★ To find analytic function when harmonic function is given.

i) For this we will use again Milne Thompson method which we used earlier. Here instead of given real part u or img. part v we have, given harmonic function u or v & from this to find f(z) & another harmonic conjugate.

1) Show that the function $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic. Find the harmonic conjugate funⁿ v & express u+iv as an analytic funⁿ of z.

→ $u = 3x^2y + 2x^2 - y^3 - 2y^2$.

$u_x = 6xy + 4x \Rightarrow u_{xx} = 6y + 4$.

$u_y = -3y^2 + 3x^2 - 4y \Rightarrow u_{yy} = -6y - 4$

$\therefore u_{xx} + u_{yy} = 6y + 4 - 6y - 4 = 0$

$\Rightarrow$ u is harmonic.

Let $f(z) = u + iv$ be analytic function where u & v are harmonic conjugate. $\Rightarrow u_x = v_y$ & $u_y = -v_x$

Also, $f'(z) = u_x + iv_y = u_x - iu_y$

$\therefore f'(z) = (6xy + 4x) - i(3x^2 - 3y^2 - 4y)$

By M.T. method put $x = z$ & $y = 0$

$\therefore f'(z) = 4z - i3z^2$

On integrating $f(z) = \int (4z - i3z^2)\,dz + C$

$= \frac{4z^2}{2} - i\frac{3z^3}{3} + C$.

$= 2z^2 - iz^3 + C$

To find v put $z = x + iy$.

$\therefore f(x+iy) = 2(x+iy)^2 - i(x+iy)^3 + C$

$= 2(x^2 + i2xy - y^2) - i(x^3 + 3x^2iy - 3xy^2 - iy^3) + C$

$= 2x^2 + i4xy - 2y^2 - ix^3 + 3x^2y + i3xy^2 - y^3 + C$

$= (2x^2 - 2y^2 + 3x^2y - y^3) + i(4xy - x^3 + 3xy^2) + C$

$\therefore v = 4xy - x^3 + 3xy^2 + C$.

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2) Show that $u = \cos x \cosh y$ is a harmonic function. find its harmonic conjugate & corresponding analytic function

$u = \cos x \cosh y$

$u_x = -\sin x \cosh y$ $\quad u_{xx} = -\cos x \cosh y$

$u_y = \cos x \sinh y$ $\quad u_{yy} = \cos x \cosh y$.

$u_{xx} + u_{yy} = -\cos x \cosh y + \cos x \cosh y = 0$.

$\Rightarrow u$ is harmonic

Let $f(z) = u + iv$ be analytic function. then u & v are harmonic conjugate $\Rightarrow u_x = v_y$ & $u_y = -v_x$.

Also $f'(z) = u_x + iv_x = u_x - iu_y$

$= -\sin x \cosh y - i \cos x \sinh y$.

By M.T. method, $f'(z) = -\sin z - i(0) = -\sin z$

On integrating, $f(z) = \cos z + c$

To find v, put $z = x + iy$.

$f(x+iy) = \cos(x+iy) + c = \cos x \cos iy - \sin x \sin iy + c$.

$= \cos x \cosh y - i \sin x \sinh y + c$

$\Rightarrow v = -\sin x \sinh y + c$

3) Show that the following functions are harmonic & find their corresponding analytic functions $f(z) = u + iv$.

1) $v = e^{2x}(y \cos 2y + x \sin 2y)$.

$v_x = e^{2x}(\sin 2y) + (y \cos 2y + x \sin 2y) 2e^{2x}$

$= e^{2x}[\sin 2y + 2y \cos 2y + 2x \sin 2y]$.

$v_{xx} = e^{2x}[2 \sin 2y] + [\sin 2y + 2y \cos 2y + 2x \sin 2y] 2e^{2x}$.

$= e^{2x}[4 \sin 2y + 4y \cos 2y + 4x \sin 2y]$ — 1)

$v_y = e^{2x}[-y 2 \sin 2y + \cos 2y + 2x \cos 2y]$

$v_{yy} = e^{2x}[-2(2y \cos 2y + \sin 2y) - 2 \sin 2y - 4x \sin 2y]$

$= e^{2x}[-4y \cos 2y - 4 \sin 2y - 4x \sin 2y]$ — 2).

from 1) & 2) $v_{xx} + v_{yy} = 0$.

