Non-Cooperative Game Theory:

1. Computing Nash Equilibria of Two-Player, Zero-Sum Games

Question: Two security firms are competing to provide cyber-security services to a large client. Both firms can choose between investing in high-end security software (high investment) or using basic software (low investment). The success of each firm is dependent on the other firm's decision. Model this competition as a two-player, zero-sum game and compute the Nash equilibrium. How does the equilibrium influence the firms' investment decisions?

Answer: In this scenario, we model the two firms as follows, where the payoff matrix reflects the gain of one firm and the loss of the other:

	High Investment (A)	**Low Investment (A)**
High Investment (B)	(0, 0)	(1, -1)
Low Investment (B)	(-1, 1)	(0, 0)

- In a zero-sum game, the goal of each firm is to minimize the maximum possible loss. By examining the strategies, neither firm gain from unilaterally switching its strategy once the equilibrium is reached. The Nash equilibrium occurs when both firms invest in high-end security software (High Investment, High Investment), ensuring that neither firm can improve its outcome without reducing the opponent's security level.

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2. Computing Nash Equilibria of Two-Player, General-Sum Games

Question: In a market for electric vehicles, two companies are competing by investing in either battery technology (high-cost, high-reward) or electric charging infrastructure (low-cost, moderate reward). The payoffs depend on the combination of investments made. Model this situation as a two-player, general-sum game, compute the Nash equilibrium, and explain the resulting strategies.

Answer: The normal form game can be structured as:

	Battery Tech (A)	**Charging Infra (A)**
Battery Tech (B)	(8, 8)	(6, 5)
Charging Infra (B)	(5, 6)	(7, 7)

- In this general-sum game, we compute the Nash equilibrium by comparing the payoffs:
  - If Company A invests in battery technology and Company B also invests in battery technology, both gain equally (8, 8).
  - If one invests in battery and the other in charging, the gains are lower.

The Nash equilibrium occurs when both companies choose Battery Tech, as they achieve the highest mutual payoff (8, 8). This suggests that the optimal strategy for both companies is to invest heavily in technology rather than infrastructure, despite the mutual benefits of charging infrastructure.

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3. Identifying Dominated Strategies

Question: In a competitive marketplace, two telecom companies must decide whether to enter a new 5G market or stay with their current 4G technology. The potential payoffs for entering the 5G market depend on whether the other company also enters. Model this decision process and identify any dominated strategies for the companies. How does eliminating dominated strategies affect the analysis?

Answer: The payoff matrix can be structured as follows:

	Enter 5G (A)	**Stay 4G (A)**
Enter 5G (B)	(7, 7)	(5, 8)
Stay 4G (B)	(8, 5)	(6, 6)

- By examining the strategies:

- For Company A, staying with 4G is dominated by entering the 5G market, as the payoff from entering is always equal to or higher than staying.
- Similarly, for Company B, entering the 5G market dominates staying with 4G.

Once we eliminate the dominated strategies (staying with 4G), the Nash equilibrium is clearly (Enter 5G, Enter 5G) with payoffs (7, 7). This result indicates that both companies will enter the new market for 5G, even though there is some incentive to free ride by staying with 4G if the other company invests in 5G.

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4. Mixed-Strategy Nash Equilibrium

- **Question:** Two rival retailers must decide whether to launch a new product in the upcoming holiday season. If both launch, the market becomes highly competitive, but if only one launches, they capture the market. Given the uncertainty, each retailer can either "Launch" or "Wait," and they are willing to randomize their strategies. Model this as a mixed-strategy Nash equilibrium problem and calculate the equilibrium strategy for each retailer.
- **Answer:** The payoff matrix is as follows:

	Launch (A)	**Wait (A)**
Launch (B)	(2, 2)	(5, 0)
Wait (B)	(0, 5)	(3, 3)

- To find the mixed-strategy Nash equilibrium, let the probability that Company A launches be p and that Company B launches be q. Each company will be indifferent to launching or waiting if the expected payoffs are equal:
- For Company A:

$p(2)+(1-p)(0)=q(5)+(1-q)(3)p(2) + (1 - p)(0) = q(5) + (1 - q)(3)p(2)+(1-p)(0)=q(5)+(1-q)(3)$

- Solving these equations for p and q, we find the mixed strategies where both companies randomize between launching and waiting. The Nash equilibrium will have both companies launching with a certain probability that keeps their expected payoffs balanced.

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5. Identifying Dominated Strategies in a General-Sum Game

- **Question:** In a scenario where two software firms must decide whether to collaborate on a new project or work independently, the potential payoffs are as follows. Identify the dominated strategies, if any, and determine the equilibrium outcome.

	Collaborate (A)	**Work Alone (A)**
Collaborate (B)	(6, 6)	(4, 7)
Work Alone (B)	(7, 4)	(5, 5)

- **Answer:** Analyzing the strategies:
  - For Company A, "Work Alone" is dominated by "Collaborate," as collaboration yields higher payoffs regardless of what B chooses.
  - For Company B, "Work Alone" is also dominated by "Collaborate."

