Q.1

a) By the use of matrices, solve the equations

$x+y+z=9,\ 2x+5y+7z=52,\ 2x+y-z=0.$

<u>Solution</u>: Given that

$$\left.\begin{aligned} x+y+z&=9 \\ 2x+5y+7z&=52 \\ 2x+y-z&=0 \end{aligned}\right\} \text{ —— (1)}$$

We write equation (1) in matrix form as

$$AX=B$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 9 \\ 52 \\ 0 \end{bmatrix} \text{ —— (2)}$$

Consider the augmented matrix:

$$[A|B]=\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \end{array}\right]$$

by $R_2 \to R_2-2R_1,\ R_3 \to R_3-2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & -1 & -3 & -18 \end{array}\right]$$

by $R_2 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -1 & -3 & -18 \\ 0 & 3 & 5 & -20 \end{array}\right]$$

by $R_3 \to R_3+3R_2$

$$[A|B] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & -1 & -3 & -18 \\ 0 & 0 & -4 & -20 \end{array}\right] \text{ —— (3)}$$

This is in Echelon form.

Since rank A = rank$[A|B]$ = 3.

$\therefore$ the given equations are Consistent.

n = Number of variable = 3.

Hence the given equation will have a unique solution.

From (3),

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -18 \\ -20 \end{bmatrix} \quad \text{——— (4)}$$

we write equation (4) in equation form.

we get,

$$x + y + z = 9 \quad \text{——— (i)}$$

$$-y - 3z = -18 \quad \text{——— (ii)}$$

$$-4z = -20 \quad \text{——— (iii)}$$

$$\boxed{z = 5}$$

put $z=5$ in (ii) we get $\boxed{y=3}$

put $y=3$ and $z=5$ in (i)

we get $x + 8 = 9$

$$\boxed{x = 1}$$

Hence $x=1$, $y=3$, $z=5$.

Q. 2

a) Find Singular Value of Decomposition of matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

Solution: Given that $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$.

· Let $A = UDV^T$ be SVD of A.

Step 1: Find V :

Consider the matrix $A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

eigenvalues of A^TA are 2, 3.

Let $\lambda_1 = 3$, $\lambda_2 = 2$

For $\lambda_1 = 3$,

we get corresponding eigenvector. $x_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

For $\lambda_2 = 2$,

we get, corresponding eigenvector. $x_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$\therefore V = \left[\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|} \right] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$V^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ ——— } (*).$$

Step 2: Find D: order of D = order of A.

$\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$, $\sigma_2 = \sqrt{\lambda_2} = \sqrt{2}$.

No. of non zero eigenvalues = Rank = 2.

$$D = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 3: Find U :

Consider the matrix $AA^T = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$\therefore$ eigenvalues of AA^T are $3, 2, 0$.

Let $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 0$

For $\lambda_1 = 3$, we get corresponding eigenvector $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda_2 = 2$, we get $x_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

For $\lambda_3 = 0$, we get $x_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$$U = \begin{bmatrix} | & | & | \\ \frac{x_1}{\|x_1\|} & \frac{x_2}{\|x_2\|} & \frac{x_3}{\|x_3\|} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

$\therefore$ $A = UDV^T$

$$\underset{3\times 2}{\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}} = \underset{3\times 3}{\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}$$

Q. 6.

(a) Four Fundamental Subspaces.

Solution: Explain (a) Null space (b) Column space.
(c) Row space (d) Left Null space.

Q2(b)

A company gave an intensive training to its salesmen to increase the sales. A random sample of 10 salesmen was selected and the value (in lakhs of Rupees) of their sales per month, made before and after the training is recorded in the following table.

Salesman	1	2	3	4	5	6	7	8	9	10
Before	15	22	6	17	12	20	18	14	10	16
After	17	23	16	20	14	21	18	20	10	11

Test whether there is any increase in mean sales at 5% level of significance.