Let $f(z) = u + iv$ be analytic funⁿ $\Rightarrow u_x = v_y$ & $u_y = -v_x$.

Also $f'(z) = u_x + iv_x = v_y + iv_x$

$= e^{2x}[-\sin 2y(2y) + \cos 2y + 2x \cos 2y]$

$+ i e^{2x}[\sin 2y + 2y \cos 2y + 2x \sin 2y]$

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By M.T. method put $x=z$, $y=0$

$f'(z) = e^{2z}[1+2z] + i\,e^{2z}[0] = e^{2z}(1+2z)$

On integrating, we get

$f(z) = \int (1+2z)e^{2z} + c$

$= (1+2z)\dfrac{e^{2z}}{2} - \int \dfrac{e^{2z}}{\cancel{2}}(\cancel{2})\,dz + c$

$= (1+2z)\dfrac{e^{2z}}{2} - \dfrac{e^{2z}}{2} + c$

$= \cancel{\dfrac{e^{2z}}{2}} + ze^{2z} - \cancel{\dfrac{e^{2z}}{2}} + c = ze^{2z} + c$

2) $u = (x-1)^3 - 3xy^2 + 3y^2$

$u_x = 3(x-1)^2 - 3y^2 \Rightarrow u_{xx} = 6(x-1) = 6x-6$

$u_y = -6xy + 6y \Rightarrow u_{yy} = -6x+6$

$\Rightarrow u_{xx} + u_{yy} = 6x-6-6x+6 = 0$.

$\Rightarrow u$ is harmonic.

Let $f(z) = u+iv$ be analytic $\Rightarrow u_x = v_y$ & $u_y = -v_x$

& $f'(z) = u_x + iv_x = u_x - iu_y$

$\therefore f'(z) = 3(x-1)^2 - 3y^2 - i(-6xy+6y)$

put $x=z$ & $y=0$. By M.T. method.

$f'(z) = 3(z-1)^2$

On integrating $f(z) = (z-1)^3 + c$.

3) $u = 2axy + b(y^2 - x^2)$

$u_x = 2ay + b(-2x) \Rightarrow u_{xx} = -2b$.

$u_y = 2ax + 2by \Rightarrow u_{yy} = 2b$

$\Rightarrow u_{xx} + u_{yy} = -2b + 2b = 0$.

$\Rightarrow u$ is harmonic.

Let $f(z) = u+iv$ be analytic $\Rightarrow u_x = v_y$ & $u_y = -v_x$

also $f'(z) = u_x + iv_x = u_x - iu_y$

$\therefore f'(z) = 2ay - 2bx - i(2ax + 2by)$

By M.T. method put $x=z$ & $y=0$

$f'(z) = -2bz - i(2az) = -2z(b+ai)$

On integrating, $f(z) = -\frac{\cancel{2}z^2}{2}(b+ai) + C$

$f(z) = -z^2(b+ai) + C$.

4) $u = y^3 - 3x^2y$. Also find its harmonic conjugate.

$\Rightarrow u_x = 0 - 6xy = -6xy \quad u_{xx} = -6y$

$u_y = 3y^2 - 3x^2, \quad u_{yy} = 6y$.

$\therefore u_{xx} + u_{yy} = -6y + 6y = 0$.

$\Rightarrow u$ is harmonic

Let $f(z) = u + iv$ be analytic $\Rightarrow u_x = v_y$ & $u_y = -v_x$,

Also, $f'(z) = u_x + iv_x = u_x - iu_y$.

$= -6xy - i(3y^2 - 3x^2)$

By M.T. method put $x = z$ & $y = 0$

$\therefore f'(z) = -i(-3z^2) = i3z^2$

On integrating $f(z) = iz^3 + C \Rightarrow f(x+iy) = i(x+iy)^3 + C$.

