INVERSE LAPLACE TRANSFORM

- **Inverse Laplace Transform**

If $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \phi(s)$ then $f(t)$ is called as the inverse Laplace transform of $\phi(s)$ and it is denoted by $L^{-1}[\phi(s)] = f(t)$.

- **Formulae**

Laplace Transform	Inverse Laplace Transform
1) $L(1) = \frac{1}{s}$	1) $L^{-1}\left(\frac{1}{s}\right) = 1$
2) $L(e^{at}) = \frac{1}{s-a}$	2) $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
3) $L(e^{-at}) = \frac{1}{s+a}$	3) $L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$
4) $L(\sin at) = \frac{a}{s^2+a^2}$	4) $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$
5) $L(\cos at) = \frac{s}{s^2+a^2}$	5) $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$
6) $L(\sinh at) = \frac{a}{s^2-a^2}$	6) $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at$
7) $L(\cosh at) = \frac{s}{s^2-a^2}$	7) $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$
8) $L(t^n) = \frac{\overline{\lvert n+1}}{s^{n+1}}$	8) $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{\overline{\lvert n}}$

Prof. Nancy Sinollin

Problems:-

1) Find $L^{-1}\left[\frac{3+2s+s^2}{s^3}\right]$

Sol^n $L^{-1}\left[\frac{3+2s+s^2}{s^3}\right] = L^{-1}\left[\frac{3}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right]$

$$= 3L^{-1}\left[\frac{1}{s^3}\right] + 2L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{1}{s}\right]$$

$$= \frac{3t^2}{\Gamma 3} + \frac{2t}{\Gamma 2} + 1$$

$$= \frac{3t^2}{2!} + \frac{2t}{1!} + 1$$

$$= \frac{3t^2}{2} + 2t + 1$$

2) If $L[f(t)] = \frac{s+3}{s^2+4}$ Find $L[f'(t)]$

Sol^n $L[f'(t)] = -f(0) + s\,L[f(t)]$

$$= -f(0) + s\left(\frac{s+3}{s^2+4}\right) \quad \text{——— (1)}$$

How to find $f(t)$

as $L[f(t)] = \frac{s+3}{s^2+4}$

$$f(t) = L^{-1}\left[\frac{s+3}{s^2+4}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+4} + \frac{3}{s^2+4}\right]$$

$$= \cos 2t + \frac{3}{2}\sin 2t$$

$\Rightarrow f(0) = \cos(0) + 3/_2 \sin(0) = 1$

Hence, $L[f'(t)] = -1 + s\left(\frac{s+3}{s^2+4}\right)$ — from (1).

• Using First Shifting Theorem

We know that,

If $L[f(t)] = \phi(s)$ then $L[e^{-at} f(t)] = \phi(s+a)$

Hence, $L^{-1}[\phi(s+a)] = e^{-at} f(t) = e^{-at} L^{-1}[\phi(s)]$

—— as $L[f(t)] = \phi(s)$

Therefore, $L^{-1}[\phi(s+a)] = e^{-at} L^{-1}[\phi(s)]$

$$L^{-1}[\phi(s-a)] = e^{at} L^{-1}[\phi(s)]$$

• Problems :-

1) Find $L^{-1}\left[\dfrac{s+2}{(s+2)^2-1}\right]$

Solⁿ $L^{-1}\left[\dfrac{(s+2)}{(s+2)^2-1}\right] = e^{-2t} L^{-1}\left[\dfrac{s}{s^2-1}\right]$ —— (we can use above formula since every s is in the form of (s+2))

$$= e^{-2t} \cosh t$$

2) Find $L^{-1}\left[\dfrac{s}{(s+1)^2+2}\right]$

Solⁿ $L^{-1}\left[\dfrac{s}{(s+1)^2+2}\right] = L^{-1}\left[\dfrac{(s+1)-1}{(s+1)^2+(\sqrt{2})^2}\right]$

$$= e^{-t} L^{-1}\left[\frac{s-1}{s^2+(\sqrt{2})^2}\right]$$

$$= e^{-t} L^{-1}\left[\frac{s}{s^2+(\sqrt{2})^2} - \frac{1}{s^2+(\sqrt{2})^2}\right]$$

