LAPLACE TRANSFORM

Laplace Transform is an integral transform method which is particularly useful in solving linear differential equations. Laplace transform techniques are widely used in engineering fields. The Laplace Transforms can be interpreted as a transformation from the time domain where inputs and outputs are functions of time(t) to the frequency domain where inputs and outputs are functions of complex angular frequency (s).

- **Def^n:** Laplace Transform (L.T.)

Let, $f(t)$ be a given function defined for all $t \geq 0$. The Laplace Transform of $f(t)$ denoted by $L[f(t)]$ is defined as

$$\boxed{L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = \phi(s)}$$

- Problems using definition of Laplace Transform

1) Find Laplace transform of $f(t) = t^2 \quad 0 < t < 3$
$\qquad\qquad = 6 \quad t > 3$

Sol^n
$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^3 e^{-st} f(t)\, dt + \int_3^\infty e^{-st} f(t)\, dt$$

$$= \int_0^3 e^{-st}\, t^2\, dt + \int_3^\infty e^{-st}\, 6\, dt$$

$$= \int_0^3 t^2 e^{-st}\, dt + 6\int_3^\infty e^{-st}\, dt$$

Prof. Nancy Sinollin

Note:- $\int u \cdot v = u \int v - \int [\int v \, u']$

Instead of using above formula we'll use the following formula.

Suppose, u has vanishing derivative. i.e. if r^{th} order derivative of u is zero.

then,

$$\int u \cdot v = u \underbrace{\int v}_{v_1} - u' \underbrace{\int v_1}_{v_2} + u'' \underbrace{\int v_2}_{v_3} - \cdots u^{(r-1)} \int v_{r-1}$$

as $u^{(r)} = 0$

$$L[f(t)] = \int_0^3 t^2 e^{-st} dt + 6 \int_3^{\infty} e^{-st} dt$$

$$= \left[(t^2) \left(\frac{e^{-st}}{-s} \right) - (2t) \left(\frac{e^{-st}}{(-s)^2} \right) + (2) \left(\frac{e^{-st}}{(-s)^3} \right) \right]_0^3 + 6 \left[\frac{e^{-st}}{(-s)} \right]_3^{\infty}$$

$$= \left[\frac{9 e^{-3s}}{(-s)} - \frac{6 e^{-3s}}{s^2} + \frac{2 e^{-3s}}{-s^3} - 0 + 0 - 2 \left(\frac{1}{-s^3} \right) \right] + 6 \left[0 - \frac{e^{-3s}}{-s} \right] \quad \because e^{-\infty} = 0$$

$$= \left[\frac{-9 e^{-3s}}{s} - \frac{6 e^{-3s}}{s^2} - \frac{2 e^{-3s}}{s^3} + \frac{2}{s^3} + \frac{6 e^{-3s}}{s} \right]$$

$$= \left[-\frac{3 e^{-3s}}{s} - \frac{6 e^{-3s}}{s^2} - \frac{2 e^{-3s}}{s^3} + \frac{2}{s^3} \right]$$

Prof. Nancy Sinollin

Note:

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$$

2) Find Laplace Transform of $f(t) = \cos t \quad 0<t<\pi$
$\qquad = \sin t \quad t>\pi$

Sol^n $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^\pi e^{-st}\cos t\,dt + \int_\pi^\infty e^{-st}\sin t\,dt$$

(from above formulae $a=-s$, $b=1$)

$$= \left[\frac{e^{-st}}{s^2+1}(-s\cos t + \sin t)\right]_0^\pi + \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_\pi^\infty$$

$$= \left[\frac{e^{-\pi s}}{s^2+1}(-s\cos\pi + \sin\pi) - \frac{1}{s^2+1}(-s(1)+0)\right]$$

$$+ \left[0 - \frac{e^{-\pi s}}{s^2+1}(-s\sin\pi - \cos\pi)\right]$$

$$= \left[\frac{e^{-\pi s}}{s^2+1}(-s(-1)+0) - \frac{(-s)}{s^2+1} - \frac{e^{-\pi s}}{s^2+1}(0-(-1))\right]$$

