Extensive Games with Imperfect Information

In strategic games, players must form beliefs about the other players' strategies, based on the presumed equilibrium being played.

In Bayesian games, players must form beliefs about the other players' strategies and their types, based on the probability distribution over types and the presumed equilibrium being played.

In extensive games with perfect information, there is the possibility that a player will face a situation that is inconsistent with the presumed equilibrium being played. However, subgame perfection takes care of this issue by requiring a form of sequential rationality, even off the equilibrium path.

In extensive games with imperfect information, when a player faces a situation that is inconsistent with the presumed equilibrium being played, she may also be forced to form beliefs about the other players' past behavior. These beliefs are often crucial in evaluating whether the ensuing play is rational.

Put another way, games with imperfect information may have few subgames, possibly only one. How then are we to eliminate Nash equilibria that involve threats that are not credible?

Definition 200.1: An extensive game consists of the following components:

1. The set of players (assumed to be a finite set), N.

2. A set, H, of sequences (histories of actions) satisfying the following properties:

(2a) The empty sequence, $\varnothing$, is an element of H.

(2b) If $(a^k)_{k=1,\ldots,K} \in H$, (where K may be infinity) and $L < K$, then $(a^k)_{k=1,\ldots,L} \in H$.

(2c) If an infinite sequence, $(a^k)_{k=1,\ldots,\infty}$ satisfies

$(a^k)_{k=1,\ldots,L} \in H$ for every positive integer L, then $(a^k)_{k=1,\ldots,\infty} \in H$.

3. A function, P , that assigns to each non-terminal history a member of $N \cup \{c\}$. (P is the player function, with $P(h)$ being the player who takes an action after the history, h. If $P(h) = c$ holds, then chance or nature takes the action after the history, h. A history, $(a^k)_{k=1,\ldots,K}$, is terminal if it is infinite or if there is no a^{K+1} such that $(a^k)_{k=1,\ldots,K+1} \in H$.)

4. A function, f_c, that associates with each $h \in H$ for which $P(h) = c$, a probability measure on $A(h)$, denoted by $f_c(\cdot \mid h)$. Each such measure is assumed to be independent of every other such measure.

5. For each player, $i \in N$, a partition $=_i$ of $\{h \in H : P(h) = i\}$ with the property that $A(h) = A(h^0)$ whenever h and h^0 are in the same element of the partition. For $I_i \in =_i$, we denote by $A(I_i)$ the set $A(h)$ and by $P(I_i)$ the player $P(h)$ for any $h \in I_i$.

6. For each player, $i \in N$, a preference relation $°_i$ on lotteries over the set of terminal histories, Z.

$I_i \in =_i$ is called an information set of player i. A player cannot distinguish two histories (nodes) in one of her information sets. Notice the requirement that the set of available actions must be the same for any two histories in an information set, or else the player would be able to distinguish between the histories.

To keep things simple, we do not allow more than one player to move after a given history. However, we can essentially model simultaneous moves with the use of information sets.

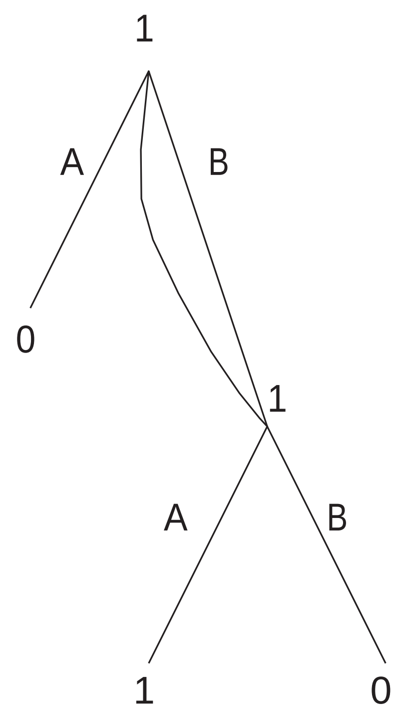
Definition 203.1: A pure strategy of player $i \in N$ in an extensive game, $\langle N, H, P, f_c, (=_i), (\succsim_i)\rangle$, is a function that assigns an action in $A(I_i)$ to each information set, $I_i \in =_i$.

Perfect and Imperfect Recall

Let $X_i(h)$ be the sequence of information sets that player i encounters in the history h, and the actions that he takes at them, in the order that these events occur.

Definition 203.3: An extensive game has perfect recall if for each player i we have $X_i(h) = X_i(h^0)$ whenever the histories h and h^0 are in the same information set of player i.

One-player games of imperfect recall include: "Where did I park?" and "Did I feed the dog?"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Mixed and Behavioral Strategies

There are two ways of thinking about how a player randomizes in extensive games.

