* NLPP with inequality Constraints (Kuhn-Tucker Conditions)

consider NLPP with $n$ variables & 1 inequality constraint

Maximise $z = f(x_1, x_2, ---, x_n)$

Subject to $g(x_1, ---, x_n) \leq b$

$x_1, x_2, ---, x_n \geq 0$

- change the inequality constraint to equality by adding slack variable $s$ in the form of $s^2$ (so that it is non negative)

$\therefore g(x_1, x_2, --, x_n) + s^2 = b$

$\therefore g(x_1, x_2, ---, x_n) - b + s^2 = 0$

$\therefore h(x_1, ---, x_n) + s^2 = 0$, where $h(x_1, --, x_n) = g(x_1, ---x_n) - b$

Now there are $(n+1)$ variables & 1 equality constraint

- construct the Lagrangian function as

$$L(x_1, x_2, --, x_n, s, \lambda) = f(x_1, ---, x_n) - \lambda [h(x_1, ---, x_n) + s^2] \quad \text{——— (1)}$$

- The necessary condition for stationary points are

$$\frac{\partial L}{\partial x_1} = 0, \quad \frac{\partial L}{\partial x_2} = 0, \quad ---, \quad \frac{\partial L}{\partial x_n} = 0, \quad \frac{\partial L}{\partial \lambda} = 0, \quad \frac{\partial L}{\partial s} = 0 \quad \text{——— (2)}$$

$(1) \Rightarrow$

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1}, \quad \frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2}, \quad -----, \quad \frac{\partial L}{\partial x_n} = \frac{\partial f}{\partial x_n} - \lambda \frac{\partial h}{\partial x_n},$$

$$\frac{\partial L}{\partial \lambda} = -[h(x_1, --, x_n) + s^2], \quad \frac{\partial L}{\partial s} = -2s\lambda$$

Using (2) we get following $(n+2)$ necessary conditions,

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0, \quad ----, \quad \frac{\partial f}{\partial x_n} - \lambda \frac{\partial h}{\partial x_n} = 0,$$

$$h(x_1, ---, x_n) + s^2 = 0, \quad -2s\lambda = 0$$

from $-2s\lambda = 0$ we get either $s=0$ or $\lambda = 0$

If $s=0$ then as $h(x_1, ---, x_n) + s^2 = 0 \Rightarrow h(x_1, --, x_n) = 0$

$\therefore$ either $\lambda = 0$ or $h(x_1, ---, x_n) = 0$

i.e. $\lambda h(x_1, ---, x_n) = 0$

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but $s^2$ is positive & $h(x_1, \ldots, x_n) + s^2 = 0$

$\therefore \; h(x_1, \ldots, x_n) < 0$

$\therefore$ when $\lambda = 0$ , $h(x_1, \ldots, x_n) < 0$

& when $\lambda > 0$ , $h(x_1, \ldots, x_n) = 0$

$\therefore$ The <u>necessary</u> <u>conditions</u> <u>for</u> <u>maxima</u> are,

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0 , \quad \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0 , \quad \ldots , \quad \frac{\partial f}{\partial x_n} - \lambda \frac{\partial h}{\partial x_n} = 0$$

$$\lambda h(x_1, x_2, \ldots, x_n) = 0 , \quad h(x_1, x_2, \ldots, x_n) \leq 0 , \quad \lambda \geq 0$$

These conditions are <u>Kuhn</u> - <u>Tucker</u> <u>conditions</u>.

- For <u>minimisation</u> problem, the last condition changes to

  $\lambda \leq 0$

<u>Note</u>:- For a <u>general</u> <u>NLPP</u>, (Kuhn - Tucker Conditions)

Consider NLPP with n variables & n inequality constraints.

