Games in normal or strategic form (simultaneous or static games)

A game in normal or strategic form is described by:

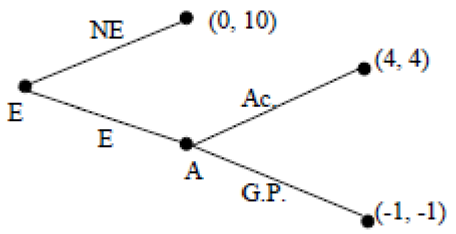
1) The players.

2) The set (or space) of strategies for each player.

3) A payoff function which assigns a payoff vector to each combination of strategies.

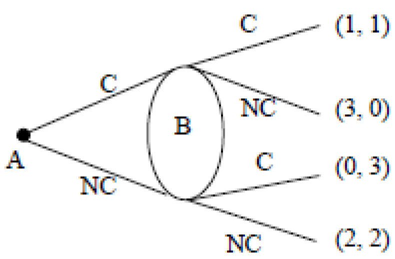
The key element of this way of representing a game is the description of the payoffs of the game as a function of the strategies of the players, without explaining the actions taken during the game. In the case of two players the usual representation is a bimatrix form game where each row corresponds to one of the strategies of one player and each column corresponds to one strategy of the other player.

EXAMPLE 1: The entry game

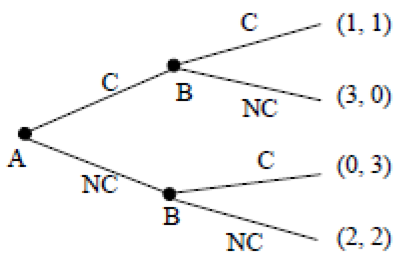


		A	
		Ac.	G.P.
E	NE	(0, 10)	(0, 10)
	E	(4, 4)	(-1, -1)

EXAMPLE 2: The Prisoner's Dilemma

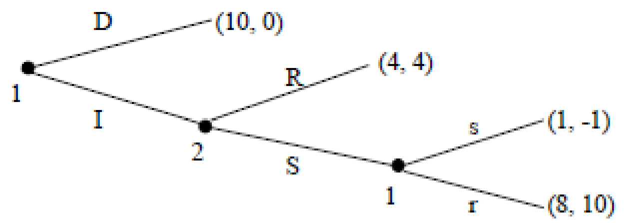


		B	
		C	NC
A	C	(1, 1)	(3, 0)
	NC	(0, 3)	(2, 2)



		B			
		CC	CNC	NCC	NCNC
A	C	(1, 1)	(1, 1)	(3, 0)	(3, 0)
	NC	(0, 3)	(2, 2)	(0, 3)	(2, 2)

EXAMPLE 3



1

	R	S
Ds	(10, 0)	(10, 0)
Dr	(10, 0)	(10, 0)
Is	(4, 4)	(1, -1)
Ir	(4, 4)	(8, 10)

Link between games in normal form and games in extensive form

a) For any game in extensive form there exists a unique corresponding game in normal form. This is due to the game in normal form being described as a function of the strategies of the players.

b) (Problem) Different games in extensive form can have the same normal (or strategic) form. (Example: in the prisoner's dilemma, PD1, if we change the order of the game then the game in extensive form also changes but the game in normal form does not change).

3. **Solution concepts (criteria) for noncooperative games**

The general objective is to predict how players are going to behave when they face a particular game. NOTE: "A solution proposal is (not a payoff vector) a combination of strategies, one for each player, which lead to a payoff vector". We are interested in predicting behaviour, not gains.

Notation

i: Representative player, $i = 1, \ldots, n$

S_i : set or space of player i's strategies.

$s_i \in S_i$: a strategy of player i.

$s_{-i} \in S_{-i}$: a strategy or combination of strategies of the other player(s).

$\Pi i\ (s_i, s_{-i})$: the profit or payoff of player i corresponding to the combination of strategies

$s \equiv (s1, s2, \ldots\ldots, sn) \equiv (si, s_{-i})$.

3.1. **Dominance criterion**

Definition 6**: Dominant strategy**

"A strategy is strictly dominant for a player if it leads to strictly better results (more payoff) than any other of his/her strategies no matter what combination of strategies is used by the other players".