$$= \frac{e^{-\pi s}}{s^2+1}s + \frac{s}{s^2+1} - \frac{e^{-\pi s}}{s^2+1}$$

Ex. Find Laplace Transform of

1) $f(t) = (t-1)^2 \quad 0<t<1$
$\qquad = 3 \quad t>1$

2) $f(t) = \sin 2t \quad 0<t<\pi$
$\qquad = 0 \quad t>\pi$

- Linearity Property of Laplace Transform

$$L[k_1 f_1(t) + k_2 f_2(t)] = k_1 L[f_1(t)] + k_2 L[f_2(t)]$$

- Laplace Transform of Standard Functions

1) $L[e^{at}] = \frac{1}{s-a}$

2) $L[e^{-at}] = \frac{1}{s+a}$

3) $L[1] = \frac{1}{s}$

4) $L[\sin at] = \frac{a}{s^2+a^2}$

5) $L[\cos at] = \frac{s}{s^2+a^2}$

6) $L[\sinh at] = \frac{a}{s^2-a^2}$

7) $L[\cosh at] = \frac{s}{s^2-a^2}$

8) $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$ Note :- $\Gamma(n) = (n-1)\,\Gamma(n-1)$
$\Gamma(n) = (n-1)!$ if n is natural no.
$\Gamma(1/2) = \sqrt{\pi}$

9) $L[\operatorname{erf}\sqrt{t}] = \frac{1}{s\sqrt{s+1}}$ — (we will prove this later on)

(Note :- One can prove all above formulae using definition of Laplace Transform.)

Note :-

1) $L[c^{at}] = L[e^{\log c^{at}}] = L[e^{at\log c}]$

$= L[e^{(a\log c)t}]$

$= \frac{1}{s - a\log c}$

2) $L[c^{-at}] = \frac{1}{s + a\log c}$ (using above method)

3) $L[k] = k\,L[1] = \frac{k}{s}$ where, k is constant

Problems :-

1) Find $L[\cos 2t + 3\sin t + 4e^{-2t} + 2]$

Sol^n $L[\cos 2t + 3\sin t + 4e^{-2t} + 2]$

$= L[\cos 2t] + L[3\sin t] + L[4e^{-2t}] + L[2]$

$= L[\cos 2t] + 3L[\sin t] + 4L[e^{-2t}] + 2L[1]$

$= \frac{s}{s^2+2^2} + 3\frac{1}{s^2+1} + 4\frac{1}{s+2} + 2\frac{1}{s}$

$= \frac{s}{s^2+4} + \frac{3}{s^2+1} + \frac{4}{s+2} + \frac{2}{s}$

2) $L[2\cosh 2t + 3e^{4t} + 4t^3 + 2^{3t}]$

Sol^n $= 2L[\cosh 2t] + 3L[e^{4t}] + 4L[t^3] + L[2^{3t}]$

$= 2\frac{s}{s^2-2^2} + 3\frac{1}{s-4} + 4\frac{\lceil 4}{s^4} + L[e^{\log 2^{3t}}]$

$= \frac{2s}{s^2-4} + \frac{3}{s-4} + \frac{4(3!)}{s^4} + L[e^{3t\log 2}]$

$= \frac{2s}{s^2-4} + \frac{3}{s-4} + \frac{4\times 6}{s^4} + L[e^{(3\log 2)t}]$

$= \frac{2s}{s^2-4} + \frac{3}{s-4} + \frac{24}{s^4} + \frac{1}{s - 3\log 2}$

3) $L\left[t^{3/2}\right]$

Sol^n

$L\left[t^{3/2}\right] = \frac{\overline{|3/2+1}}{s^{3/2+1}} = \frac{\overline{|5/2}}{s^{5/2}}$

but $\overline{|n} = (n-1)\overline{|n-1}$

$\therefore L\left[t^{3/2}\right] = \frac{(5/2-1)\overline{|5/2-1}}{s^{5/2}}$

$= \frac{3/2\,\overline{|3/2}}{s^{5/2}} = \frac{3/2(3/2-1)\overline{|3/2-1}}{s^{5/2}} = \frac{3/2\cdot 1/2\,\overline{|1/2}}{s^{5/2}}$

$= \frac{3\sqrt{\pi}}{4s^{5/2}}$

4) $L\left[\frac{1}{\sqrt{\pi t}}\right]$

Sol^n $L\left[\frac{1}{\sqrt{\pi t}}\right] = L\left[\frac{1}{\sqrt{\pi}\sqrt{t}}\right] = \frac{1}{\sqrt{\pi}} L\left[t^{-1/2}\right] = \frac{1}{\sqrt{\pi}} \frac{\overline{|-1/2+1}}{s^{-1/2+1}}$

$= \frac{1}{\sqrt{\pi}} \frac{\overline{|1/2}}{s^{1/2}} = \frac{1}{\cancel{\sqrt{\pi}}} \times \frac{\cancel{\sqrt{\pi}}}{s^{1/2}} = \frac{1}{\sqrt{s}}$

$L\left[\frac{1}{\sqrt{\pi t}}\right] = \frac{1}{\sqrt{s}}$.

