11) Find Fourier Series of $f(x) = x\sin x$ in $(-\pi, \pi)$
Deduce that, $\frac{\pi-2}{4} = \frac{1}{1\cdot3} - \frac{1}{3\cdot5} + \frac{1}{5\cdot7} \dots$

Sol^n^ Given interval is $(-\pi, \pi)$ which is symmetric (we'll check that the function is even or odd)

$$f(-x) = -x\sin(-x)$$
$$= (-x)(-\sin x)$$
$$= x\sin x$$
$$= f(x)$$

$\Rightarrow f(x)$ is even function

$\Rightarrow b_n = 0$ & Here, $l = \pi$.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{——— (1)}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$
$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx$$
$$= \frac{2}{\pi}\Big[x(-\cos x) - (1)(-\sin x)\Big]_0^{\pi}$$
$$= \frac{2}{\pi}\Big[\pi(-\cos\pi) - 0 - 0 + 0\Big]$$
$$= \frac{2}{\pi}\Big[-\pi(-1)\Big]$$
$$= 2$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$
$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx.$$

$$= \frac{\cancel{2}}{\pi}\int_0^{\pi} \frac{x}{\cancel{2}}\left[\sin(x+nx) + \sin(x-nx)\right]dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left[\sin(1+n)x + \sin(1-n)x\right]dx.$$

$$= \frac{1}{\pi}\left[x\left[\frac{-\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right] - (1)\left[\frac{-\sin(1+n)x}{(1+n)^2} - \frac{\sin(1-n)x}{(1-n)^2}\right]\right]_0^{\pi} \quad — n \neq 1$$

$$= \frac{1}{\pi}\left[\pi\left[\frac{-\cos(1+n)\pi}{1+n} - \frac{\cos(1-n)\pi}{1-n}\right] - \left[0 - 0\right] - 0 + 0\right]$$

$— n \neq 1$

$$= \frac{1}{\cancel{\pi}}\cancel{\pi}\left[\frac{-(-1)^{1+n}}{1+n} - \frac{(-1)^{1-n}}{1-n}\right]$$

$$= -\frac{(-1)(-1)^n}{1+n} - \frac{(-1)(-1)^{-n}}{1-n}$$

$$= \frac{(-1)^n}{1+n} + \frac{(-1)^n}{1-n} \qquad (-1)^n = (-1)^{-n}$$

$$= (-1)^n\left[\frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= (-1)^n\left[\frac{1-\cancel{n}+1+\cancel{n}}{1-n^2}\right]$$

$$= \frac{(-1)^n 2}{1-n^2} \quad — n \neq 1.$$

How to find a_n for $n=1$ i.e to find a_1

$$a_1 = \frac{2}{l}\int_0^{l} f(x)\cos x\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx.$$

$$= \frac{\cancel{2}}{\pi}\int_0^{\pi} \frac{x}{\cancel{2}}\sin 2x\,dx.$$

Prof. Nancy Sinollin

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{2^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left(-\frac{\cos 2\pi}{2}\right) - 0 - 0 + 0\right]$$

$$= \frac{1}{\cancel{\pi}}\left[\cancel{\pi}\frac{(-1)}{2}\right]$$

$$= \frac{-1}{2}$$

$\therefore$ from (1)

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx.$$

$$= \frac{+2}{2} + \left(\frac{-1}{2}\right)\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^n}{1-n^2}\cos nx$$

$$= 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty} \frac{(-1)^n}{1-n^2}\cos nx.$$

$$x\sin x = 1 - \frac{1}{2}\cos x + 2\left[\frac{1}{(-3)}\cos 2x + \frac{(-1)}{(-8)}\cos 3x + \frac{1}{(-15)}\cos 4x + \dots\right]$$

$$= 1 - \frac{1}{2}\cos x + 2\left[-\frac{\cos 2x}{3} + \frac{\cos 3x}{8} - \frac{\cos 4x}{15} + \dots\right]$$

put $x = \frac{\pi}{2}$

$$\frac{\pi}{2}(1) = 1 - \frac{1}{2}(0) + 2\left[-\frac{\cos \pi}{3} + \frac{\cos(3\pi/2)}{8} - \frac{\cos 2\pi}{15} + \dots\right]$$

$$\frac{\pi}{2} = 1 + 2\left[\frac{-(-1)}{3} + 0 - \frac{(1)}{15} + \dots\right]$$

$$\frac{\pi}{2} - 1 = 2\left[\frac{1}{3} - \frac{1}{15} + \dots\right]$$

$$\frac{\pi - 2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \dots$$

Prof. Nancy Sinollin

12) Find Fourier Series for $f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$

Also deduce $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

Solⁿ Given interval is $(-\pi, \pi)$ i.e symmetric interval

Hence we can check the function is even or odd.

as $f(-x) = \begin{cases} -\pi & -\pi < -x < 0 \\ -x & 0 < -x < \pi \end{cases}$

$= \begin{cases} -\pi & \pi > x > 0 \\ -x & 0 > x > -\pi \end{cases}$

$\neq f(x)$

also $f(-x) \neq -f(x)$ $\quad \because -f(x) = \begin{cases} \pi & -\pi < x < 0 \\ -x & 0 < x < \pi \end{cases}$

$\therefore$ Given function is neither even nor odd.