Maximise $Z = f(x_1, \ldots, x_n)$

subject to $g_i(x_1, \ldots, x_n) \leq b_i$ , $i = 1, 2, \ldots, n$

$x_1, \ldots, x_n \geq 0$

<u>Kuhn</u> - <u>Tucker</u> <u>conditions</u> are:-

$$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial h_1}{\partial x_1} - \lambda_2 \frac{\partial h_2}{\partial x_1} - \cdots - \lambda_n \frac{\partial h_n}{\partial x_1} = 0 , \quad \frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial h_1}{\partial x_2} - \lambda_2 \frac{\partial h_2}{\partial x_2} - \cdots - \lambda_n \frac{\partial h_n}{\partial x_2} = 0$$

$$\ldots \quad \frac{\partial f}{\partial x_n} - \lambda_1 \frac{\partial h_1}{\partial x_n} - \lambda_2 \frac{\partial h_2}{\partial x_n} - \cdots - \lambda_n \frac{\partial h_n}{\partial x_n} = 0 ,$$

$$\lambda_1 h_1(x_1, \ldots, x_n) = 0 , \quad \lambda_2 h_2(x_1, \ldots, x_n) = 0 , \quad \ldots , \quad \lambda_n h_n(x_1, \ldots, x_n) = 0$$

$$h_1(x_1, \ldots, x_n) \leq 0 , \quad h_2(x_1, \ldots, x_n) \leq 0 , \quad \ldots , \quad h_n(x_1, \ldots, x_n) \leq 0 ,$$

$$x_1, \ldots, x_n \geq 0 , \quad \lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$$

- For <u>minimisation</u> <u>problem</u>, the last condition changes to

  $\lambda_1, \lambda_2, \ldots, \lambda_n \leq 0$

Examples:- Solve the following NLPP using Kuhn Tucker method 

1) Maximise $Z = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$

Subject to $2x_1 + x_2 \le 5$

$x_1, x_2 \geq 0$

→ Rewrite the given problem as

$f(x_1,x_2) = 10x_1 + 4x_2 - 2x_1^2 - x_2^2$

& $h(x_1,x_2) = 2x_1 + x_2 - 5$

The Kuhn-Tucker conditions are,

$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0, \quad \lambda h(x_1,x_2) = 0, \quad h(x_1,x_2) \le 0, \lambda \geq 0$

∴ we get $10 - 4x_1 - 2\lambda = 0$ —— (1) $\quad 4 - 2x_2 - \lambda = 0$ —— (2)

$\lambda(2x_1 + x_2 - 5) = 0$ —— (3) $\quad 2x_1 + x_2 - 5 \le 0$ —— (4)

$x_1, x_2, \lambda \geq 0$ —— (5)

(3) ⇒ either $\lambda = 0$ or $2x_1 + x_2 - 5 = 0$

Case (1) If $\lambda = 0$ then from (1) & (2) we get

$4x_1 = 10$ & $2x_2 = 4$ ⇒ $x_1 = 5/2$ & $x_2 = 2$

∴ LHS of (4) becomes, $2(5/2) + 2 - 5 = 5 + 2 - 5 = 2 \nleq 0$

∴ It doesn't satisfy (4). ∴ $\lambda \neq 0$

Case (2) If $\lambda \neq 0$ & $2x_1 + x_2 - 5 = 0$ —— (6)

Solving (1), (2) & (6)

∴ (1) − 2×(2) ⇒
$$\begin{array}{r} 10 - 4x_1 - \cancel{2\lambda} = 0 \\ -\; 8 - 4x_2 - 2\lambda = 0 \\ \hline 2 - 4(x_1 - x_2) = 0 \end{array} \Rightarrow 2x_1 - 2x_2 = 1 \text{ —— (7)}$$

multiply (6) by 2 ⇒ $4x_1 + 2x_2 = 10$ —— (8)

