1(A). Find the Fourier series for $f(x)=\begin{cases} x+\dfrac{\pi}{2}, & -\pi<x<0 \\ \dfrac{\pi}{2}-x, & 0<x<\pi \end{cases}$.

Hence deduce that $\dfrac{\pi^2}{8}=\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\cdots$

Solution: Check whether f(x) is even or odd

Analytical Method:

Let $f(x)=\begin{cases} \frac{\pi}{2}+x & -\pi<x<0 \\ \frac{\pi}{2}-x & 0<x<\pi \end{cases}$

Replace $-x \to x$

$$f(-x)=\begin{cases} \frac{\pi}{2}-x & -\pi<-x<0 \\ \frac{\pi}{2}+x & 0<-x<\pi \end{cases}$$

$$=\begin{cases} \frac{\pi}{2}-x & \pi>x>0 \\ \frac{\pi}{2}+x & 0>x>-\pi \end{cases}$$

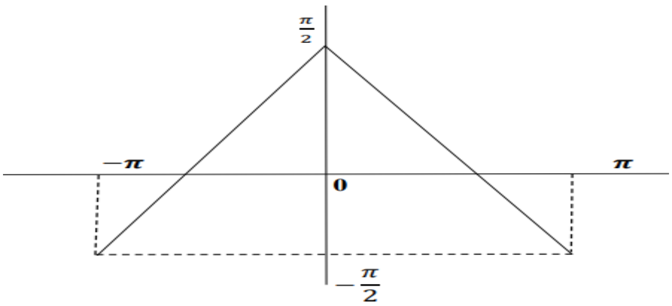
$$=\begin{cases} \frac{\pi}{2}-x & 0<x<\pi \\ \frac{\pi}{2}+x & -\pi<x<0 \end{cases}$$

$$=\begin{cases} \frac{\pi}{2}+x & -\pi<x<0 \\ \frac{\pi}{2}-x & 0<x<\pi \end{cases}$$

$= f(x)$

∴ f(x) is even function.

Graphical Method:



Since graph is symmetric about is y-axis, thus f(x) is even

$\therefore b_n = 0$

Now, find the remaining Fourier coefficient a_0, a_n

$a_0 = \frac{2}{a}\int_0^a f(x)dx = \frac{2}{\pi}\int_0^\pi f(x)dx$

$a_n = \frac{2}{a}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)dx = \frac{2}{\pi}\int_0^\pi f(x)\cos\left(\frac{n\pi x}{a}\right)dx = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx$

Fourier series of f(x) as even function in $(-a, a)$ is

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{a}\right)$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$

$$a_0 = \frac{2}{\pi}\int_0^\pi \left(\frac{\pi}{2} - x\right)dx$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2}x - \frac{x^2}{2}\right]_0^\pi$$

$$= \frac{2}{\pi}\left\{\left[\frac{\pi}{2}\pi - \frac{\pi^2}{2}\right] - \left[\frac{\pi}{2}\times 0 - \frac{0^2}{2}\right]\right\} = 0$$

$$a_n = \frac{2}{\pi}\int_0^\pi \left(\frac{\pi}{2} - x\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2} - x\right)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left\{\begin{array}{l}\left[\left(\frac{\pi}{2} - \pi\right)\left(\frac{\sin n\pi}{n}\right) - (-1)\left(-\frac{\cos n\pi}{n^2}\right)\right] \\ -\left[\left(\frac{\pi}{2} - 0\right)\left(\frac{\sin 0}{n}\right) - (-1)\left(-\frac{\cos 0}{n^2}\right)\right]\end{array}\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{(-1)^n}{n^2} + \frac{1}{n^2}\right\}$$

$$a_n = \frac{2}{\pi}\left\{\frac{1-(-1)^n}{n^2}\right\}$$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$

$\therefore f(x) = \sum_{n=1}^{\infty} \frac{1}{\pi}\left\{\frac{1-(-1)^n}{n^2}\right\}\cos(nx)$

