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A. P. SHAH INSTITUTE OF TECHNOLOGY
(Approved by AICTE New Delhi & Govt. of Maharashtra, Affiliated to University of Mumbai)
(Religious Jain Minority)

**Subject: Applied Mathematics III** **SEM: III**

## # To find Inverse Laplace using convolution Theorem.

If $f(t)$ and $g(t)$ are two functions then the following integral $\int_0^t f(u)\cdot g(t-u)\cdot du$ is called the convolution of $f(t)$ and $g(t)$ and it is denoted as, $f(t) * g(t)$ & given by,

$$f(t) * g(t) = \int_0^t f(u)\cdot g(t-u)\, du.$$

## # Convolution Theorem :

let, $f(t)$ and $g(t)$ be two functions and

$L[f(t)] = \phi(s)$ & $L[g(t)] = \psi(s)$

$\therefore \; f(t) = L^{-1}[\phi(s)]$ & $g(t) = L^{-1}[\psi(s)]$.

then $L^{-1}[\phi(s)\cdot\psi(s)] = \int_0^t f(u)\cdot g(t-u)\, du.$

Taking Laplace transforms of both sides

$$\therefore \; \phi(s)\cdot\psi(s) = L\left[\int_0^t f(u)\cdot g(t-u)\, du\right]$$

$$\therefore \; L[f(t)]\cdot L[g(t)] = L\left[\int_0^t f(u)\cdot g(t-u)\, du\right]$$

$\therefore$ The Laplace transform of the convolution of two functions is equal to the product of Laplace transforms of the two functions.

One can state above theorem, by using another notation :

$$L^{-1}[\phi_1(s)\cdot\phi_2(s)] = \int_0^t f_1(u)\cdot f_2(t-u)\cdot du.$$

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Note: Always consider smaller function as $\psi(s)$ while choosing $\phi(s)$ & $\psi(s)$.

# Problems: solve following using convolution theorem.

1) $L^{-1}\left[\frac{1}{s(s+a)}\right]$

Soln: let $\phi(s) = \frac{1}{s+a}$, $\psi(s) = \frac{1}{s}$.

$\therefore L^{-1}[\phi(s)] = L^{-1}\left[\frac{1}{s+a}\right] = e^{-at} = f(t)$

$L^{-1}[\psi(s)] = L^{-1}\left[\frac{1}{s}\right] = 1 = g(t)$.

By using convolution theorem.

$$L^{-1}[\phi(s)\cdot\psi(s)] = \int_0^t f(u)\cdot g(t-u)\,du$$

$$= \int_0^t e^{-au}\cdot 1\cdot du$$

$$= \left[\frac{e^{-au}}{-a}\right]_0^t$$

$$\therefore L^{-1}\left[\frac{1}{s(s+a)}\right] = \frac{-e^{-at}}{a} + \frac{1}{a}.$$

2) $L^{-1}\left[\frac{1}{s(s+a)^2}\right]$

Soln: let $\phi(s) = \frac{1}{(s+a)^2}$, $\psi(s) = \frac{1}{s}$.

$L^{-1}[\phi(s)] = e^{-at}\cdot t$, $L^{-1}[\psi(s)] = 1$

By convolution theorem

$$L^{-1}[\phi(s)\cdot\psi(s)] = \int_0^t f(u)\cdot g(t-u)\,du$$

$$= \int_0^t u\cdot e^{-au}\cdot 1\,du$$

$$= \left[u\cdot\frac{e^{-au}}{-a} - (+1)\left(\frac{e^{-au}}{a^2}\right)\right]_0^t$$

$$= \frac{t\cdot e^{-at}}{-a} - \frac{e^{-at}}{a^2} - 0 + \frac{1}{a^2}$$

$$= -\frac{t\cdot e^{-at}}{a} - \frac{e^{-at}}{a^2} + \frac{1}{a^2}.$$

3) $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$

**Soln**: let, $\phi(s) = \frac{1}{s^2+a^2}$ & $\psi(s) = \frac{1}{s}$

$$L^{-1}[\phi(s)] = \frac{1}{a}\sin at \quad \& \quad L^{-1}[\psi(s)] = 1$$

By convolution theorem,

$$L^{-1}[\phi(s)\cdot\psi(s)] = \int_0^t f(u)\cdot g(t-u)\,du$$

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_0^t \frac{1}{a}\sin au\cdot 1\,du$$

$$= \frac{-1}{a}\left[\frac{\cos au}{a}\right]_0^t$$

$$= \frac{-1}{a^2}[\cos at - 1]$$

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1-\cos at}{a^2}$$

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4) $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)^2}\right]$

<u>Sol<sup></sup></u>$^n$: consider, $\phi(s) = \dfrac{s}{s^2+a^2}$ & $\psi(s) = \dfrac{s}{s^2+a^2}$

$\because L^{-1}[\phi(s)] = \cos at$ & $L^{-1}[\psi(s)] = \cos at$

$\Rightarrow f(t) = \cos at$ & $g(t) = \cos at$

$\Rightarrow f(u) = \cos au$ & $g(t-u) = \cos a(t-u)$.

