Moments and Moment generating function

Definition: rth raw moment:

Suppose X is a random variable, discrete or continuous. Then the rth raw moment about the origin is given by

$$\mu_r' = E(X^r), \quad r = 0, 1, 2, \ldots$$
$$= \begin{cases} \sum_x x^r p_X & \text{if X is discrete with probability mass function } p_X \\ \int_{-\infty}^{\infty} x^r f(x)dx & \text{if X is continuous with probability density function } f(x) \end{cases}$$

Definition: rth central moment:

Suppose X is a random variable, discrete or continuous with mean μ.. Then the rth central moment (about the mean) is given by

$$\mu_r = E((X-\mu)^r), \quad r = 0, 1, 2, \ldots$$
$$= \begin{cases} \sum_x (x-\mu)^r p_X & \text{if X is discrete with probability mass function } p_X \\ \int_{-\infty}^{\infty} (x-\mu)^r f(x)dx & \text{if X is continuous with probability density function } f(x) \end{cases}$$

*From the above definitions, you can already infer that

- mean = $E(X) = \mu_1'$
- Variance = $E(X-\mu)^2 = \mu_2$

This means that mean or expectation is a moment of the first degree, i.e. the first raw moment, and variance is the second central moment. What about moments of higher degrees? What purpose do they serve?
Since the mean alone is not an accurate representative of the data, we use variance or standard deviation to check how scattered the data is about the mean. The moments of the third degree tell us about the ***skewness*** of the data, I.e, whether more values are towards the right or the left of the mean. The moments of the fourth degree lead us to ***kurtosis***, which is the flatness of the curve once you trace the graph.

Relation between raw and central moments:

We have

- Variance = $E(X^2)-[E(X)]^2=\text{¿}\quad E(X-\mu)^2=\mu_2$, i.e. $\mu_2=\mu_2'-\left(\mu_1'\right)^2$
- $\mu_1=E(X-\mu)=0$

Now, let us attempt to generalize this connection between raw and central moments to the rth moment.

$$\begin{aligned}
\mu_r &= E((X-\mu)^r),\quad r=0,1,2,\ldots \\
&= E(X^r - rC_1X^{r-1}\mu + rC_2X^{r-2}\mu^2 - rC_3X^{r-3}\mu^3+\ldots+(-1)^r\mu^r) \\
&= E(X^r) - rC_1E(X^{r-1}\mu) + rC_2E(X^{r-2}\mu^2) - rC_3E(X^{r-3}\mu^3)+\ldots+(-1)^rE(\mu^r)
\end{aligned}$$

$$= E(X^r) - rC_1E(X^{r-1})\mu + rC_2E(X^{r-2})\mu^2 - rC_3E(X^{r-3})\mu^3+\ldots+(-1)^rE(\mu^r)$$

$$(\text{Q } \mu \text{ is a constant and } E(k)=k, \text{ for all constants } k)$$

$$\Rightarrow \mu_r = \mu_r' - rC_1\mu_{r-1}'\mu + rC_2\mu_{r-2}'\mu - rC_3\mu_{r-3}'\mu+\ldots+(-1)^r\mu^r$$

Since it is easier to find the raw moments than the central moments, this relation can be very helpful, especially after the next part, which is a still easier method to find the raw moments.

Finding raw moments can still be tedious using a table, as you have to calculate a lot of values. It would be much easier if we could just develop a formula (or function!) that gives us all the moments at one go. That is what we shall look at now.

Moment Generating Function:

Suppose X is a random variable, discrete or continuous. The Moment generating function (mgf or MGF) is denoted by $M_X(t)$ and is defined as:

$$M_X(t) = E(e^{tX})$$

$$= \begin{cases} \sum_x e^{tx} p_x, \text{ if X is a } \textbf{discrete} \text{ r.v. with probability mass function } P(X=x) = p_x \\ \int_{-\infty}^{\infty} e^{tx} f(x)dx, \text{ if X is a } \textit{continuous} \text{ r.v. with probability density function } f(x) \end{cases}$$

Remark: If the mgf exists for a random variable X, we will be able to obtain all the moments of X. It is very plainly put, one function that generates all the moments of X.