Hence, deduce that (i) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Proof: Let $f(x) = \sum_{n=1}^{\infty} \frac{1}{\pi}\left\{\frac{1-(-1)^n}{n^2}\right\}\cos(nx)$

Put $x = 0$

$f(0) = \sum_{n=1}^{\infty} \frac{2}{\pi}\left\{\frac{1-(-1)^n}{n^2}\right\}\cos(0)$

$$f(0) = \frac{2}{\pi}\left\{\frac{1-(-1)^1}{1^2} + \frac{1-(-1)^2}{2^2} + \frac{1-(-1)^3}{3^2} + \frac{1-(-1)^4}{4^2} + \frac{1-(-1)^5}{5^2} + \cdots\right\}$$

$f(0) = \frac{2}{\pi}\left\{\frac{2}{1^2} + \frac{0}{2^2} + \frac{2}{3^2} + \frac{0}{4^2} + \frac{2}{5^2} + \frac{0}{6^2} + \cdots\right\}$

$\frac{\pi}{4}f(0) = \left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right\}$

Now find $f(0)$

Let $f(x) = \begin{cases} \frac{\pi}{2} + x & -\pi < x < 0 \\ \frac{\pi}{2} - x & 0 < x < \pi \end{cases}$

Put $x = 0$

$$f(0) = \frac{1}{2}\begin{cases} \frac{\pi}{2} + 0 & -\pi < x < 0 \\ \frac{\pi}{2} - 0 & 0 < x < \pi \end{cases}$$

$$f(0) = \frac{\pi}{2}$$

$$\therefore \frac{\pi}{4} \; \frac{\pi}{2} = \left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right\}$$

$$\therefore \left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right\} = \frac{\pi^2}{8}$$

1(A). Expand $f(x) = lx - x^2,\ 0 < x < l$ in a half range sine series. Hence deduce that

$$\frac{\pi^6}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \cdots$$

Solution:

Now, find half range sine coefficient (a_0, a_n, b_n)

$$a_0 = 0$$
$$a_n = 0$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{2n\pi x}{l}\right)dx$$

Half range sine series of f(x) in $(0, l)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{2}{l}\int_0^l (lx - x^2)\sin\left(\frac{\pi n x}{l}\right)dx$$

Using $\int U \cdot V dx = U\int V dx - U'\int V_1 dx + U''\int V_2 dx - \cdots$

$$= \frac{2}{l}\left[(lx - x^2)\left(\frac{-\cos\left(\frac{\pi n x}{l}\right)}{\left(\frac{\pi n}{l}\right)}\right) - [l - 2x]\left(-\frac{\sin\left(\frac{\pi n x}{l}\right)}{\left(\frac{\pi n}{l}\right)^2}\right) + [-2]\left(\frac{\cos\left(\frac{\pi n x}{l}\right)}{\left(\frac{\pi n}{l}\right)^3}\right)\right]_0^l$$

Substituting the limits

$$a_n = \frac{2}{l}\left\{\left[0 - 0 + [-2]\left(\frac{(-1)^n}{\left(\frac{\pi n}{l}\right)^3}\right)\right] - \left[0 - 0 + [-2]\left(\frac{1}{\left(\frac{\pi n}{l}\right)^3}\right)\right]\right\}$$

$$= -2\,\frac{2l^3}{l}\left\{\frac{(-1)^n - 1}{n^3\pi^3}\right\}$$

$$a_n = -4l^2\left\{\frac{(-1)^n - 1}{n^3\pi^3}\right\}$$

Substituting values of a_0, a_n and b_n in half range sine series

$$\therefore lx - x^2 = \sum_{n=1}^{\infty} -4l^2\left\{\frac{(-1)^n - 1}{n^2\pi^2}\right\}\sin\left(\frac{n\pi x}{l}\right)$$

By using Parseval's Identity

$\frac{2}{l}\int_0^l \left(\mathrm{f(x)}\right)^2 \mathrm{dx} = \sum_{\mathrm{n=1}}^{\infty} \mathrm{b_n^2}$