"s_i^D is a strictly dominant strategy for player i if $\Pi i\ (s_i^D, s_{-i}) > \Pi i\ (s_i, s_{-i}), \forall s_i \in S_i, s_i \neq s_i^D; \forall s_{-i}$"

EXAMPLE 2: The Prisoner's Dilemma

In game PD1 "confess", C, is a (strictly) dominant strategy for each player. Independently of the behavior of the other player the best each player can do is "confess".

The presence of dominant strategies leads to a solution of the game. We should expect each player to use his/her dominant strategy. The solution proposal for game DP1 is the combination of strategies (C, C).

Definition 7**: Strict dominance**

"One strategy strictly dominates another when it leads to strictly better results (more payoff) than the other no matter what combination of strategies is used by the other players".

"If $\Pi_i(s_i^d, s_{-i}) > \Pi_i(s_i^{dd}, s_{-i}), \forall s_{-i}$, then s_i^d strictly dominates s_i^{dd}".

Obviously, one strategy is strictly dominated for a player when there is another strategy which dominates it. The dominance criterion consists of the iterated deletion of strictly dominated strategies.

EXAMPLE 4

		t_1	t_2	t_3
1	s_1	(4, 3)	(2, 7)	(0, 4)
	s_2	(5, 5)	(5, -1)	(-4, -2)

In this game there are no dominant strategies. However, the existence of dominated strategies allows us to propose a solution. We next apply the dominance criterion. Strategy t3 is strictly dominated by strategy t2 so player 1 can conjecture (predict) that player 2 will never use that strategy. Given that conjecture, which assumes rationality on the part of player 2, strategy s2 is better than strategy s1 for player 1. Strategy s1 would be only used in the event that player 2 used strategy t3. If player 1 thinks player 2 is racional then he/she assigns zero probability to the event of player 2 playing t3. In that case, player 1 should play s2 and if player 2 is rational the best he/she can do is t1. The criterion of iterated deletion of strictly dominated strategies (by eliminating dominated strategies and by computing the reduced games) allows us to solve the game.

EXAMPLE 5

	t_1 (2)	t_2
s_1	(10, 0)	(5, 2)
s_2	(10, 1)	(2, 0)

In this game there are neither dominant strategies nor (strictly) dominated strategies.

Definition 8**: Weak dominance**

"One strategy weakly dominates another for a player if the first leads to results at least as good as those of the second no matter what combination of strategies is used by the other players and

to strictly better results for any combination of strategies of the other players".

"If $\Pi_i(s_i^{db}, s_{-i}) \geq \Pi_i(s_i^{ddb}, s_{-i}), \forall s_{-i}$, and $\exists s_{-i}$ such that $\Pi_i(s_i^{db}, s_{-i}) > \Pi_i(s_i^{ddb}, s_{-i})$, then s_i^{db} weakly dominates s_i^{ddb}".

Thus, a strategy is weakly dominated if another strategy does at least as well for all s_{-i} and strictly better for some s_{-i} .

In example 5, strategy s_1 weakly dominates s2. Player 2 can conjecture that player 1 will play s_1 and given this conjecture the best he/she can do would be to play t_2. By following the criterion of weak dominance (iterated deletion of weakly dominated strategies) the solution proposal would be (s_1,t_2).

However, the criterion of weak dominance may lead to problematic results, as occurs in example

6, or to no solution proposal as occurs in example 7 (because there are no dominant strategies, no dominated strategies and no weakly dominated strategies).

EXAMPLE 6

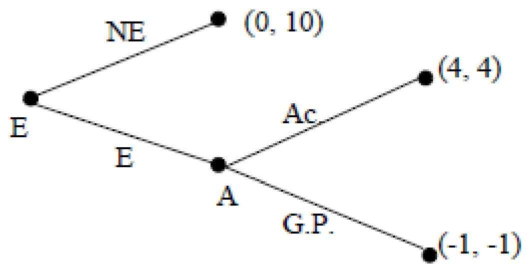
		2		
		t_1	t_2	t_3
1	s_1	(10, 0)	(5, 1)	(4, -200)
	s_2	(10, 100)	(5, 0)	(0, -100)

EXAMPLE 7

		2		
		t_1	t_2	t_3
1	s_1	(4, 10)	(3, 0)	(1, 3)
	s_2	(0, 0)	(2, 10)	(10, 3)

3.2. Backward induction criterion

EXAMPLE 1: The entry game



		A	
		Ac.	G.P.
E	NE	(0, 10)	(0, 10)
	E	(4, 4)	(-1, -1)

In the game in normal form, player A has a weakly dominated strategy: G.P.. Player E might conjecture that and play E. However, player E might also have chosen NE in order to obtain a certain payoff against the possibility of player A playing G.P..

