EM-III_COPM/IT/DS/AI-ML

1(a). Find LT of $\frac{\cos\sqrt{t}}{\sqrt{t}}$ given that $L(\sin\sqrt{t}) = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}e^{-\left(\frac{1}{4s}\right)}$.

Solution:

Since $L(\sin\sqrt{t}) = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}e^{-\left(\frac{1}{4s}\right)}$

Apply differentiation properties

$$L\left(\frac{d}{dt}\sin\sqrt{t}\right) = s\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}e^{-\left(\frac{1}{4s}\right)} - \sin 0$$

$$L\left(\cos\sqrt{t}\frac{1}{2\sqrt{t}}\right) = s\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}e^{-\left(\frac{1}{4s}\right)} - \sin 0$$

$$L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = s\frac{\sqrt{\pi}}{s^{\frac{3}{2}}}e^{-\left(\frac{1}{4s}\right)}$$

1(b). Calculate Spearman's rank correlation coefficients.

Solution:

x	y	Rx	Ry	d=Rx-Ry	d^2
32	40	8	6	2	4
55	30	2	7 7.5	-5.5	30.25
49	70	4 (3.5)	2	1.5	2.25
60	20	1	10	-9	81
43	30	6 5.5	8 7.5	-2	4
37	50	7	4	3	9
43	72	5 5.5	1	4.5	20.25
49	60	3 (3.5)	3	0.5	0.25
10	45	10	5	5	25
20	25	9	9	0	0
					$\sum d^2 = 176$

To find $c.f.(x = 49) = \frac{2(2^2-1)}{12} = 0.5$

$$c.f.(x = 43) = \frac{2(2^2-1)}{12} = 0.5$$

$$c.f.(y = 30) = \frac{2(2^2-1)}{12} = 0.5$$

$$R = 1 - \left\{\frac{6(\sum d^2 + \sum c.f)}{n(n^2-1)}\right\}$$

$$= 1 - \left\{\frac{6(176 + 1.5)}{10(10^2 - 1)}\right\}$$

$$R = -\frac{5}{66} = -0.07575$$

1(c). Find ILT of $\frac{2s-1}{s^2+8s+29}$.

Solution:

$$L^{-1}\left[\frac{2s-1}{s^2+8s+29}\right]$$

$$= L^{-1}\left[\frac{2s-1}{(s+4)^2 - 4^2 + 29}\right]$$

$$= L^{-1}\left[\frac{2(s+4-4)-1}{(s+4)^2 - 4^2 + 29}\right]$$

$$= e^{-4t}L^{-1}\left[\frac{2(s-4)-1}{(s)^2+13}\right]$$

$$= e^{-4t}L^{-1}\left[\frac{2(s)-8-1}{(s)^2+13}\right]$$

$$= e^{-4t}L^{-1}\left[\frac{2(s)}{(s)^2+13} + \frac{-9}{(s)^2+13}\right]$$

$$= e^{-4t}\left[2\cos\sqrt{13t} - \frac{9}{\sqrt{13}}\sin\sqrt{13t}\right]$$

1(d). IF $f(z) = qx^2y + 2x^2 + ry^3 - 2y^2 - i(px^3 - 4xy - 3xy^2)$ is analytic, find the value of p, q, r.

Solution:

let $u = qx^2y + 2x^2 + ry^3 - 2y^2$ & $v = -(px^3 - 4xy - 3xy^2)$

Since $f(z)$ is analytic function

$$\therefore\ u_x = v_y \ \&\ u_y = -v_x$$

$$\therefore 2qxy + 4x = 4x + 6xy \quad \& \quad qx^2 + 3ry^2 - 4y = -(3px^2 + 4y + 3y^2)$$

Compare coefficients

$$2q = 6, q = -3p, 3r = -3$$

$$\therefore q = 3, p = -1, r = -1$$

2(a). Find LT of $e^{3t}f(t)$ where $f(t)\begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \\ 0, & otherwise \end{cases}$

Solution:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt = \int_1^2 e^{-st}(t-1)dt + \int_2^3 e^{-st}(3-t)dt$$

$$= \left[(t-1)\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right]_1^2 + \left[(3-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right)\right]_2^3$$

$$= \left[(2-1)\left(\frac{e^{-2s}}{-s}\right) - (1)\left(\frac{e^{-2s}}{s^2}\right)\right] - \left[(1-1)\left(\frac{e^{-s}}{-s}\right) - (1)\left(\frac{e^{-s}}{s^2}\right)\right]$$

$$+ \left[(3-3)\left(\frac{e^{-3s}}{-s}\right) - (-1)\left(\frac{e^{-3s}}{s^2}\right)\right] - \left[(3-2)\left(\frac{e^{-2s}}{-s}\right) - (-1)\left(\frac{e^{-2s}}{s^2}\right)\right]$$

$$= -\frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s^2} + \frac{e^{-3s}}{s^2} + \frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2}$$

$$= -\frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s^2} + \frac{e^{-3s}}{s^2} - \frac{e^{-2s}}{s^2}$$

$$L[f(t)] = \frac{1}{s^2}[e^{-s} - 2e^{-2s} + e^{-3s}]$$

$$\therefore L[e^{3t} f(t)] = \frac{1}{(s-3)^2}\left[e^{-(s-3)} - 2e^{-2(s-3)} + e^{-3(s-3)}\right]$$

2(b). Two unbiased dice are thrown. If X represents sum of the numbers on the two dice.Write probability distribution of the random variable X and find mean, SD, and $\boldsymbol{P(|x-7| \geq 3)}$.