Definition 212.1: A mixed strategy of player i in an extensive game, $\langle N, H, P, f_c, (=_i), (\circ_i)\rangle$, is a probability measure over the set of player i's pure strategies. A behavioral strategy is a collection $(\beta_i(I_i))_{I_i \in =_i}$ of independent probability measures, where $\beta_i(I_i)$ is a measure over $A(I_i)$.

For any history, $h \in I_i \in =_i$, and action $a \in A(h)$, we let $\beta_i(h)(a)$ denote the probability assigned by $\beta_i(I_i)$ to the action a.

For any profile of (mixed or behavioral) strategies $\sigma = (\sigma_i)_{i \in N}$ in an extensive game, the outcome $O(\sigma)$ is defined to be the probability distribution over the terminal histories that results when player i follows σ_i.

Two (mixed or behavioral) strategies are outcome equivalent if, for every collection of pure strategies for the other players, the two strategies induce the same probability distribution over terminal histories.

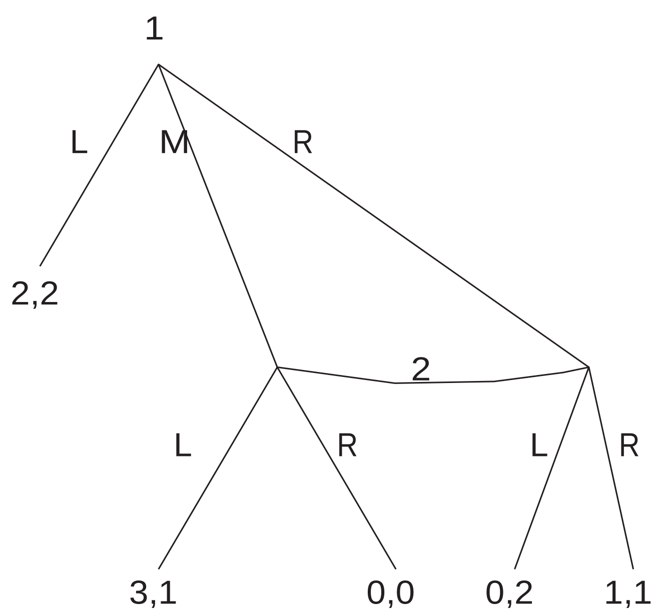
Proposition 214.1: For any mixed strategy of a player in a finite extensive game with perfect recall, there is an outcome-equivalent behavioral strategy, and vice-versa.

For the "Did I feed the dog?" game, mixed and behavioral strategies are not outcome equivalent. There are two pure strategies, A = don't feed and B = feed. All mixed strategies yield a payoff of zero, because the player chooses A with probability p and B with probability $1-p$. The randomization is only done once, so if B is chosen after the null history, B is also chosen after the history, (B). You remember the outcome of your randomization about whether to feed the dog, but not whether you have previously had an opportunity to feed the dog.

With behavioral strategies, a positive payoff is possible. Letting $\beta(I)(a) = p$ for the only information set, I, and the action $a = A$, the terminal history is A with probability p, (B,A) with probability $(1 - p)p$, and (B,B) with probability $(1 - p)^2$. You do not remember whether you have previously had an opportunity to feed the dog, or the outcome of any previous randomization.

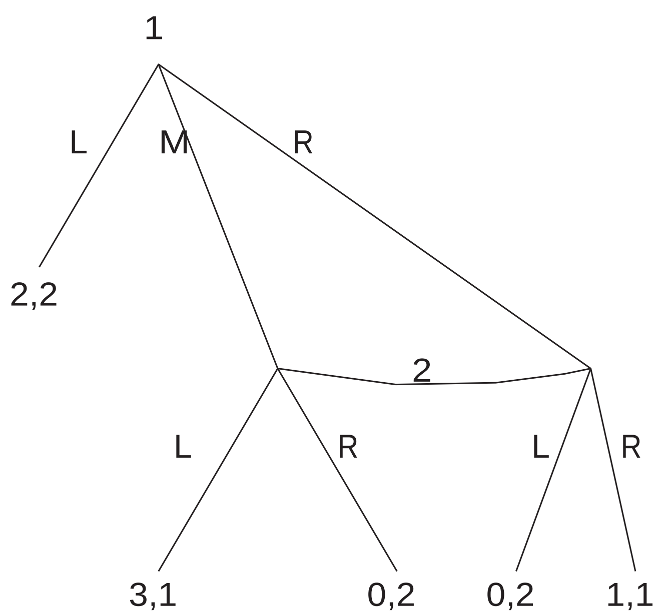
Definition: A Nash equilibrium in mixed (behavioral) strategies of an extensive game is a profile σ^* of mixed (behavioral) strategies with the property that, for every player $i \in N$, we have $O(\sigma^*_{-i}, \sigma^*_i) \succsim_i O(\sigma^*_{-i}, \sigma_i)$ for every mixed (behavioral) strategy σ_i of player i.

