- **Lattices:**

Let L be a non-empty set closed under two binary operations called meet and join, denoted by $\wedge$ and $\vee$. Then L is called a lattice if the following axioms hold where a, b, c are elements in L:

1) Commutative Law: -

(a) $a \wedge b = b \wedge a$ (b) $a \vee b = b \vee a$

2) Associative Law:-

(a) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (b) $(a \vee b) \vee c = a \vee (b \vee c)$

3) Absorption Law: -

(a) $a \wedge (a \vee b) = a$ (b) $a \vee (a \wedge b) = a$

- **Duality:**

The dual of any statement in a lattice $(L, \wedge, \vee)$ is defined to be a statement that is obtained by interchanging $\wedge$ an $\vee$.

For example, the dual of $a \wedge (b \vee a) = a \vee a$ is $a \vee (b \wedge a) = a \wedge a$

Bounded Lattices:

A lattice L is called a bounded lattice if it has greatest element 1 and a least element 0.

Example:

1. The power set P(S) of the set S under the operations of intersection and union is a bounded lattice since $\emptyset$ is the least element of P(S) and the set S is the greatest element of P(S).
2. The set of +ve integer I_+ under the usual order of $\leq$ is not a bounded lattice since it has a least element 1 but the greatest element does not exist.

Properties of Bounded Lattices:

If L is a bounded lattice, then for any element $a \in L$, we have the following identities:

1. $a \vee 1 = 1$
2. $a \wedge 1 = a$
3. $a \vee 0 = a$
4. $a \wedge 0 = 0$

Theorem: Prove that every finite lattice $L = \{a_1, a_2, a_3 \ldots a_n\}$ is bounded.

Proof: We have given the finite lattice:

$L = \{a_1, a_2, a_3 \ldots a_n\}$

Thus, the greatest element of Lattices L is $a_1 \vee a_2 \vee a_3 \vee \ldots \vee a_n$.

Also, the least element of lattice L is $a_1 \wedge a_2 \wedge a_3 \wedge \ldots \wedge a_n$.

Since, the greatest and least elements exist for every finite lattice. Hence, L is bounded.

- ## Sub-Lattices:

Consider a non-empty subset L_1 of a lattice L. Then L_1 is called a sub-lattice of L if L_1 itself is a lattice i.e., the operation of L i.e., $a \vee b \in L_1$ and $a \wedge b \in L_1$ whenever $a \in L_1$ and $b \in L_1$.

Example: Consider the lattice of all +ve integers I_+ under the operation of divisibility. The lattice D_n of all divisors of $n > 1$ is a sub-lattice of I_+.

Determine all the sub-lattices of D_{30} that contain at least four elements, $D_{30}=\{1,2,3,5,6,10,15,30\}$.

Solution: The sub-lattices of D_{30} that contain at least four elements are as follows:

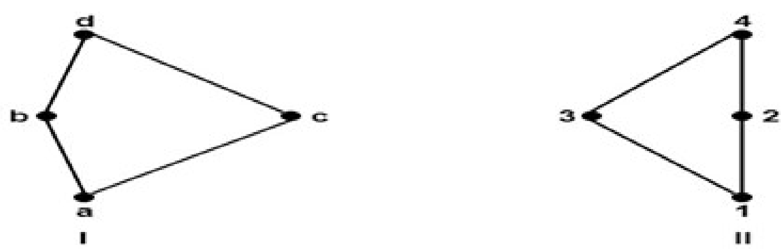
1. {1, 2, 6, 30}
2. {1, 2, 3, 30}
3. {1, 5, 15, 30}
4. {1, 3, 6, 30}
5. {1, 5, 10, 30}
6. {1, 3, 15, 30}
7. {2, 6, 10, 30}

- ## Isomorphic Lattices:

Two lattices L_1 and L_2 are called isomorphic lattices if there is a bijection from L_1 to L_2 i.e., f: $L_1 \longrightarrow L_2$, such that $f(a \wedge b) = f(a) \wedge f(b)$ and $f(a \vee b) = f(a) \vee f(b)$

Example: Determine whether the lattices shown in fig are isomorphic.

Solution: The lattices shown in fig are isomorphic. Consider the mapping f = {(a, 1), (b, 2), (c, 3), (d, 4)}.For example $f(b \wedge c) = f(a) = 1$. Also, we have $f(b) \wedge f(c) = 2 \wedge 3 = 1$



- ## Distributive Lattice:

A lattice L is called distributive lattice if for any elements a, b and c of L,it satisfies following distributive properties:

1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

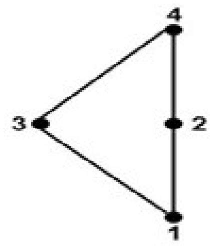
If the lattice L does not satisfies the above properties, it is called a non-distributive lattice.

Example:

1. The power set P (S) of the set S under the operation of intersection and union is a distributive function. Since,

   $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$

   and, also $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$ for any sets a, b and c of P(S).
2. The lattice shown in fig II is a distributive. Since, it satisfies the distributive properties for all ordered triples which are taken from 1, 2, 3, and 4.