In the game in extensive form, the solution is obtained more naturally by applying backward induction. As he/she moves first, Player E may conjecture, correctly, that if he/she plays E then player A (if rational) is sure to choose Ac.. By playing before A, player E may anticipate the rational behavior of player A. In the extensive form of the game we have more information because when player A has to move he already knows the movement of player E. The criterion of backward induction lies in applying the criterion of iterated dominance backwards starting from the last subgame(s). In example 1 in extensive form the criterion of backward induction proposes the combination of strategies (E, Ac.) as a solution.

Result: In perfect information games with no ties, the criterion of backward induction leads to a unique solution proposal.

Problems

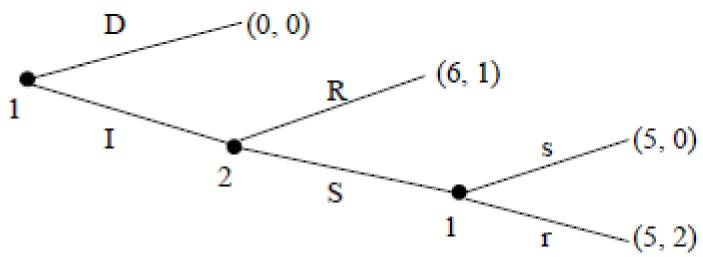
(i) Ties.

(ii) Imperfect information. Existence of information sets with two or more nodes.

(iii) The success of backward induction is based on all conjectures about the rationality of agents

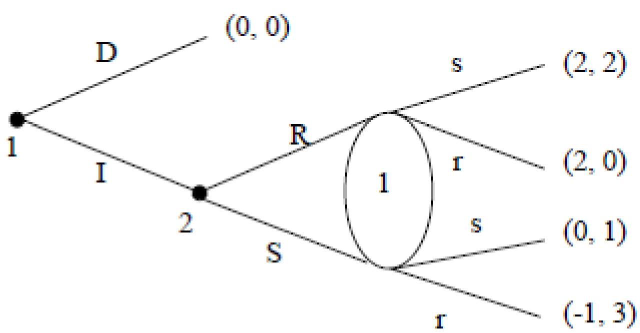
checking out exactly with independence of how long the backward path is. (It may require unbounded rationality).

EXAMPLE 8



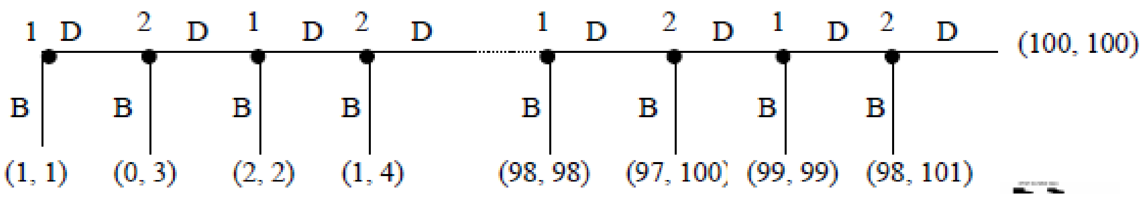
Backward induction does not propose a solution because in the last subgame player 1 is indifferent between s and r. In the previous subgame, player 2 would not have a dominated action (because he/she is unable to predict the behavior of player 1 in the last subgame).

EXAMPLE 9



We cannot apply the criterion of backward induction

EXAMPLE 10: Rosenthal's (1981) centipede game



In the backward induction solution the payoffs are (1, 1). ¿Is another rationality possible?

3.3. **Nash equilibrium**

Player i, $i = 1,\ldots, n$, is characterized by:

(i) A set of strategies: S_i.