$$= e^{-t}\left(\cos\sqrt{2}t - \frac{1}{\sqrt{2}}\sin\sqrt{2}t\right)$$

3) Evaluate $L^{-1}\left[\frac{1}{(s+3)^{3/2}}\right]$

Sol^n $L^{-1}\left[\frac{1}{(s+3)^{3/2}}\right] = e^{-3t} L^{-1}\left[\frac{1}{s^{3/2}}\right]$

$$= e^{-3t} \frac{t^{3/2-1}}{\Gamma 3/2}$$

$$= e^{-3t} \frac{t^{1/2}}{\frac{1}{2}\Gamma \frac{1}{2}}$$

$$= \frac{2e^{-3t} t^{1/2}}{\sqrt{\pi}}$$

$$= \frac{2e^{-3t}\sqrt{t}}{\sqrt{\pi}}$$

4) $L^{-1}\left[\frac{s}{(s-2)^6}\right]$

Sol^n $L^{-1}\left[\frac{s}{(s-2)^6}\right] = L^{-1}\left[\frac{s-2+2}{(s-2)^6}\right]$

$$= e^{2t} L^{-1}\left[\frac{s+2}{s^6}\right]$$

$$= e^{2t} L^{-1}\left[\frac{1}{s^5} + \frac{2}{s^6}\right]$$

$$= e^{2t}\left[\frac{t^4}{\Gamma 5} + \frac{2t^5}{\Gamma 6}\right]$$

$$= e^{2t}\left[\frac{t^4}{4!} + \frac{2t^5}{5!}\right]$$

5) Find $L^{-1}\left[\dfrac{6s-4}{s^2-4s+20}\right]$

Sol^n $L^{-1}\left[\dfrac{6s-4}{s^2-4s+20}\right] = L^{-1}\left[\dfrac{6s-4}{s^2-4s+4-4+20}\right]$

$$= L^{-1}\left[\frac{6(s-4/6)}{(s-2)^2+16}\right]$$

$$= 6\,L^{-1}\left[\frac{s-2+2-2/3}{(s-2)^2+16}\right]$$

$$= 6e^{2t}\,L^{-1}\left[\frac{s+4/3}{s^2+16}\right]$$

$$= 6e^{2t}\,L^{-1}\left[\frac{s}{s^2+16}+\frac{4}{3}\,\frac{1}{s^2+16}\right]$$

$$= 6e^{2t}\left[\cos 4t + \frac{4}{3}\times\frac{1}{4}\sin 4t\right]$$

$$= 6e^{2t}\left[\cos 4t + \frac{1}{3}\sin 4t\right]$$

- **Note:-**

We will use above method only when, we have factors of denominator as not in integer form. otherwise we will use method of partial fraction.

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Method of Partial Fraction :-

We can use partial fraction directly when degree of polynomial in numerator is less than the degree of polynomial in denominator.

How to apply partial fraction

Consider degree of f(x) is less than degree of polynomial in denominator

1) $$\frac{f(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

2) $$\frac{f(x)}{(x-a)^r(x-b)} = \frac{A_1}{(x-a)^r} + \frac{A_2}{(x-a)^{r-1}} + \cdots + \frac{A_r}{(x-a)} + \frac{B}{x-b}$$

3) $$\frac{f(x)}{(x^2+a^2)(x+b)(x-c)} = \frac{Ax+B}{(x^2+a^2)} + \frac{C}{x+b} + \frac{D}{x-c}$$

Problems :

Find

1) $$L^{-1}\left[\frac{3s+7}{s^2-2s-3}\right]$$

Soln $$L^{-1}\left[\frac{3s+7}{s^2-2s-3}\right] = L^{-1}\left[\frac{3s+7}{(s-3)(s+1)}\right]$$

$$\frac{3s+7}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1} = \frac{A(s+1)+B(s-3)}{(s-3)(s+1)}$$

$$3s+7 = A(s+1) + B(s-3)$$

put $s=3$ $\quad 9+7 = 4A \Rightarrow 16 = 4A \Rightarrow \boxed{A=4}$

put $s=-1$ $\quad -3+7 = -4B \Rightarrow 4 = -4B \Rightarrow \boxed{B=-1}$

$$L^{-1}\left[\frac{3s+7}{s^2-2s-3}\right] = L^{-1}\left[\frac{4}{s-3} + \frac{(-1)}{s+1}\right] = 4e^{3t} - e^{-t}$$