Table Values:t(α,df,test type)

t(0.05,10,one-tailed)=1.812

t(0.05,9,one-tailed)=1.833

t(0.05,10,two-tailed)=2.228

t(0.05,9,two-tailed)=2.262

Sol:

step 1: Null Hypothesis $H_0: \mu_D = 0$

i.e. There is no significant increase in the mean sales after the training

Alternate Hypothesis $H_a: \mu_D > 0$

i.e. There is significant increase in the mean sales after the training. It is a one sided alternate Hypothesis

Step2: Level of significance $\alpha = 5\%$.

step3: Test statistics

Before (x_i)	15	22	6	17	12	20	18	14	10	16
After (y_i)	17	23	16	20	14	21	18	20	10	11
$d_i = y_i - x_i$	2	1	10	3	2	1	0	6	0	-5

$$\bar{d_i} = \frac{\Sigma d_i}{n} = \frac{20}{10} = 2$$

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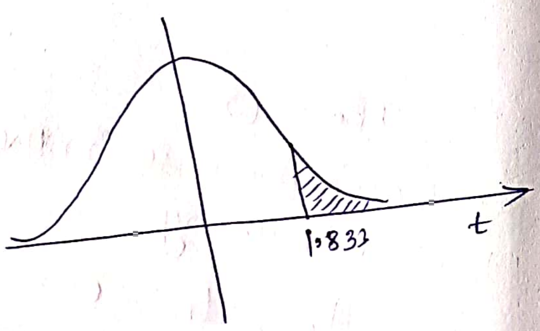
Also $s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (d_i - \bar{d})^2} = 3.94$

$$t = \frac{\bar{d}}{\frac{s}{\sqrt{n}}} = \frac{2}{\frac{3.94}{\sqrt{10}}} = 1.6052$$

Step 4: critical value

The critical value at 5% L.O.S. at $n-1 = 10-1 = 9$ degrees of freedom for one tailed test is 1.833

Step 5: Decision



$\because$ t (computed) = 1.6052 lies in the region of acceptance of H_0

$\therefore$ H_0 is accepted.

$\therefore$ There is no significant evidence that mean sales is increased

Q 3(a)

A survey was conducted with 500 female students of which 60% were intelligent, 40% had uneducated fathers, while 30 % of the not intelligent female students had educated fathers. Test the hypothesis that the education of fathers and intelligence of female students are independent at 5% level of significance. (Given $\chi^2(1,0.05) = 3.841$)

Sol:

	Intelligent female	Non Intelligent female	Row Total
Educated father	180	60	240
Uneducated father	120	140	260
Total	300	200	500

Step 1: Ho: There is no association between education of father & intelligence of female student.

Ha: There is association between education of father & intelligence of female student

Step 2: Level of significance : $\alpha = 5\%$

Step 3: Test statistic

O	E	O−E	$(O-E)^2$	$(O-E)^2/E$
180	$\frac{240 \times 300}{500} = 144$	36	1296	1296/144
60	$\frac{200 \times 240}{500} = 96$	−36	1296	1296/96
120	$\frac{300 \times 260}{500} = 156$	−36	1296	1296/156
140	$\frac{200 \times 260}{500} = 104$	36	1296	1296/104

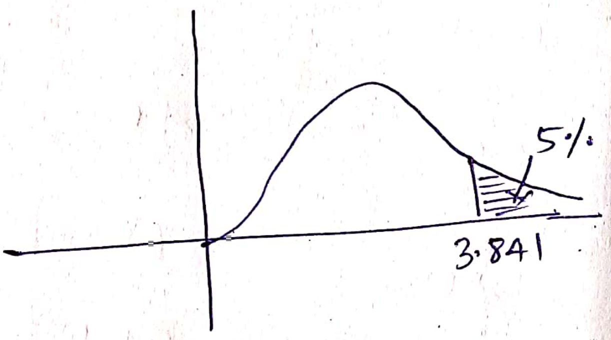
$\chi^2 = 43.2676$

Step 4: Critical value:

The critical value at 5% Level of significance with $(r-1)(c-1) = (2-1)(2-1) = 1$ degree of freedom is

$$\chi^2_{1, 0.05} = 3.841$$

Step 5: Decision:



$\because$ $\chi^2_{(calculated)} = 43.26$ lies in the region of rejection of Ho

$\therefore$ Ho is rejected

Q 5 (a)