(ii) A profit function, $\Pi_i(s_i, s_{-i})$ where $s_i \in S_i$ and $s_{-i} \in S_{-i}$.

Each player will try to maximize his/her profit (utility or payoff) function by choosing an appropriate strategy with knowledge of the strategy space and profit functions of the other players but with no information concerning the current strategy used by rivals. Therefore, each player must conjecture the strategy(ies) used by his/her rival(s).

Definition 9**: Nash equilibrium**

"A combination of strategies or strategy profile $s^* \equiv (s_1^*,\ldots,s_n^*)$ constitutes a Nash equilibrium if the result for each player is better than or equal to the result which would be obtained by playing another strategy, with the behaviour of the other players remaining constant

$s^* \equiv (s_1^*,\ldots,s_n^*)$ is a Nash equilibrium if: $\Pi_i(s_i^*, s_{-i}^*) \geq \Pi_i(s_i, s_{-i}^*)\ \forall s_i \in S_i, \forall i, i = 1,\ldots,n$."

At equilibrium two conditions must be satisfied:

(i) The conjectures of players concerning how their rivals are going to play must be correct.

(ii) No player has incentives to change his/her strategy given the strategies of the other players.

This is an element of individual rationality: do it as well as possible given what the rivals do.

Put differently, no player increases his/her profits by unilateral deviation

Being a Nash equilibrium is a necessary condition or minimum requisite for a solution proposal to be a reasonable prediction of rational behaviour by players. However, as we shall see it is not a sufficient condition. That is, being a Nash equilibrium is not in itself sufficient for a combination of strategies to be a prediction of the outcome for a game.

Definition10: **Nash equilibrium**

"A combination of strategies or strategy profile $s^* \equiv (s_1^*,\dots,s_n^*)$ constitutes a Nash equilibrium if each player's strategy choice is a best response to the strategies actually played by his/her rivals".

That is,

$s^* \equiv (s_1^*, \dots, s_n^*)$ is a Nash equilibrium if: $s_i^* \in MR_i(s_{-i}^*) \forall i, i = 1, \dots, n$

Where

$$MR_i(s_{-i}^*) = \{s_i' \in S_i \mid : \Pi_i(s_i', s_{-i}^*) \geq \Pi_i(s_i, s_{-i}^*),\ \forall s_i \in S_i, s_i \neq s_i'\}$$

A simple way of obtaining the Nash equilibria for a game is to build the best response sets of each player to the strategies (or combinations of strategies) of the other(s) player(s) and then look for those combinations of strategies being mutually best responses.

EXAMPLE 11

		2: h	i	j
1	a	(5, 3)	(5, 11)	(20, 5)
	b	(9, 11)	(2, 8)	(15, 6)
	c	(3, 10)	(10, 2)	(0, 5)

s_1	MR_2	s_2	MR_1
a	i	h	b
b	h	i	c
c	h	j	a

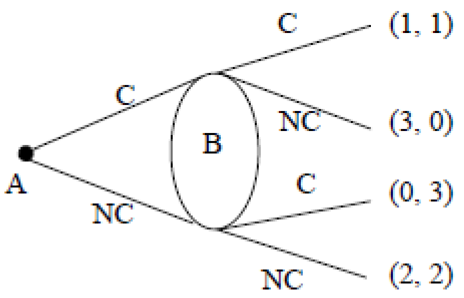
The strategy profile (b, h) constitutes the unique Nash equilibrium.

3.4. **Problems and refinements of the Nash equilibrium**

3.4.1. **The possibility of inefficiency**

It is usual to find games where Nash equilibria are not Pareto optimal (efficient).

EXAMPLE 2: The Prisoner's Dilemma



		B: C	B: NC
A	C	(1, 1)	(3, 0)
	NC	(0, 3)	(2, 2)

(C, C) is a Nash equilibrium based on dominant strategies. However, that strategy profile is the only profile which is not Pareto optimal. In particular, there is another combination of strategies, (NC, NC), where both players obtain greater payoffs.

