DSGT

Q1.] • A recurrence relation is an equation which represents a sequence based on some rule.

• It helps in finding the subsequent term(next term) dependent upon the preceding term (previous term).

• If we know the previous term in a given series, then we can easily determine the next term.

• The sequence or series generated by recurrence rel^n is called Recurrence ~~relation~~ sequence.

• Recurrence Relation formula

Let us assume x_n is n^{th} term of series. Then the recurrence rel^n is shown in the form of

$$x_n + 1 = f(x_n) \quad n > 0$$

where $f(x_n)$ is the function.

To write recurrence relation of first order, say order K

$$x_n = f(n, x_{n-1}, x_{n-2}, \ldots x_{n-k}) ; n-k > 0.$$

• Solving Recurrence relations.

Solve the recurrence relation $a_n = a_{n-1} - n$ with initial term $a_0 = 4$.

→ Sol:

Let us write the sequence based on the equation given starting with initial number

The sequence will be 4, 5, 7, 10, 14, 19, ...

Now the difference between each term

$$a_1 - a_0 = 1$$
$$a_2 - a_1 = 2$$
$$a_3 - a_2 = 3$$
$$\vdots$$
$$a_n - a_{n-1} = n.$$

Adding all these ~~eq^ns~~ equations

$$1 + 2 + 3 + 4 + \ldots n = \frac{1}{2}(n(n+1))$$

$$(a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_{n-1} - a_{n-2}) = \frac{1}{2} n(n+1)$$

$$a_n - a_0 = \frac{1}{2}(n(n+1))$$

$$a_n = \frac{1}{2} n(n+1) + a_0$$

Hence the solⁿ to recurrence relⁿ with $a_0 = 4$ is

$$a_n = \frac{1}{2}(n(n+1)) + 4.$$

2.] $n(S)$ = no. of integers $= 100$

$n(A)$ = no. of integers divisible by 2

$= \frac{100}{2} = 50$

$n(B)$ = no. of integers divisible by 3

$= \frac{100}{3} = 33$

$n(C)$ = no of integers divisible by 5

$= \frac{100}{5} = 20$

$n(A \cap B)$ = no. of integers divisible by 2 and 3

$= \frac{100}{2 \times 3} = 16.$

$n(B \cap C) = \frac{100}{3 \times 5} = 6.$

$n(A \cap C) = \frac{100}{2 \times 5} = 10.$

$n(A \cap B \cap C) = \frac{100}{2 \times 3 \times 5} = 3.$

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

$$= 50 + \cancel{20}^{33} + 20 - 16 - 10 - 6 + 3$$

$$= 74.$$

Not divisible by 2, 3 or 5

$$= n(\overline{A \cup B \cup C})$$

$$= n(S) - n(A \cup B \cup C)$$

$$= 100 - 74$$

$$= 26$$

3] ~~Suppose we have n people in a room.~~

- ~~The first person shakes hands with everybody in the room except for himself. His total no. of handshakes is therefore one lower than the total no. of people.~~
- ~~The second person has now shaken hands with the first person but still needs to shake hands with everybody else. The no. of people left is therefore, two lower than the total no. of people in a room.~~
- ~~This continues with each person having one less handshake to make until we get the penultimate person who has to shake hands with the last person.~~

Pigeonhole principle

Theorem - If n pigeons are assigned to m pigeonholes, & $m < n$ then at least one pigeonhole contains two or more pigeons.

Proof -

- Consider labelling m pigeonholes with the numbers 1 to m & n pigeons with numbers 1 to n.

- Now beginning with pigeon 1, assign each pigeon in order to pigeonhole with same number.
- This assigns as many pigeons as possible to individual pigeon holes but because $m < n$, there are $n-m$ pigeons that have not yet been assigned to a pigeonhole.
- At least one pigeonhole will be assigned a second pigeon.

n = pigeons
m = pigeonhole
$m < n$

~~Suppose~~ Extended pigeonhole principle.

- If there are m pigeonhole and $2m^{n}$ pigeons, then three or more pigeons will have to be assigned to at least of the pigeonholes.
- If n and m are positive integers, then $\lfloor n/m \rfloor$ stands for the largest integer less than or equal to rational number n/m.

Thus $\lfloor 3/2 \rfloor$ is 1.
$\lfloor 9/4 \rfloor$ is 2
$\lfloor 6/3 \rfloor$ is 2

Theorem:

If n pigeons are assigned to m pigeonholes then one of the pigeonhole must contain atleast $\lfloor (n-1)/m \rfloor + 1$ pigeons.

