Two-player, general-sum games

Two-player, general-sum games involve two players making decisions where the sum of their payoffs can vary depending on the outcome. Unlike zero-sum games, where one player's gain is exactly the other player's loss, general-sum games allow for cooperation, competition, or a mix of both, meaning that the players' interests may align or conflict to various degrees.

- **Payoff Matrix**: In a two-player general-sum game, each player has a set of possible strategies. The game can be represented as a matrix where each cell contains the payoffs for both players for a given strategy combination.
- **Nash Equilibrium (NE)**: A pair of strategies, one for each player, where no player can improve their payoff by changing their strategy while the other player's strategy remains unchanged.

Structure of a Two-Player General-Sum Game:

- **Players**: Two players (Player 1 and Player 2).
- **Strategies**: Each player has a set of strategies to choose from. For example:
    - Player 1 can choose between strategies A and B.
    - Player 2 can choose between strategies X and Y.
- **Payoffs**: The payoffs depend on the combination of strategies chosen by both players. The payoff matrix is of the form:

	Player 2: X	Player 2: Y
Player 1: A	(a_{11}, b_{11})	(a_{12}, b_{12})
Player 1: B	(a_{21}, b_{21})	(a_{22}, b_{22})

Where (a_{ij}, b_{ij}) represents the payoffs to Player 1 and Player 2 when Player 1 plays strategy i and Player 2 plays strategy j.

1. Pure Strategy Nash Equilibrium (PSNE)

A pure strategy Nash equilibrium occurs when both players choose a specific strategy, and no player can increase their payoff by deviating unilaterally.

Example:

Consider the following payoff matrix:

	Player 2: X	Player 2: Y
Player 1: A	$(3, 2)$	$(1, 4)$
Player 1: B	$(2, 3)$	$(4, 1)$

Step 1: Identify the best responses for each player.

- Player 1's best response:
    - If Player 2 plays X, Player 1 prefers A ($3 > 2$).

- If Player 2 plays Y, Player 1 prefers B (4 > 1).

- Player 2's best response:
  - If Player 1 plays A, Player 2 prefers Y (4 > 2).
  - If Player 1 plays B, Player 2 prefers X (3 > 1).

Step 2: Find the mutual best responses:

- No cell in the payoff matrix is a mutual best response, so there is **no pure strategy Nash equilibrium** in this game.

2. Mixed Strategy Nash Equilibrium (MSNE)

A mixed strategy Nash equilibrium occurs when players randomize over their strategies, assigning probabilities to each strategy to maximize their expected payoffs.

Example (continued from above):

- Let Player 1 play A with probability p and B with probability 1−p1.
- Let Player 2 play X with probability q and Y with probability 1−q.

Step 1: Calculate the expected payoffs for each player.

- **Player 1's expected payoff**:
  - From A: $3q+1(1-q)=3q+1-q=2q+1$
  - From B: $2q+4(1-q)=2q+4-4q=4-2q$

  For Player 1 to be indifferent between A and B (as required in a mixed strategy equilibrium), the expected payoffs from A and B must be equal:

  $2q+1=4-2q$

Solving this gives $q=3/4$

- **Player 2's expected payoff**:
  - From X: $2p+3(1-p)=2p+3-3p$
  - From Y: $4p+1(1-p)=4p+1-p=3p+1$

  For Player 2 to be indifferent between X and Y:

  $3-p=3p+1$

Solving this gives $p=1$.

Step 2:

- In the mixed strategy Nash equilibrium, Player 1 always plays A (with probability $p=1$), and Player 2 randomizes between X and Y, playing X with probability $q=3/4$ and Y with probability 1/4.
- **Multiple Nash equilibria**: General-sum games often have multiple Nash equilibria, including both pure and mixed strategies.

- **Non-uniqueness**: Some games may have more than one equilibrium, which could make predicting players' behavior challenging.
- **Efficiency and fairness**: The Nash equilibrium is not always socially efficient or fair, as it depends only on individual incentives.

Two-player, general-sum games are more complex than zero-sum games because they allow for a variety of outcomes, including cooperative and competitive interactions. Finding Nash equilibria, especially mixed strategies, involves solving for the optimal randomization that balances the expected payoffs for each player. These games are central in economics, game theory, and multi-agent systems, modeling scenarios where agents have potentially conflicting or aligned interests.