5) $L\left[\sqrt{1+\sin t}\right]$

Sol^n $L\left[\sqrt{1+\sin t}\right]$

$= L\left[\sqrt{\sin^2 t/2 + \cos^2 t/2 + 2\sin t/2 \cos t/2}\right]$

$= L\left[\sqrt{(\sin t/2 + \cos t/2)^2}\right]$

$= L\left[(\sin t/2 + \cos t/2)\right]$

$= \frac{1/2}{s^2+(1/2)^2} + \frac{s}{s^2+(1/2)^2} = \frac{1}{2\left[s^2+1/4\right]} + \frac{s}{s^2+1/4}$

Prof. Nancy Sinollin

6) $L[\cos t \; \cos 2t \; \cos 3t]$

Sol^n

$L[\cos t(\cos 2t \cos 3t)]$

$= L\left[\cos t\left(\frac{\cos(2t+3t)+\cos(2t-3t)}{2}\right)\right]$

$= \frac{1}{2} L[\cos t \cos 5t + \cos t \cos(-t)]$

$= \frac{1}{2} L[\cos t \cos 5t + \cos t \cos t]$

$= \frac{1}{2} L\left[\frac{\cos 6t + \cos(-4t)}{2} + \frac{\cos 2t + \cos(0)}{2}\right]$

$= \frac{1}{4} L[\cos 6t + \cos 4t + \cos 2t + 1]$

$= \frac{1}{4}\left[\frac{s}{s^2+6^2} + \frac{s}{s^2+4^2} + \frac{s}{s^2+2^2} + \frac{1}{s}\right]$

$= \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{s}{s^2+4} + \frac{1}{s}\right]$

7) $L[\sin t \; \cos 4t \; \cos 3t]$

Sol^n $L[\sin t(\cos 4t \cos 3t)]$

$= L\left[\sin t\left(\frac{\cos 7t + \cos t}{2}\right)\right]$

$= \frac{1}{2} L[\sin t \cos 7t + \sin t \cos t]$

$= \frac{1}{2} L\left[\frac{\sin 8t + \sin(-6t)}{2} + \frac{\sin(2t)+\sin(0)}{2}\right]$

$= \frac{1}{4} L[\sin 8t - \sin 6t + \sin 2t]$

$= \frac{1}{4}\left[\frac{8}{s^2+8^2} - \frac{6}{s^2+6^2} + \frac{2}{s^2+2^2}\right]$

$= \frac{1}{4}\left[\frac{8}{s^2+64} - \frac{6}{s^2+36} + \frac{2}{s^2+4}\right]$

Prof. Nancy Sinollin

Note:-

- $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

  $\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$ $\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$

- We have, Binomial expansion as

  $(a\pm b)^n = a^n \pm {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 \pm \cdots\cdots \pm b^n$

One can also find ${}^nC_1, {}^nC_2, \cdots$ using Pascal triangle, which is as follows.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \rightarrow (a+b) \\
 & & & 1 & & 2 & & 1 & & & \rightarrow (a+b)^2 = a^2+2ab+b^2 \\
 & & 1 & & 3 & & 3 & & 1 & & \rightarrow (a+b)^3 = a^3+3a^2b+3ab^2+b^3 \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \rightarrow (a+b)^4 \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \rightarrow (a+b)^5
\end{array}
$$

& so on.

- We'll use above formulae when we need to find Laplace transform of powers of cosine, sine & also hyperbolic cosine & sine.