Proof:

Assume that each pigeonhole does not contain more than $\lceil (n-1)/m \rceil$ pigeons.

Then there will be at must $m \lceil (n-1)/m \rceil \leq m(n-1)/m = n-1$ pigeons in all.

- Suppose there are n people in a room.
- In the given case, the pigeonhole is hands shaken & pigeons are people.
- Since you can never shake hands with yourself, you only shake $(n-1)$ other people's hand and for a total of at most $(n-1)$ handshakes.
- So there are $(n-1)$ possible numbers of handshakes for a given person and n possible people i.e. more pigeons than pigeonholes.
- Acc. to pigeonhole principle, average value $\frac{\text{pigeons}}{\text{pigeonholes}} = \frac{n}{n-1}$ is greater than 1 but smaller than 2, so maximum must be atleast 2
- Therefore atleast two people have the same handshake numbers.

4.) let A be set of people
B be the set of seconds of one day.

$|A| = 100000 = n$
$|B| = 24 \times 3600 = 86400 = m.$

Then $k = \lfloor (n-1)/m \rfloor + 1$

$= \lfloor (100000-1)/86400 \rfloor + 1$
$= 1 + 1$
$= 2.$

Hence atleast 2 are born ~~a~~ on same day.

5.] By extended pigeonhole principle
$\lfloor (n-1)/m \rfloor + 1 = 6$

$\lfloor (n-1)/5 \rfloor + 1 = 6$

$\frac{(n-1)}{5} = 5$

$n-1 = 25$

$n = 26.$

$\therefore$ 26 students ~~are required to~~ will receive same grades.

Q6] $G = \{1, 2, 3, 4, 5, 6\}.$
order $= 6$
multipliation mod 7 with order 6.

* mod 7	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

(i) closure property
All entries in composition table exists in set G.
$a = 5$, $b = 2$
$a * b \in G$
$5 \times_7 2 = 3 \in G$
$\therefore$ It is algebraic structure.

(ii) Associative

$a * (b * c) = (a * b) * c$

e.g. $a = 3$, $b = 2$, $c = 4$

$3 \times_7 (2 \times_7 4) = (3 \times_7 2) \times_7 4$

$3 \times_7 1 = 6 \times_7 4$

$3 = 3$

L.H.S. = R.H.S.

$\therefore$ It is semigroup

(iii) Identity

$a * e = a$

let $a = 3$

$3 \times_7 e = 3$

$e = 1$

$\therefore$ It is monoid

(iv) Inverse

$a * b = e$ & $b * a = e$

$a = 3$

$3 \times_7 b = 1$

$b = \cancel{\frac{1}{3}}\ 5$

Inverse of 1, 2, 3, 4, 5, 6 is 1, 4, 5, 2, 3, 6.

$\therefore$ It is group.

(v) Commutative

$a * b = b * a$

$a = 3$, $b = 2$

$3 \times_7 2 = 2 \times_7 3$

$6 = 6$

~~Hence Proved.~~

$\because$ operation is commutative

It is abelian grp.

7) $H = \begin{vmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{vmatrix}$

- In given parity check matrix all columns are distinct and non-zero, $d \geq 3$
- Use the property that minimum distance of a ~~binary~~ linear code is equal to the smallest number of columns of parity check matrix H that sum upto 0.
- Sum of first 3 columns is zero.

  so minimum distance $d_{min} = 3$.

It can detect

$$d_{min} - 1 = 3 - 1 = 2 \text{ errors}$$

It can correct

$$(d_{min} - 1)/2 = 1 \text{ error.}$$

7] $H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

- In given parity check matrix all columns are distinct and non-zero, $d \geq 3$
- Use the property that minimum distance of a binary linear code is equal to the smallest number of columns of parity check matrix H that sum upto 0.
- Sum of first 3 columns is zero.

so minimum distance $d_{min} = 3$.

It can detect

$d_{min} - 1 = 3 - 1 = 2$ errors

It can correct

$(d_{min} - 1)/2 = 1$ error.

Prove that set G ~~is~~ = {0, 1, 2, 3, 4, 5} is an Abelian grp of order 6 with respect to addition modulo 6.

Composition table

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\frac{-6}{2}$

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i) $0+0 \mod 6$

$0 \mod 6$

0

Closure property

i) all entries in composition table belongs to or exists in set G.

if $a = 1 \quad b = 2$

$* = +_6$

$1 +_6 2 = 3$

$1, 2, 3 \in G$ (Hence it is closed w.r.t. $op^n +_6$)

Associative

ii) $a * (b * c) = (a * b) * c$

$a = 2 \quad b = 3 \quad c = 4$

$2 +_6 (3 +_6 4) = (2 +_6 3) +_6 4$

$2 +_6 1 = 5 +_6 4$

$3 = 3$

LHS = R.H.S.