3.4.2. **Inexistence of Nash equilibrium (in pure strategies)**

EXAMPLE 12

		2: t_1	2: t_2
1	s_1	($\underline{1}$, 0)	(0, $\underline{1}$)
	s_2	(0, $\underline{1}$)	($\underline{1}$, 0)

This game does not have Nash equilibria in pure strategies. However, if we allow players to use mixed strategies (probability distributions on the space of pure strategies) the result obtained is that "for any finite game there is always at least one mixed strategy Nash equilibrium".

3.4.3. **Multiplicity of Nash equilibria**

We distinguish two types of games.

3.4.3.1. ***With no possibility of refinement or selection***

EXAMPLE 13: The Battle of the Sexes

		Na: C	Na: T
No	C	($\underline{3}$, $\underline{2}$)	(1, 1)
	T	(1, 1)	($\underline{2}$, $\underline{3}$)

This game has two Nash equilibria: (C, C) and (T, T). There is a *pure coordination problem*.

3.4.3.2. ***With possibility of refinement or selection***

a) *Efficiency criterion*

This criterion consists of choosing the Nash equilibrium which maximizes the payoff of players.

In general this is not a good criterion for selection.

b) *Weak dominance criterion*

This criterion consists of eliminating Nash equilibria based on weakly dominated strategies.

Although as a solution concept it is not good, the weak dominance criterion allows us to select among the Nash equilibria.

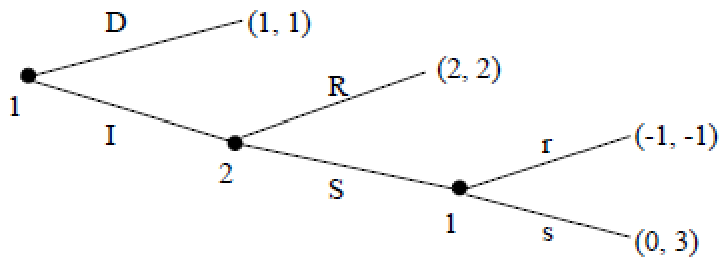
EXAMPLE 14

		2: D	2: I
1	D	(1, 1)	(0, 0)
	I	(0, 0)	(0, 0)

Nash equilibria: (*D*, *D*) and (*I*, *I*). Strategy *I* is a weakly dominated strategy for each player. By playing strategy *D* each player guarantees a payoff at least as high (and sometimes a higher) than that obtained by playing *I*. So we eliminate equilibrium (*I*, *I*) because it is based on weakly dominated strategies. So we propose the strategy profile (*D*, *D*) as the outcome of the game.

c) *Backward induction criterion and subgame perfect equilibrium*

EXAMPLE 15



s_1	MR_2	s_2	MR_1
Dr	R, S	R	Ir, Is
Ds	R, S	S	Dr, Ds
Ir	R		
Is	S		

There are three Nash equilibria: (*Dr*, *S*), (*Ds*, *S*) and (*Ir*, *R*). We start by looking at the efficient profile: (*Ir*, *R*). This Nash equilibrium has a problem: at his/her second decision node, although it is an unattanaible given the behavior of the other player, player 1 announces that he/she would play *r*. By threatening him/her with *r* player 1 tries to make player 2 play *R* and so obtain more profits. However, that equilibrium is based on a non credible threat: if player 1 were called on to play at his/her second node he/she would not choose *r* because it is an action (a non credible threat) dominated by *s*. The refinement we are going to use consists of eliminationg those equilibria based on non credible threats (that is, based on actions dominated in one subgame).

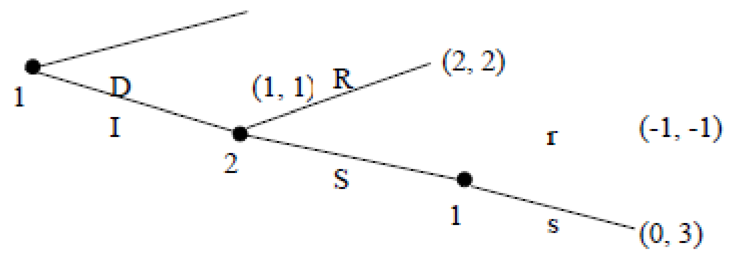
From the joint use of the notion of Nash equilibrium and the backward induction criterion the following notion arises:

Definition 11: **Subgame perfect equilibrium**

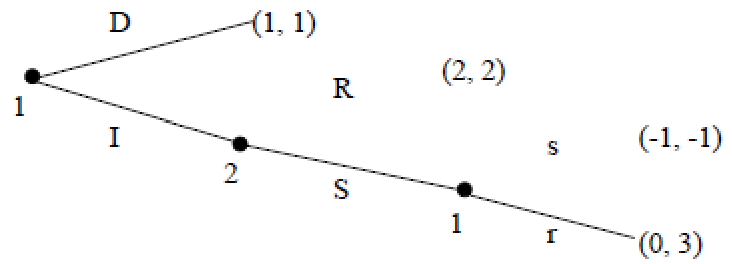
"A combination of strategies or strategy profile $s^* \equiv (s1^*,\ldots,sn^*)$, which is a Nash equilibrium,

constitutes a *subgame perfect equilibrium* if the relevant parts of the equilibrium strategies of each player are also an equilibrium in each of the subgames".

In example 15 (*Dr*, *S*) and (*Ir*, *R*) are not subgame perfect equilibria. Subgame perfect equilibria may be obtained by backward induction. We start at the last subgame. In this subgame *r* is a dominated action (a non credible threat); therefore, it cannot form part of player 1's strategy in the subgame perfect equilibrium, so we eliminate it and compute the reduced game



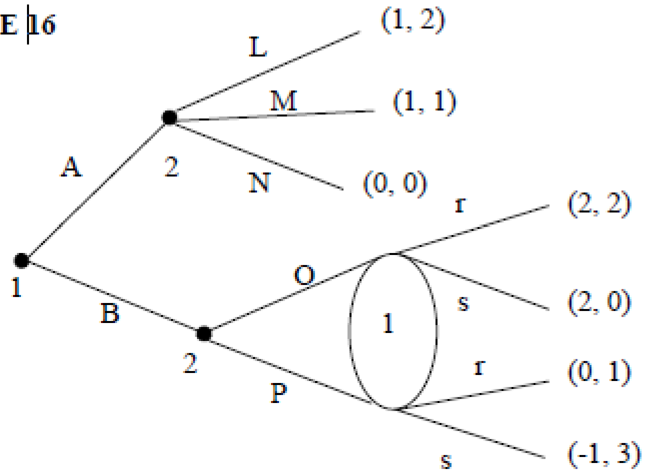
In the second stage of the backward induction we go to the previous subgame which starts at the decision node of player 2. In this subgame *R* is a dominated action for player 2. Given that player 2 anticipates that player 1 is not going to play *r* then *R* is a dominated action or non credible threat. We therefore eliminate it and compute the reduced game



At his/her first node player 1 has I as a dominated action (in the reduced game) and, therefore, he/she will play D. Then the subgame perfect equilibrium is (Ds, S). We can interpret the logic of backward induction in the following way. When player 2 has to choose he/she should conjecture that if he/she plays S player 1 is sure to play s. Player 2 is able to predict the rational
At his/her first node player 1 has *I* as a dominated action (in the reduced game) and, therefore, he/she will play *D*. Then the subgame perfect equilibrium is (*Ds*, *S*). We can interpret the logic

of backward induction in the following way. When player 2 has to choose he/she should conjecture that if he/she plays *S* player 1 is sure to play *s*. Player 2 is able to predict the rational.

EXAMPLE 16



In this game there is a multiplicity of Nash equilibria and we cannot apply backward induction because there is a subgame with imperfect information. We shall use the definition of subgame perfect equilibrium and we shall require that the relevant part of the equilibrium strategies to be an equilibrium at the subgames. What we can do is solve the lower subgame (which starts at the lower decision node of player 2) and it is straightforward to check that the Nash equilibrium is *O*, *r*. At the upper subgame the only credible threat by player 2 is *L*. At his/her first decision node player 1 therefore has to choose between *A* and *B* anticipating that if he/she chooses *A* then player 2 will play *L* and if he/she chooses *B*, then they will both play the Nash equilibrium (of the subgame) *O*, *r*. Therefore, the subgame perfect equilibrium is (*Br*, *LO*): the relevant part of the equilibrium strategies are also an equilibrium at each of the subgames.