$\frac{2}{l}\int_0^l (lx - x^2)^2 \mathrm{dx} = \sum_{\mathrm{n=1}}^{\infty} \left(-4l^2\left\{\frac{(-1)^n-1}{n^3\pi^3}\right\}\right)^2$

$$\frac{2}{l}\int_0^l (l^2x^2 - 2lx^3 + x^4)\mathrm{dx} = \sum_{\mathrm{n=1}}^{\infty}\left(-4l^2\left\{\frac{(-1)^n-1}{n^3\pi^3}\right\}\right)^2$$

$\frac{2}{l}\left[\frac{l^2x^3}{3} - \frac{2lx^4}{4} + \frac{x^5}{5}\right]_0^l = \frac{16l^4}{\pi^6}\sum_{\mathrm{n=1}}^{\infty}\left(\left\{\frac{(-1)^n-1}{n^3}\right\}\right)^2$

$\frac{2}{l}\left\{\left[\frac{l^5}{3} - \frac{2l^5}{4} + \frac{l^5}{5}\right] - [0]\right\} = \frac{16l^4}{\pi^6}\left\{\left(\left\{\frac{-2}{1^3}\right\}\right)^2 + \left(\left\{\frac{0}{2^3}\right\}\right)^2 + \left(\left\{\frac{-2}{3^3}\right\}\right)^2\right\}$

$\frac{2}{l}\left[\frac{l^5}{30}\right] = \frac{64l^4}{\pi^6}\left\{\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \cdots\right\}$

$\frac{\pi^6}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \cdots$

1(B). Find Fourier series of $f(x) = x^2, -\pi < x < \pi$. Hence deduce that

$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$

Solution: Check whether f(x) is even or odd

Analytical Method:

Let $\mathrm{f(x)} = \mathrm{x}^2$

$\mathrm{f(-x)} = (-x)^2$

$= (x)^2$

$= \mathrm{f(x)}$

Therefore f(x) is even function

$\therefore \mathrm{b_n} = 0$

Now, find the remaining Fourier coefficient $\mathrm{a_0}, \mathrm{a_n}$

$\mathrm{a_0} = \frac{2}{\mathrm{a}}\int_0^{\mathrm{a}} \mathrm{f(x)dx} = \frac{2}{\pi}\int_0^{\pi} \mathrm{f(x)dx}$

$\mathrm{a_n} = \frac{2}{\mathrm{a}}\int_0^{\mathrm{a}} \mathrm{f(x)}\cos\left(\frac{\mathrm{n\pi x}}{\mathrm{a}}\right)\mathrm{dx} = \frac{2}{\pi}\int_0^{\pi} \mathrm{f(x)}\cos\left(\frac{\mathrm{n\pi x}}{\pi}\right)\mathrm{dx}$

Fourier series of f(x) as even function in $(-\mathrm{a}, \mathrm{a})$ is

$\mathrm{f(x)} = \frac{\mathrm{a_0}}{2} + \sum_{\mathrm{n=1}}^{\infty} \mathrm{a_n}\cos(nx)$

$\mathrm{a_0} = \frac{2}{\pi}\int_0^{\pi} x^2 \ \mathrm{dx}$

$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$

Substituting the limits

$$\therefore a_0 = \frac{2}{3\pi}[(\pi)^3 - (0)^3]$$

$$\therefore a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos(nx)dx$$

Using $\int U \cdot Vdx = U\int Vdx - U'\int V_1 dx + U''\int V_2 dx - \cdots$

$$= \frac{2}{\pi}\left[(x)^2\left(\frac{\sin nx}{n}\right) - [2x]\left(\frac{-\cos nx}{n^2}\right) + [2]\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

Substituting the limits

$\because \sin n\pi = \sin(0) = 0$

$$a_n = \frac{2}{\pi}\left[-(2x)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(-2\pi)\left(\frac{-\cos n\pi}{n^2}\right) - (0)\left(\frac{-\cos(0)}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[2\pi\left(\frac{(-1)^n}{n^2}\right)\right]$$

$$a_n = \frac{4(-1)^n}{n^2}$$

Substituting values of a_0, a_n and b_n in Fourier series

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx)$$

Put $x = \pi$

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{-4(-1)^n(-1)^n}{n^2}$$

Now Expand

$$\pi^2 - \frac{\pi^2}{3} = 4\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right\}$$

$$\frac{2\pi^2}{3} = 4\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right\}$$

$$\frac{\pi^2}{6} = \left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right\}$$

