Markov chains

Properties of Markov chains:-

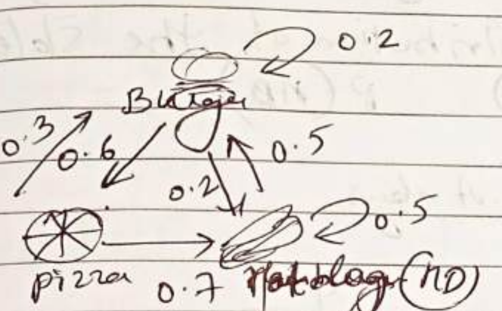
1. Future steps depends only on current steps not complete previous states.

ie $P(X_{n+1} = x \mid X_1 = x_1, X_2 = x_2 \ldots X_n = x_n)$ — step

probability of $X_{n+1} = x$ depends only on n^{th} step

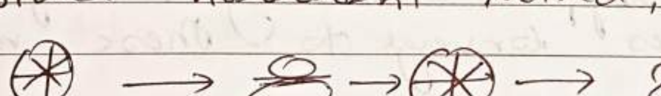
$$\therefore P(X_{n+1} = x \mid X_n = x_n)$$

Eg.



Arrow → transition from 1 state to other

Consider Restaurant served,



on Day 1 → 2nd day → 3rd → ? Now, what is the probability that they will serve hamburger on 4th day

∴ Looking at the 3rd day, we can say

$$P(X_4 = HD \mid X_3 = \text{Pizza}) = 0.7$$

2. The sum of the weights of the outgoing arrows from any state is = 1.

⇒ Doing a Random walk along a chain: ①

Random walk:-

$B \rightarrow P \rightarrow B \rightarrow MB \rightarrow B \rightarrow MB \rightarrow MB \rightarrow MB \rightarrow B \rightarrow P$

→ After 10 steps :-

→ Now find probability corresponding to each of them
ie. Prob. distribution of the states.

$P(B)$ $P(P)$ $P(MB)$

ie occurrence / total no. of days.

$\frac{4}{10}$ $\frac{2}{10}$ $\frac{4}{10}$

what happens in long runs for eg: 1000 steps
Lets say, probabilities converge to these values

0.35191 0.21245 0.43564.

* This probability distribution is called as "stationary distribution or equilibrium / state".
ie
this prob. distribution doesn't change for this Markov chain.

∴ To find this state we use linear algebra.

① Given as, Transition probabilities Matrix :-

$$A = \begin{array}{c c} & \begin{array}{ccc} B & P & NB \end{array} \\ \begin{array}{c} B \\ P \\ NB \end{array} & \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0 & 0.7 \\ 0.5 & 0 & 0.5 \end{bmatrix} \end{array}$$

Goal is to find probabilities of each state ?

∴ → Take a Row vector 'π' whose elements represent the probabilities of the state.

& In the beginning Assume we are on a Pizza day

$\pi_0 = [0 \quad 1 \quad 0]$

$$\pi_0 \cdot A = [0 \quad 1 \quad 0] \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0 & 0.7 \\ 0.5 & 0 & 0.5 \end{bmatrix} = \underbrace{[0.3 \quad 0 \quad 0.7]}_{\pi_1}$$

future probabilities Corresponding to Pizza state

Again take this result &,

$$\pi_1 \cdot A = [0.3 \quad 0 \quad 0.7] \begin{bmatrix} 0.2 & 0.6 & 0.2 \\ 0.3 & 0 & 0.7 \\ 0.5 & 0 & 0.5 \end{bmatrix} = [0.34 \quad 0.25 \quad 0.41]$$

⇒ If there exist a stationary state, then at some point the output Row vector = i/p vector. ets

i.e. $\pi A = \pi$

$Av = \lambda v$. - eigenvector eqn.