Solution:

$X =$ sum of numbers appear on top of dice $= \{2,3,4,5,6,7,8,9,10,11,12\}$

	1	2	3	4	5	6
1	(1,1)	1,2	1,3	1,4	1,5	1,6
2	2,1	2,2	2,3	2,4	2,5	2,6
3	3,1	3,2	3,3	3,4	3,5	3,6
4	4,1	4,2	4,3	4,4	4,5	4,6
5	5,1	5,2	5,3	5,4	5,5	5,6
6	6,1	6,2	6,3	6,4	6,5	6,6

$n(s) = 6^2 = 36$

X	2	3	4	5	6	7	8	9	10	11	12
$P(X)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Mean$= E(x) = \sum_2^{12} xp(x)$

$$= \frac{1}{36}\begin{bmatrix} 2\times 1+3\times 2+4\times 3+5\times 4+6\times 5+7\times 6 \\ +8\times 5+9\times 4+10\times 3+11\times 2+12\times 1 \end{bmatrix}$$

$$= \frac{252}{36} = 7$$

$$E(x^2) = \sum_{2}^{12} x^2 p(x) = \frac{1}{36}\begin{bmatrix} 2^2\times 1+3^2\times 2+4^2\times 3+5^2\times 4+6^2\times 5+7^2\times 6 \\ +8^2\times 5+9^2\times 4+10^2\times 3+11^2\times 2+12^2\times 1 \end{bmatrix}$$

$$= \frac{1975}{36} = 54.833$$

$$v(x) = E(X^2) - \left(E(X)\right)^2 = 54.833 - 7^2 = 5.833$$

$$\sigma_x = \sqrt{v(x)} = \sqrt{5.833} = 2.415$$

$$P(|x-7| \geq 3)$$

$$= P(x-7 \leq -3) + P(x-7 \geq 3)$$

$$= P(x \leq 7-3) + P(x \geq 7+3)$$

$$= P(x \leq 4) + P(x \geq 10)$$

$$= P(2) + P(3) + P(4) + P(10) + P(11) + P(12)$$

$$= \frac{1}{36}[1+2+3+3+2+1] = \frac{12}{36} = \frac{1}{3}$$

2(c). Obtain F.S. for $f(x) = x\sin x \quad in\ (0, 2\pi)$

Solution:

$a = 2\pi$

Now, find Fourier coefficient (a_0, a_n, b_n)

$a_0 = \frac{2}{a}\int_0^a f(x)dx = \frac{2}{2\pi}\int_0^{2\pi} f(x)\,dx$

$a_n = \frac{2}{a}\int_0^a f(x)\cos\left(\frac{2n\pi x}{a}\right)dx = \frac{2}{2\pi}\int_0^{2\pi} f(x)\cos\left(\frac{2n\pi x}{2\pi}\right)dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx$

$b_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{2n\pi x}{a}\right)dx = \frac{2}{2\pi}\int_0^{2\pi} f(x)\sin\left(\frac{2n\pi x}{2\pi}\right)dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx$

Fourier series of f(x) in $(0, a)$ is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{2n\pi x}{a}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{2n\pi x}{a}\right)$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$

$a_0 = \frac{2}{2\pi}\int_0^{2\pi} x\sin x\ dx$

$= \frac{2}{2\pi}[(x)(-\cos x) - (1)(-\sin x)]_0^{2\pi}$

Substituting the limits

$\therefore a_0 = \frac{2}{2\pi}[-2\pi - 0]$

$\therefore a_0 = -2$

$a_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos(nx)dx$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} x\{\sin(x+nx) + \sin(x-nx)\}dx$$
$$a_n = \frac{1}{2\pi}\int_0^{2\pi} x\{\sin x(1+n) + \sin x(1-n)\}dx$$

Using $\int U\cdot V dx = U\int V dx - U'\int V_1 dx + U''\int V_2 dx - \cdots$

$$a_n = \frac{1}{2\pi}\left[(x)\left(-\frac{\cos x(1+n)}{1+n} - \frac{\cos x(1-n)}{1-n}\right) - (1)\left(-\frac{\sin x(1+n)}{1+n} - \frac{\sin x(1-n)}{1-n}\right)\right]_0^{2\pi}$$

Substituting the limits

$$a_n = \frac{1}{2\pi}\left\{\begin{array}{l}\left[\begin{array}{l}(2\pi)\left(-\frac{\cos 2\pi(1+n)}{1+n} - \frac{\cos 2\pi(1-n)}{1-n}\right)\\ -(1)\left(-\frac{\sin 2\pi(1+n)}{1+n} - \frac{\sin 2\pi(1-n)}{1-n}\right)\end{array}\right]\\ -\left[\begin{array}{l}(0)\left(-\frac{\cos 0(1+n)}{1+n} - \frac{\cos 0(1-n)}{1-n}\right)\\ -(1)\left(-\frac{\sin 0(1+n)}{1+n} - \frac{\sin 0(1-n)}{1-n}\right)\end{array}\right]\end{array}\right\}$$

$$a_n = \frac{1}{2\pi}\left\{(2\pi)\left(-\frac{1}{1+n} - \frac{1}{1-n}\right)\right\}$$
$$a_n = -\left\{\frac{1}{1+n} + \frac{1}{1-n}\right\}$$
$$= -\left\{\frac{1-n+1+n}{(1+n)(1-n)}\right\}$$
$$a_n = \frac{2}{n^2-1}\ ; n \neq 1$$
$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos(x)dx$$
$$a_1 = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\, dx$$
$$= \frac{1}{2\pi}\left[(x)\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{2^2}\right)\right]_0^{2\pi}$$
$$= \frac{1}{2\pi}\left\{\begin{array}{l}\left[(2\pi)\left(-\frac{\cos 4\pi}{2}\right) - (1)\left(-\frac{\sin 4\pi}{2^2}\right)\right]\\ -\left[(0)\left(-\frac{\cos 2(0)}{2}\right) - (1)\left(-\frac{\sin 2(0)}{2^2}\right)\right]\end{array}\right\}$$
$$= \frac{1}{2\pi}\{-\pi + 0 - 0 - 0\}$$
$$= \frac{\pi}{2}$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin(nx)dx$$
$$b_n = -\frac{1}{2\pi}\int_0^{2\pi} x\{\cos(x+nx) - \cos(x-nx)\}dx$$
$$b_n = -\frac{1}{2\pi}\int_0^{2\pi} x\{\cos(x+nx) - \cos(x-nx)\}dx$$

Using $\int U\cdot V dx = U\int V dx - U'\int V_1 dx + U''\int V_2 dx - \cdots$

$$b_n = \frac{1}{2\pi}\left[(x)\left(\frac{\sin x(1+n)}{1+n} - \frac{\sin x(1-n)}{1-n}\right) - (1)\left(-\frac{\cos x(1+n)}{(1+n)^2} + \frac{\cos x(1-n)}{(1-n)^2}\right)\right]_0^{2\pi}$$