Therefore, the dominated strategy for both companies is "Work Alone." After eliminating dominated strategies, the Nash equilibrium is (Collaborate, Collaborate), yielding payoffs (6, 6). This outcome suggests that both firms will prefer collaboration to working alone, as it maximizes their joint and individual payoffs.

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6. Computing Solution Concepts of Normal-Form Games

- **Question:** Two pharmaceutical companies are deciding how much to invest in research and development (R&D) for a new vaccine. The amount of R&D investment impacts the likelihood of success, but each firm wants to minimize their costs. Model this situation as a normal-form game, and compute the Nash equilibrium if both firms choose their R&D investment levels simultaneously.

- **Answer:** The normal-form payoff matrix could look like this:

	High Investment (A)	**Low Investment (A)**
High Investment (B)	(10, 10)	(5, 12)
Low Investment (B)	(12, 5)	(8, 8)

- Both firms face trade-offs between maximizing their own profit and reducing costs. The Nash equilibrium is found when neither firm can unilaterally change its investment to improve its payoff. In this case, the Nash equilibrium is (High Investment, High Investment) with payoffs (10, 10), indicating that both firms will opt for a high level of R&D investment, leading to the most likely success in vaccine development.

7. **A company is deciding whether to enter a new market. If they enter, they must decide how much to produce, and they know that another firm is already in the market. The market price is determined by the total output of both firms. How can Cournot's model of oligopoly be applied to predict the equilibrium quantities produced by both firms? What strategy should the new entrant adopt?**

Answer: Cournot's model applies to this scenario where two firms compete by deciding how much quantity to produce. According to the model, each firm assumes that the other firm's output is fixed and chooses its own output to maximize profit. The Nash Equilibrium in Cournot competition occurs when neither firm can increase its profit by unilaterally changing its own output.

To predict the equilibrium:

1. Let the market price be determined by $P(Q) = a - bQ$, where $Q = q_1 + q_2$ is the total quantity produced by both firms.
2. Each firm maximizes its profit: $\pi_1 = q_1(P(Q) - c)$ for firm 1 and similarly for firm 2.
3. Solve the reaction functions for each firm's output:
   $q_1^*(q_2) = \frac{a-bq_2-c}{2b}$
   $q_2^*(q_1) = \frac{a-bq_1-c}{2b}$
4. At equilibrium, both firms' outputs are determined by solving the system of reaction functions.

The new entrant should carefully calculate its output based on the incumbent firm's output. In equilibrium, the new entrant should produce a quantity where their marginal cost equals their marginal revenue, factoring in the impact on market price from total output.

8. **In an extensive game with perfect information, two technology firms are deciding whether to collaborate on a new project. Firm A moves first, choosing whether to invest in joint research or not. Firm B observes this and then decides whether to collaborate or pursue a solo project. How can Subgame Perfect Nash Equilibrium (SPNE) be used to analyze this decision-making process? Illustrate the steps involved in backward induction to find the SPNE.**

Answer: Subgame Perfect Nash Equilibrium (SPNE) is found by analyzing the game backward (backward induction). Since Firm B moves after observing Firm A's choice, Firm A anticipates Firm B's reaction when making its decision.

Steps to find the SPNE:

1. **Backward Induction:** Start by analyzing Firm B's choices at each possible decision point. If Firm A invests, Firm B will choose the best response (collaborate or solo project) based on its own payoff.
2. If Firm A does not invest, Firm B again chooses the best response (likely solo project).
3. Based on Firm B's response, Firm A chooses whether to invest, knowing how Firm B will react.

To find the SPNE:

- Firm B's optimal choice after observing Firm A's decision is determined by comparing payoffs in each subgame (whether Firm A invests or not).

- Firm A then selects its move (invest or not) based on which outcome gives it the highest payoff, factoring in Firm B's optimal response.

In this case, the SPNE ensures that both firms are making decisions that are optimal at every stage of the game.

1. **Players:** Firm A and Firm B.
2. **Actions:**
   - **Firm A** (First mover) chooses between:
     - **Invest** in joint research (denote this as **I**).
     - **Not Invest** (denote this as **N**).
   - **Firm B** (Second mover) observes Firm A's action and then chooses:
     - If Firm A **Invests** (**I**), Firm B chooses between:
       - **Collaborate** (denote this as **C**).
       - **Solo Project** (denote this as **S**).
     - If Firm A **Does Not Invest** (**N**), Firm B chooses:
       - **Collaborate** (denote this as **C**).
       - **Solo Project** (denote this as **S**).

Payoffs

To analyze the game, we need to define the payoffs for each outcome. Let's assume the following hypothetical payoffs:

- If **A invests (I)** and **B collaborates (C)**: Payoff (A: 4, B: 4).
- If **A invests (I)** and **B chooses solo (S)**: Payoff (A: 0, B: 5).
- If **A does not invest (N)** and **B collaborates (C)**: Payoff (A: 1, B: 2).
- If **A does not invest (N)** and **B chooses solo (S)**: Payoff (A: 2, B: 3).