$= i(x^3 + 3ix^2y - 3xy^2 - iy^3) + C$

$\Rightarrow v = x^3 - 3xy^2$

5) $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$

$\Rightarrow u_x = \cos x \cosh y - 2\sin x \sinh y + 2x + 4y$

$\Rightarrow u_{xx} = -\sin x \cosh y - 2\cos x \sinh y + 2 \longrightarrow 1)$

$u_y = \sin x \sinh y + 2\cos x \cosh y - 2y + 4x$.

$\Rightarrow u_{yy} = \sin x \cosh y + 2\cos x \sinh y - 2 \longrightarrow 2)$

$u_{xx} + u_{yy} = 0$ from 1) & 2).

Hence, u is harmonic funⁿ.

Let, $f(z) = u + iv$ be analytic funⁿ $\Rightarrow u_x = v_y$ & $u_y = -v_x$.

Also, $f'(z) = u_x + iv_x = u_x - iu_y$

$\Rightarrow f'(z) = \cos x \cosh y - 2\sin x \sinh y + 2x + 4y$

$- i(\sin x \sinh y + 2\cos x \cosh y - 2y + 4x)$.

By M.T. method put $x = z$ & $y = 0$.

$\Rightarrow f'(z) = (\cos z + 2z) - i(2\cos z + 4z)$.

On integrating, $f(z) = \int [\cos z + 2z - i(2\cos z + 4z)]\,dz + C$.

$\Rightarrow f(z) = \sin z + z^2 - i(2\sin z + 2z^2) + C$.

6) $V = 3x^2y + 6xy - y^3$

$V_x = 6xy + 6y \Rightarrow V_{xx} = 6y \longrightarrow 1)$

$V_y = 3x^2 + 6x - 3y^2 \Rightarrow V_{yy} = -6y \longrightarrow 2)$

$\Rightarrow V_{xx} + V_{yy} = 6y - 6y = 0$ from 1) & 2)

Let $f(z) = u + iv$ be analytic $\Rightarrow u_x = v_y$ & $u_y = -v_x$

Also $f'(z) = u_x + iv_x = v_y + iv_x = (3x^2 + 6x - 3y^2) + i(6xy + 6y)$

~~On inte~~ By M.T. method put $x = z$ & $y = 0$

$f'(z) = 3z^2 + 6z$

On integrating, $f(z) = z^3 + \frac{6z^2}{2} + c = z^3 + 3z^2 + c$

7) $u = \frac{1}{2}\log(x^2 + y^2)$ also find it's harmonic conjugate

already done

$u_x = \frac{x}{x^2 + y^2}$ $\quad u_{xx} = \frac{-x^2 + y^2}{(x^2 + y^2)^2} \longrightarrow 1)$

$u_y = \frac{y}{x^2 + y^2}$ $\quad u_{yy} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \longrightarrow 2)$

from 1) & 2) $u_{xx} + u_{yy} = 0$

Let $f(z) = u + iV$ be analytic $\Rightarrow u_x = v_y$ & $u_y = -v_x$

Also $f'(z) = u_x + iv_x = u_x - iu_y$

$\therefore f'(z) = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$

By M.T. method put $x = z$ & $y = 0$.

$\Rightarrow f'(z) = \frac{z}{z^2} = \frac{1}{z}$.

On Integrating, $f(z) = \log z + c$.

put $z = x + iy$ to find harmonic conjugate v

$f(z) = \log(x + iy) + c$

$= \log(re^{i\theta}) + c = \log r + i\theta + c$

$= \log\sqrt{x^2 + y^2} + i\tan^{-1}\left(\frac{y}{x}\right) + c$

$= \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\left(\frac{y}{x}\right) + c$

$\Rightarrow V = \tan^{-1}\frac{y}{x}$.