Hence op^n is associative

Hence it is semigroup

Identity

iii) $a * e = a$

$a = 3$

$3 +_6 e = 3$

$\boxed{e = 0}$

$a = 5 \quad 5 +_6 e = 5$

$\boxed{e = 0}$

Hence it is monoid

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Inverse
(iv) $a * b = e$ & $b * a = e$
$a = 5$

0, 3, 2 & 4, 1 & 5
inverse of each other

$5 * b = 0$
$5 +_6 b = 0$

Inverse of 0, 1, 2, 3, 4, 5   0, 5, 4, 3, 2, 1
Hence it is group

Commutative
(v) $a * b = b * a$
$a = 5, b = 2$
$5 +_6 2 = 2 +_6 5$
$1 = 1$

$a = 3, b = 0$
$3 +_6 0 = 0 +_6 3$
$3 = 3$

wrt opⁿ raddian mod 6 is operation is commutative
Hence it is an Abelian grp.

Let $H = \{[0]_6, [3]_6\}$ find left and right coset in group Z_6.

Is H a normal subgroup of group Z_6.

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

left coset

aH

$0H = \{0 +_6 0, 0 +_6 3\} = \{0, 3\}$

$1H = \{1 +_6 0, 1 +_6 3\} = \{1, 4\}$

$2H = \{2 +_6 0, 2 +_6 3\} = \{2, 5\}$

$3H = \{3 +_6 0, 3 +_6 3\} = \{3, 0\}$

$4H = \{4 +_6 0, 4 +_6 3\} = \{4, 1\}$

$5H = \{5 +_6 0, 5 +_6 3\} = \{5, 2\}$

Right coset

Ha

$H0 = \{0 +_6 0, 3 +_6 0\} = \{0, 3\}$

$H1 = \{0 +_6 1, 3 +_6 1\} = \{1, 4\}$

$H2 = \{0 +_6 2, 3 +_6 2\} = \{2, 5\}$

$H3 = \{0 +_6 3, 3 +_6 3\} = \{3, 0\}$

$H4 = \{0 +_6 4, 3 +_6 4\} = \{4, 1\}$

$H5 = \{0 +_6 5, 3 +_6 5\} = \{5, 2\}$

Hence proved $aH = Ha$

Sonal Sonarghare.

DSGT- UT-II

5. How many ways a comittee of three faculty members and 2 students can be formed from 7 faculty members and 8 students:

Solution:

Given: 7 faculty members and 8 students.

$\therefore$ out of 7 faculty members 3 faculty members are selected: $\therefore {}^7C_3$

Out of 8 students members 2 students members are selected: ${}^8C_2$.

$\therefore$ Total Number of ways forming a committee

$$= {}^7C_3 \times {}^8C_2$$

$$= \frac{7!}{3!(7-3)!} \times \frac{8!}{2!(8-2)!} \quad \ldots \; {}^nC_r = \frac{n!}{r!(n-r)!}$$

$$= \frac{7!}{3! \times 4!} \times \frac{8!}{2! \times 6!}$$

$$= \frac{7 \times 6! \times 8 \times 7 \times 6 \times 5 \times 4!}{3! \times 4! \times 2! \times 6!}$$

$$= \frac{7 \times 8 \times 7 \times 6 \times 5}{3 \times 2 \times 1 \times 2 \times 1}$$

$$= 980 \text{ ways.}$$

$\lfloor 2.4 \rfloor = 2$ $\lceil \ldots \rceil \ldots$ smallest int greter than or equal to ... round up

$\lceil 2.4 \rceil = 3$ Note $\lfloor \ldots \rfloor$ = floor ... greats int less than or equal tox (nearest down)

Q17. How many friends must you have to gaurantee that at least five of them will have birthdays in the same month.

Solution:

Let n be the no. of friends.

If no. of month are pigeonholes then ... m

no. of friends will be pigeons ... n

$K = 5$

$\therefore$ By Extended pigeonhole Principle.

$$\lfloor (n-1)/m \rfloor + 1 = K.$$

$$\lfloor (n-1/12) \rfloor + 1 = 5$$

$$\lfloor (n-1/12) \rfloor = 4$$

$$\frac{n-1}{12} = 4$$

$$n-1 = 48$$

$$n = 49$$

Therefore among 49 friends at least five of them will have birthdays in the same month.