Step 1: Analyze Firm B's Decisions (Second mover)

We will start by analyzing Firm B's choices after observing Firm A's action.

If Firm A Invests (I):

- **Firm B's Options:**
  - If **B Collaborates (C)**: Payoff is (4, 4).
  - If **B Chooses Solo (S)**: Payoff is (0, 5).

Best Response: Firm B will choose to **Collaborate (C)** because it provides a higher payoff for both (4 > 0).

If Firm A Does Not Invest (N):

- **Firm B's Options:**
  - If **B Collaborates (C)**: Payoff is (1, 2).
  - If **B Chooses Solo (S)**: Payoff is (2, 3).

Best Response: Firm B will choose to **Choose Solo (S)** because it provides a higher payoff for both (2 > 1).

Step 2: Analyze Firm A's Decision (First mover)

Now, we will determine Firm A's best action based on Firm B's responses.

If Firm A Invests (I):

- Firm B will collaborate, leading to a payoff of (4, 4) for A and B.

If Firm A Does Not Invest (N):

- Firm B will choose solo, leading to a payoff of (2, 3) for A and B.

Step 3: Determine Firm A's Best Response

- **If A chooses I:** A gets a payoff of 4.
- **If A chooses N:** A gets a payoff of 2.

Best Response for Firm A: Firm A will choose to **Invest (I)**, as it maximizes its payoff (4 > 2).

Step 4: Conclusion and SPNE

The SPNE of this game can be represented as follows:

1. Firm A chooses to **Invest (I)**.
2. Firm B observes this and chooses to **Collaborate (C)**.

Illustration of Backward Induction Steps:

1. **Analyze Firm B's Best Responses:**
   - After A invests (I), B will collaborate (C).
   - After A does not invest (N), B will choose solo (S).
2. **Determine Firm A's Best Initial Move:**
   - Comparing the payoffs for A:
     - If A chooses I: (4, 4).
     - If A chooses N: (2, 3).

- Thus, A chooses to **Invest (I)**.

3. **Establish SPNE:**
    - The SPNE of the game is: (I, C) leading to payoffs of (4, 4).

9. **Two companies are bidding for a government contract through a sealed-bid first-price auction. Both companies have private valuations of the contract, and they must choose their bid without knowing the bid of the other. How should each company decide on their bid to maximize their chances of winning while still earning a profit? What strategic considerations apply in this auction setting?**

Answer: In a sealed-bid first-price auction, each company must balance the desire to win the auction against the risk of bidding too high and eroding their profit margin. The optimal strategy involves bidding less than their private valuation of the contract but still high enough to outbid competitors.

In a first-price auction, each company wants to maximize their expected utility, which can be formulated as:

- $U_A = P_A \times (v_A - b_A)$
- $U_B = P_B \times (v_B - b_B)$

Where:

- P_A and P_B are the probabilities of winning for Company A and Company B, respectively.
- b_A and b_B are the bids submitted by Company A and Company B.

1. **Bid Shading:** Each company should "shade" its bid—bid less than its true valuation to preserve some profit margin if it wins.
2. The optimal bid, b_i*, is generally lower than the true valuation vi, and depends on beliefs about the competitor's valuation. A common approach is to bid the expected second-highest bid, slightly below the competitor's valuation.
3. **Strategic Consideration:** Each firm needs to estimate the distribution of the opponent's bid based on market conditions and adjust its own bid to balance the trade-off between maximizing profit and winning the auction. Risk-averse firms may shade their bids less than risk-neutral ones.

Thus, each firm calculates an optimal bid based on its valuation and estimates about its rival's strategy.

10. **A tech company is considering adopting a new technology but is unsure about the market demand. They know that another competitor is also considering the same technology. Using Bertrand's model of price competition, analyze how the companies should set their prices in order to maximize profit, given that both will release the technology in the market.**

Answer: In Bertrand competition, firms compete on price rather than quantity. In this scenario, both companies choose their prices simultaneously, and consumers buy from the company offering the lowest price. The key insight is that in a Bertrand model, price competition leads to prices being driven down to the marginal cost of production.

- Profit of Company A:

$$\pi_A = (P_A - C) \cdot Q_A$$

- Profit of Company B:

$$\pi_B = (P_B - C) \cdot Q_B$$

In the Bertrand model, firms will engage in a price war to capture the entire market share. The key points are:

- If **P_A** > **P_B**, then Company B captures the entire market.
- If **P_A** < **P_B**, then Company A captures the entire market.
- If **P_A** = **P_B**, they share the market evenly.

This leads to the following strategic outcomes:

1. **Price Competition:** Each firm will try to undercut the other by offering a slightly lower price to capture the entire market. The equilibrium occurs when prices are equal to marginal cost because any higher price would allow the competitor to undercut and steal the market share.
2. **Outcome:** At equilibrium, both firms will set their prices equal to marginal cost, and neither can raise their price without losing all customers to the competitor. Profit margins become thin, and the market reaches a highly competitive state.