Substituting the limits

$$b_n = \frac{1}{2\pi}\left\{\begin{bmatrix} (2\pi)\left(\frac{\sin 2\pi(1+n)}{1+n} - \frac{\sin 2\pi(1-n)}{1-n}\right) \\ -(1)\left(-\frac{\cos 2\pi(1+n)}{(1+n)^2} + \frac{\cos 2\pi(1-n)}{(1-n)^2}\right)\end{bmatrix} - \begin{bmatrix} (0)\left(\frac{\sin 0(1+n)}{1+n} - \frac{\sin 0(1-n)}{1-n}\right) \\ -(1)\left(-\frac{\cos 0(1+n)}{(1+n)^2} + \frac{\cos 0(1-n)}{(1-n)^2}\right)\end{bmatrix}\right\}$$

$$b_n = \frac{1}{2\pi}\left\{\begin{matrix}\left[(2\pi)(0) - \left(-\frac{1}{(1+n)^2} + \frac{1}{(1-n)^2}\right)\right] \\ -\left[(0)(0) - \left(-\frac{1}{(1+n)^2} + \frac{1}{(1-n)^2}\right)\right]\end{matrix}\right\}$$

$$B_n = \frac{1}{2\pi}\left[-\frac{1}{(1+n)^2} + \frac{1}{(1-n)^2}\right) + \left(-\frac{1}{(1+n)^2} + \frac{1}{(1-n)^2}\right)$$

$$b_n = 0\,; n \neq 1$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x \, dx$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin^2 2x \, dx$$

$$b_1 = \frac{1}{2\pi}\int_0^{2\pi} x(1 - \cos 2x)\, dx$$

$$= \frac{1}{2\pi}\left[(x)\left(x - \frac{\sin 2x}{2}\right) - (1)\left(\frac{x^2}{2} + \frac{\cos 2x}{2^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left\{\begin{matrix}\left[(2\pi)\left(2\pi - \frac{\sin 4\pi}{2}\right) - (1)\left(\frac{(2\pi)^2}{2} + \frac{\cos 4\pi}{2^2}\right)\right] \\ -\left[(0)\left(x - \frac{\sin 0}{2}\right) - (1)\left(\frac{0^2}{2} + \frac{\cos 0}{2^2}\right)\right]\end{matrix}\right\}$$

$$= \frac{1}{2\pi}\left\{\left[(2\pi)(2\pi - 0) - (1)\left(\frac{(2\pi)^2}{2} + \frac{1}{4}\right)\right] - \left[0 - (1)\left(\frac{1}{4}\right)\right]\right\}$$

$$= \frac{1}{2\pi}\left[4\pi^2 - 2\pi^2 - \frac{1}{4} + \frac{1}{4}\right]$$

$$= \frac{1}{2\pi}[2\pi^2]$$

$$b_1 = \pi$$

Substituting values of a_0, a_n and b_n in Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos(nx) + b_1 \sin x + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$\therefore x \sin x = -\frac{2}{2} + \frac{\pi}{2}\cos x + \sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx + \pi \sin x + \sum_{n=2}^{\infty} 0 \sin nx$$

3(a). Using Milne Thomson Method find $f(z)$ in terms of z, where

$$\boldsymbol{u + v = e^x(\cos y + \sin y) + \frac{x - y}{x^2 + y^2}}$$

Solution:

Let $u + v = e^x(\cos y + \sin y) + \frac{x-y}{x^2+y^2}$

Step:1 Partially differentiating $w.r.t.x.$ put $x = z \; and \; y = 0$

$$u_x + v_x = e^x(\cos y + \sin y) + \frac{(x^2+y^2)(1) - (x-y)(2x)}{(x^2+y^2)^2}$$

$u_x + v_x = e^z(1+0) + \frac{(z^2+0^2)(1)-(z-0)(2z)}{(z^2+0^2)^2}$

$$u_x + v_x = e^z + \frac{z^2 - 2z^2}{(z^2)^2}$$

$$u_x + v_x = e^z - \frac{1}{z^2} \dots (i)$$

Step:2 Partially differentiating $w.r.t.x.$ put $x = z \ and \ y = 0$

$u_y + v_y = e^x(-\sin y + \cos y) + \frac{(x^2+y^2)(-1)-(x-y)(2y)}{(x^2+y^2)^2}$

$u_y + v_y = e^z + \frac{(z^2+0^2)(-1)}{(z^2+0^2)^2}$

$$u_y + v_y = e^z - \frac{1}{z^2} \dots (ii)$$

Step:3 Using CR equations replace $\mathrm{u_y} = -v_{\mathrm{x}}$& $\mathrm{v_y} = u_{\mathrm{x}}$ in (ii)

$\therefore -v_x + u_x = e^z - \frac{1}{z^2} \dots (iii)$

Step:4 To find value of from step: -1 and step: -3

$u_x = \mathrm{e^z} - \frac{1}{z^2}$ and $\mathrm{v_x} = 0$

Step: -5 Put value of $\mathrm{u_x}$ & $\mathrm{v_x}$ in $f(z) = \int(\mathrm{u_x} + \mathrm{iv_x})\mathrm{dz}$ and integrate.

$\therefore \ f(z) = \int\left(\mathrm{e^z} - \frac{1}{z^2}\right)\mathrm{dz}$

$f(z) = e^z + \frac{1}{z} + c$

3(b). Find $\mathbf{L^{-1}}\left[\frac{(\mathbf{s+3})^2}{(\mathbf{s^2+6s+5})^2}\right]$by using convolution theorem.