$(c, c+2l) = (-\pi, \pi)$

$\Rightarrow c = -\pi$ & $c + 2l = \pi \Rightarrow l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{—— (1)}$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-\pi)\,dx + \int_0^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)(x)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[(-\pi)(0-(-\pi)) + \frac{\pi^2}{2}\right]$$

Prof. Nancy Sinollin

$$= \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{2}\right]$$

$$= -\frac{\pi}{2}$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\, dx + \int_0^{\pi} x\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[(-\pi)(0-0) + \left[\pi(0) + \frac{\cos n\pi}{n^2} - 0 + \frac{(-1)}{n^2}\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\, dx + \int_0^{\pi} x\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} + \left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left[\frac{-1}{n} + \frac{\cos(-n\pi)}{n}\right] + \left[\pi\left(\frac{-\cos n\pi}{n}\right) - 0 - 0 + 0\right]\right]$$

$$= \frac{1}{\pi}\left[+\frac{\pi}{n} - \frac{\pi\cos n\pi}{n} - \frac{\pi\cos n\pi}{n}\right]$$

$$= \frac{\pi}{\pi}\left[\frac{1}{n} - \frac{2\cos n\pi}{n}\right]$$

$$= \frac{1-2(-1)^n}{n}$$

from (1)

$$f(x) = \frac{-\pi}{2\times 2} + \sum_{n=1}^{\infty} \frac{1}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)\cos nx + \sum_{n=1}^{\infty}\left(\frac{1-2(-1)^n}{n}\right)\sin nx$$

$$f(x) = \frac{-\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n^2}\right)\cos nx + \sum_{n=1}^{\infty}\left(\frac{1-2(-1)^n}{n}\right)\sin nx$$

put $x = 0$

$$f(0) = \frac{-\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}(1) + 0$$

$$f(0) = \frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right]$$

$$= \frac{1}{2}\left[\lim_{x\to 0^-}(-\pi) + \lim_{x\to 0^+} x\right]$$

$$= \frac{-\pi}{2}.$$

$$\therefore -\frac{\pi}{2} = \frac{-\pi}{4} + \frac{1}{\pi}\left[\frac{(-2)}{1^2} + 0 + \frac{(-2)}{3^2} + 0 \cdots\right]$$

$$\Rightarrow \frac{\pi}{4} - \frac{\pi}{2} = \frac{(-2)}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right]$$

$$\Rightarrow -\frac{\pi}{4} \times \frac{\pi}{(-2)} = \frac{1}{1^2} + \frac{1}{3^2} + \cdots$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \cdots$$

$$\Rightarrow \frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$ ---- $(2n-1)$ is odd number.

Prof. Nancy Sinollin

13) Find Fourier series of $f(x) = |\sin x|$ in $(-\pi, \pi)$.

Solⁿ Since given interval is $(-\pi, \pi)$ i.e symmetric we'll check the function is even or odd

$f(-x) = |\sin(-x)| = |-\sin x| = |\sin x| = f(x)$

$\Rightarrow$ Given function is even function

$\Rightarrow b_n = 0$, & here $l = \pi$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{——— (1)}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$\text{as } f(x) = |\sin x| \quad -\pi < x < \pi$$

$$= \begin{cases} -\sin x & -\pi < x < 0 \\ +\sin x & 0 < x < \pi \end{cases}$$

$$\therefore a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x\,dx$$

$$= \frac{2}{\pi}\left[-\cos x\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\cos\pi + 1\right]$$

$$= \frac{2}{\pi}\left[-(-1) + 1\right]$$

$$= \frac{4}{\pi}$$

Prof. Nancy Sinollin

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{\pi}\int_0^\pi \sin x \cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^\pi \frac{1}{2}\left[\sin(x+nx) + \sin(x-nx)\right]dx$$

$$= \frac{\cancel{2}}{\pi}\times\frac{1}{\cancel{2}}\int_0^\pi \left[\sin(1+n)x + \sin(1-n)x\right]dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right]_0^\pi \quad n \neq 1$$

$$= \frac{1}{\pi}\left[\frac{-\cos(1+n)\pi}{1+n} - \frac{\cos(1-n)\pi}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= \frac{1}{\pi}\left[\frac{-(-1)^{1+n}}{1+n} - \frac{(-1)^{1-n}}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= \frac{1}{\pi}\left[\frac{-(-1)(-1)^n}{1+n} - \frac{(-1)(-1)^{-n}}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{1+n} + \frac{(-1)^n}{1-n} + \frac{1-\cancel{n}+1+\cancel{n}}{1-n^2}\right] \quad (-1)^{-n} = (-1)^n$$

$$= \frac{1}{\pi}\left[(-1)^n\left(\frac{1}{1+n} + \frac{1}{1-n}\right) + \frac{2}{1-n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n 2}{1-n^2} + \frac{2}{1-n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{1+(-1)^n}{1-n^2}\right] \qquad n \neq 1.$$

