Computing Nash equilibria of two-player, zero-sum games

In **two-player, zero-sum games**, the interaction between two players is entirely competitive: one player's gain is exactly equal to the other player's loss, making the sum of their payoffs zero. The goal is to compute the **Nash equilibrium**, which represents the optimal strategy for both players, ensuring that no player can improve their outcome by unilaterally changing their strategy.

- **Players**: Player 1 and Player 2.
- **Strategies**: Each player has a set of strategies they can choose from.
- **Payoffs**: If Player 1 wins a certain amount, Player 2 loses the exact same amount. The payoff matrix only needs to display Player 1's payoffs, since Player 2's payoffs are simply the negative of Player 1's.

Example Payoff Matrix:

	Player 2: X	Player 2: Y	Player 2: Z
Player 1: A	3	-1	2
Player 1: B	-2	0	4

In this example, if Player 1 chooses A and Player 2 chooses X, Player 1 wins 3 points and Player 2 loses 3 points (hence Player 2's payoff is -3).

Nash Equilibria in Zero-Sum Games:

There are two types of Nash equilibria in zero-sum games:

1. **Pure Strategy Nash Equilibrium (PSNE)**: Both players choose a single strategy deterministically.
2. **Mixed Strategy Nash Equilibrium (MSNE)**: Players randomize over their strategies with certain probabilities to maximize their expected payoffs.

Finding Nash Equilibrium (Pure Strategy):

A **pure strategy Nash equilibrium** exists when one strategy for each player is a **best response** to the other player's strategy. In a zero-sum game, a **saddle point** occurs when Player 1's minimax equals Player 2's maximin.

- **Minimax for Player 1**: The strategy that maximizes Player 1's worst possible outcome.
- **Maximin for Player 2**: The strategy that minimizes Player 2's worst possible outcome.

Steps to Find Pure Strategy Nash Equilibrium:

1. **Minimax**: For each row (Player 1's strategies), find the **minimum** value (Player 1's worst-case scenario).
2. **Maximin**: For each column (Player 2's strategies), find the **maximum** value (Player 2's worst-case scenario).

3. **Saddle Point**: If the minimax value (Player 1's best of the worst-case scenarios) equals the maximin value (Player 2's worst of the best-case scenarios), that pair of strategies forms a pure strategy Nash equilibrium.

Example: Pure Strategy Nash Equilibrium

Consider the payoff matrix again:

	Player 2: X	Player 2: Y	Player 2: Z
Player 1: A	3	−1	2
Player 1: B	−2	0	4

- Player 1's **row minima** (worst-case outcomes):
  - For strategy A: $\min(3,-1,2)=-1$
  - For strategy B: $\min(-2,0,4)=-2$

  Player 1's minimax is $\max(-1,-2)=-1$

- Player 2's **column maxima** (worst-case outcomes):
  - For strategy X: $\max(3,-2)=3$
  - For strategy Y: $\max(-1,0)=0$
- For strategy Z: $\max(2,4)=4$

  Player 2's maximin is $\min(3,0,4)=0$

  There is **no pure strategy Nash equilibrium** here because the minimax and maximin values do not match.

Mixed Strategy Nash Equilibrium (MSNE):

If no pure strategy equilibrium exists, we turn to **mixed strategies**, where players randomize over their available strategies.

In a **mixed strategy Nash equilibrium**:

- Each player selects strategies probabilistically, and the expected payoff for each player is maximized.

Steps to Find Mixed Strategy Nash Equilibrium:

1. **Assign probabilities** to the strategies for both players.
   - Let Player 1 play A with probability p and B with probability $1-p$.
   - Let Player 2 play X, Y, and Z with probabilities $q1$,$q2$, and $q3$, respectively.
2. **Calculate expected payoffs**:
   - For Player 1: The expected payoffs from choosing A and B should be equal (because Player 1 should be indifferent between these two strategies in equilibrium).
   - For Player 2: The expected payoffs from choosing X, Y, and Z should be equal (because Player 2 should be indifferent between these strategies in equilibrium).

3. **Solve the system of equations** formed by setting the expected payoffs equal for each player.

Example: Mixed Strategy Nash Equilibrium

For the matrix:

	Player 2: X	Player 2: Y	Player 2: Z
Player 1: A	3	−1	2
Player 1: B	−2	0	4

Let Player 1 play A with probability p and B with probability 1−p.

Let Player 2 play:

- X with probability q_1
- Y with probability q_2
- Z with probability q_3

Player 1's Expected Payoff:

- Expected payoff from A: $3q_1+(-1)q_2+2q_3$
- Expected payoff from B: $(-2)q_1+0q_2+4q_3$
- Since Player 1 should be indifferent between AAA and BBB in the mixed strategy Nash equilibrium:

$3q_1-q_2+2q_3=-2q_1+0q_2+4q_3$

Player 2's Expected Payoff:

- Expected payoff from X: $3p+(-2)(1-p)$
- Expected payoff from Y: $(-1)p+0(1-p)$
- Expected payoff from Z: $2p+4(1-p)$

  Similarly, Player 2 should be indifferent between X, Y, and Z, which leads to another system of equations.

Pure Strategy Equilibrium:

- Check for a **saddle point** by identifying minimax and maximin values. If these values are equal, you have a pure strategy Nash equilibrium.

Mixed Strategy Equilibrium:

- If no pure strategy equilibrium exists, assign probabilities to each player's strategies.
- Set up equations based on expected payoffs and solve for the equilibrium probabilities.