Q16: A box contains 6 white balls and 5 red balls. In many ways 4 balls can be drawn from the box. if i) they are to be of any color
ii) all the balls to be of same color.

Solution:

White balls: 6 Red balls: 5

$\therefore$ Total balls: $6+5=11$

Number of balls to be drawn: 4

i) When they are to be of any color:

$${}^nC_r = n!/r!(n-r)!$$

$${}^{11}C_4 = \frac{11!}{4!(11-4)!} = \frac{11!}{4!\,7!} = \frac{11\times 10\times 9\times 8}{4\times 3\times 2\times 1}$$

$$= 330 \text{ balls.}$$

ii) All balls to be of same color

$${}^6C_4 + {}^5C_4 \quad \ldots \text{ 6 white balls, 5 Red balls.}$$

$$= \frac{6!}{4!(6-4)!} + \frac{5!}{4!(5-4)!}$$

$$= \frac{6!}{4!\cdot 2!} + \frac{5!}{4!\cdot 1!}$$

$$= \frac{\cancel{6}^{3}\times 5}{\cancel{2}\times 1} + 5$$

$$= 15 + 5$$

$$= 20 \text{ balls.}$$

Q15) In a grp of 6 boys and 4 girls, four children are to be selected. In how many different ways can be selected such that at least one boy should be there.
Solution:

Boys: 6 Girls: 4

No. of children to be selected: 4.

At least one boy should be there
= Total selection − NO of boys selection.

∴ Total selection =
Total no. of children = 6+4 = 10.
∴ $^{10}C_4$ ..... As 4 children are to be selected

$= \frac{10!}{4!(10-4)!}$ ..... $^nC_r = \frac{n!}{r!(n-r)!}$

$= \frac{10!}{4! \times 6!} = \frac{10 \times \overset{3}{9} \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$ children.

No Boys are selected:
Girls: $^4C_4 = 1$.

∴ At least one boy should be selected
= Total selection − No boys selected
= 210 − 1
= 209

∴ There are 209 ways at least one boy can be selected.

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Note: By default: +

Q14) Let $G = Z_8$, determine all left cosets of $H = \{[0], [4]\}$ in G.

Solution: subgroup of H of G

Coset consist of all the products obtained by multiplying fixed element of group by each of element of given sub group either right or Left.

aH ... Left coset   Ha ... Right coset.

$+_8$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

Left coset of H with respect to a is

$$aH = \{ a * h \mid h \in H \}.$$

$0H = \{0 +_8 0, \ 0 +_8 4\} = \{0, 4\}$

$1H = \{1 +_8 0, \ 1 +_8 4\} = \{1, 5\}$

$2H = \{2 +_8 0, \ 2 +_8 4\} = \{2, 6\}$

$3H = \{3 +_8 0, \ 3 +_8 4\} = \{3, 7\}$

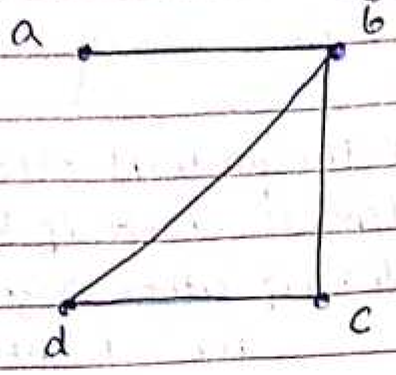
$4H = \{4 +_8 0, \ 4 +_8 4\} = \{4, 0\}$

$5H = \{5 +_8 0, \ 5 +_8 4\} = \{5, 1\}$

$6H = \{6 +_8 0, \ 6 +_8 4\} = \{6, 2\}$

$7H = \{7 +_8 0, \ 7 +_8 4\} = \{7, 2\}$

Q13) Determining Hamiltonian cycle and path in graph



Hamiltonian Graph:

A connected graph G is called Hamiltonian graph if there is a cycle which includes every vertex of G and the cycle is called H^Cycle closed path

start from same vertex & end from same vertex

Hamiltonian Path: that visits each vertex exactly once

↓

ABCD OR ABDC

Hamiltonian cycle:

A graph contains Hamiltonian circuit if there is a path that starts and ends at same vertex.

We start from A, then we go to B C D.

To reach to the same vertex A we have to again go to B.

Therefore B is repeated.

That's why we can say that this graph does not contain a Hamiltonian cycle.