8) $L[\cosh^4 t]$

Sol^n $L[\cosh^4 t] = L[(\cosh t)^4]$

$$= L\left[\left(\frac{e^t+e^{-t}}{2}\right)^4\right]$$

$$= \frac{1}{2^4} L[(e^t+e^{-t})^4]$$

$$= \frac{1}{16} L[(e^t)^4 + 4(e^t)^3(e^{-t}) + 6(e^t)^2(e^{-t})^2 + 4(e^t)(e^{-t})^3 + (e^{-t})^4]$$

$$L[\cosh^4 t] = \frac{1}{16} L\left[e^{4t} + 4e^{3t}e^{-t} + 6e^{2t}e^{-2t} + 4e^{t}e^{-3t} + e^{-4t}\right]$$

$$= \frac{1}{16} L\left[e^{4t} + 4e^{2t} + 6 + 4e^{-2t} + e^{-4t}\right]$$

$$= \frac{1}{16}\left[\frac{1}{s-4} + \frac{4}{s-2} + \frac{6}{s} + \frac{4}{s+2} + \frac{1}{s+4}\right]$$

9) $L[\sinh^5 2t]$

Sol^n $L[\sinh^5 2t] = L\left[(\sinh 2t)^5\right]$

$$= L\left[\left(\frac{e^{2t} - e^{-2t}}{2}\right)^5\right]$$

$$= \frac{1}{2^5} L\left[\left(e^{2t} - e^{-2t}\right)^5\right]$$

$$= \frac{1}{32} L\left[(e^{2t})^5 - 5(e^{2t})^4(e^{-2t}) + 10(e^{2t})^3(e^{-2t})^2 - 10(e^{2t})^2(e^{-2t})^3 + 5(e^{2t})(e^{-2t})^4 - (e^{-2t})^5\right]$$

$$= \frac{1}{32} L\left[e^{10t} - 5e^{8t}e^{-2t} + 10e^{6t}e^{-4t} - 10e^{4t}e^{-6t} + 5e^{2t}e^{-8t} - e^{-10t}\right]$$

$$= \frac{1}{32} L\left[e^{10t} - 5e^{6t} + 10e^{2t} - 10e^{-2t} + 5e^{-6t} - e^{-10t}\right]$$

$$= \frac{1}{32}\left[\frac{1}{s-10} - \frac{5}{s-6} + \frac{10}{s-2} - \frac{10}{s+2} + \frac{5}{s+6} - \frac{1}{s+10}\right]$$

10) $L[\sin^3 t]$

Sol^n $L[\sin^3 t] = L\left[(\sin t)^3\right]$

$$= L\left[\left(\frac{e^{it} - e^{-it}}{2i}\right)^3\right]$$

$$= \frac{1}{(2i)^3} L\left[\left(e^{it} - e^{-it}\right)^3\right]$$

$$= \frac{1}{8i^3} L\left[(e^{it})^3 - 3(e^{it})^2(e^{-it}) + 3(e^{it})(e^{-it})^2 - (e^{-it})^3\right]$$

$$= -\frac{1}{8i} L\left[e^{i3t} - 3e^{i2t}e^{-it} + 3e^{it}e^{-i2t} - e^{-i3t}\right]$$

$$= -\frac{1}{8i} L\left[e^{i3t} - 3e^{it} + 3e^{-it} - e^{-i3t}\right]$$

$$= -\frac{1}{8i}\left[\frac{1}{s-i3} - \frac{3}{s-i} + \frac{3}{s+i} - \frac{1}{s+3i}\right]$$

Ex. 1) $L[\cos^4 2t]$ 2) $L[\cosh^3 3t]$ 3) $L[\sinh^3 5t]$

Note:-

i) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

ii) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

iii) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

11) Find $L[\sin\sqrt{t}]$

Solⁿ as $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$$\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \cdots$$

$$\Rightarrow L(\sin\sqrt{t}) = L\left[t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \cdots\right]$$

$$= L(t^{1/2}) - \frac{1}{3!}L(t^{3/2}) + \frac{1}{5!}L(t^{5/2}) - \cdots$$

$$= \frac{\overline{|\tfrac{1}{2}+1}}{s^{1/2+1}} - \frac{1}{3!}\frac{\overline{|\tfrac{3}{2}+1}}{s^{3/2+1}} + \frac{1}{5!}\frac{\overline{|5/2+1}}{s^{5/2+1}} - \cdots$$