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8) $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$. Also find its harmonic conjugate.

already done

$$u_x = \frac{(\cosh 2y + \cos 2x) 2\cos 2x + \sin 2x (2 \sin 2x)}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{2\cosh 2y \cos 2x + 2}{(\cosh 2y + \cos 2x)^2}$$

$$u_{xx} = \frac{(\cosh 2y + \cos 2x)^2 (-4\cosh 2y \sin 2x) + 4(2\cosh 2y \cos 2x + 2)(\cosh 2y + \cos 2x)\sin 2x}{(\cosh 2y + \cos 2x)^4}$$

$$= \frac{(\cosh 2y + \cos 2x)[-4\cosh^2 2y \sin 2x - 4\cosh 2y \sin 2x \cos 2x + 8\cosh 2y \cos 2x \sin 2x + 8\sin 2x]}{(\cosh 2y + \cos 2x)^4}$$

$$= \frac{[-4\cosh^2 2y \sin 2x + 4\cosh 2y \cos 2x \sin 2x + 8\sin 2x]}{(\cosh 2y + \cos 2x)^3} \longrightarrow 1)$$

$$u_y = -\frac{2\sin 2x \sinh 2y}{(\cosh 2y + \cos 2x)^2}$$

$$u_{yy} = \frac{(\cosh 2y + \cos 2x)^2(-4\sin 2x \cosh 2y) + 4\sin 2x \sinh 2y (\cosh 2y + \cos 2x) 2\sinh 2y}{(\cosh 2y + \cos 2x)^4}$$

$$= \frac{(\cosh 2y + \cos 2x)[-4\sin 2x \cosh^2 2y - 4\sin 2x \cosh 2y \cos 2x + 8\sin 2x \sinh^2 2y]}{(\cosh 2y + \cos 2x)^4}$$

$$= \frac{[-4\sin 2x \cosh^2 2y - 4\sin 2x \cosh 2y \cos 2x + 8\sin 2x \sinh^2 2y]}{(\cosh 2y + \cos 2x)^3} \quad — 2)$$

$\therefore u_{xx} + u_{yy}$

$$= \frac{-8\cosh^2 2y \sin 2x + 8\sin 2x + 8\sin 2x \sinh^2 2y}{(\cosh 2y + \cos 2x)^3}$$

$$= \frac{-8\sin 2x(\cosh^2 2y - \sinh^2 2y) + 8\sin 2x}{(\cosh 2y + \cos 2x)^2}$$

$= 0$. $\quad \because \cosh^2 2y - \sinh^2 2y = 1$.

Hence u is harmonic.

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Let $f(z) = u + iv$ be analytic $\exists$ $u_x = v_y$ & $u_y = -v_x$

Also $f'(z) = u_x + iv_x = u_x - iu_y$.

$$= \frac{2\cosh 2y \cos 2x + 2}{(\cosh 2y + \cos 2x)^2} + i \frac{2\sin 2x \sinh 2y}{(\cosh 2y + \cos 2x)^2}$$

By M.T. method put $x = z$ & $y = 0$.

$$f'(z) = \frac{2\cos 2z + 2}{(1 + \cos 2z)^2} + i(0)$$

$$= \frac{2(1 + \cos 2z)}{(1 + \cos 2z)^2} = \frac{\cancel{2}}{\cancel{2}\cos^2 z} = \sec^2 z.$$

$\therefore$ On integrating $f(z) = \tan z + C$.

To find harmonic conjugate v put $z = x + iy$.

$$\therefore f(z) = \tan(x+iy) + C$$

$$= \frac{\sin(x+iy)}{\cos(x+iy)} + C$$

$$= \frac{2\sin(x+iy)\cos(x-iy)}{2\cos(x+iy)\cos(x-iy)} + C.$$

$$= \frac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy} + C$$

$$= \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y} + C.$$

$$= \frac{\sin 2x}{\cos 2x + \cosh 2y} + i\frac{\sinh 2y}{\cos 2x + \cosh 2y} + C$$

$$\therefore v = \frac{\sinh 2y}{\cos 2x + \cosh 2y} + C.$$

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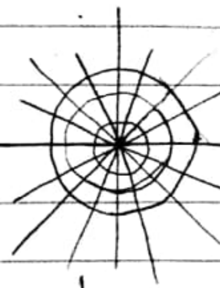
★ Orthogonal Curves:-

Thm: If $f(z) = u + iv$ is an analytic function then the curves $u = c_1$ & $v = c_2$ intersect orthogonally.