1(B). Find half range cosine series of $f(x) = e^x, \;\; 0 < x < 1$

Solution:

Now, find Half range cosine coefficient (a_0, a_n, b_n)

$$a_0 = \frac{2}{a}\int_0^a f(x)dx$$

$$a_n = \frac{2}{a}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)dx$$

$$b_n = 0$$

Half range cosine series of f(x) in $(0, a)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{a}\right)$$

$$a_0 = \frac{2}{a}\int_0^a f(x)dx$$

$= \frac{2}{1}\int_0^1 e^x dx$

$= 2[e^x]_0^1$

Substituting the limits

$\therefore a_0 = 2[e^1 - 1]$

$a_n = \frac{2}{1}\int_0^1 (e^x)\cos(n\pi x)\,dx$

Using $\int U \cdot V dx = U\int V dx - U'\int V_1 dx + U''\int V_2 dx - \cdots$

$= 2\left[\frac{(e^x)\{\cos(n\pi x) + n\pi\sin(n\pi x)\}}{1+(n\pi)^2}\right]_0^1$

Substituting the limits

$a_n = 2\left\{\left[\frac{(e^1)\{\cos(n\pi) + n\pi\sin(n\pi)\}}{1+(n\pi)^2}\right] - \left[\frac{(e^0)\{\cos(0) + n\pi\sin(0)\}}{1+(n\pi)^2}\right]\right\}$

$= 2\left\{\left[\frac{(e^1)\{(-1)^n\}}{1+(n\pi)^2}\right] - \left[\frac{1}{1+(n\pi)^2}\right]\right\}$

$a_n = 2\left[\frac{(-1)^n e - 1}{1+(n\pi)^2}\right]$

$b_n = 0$

Substituting values of a_0, a_n and b_n in half range cosine series

$e^x = \frac{2[e^1-1]}{2} + \sum_{n=1}^{\infty} 2\left[\frac{(-1)^n e - 1}{1+(n\pi)^2}\right]\cos(n\pi x)$

$e^x = (e-1) + \sum_{n=1}^{\infty} 2\left[\frac{(-1)^n e - 1}{1+(n\pi)^2}\right]\cos(n\pi x)$

2(A). If imaginary part of the analytic function is $v = x^2 - y^2 + \frac{x}{x^2+y^2}$ Then find real part.

Solution:

Let $v = x^2 - y^2 + \frac{x}{x^2+y^2}$

Step:1 Partially differentiating $w.r.t.x.$ put $x = z$ and $y = 0$

$v_x = \frac{\partial v}{\partial x} = 2x - 0 + \frac{(x^2+y^2)1 - x(2x)}{(x^2+y^2)^2}$

$= 2z - 0 + \frac{(z^2+0)1 - z(2z)}{(z^2+0)^2}$

$v_x = 2z - \frac{1}{z^2}$

Step:2 Partially differentiating $w.r.t.x.$ put $x = z$ and $y = 0$

$v_y = \frac{\partial v}{\partial y} = 2y + x\left(\frac{-2y}{(x^2+y^2)^2}\right)$

$= 0$

$v_y = 0$

Step:3 Using CR equations replace $v_y = u_x$

$\therefore u_x = 0$

Step:4 To find value of from step: -1 and step: -3

$u_x = 0$and $\mathrm{v_x} = 2\mathrm{z} - \frac{1}{\mathrm{z}^2}$

Step: -5 Put value of $\mathrm{u_x}$ & $\mathrm{v_x}$ in $f(z) = \int(\mathrm{u_x} + \mathrm{i}\mathrm{v_x})\mathrm{dz}$ and integrate.