The optimal strategy for each firm in Bertrand competition is to price as close to marginal cost as possible, as any attempt to price higher will be met with undercutting by the rival.

11. In the context of combinatorial auctions like those used in online advertising (e.g., Google's sponsored search), explain how multiple companies can bid on combinations of ad slots. How can the Vickrey-Clarke-Groves (VCG) mechanism be applied to ensure efficient allocation of these ad slots? What is the role of truthful bidding in this setting?

Answer: In combinatorial auctions, bidders can bid on bundles of items (ad slots in this case), and the VCG mechanism is designed to allocate items efficiently while encouraging truthful bidding.

1. **VCG Mechanism:** Each bidder submits a bid for a combination of ad slots. The VCG mechanism allocates the slots to maximize total social welfare—i.e., the sum of the valuations of the winning bids.

2. **Truthful Bidding:** The VCG mechanism incentivizes truthful bidding because each bidder pays an amount equal to the harm they cause to other bidders by winning (the difference between the total welfare with and without their participation). Since the payment is not based on their own bid, bidders have no incentive to misreport their valuations.
3. **Efficiency:** The VCG mechanism ensures that ad slots are allocated to the bidders who value them the most, leading to an efficient outcome.

Thus, in online platforms like Google's sponsored search, the VCG mechanism ensures both efficiency in allocation and truthfulness in bidding behavior.

12. Consider a repeated Prisoner's Dilemma scenario where two competing businesses are deciding whether to cooperate by maintaining high prices or defect by lowering prices to capture more market share. How do strategies in infinitely repeated games help to sustain cooperation between these businesses? Discuss how the concept of Nash equilibria in repeated games applies here.

Answer: In infinitely repeated games, cooperation can be sustained because businesses can punish each other for defecting, using strategies like **grim-trigger** or **tit-for-tat**.

1. **Grim-Trigger Strategy:** Each firm cooperates (maintains high prices) as long as the other firm cooperates. If one firm defects (lowers prices), the other will retaliate by defecting in all future rounds. The fear of permanent price wars keeps both firms cooperating.
2. **Nash Equilibria:** In infinitely repeated games, there can be many Nash equilibria, including cooperative outcomes. The threat of future punishment (e.g., a price war) makes it rational for firms to cooperate today, as the long-term benefits of cooperation outweigh the short-term gain from defecting.
3. **Sustainability of Cooperation:** If both firms are sufficiently patient (i.e., they value future payoffs highly), cooperation can be a Nash equilibrium because each firm prefers the ongoing profit from high prices over the one-time gain from defection followed by retaliation.

Thus, repeated interactions enable businesses to sustain cooperation through strategies that enforce mutual cooperation.

Mechanism Design and Aggregating Preferences

1. Designing an Efficient Voting Mechanism

Question: Consider a group of individuals with distinct preferences over three possible outcomes: **X**, **Y**, and **Z**. Using the **Borda Count** voting system, determine the winner if the preferences of the individuals are as follows:

- Voter 1: X > Y > Z
- Voter 2: Y > Z > X
- Voter 3: Z > X > Y

Show the calculation for each outcome and explain why this voting system might lead to different results compared to a **Plurality Voting** system.

Answer:

- **Borda Count Calculation:**
    - Voter 1: X gets 2 points, Y gets 1 point, Z gets 0 points.
    - Voter 2: Y gets 2 points, Z gets 1 point, X gets 0 points.
    - Voter 3: Z gets 2 points, X gets 1 point, Y gets 0 points.

  Total points:

    - X: 2 + 0 + 1 = 3 points
    - Y: 1 + 2 + 0 = 3 points
    - Z: 0 + 1 + 2 = 3 points

  There is a tie between X, Y, and Z.

- **Plurality Voting Outcome:** X gets 1 vote, Y gets 1 vote, and Z gets 1 vote (tie again).
- **Explanation:** In Borda count, the rankings of all candidates influence the outcome, whereas plurality voting only considers the top preference. This illustrates how different voting systems aggregate preferences differently, potentially leading to different social choices.

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2. Applying Vickrey-Clarke-Groves (VCG) Mechanism

Question: A city wants to construct a new road connecting three locations: **A**, **B**, and **C**. Each resident has a private value for the construction of the road. The reported values for each location are as follows:

- Resident 1: A = 10, B = 5, C = 8
- Resident 2: A = 7, B = 8, C = 5
- Resident 3: A = 5, B = 6, C = 9

Apply the **VCG mechanism** to determine the location that should be selected for the road, and compute the payments each resident should make under this mechanism.

Answer:

- **Step 1: Total values for each location:**
  - A: 10 + 7 + 5 = 22
  - B: 5 + 8 + 6 = 19
  - C: 8 + 5 + 9 = 22

  Since A and C tie for the highest total value, A is selected based on a tiebreaker.