★ Orthogonal Trajectories:

By an orthogonal trajectory of a family of curves we mean a curve which cuts every member of the given family at right angles.

eg. Family of straight lines passing thro' origin given by $y = mx$ where m is const.



These st. lines cut by a circle with centre at origin at right angles at each point of intersection. Eqⁿ of this family of curves is $x^2 + y^2 = a^2$ where a is const.

$\therefore$ Family of circles $x^2 + y^2 = a^2$ represents the family of orthogonal trajectories to the family of st. lines $y = mx$.

★ To find orthogonal trajectories of the family of curves $u = c_1$

From thm. we know that If $f(z) = u + iv$ is an analytic funⁿ then the curves $u = c_1$ & $v = c_2$ intersects orthogonally i.e. $v = c_2$ is the family of orthogonal trajectory of the family of curves $u = c_1$

Hence to find orthogonal trajectories of $u = c_1$ (or $v = c_2$) we again find harmonic conjugate $v = c_2$ (or $u = c_1$).

Find orthogonal trajectories of family of curves.

1) $3x^2y - y^3 = c$

Let $u = 3x^2y - y^3 = c$.

Let $f(z) = u + iv$ be analytic funⁿ s.t. $u = 3x^2y - y^3$.

$\Rightarrow u_x = v_y$ & $u_y = -v_x$.

Here $u_x = 6xy$ $\quad u_y = 3x^2 - 3y^2$.

Also $f'(z) = u_x + iv_x = u_x - iu_y$

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$\Rightarrow f'(z) = 6xy - i(3x^2 - 3y^2)$.

By M.T. method put $x = z$ & $y = 0$

$\Rightarrow f'(z) = -i(3z^2)$.

On integrating, $f(z) = -iz^3 + c$.

To find v, put $z = x+iy$

$f(z) = -i(x+iy)^3 + c$

$= -i(x^3 + i3x^2y - 3xy^2 - iy^3) + c$

$= -ix^3 + 3x^2y + i3xy^2 - y^3 + c$

$= (3x^2y - y^3) + i(3xy^2 - x^3) + c$

$v = 3xy^2 - x^3$

Hence required orthogonal trajectory is $3xy^2 - x^3 = c'$

H.W 2) $e^x\cos y - xy = c$.

Let $u = e^x\cos y - xy = c$.

Let $f(z) = u + iv$ be analytic s.t. $u = e^x\cos y - xy$.

$\Rightarrow u_x = v_y$ & $u_y = -v_x$

$u_x = e^x\cos y - y \qquad u_y = -e^x\sin y - x = -(e^x\sin y + x)$

Also $f'(z) = u_x + iv_x = u_x - iu_y$

$\Rightarrow f'(z) = (e^x\cos y - y) + i(e^x\sin y + x)$

By M.T. method put $x = z$ & $y = 0$.

$\Rightarrow f'(z) = e^z + iz$

On integrating, $f(z) = e^z + i\frac{z^2}{2} + c$

To find v, put $z = x+iy$

$\therefore f(z) = e^{(x+iy)} + i\frac{(x+iy)^2}{2} + c$

$= e^x e^{iy} + i\left(\frac{x^2 + i2xy - y^2}{2}\right) + c$.

$= e^x(\cos y + i\sin y) + i\frac{(x^2 - y^2)}{2} - xy + c$.

$= (e^x\cos y - xy) + i\left(e^x\sin y + \frac{(x^2 - y^2)}{2}\right) + c$

$v = e^x\sin y + \frac{x^2 - y^2}{2}$

Hence required orthogonal ~~trajectory~~ is $e^x\sin y + \frac{x^2 - y^2}{2}$

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H.W 3) $x^2-y^2-2xy+2x-3y=c$.

$\rightarrow$ Let $u=x^2-y^2-2xy+2x-3y=c$.

Let $f(z)=u+iv$ be analytic s.t. $u=x^2-y^2-2xy+2x-3y$.

$\Rightarrow u_x=v_y$ & $u_y=-v_x$

$u_x=2x-2y+2$ $\quad u_y=-2y-2x-3$

Also $f'(z)=u_x+iv_x=u_x-iu_y$.