$\therefore\ f(z) = \int i\left(2\mathrm{z} - \frac{1}{\mathrm{z}^2}\right)\mathrm{dz}$

$f(z) = i\left(z^2 + \frac{1}{z}\right)$

Put $z = x + iy$ and separate real and imaginary part

$f(x+iy) = i\left((x+iy)^2 + \frac{1}{x+iy}\right)$

$= i\left(x^2 + 2xiy - y^2 + \frac{x-iy}{x^2+y^2}\right)$

$= ix^2 - 2xy - iy^2 + \frac{ix+y}{x^2+y^2}$

$= \left(-2xy + \frac{y}{x^2+y^2}\right) + i\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right)$

$\therefore u = \left(-2xy + \frac{y}{x^2+y^2}\right)$

2(B). Find an analytic function $f(z) = u + iv$, where $u + v = e^x(\cos y + \sin y)$.

Solution:

Let $u + v = e^x(\cos y + \sin y)$.

Step:1 Partially differentiating $w.r.t.x.$ put $x = z\ and\ y = 0$

$u_x + v_x = e^x(\cos y + \sin y)$.

$u_x + v_x = e^z \ldots (i)$

Step:2 Partially differentiating $w.r.t.x.$ put $x = z\ and\ y = 0$

$u_y + v_y = e^x(-\sin y + \cos y)$.

$u_y + v_y = e^z \ldots (ii)$

Step:3 Using CR equations replace $\mathrm{u_y} = -v_\mathrm{x}$& $\mathrm{v_y} = u_\mathrm{x}$ in (ii)

$\therefore -v_x + u_x = e^z \ldots (iii)$

Step:4 To find value of from step: -1 and step: -3

$u_x = \mathrm{e^z}$ and $\mathrm{v_x} = 0$

Step: -5 Put value of $\mathrm{u_x}$ & $\mathrm{v_x}$ in $f(z) = \int(\mathrm{u_x} + \mathrm{i}\mathrm{v_x})\mathrm{dz}$ and integrate.

$\therefore\ f(z) = \int e^z\mathrm{dz}$

$f(z) = e^z + c$

2(B). Construct an analytic function whose real part is $\frac{\sin 2x}{\cosh 2y + \cos 2x}$.

Solution:

Let $u = \frac{\sin 2x}{\cosh 2y + \cos 2x}$

Step:1 Partially differentiating $w.r.t.x.$ put $x = z\ and\ y = 0$

$$u_x = \frac{\partial u}{\partial x} = \frac{(\cosh 2y+\cos 2x)(2\cos 2x)-(\sin 2x)(-2\sin 2x)}{(\cosh 2y+\cos 2x)^2}$$

$$= \frac{(\cos 0+\cos 2z)(2\cos 2z)-(\sin 2z)(-2\sin 2z)}{(\cos 0+\cos 2z)^2}$$
$$= \frac{(1+\cos 2z)(2\cos 2z)-(\sin 2z)(-2\sin 2z)}{(1+\cos 2z)^2}$$
$$= \frac{2\cos 2z+2\cos^2 2z+2\sin^2 2z}{(1+\cos 2z)^2}$$
$$= 2\left\{\frac{\cos 2z+\cos^2 2z+\sin^2 2z}{(1+\cos 2z)^2}\right\}$$
$$= 2\left\{\frac{\cos 2z+1}{(1+\cos 2z)^2}\right\}$$
$$= \left\{\frac{2}{1+\cos 2z}\right\}$$
$$= \frac{2}{2\cos^2 z}$$
$$u_x = \sec^2 z$$

Step:2 Partially differentiating $w.r.t.x.$ put $x = z\ and\ y = 0$

$$u_y = \frac{\partial u}{\partial y} = \frac{(\cosh 2y+\cos 2x)(0)-(\sin 2x)(2\sinh 2y)}{(\cosh 2y+\cos 2x)^2}$$
$$= \frac{0-(\sin 2z)(2\sin 0)}{(\cos 0+\cos 2z)^2}$$
$$u_y = 0$$

Step:3 Using CR equations replace $u_y = -v_x$

$\therefore -v_x = 0$

$\therefore v_x = 0$

Step:4 To find value of from step: -1 and step: -3

$u_x = \sec^2 z$ and $v_x = 0$

Step: -5 Put value of u_x & v_x in $f(z) = \int (u_x + iv_x)dz$ and integrate.