Now to find a_n for $n=1$

$$\therefore a_1 = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{\pi x}{l}\right)dx$$

$$= \frac{2}{\pi}\int_0^\pi \sin x \cos x \, dx$$

$$= \frac{\cancel{2}}{\pi}\int_0^{\pi} \frac{1}{\cancel{2}} \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\cos 2\pi}{2} + \frac{1}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{-1}{2} + \frac{1}{2}\right]$$

$$= 0$$

$\therefore$ from (1)

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= \frac{\cancel{4}\,2}{\pi \times \cancel{2}_1} + 0 + \sum_{n=2}^{\infty} \frac{2}{\pi}\left[\frac{1+(-1)^n}{1-n^2}\right] \cos nx$$

$$= \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=2}^{\infty}\left(\frac{1+(-1)^n}{1-n^2}\right) \cos nx .$$

14) Ex.

Find Fourier series of $f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases}$

Hint:- Given function is neither even nor odd.

15) Ex.

Find Fourier series of $f(x) = \begin{cases} 2 & -2 < x < 0 \\ x & 0 < x < 2 \end{cases}$

Hint:- As interval is $(-2, 2)$ but given function is neither even nor odd.

Prof. Nancy Sinollin

16) Ex:

Find Fourier Series of $f(x) = |x|$ in $(-1,1)$

Hint: $f(x)=|x|$ is even function

Also $f(x) = |x| \quad (-1,1)$

$$= \begin{cases} -x & -1 < x < 0 \\ +x & 0 < x < 1 \end{cases}$$

17) Ex:

Find Fourier series of $f(x) = \begin{cases} \pi x & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$

18) Find Fourier Series of $f(x) = \dfrac{3x^2 - 6x\pi + 2\pi^2}{12}$ in $(0, 2\pi)$ also deduce $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots$

Solⁿ $(c, c+2l) = (0, 2\pi)$

$\Rightarrow c = 0$ & $c + 2l = 2\pi \Rightarrow l = \pi$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \left(\frac{3x^2 - 6x\pi + 2\pi^2}{12}\right) dx$$

$$= \frac{1}{12\pi}\int_0^{2\pi} (3x^2 - 6x\pi + 2\pi^2)\,dx$$

$$= \frac{1}{12\pi}\left[\frac{\cancel{3}x^3}{\cancel{3}} - \frac{\overset{3}{\cancel{6}}x^2\pi}{\underset{1}{\cancel{2}}} + 2\pi^2 x\right]_0^{2\pi}$$

$$= \frac{1}{12\pi}\left[8\pi^3 - 12\pi^3 + 4\pi^3 - 0\right] = 0$$

$a_0 = 0$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{3x^2 - 6x\pi + 2\pi^2}{12}\right)\cos nx\,dx$$

$$= \frac{1}{12\pi}\int_0^{2\pi}(3x^2 - 6x\pi + 2\pi^2)\cos nx\,dx$$

$$= \frac{1}{12\pi}\left[(3x^2 - 6x\pi + 2\pi^2)\left(\frac{\sin nx}{n}\right) - (6x - 6\pi)\left(-\frac{\cos nx}{n^2}\right) + 6\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{12\pi}\left[0 - (12\pi - 6\pi)\left(-\frac{\cos 2n\pi}{n^2}\right) + 0 - 0 + (-6\pi)\left(\frac{-1}{n^2}\right) - 0\right]$$

$$= \frac{1}{12\pi}\left[\frac{6\pi}{n^2} + \frac{6\pi}{n^2}\right] = \frac{1}{12\pi}\left[\frac{12\pi}{n^2}\right]$$

$$a_n = \frac{1}{n^2}$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{3x^2 - 6x\pi + 2\pi^2}{12}\right)\sin nx\,dx$$

$$= \frac{1}{12\pi}\int_0^{2\pi}(3x^2 - 6x\pi + 2\pi^2)\sin nx\,dx$$

$$= \frac{1}{12\pi}\left[(3x^2 - 6x\pi + 2\pi^2)\left(-\frac{\cos nx}{n}\right) - (6x - 6\pi)\left(-\frac{\sin nx}{n^2}\right) + 6\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{12\pi}\left[(12\pi^2 - \cancel{12\pi^2} + 2\pi^2)\left(-\frac{\cos 2n\pi}{n}\right) - 0 + \frac{6\cos 2n\pi}{n^3} - 2\pi^2\left(-\frac{1}{n}\right) + 0 - \frac{6}{n^3}\right]$$

Prof. Nancy Sinollin

$$b_n = \frac{1}{12\pi}\left[-\frac{2\pi^2}{n} + \frac{6}{n^3} + \frac{2\pi^2}{n} - \frac{6}{n^3}\right]$$

$$= \frac{1}{12\pi}(0)$$

$$b_n = 0$$

$\therefore$ from (i)

$$f(x) = 0 + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx + 0$$

$$= \left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots\right]$$

$$\frac{3x^2 - 6x\pi + 2\pi^2}{12} = \frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots$$

put $x = 0$

$$\frac{\cancel{2}\pi^2}{\cancel{12}\,6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Prof. Nancy Sinollin