of **transformations** must not be changed.
- More complex geometric & coordinate transformations can be built from the basic transformation by using the process of composition of function.

In Translation (Two successive translations are additive)

- $P' = T(t_x, t_y)\, P$
- $P'' = T_2(t_x, t_y)\, [T_1(t_x, t_y)\, P]$
- $$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & T_{x1} \\ 0 & 1 & T_{y1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & T_{x2} \\ 0 & 1 & T_{y2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- $P' = T(t_{x1} + t_{x2}, t_y + t_{y2})\, P$
- $$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & T_{x1} + T_{x2} \\ 0 & 1 & T_{x1} + T_{y2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

In Scaling (Two successive scaling is multiplicative)

- $P' = S(S_x, S_y) \cdot P$
- $P'' = S_2(S_x, S_y)\ [S_1(S_x, S_y) \cdot P]$
- $$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_{x2} & 0 & 0 \\ 0 & S_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_{x1} & 0 & 0 \\ 0 & S_{y1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$
- **$P' = S(S_{x1}S_{x2}, S_{y1}S_{y2}) \cdot P$**
- $$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} S_{x2}S_{x1} & 0 & 0 \\ 0 & S_{y2}S_{y1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

In Rotation (Two successive rotations are additive)

- $P' = R(\theta) . P$
- $P'' = R(\theta_2)\ [R(\theta_1) . P]$
-

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 \\ -\sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

-

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_2\cos\theta_1 - \sin\theta_2\sin\theta_1 & \cos\theta_2\sin\theta_1 + \sin\theta_2\cos\theta_1 & 0 \\ -\sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1 & -\sin\theta_2\sin\theta_1 + \cos\theta_2\cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

-

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta_1+\theta_2) & \sin(\theta_1+\theta_2) & 0 \\ -\sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

-

Rotation about an arbitrary point (Xr, Yr)

1. Translate the reference point (x_r, y_r) to origin : Translation vector is given as,

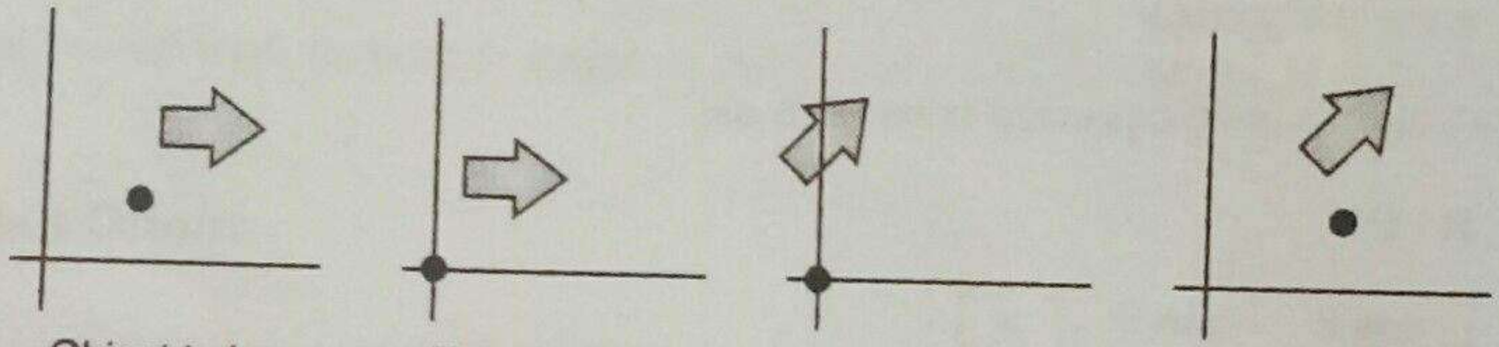
$$T = \begin{bmatrix} -x_r \\ -y_r \end{bmatrix}$$

2. Apply rotation by given angle θ. The rotation matrix is given as,

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

3. Translate reference point back to its actual location.

$$T^{-1} = \begin{bmatrix} x_r \\ y_r \end{bmatrix}$$



Object to be rotated and reference point

Translate the reference point to origin

Rotate object about origin by given angle

Translate back the reference point to original place

For column major representation, the sequence of operations is defined as :

$$P' = [T^{-1} + [R[T + P]]]$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \left\{\begin{bmatrix} x_r \\ y_r \end{bmatrix} + \left\{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \left\{\begin{bmatrix} -x_r \\ -y_r \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix}\right\}\right\}\right\}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \left\{\begin{bmatrix} x_r \\ y_r \end{bmatrix} + \left\{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x - x_r \\ y - y_r \end{bmatrix}\right\}\right\}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x_r \\ y_r \end{bmatrix} + \begin{bmatrix} (x - x_r)\cos\theta - (y - y_r)\sin\theta \\ (x - x_r)\sin\theta + (y - y_r)\cos\theta \end{bmatrix}$$

So,

$$x' = x_r + (x - x_r)\cos\theta - (y - y_r)\sin\theta$$

$$y' = y_r + (x - x_r)\sin\theta + (y - y_r)\cos\theta$$

Scaling about an arbitrary point (Xr, Yr)