Result: Suppose X is a random variable (discrete or continuous) with moment generating function $M_X(t)$ then the rth raw moment is given by

$$\mu_r' = \begin{cases} \text{coefficient of } \frac{t^r}{r!} \text{ in the expansion of } M_X(t) \\ \frac{d^r}{dt^r}\left[M_X(t)\right]_{t=0} \end{cases}, \quad r = 1, 2, \ldots$$

Proof: We have,

$$M_X(t) = E(e^{tX})$$

$$= E\left(1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \ldots + \frac{t^rX^r}{r!} + \ldots\right)$$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \ldots + \frac{t^r}{r!}E(X^r) + \ldots$$

$$i.e. M_X(t) = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \ldots + \frac{t^r}{r!}\mu_r' + \ldots \quad \_\_\_\_(i)$$

$$\Rightarrow \mu_r' = \text{coefficient of } \frac{t^r}{r!} \text{ in the expansion of } M_X(t) \quad \_\_\_\_(A)$$

Now, differentiating (i) w.r.t t we get

$$\frac{d}{dt}(M_X(t)) = \mu_1' + \frac{2t}{2}\mu_2' + \frac{3}{6}t^2\mu_3' + \ldots + \frac{r}{r!}t^{r-1}\mu_r' + \ldots \quad \_\_\_\_(ii)$$

Putting $t=0$ in (ii) we get,

$$\frac{d}{dt}\left[M_X(t)\right]_{t=0} = \mu_1'$$

Now, differentiating (ii) w.r.t t we get

$$\frac{d}{dt}\left(\frac{d}{dt}M_X(t)\right) = \frac{d^2}{dt^2}M_X(t)$$

$$= t^2\mu_2' + \frac{3(2)}{6}t\mu_3' + \ldots + \frac{r(r-1)}{r!}t^{r-2}\mu_r' + \ldots \quad \_\_\_\_(iii)$$

Putting $t=0$ in (iii) we get,

$$\frac{d^2}{dt^2}(M_X(t)) = \mu_2'$$

$$i.e.\ \mu_2' = \frac{d^2}{dt^2}M_X(t)]_{t=0} \quad \_\_\_\_\_\_(II)$$

Generalizing (I) and (II) we get,

$$\mu_r' = \left[\frac{d^r}{dt^r}M_X(t)\right]_{t=0} \quad \_\_\_\_\_\_ (B)$$

(A) and (B) give the required result.

Examples:

1. A random variable takes values 1 and -1 with probabilities 1/2 each. Find its moment generating function

Solution: We have the probability distribution of X to be given by

X	-1	1
P(X=x)	1/2	1/2

We have the moment generating function

$$M_X(t) = E(e^{tX})$$
$$= \sum_x e^{tx} p_x$$
$$= e^{t(-1)}\left(\frac{1}{2}\right) + e^{t(+1)}\left(\frac{1}{2}\right)$$
$$= \frac{e^{-t} + e^{t}}{2}$$
$$\Rightarrow M_X(t) = \cosh t$$

So what this means is that if you were to write the expansion of cosh t, the coefficients of $\frac{t^r}{r!}$ *would give you all the moments of the above probability distribution.*

2. A r.v. *X* takes values 0 and 1 with probabilities q and p respectively with q+p=1. Find the mgf of *X* and show that all the moments about the origin equal p. (Bernoulli distribution)

Solution: We have the probability distribution of X to be given by:

X	0	1
P(X=x)	q	p

Therefore the mgf of X is given by:

$$M_X(t) = E(e^{tX})$$
$$= \sum_x e^{tx} p_x$$
$$= \sum_{x=0}^{1} e^{tx} p_x$$
$$= e^{t*0} q + e^{t*1} p$$
$$i.e. M_X(t) = q + pe^t$$

We have the r^th raw moment to be given by

$$\mu_r' = \begin{cases} \text{coefficient of } \dfrac{t^r}{r!} \text{ in the expansion of } M_X(t) \\ \dfrac{d^r}{dt^r}\left[M_X(t)\right]_{t=0} \end{cases}, \quad r = 1, 2, \ldots$$

Now

$$M_X(t) = q + pe^t$$
$$\Rightarrow M_X(t) = q + p\left\{1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots + \frac{t^r}{r!} + \ldots\right\} \ldots\ldots(1)$$

$$\Rightarrow \mu_r' = \text{coefficient of } \frac{t^r}{r!} \text{ in } M_X(t)$$
$$\Rightarrow \mu_r' = p \ (\text{from } (1))$$

Hence all the moments about the origin equal *p*.

3. A r.v. *X* has the probability distribution $P(X = x) = \frac{1}{8} 3C_X,\ x = 0, 1, 2, 3$ Find the mgf of X and hence find the mean and variance.