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2) $L^{-1}\left[\dfrac{3s-7}{s^2-6s+8}\right]$

Sol^n $L^{-1}\left[\dfrac{3s-7}{s^2-6s+8}\right] = L^{-1}\left[\dfrac{3s-7}{(s-4)(s-2)}\right]$

$$\frac{3s-7}{(s-4)(s-2)} = \frac{A}{s-4} + \frac{B}{s-2} = \frac{A(s-2)+B(s-4)}{(s-4)(s-2)}$$

$\Rightarrow 3s-7 = A(s-2) + B(s-4)$

put $s=4 \Rightarrow 12-7 = 2A \Rightarrow 5 = 2A \Rightarrow \boxed{A = 5/2}$

put $s=2 \Rightarrow 6-7 = -2B \Rightarrow -1 = -2B \Rightarrow \boxed{B = 1/2}$

$$\therefore L^{-1}\left[\frac{3s-7}{s^2-6s+8}\right] = L^{-1}\left[\frac{5/2}{s-4} + \frac{1/2}{s-2}\right]$$

$$= L^{-1}\left[\frac{5}{2}\frac{1}{s-4} + \frac{1}{2}\frac{1}{s-2}\right]$$

$$= \frac{5}{2}e^{4t} + \frac{1}{2}e^{2t}$$

3) $L^{-1}\left[\dfrac{1}{(s-2)(s+2)^2}\right]$

Sol^n $L^{-1}\left[\dfrac{1}{(s-2)(s+2)^2}\right]$

$$\frac{1}{(s-2)(s+2)^2} = \frac{A}{s-2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)}$$

$$= \frac{A(s+2)^2 + B(s-2) + C(s-2)(s+2)}{(s-2)(s+2)^2}$$

$1 = A(s+2)^2 + B(s-2) + C(s-2)(s+2)$ —— ①

put $s=2$ $\quad 1 = A(4)^2 \Rightarrow 1 = 16A \Rightarrow \boxed{A = 1/16}$

put $s=-2$ $\quad 1 = B(-4) \Rightarrow 1 = -4B \Rightarrow \boxed{B = -1/4}$

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put $s=0$ in ①, we get

$$1 = A(2)^2 + B(-2) + C(-2)(2)$$

$$1 = 4A - 2B - 4C$$

$$\Rightarrow 1 = 4\frac{1}{16} - 2\left(\frac{-1}{4}\right) - 4C$$

$$\Rightarrow 4C = \frac{1}{4} + \frac{1}{2} - 1 = \frac{1+\cancel{2-4}}{4} = \frac{-1}{4}$$

$$\Rightarrow \boxed{C = -\frac{1}{16}}$$

$$\therefore L^{-1}\left[\frac{1}{(s-2)(s+2)^2}\right] = L^{-1}\left[\frac{1/16}{s-2} + \frac{(-1/4)}{(s+2)^2} + \frac{(-1/16)}{s+2}\right]$$

$$= L^{-1}\left[\frac{1}{16}\frac{1}{(s-2)} - \frac{1}{4}\frac{1}{(s+2)^2} - \frac{1}{16}\frac{1}{(s+2)}\right]$$

$$= \frac{1}{16}e^{2t} - \frac{1}{4}L^{-1}\left[\frac{1}{(s+2)^2}\right] - \frac{1}{16}e^{-2t}$$

$$= \frac{1}{16}e^{2t} - \frac{1}{4}e^{-2t}L^{-1}\left(\frac{1}{s^2}\right) - \frac{1}{16}e^{-2t}$$

$$= \frac{1}{16}e^{2t} - \frac{1}{4}e^{-2t}\frac{t}{\lceil 2} - \frac{1}{16}e^{-2t}$$

$$= \frac{1}{16}e^{2t} - \frac{1}{4}te^{-2t} - \frac{1}{16}e^{-2t}$$

4) $L^{-1}\left[\dfrac{s+29}{(s+4)(s^2+9)}\right]$

Soln $L^{-1}\left[\dfrac{s+29}{(s+4)(s^2+9)}\right]$

$$\frac{s+29}{(s+4)(s^2+9)} = \frac{A}{s+4} + \frac{Bs+C}{s^2+9} = \frac{A(s^2+9)+(Bs+C)(s+4)}{(s+4)(s^2+9)}$$

