Mathematical Induction:

Mathematical Induction is a technique of proving a statement, theorem or formula which is thought to be true, for each and every natural number n. By generalizing this in form of a principle which we would use to prove any mathematical statement is **"**Principle of Mathematical Induction".

For example: $1^3 + 2^3 + 3^3 + \ldots.. + n^3 = (n(n+1) / 2)^2$, the statement is considered here as true for all the values of natural numbers.

Principle of Mathematical Induction Solution and Proof

Consider a statement P(n), where n is a natural number. Then to determine the validity of P(n) for every n, use the following principle:

Step 1: Check whether the given statement is true for n = 1.

Step 2: Assume that given statement P(n) is also true for n = k, where k is any positive integer.

Step 3: Prove that the result is true for P(k+1) for any positive integer k.

If the above-mentioned conditions are satisfied, then it can be concluded that P(n) is true for all n natural numbers.

Proof:

The first step of the principle is a *factual statement* and the second step is a *conditional one*. According to this if the given statement is true for some positive integer k only then it can be concluded that the statement P(n) is valid for n = k + 1.

This is also known as the *inductive step* and the assumption that P(n) is true for n=k is known as the *inductive hypothesis.*

Example 1.

Prove that the sum of cubes of n natural numbers is equal to ($n(n+1)^2)^2$ for all n natural numbers.

Solution:

In the given statement we are asked to prove:

$1^3+2^3+3^3+\cdots+n^3 = (n(n+1)^2)^2$

Step 1: Now with the help of the principle of induction in math let us check the validity of the given statement P(n) for n=1.

$P(1)=(1(1+1)^2)^2 = 1$ This is true.

Step 2: Now as the given statement is true for n=1 we shall move forward and try proving this for n=k, i.e.,

$1^3+2^3+3^3+\cdots+k^3= (k(k+1)^2)^2$.

Step 3: Let us now try to establish that P(k+1) is also true.

$$1^3+2^3+3^3+\cdots+k^3+(k+1)^3 = (k(k+1)^2)^2+(k+1)^3$$
$$\Rightarrow 1^3+2^3+3^3+\cdots+k^3+(k+1)^3=k^2(k+1)^4+(k+1)^3$$
$$= k^2(k+1)^2+4((k+1)^3)^4$$
$$=(k+1)^2(k^2+4(k+1))^4$$
$$=(k+1)^2(k^2+4k+4)^4$$
$$= (k+1)^2((k+2)^2)^4$$
$$=(k+1)^2(k+1+1)^2)^4$$
$$=(k+1)^2((k+1)+1)^2)^4$$

Example 2:

Show that 1 + 3 + 5 + … + (2n−1) = n^2

Solution:

Step 1: Result is true for n = 1

That is $1 = (1)^2$ (True)

Step 2: Assume that result is true for n = k

$1 + 3 + 5 + \ldots + (2k-1) = k^2$

Step 3: Check for n = k + 1

i.e. $1 + 3 + 5 + \ldots + (2(k+1)-1) = (k+1)^2$

We can write the above equation as,

$1 + 3 + 5 + \ldots + (2k-1) + (2(k+1)-1) = (k+1)^2$

Using step 2 result, we get

$k^2 + (2(k+1)-1) = (k+1)^2$

$k^2 + 2k + 2 - 1 = (k+1)^2$

$k^2 + 2k + 1 = (k+1)^2$

$(k+1)^2 = (k+1)^2$

L.H.S. and R.H.S. are same.

So the result is true for n = k+1

By mathematical induction, the statement is true.

We see that the given statement is also true for n=k+1. Hence we can say that by the principle of mathematical induction this statement is valid for all natural numbers n.