Minimize the function $f(x_1, x_2) = 4x_1 + 8x_2 - x_1^2 - x_2^2$

subject to $x_1 + x_2 = 4, \quad x_1, x_2 \geq 0$

Sol: The Langrangse's function is given by

$$L(x, \lambda) = 4x_1 + 8x_2 - x_1^2 - x_2^2 - \lambda(x_1 + x_2 - 4)$$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 4 - 2x_1 - \lambda = 0 \quad \text{------ (1)}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 8 - 2x_2 - \lambda = 0 \quad \text{------ (2)}$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x_1 + x_2 = 4 \quad \text{------ (3)}$$

solving (1), (2) & (3), we get

$$x_1 = 1, \quad x_2 = 3, \quad \lambda = 2$$

$$\therefore \quad p \equiv (1, 3)$$

$$Q = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

O = Null matrix of size $1 \times 1 = [0]$

$$P = [\nabla g_1(x)] = \left[\frac{\partial g_1(x)}{\partial x_1} \quad \frac{\partial g_2(x)}{\partial x_2}\right] = [1 \quad 1]$$

$$\therefore \quad H_B = \begin{bmatrix} O & P \\ P^t & Q \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

starting order $= 2m+1 = 2 \times 1 + 1 = 3$

No. of principle minor determinants $= n - m$
$= 2 - 1 = 1$

$$\Delta = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = 4 > 0$$

$\because \Delta > 0$

$\therefore$ $f(x_1, x_2)$ has maxima at $p \equiv (1, 3)$

$\therefore$ $Z_{max} = 18$ at $x_1 = 1, x_2 = 3$

Q5 (b)

Find the minimizer of $f(x) = x^2 + \frac{54}{x}$ using bisection method in (2,5) within a range of 0.3

Sol:
$$f(x) = x^2 + \frac{54}{x}$$
$$f'(x) = 2x - \frac{54}{x^2}$$

Iteration 1

$2 \quad - \qquad 5 \quad +$

Let $a = x_1 = 2$
$b = x_2 = 5$
$$f'(2) = 4 - \frac{54}{4} = -9.5 < 0$$
$$f'(5) = 10 - \frac{54}{\cancel{100}\ 25} = 7.84 > 0$$
$$\therefore z = \frac{x_1 + x_2}{2} = \frac{2+5}{2} = 3.5$$
$$f'(3.5) = 2(3.5) - \frac{54}{(3.5)^2} = 2.5918$$
$$|f'(3.5)| = 2.5918 \neq 0.3$$

Iteration 2

$2 \quad - \qquad 3.5 \quad +$

$\therefore f'(3.5) = 2.5918 > 0$

Let $x_1 = 2$ & $x_2 = 3.5$
$$\therefore z = \frac{x_1 + x_2}{2} = \frac{2+3.5}{2} = 2.75$$
$$f'(2.75) = 2(2.75) - \frac{54}{(2.75)^2} = -1.6405$$
$$|f'(2.75)| = |-1.6405| \neq 0.3.$$

Iteration 3

$2 \quad - \qquad 2.75 \quad - \qquad 3.5 \quad +$

$\therefore f'(2.75) < 0$

Let $x_1 = 2.75$ & $x_2 = 3.5$
$$\therefore z = \frac{x_1 + x_2}{2} = \frac{2.75 + 3.5}{2} = 3.125$$

$$f'(3.125) = 2(3.125) - \frac{54}{(3.125)^2} = 0.7204$$

$$|f'(3.125)| = |0.7204| \nleq 0.3$$

<u>Iteration 4</u>

2.75   3.125   3.5

$-$ $+$ $+$

$\because f'(3.125) > 0$

Let $x_1 = 2.75$, $x_2 = 3.125$

$$z = \frac{x_1 + x_2}{2} = \frac{2.75 + 3.125}{2} = 2.9375$$

$$f'(z) = f'(2.9375) = 2(2.9375) - \frac{54}{(2.9375)^2} = -0.3830$$

$$\therefore |f'(2.9375)| = |-0.3830| \nleq 0.3$$

<u>Iteration 5</u>

$\because f'(2.9375) < 0$

2.75   2.9375   3.125

$-$ $-$ $+$

Let $x_1 = 2.9375$, $x_2 = 3.125$

$$z = \frac{x_1 + x_2}{2} = \frac{2.9375 + 3.125}{2} = 3.03125$$

$$f'(z) = f'(3.03125) = 2(3.03125) - \frac{54}{(3.03125)^2}$$

$$= 0.1856$$

$$|f'(3.03125)| = |0.1856| \leq 0.3$$

$\therefore$ The ~~root~~ minimum is $z = 3.03125$