(7) + (8) ⇒ $6x_1 = 11$ ⇒ $\boxed{x_1 = 11/6}$

(6) ⇒ $x_2 = 5 - 2x_1 = 5 - 2(11/6) = 4/3$ ⇒ $\boxed{x_2 = 4/3}$

(2) ⇒ $\lambda = 4 - 2x_2 = 4 - 2(4/3) = 4/3$ ⇒ $\boxed{\lambda = 4/3}$

∴ $Z_{max} = 10\left(\frac{11}{6}\right) + 4\left(\frac{4}{3}\right) - 2\left(\frac{11}{6}\right)^2 - \left(\frac{4}{3}\right)^2 = \frac{91}{6}$ ⇒ $\boxed{Z_{max} = \frac{91}{6}}$

2) Maximise $Z = 2x_1^2 - 7x_2^2 - 16x_1 + 2x_2 + 12x_1x_2 + 7$

subject to $2x_1 + 5x_2 \le 105$

$x_1, x_2 \geq 0$

$\rightarrow$ Rewrite the problem as

$f(x_1, x_2) = 2x_1^2 - 7x_2^2 - 16x_1 + 2x_2 + 12x_1x_2 + 7$

& $h(x_1, x_2) = 2x_1 + 5x_2 - 105$

The Kuhn-Tucker conditions are

$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial h}{\partial x_1} = 0$, $\frac{\partial f}{\partial x_2} - \lambda \frac{\partial h}{\partial x_2} = 0$, $\lambda h(x_1, x_2) = 0$, $h(x_1, x_2) \le 0$, $\lambda \geq 0$

$\therefore$ $4x_1 - 16 + 12x_2 - 2\lambda = 0$ —— (1) $\quad -14x_2 + 2 + 12x_1 - 5\lambda = 0$ —— (2)

$\lambda(2x_1 + 5x_2 - 105) = 0$ —— (3) $\quad 2x_1 + 5x_2 - 105 \le 0$ —— (4)

$x_1, x_2, \lambda \geq 0$ —— (5)

from (3) we get either $\lambda = 0$ or $2x_1 + 5x_2 - 105 = 0$

<u>Case (1)</u> : If $\lambda = 0$ then from (1) & (2) we get

$4x_1 + 12x_2 = 16$ & $12x_1 - 14x_2 = -2$

$$\therefore \begin{array}{r} 12x_1 + 36x_2 = 48 \\ -\,12x_1 - 14x_2 = -2 \\ \hline 50x_2 = 50 \end{array} \Rightarrow x_2 = 1 \quad \& \quad x_1 = 1$$

$\therefore$ LHS of (4) $= 2(1) + 5(1) - 105 = -98 < 0$

For $x_1 = 1$, $x_2 = 1$, $Z = 0$ $\therefore$ for $\lambda = 0$, feasible solution is not obtaine

$\therefore$ reject these values.

<u>Case (2)</u> : If $\lambda \neq 0$ then $2x_1 + 5x_2 = 105$ — (6)

Now eliminate $\lambda$ from (1) & (2)

$\therefore$ $5\times(1) - 2\times(2) \Rightarrow -4x_1 - 84 + 88x_2 = 0 \Rightarrow -x_1 + 22x_2 = 21$ — (7)

(6) $2\times(7) \Rightarrow 49x_2 = 147 \Rightarrow \boxed{x_2 = 3}$

(6) $\Rightarrow 2x_1 = 105 - 5\times 3 = 90 \Rightarrow \boxed{x_1 = 45}$

(1) $\Rightarrow 2\lambda = 4(45) - 16 + 12(3) = 200 \Rightarrow \boxed{\lambda = 100}$

$\therefore Z_{max} = 2(45)^2 - 7(3)^2 - 16(45) + 2(3) + 12(3)(45) + 7$

$\therefore \boxed{Z_{max} = 4900}$

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3) Maximise $z = 2x_1 + 3x_2 - x_1^2 - 2x_2^2$