$\therefore\ f(z) = \int \left(\sec^2 z + (0)\right)dz$

$\therefore\ f(z) = \tan z + c$

2(B). Determine the constants a, b, c, d, e if

$f(z) = ax^3 + bxy^2 + 3x^2 + cy^2 + x + i(dx^2y - 2y^3 + exy + y)$ is analytic.

Solution:

Let $f(z) = (ax^3 + bxy^2 + 3x^2 + cy^2 + x) + i(dx^2y - 2y^3 + exy + y)$

$\therefore u = (ax^3 + bxy^2 + 3x^2 + cy^2 + x)\ \ \&\ \ v = (dx^2y - 2y^3 + exy + y)$

To find $u_x = \frac{\partial u}{\partial x} = 3ax^2 + by^2 + 6x + 1$

$u_y = \frac{\partial u}{\partial y} = 2bxy + 2cy$

$v_x = \frac{\partial v}{\partial x} = 2dxy + ey$

$v_y = \frac{\partial v}{\partial y} = dx^2 - 6y^2 + ex + 1$

By using Cauchy -Reimann Theorem

$u_x = v_y$
$3ax^2 + by^2 + 6x + 1 = dx^2 - 6y^2 + ex + 1$
Compare coefficients
$a = d,$
$b = -6,$
$6 = e$
Now
$u_y = -v_x$
$2bxy + 2cy = 2dxy + ey$
Compare coefficients
$2b = 2d$
$2c = e$
$\therefore a = -6, b = -6, c = 3, d = -6, e = 6$

3(A). Given the following probability function of a discrete random variable X

X	0	1	2	3	4	5	6	7
$P(X = x)$	0	c	$2c$	$2c$	$3c$	c^2	$2c^2$	$7c^2 + c$

Find (i) c (ii) $P(X \geq 6)$ (iii) $P(X < 6)$
(iv) $P(1.5 < X < 4.5/X > 2)$.

Solution:

i. To find K

$\sum p(x) = 1$

$0 + c + 2c + 2c + 3c + c^2 + 2c^2 + 7c^2 + c = 1$

$10c^2 + 9c - 1 = 0$

$c = \left\{0.1 = \frac{1}{10}\right\}$

ii. $P(X \geq 6) = p(6) + p(7) = 2c^2 + 7c^2 + c = \frac{2}{100} + \frac{7}{100} + \frac{1}{10} = 0.19$

iii. $P(X < 6) = p(0) + p(1) + p(2) + p(3) + p(4) + p(5)$
$= 0 + c + 2c + 2c + 3c + c^2 = \frac{8}{10} + \frac{1}{100} = 0.81$

iv. $P\left(\frac{1.5<X<4.5}{x>2}\right) = ?$

We know that $p\left(\frac{A}{B}\right) = \frac{P(A\cap B)}{P(B)}$

Lert $A = \{1.5 < X < 4.5\} = \{2, 3, 4\}$

$B = \{x > 2\} = \{3, 4, 5, 6, 7\}$

$A \cap B = \{3, 4\}$

$P\left(\frac{1.5<X<4.5}{x>2}\right)$

$= \frac{p(3)+p(4)}{p(3)+p(4)+p(5)+p(6)+p(7)}$

$= \frac{2c+3c}{2c+3c+c^2+2c^2+7c^2+c}$

$= \frac{0.5}{0.6+0.1} = \frac{5}{7}$

3(A). Suppose thar in a certain region the daily rainfall (in inches) is a continuous random variable X with probability density function $f(x) = \begin{cases} \frac{3}{4}(2x - x^2)\,, & 0 < x < 2 \\ 0 & , \text{ elsewhere} \end{cases}$. Find the probability that on a given day in this region, the rainfall is

(i) not more than 1 inch

(ii) greater than 1.5 inches

(iii) between 1 and 1.5 inches

Solution:

i. The probability that on a given day in this region, the rainfall is not more than one

