Poisson Distribution

Poisson distribution was discovered by the French mathematician & physicist Simeon Denis Poisson (1781-1840) who published it in 1837. Poisson distribution is found in cases of events that occur rarely (Poisson distribution is the distribution of rare events).

The Poisson distribution is a limiting case of the binomial distribution under the following conditions:

1. n, the no of trails is indefinitely large, i.e. n → ∞.
2. p ,the constant probability for the success of each trail is indefinitely small, i.e. p → 0.
3. np = λ(say) is finite.

Poisson distribution as a limiting case of the Binomial distribution

Let $X \sim B(n, p)$
Then

$$P(X=x) = p_x = nC_x p^x q^{n-x},\ x=0,1,2,3,\dots n;\ \ q=1-p$$

$$i.e\, P(X=x) = \frac{n!}{x!(n-x)!}\left(\frac{np}{n}\right)^x (1-p)^{n-x},\ x=0,1,2,3,\dots n$$

$$= \frac{(n-(x-1))(n-(x-2))(n-(x-3))\dots(n-1)n}{x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{np}{n}\right)^{n-x}$$

$$= \frac{n(1-(x-1/n)n(1-(x-2)/n)n(1-(x-3)/n)\dots n(1-(n-1)/n)n}{x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n^x(1-(x-1)/n)(1-(x-2)/n)(1-(x-3)/n)\dots(1-(n-1)/n)(1)}{x!}\left(\frac{np}{n}\right)^x\frac{\left(1-\frac{np}{n}\right)^n}{\left(1-\frac{np}{n}\right)^x}$$

$$i.e\, P(X=x) = \frac{(1-(x-1)/n)(1-(x-2)/n)(1-(x-3)/n)\dots(1-(n-1)/n)(1)}{x!}(np)^x\frac{\left(1-\frac{np}{n}\right)^n}{\left(1-\frac{np}{n}\right)^x} \qquad \dots.(I)$$

Now let $n \to \infty$ such that $np=\lambda$ is finite. Using this in *(I)* we get

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!},\ \ x=0,1,2,\dots(\because \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}\ \&\ \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^x = 1)$$

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Definition: A random variable X is said to follow a Poisson distribution with parameter 'λ' if it assumes only non-negative, integral values & its probability distribution is given by:

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!},\ x = 0,1,2,3,\ldots\ \&\ \lambda > 0\cdot$$

We shall use the notation $X \sim P(\lambda)$ to denote that X is a Poisson variate with parameter 'λ' .

Solved Examples:

1. A transmission channel has a per-digit error probability $p = 0.01$ Calculate the probability of more than 1 error in 10 received digits using (i) Binomial distribution (ii) Poisson distribution

   Solution: (i) Let X denote the number of errors in 10 received digits

   Then $X \sim B(n = 10, p = 0.01)$ Therefore

$$\begin{aligned} P(\text{ more than 1 error in 10 received digits}) &= P(X > 1) \\ &= 1 - P(X \le 1) \\ &= 1 - \left[10C_0(0.01)^0(0.99)^{10} + 10C_1(0.01)^1(0.99)^9\right] \\ &= 0.004 \end{aligned}$$

   (ii) Let X denote the number of errors

   Then X *follows a* $P(\lambda = np = 10(0.01) = 0.1)$

   Therefore

$$\begin{aligned} P(\text{ more than 1 error in 10 received digits}) &= P(X > 1) \\ &= 1 - P(X \le 1) \end{aligned}$$

$$= 1 - \left[\frac{e^{-0.1}(0.1)^0}{0!} + \frac{e^{-0.1}(0.1)^{01}}{1!}\right] = 0.004$$

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2. Find the probability that atmost 4 defective bulbs will be found in a box of 200 bulbs if it is known that 2% of the bulbs are defective.

Solution: We do Poisson approximation to Binomial here.

Let X denote number of defective bulbs in the box

We have $\lambda = np = 200(\frac{2}{100}) = 4$

Therefore $X \sim P(\lambda = 4)$

Hence

$$P(\text{atmost 4 defective bulbs}) = P(X \le 4)$$
$$= P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4)$$
$$= \frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!} + \frac{e^{-4}4^2}{2!} + \frac{e^{-4}4^3}{3!} + \frac{e^{-4}4^4}{4!}$$
$$= 0.6283$$

3. If X is a Poisson Variate such that $P(X=1) = P(X=2)$ find $P(X=3)$ and $E(X^2)$.

Solution: Given: $P(X=1) = P(X=2)$ Since $X \sim P(\lambda)$

This implies $\frac{e^{-\lambda}\lambda}{1!} = \frac{e^{-\lambda}\lambda^2}{2!}$

$\Rightarrow \lambda = 2$

$$E(X^2) = \lambda^2 + \lambda$$
$$= 4 + 2 = 6$$

Now $P(X=3) = \frac{e^{-\lambda}\lambda^3}{3!}$

$$= \frac{8e^{-2}}{6}$$

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