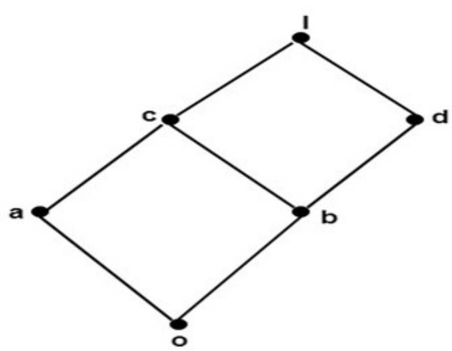
II

- **Complements and complemented lattices:**

Let L be a bounded lattice with lower bound o and upper bound I. Let a be an element if L. An element x in L is called a complement of a if $a \vee x = I$ and $a \wedge x = 0$

A lattice L is said to be complemented if L is bounded and every element in L has a complement.

Example: Determine the complement of a and c in fig:



Solution: The complement of a is d. Since, $a \vee d = 1$ and $a \wedge d = 0$

The complement of c does not exist. Since, there does not exist any element c such that $c \vee c'=1$ and $c \wedge c'= 0$.

Modular Lattice:

A lattice $(L, \wedge,\vee)$ is called a modular lattice if $a \vee (b \wedge c) = (a \vee b) \wedge c$ whenever $a \leq c$.

Direct Product of Lattices:

Let (L_1 $\vee_1$ $\wedge_1$)and (L_2 $\vee_2$ $\wedge_2$) be two lattices. Then (L, ∧,∨) is the direct product of lattices, where L = L_1 x L_2 in which the binary operation ∨(join) and ∧(meet) on L are such that for any (a_1,b_1)and (a_2,b_2) in L.

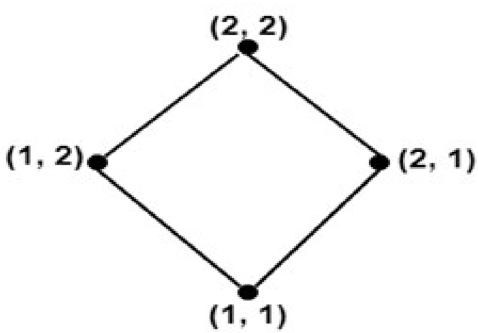
$(a_1,b_1)\vee(a_2,b_2)=(a_1 \vee_1 a_2,b_1 \vee_2 b_2)$

and $(a_1,b_1) \wedge (a_2,b_2)=(a_1 \wedge_1 a_2,b_1 \wedge_2 b_2)$.

Example: Consider a lattice (L, ≤) as shown in fig. where L = {1, 2}. Determine the lattices (L^2, ≤), where L^2=L x L.

(L, <=)

Solution: The lattice (L^2, ≤) is shown in fig:



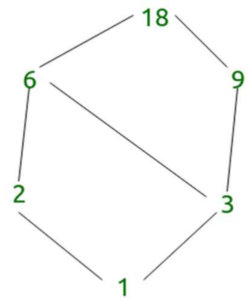
(L_2, <=)

Types of Lattice:-

1. Bounded Lattice:

A lattice L is said to be bounded if it has the greatest element I and a least element 0.

E.g. – D_{18}= {1, 2, 3, 6, 9, 18} is a bounded lattice.



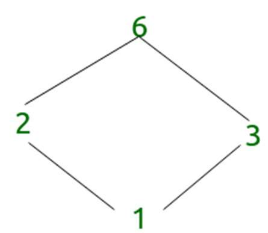
Hasse Diagram of D_{18}

Note: Every Finite lattice is always bounded.

2. Complemented Lattice:

A lattice L is said to be complemented if it is bounded and if every element in L has a complement. Here, each element should have at least one complement.

E.g. – D_6 {1, 2, 3, 6} is a complemented lattice.



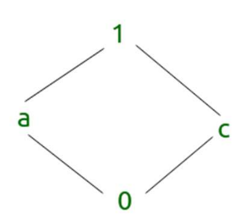
Hasse Diagram of D_6

In the above diagram every element has a complement.

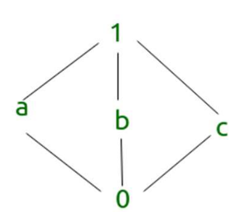
3.Distributive Lattice:

If a lattice satisfies the following two distribute properties, it is called a distributive lattice.

- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
- $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$



A distributive lattice



A non-distributive lattice

N_5 M_3

Both are diamond and pentagon are non-distributive lattice.

- A complemented distributive lattice is a boolean algebra or boolean lattice.
- A lattice is distributive if and only if none of its sublattices is isomorphic to N_5 or M_3.
- For distributive lattice each element has unique complement. This can be used as a theorem to prove that a lattice is not distributive.

4.Modular Lattice

If a lattice satisfies the following property, it is called a modular lattice. a^(b∨(a^d)) = (a^b)(a^d).

Example-