- **Step 2: Determine each resident's payment:** Resident 1's value for the chosen outcome (A) is 10, but they must pay the difference between the total welfare without their participation and the total welfare with their participation.

  For example, Resident 1's payment is the value of choosing A without their bid (7 + 5 = 12), minus the value of the chosen outcome (A with them, 22), meaning they pay (22 - 12) = 10.

  Payments for Resident 2 and 3 can be similarly calculated.

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3. Combinatorial Auctions Application

Question: Suppose an auction is being held for two items, **X** and **Y**, with three bidders. The bids are as follows:

- Bidder 1: $5 for X, $8 for Y
- Bidder 2: $7 for X, $10 for Y
- Bidder 3: $10 for the bundle {X, Y}

Use the **Vickrey Auction** approach to determine the winners and the payments each bidder makes.

Answer:

- **Step 1: Identifying the winning bid:** The highest bid is $10 by Bidder 3 for the bundle {X, Y}, so they win the auction.
- **Step 2: Payment under Vickrey rules:** In a Vickrey auction, the winner pays the second-highest bid. The second-highest bid in this case is $7 for X + $10 for Y = $17 (if items were sold separately). However, since Bidder 3 won the bundle, they only pay the highest non-winning bid for the entire bundle, which is $8 + $7 = $15.
- **Result:** Bidder 3 wins the bundle {X, Y} and pays $15.

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4. Profit Maximization and VCG Mechanism in Sponsored Search

Question: Google uses the **VCG mechanism** for sponsored search auctions. Suppose three advertisers are bidding for two ad slots. Their bids per click are as follows:

- Advertiser 1: $4
- Advertiser 2: $3
- Advertiser 3: $2

Each advertiser values the top ad slot twice as much as the second ad slot. Apply the **VCG mechanism** to determine which advertisers get which slots and what they will pay.

Answer:

- **Step 1: Assign slots based on bids:** Advertiser 1 gets the top slot, and Advertiser 2 gets the second slot.
- **Step 2: Payments using VCG:**
    - Advertiser 1 pays the bid of Advertiser 2 (the second-highest bid), which is $3.
    - Advertiser 2 pays the bid of Advertiser 3 (the third-highest bid), which is $2.

  This reflects each advertiser's payment as per the VCG mechanism, where the payment is based on the externality they impose on other advertisers.

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5. K-Armed Bandits in Mechanism Design

Question: A company is testing multiple advertising strategies and models this scenario as a **K-armed bandit** problem, where each strategy (arm) provides uncertain rewards. The company wants to maximize profit over time while exploring different strategies. Explain how the **explore-exploit trade-off** can be handled in this case and describe an algorithm (such as **Upper Confidence Bound - UCB**) to balance exploration and exploitation in this context.

Answer:

- **Explore-Exploit Trade-off:** The company must decide whether to exploit the current best-known strategy (arm) or explore other strategies that may lead to higher long-term rewards.
- **UCB Algorithm:** The UCB algorithm addresses this by selecting the strategy (arm) that maximizes the upper confidence bound on the expected reward. The selection formula is:

$$\text{Select arm } k = \arg\max_k \left(\hat{\mu}_k + \sqrt{\frac{2 \log n}{n_k}} \right)$$

where $\hat{\mu}_k$ is the average reward of arm k, n is the total number of trials, and n_k is the number of times arm k has been selected. This ensures that arms with fewer trials get more exploration.

- **Application:** The company uses this algorithm to balance exploring new strategies and exploiting the current best-performing one, maximizing its profit over time by refining its advertising strategies.

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6. Computational Applications in eBay Auctions

Question: eBay auctions use a **proxy bidding system** similar to the Vickrey auction mechanism. Explain how this system works and how it prevents overbidding. Provide a real-world example where this mechanism benefits both the buyer and the seller.

Answer:

- **Proxy Bidding System:** In eBay's system, buyers submit their maximum bid, and the system automatically increases their bid in small increments to outbid others, up to their maximum bid. The winner pays the second-highest bid plus an increment, akin to a Vickrey auction.
- **Prevention of Overbidding:** Since bidders only pay the second-highest price, there is no incentive to bid more than their true value, preventing overbidding.
- **Example:** A bidder sets a maximum bid of $100 for an item. Another bidder sets a maximum of $90. The first bidder wins the auction but only pays $91 (second-highest bid + increment). This benefits the buyer by not paying their full bid and benefits the seller by ensuring competitive bidding.

7. Designing an Efficient Voting Mechanism and Its Limitations

a) Consider a group of individuals with distinct preferences over three outcomes: **X**, **Y**, and **Z**. Using the **Borda Count** voting system, determine the winner if the preferences of the individuals are as follows:

- Voter 1: X > Y > Z
- Voter 2: Y > Z > X
- Voter 3: Z > X > Y

Show the calculation for each outcome and compare the results with the **Plurality Voting** system.

b) Explain how different voting mechanisms might lead to different social choices. Discuss the limitations of the **Borda Count** and **Plurality Voting** systems in aggregating preferences.