$$\Rightarrow L[\sin\sqrt{t}] = \frac{\Gamma 3/2}{s^{3/2}} - \frac{1}{3!}\frac{\Gamma 5/2}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma 7/2}{s^{7/2}} - - - -$$

as $\Gamma n = (n-1)\,\Gamma n-1$

$$\Gamma 3/2 = 1/2\,\Gamma 1/2 = \tfrac{1}{2}\sqrt{\pi} = \frac{\sqrt{\pi}}{2}$$

$$\Gamma 5/2 = 3/2\,\Gamma\tfrac{3}{2} = \frac{3}{2}\frac{\sqrt{\pi}}{2} = \frac{3\sqrt{\pi}}{4}$$

$$\Gamma 7/2 = 5/2\,\Gamma 5/2 = 5/2\,\frac{3\sqrt{\pi}}{4} = \frac{15}{8}\sqrt{\pi}$$

& so on.

$$\therefore L[\sin\sqrt{t}] = \frac{\sqrt{\pi}/2}{s^{3/2}} - \frac{1}{3!}\frac{3\sqrt{\pi}/4}{s^{5/2}} + \frac{1}{5!}\frac{15\sqrt{\pi}/8}{s^{7/2}} - - -$$

$$= \frac{\sqrt{\pi}}{2}\left[\frac{1}{s^{3/2}} - \frac{1}{3!}\frac{3}{(2)s^{5/2}} + \frac{1}{5!}\frac{15}{(4)s^{7/2}} - - -\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{\cancel{3}}{2\times\cancel{3}\times 2s} + \frac{\cancel{15}}{2\times\cancel{3}\times 4\times\cancel{5}\times 4\,s^2} - - - -\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{2!}\frac{1}{(4s)^2} - - - -\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\,e^{-1/4s}$$ — from i) (here $x = -1/4s$)

12) Prove that $L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \frac{\sqrt{\pi}}{s^{1/2}}\,e^{-1/4s}$

<u>Soln</u> as $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - - - - -$

$$\cos\sqrt{t} = 1 - \frac{(\sqrt{t})^2}{2!} + \frac{(\sqrt{t})^4}{4!} - - - - -$$

$$= 1 - \frac{t}{2!} + \frac{t^2}{4!} - - - - -$$

$$\therefore \frac{\cos\sqrt{t}}{\sqrt{t}} = \frac{1}{\sqrt{t}} - \frac{t}{2!\sqrt{t}} + \frac{t^2}{4!\sqrt{t}} - - - -$$

$$\therefore \frac{\cos\sqrt{t}}{\sqrt{t}} = t^{-1/2} - \frac{t\, t^{-1/2}}{2!} + \frac{t^2 t^{-1/2}}{4!} - \cdots$$

$$= t^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \cdots$$

$$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = L\left[t^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \cdots\right]$$

$$= L(t^{-1/2}) - \frac{1}{2!}L(t^{1/2}) + \frac{1}{4!}L(t^{3/2}) - \cdots$$

$$= \frac{\Gamma(1/2)}{s^{1/2}} - \frac{1}{2!}\frac{\Gamma(3/2)}{s^{3/2}} + \frac{1}{4!}\frac{\Gamma(5/2)}{s^{5/2}} - \cdots$$

$$= \frac{\sqrt{\pi}}{s^{1/2}} - \frac{1}{2!}\frac{\sqrt{\pi}/2}{s^{3/2}} + \frac{1}{4!}\frac{3\sqrt{\pi}/4}{s^{5/2}} - \cdots$$

— as we found in earlier problem.

$$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \frac{\sqrt{\pi}}{s^{1/2}} - \frac{\sqrt{\pi}}{2!\,(2)\,s^{3/2}} + \frac{3\sqrt{\pi}}{4!\,4\,s^{5/2}} - \cdots$$

$$= \frac{\sqrt{\pi}}{s^{1/2}}\left[1 - \frac{1}{4s} + \frac{\cancel{3}}{2\times\cancel{3}\times4\times4s^2} - \cdots\right]$$

$$= \frac{\sqrt{\pi}}{s^{1/2}}\left[1 - \frac{1}{4s} + \frac{1}{2!}\frac{1}{(4s)^2} - \cdots\right]$$

$$= \sqrt{\frac{\pi}{s}}\, e^{-1/4s}$$ — from i) Here $x = -\frac{1}{4s}$

Prof. Nancy Sinollin