Solution: We have the probability distribution of X to be given by:

X	0	1	2	3
P(X=x)	$(1/8)3C_0 = 1/8$	$(1/8)3C_1 = 3/8$	$(1/8)3C_2 = 3/8$	$(1/8)3C_3 = 1/8$

Therefore the mgf of X is given by:

$$M_X(t) = E(e^{tX})$$
$$= \sum_x e^{tx} p_x$$
$$= \sum_{x=0}^{3} e^{tx} p_x$$

$$\Rightarrow M_X(t) = e^{t*0} * \frac{1}{8} + e^{t*1} * \frac{3}{8} + e^{t*2} * \frac{3}{8} + e^{t*3} * \frac{1}{8}$$
$$i.e. M_X(t) = \frac{1}{8}\left(1 + 3e^{t} + 3e^{2t} + e^{3t}\right)$$

We have the rth raw moment to be given by

$$\mu_r' = \begin{cases} coefficient\ of\ \ \dfrac{t^r}{r!}\ in\ the\ \exp ansion\ of\ \ M_X(t) \\ \dfrac{d^r}{dt^r}\left[M_X(t)\right]_{t=0} \end{cases}, \quad r = 1, 2, \ldots$$

Now

$$M_X(t) = \frac{1}{8}\left(1 + 3e^{t} + 3e^{2t} + e^{3t}\right) \ldots\ldots(1)$$

$$\Rightarrow \mu_r' = \frac{d^r}{dt^r}\left[M_X(t)\right]_{t=0}$$
$$\Rightarrow Mean = \mu_1' = \frac{d}{dt}\left[\frac{1}{8}\left(1 + 3e^{t} + 3e^{2t} + e^{3t}\right)\right]_{t=0}$$
$$i.e\ \mu_1' = \frac{1}{8}\left(0 + 3e^{t} + 6e^{2t} + 3e^{3t}\right)_{t=0}$$
$$\Rightarrow \mu_1' = \frac{1}{8}(3 + 6 + 3) = \frac{12}{8}$$
$$i.e\ \mu_1' = \frac{3}{2}$$

Again,

$$E(X^2)=\mu_2' = \frac{d^2}{dt^2}\left[M_X(t)\right]_{t=0}$$

$$i.e\ \mu_2' = \frac{d^2}{dt^2}\left[\frac{1}{8}\left(1+3e^t+3e^{2t}+e^{3t}\right)\right]_{t=0}$$

$$=\frac{d}{dt}\left[\frac{1}{8}\left(3e^t+6e^{2t}+3e^{3t}\right)\right]_{t=0}$$

$$i.e\ \mu_2' = \frac{1}{8}\left(3e^t+12e^{2t}+9e^{3t}\right)_{t=0}$$

$$\Rightarrow \mu_1' = \frac{1}{8}(3+12+9) = \frac{24}{8}$$

$$i.e\ \mu_2' = 3$$

$$\therefore Variance = \mu_2 = \mu_2' - \left(\mu_1'\right)^2 = 3-\frac{9}{4} \Rightarrow Variance = \frac{3}{4}$$

4. Suppose a r.v. X has the mgf: $M_X(t) = \frac{3}{3-t}$. Obtain the mean and standard deviation of X.

Solution: We have the r[th] raw moment to be given by

$$\mu_r' = \begin{cases} coefficient\ of\ \frac{t^r}{r!}\ in\ the\ expansion\ of\ M_X(t) \\ \frac{d^r}{dt^r}\left[M_X(t)\right]_{t=0} \end{cases},\ r=1,2,\ldots$$

Now $M_X(t) = \frac{3}{3-t}$ ....(1)

$$\Rightarrow \mu_r' = \frac{d^r}{dt^r}\left[M_X(t)\right]_{t=0}$$

$$\Rightarrow Mean = \mu_1' = \frac{d}{dt}\left[\frac{3}{3-t}\right]_{t=0}$$

$$i.e\ \mu_1' = \left(\frac{-3(-1)}{(3-t)^2}\right)_{t=0}$$

$$\Rightarrow Mean = \mu_1' = \frac{1}{3}$$

Again,

$$E(X^2) = \mu_2' = \frac{d^2}{dt^2}\left[M_X(t)\right]_{t=0}$$

$$i.e\ \mu_2' = \frac{d}{dt}\left[\frac{3}{(3-t)^2}\right]_{t=0}$$

$$i.e\ \mu_2' = \left[\frac{-3(2(3-t)(-1))}{(3-t)^4}\right]_{t=0}$$

$$\Rightarrow \mu_2' = \left[\frac{2}{9}\right]$$

Therefore,

$$Variance = \mu_2 = \mu_2' - \left(\mu_1'\right)^2 = \frac{2}{9} - \left(\frac{1}{3}\right)^2$$

$$\Rightarrow Variance = \frac{1}{9}$$

$$\Rightarrow S\tan dard\ deviation = \sqrt{\frac{1}{9}} = \frac{1}{3}$$