$\therefore f'(z)=(2x-2y+2)-i(-2y-2x-3)$

By M.T. method put $x=z$ & $y=0$.

$\Rightarrow f'(z)= 2z+2-i(-2z-3)$

$= 2(z+1)+i(2z+3)$

On integrating, $f(z)=\cancel{2}\frac{z^2}{\cancel{2}}+2z+i\left(\frac{\cancel{2}z^2}{\cancel{2}}+3z\right)+c$.

$\therefore f(z)=(1+i)z^2+(2+3i)z+c$.

To find v put $z=x+iy$.

$f(z)=(1+i)(x+iy)^2+(2+3i)(x+iy)+c$

$=(1+i)(x^2+2ixy-y^2)+2x+i2y+i3x-3y+c$

$=x^2+2ixy-y^2+ix^2-2xy-iy^2+2x+i2y+i3x-3y+c$

$=(x^2-y^2-2xy+2x-3y)+i(2xy+x^2-y^2+2y+3x)+c$

$\therefore v=2xy+x^2-y^2+2y+3x$

Hence, required orthogonal trajectory is

$2xy+x^2-y^2+2y+3x=c'$

4) $x^3y-xy^3=c$.

$\rightarrow$ Let $u=x^3y-xy^3=c$

Let $f(z)=u+iv$ be analytic s.t. $u=x^3y-xy^3$

$\Rightarrow u_x=v_y$ & $u_y=-v_x$

$u_x=3x^2y-y^3$ & $u_y=x^3-3xy^2$

Also, $f'(z)=u_x+iv_x=u_x-iu_y=(3x^2y-y^3)-i(x^3-xy^2)$

By M.T. method put $x=z$, $y=0$

$f'(z)=-iz^3$

On integrating, $f(z)=\frac{-iz^4}{4}$

$\therefore$ ... $=(3x^2y-y^3)+i(3xy^2-x^3)+c$

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To find v, put $z = x+iy$

$f(z) = -\frac{i}{4}(x+iy)^4 + c$

$= -\frac{i}{4}(x^4 + 4x^3iy - 6x^2y^2 - 4iy^3x + y^4) + c$

$\therefore = -\frac{i}{4}x^4 + \frac{\cancel{4}x^3y}{\cancel{4}} + i\frac{\cancel{6}x^2y^2}{\cancel{4}} - xy^3 - \frac{iy^4}{4} + c$

$= (x^3y - xy^3) - \frac{i}{4}(x^4 - 6x^2y^2 + y^4) + c$

$\therefore V = -\frac{1}{4}(x^4 - 6x^2y^2 + y^4)$.

Hence required trajectory is $-\frac{1}{4}(x^4 - 6x^2y^2 + y^4) = c'$

5) $2x - x^3 + 3xy^2 = a$.

Let $u = 2x - x^3 + 3xy^2 = a$

Let $f(z) = u + iv$ be analytic s.t. $u = 2x - x^3 + 3xy^2$

$\Rightarrow u_x = v_y$ & $u_y = -v_x$.

$u_x = 2 - 3x^2 + 3y^2$, $u_y = 6xy$.

Also, $f'(z) = u_x + iv_x = u_x - iu_y$

$f'(z) = (2 - 3x^2 + 3y^2) - i6xy$

By M.T. method, put $x = z$ & $y = 0$

$f'(z) = (2 - 3z^2)$

On integrating, $f(z) = 2z - z^3 + c$.

To find v put $z = x+iy$.

$f(z) = 2(x+iy) - (x+iy)^3 + c$.

$= 2x + i2y - (x^3 + 3ix^2y - 3xy^2 - iy^3) + c$.

$= (2x - x^3 + 3xy^2) + i(2y - 3x^2y + y^3) + c$

$\therefore V = (2y - 3x^2y + y^3)$.

$\therefore$ Required trajectory is $2y - 3x^2y + y^3 = c$.

6) $e^x \cos y + xy = \alpha$, α is const (same as 2)

7) $3x^2y + 2x^2 - y^3 - 2y^2 = c$. (already done) in harmonic