By convolution theorem,

$$L^{-1}[\phi(s)\cdot\psi(s)] = \int_0^t f(u)\cdot g(t-u)\,du$$

$$= \int_0^t \cos au \cdot \cos(at-au)\,du$$

$$= \frac{1}{2}\int_0^t \left(\cos at + \cos(2au-at)\right)du$$

— (using, $2\cos A\cdot\cos B = \cos(A+B) + \cos(A-B)$)

$$= \frac{1}{2}\left[u\cdot\cos at + \frac{\sin(2au-at)}{2a}\right]_0^t$$

$$= \frac{1}{2}\left[t\cos at + \frac{\sin at}{2a} + \frac{\sin at}{2a}\right]$$

$$= \frac{1}{2}\left[t\cos at + \frac{\sin at}{a}\right].$$

5) $L^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right]$

<u>Sol</u>$^n$: consider, $\phi(s) = \dfrac{s}{s^2+a^2}$ & $\psi(s) = \dfrac{1}{s^2+a^2}$.

$L^{-1}[\phi(s)] = \cos at = f(t)$ & $L^{-1}[\psi(s)] = \frac{1}{a}\sin at = g(t)$

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$$\therefore L^{-1}[\phi(s).\psi(s)] = \int_0^t f(u).\,g(t-u)\,du$$

$$= \int_0^t \cos au.\,\frac{1}{a}.\sin a(t-u)\,du$$

$$= \frac{1}{a}\int_0^t \cos au.\,\sin(at-au)\,du$$

$$= \frac{1}{2a}\int_0^t [\sin(at) - \sin(2au-at)]\,du$$

$$= \frac{1}{2a}\left[u.\sin at + \frac{\cos(2au-at)}{2a}\right]_0^t$$

$$= \frac{1}{2a}\left[t\sin at + \frac{\cos at}{2a} - \frac{\cos at}{2a}\right]$$

$$= \frac{1}{2a}.\,t\sin at.$$

- Examples for practice :

6) $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$ 7) $L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$

8) $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$ 9) $L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right]$.

10) $L^{-1}\left[\frac{1}{(s^2+9)(s^2+1)}\right]$

Sol$^n$: let, $\phi(s) = \frac{1}{s^2+9}$ & $\psi(s) = \frac{1}{s^2+1}$

$\therefore L^{-1}[\phi(s)] = \frac{\sin 3t}{3}$ & $L^{-1}[\psi(s)] = \sin t$

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$\therefore f(u) = \dfrac{\sin 3u}{3}$ & $g(t-u) = \sin(t-u)$

By convolution theorem

$$L^{-1}[\phi(s)\cdot\psi(s)] = \int_0^t f(u)\cdot g(t-u)\,du$$

$$L^{-1}\left[\frac{1}{s^2+9}\cdot\frac{1}{s^2+4}\right] = \int_0^t \frac{1}{3}\sin 3u\cdot\sin(t-u)\,du.$$

$$= \frac{1}{3}\cdot\frac{1}{2}\cdot\int_0^t \cos(3u-(t-u)) - \cos(3u+(t-u))\,du$$

$$= \frac{1}{6}\int_0^t (\cos(4u-t) - \cos(2u+t))\,du$$

$$= \frac{1}{6}\left[\frac{\sin(4u-t)}{4} - \frac{\sin(2u+t)}{2}\right]_0^t$$

$$= \frac{1}{6}\left[\frac{\sin 3t}{4} - \frac{\sin 3t}{2} - \frac{\sin(-t)}{4} + \frac{\sin t}{2}\right]$$

$$= \frac{1}{6}\left[-\frac{\sin 3t}{4} + \frac{3\sin t}{4}\right]$$

ii] $L^{-1}\left[\dfrac{s^2+s}{(s^2+1)(s^2+2s+2)}\right]$

Sol$^n$: $L^{-1}\left[\dfrac{s^2+s}{(s^2+1)(s^2+2s+2)}\right] = L^{-1}\left[\dfrac{s(s+1)}{(s^2+1)(s^2+2s+1+1)}\right]$

$$= L^{-1}\left[\frac{s(s+1)}{(s^2+1)((s+1)^2+1)}\right]$$

Let $\phi(s) = \dfrac{s+1}{(s+1)^2+1}$ & $\psi(s) = \dfrac{s}{s^2+1}$

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$\therefore L^{-1}[\phi(s)] = e^{-t}\cos t$ & $L^{-1}[\psi(s)] = \cos t$

$f(t) = e^{-t}\cos t$ & $g(t) = \cos t$

By convolution theorem

$$L^{-1}[\phi(s)\cdot\psi(s)] = \int_0^t f(u)\cdot g(t-u)\,du$$

$$L^{-1}\left[\frac{s+1}{(s+1)^2+1}\cdot\frac{s}{s^2+1}\right] = \int_0^t e^{-u}\cos u\cdot\cos(t-u)\,du$$

$$= \frac{1}{2}\int_0^t e^{-u}\left[\cos t + \cos(2u-t)\right]du$$

$$= \frac{1}{2}\int_0^t \left(e^{-u}\cos t + e^{-u}\cos(2u-t)\right)du$$

$$= \frac{1}{2}\left[\cos t\cdot\frac{e^{-u}}{(-1)} + \frac{e^{-u}}{1+4}\left[-\cos(2u-t) + 2\sin(2u-t)\right]\right]_0^t$$

$$= \frac{1}{2}\left[-\cos t\cdot e^{-t} + \frac{e^{-t}}{5}(-\cos t + 2\sin t) + \cos t - \frac{1}{5}(-\cos t + 2\sin(-t))\right]$$

$$= \frac{1}{2}\left[-\cos t\, e^{-t} + \frac{e^{-t}}{5}(-\cos t + 2\sin t) + \cos t - \frac{1}{5}(-\cos t - 2\sin t)\right]$$

$$= \frac{1}{2}\left[\frac{-6}{5}\cos t\, e^{-t} + \frac{2}{5}e^{-t}\sin t + \frac{6\cos t}{5} + \frac{2\sin t}{5}\right]$$

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