Solution:

Let $\mathrm{L^{-1}}\left[\frac{(\mathrm{s+3})^2}{(\mathrm{s^2+6s+5})^2}\right] = \mathrm{L^{-1}}\left[\frac{(\mathrm{s+3})^2}{((\mathrm{s+3})^2-3^2+5)^2}\right] = e^{-3t}\mathrm{L^{-1}}\left[\frac{(\mathrm{s})^2}{((\mathrm{s})^2-4)^2}\right]$

$$= e^{-3t}\mathrm{L^{-1}}\left[\frac{\mathrm{s}^2}{(\mathrm{s}^2-4)^2}\right]$$

$$= e^{-3t}\mathrm{L^{-1}}\left[\frac{\mathrm{s}}{(\mathrm{s}^2-4)}\frac{\mathrm{s}}{(\mathrm{s}^2-4)}\right]$$

$$f_1(t) = \mathrm{L^{-1}}\left[\frac{\mathrm{s}}{(\mathrm{s}^2-4)}\right] = \cosh 2t$$

$$f_2(t) = \mathrm{L^{-1}}\left[\frac{\mathrm{s}}{(\mathrm{s}^2-4)}\right] = \cosh 2t$$

$$f_1(u) = \cosh 2u$$

$$f_2(t-u) = \cosh 2(t-u)$$

By using convolution theorem

$$\mathrm{L^{-1}}[\phi_1(s).\phi_1(s)] = \int_0^t f_1(u)f_2(t-u)du$$

$$= e^{-3t}\mathrm{L}^{-1}\left[\frac{\mathrm{s}}{(\mathrm{s}^2-4)}\frac{s}{(\mathrm{s}^2-4)}\right]$$

$$= e^{-3t}\int_0^t \cosh 2u \cosh 2(t-u)\,du$$

$$= e^{-3t}\int_0^t \cosh 2t \cosh 2(t-u)\,du$$

$$= \frac{e^{-3t}}{2}\int_0^t \{\cosh(2u+2t-2u)+\cosh(2u-2t+2u)\}du$$

$$= \frac{1}{2}\int_0^t \{\cosh(2t)+\cosh(4u-2t)\}du$$

$$= \frac{e^{-3t}}{2}\left\{\int_0^t \{\cosh(2t)\}du + \int_0^t \{\cosh(4u-2t)\}du\right\}$$

$$= \frac{e^{-3t}}{2}\left\{\cosh(2t)\int_0^t du + \int_0^t \{\cosh(4u-2t)\}du\right\}$$

$$= \frac{e^{-3t}}{2}\left\{\cosh(2t)\,[u]_0^t + \left[\frac{\sinh(4u-2t)}{4}\right]_0^t\right\}$$

$$= \frac{e^{-3t}}{2}\left\{\cosh 2t\,[t-0] + \left[\frac{\sinh(4t-2t)}{4} - \frac{\sin(4\times 0-2t)}{4}\right]\right\}$$

$$= \frac{e^{-3t}}{2}\left\{t\cosh 2t + \left[\frac{\sinh(2t)}{4} - \frac{\sinh(-2t)}{4}\right]\right\}$$

$$= \frac{e^{-3t}}{2}\left\{t\cosh 2t + \left[\frac{\sinh(2t)}{4} + \frac{\sinh(2t)}{4}\right]\right\}$$

$$= \frac{e^{-3t}}{2}\left\{t\cosh 2t + \left[2\frac{\sinh(2t)}{4}\right]\right\}$$

$$= \frac{e^{-3t}}{2}\left\{t\cosh 2t + \frac{\sinh(2t)}{2}\right\}$$

$$\mathrm{L}^{-1}\left[\frac{(\mathrm{s}+3)^2}{(\mathrm{s}^2+6\mathrm{s}+5)^2}\right] = \frac{e^{-3t}}{2}\left\{t\cosh 2t + \frac{\sinh(2t)}{2}\right\}$$

3(c). Fit Parabola $y = a + bx + cx^2$ to the following data and estimate $y\ at\ x = 10$

Solution:

Step-1: Write equation of parabolic curve

$y = a + bx + cx^2 \ldots\ldots(*)$

$Taking \sum\ on\ both\ sides$

$\sum y = \sum a + \sum bx + \sum cx^2$

$\sum y = a\sum 1 + b\sum x + c\sum x^2$

$\sum y = na + b\sum x + c\sum x^2 \ldots\ldots(i)$

Multiplying by x on both side in equation $(*)$

$xy = ax + bx^2 + cx^3$

$Taking \sum\ on\ both\ sides$

$\sum xy = \sum ax + \sum bx^2 + \sum cx^3$

$\sum xy = a\sum x + b\sum x^2 + c\sum x^3 \ldots\ldots(ii)$

multiplying by x^2 on both sides in euation $(*)$

$$x^2y = ax^2 + bx^3 + cx^4$$

$Taking \sum\ on\ both\ sides$

$$\sum x^2y = \sum ax^2 + \sum bx^3 + \sum cx^4$$

$$\sum x^2y = a\sum x^2 + b\sum x^3 + c\sum x^4 \ldots\ldots (iii)$$

Solve Equations $(i), (ii) and\ (iii)$

$$\begin{bmatrix} n & \sum x & \sum x^2 \\ \sum x & \sum x^2 & \sum x^3 \\ \sum x^2 & \sum x^3 & \sum x^4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum xy \\ \sum x^2y \end{bmatrix}$$

Step-2: Prepare the table

x	Y	x^2	x^3	x^4	xy	x^2y
1	2	1	1	1	2	2
2	6	4	8	16	12	24
3	7	9	27	81	21	63
4	8	16	64	256	32	128
5	10	25	125	625	50	250
6	11	36	216	1296	66	396
7	11	49	343	2401	77	539
8	10	64	512	4096	80	640
9	9	81	729	6561	81	729
$\sum x = 280$	$\sum y = 74$	$\sum x^2 = 285$	$\sum x^3 = 2025$	$\sum x^4 = 15333$	$\sum xy = 421$	$\sum x^2y = 2771$

Step-3: Put Values of

$\sum \mathrm{x} = 45$

$\sum \mathrm{x}^2 = 285$

$\sum \mathrm{x}^3 = 2025$

$\sum \mathrm{x}^4 = 153333$

$\sum \mathrm{y} = 74$

$\sum \mathrm{xy} = 421$

$\sum \mathrm{x}^2\mathrm{y} = 2771$

in equation (i), (ii)&(iii)and find values of a, b & c

$$\begin{bmatrix} 9 & 45 & 285 \\ 45 & 285 & 2025 \\ 285 & 2045 & 15333 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 74 \\ 421 \\ 2771 \end{bmatrix}$$

$\mathrm{a} = -0.9285, \mathrm{b} = 3.5231\ \&\ \mathrm{c} = -0.2673$

Step-4: Put Values $\mathrm{a} = -28.5, \mathrm{b} = 5.7\ \&\ \mathrm{c} = -0.07$ in Equation $(*)$

$y = a + bx + cx^2 \ldots\ldots (*)$

$y = -0.9285 + 3.5231\mathrm{x} - 0.2673\mathrm{x}^2$

$$y(at\ x = 10) = -0.9285 + 3.5231(10) - 0.2673(10)^2 =$$

4(a). Find $L\left[e^{-\frac{1}{2}t}f(3t)\right]$ If $L[f(t)] = \frac{1}{s\sqrt{s+1}}$.