$$s+29 = A(s^2+9) + (Bs+C)(s+4)$$

$$s+29 = As^2 + 9A + Bs^2 + 4Bs + Cs + 4C$$

$$= As^2 + Bs^2 + 4Bs + Cs + 9A + 4C$$

$\Rightarrow s^2+29 = s^2(A+B) + s(4B+C) + 9A+4C$

$\therefore$ On comparing,

$A+B=1$, $4B+C=0$, $9A+4C=29$

$A+B+0=1$

$0+4B+C=0$

$9A+0+4C=29$

On calulating these equations we get,

$A=1$, $B=-1$, $C=5$

$$\therefore L^{-1}\left[\frac{s+29}{(s+4)(s^2+9)}\right] = L^{-1}\left[\frac{1}{s+4} + \frac{(-1)s+5}{s^2+9}\right]$$

$$= L^{-1}\left[\frac{1}{s+4} - \frac{s}{s^2+9} + \frac{5}{s^2+9}\right]$$

$$= e^{-4t} - \cos 3t + \frac{5}{3}\sin 3t$$

5) $L^{-1}\left[\frac{s}{(s^2+16)(s^2+4)}\right]$

Soln $L^{-1}\left[\frac{s}{(s^2+16)(s^2+4)}\right] = L^{-1}\left[s\left(\frac{1}{(s^2+16)(s^2+4)}\right)\right]$ —— (1)

$\frac{1}{(s^2+16)(s^2+4)}$

put $s^2=x$, we get

$$\frac{1}{(x+16)(x+4)} = \frac{A}{x+16} + \frac{B}{x+4} = \frac{A(x+4)+B(x+16)}{(x+16)(x+4)}$$

$1 = A(x+4) + B(x+16)$

put $x=-16 \Rightarrow 1 = A(-16+4) \Rightarrow 1=(-12)A \Rightarrow \boxed{A=-1/12}$

put $x=-4 \Rightarrow 1 = B(-4+16) \Rightarrow 1=(12)B \Rightarrow \boxed{B=\frac{1}{12}}$

$$\therefore \frac{1}{(x+16)(x+4)} = \frac{-1/12}{x+16} + \frac{1/12}{x+4}$$

resubstitute $x=s^2$ $\frac{1}{(s^2+16)(s^2+4)} = \frac{-1}{12}\frac{1}{s^2+16} + \frac{1}{12}\frac{1}{s^2+4}$

$\therefore$ from ①

$$L^{-1}\left[\frac{s}{(s^2+16)(s^2+4)}\right] = L^{-1}\left[s\left(-\frac{1}{12}\frac{1}{s^2+16}+\frac{1}{12}\frac{1}{s^2+4}\right)\right]$$

$$= L^{-1}\left[-\frac{1}{12}\frac{s}{s^2+16}+\frac{1}{12}\frac{s}{s^2+4}\right]$$

$$= -\frac{1}{12}\cos 4t+\frac{1}{12}\cos 2t$$

6) $L^{-1}\left[\frac{1}{(s^2+1)(s^2+36)}\right]$

Soln

$$\frac{1}{(s^2+1)(s^2+36)}$$

put $s^2=x$, we get

$$\frac{1}{(x+1)(x+36)} = \frac{A}{x+1}+\frac{B}{x+36} = \frac{A(x+36)+B(x+1)}{(x+1)(x+36)}$$

$\Rightarrow 1 = A(x+36)+B(x+1)$

put $x=-1 \Rightarrow 1 = A(35) \Rightarrow \boxed{A = 1/35}$

put $x=-36 \Rightarrow 1 = B(-35) \Rightarrow \boxed{B = -\frac{1}{35}}$

$$\Rightarrow \frac{1}{(x+1)(x+36)} = \frac{1/35}{x+1}+\frac{(-1/35)}{x+36}$$

resubstitute $x=s^2$, we get

$$\frac{1}{(s^2+1)(s^2+36)} = \frac{1}{35}\frac{1}{s^2+1}-\frac{1}{35}\frac{1}{s^2+36}$$

$$\therefore L^{-1}\left[\frac{1}{(s^2+1)(s^2+36)}\right] = L^{-1}\left[\frac{1}{35}\frac{1}{s^2+1}-\frac{1}{35}\frac{1}{s^2+36}\right]$$