Subject to $x_1 + 3x_2 \le 6$

$5x_1 + 2x_2 \le 10$

$x_1, x_2 \geq 0$

→ Rewrite the problem as,

$f(x_1, x_2) = 2x_1 + 3x_2 - x_1^2 - 2x_2^2$

$h_1(x_1, x_2) = x_1 + 3x_2 - 6$ , $h_2(x_1, x_2) = 5x_1 + 2x_2 - 10$

Kuhn - Tucker Conditions for maxima are,

$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial h_1}{\partial x_1} - \lambda_2 \frac{\partial h_2}{\partial x_1} = 0$ , $\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial h_1}{\partial x_2} - \lambda_2 \frac{\partial h_2}{\partial x_2} = 0$

$\lambda_1 h_1(x_1, x_2) = 0$ , $\lambda_2 h_2(x_1, x_2) = 0$ , $h_1(x_1, x_2) \le 0$, $h_2(x_1, x_2) \le 0$,

$x_1, x_2, \lambda_1, \lambda_2 \geq 0$

$\therefore$ $2 - 2x_1 - \lambda_1 - 5\lambda_2 = 0$ —— (1) $3 - 4x_2 - 3\lambda_1 - 2\lambda_2 = 0$ —— (2)

$\lambda_1(x_1 + 3x_2 - 6) = 0$ —— (3) $\lambda_2(5x_1 + 2x_2 - 10) = 0$ —— (4)

$x_1 + 3x_2 - 6 \le 0$ —— (5) $5x_1 + 2x_2 - 10 \le 0$ —— (6)

$x_1, x_2 \geq 0$ —— (7) , $\lambda_1, \lambda_2 \geq 0$ —— (8)

depending upon values of $\lambda_1, \lambda_2$, consider the following cases

Case 1: If $\lambda_1 = 0$ & $\lambda_2 = 0$

$\therefore$ (1) & (2) $\Rightarrow$ $2 = 2x_1$ & $3 = 4x_2$ $\Rightarrow$ $x_1 = 1$ , $x_2 = 3/4$

These values satisfy (5), (6), (7) & (8), But we cannot immediately conclude that $x_1 = 1$, $x_2 = 3/4$ is a maxima (because $\lambda_1 = 0$, $\lambda_2 = 0$ can also give minima, $\because$ condition for minima is $\lambda_1, \lambda_2 \le 0$)

$\therefore$ Test the Hessian matrix for the objective function

$$H = \begin{bmatrix} \frac{\partial^2 z}{\partial x_1^2} & \frac{\partial^2 z}{\partial x_1 \partial x_2} \\ \frac{\partial^2 z}{\partial x_2 \partial x_1} & \frac{\partial^2 z}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$$

$\therefore A_1 = [-2]$ & $A_2 = \begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$ $\therefore D_1 = -2$ & $D_2 = 8$

$\therefore$ the principal minors are alternately negative, positive

$\therefore$ $\boxed{x_1 = 1}$, $\boxed{x_2 = 3/4}$ gives maxima

$\therefore$ $Z_{max} = 2 + 3\left(\frac{3}{4}\right) - 1 - 2\left(\frac{9}{16}\right) = \frac{17}{8} \Rightarrow \boxed{Z_{max} = \frac{17}{8}}$

Case(2):- If $\lambda_1 = 0$ & $\lambda_2 \neq 0$

To find $x_1, x_2$, first we eliminate $\lambda_2$ from (1) & (2)

$\therefore$ $2\times(1) - 5\times(2) \Rightarrow 4 - 4x_1 - 15 + 20x_2 = 0 \Rightarrow 4x_1 - 20x_2 = -11$

Since $\lambda_2 \neq 0$, $(4) \Rightarrow 5x_1 + 2x_2 = 10$

Solving these two equations, $x_1 = \frac{89}{54}$, $x_2 = \frac{95}{108}$

$\therefore$ For $x_1 = \frac{89}{54}$, $x_2 = \frac{95}{108}$, $\lambda_1 = 0$, we get from (1),

$2 - 2\times\frac{89}{54} - 5\lambda_2 = 0 \Rightarrow 5\lambda_2 = \frac{-70}{54}$ but $\lambda$ is not negative

$\therefore$ case (2) is not possible.