Solution:

$$L[f(t)] = \frac{1}{s\sqrt{s+1}}$$

$$\therefore L[f(3t)] = \frac{1}{3}\frac{1}{\frac{s}{3}\sqrt{\frac{s}{3}+1}} = \frac{\sqrt{3}}{s\sqrt{s+3}} = \frac{\sqrt{3}}{\sqrt{s^3+3s^2}}$$

$$\therefore L[tf(3t)] = -\frac{d}{ds}\frac{\sqrt{3}}{\sqrt{s^3+3s^2}}$$

$$= -\sqrt{3}\frac{d}{ds}(s^3+3s^2)^{-\frac{1}{2}}$$

$$= \frac{\sqrt{3}}{2}(s^3+3s^2)^{-\frac{1}{2}-1}(3s^2+6s)$$

$$L[tf(3t)] = \frac{\sqrt{3}}{2}(s^3+3s^2)^{-\frac{3}{2}}(3s^2+6s)$$

$$\therefore L\left[e^{-\frac{1}{2}t}f(3t)\right] = \frac{\sqrt{3}}{2}\left(\left(s+\frac{1}{2}\right)^3 + 3\left(s+\frac{1}{2}\right)^2\right)^{-\frac{3}{2}}\left(3\left(s+\frac{1}{2}\right)^2 + 6\left(s+\frac{1}{2}\right)\right)$$

4(b). Find half range sine series of $\mathbf{f(x) = x - x^2}$, $\mathbf{0 < x < 1}$, hence find

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}$$

Sol: $\mathrm{f(x) = x - x^2}$

Now, find Fourier coefficient for half range sine series$(\mathrm{b_n})$

$\mathrm{b_n} = \frac{2}{a}\int_0^a \mathrm{f(x)}\sin\left(\frac{n\pi x}{a}\right)dx = \frac{2}{1}\int_0^1 \mathrm{f(x)}\sin\left(\frac{n\pi x}{1}\right)dx = \frac{2}{1}\int_0^1 \mathrm{f(x)}\sin(n\pi x)dx$

Fourier series of f(x) as half range sine series is

$\mathrm{f(x)} = \sum_{n=1}^{\infty} \mathrm{b_n}\sin\left(\frac{n\pi x}{a}\right) = \sum_{n=1}^{\infty}\mathrm{b_n}\sin(n\pi x)$

$\mathrm{b_n} = \frac{2}{1}\int_0^1 (x - x^2)\ \sin(nx)dx$

Using $\int U \cdot V dx = U\int V dx - U'\int V_1 dx + U''\int V_2 dx - \cdots$

$\mathrm{b_n} = \frac{2}{1}\left[(x-x^2)\left(\frac{-\cos(n\pi x)}{n\pi}\right) - (1-2x)\left(\frac{-\sin(n\pi x)}{n^2\pi^2}\right) + (-2)\left(\frac{\cos(n\pi x)}{n^3\pi^3}\right)\right]_0^1$

Substituting the limits

$\because \sin n\pi = \sin 0 = 0$

$\therefore\ \mathrm{b_n} = \frac{2}{1}\left[(x-x^2)\left(\frac{-\cos(n\pi x)}{n\pi}\right) - \frac{2\cos(nx)}{n^3\pi^3}\right]_0^1$

$= \frac{2}{1}\left[\left(((1)-1^2)\left(\frac{-\cos(n\pi)}{n\pi}\right) - \frac{2\cos(n\pi)}{n^3\pi^3}\right) - \left(((0)-0^2)\left(\frac{-\cos(0)}{n\pi}\right) - \frac{2\cos(0)}{n^3\pi^3}\right)\right]$

$\because \cos(n\pi) = (-1)^n$

$$\therefore\ b_n = \frac{2}{1}\left[\left((0)\left(\frac{-(-1)^n}{n}\right) - \frac{2(-1)^n}{n^3}\right) - \left((0-0)\left(\frac{-1}{n}\right) - \frac{2(1)}{n^3}\right)\right]$$

$$= \frac{2}{1}\left[\left(0 - \frac{2(-1)^n}{n^3\pi^3}\right) - \left(0 - \frac{2(1)}{n^3\pi^3}\right)\right]$$

$$= \frac{2}{1}\left[-\frac{2(-1)^n}{n^3\pi^3} + \frac{2(1)}{n^3\pi^3}\right]$$

$$= \frac{4}{1}\left[\frac{1-(-1)^n}{n^3\pi^3}\right]$$

$$\therefore\ b_n = \begin{cases} 0 & \text{if n is even} \\ \frac{8}{\pi^3 n^3} & \text{if n is odd i.e. } n = 2n-1 \end{cases}$$

$$\therefore\ b_n = \frac{8}{\pi^3(2n-1)^3}$$

Substituting values of b_n in Fourier series

$$f(x) = \sum_{n=2}^{\infty} \frac{8}{\pi^3(2n-1)^3} \sin\big((2n-1)\pi x\big)$$

OR

$$\therefore\ x - x^2 = \frac{8}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin(2n+1)x}{(2n-1)^3}$$

Put $x = \frac{1}{2}$

$$\therefore \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{8}{\pi^3}\sum_{n=1}^{\infty} \frac{\sin\frac{(2n+1)\pi}{2}}{(2n-1)^3}$$

$$\therefore \frac{1}{4} \times \frac{\pi^3}{8} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$$

$$\therefore \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32}$$

4©. Given line of regressions $6y = 5x + 90$, $15x = 8y + 130$, $\sigma_x^2 = 16$. Find

(i). To find $\bar{x}$ & $\bar{y}$

(ii). To find r

(iii). To find σ_y^2

(iv). To find angle

Solution:

(i). To find $\bar{x}$ & $\bar{y}$

Solve following equations

$$6y = 5x + 90 \ldots (i)$$

$$15x = 8y + 130 \ldots (ii)$$

$$\bar{x} = 30\ \ \&\ \bar{y} = 40$$

(ii). To find r

Let Line of regression of y on x is $6y = 5x + 90$

$$\therefore y = \frac{5}{6}x + \frac{90}{6}$$

$$\therefore b_{yx} = \frac{5}{6}$$

Let Line of regression of y on x is $15x = 8y + 130$

$$\therefore x = \frac{8}{15}y + \frac{130}{15}$$

$$\therefore b_{xy} = \frac{8}{15}$$

$$\therefore r = \sqrt{b_{xy}b_{yx}} = \sqrt{\frac{8}{15} \times \frac{5}{6}} = \frac{2}{3} = 0.67$$

(iii). To find σ_y^2

$$Since\ b_{xy} = r\frac{\sigma_x}{\sigma_y}$$

$$\therefore \frac{8}{15} = \frac{2}{3} \times \frac{4}{\sigma_y}$$

$$\therefore \sigma_y = 5$$

$$\therefore \sigma_y^2 = 25$$

(iv). To find angle

Since Angle between two lines= $\pm\tan^{-1}\left(\frac{b_{yx} - \frac{1}{b_{xy}}}{1 - b_{yx} * \frac{1}{b_{xy}}}\right)$

$$= \pm\tan^{-1}\left(\frac{\frac{5}{6} - \frac{15}{8}}{1 - \frac{5}{6} \times \frac{15}{8}}\right)$$

$$= \pm 61.63^0$$

5(a). Find analytic function if $u = \left(r + \frac{a^2}{r}\right)\cos\theta$

Solution:

Let $u = \left(r + \frac{a^2}{r}\right)\cos\theta$

Step:1 Partially differentiating $w.r.t.r$ put $r = z\ and\ \theta = 0$

$u_r = \frac{\partial u}{\partial r} = \left(1 - \frac{a^2}{r^2}\right)\cos\theta$

$u_r = \left(1 - \frac{a^2}{z^2}\right)$

Step:2 Partially differentiating $w.r.t.\theta$ put $x = z\ and\ y = 0$

$u_\theta = \frac{\partial u}{\partial \theta} = -\left(r + \frac{a^2}{r}\right)\ \sin\theta$

$u_\theta = 0$

Step:3 Using CR equations replace $\mathrm{u}_\theta = -r\mathrm{v}_\mathrm{r}$

$\therefore \mathrm{v}_\mathrm{r} = 0$

Step:4 To find value of from step: -1 and step: -3

$u_r = \left(1 - \frac{a^2}{z^2}\right)$ and $\mathrm{v}_\theta = 0$

Step: -5 Put value of $\mathrm{u_x}$ & $\mathrm{v_x}$ in $f(z) = \int(\mathrm{u_r} + \mathrm{iv_r})\mathrm{dz}$ and integrate.

$\therefore\ f(z) = \int\left(\left(1 - \frac{a^2}{z^2}\right) + (0)\right)\mathrm{dz}$

$\therefore\ f(z) = z + \frac{a^2}{z}$

Put $z = re^{i\theta}$

$$\therefore\ f(re^{i\theta}) = re^{i\theta} + \frac{a^2}{re^{i\theta}}$$

$$= re^{i\theta} + \frac{a^2}{r}e^{-i\theta}$$

$$= r(\cos\theta + i\sin\theta) + \frac{a^2}{r}(\cos\theta - i\sin\theta)$$

$$= \left(r + \frac{a^2}{r}\right)\cos\theta + i\left(r - \frac{a^2}{r}\right)\sin\theta$$

$$\therefore u = \left(r + \frac{a^2}{r}\right)\cos\theta\ \ \&\ v = \left(r - \frac{a^2}{r}\right)\sin\theta$$

5(b). An unbiased coin is tosses three times. If X denotes the absolute difference between head and tails, Find Moment generating function of X and hence obtained the first moment about origin and 2nd moment about mean.

Solution: Sample Space

$s = \{H,T\}\{H,T\}\{H,T\}$

$= \{HH, HT, TH, TT\}\{H,T\}$

$= \{HHH, H\,TH, THH, TTH, HHT, HTT, THT, TTT\}$

$n(s) = 2^3 = 8$

$X = |H - T| = \{3, 1, 1,1,1,1,1,3\}$

x	1	3	
$P(x)$	$\frac{6}{8}$	$\frac{2}{8}$	
$x\,P(x)$	$\frac{6}{8}$	$\frac{6}{8}$	$E(x) = \sum x\,P(x) = \frac{12}{8}$
$x^2\,P(x)$	$\frac{6}{8}$	$\frac{18}{8}$	$E(x^2) = \sum x^2\,P(x) = \frac{24}{8}$

$$MGF = E(e^{tx}) = \sum e^{tx}\, P(x) = \frac{6}{8}e^{t} + \frac{2}{8}e^{3t}$$

To find first moment about origin $= m_1 = \frac{d}{dt} MGF\ (at\ t = 0) = \frac{6}{8}e^{t} + \frac{6}{8}e^{3t} = \frac{12}{8} = 1.5$

To find 2nd moment about origin $= m_2 = \frac{d^2}{dt^2} MGF\ (at\ t = 0) = \frac{6}{8}e^{t} + \frac{18}{8}e^{3t} = \frac{24}{8}$

To find 2nd moment about mean $= \mu_2 = E(x^2) - \{E(x)\}^2 = \frac{24}{8} - \left(\frac{12}{8}\right)^2 = \frac{3}{4} = 0.75$

2nd Method

To find first moment about origin $= m_1 = E(x) = \sum x\, P(x) = \frac{12}{8} = \frac{3}{2} = 1.5$

To find 2nd moment about mean $= \mu_2 = E(x^2) - \{E(x)\}^2 = \frac{24}{8} - \left(\frac{12}{8}\right)^2 = \frac{3}{4} = 0.75$

5(c). Find $\int_0^\infty e^{-2t} \cosh t \int_0^t u^2 \sinh u \cosh u\, du\ dt$

Solution:

Let $\int_0^\infty e^{-2t} \cosh t \int_0^t u^2 \sinh u \cosh u\, du\ dt$

$$= \int_0^\infty e^{-2t} \left(\frac{e^{t} + e^{-t}}{2}\right) \int_0^t u^2 \frac{\sinh 2u}{2}\, du\ dt$$

$$= \int_0^\infty \left(\frac{e^{-t} + e^{-3t}}{2}\right) \int_0^t u^2 \frac{\sinh 2u}{2}\, du\ dt$$

$$= \frac{1}{2}\left\{\int_0^\infty e^{-t} \int_0^t u^2 \frac{\sinh 2u}{2}\, du\ dt + \int_0^\infty e^{-3t} \int_0^t u^2 \frac{\sinh 2u}{2}\, du\ dt\right\}$$

$$= \frac{1}{4}\left\{L\left[\int_0^t u^2 \sinh 2u\ du\right]_{s=1} + L\left[\int_0^t u^2 \sinh 2u\ du\right]_{s=3}\right\}$$

Since $L[\sinh 2u] = \frac{2}{s^2-4}$

$$\therefore L[u \sinh 2u] = -\frac{d}{ds}\frac{2}{s^2 - 4}$$

$$= -2\left(-\frac{2s}{(s^2-4)^2}\right)$$

$$= \left(\frac{4s}{(s^2-4)^2}\right)$$

$$\therefore L[u\, u \sinh 2u] = -\frac{d}{ds}\left(\frac{4s}{(s^2-4)^2}\right) = -\frac{(s^2-4)^2(4) - (4s)2(s^2-4)2s}{((s^2-4)^2)^2}$$

$$= -(s^2-4)\left\{\frac{(4s^2-16) - 16s^2}{((s^2-4)^2)^2}\right\}$$

$$= -\left\{\frac{(-12s^2 - 16)}{(s^2 - 4)^3}\right\}$$

$$= \left\{\frac{(12s^2 + 16)}{(s^2 - 4)^3}\right\}$$

Now $\int_0^\infty e^{-2t} \cosh t \int_0^t u^2 \sinh u \cosh u \, du \;\; dt$

$$= \frac{1}{4}\left\{L\left[\int_0^t u^2 \sinh 2u \;\; du\right]_{s=1} + L\left[\int_0^t u^2 \sinh 2u \;\; du\right]_{s=3}\right\}$$

$$= \frac{1}{4}\left\{\left[\frac{(12s^2 + 16)}{(s^2 - 4)^3}\right]_{s=1} + L\left[\frac{(12s^2 + 16)}{(s^2 - 4)^3}\right]_{s=3}\right\}$$

$$= \frac{1}{4}\left\{\frac{(12 + 16)}{(1 - 4)^3} + \frac{(12 \times 9 + 16)}{(9 - 4)^3}\right\} = -\frac{76}{3375} = -0.0225$$

6(a). Find $L^{-1}\left[\frac{1}{(s-2)^4(s+3)}\right]$ by using partial fraction method.

Solution:

Let $L^{-1}\left[\frac{1}{(s-2)^4(s+3)}\right] = L^{-1}\left[\frac{A}{(s-2)^1} + \frac{B}{(s-2)^2} + \frac{C}{(s-2)^3} + \frac{D}{(s-2)^4} + \frac{E}{(s+3)}\right]$

$$= Ae^{2t} + Bte^{2t} + C\frac{t^2}{2!}\, e^{2t} + D\frac{t^3}{3!}\, e^{2t} + Ee^{-3t}$$

Now find value of D and E

$$E = \left[\frac{1}{(s - 2)^4}\right]_{s=-3} = \frac{1}{(-3 - 2)^4} = \frac{1}{625}$$

$$D = \left[\frac{1}{s + 3}\right]_{s=2} = \frac{1}{2 + 3} = \frac{1}{5}$$

Now find value of A, B and C

Let $\frac{A}{(s-2)^1} + \frac{B}{(s-2)^2} + \frac{C}{(s-2)^3} + \frac{D}{(s-2)^4} + \frac{E}{(s+3)} = \frac{1}{(s-2)^4(s+3)}$

Put any real three real numbers in above expression except {-3, 2}

Now Put $s = 0,1 \; and - 1$ one by one

$$@\; s = 0, \frac{A}{(0 - 2)^1} + \frac{B}{(0 - 2)^2} + \frac{C}{(0 - 2)^3} + \frac{D}{(0 - 2)^4} + \frac{E}{(0 + 3)} = \frac{1}{(0 - 2)^4(0 + 3)}$$

$$-\frac{A}{2} + \frac{B}{4} - \frac{C}{8} + \frac{D}{16} + \frac{E}{3} = \frac{1}{48}$$

$$-\frac{A}{2} + \frac{B}{4} - \frac{C}{8} = \frac{1}{48} - \frac{D}{16} - \frac{E}{3}$$

$$-\frac{A}{2}+\frac{B}{4}-\frac{C}{8}=\frac{1}{48}-\frac{1}{16\times 5}-\frac{1}{3\times 625}$$

$$-\frac{A}{2}+\frac{B}{4}-\frac{C}{8}=\frac{39}{5000}\dots(i)$$

$$@\, s=1, \frac{A}{(1-2)^1}+\frac{B}{(1-2)^2}+\frac{C}{(1-2)^3}+\frac{D}{(1-2)^4}+\frac{E}{(1+3)}=\frac{1}{(1-2)^4(1+3)}$$

$$-A+B-C+D+\frac{E}{4}=\frac{1}{4}$$

$$-A+B-C=\frac{1}{4}-D-\frac{E}{4}$$

$$-A+B-C=\frac{1}{4}-\frac{1}{5}-\frac{1}{4\times 625}$$

$$-A+B-C=\frac{31}{625}\dots(ii)$$

$$@\, s=-1, \frac{A}{(-1-2)^1}+\frac{B}{(-1-2)^2}+\frac{C}{(-1-2)^3}+\frac{D}{(-1-2)^4}+\frac{E}{(-1+3)}$$
$$=\frac{1}{(-1-2)^4(-1+3)}$$