$\pi[1] + \pi[2] + \pi[3] = 1$ property

After solving, we get the stationary state

$$\pi = \left[\frac{25}{71} \quad \frac{15}{71} \quad \frac{31}{71}\right]$$

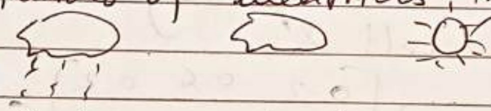
$$= [0.35211 \quad 0.21127 \quad 0.43662]$$

ie it tells us that, the restaurant will serve Burger about 35% of the days, pizza with 21% of the days & so on.

(Compare the result got from previous simulation ie π. its almost similar)

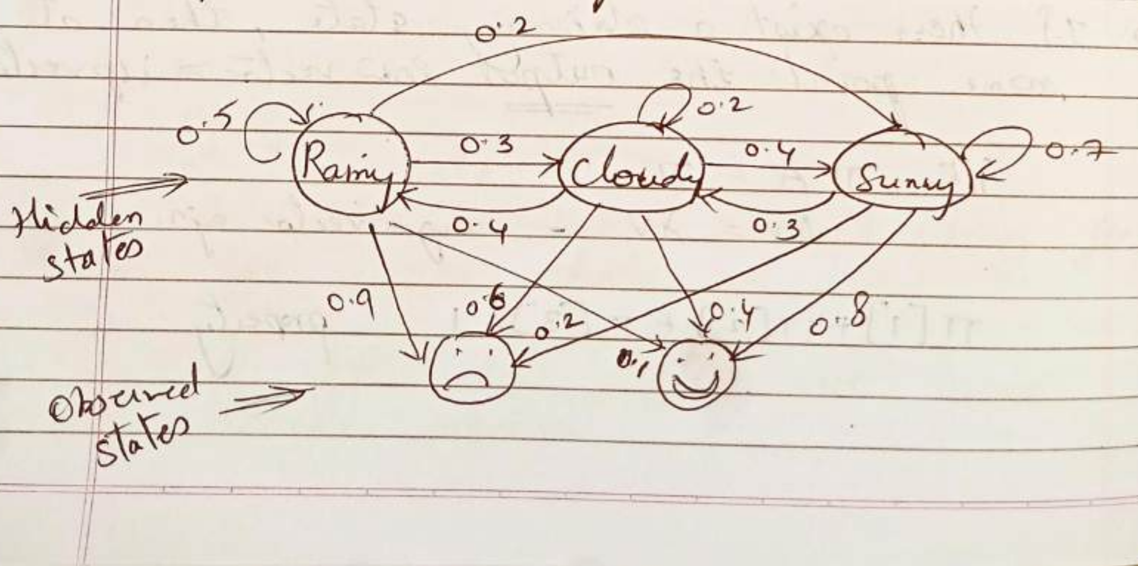
Eg (2) HMM

→ 3 kinds of weathers, in a town



Rainy Cloudy Sunny

→ person Mood depends on weather



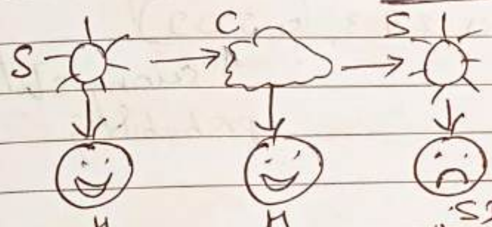
- we can't know weather in that Country but can get the details from the person w.r.t. mood

HMM = Hidden MC + observed variables.

→ write probabilities as:
Transition & emission Matrix

$$\begin{array}{c|ccc} & R & C & S \\ \hline \text{Rainy} & 0.5 & 0.3 & 0.2 \\ \text{cloudy} & 0.4 & 0.2 & 0.4 \\ \text{sunny} & 0.0 & 0.3 & 0.7 \end{array} \qquad \begin{array}{cc} \text{Sad} & \text{Happy} \\ \hline 0.9 & 0.1 \\ 0.6 & 0.4 \\ 0.2 & 0.8 \end{array}$$

- Consider a Scenario :-



what is the probability of this scenario occuring?

ie. what is the "joint probability" of the observed mood sequence & the weather sequence

$P(Y = 😊😊☹, X = ☀☁☀)$

* using Markov property, we can compute this as a product of six terms :-

$P(X_1 = ☀)$?