Sequential Equilibrium



This game has two Nash equilibria: (L,R) and (M,L)

For this game, the Nash equilibrium (L,R) relies on player 2 playing a strategy that is strictly dominated (given that she is asked to move). However, subgame perfection does not rule this out. Here is a trickier example.



Now (L,R) is a NE, but seeing whether R is rational for player 2 must depend on what counterfactual choice is made by player 1: M or R.

In the (L,R) equilibrium, the optimal choice for player 2 depends on her beliefs about whether M or R is more likely. However, these beliefs cannot depend on the equilibrium strategies, because both M and R occur with probability zero.

The concept, sequential equilibrium, is able to evaluate the rationality of strategies at any information set by requiring us to specify a belief system as part of the equilibrium.

A belief system specifies, for each information set, a probability measure on the set of histories in the information set.

For extensive games of perfect information, beliefs about the future play of the game are specified in the continuation strategies. Subgame perfection requires sequential rationality, given beliefs about future play. For games of imperfect information, sequential rationality requires us to specify beliefs about the past as well as the future.

Definition 222.1: An assessment in an extensive game is a pair (β, μ), where β is a profile of behavioral strategies and μ is a function that assigns to each information set a probability measure on the set of histories in the information set.

$\mu(I)(h)$ is the probability that the player $P(I)$ assigns to the history $h \in I$, conditional on I being reached.

The outcome of (β, μ) conditional on I is the distribution of terminal histories, conditional on I being reached. For a terminal history $h^* = (a^1, \ldots, a^K)$, we have

$O(\beta, \mu \mid I)(h^*) = 0$ if there is no *subhistory* of h^* in I

$O(\beta, \mu \mid I)(h^*) =$

$$\mu(I)(h) \mathop{}\limits_{k=L}^{K-1} \beta_{P((a^1,\ldots,a^k))}(a^1, \ldots, a^k)(a^{k+1}),$$

if the subhistory of h^*, $h = (a^1, \ldots, a^L)$ is in I.

Definition 224.1: Let $\Gamma = \langle N, H, P, f_c, (\mathcal{I}_i), (\succsim_i)\rangle$ be an extensive game with perfect recall. The assessment, (β, μ), is sequentially rational if for every player $i \in N$ and every information set $I_i \in \mathcal{I}_i$, we have

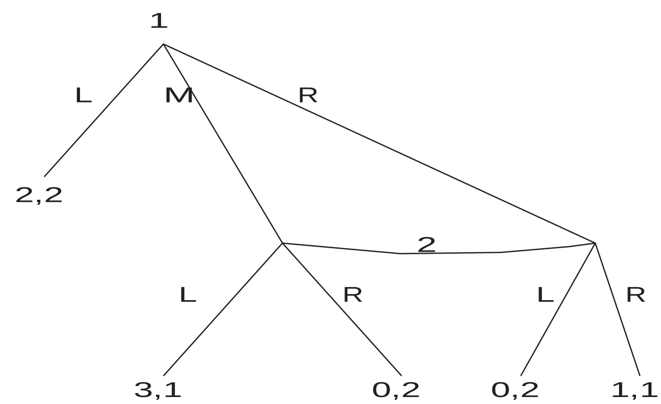
$$O(\beta, \mu \mid I) \succsim_i O((\beta_{-i}, \beta_i'), \mu \mid I)$$

for every strategy β_i'.

Definition 224.2: An assessment, (β, μ), is consistent if there is a sequence $((\beta^n, \mu^n))_{n=1}^{\infty}$ that converges to (β, μ), and has the properties that each strategy profile β^n is completely mixed (every action at every information set has positive probability) and each belief system μ^n is derived from β^n using Bayes' rule.

Definition 225.1: An assessment is a sequential equilibrium of a finite extensive game with perfect recall if it is sequentially rational and consistent.

The purpose of the sequence of completely mixed strategy profiles is to be able to apply Bayes' rule. All information sets are reached with positive probability, although some could have a limiting probability of zero.