$$= \frac{1}{35}\sin t-\frac{1}{35\times 6}\sin 6t$$

$$= \frac{1}{35}\sin t-\frac{1}{210}\sin 6t$$

7) $L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right]$

Soln $L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right] = L^{-1}\left[\frac{s^2+2s+1+2}{(s^2+2s+1+4)(s^2+2s+1+1)}\right]$

$$= L^{-1}\left[\frac{(s+1)^2+2}{[(s+1)^2+4][(s+1)^2+1]}\right]$$

$$= e^{-t} L^{-1}\left[\frac{s^2+2}{(s^2+4)(s^2+1)}\right] \longleftarrow (1)$$

$\frac{s^2+2}{(s^2+4)(s^2+1)}$

put $s^2 = x$, we get

$$\frac{x+2}{(x+4)(x+1)} = \frac{A}{x+4} + \frac{B}{x+1} = \frac{A(x+1)+B(x+4)}{(x+4)(x+1)}$$

$x+2 = A(x+1) + B(x+4)$

put $x=-4$, $-2 = A(-3)$ $\Rightarrow \boxed{A = 2/3}$

put $x=-1$, $1 = B(3)$ $\Rightarrow \boxed{B = 1/3}$

$$\frac{x+2}{(x+4)(x+1)} = \frac{2/3}{x+4} + \frac{1/3}{x+1}$$

Resubstitute $x = s^2$

$$\frac{s^2+2}{(s^2+4)(s^2+1)} = \frac{2/3}{s^2+4} + \frac{1/3}{s^2+1}$$

$\therefore$ from (1)

$$L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right] = e^{-t} L^{-1}\left[\frac{s^2+2}{(s^2+4)(s^2+1)}\right]$$

$$= e^{-t} L^{-1}\left[\frac{2}{3}\frac{1}{s^2+4} + \frac{1}{3}\frac{1}{s^2+1}\right]$$

$$= e^{-t}\left[\frac{\cancel{2}}{3}\times\frac{1}{\cancel{2}}\sin 2t + \frac{1}{3}\sin t\right]$$

$$= \frac{e^{-t}}{3}\left(\sin 2t + \sin t\right).$$

8) $L^{-1}\left[\frac{s}{s^4+4a^4}\right]$

<u>Sol</u>

Soln $L^{-1}\left[\frac{s}{s^4+4a^4}\right] = L^{-1}\left[\frac{s}{(s^2)^2+(2a^2)^2}\right]$

$= L^{-1}\left[\frac{s}{(s^2)^2+(2a^2)^2+2s^2(2a^2)-2s^2(2a^2)}\right]$

$= L^{-1}\left[\frac{s}{(s^2+2a^2)^2-(2as)^2}\right]$

$= L^{-1}\left[\frac{s}{(s^2+2a^2-2as)(s^2+2a^2+2as)}\right]$

$= \frac{1}{4a} L^{-1}\left[\frac{1}{s^2+2a^2-2as} - \frac{1}{s^2+2a^2+2as}\right]$

$= \frac{1}{4a} L^{-1}\left[\frac{1}{s^2-2as+a^2+a^2} - \frac{1}{s^2+2as+a^2+a^2}\right]$

$= \frac{1}{4a} L^{-1}\left[\frac{1}{(s-a)^2+a^2} - \frac{1}{(s+a)^2+a^2}\right]$

$= \frac{1}{4a}\left\{L^{-1}\left[\frac{1}{(s-a)^2+a^2}\right] - L^{-1}\left[\frac{1}{(s+a)^2+a^2}\right]\right\}$

$= \frac{1}{4a}\left[e^{at} L^{-1}\left[\frac{1}{s^2+a^2}\right] - e^{-at} L^{-1}\left[\frac{1}{s^2+a^2}\right]\right]$

$= \frac{1}{4a}\left[\frac{e^{at}}{a}\sin at - \frac{e^{-at}}{a}\sin at\right]$

$= \frac{1}{4a^2}\sin at\,(e^{at}-e^{-at})$

Ex. Find $L^{-1}\left[\frac{(s^2+2a)^2 a}{s^4+4a^4}\right]$

Prof. Nancy Sinollin