$P(Y_1 = 😊 \mid X_1 = ☀)$ 0.8

$P(X_2 = ☁ \mid X_1 = ☀)$ 0.3

$P(Y_2 = 😊 \mid X_2 = ☁)$ 0.4

$P(X_3 = ☀ \mid X_2 = ☁)$ 0.4

$P(Y_3 = ☹ \mid X_3 = ☀)$ 0.2

Transition probabilities

Emission probabilities.

→ To get the values of above probabili..

* we get it from Transmission & emission Matrices. *

But, what is 1st term ⇒ ?

* ⇒ To find the probability of 1st state we need the "stationary" distribution of the Markov chain.

→ using eigenvectors,

$$\pi A = \pi$$

$$\pi = [0.218, 0.273, 0.509]$$

↑ sunny state has probability

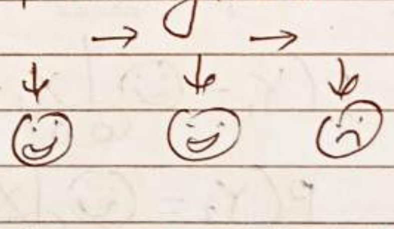
$\therefore P(X_1 =$  $)$
$= \underline{0.509}$

→ Now, Compute the product

$$0.509 * 0.8 * 0.3 * 0.4 * 0.4 * 0.2$$
$$= \underline{0.00391}$$

(i.e. probability of this scenario occurring)

⇒ Now, if given:

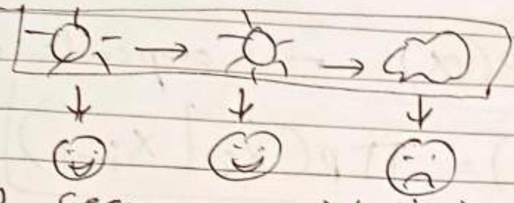


* { What is the "most likely" weather sequence for the observed mood sequence

There are many Combinations

* → To find most likely sequence, we need to Calculate probability corresponding to each sequence.
And find the one with Maximum probability.

Ans is:



This sequence Maximizes the probability

P(Y = ☺ ☺ ☹ , X = ☼ ☼ ☁)
= 0.04105

⇒ So how to get this,

$$\underset{X = X_1, X_2 \ldots X_n}{\operatorname{argmax}} P(\underbrace{X = X_1, X_2 \ldots X_n}_{\text{hidden States}} \mid \underbrace{Y = Y_1, Y_2 \ldots Y_n}_{\text{observed variables}})$$

* → find that sequence of X, for which the prob of X given Y is Maximum.

~~Direct way~~ to find this, so using

Bayes Theorem :-

Rewrite this, P(X|Y) =

$$\underset{X = X_1, X_2 \ldots X_n}{\operatorname{arg\,max}} \frac{P(Y|X)\,P(X)}{P(Y)} \quad \text{— (1)}$$

joint probability distribution of X & Y.

neglect the denominator

① Consider, $P(Y|X)$

$\rightarrow P(Y|X) = P(Y_1|X_1) * P(Y_2|X_2) * \cdots P(Y_n|X_n)$

we can fill this from weight Matrix

$\rightarrow \boxed{P(Y|X) = \prod P(Y_i, X_i)}$ — ②

② Consider, $P(X)$ — depends only on X_{i-1}.

$\boxed{P(X) = \prod P(X_i | X_{i-1})}$ — ③

③ For $X_0 \rightarrow$ we need to use the stationary distribution vector

④ ~~the~~ ~~class~~ Substitute in eqn ①

$$\underset{X = X_1, X_2 \cdots X_n}{\arg\max} \prod \{ P(Y_i|X_i) \cdot P(X_i|X_{i-1})$$

This is the equation, we want to Maximize.