Answer:

a) Borda Count and Plurality Voting Calculation

Borda Count Voting System:

- In the Borda count system, each voter ranks the candidates. The first choice gets 2 points (since there are 3 outcomes), the second choice gets 1 point, and the third choice gets 0 points. We calculate the scores for **X**, **Y**, and **Z**:

Voter	First Choice	Second Choice	Third Choice
Voter 1	X (2)	Y (1)	Z (0)
Voter 2	Y (2)	Z (1)	X (0)
Voter 3	Z (2)	X (1)	Y (0)

Total Points for Each Outcome:

- X = 2 (Voter 1) + 0 (Voter 2) + 1 (Voter 3) = **3 points**
- Y = 1 (Voter 1) + 2 (Voter 2) + 0 (Voter 3) = **3 points**
- Z = 0 (Voter 1) + 1 (Voter 2) + 2 (Voter 3) = **3 points**

Since all outcomes have the same score, the result is a tie in the Borda count.

Plurality Voting System:

In the plurality voting system, only the first-choice votes are counted.

- X: 1 vote (Voter 1)
- Y: 1 vote (Voter 2)
- Z: 1 vote (Voter 3)

In this case, all outcomes also receive 1 vote, resulting in a tie.

b) Limitations of Voting Mechanisms

- **Borda Count:** While Borda count considers more information about voters' preferences, it can lead to ties when preferences are evenly distributed, as seen here. It is also vulnerable to manipulation, where a voter might misrepresent their preferences to influence the outcome.
- **Plurality Voting:** Plurality voting ignores the intensity of preferences and only considers first-choice votes, which can result in outcomes that do not reflect the overall preference of the voters. It is also susceptible to the "spoiler effect," where a less popular candidate can take votes away from a similar but stronger candidate, leading to an undesirable result

8. **Applying Vickrey-Clarke-Groves (VCG) Mechanism and Combinatorial Auctions**

a) A city wants to construct a new road connecting three locations: A, B, and C. Each resident has a private value for the construction of the road. The reported values for each location are:

- Resident 1: A = 10, B = 5, C = 8
- Resident 2: A = 7, B = 8, C = 5
- Resident 3: A = 5, B = 6, C = 9

Apply the **VCG mechanism** to determine the location that should be selected for the road, and compute the payments each resident should make under this mechanism.

b) Suppose an auction is being held for two items, **X** and **Y**, with three bidders. The bids are as follows:

- Bidder 1: $5 for X, $8 for Y
- Bidder 2: $7 for X, $10 for Y
- Bidder 3: $10 for the bundle {X, Y}

Use the **Vickrey Auction** approach to determine the winners and the payments each bidder makes.

Answer:

a) VCG Mechanism for Road Construction

Each resident reports their value for constructing the road to each location (A, B, or C). The VCG mechanism selects the option that maximizes the total welfare (sum of reported values) and computes payments to ensure truthfulness.

- **Total Value for A:** 10 (Resident 1) + 7 (Resident 2) + 5 (Resident 3) = **22**
- **Total Value for B:** 5 (Resident 1) + 8 (Resident 2) + 6 (Resident 3) = **19**
- **Total Value for C:** 8 (Resident 1) + 5 (Resident 2) + 9 (Resident 3) = **22**

Both **A** and **C** maximize welfare with a total value of 22. Let's assume location **A** is selected.

Now, compute payments using the VCG payment rule:

- **Resident 1's Payment:** The welfare of others if A is not chosen = max(7 + 6, 8 + 9) = 17. Resident 1's value = 10. Payment = 17 - (22 - 10) = **5**
- **Resident 2's Payment:** The welfare of others if A is not chosen = max(10 + 6, 8 + 9) = 19. Resident 2's value = 7. Payment = 19 - (22 - 7) = **4**
- **Resident 3's Payment:** The welfare of others if A is not chosen = max(10 + 7, 8 + 5) = 17. Resident 3's value = 5. Payment = 17 - (22 - 5) = **0**

So, **Resident 1 pays 5**, **Resident 2 pays 4**, and **Resident 3 pays 0**.

b) Combinatorial Auction Using Vickrey Auction

- **Bids for Items:**
  - Bidder 1: X = $5, Y = $8
  - Bidder 2: X = $7, Y = $10
  - Bidder 3: Bundle {X, Y} = $10

Step 1: Determine winners using the Vickrey Auction:

- Bidder 2 wins **Y** with a bid of $10.
- Bidder 1 wins **X** with a bid of $5.

Step 2: Compute payments:

- **Bidder 2's payment for Y**: The highest losing bid for Y is $8 (Bidder 1's bid), so Bidder 2 pays $8 for Y.
- **Bidder 1's payment for X**: The highest losing bid for X is $7 (Bidder 2's bid), so Bidder 1 pays $7 for X.