1. Translate the reference point to the origin.

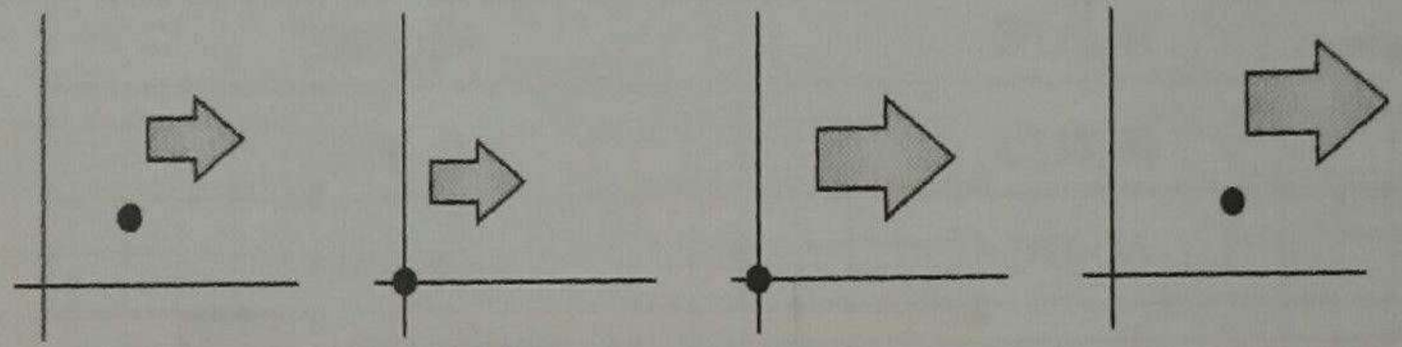
$$T = \begin{bmatrix} -x_r \\ -y_r \end{bmatrix}$$

2. Apply scaling on the translated object.

$$S = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix}$$

3. Translate reference point back to its actual location.

$$T^{-1} = \begin{bmatrix} x_r \\ y_r \end{bmatrix}$$



Scale arrow with respect to reference point

Translate reference point to origin

Apply scaling
$x' = x, S_x$
$y' = y, S_y$

Translate back the reference to the original place

- Recall that the transformation sequence is written from right to left for column-major representation. So, the transformed coordinates of an object are given by,

$$P' = [T^{-1} + [S[T + P]]]$$

$$P' = \left\{ \begin{bmatrix} x_r \\ y_r \end{bmatrix} + \left\{ \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \cdot \left\{ \begin{bmatrix} -x_r \\ -y_r \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} \right\} \right\} \right\}$$

$$= \left\{ \begin{bmatrix} x_r \\ y_r \end{bmatrix} + \left\{ \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \cdot \begin{bmatrix} x - x_r \\ y - y_r \end{bmatrix} \right\} \right\} = \begin{bmatrix} x_r \\ y_r \end{bmatrix} + \begin{bmatrix} (x - x_r)\, S_x \\ (y - y_r)\, S_y \end{bmatrix}$$

$$x' = x_r + (x - x_r)S_x = S_x \cdot x + x_r \cdot (1 - S_x)$$

$$y' = y_r + (y - y_r)S_y = S_y \cdot y + y_r \cdot (1 - S_y)$$

- As scaling parameters and reference points are fixed, the terms $(x_r \cdot (1 - S_x))$ and $(y_r \cdot (1 - S_y))$ are constant for all the vertices of the object.

Shear about a line parallel to the X-axis

We can derive generalized matrix for any reference line $Y = Y_{ref}$ parallel to X-axis using the following steps :

1. Translate line by $-Y_{ref}$ so the line gets aligned with X-axis.
2. Apply pure shear operation in the X-direction.
3. Translate line by $+Y_{ref}$ to bring it back to its original position.

$\therefore$ The composite transformation matrix would be, $M = T^{-1} \cdot SH_x \cdot T$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & y_{ref} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & Sh_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -y_{ref} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & y_{ref} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & Sh_x & -y_{ref} \cdot Sh_x \\ 0 & 1 & -y_{ref} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & Sh_x & -y_{ref} \cdot Sh_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Shear about a line parallel to the Y-axis

4.9.4 Shearing about a Line Parallel to Y-axis

We can derive generalized matrix for shearing about any reference line $X = X_{ref}$ parallel to Y-axis as follows:

1. Translate line by $-X_{ref}$ so the line will align with Y-axis.
2. Apply pure shear operation in the Y direction.
3. Translate line by $+X_{ref}$ to bring it back to the original place.

$\therefore$ The composite transformation matrix $M = T^{-1} \cdot Sh_y \cdot T$

$$M = \begin{bmatrix} 1 & 0 & x_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ Sh_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & x_{ref} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_{ref} \\ Sh_y & 1 & -Sh_y \cdot x_{ref} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ Sh_y & 1 & -Sh_y \cdot x_{ref} \\ 0 & 0 & 1 \end{bmatrix}$$

Y-direction shear relative to line $X = x_{ref}$ can be produced with transformation matrix :

$$M = \begin{bmatrix} 1 & 0 & 0 \\ Sh_y & 1 & -Sh_y \cdot x_{ref} \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection Transformation Matrices

Reflection about X-axis (Y=0 Line)

X’	**= 1**	**0**	**0**	**[x**
Y’	**= 0**	**-1**	**0**	**y**
1	**= 0**	**0**	**1**	**1]**

Reflection about Y-axis (X=0 Line)

-1	**0**	**0**
0	**1**	**0**
0	**0**	**1**

Reflection about X=Y

0	**1**	**0**
1	**0**	**0**
0	**0**	**1**

Reflection Transformation Matrices

Reflection about X=-Y

0	**-1**	**0**
-1	**0**	**0**
0	**0**	**1**

Reflection about origin

-1	**0**	**0**
0	**-1**	**0**
0	**0**	**1**

References

- Hearn & Baker, “Computer Graphics C version”, 2nd Edition, Pearson Publication