Case (3):- If $\lambda_1 \neq 0$, & $\lambda_2 = 0$

To find $x_1, x_2$, first we eliminate $\lambda_1$ from (1) & (2)

$\therefore$ $3\times(1) - (2) \Rightarrow 6 - 6x_1 - 3 + 4x_2 = 0 \Rightarrow 6x_1 - 4x_2 = 3$

Since $\lambda_1 \neq 0$, $(3) \Rightarrow x_1 + 3x_2 = 6$

Solving these equations, $x_1 = 3/2$, $x_2 = 3/2$

But it doesn't satisfy (6) $\therefore$ case (3) is not possible.

Case (4):- If $\lambda_1 \neq 0$ & $\lambda_2 \neq 0$

$\therefore$ (3) & (4) $\Rightarrow$ $x_1 + 3x_2 = 6$ & $5x_1 + 2x_2 = 10$

Solving these equations, $x_1 = \frac{20}{13}$, $x_2 = \frac{18}{13}$

$\therefore$ $(1) \Rightarrow \lambda_1 + 5\lambda_2 = -\frac{14}{13}$ $\quad (2) \Rightarrow 3\lambda_1 + 2\lambda_2 = \frac{-33}{13}$

Solving these equations $\lambda_1 = \frac{-137}{169}$, $\lambda_2 = \frac{-9}{169}$

i.e. $\lambda_1, \lambda_2$ are negative

$\therefore$ case (4) is not possible.

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4) Maximise $Z = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$

Subject to $x_1 + x_2 \le 2$

$2x_1 + 3x_2 \le 12$

$x_1, x_2, x_3 \geq 0$

→ Rewrite the problem as

$f(x_1, x_2, x_3) = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$

$h_1(x_1, x_2, x_3) = x_1 + x_2 - 2$

$h_2(x_1, x_2, x_3) = 2x_1 + 3x_2 - 12$

Kuhn - Tucker conditions for maxima are

$\frac{\partial f}{\partial x_1} - \lambda_1 \frac{\partial h_1}{\partial x_1} - \lambda_2 \frac{\partial h_2}{\partial x_1} = 0$ , $\frac{\partial f}{\partial x_2} - \lambda_1 \frac{\partial h_1}{\partial x_2} - \lambda_2 \frac{\partial h_2}{\partial x_2} = 0$,

$\frac{\partial f}{\partial x_3} - \lambda_1 \frac{\partial h_1}{\partial x_3} - \lambda_2 \frac{\partial h_2}{\partial x_3} = 0$ , $\lambda_1 h_1(x_1, x_2, x_3) = 0$ , $\lambda_2 h_2(x_1, x_2, x_3) = 0$

$h_1(x_1, x_2, x_3) \le 0$ , $h_2(x_1, x_2, x_3) \le 0$ , $x_1, x_2, x_3 \geq 0$ , $\lambda_1, \lambda_2 \geq 0$

$\therefore -2x_1 + 4 - \lambda_1 - 2\lambda_2 = 0$ —— (1)   $-2x_2 + 6 - \lambda_1 - 3\lambda_2 = 0$ —— (2)

$x_3 = 0$ —— (3)   $\lambda_1(x_1 + x_2 - 2) = 0$ —— (4)

$\lambda_2(2x_1 + 3x_2 - 12) = 0$ —— (5)   $x_1 + x_2 - 2 \le 0$ —— (6)

$2x_1 + 3x_2 - 12 \le 0$ —— (7)   $x_1, x_2, x_3 \geq 0$ —— (8)   $\lambda_1, \lambda_2 \geq 0$ —— (9)

Case (1):- If $\lambda_1 = 0$ & $\lambda_2 = 0$

$\therefore$ (1), (2) & (3) $\Rightarrow$ $-2x_1 + 4 = 0$ , $-2x_2 + 6 = 0$ , $x_3 = 0$

$\therefore$ $x_1 = 2$ , $x_2 = 3$, $x_3 = 0$

But it do not satisfy (6) & (7)

$\therefore$ Reject this pair.