Bidder 1 pays **$7** for X, and Bidder 2 pays **$8** for Y.

9. Profit Maximization in Sponsored Search and K-Armed Bandits

a) Google uses the **VCG mechanism** for sponsored search auctions. Suppose three advertisers are bidding for two ad slots. Their bids per click are as follows:

- Advertiser 1: $4
- Advertiser 2: $3
- Advertiser 3: $2

Each advertiser values the top ad slot twice as much as the second ad slot. Apply the **VCG mechanism** to determine which advertisers get which slots and what they will pay.

b) A company is testing multiple advertising strategies and models this scenario as a **K-armed bandit** problem. Explain how the **explore-exploit trade-off** can be handled using an algorithm like **Upper Confidence Bound (UCB)** to balance exploration and exploitation in this context.

Answer:

a) VCG Mechanism for Sponsored Search Ads

Advertisers are bidding for two ad slots with the following bids per click:

- **Advertiser 1:** $4
- **Advertiser 2:** $3
- **Advertiser 3:** $2

The value for the top slot is twice that of the second slot.

Step 1: Assign slots using the VCG mechanism:

- Advertiser 1 gets the top slot.
- Advertiser 2 gets the second slot.

Step 2: Compute payments:

- **Advertiser 1's payment:** If Advertiser 1 were removed, Advertiser 2 would get the top slot and pay $2. Hence, Advertiser 1 pays $2 per click.
- **Advertiser 2's payment:** If Advertiser 2 were removed, Advertiser 3 would get the second slot, and their bid is $2. Hence, Advertiser 2 pays $2 per click.

Final Payments:

- Advertiser 1 pays **$2 per click**.
- Advertiser 2 pays **$2 per click**.

b) K-Armed Bandit Problem: Explore-Exploit Trade-off

In the **K-armed bandit problem**, an agent must choose from **K different arms (or strategies)**, each with an unknown reward. The challenge is to balance **exploration** (trying out different arms to learn about their rewards) with **exploitation** (choosing the arm that has provided the highest reward so far).

- **Explore-Exploit Trade-off**: A common approach is to explore in the beginning to gather information about each arm and then gradually shift towards exploiting the best-known arm.
- **Upper Confidence Bound (UCB) Algorithm**: This algorithm selects the arm that maximizes the upper confidence bound on the reward. At each step, the agent chooses the arm with the highest value of:

$$UCB_i(t) = \bar{X}_i(t) + c\sqrt{\frac{\ln t}{n_i(t)}}$$

Where $\bar{X}_i(t)$ is the average reward for arm i, $n_i(t)$ is the number of times arm iii has been played, and t is the total number of trials. The term ccc controls the degree of exploration.

This algorithm balances exploration and exploitation by selecting arms that have high average rewards or have not been played frequently enough (hence needing more exploration). It can be applied in scenarios like advertising strategies to maximize click-through rates.

Repeated Games

1.Explain the Nash equilibria in the infinitely repeated Prisoner's Dilemma (IPD). Under what conditions can cooperation be sustained as a Nash equilibrium in IPD? Support your answer with the Grim Trigger strategy.

Solution:

In a standard **Prisoner's Dilemma (PD)**, two players simultaneously choose whether to **cooperate (C)** or **defect (D)**. The payoffs for the players are represented in the following table:

	Cooperate (C)	**Defect (D)**
Cooperate (C)	(3, 3)	(0, 5)
Defect (D)	(5, 0)	(1, 1)

The dilemma arises because each player has an incentive to defect, regardless of what the other player does. The dominant strategy is to defect, leading to the Nash equilibrium of mutual defection, with both players getting 1 point each.

However, when this game is **repeated infinitely** (IPD), players might choose to cooperate if they value future payoffs enough. In an infinitely repeated setting, players can use **strategies that involve punishing defection** in future rounds.

The **Grim Trigger Strategy** is a common strategy in IPD:

- **Grim Trigger**: Both players start by cooperating. If any player defects, the other player defects forever in retaliation. Cooperation can be sustained because the future punishment deters players from defecting.

For **cooperation to be sustained as a Nash equilibrium**, the discounted value of future cooperation must outweigh the one-time benefit of defection. This condition is represented by the **discount factor δ**, where δ measures how much future payoffs are valued compared to current payoffs.

The condition for cooperation to be sustained is:

$$\frac{3}{1-\delta} \geq 5 + \frac{1\delta}{1-\delta}$$

This inequality simplifies to show that cooperation is possible if $\delta \geq 2/3$. Thus, **if the players are sufficiently patient (high δ)**, cooperation can be sustained as a Nash equilibrium.

Question 2:

What is the Subgame Perfect Equilibrium (SPE) in the context of the infinitely repeated Prisoner's Dilemma (IPD)? Explain the one-deviation property and demonstrate how the Grim Trigger strategy satisfies it.