$= p(x \le 1)$

$= \int_0^1 f(x)dx$

$= \int_0^1 \frac{3}{4}(2x - x^2)dx$

$= \frac{3}{4}\left[x^2 - \frac{x^3}{3}\right]_0^1$

$= \frac{3}{4}\left\{\left[1 - \frac{1}{3}\right] - [0 - 0]\right\}$

$= \frac{1}{2}$

ii. The probability that on a given day in this region, the rainfall is greater than 1.5

$= p(x \ge 1.5)$

$= \int_{1.5}^2 f(x)dx$

$= \int_{1.5}^2 \frac{3}{4}(2x - x^2)dx$

$= \frac{3}{4}\left[x^2 - \frac{x^3}{3}\right]_{1.5}^2 = \frac{3}{4}\left\{\left[4 - \frac{8}{3}\right] - \left[1.5^2 - \frac{1.5^3}{3}\right]\right\}$

$= \frac{5}{32}$

iii. The probability that on a given day in this region, the rainfall between 1 and 1.5

$= p(x \ge 1.5)$

$= \int_1^{1.5} f(x)dx$

$= \int_1^{1.5} \frac{3}{4}(2x - x^2)dx$

$= \frac{3}{4}\left[x^2 - \frac{x^3}{3}\right]_1^{1.5} = \frac{3}{4}\left\{\left[1.5^2 - \frac{1.5^3}{3}\right] - \left[1 - \frac{1}{3}\right]\right\}$

$= \frac{11}{32}$

3(B). Find k & Expectation & Variance if X has the p.d.f.

$$f(x) = \begin{cases} k\,x(2-x)\,, & 0 < x < 2 \\ 0 & , \text{ elsewhere} \end{cases}$$

Solution:

To find k,

$\int_0^2 f(x)dx = 1$

$\int_0^2 k(2x - x^2)dx = 1$

$k\left[x^2 - \frac{x^3}{3}\right]_0^2 = 1$

$k\left\{\left[2^2 - \frac{8}{3}\right] - [0 - 0]\right\} = 1$

$k\left\{\frac{4}{3}\right\} = 1$

$k = \frac{3}{4}$

To find Expectation $= E(x) = \int_0^2 xf(x)dx$

$= \int_0^2 \frac{3}{4}(2x^2 - x^3)dx$

$= \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{4}\left\{\left[\frac{16}{3} - \frac{16}{4}\right] - [0 - 0]\right\} = 1$

To find Variance $= V(x) = E(x^2) - \{E(x)\}^2$

$= \int_0^2 x^2 f(x)dx - 1^2$

$= \int_0^2 \frac{3}{4}(2x^3 - x^4)dx - 1$

$= \frac{3}{4}\left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2 - 1$

$= \frac{3}{4}\left\{\left[\frac{32}{4} - \frac{32}{5}\right] - [0 - 0]\right\} - 1$

$= \frac{6}{5} - 1$

$= \frac{1}{5}$

3(B). If mean of following distribution is 16, find m, n.

X	8	12	16	20	24
$P(X = x)$	$\frac{1}{8}$	m	n	$\frac{1}{4}$	$\frac{1}{12}$

Solution:

We know that $\sum P(X) = 1$

$\frac{1}{8} + m + n + \frac{1}{4} + \frac{1}{12} = 1$

$m + n = 1 - \left(\frac{1}{8} + \frac{1}{4} + \frac{1}{12}\right) = \frac{13}{24}$

$m + n = \frac{13}{24} \ldots (i)$

Mean= $E(x) = \sum xP(X) = 8\frac{1}{8} + 12m + 16n + 20\frac{1}{4} + 24\frac{1}{12} = 16$

$12m + 16n = 16 - (1 + 5 + 2)$

$12m + 16n = 8 \ldots (ii)$

Solve (i)and (ii)

$\begin{bmatrix} 1 & 1 \\ 12 & 16 \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} \frac{13}{24} \\ 8 \end{bmatrix}$

$m = \frac{1}{6},\ n = \frac{3}{8}$