Case (2):- If $\lambda_1 = 0$ & $\lambda_2 \neq 0$

So first we eliminate $\lambda_2$ from (1) & (2)

$\therefore 3 \times (1) - 2 \times (2) \Rightarrow -6x_1 + 12 + 4x_2 - 12 = 0 \Rightarrow 2x_2 = 3x_1$

from (5) , $2x_1 + 3x_2 = 12$ $(\because \lambda_2 \neq 0)$

$\therefore 2x_1 + 3\left(\frac{3x_1}{2}\right) = 12 \Rightarrow \frac{13x_1}{2} = 12 \Rightarrow x_1 = \frac{24}{13}$

& $x_2 = 3/2$

But for $x_1 = \frac{36}{13}$, $x_2 = \frac{3}{2}$, $x_3 = 0$, condition (6) is not satisfy

$\therefore$ Reject these values.

Case (3) :- If $\lambda_1 \neq 0$ & $\lambda_2 = 0$

First eliminate $\lambda_1$ from (1) & (2)

$\therefore$ (2) - (1) $\Rightarrow$ $-2x_1 + 4 + 2x_2 - 6 = 0$ $\Rightarrow$ $-x_1 + x_2 = 1$

Since $\lambda_1 \neq 0$, (4) $\Rightarrow$ $x_1 + x_2 = 2$

add these equations, $\Rightarrow$ $2x_2 = 3$ $\Rightarrow$ $\boxed{x_2 = 3/2}$ & $\boxed{x_1 = 1/2}$

from (1), $\boxed{\lambda_1 = 3}$

These values satisfy the conditions (6), (7), (8), (9)

$\therefore$ $\boxed{x_1 = 1/2}$ $\boxed{x_2 = 3/2}$ $\boxed{x_3 = 0}$ is a feasible solution

& $Z_{max} = -\left(\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 4\left(\frac{1}{2}\right) + 6\left(\frac{3}{2}\right) = \frac{17}{2}$ $\Rightarrow$ $\boxed{Z_{max} = \frac{17}{2}}$

Case (4) :- If $\lambda_1 \neq 0$ & $\lambda_2 \neq 0$

$\therefore$ (3) & (4) $\Rightarrow$ $x_1 + x_2 = 2$ & $2x_1 + 3x_2 = 12$

Solving these equations $\Rightarrow$ $x_1 = -6$, $x_2 = 8$

i.e. $x_2 < 0$ which is not possible.

$\therefore$ reject these values.

$\therefore$ The solution is $x_1 = 1/2$, $x_2 = 3/2$, $x_3 = 0$, $Z_{max} = \frac{17}{2}$

* Practice Problems:-

Using Kuhn-Tucker condition solve the following NLPP

(i) Maximise $z = 2x_1 + 3x_2 - x_1^2 - x_2^2$

Subject to $x_1 + x_2 \leq 1$

$2x_1 + 3x_2 \leq 6$

$x_1, x_2 \geq 0$

(ii) Minimise $z = 7x_1^2 + 5x_2^2 - 6x_1$

Subject to $x_1 + 2x_2 \leq 10$

$x_1 + 3x_2 \leq 9$

$x_1, x_2 \geq 0$

(iii) Maximise $z = x_1 + 3x_1^2 - x_2^3 + 2x_2 + 10$

Subject to $x_1 + x_2 \leq 0$

$x_1, x_2 \geq 0$

(ii) Maximise $z = 6x_1^2 + 5x_2^2$

Subject to $x_1 + 5x_2 \leq 3$

$x_1, x_2 \geq 0$

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