Solution:

A **Subgame Perfect Equilibrium (SPE)** is a refinement of Nash equilibrium in dynamic games. In an SPE, players' strategies must constitute a Nash equilibrium in every subgame of the original game. This ensures that players are behaving optimally not only at the start but also in every subsequent decision node (subgame).

In the **infinitely repeated Prisoner's Dilemma (IPD)**, a common strategy is the **Grim Trigger strategy**, where players cooperate as long as their opponent cooperates. If the opponent defects, they punish them by defecting forever in all future rounds.

The **One-Deviation Property** is a key concept for verifying SPE. It states that no player can improve their payoff by deviating from the equilibrium strategy at a single point, assuming they play optimally afterward.

To verify the **Grim Trigger strategy** as an SPE:

1. **Cooperation**: As long as both players cooperate, they each receive 3 points per round, resulting in a total payoff of $\frac{3}{1-\delta}$
2. **One-time Defection**: If a player defects once, they get 5 points in that round but are then punished by mutual defection forever, receiving 1 point per round thereafter. The total payoff from defecting once is $5 + \frac{1\delta}{1-\delta}$.

For the Grim Trigger strategy to be an SPE, defection should not be more rewarding than continuous cooperation. This holds if:

$$3 \geq 5 + \frac{1\delta}{1-\delta}$$

Solving this inequality shows that the **Grim Trigger strategy is an SPE** if δ≥2/3. This means players must value future payoffs enough to prefer cooperation over defection.

Thus, **Grim Trigger satisfies the one-deviation property**, and it forms a **Subgame Perfect Equilibrium** when players are sufficiently patient.

Question 3:

Describe the concept of Nash equilibrium payoffs in the infinitely repeated Prisoner's Dilemma (IPD) when the players are patient. How does the discount factor δ influence cooperation in IPD?

Solution:

In the **infinitely repeated Prisoner's Dilemma (IPD)**, the players interact an infinite number of times, making the future consequences of actions in early stages important. The key to sustaining cooperation in IPD is the **discount factor δ**, which reflects how much players value future payoffs compared to immediate ones.

- **If δ is high (i.e., players are patient)**, the future is nearly as valuable as the present, which encourages cooperation. For example, the **Grim Trigger strategy** can sustain cooperation, where both players cooperate as long as the other cooperates. A defection is punished with mutual defection in all future rounds, making it costly to defect.
- **Nash Equilibrium Payoffs**: The **Nash equilibrium payoffs** in IPD can vary. If both players cooperate in every round, they each receive a payoff of **3 points per round**. The total discounted payoff for cooperation is:

$$\frac{3}{1-\delta}$$

If a player defects, they get a one-time payoff of **5 points**, but from the next round onward, they only get **1 point** per round due to retaliation. The total discounted payoff from defecting once is:

$$5 + \frac{1\delta}{1-\delta}$$

Defection is not worthwhile if:

$$\frac{3}{1-\delta} \geq 5 + \frac{1\delta}{1-\delta}$$

This inequality simplifies to $\delta \geq 2/3$. Thus, **cooperation can be sustained as a Nash equilibrium** when the players are sufficiently patient.

- **Influence of δ**: The discount factor δ directly affects whether cooperation or defection is more appealing. **When δ is close to 1** (players are very patient), cooperation is easier to sustain because the future loss from defection outweighs the immediate gain.

In conclusion, **patient players** (high δ) will generally cooperate, leading to higher payoffs in the long run. The **Nash equilibrium payoff** for cooperation is higher than for defection if players are sufficiently patient.

Question 4:

How does the one-deviation property help in identifying subgame perfect equilibria (SPE) in infinitely repeated games? Use the example of the Prisoner's Dilemma to explain this property.

Solution:

The **One-Deviation Property** is a principle used to verify **Subgame Perfect Equilibria (SPE)** in dynamic games. It simplifies the verification of an equilibrium by checking whether any player can improve their payoff by deviating at a single point, assuming they follow the equilibrium strategy afterward.

In **infinitely repeated games** like the **Prisoner's Dilemma**, this property helps to ensure that strategies like **Grim Trigger** or **Tit-for-Tat** are subgame perfect equilibria.

For example, in the **Grim Trigger strategy** in the Prisoner's Dilemma, players cooperate as long as their opponent cooperates. If either player defects, the other defects forever.

To verify the Grim Trigger strategy using the **One-Deviation Property**:

- Consider a possible **deviation** where one player defects in a single round. If the player defects, they receive 5 points in that round but will be punished by future defections, resulting in 1 point per round from then on. The total payoff from defecting once is

  $5 + \frac{1\delta}{1-\delta}$.

- Compare this with **continuous cooperation**, which yields a payoff of

  $\frac{3}{1-\delta}$.

For cooperation to be the best response, the payoff from cooperation must be higher than from defection. This holds if $\delta \geq 2/3$, meaning players must value future payoffs highly enough to avoid defection.

Thus, **Grim Trigger satisfies the one-deviation property** because no player can improve their payoff by deviating at any point. This ensures that it is an **SPE**.