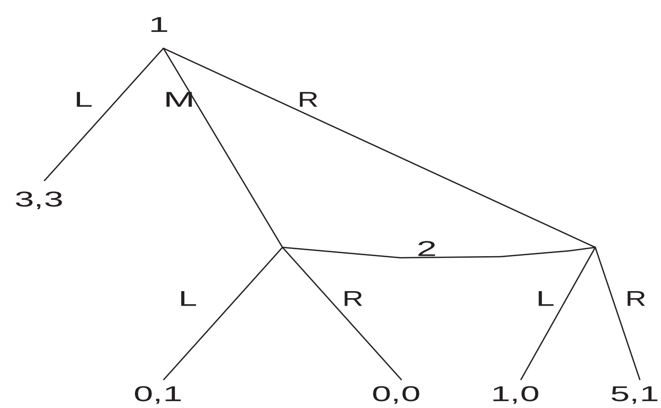
Now (L,R) is a NE, but seeing whether R is rational for player 2 must depend on what counterfactual choice is made by player 1: M or R.

$\beta_1 = (1, 0, 0)$, $\beta_2 = (0, 1)$, $\mu(\{M, R\})(M) = 0.9$ is a sequential equilibrium, based on the sequence

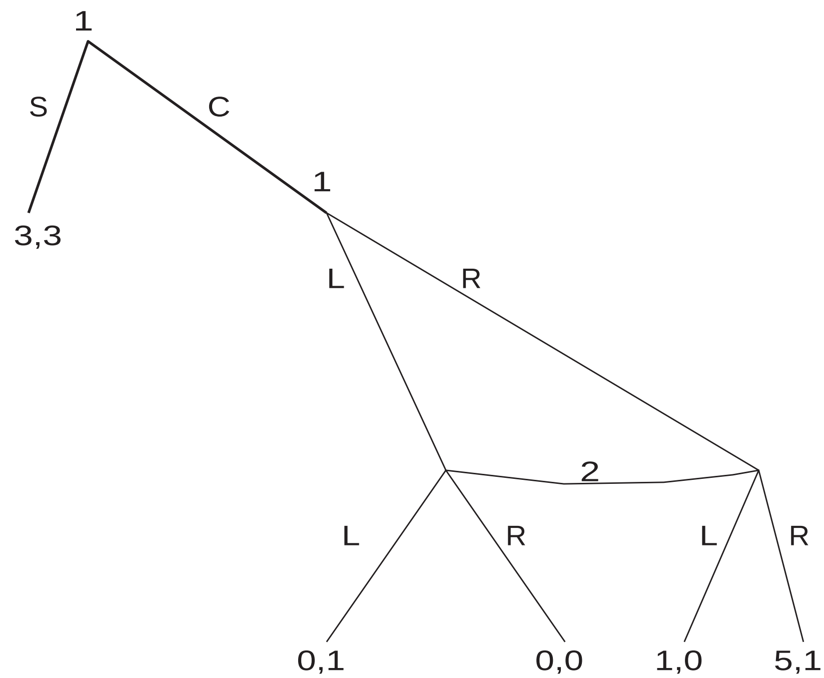
$\beta_1^{\varepsilon} = (1 - \varepsilon, 0.9\varepsilon, 0.1\varepsilon)$, $\beta_2^{\varepsilon} = (\varepsilon, 1 - \varepsilon)$,

$\mu^{\varepsilon}(\{M, R\})(M) = 0.9.$

Sequential equilibrium is not invariant to seemingly inessential changes to a game. In the first game, player 1's choice is between L,M, and R. In the second game, player 1 first chooses S (corresponding to L) or C (corresponding to M or R), and follows C with a choice between the remaining two options.



(L, L) and $\mu(\{M, R\})(R) = 0$ is a sequential equilibrium.



Claim: The the only sequential equilibrium payoff profile is $(5, 1)$.

In the second game, sequential rationality after the history, C, requires player 1 to play R, because that strictly dominates L. Therefore, at the information set $\{(C, L), (C, R)\}$, consistency requires player 2 to put probability 1 on (C, R), so by sequential rationality, player 2 must choose R. Hence, player 1 must choose (C, R), so payoffs are $(5, 1)$.

(L, L) and $\mu(\{M, R\})(R) = 0$ is a sequential equilibrium in the first game, because player 2 believes that (C, L) is much more likely than (C, R). These beliefs are a bit strange, but the relative probabilities are not constrained by any choice of player 1, since he never has to choose between the two. In the second game, he must choose after the history, C.

It is sometimes difficult to check whether an assessment is a sequential equilibrium. It would be nice to have a consistency notion based only on μ, without requiring ε perturbations.

Definition 228.1: The belief system μ in an extensive game with perfect recall is structurally consistent if for each information set I there is a strategy profile β such that I is reached with positive probability under β and $\mu(I)$ is derived from β using Bayes' rule.

In many games, any consistent assessment is structurally consistent. However, there are examples of sequential equilibrium assessments that are not structurally consistent.

There are other examples of a sequentially rational assessment (β, μ) in which μ is structurally consistent but not consistent.