$$-\frac{A}{3}+\frac{B}{9}-\frac{C}{27}+\frac{D}{81}+\frac{E}{2}=\frac{1}{162}$$

$$-\frac{A}{3}+\frac{B}{9}-\frac{C}{27}=\frac{1}{162}-\frac{D}{81}-\frac{E}{2}$$

$$-\frac{A}{3}+\frac{B}{9}-\frac{C}{27}=\frac{1}{162}-\frac{1}{81\times 5}-\frac{1}{2\times 625}$$

$$-\frac{A}{3}+\frac{B}{9}-\frac{C}{27}=\frac{49}{16875}\dots(iii)$$

Solve following equations

$$-\frac{A}{2}+\frac{B}{4}-\frac{C}{8}=\frac{39}{5000}\dots(i)$$

$$-A+B-C=\frac{31}{625}\dots(ii)$$

$$-\frac{A}{3}+\frac{B}{9}-\frac{C}{27}=\frac{49}{16875}\dots(iii)$$

$$A=-\frac{1}{625} \qquad B=\frac{1}{125} \qquad C=-\frac{1}{25}$$

$$L^{-1}\left[\frac{1}{(s-2)^4(s+3)}\right]=-\frac{1}{625}e^{2t}+\frac{1}{125}te^{2t}-\frac{1}{25}\frac{t^2}{2!}e^{2t}+\frac{1}{5}\frac{t^3}{3!}e^{2t}+\frac{1}{625}e^{-3t}$$

6(b). If $f(x) = \begin{cases} ke^{-\frac{x}{4}} & ; x. > 0 \\ 0 & ; else \end{cases}$ find k, mean and variance

Solution:

Let $f(x) = \begin{cases} ke^{-\frac{x}{4}} & ; x. > 0 \\ 0 & ; else \end{cases}$

To find K.

Since $\int_0^\infty f(x)dx = 1$

$$\int_0^\infty ke^{-\frac{x}{4}}dx = 1$$

$$\left[\frac{ke^{-\frac{x}{4}}}{-\frac{1}{4}}\right]_0^\infty = 1$$

$$-4k[0-1] = 1$$

$$k = \frac{1}{4}$$

To find mean $m = E(x) = \int_0^\infty xf(x)dx$

$$= \int_0^\infty xke^{-\frac{x}{4}}dx$$

$$= k\left[x\frac{e^{-\frac{x}{4}}}{-\frac{1}{4}} - \frac{e^{-\frac{x}{4}}}{\left(-\frac{1}{4}\right)^2}\right]_0^\infty$$

$$= \frac{1}{4}\{[0-0]-[0-\ 16]\} = 4$$

To find $E(x^2) = \int_0^\infty x^2f(x)dx$

$$= \int_0^\infty x^2ke^{-\frac{x}{4}}dx$$

$$= k\left[x^2\frac{e^{-\frac{x}{4}}}{-\frac{1}{4}} - 2x\frac{e^{-\frac{x}{4}}}{\left(-\frac{1}{4}\right)^2} + 2\frac{e^{-\frac{x}{4}}}{\left(-\frac{1}{4}\right)^3}\right]_0^\infty$$

$$= \frac{1}{4}\{[0-0+0]-[0-\ 0-128]\} = 32$$

To find Variance $v(x) = E(x^2) - \{E(x)\}^2 = 32 - 4^2 = 16$

6(c). Find Half Range Cosine Series if $\mathbf{f(x) = x}$, $\mathbf{0 < x < 2}$ .Hence deduce that

$$\frac{\pi^4}{96} = \left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{2}{7^4} + \cdots\right]$$

Solution:

$\mathrm{f(x)} = x$

Now, find fourier coefficient (a_0, a_n)

$a_0 = \frac{2}{a}\int_0^a f(x)dx = \frac{2}{2}\int_0^2 f(x)\,dx$

$$a_n = \frac{2}{a}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)dx = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

Fourier series of f(x) as half range cosine series is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$

$a_0 = \int_0^2 x\,dx$

$= \left[\frac{x^2}{2}\right]_0^2$

$= \left[\frac{4}{2} - 0\right]$

$\therefore a_0 = 2$

$a_n = \int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx$

Using $\int U\cdot V dx = U\int V dx - U'\int V_1 dx + U''\int V_2 dx - \cdots$

$a_n = \left[(x)\left(\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\right) - (1)\left(\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{4}}\right)\right]_0^2$

Substituting the limits

$\because \sin n\pi = \sin(0) = 0$

$a_n = \left[-(1)\left(\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{4}}\right)\right]_0^2$

$= \frac{4}{\pi^2 n^2}[(\cos n\pi) - 1]$

$\therefore a_n = \frac{4}{\pi^2 n^2}[(-1)^n - 1]$

Substituting values of a_0 and a_n in Fourier series

$f(x) = \frac{2}{2} + \sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}[(-1)^n - 1]\cos\left(\frac{n\pi x}{2}\right)$

$\therefore x = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}[(-1)^n - 1]\frac{\cos\left(\frac{n\pi x}{2}\right)}{n^2}$

i) Using Parseval' s identity

$\frac{2}{a}\int_0^a [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2)$

$\because \frac{2}{2}\int_0^2 [f(x)]^2 dx = \frac{1}{2}(2)^2 + \sum_{n=0}^{\infty}\left(\frac{4}{\pi^2 n^2}[(-1)^n - 1]\right)^2$

$$\therefore \int_0^2 x^2 \mathrm{dx} = 2 + \sum_{\mathrm{n}=1}^{\infty} \frac{16[(-1)^{\mathrm{n}}-1]^2}{(\pi^2 \mathrm{n}^2)^2}$$

$$\therefore \left[\frac{x^3}{3}\right]_0^2 = 2 + \frac{16}{\pi^4}\left[\frac{4}{1^4} + \frac{4}{3^4} + \frac{4}{5^4} + \frac{4}{7^4} + \cdots\right]$$

$$\therefore \left[\frac{8}{3}\right] - 2 = \frac{16}{\pi^4}\left[\frac{4}{1^4} + \frac{4}{3^4} + \frac{4}{5^4} + \frac{4}{7^4} + \cdots\right]$$

$$\therefore \frac{2}{3} \times \frac{\pi^4}{64} = \left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{2}{7^4} + \cdots\right]$$

$$\therefore \frac{\pi^4}{96} = \left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{2}{7^4} + \cdots\right]$$