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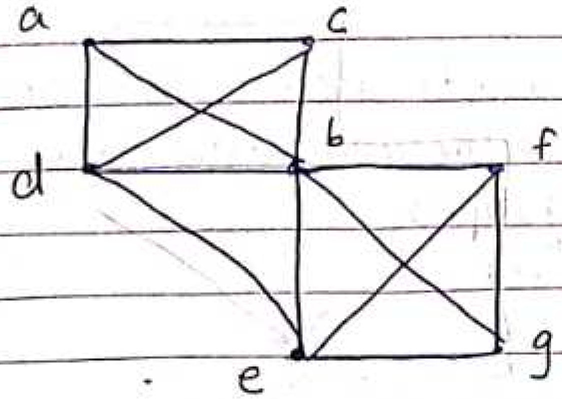
Q12) Determine Euler cycle and path in graph.

a)



Euler graph:
Closed trail includes
every edge of graph.

Euler path:
It is a path that uses every edge of a graph
exactly once.
Euler path starts and ends at different vertices.

A C B F G E D

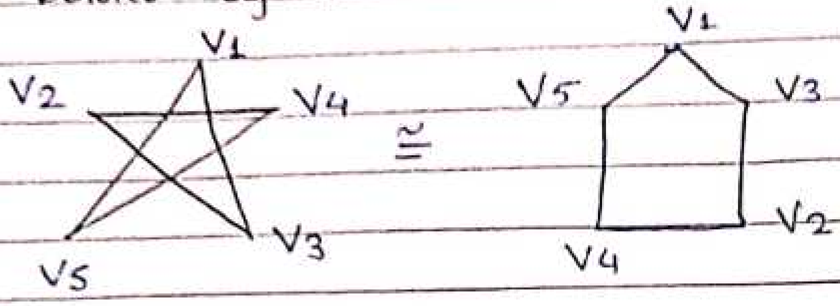
Euler cycle:
starts and ends at same vertex

A C B F G E D A

Q11) Define isomorphic graph and Homomorphic graph.

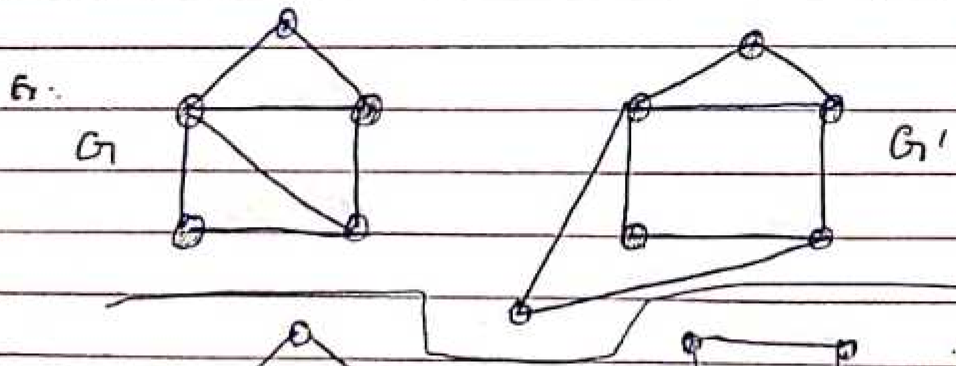
- ISOMORPHIC GRAPHS:

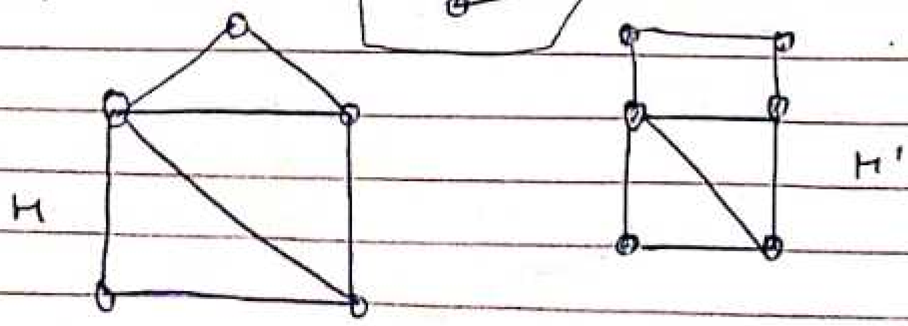
If two graphs G and H contain the same number of vertices, edges and are connected in the same way. they are called isomorphic graphs.
Denoted by $G \cong H$.



- HOMOMORHIC GRAPHS:

If other graph G' can be formed by dividing the edges of G with additional vertices or if G' can be obtained by indroducing vertices of degree 2 in any edge of Grap G then the graph G' is complete.
Both G' and G are known as Homeomorphic Graph





intro - introducing vertex of degree 2.