SVM.

Example:

suppose we are given with following positive labled data points:

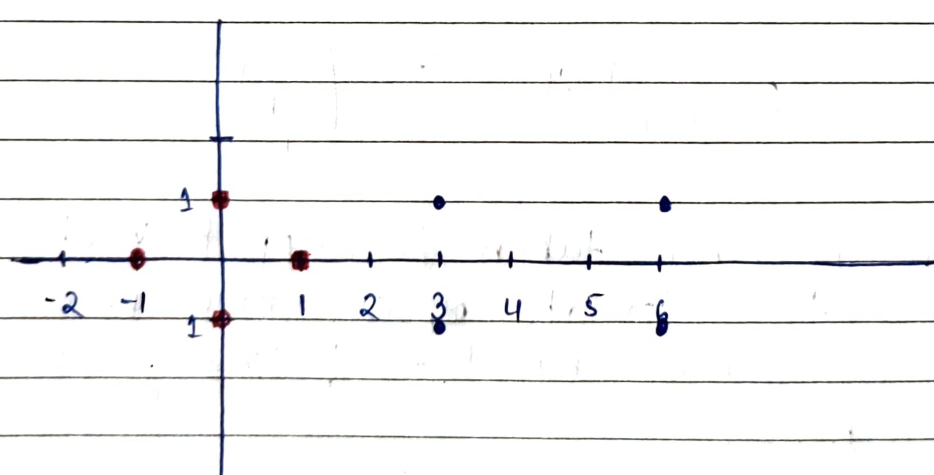
(3,1), (3,-1), (6,1), (6,-1)

& following negative labelled data points.

(1,0), (0,1), (0,-1), (-1,0)

find optimal hyperplane.

→ The plot of the points belonging to two classes is shown as -



now, we need to identify the nearest data points on either side of these particular datapoints, so, we select support vectors as →

(1,0) → from negatively labeled data points
(3,1) } —— positively ——
(3,-1) }

So, support vectors,

$$s_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad s_2 = \begin{pmatrix}3\\1\end{pmatrix}, \quad s_3 = \begin{pmatrix}3\\-1\end{pmatrix}$$

Here each vector is augmented with a 1 as bias input.

So, $s_1 = \begin{pmatrix}1\\0\end{pmatrix}$, then $\tilde{s}_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}$

Similarly

$s_2 = \begin{pmatrix}3\\1\end{pmatrix}$, then $\tilde{s}_2 = \begin{pmatrix}3\\1\\1\end{pmatrix}$

$s_3 = \begin{pmatrix}3\\-1\end{pmatrix}$, ~~then~~ $\tilde{s}_3$ $\begin{pmatrix}3\\-1\\1\end{pmatrix}$

Here we have taken 3 support vectors, so we need to calculate three parameters α_1, α_2 and α_3 based on following three linear equations:

$$\alpha_1 \tilde{s}_1 \cdot \tilde{s}_1 + \alpha_2 \tilde{s}_2 \cdot \tilde{s}_1 + \alpha_3 \tilde{s}_3 \cdot \tilde{s}_1 = -1 \quad \{s_1 \text{ neg. class}\}$$

$$\alpha_1 \tilde{s}_1 \cdot \tilde{s}_2 + \alpha_2 \tilde{s}_2 \cdot \tilde{s}_2 + \alpha_3 \tilde{s}_3 \cdot \tilde{s}_2 = 1$$

$$\alpha_1 \tilde{s}_1 \cdot \tilde{s}_3 + \alpha_2 \tilde{s}_2 \cdot \tilde{s}_3 + \alpha_3 \cdot \tilde{s}_3 \cdot \tilde{s}_3 = 1$$

$\{s_2, s_3$ pos. class$\}$

substitute $\tilde{s}_1, \tilde{s}_2, \tilde{s}_3$ in above eq^n's →

$$\alpha_1 \begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}3\\1\\1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} + \alpha_3 \begin{pmatrix}3\\-1\\1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} = -1$$

$$\alpha_1 \begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}3\\1\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}3\\1\\1\end{pmatrix}\begin{pmatrix}3\\1\\1\end{pmatrix} + \alpha_3 \begin{pmatrix}3\\-1\\+1\end{pmatrix}\begin{pmatrix}3\\1\\1\end{pmatrix} = 1$$

$$\alpha_1 \begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}3\\-1\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}3\\1\\1\end{pmatrix}\begin{pmatrix}3\\-1\\1\end{pmatrix} + \alpha_3 \begin{pmatrix}3\\-1\\1\end{pmatrix}\begin{pmatrix}3\\-1\\1\end{pmatrix} = 1$$

after simplification, we get

$$2\alpha_1 + 4\alpha_2 + 4\alpha_3 = -1$$
$$4\alpha_1 + 11\alpha_2 + 9\alpha_3 = 1$$
$$4\alpha_1 + 9\alpha_2 + 11\alpha_3 = 1$$

on solving, we get

$$\alpha_1 = -3.5$$
$$\alpha_2 = 0.75$$
$$\alpha_3 = 0.75$$

The hyperplane that discriminate the positive class from negative class is given by

$$\tilde{w} = \sum \alpha_i \tilde{s}_i$$

$$= -3.5 \begin{pmatrix}1\\0\\1\end{pmatrix} + 0.75 \begin{pmatrix}3\\1\\1\end{pmatrix} + 0.75 \begin{pmatrix}3\\-1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\-2\end{pmatrix}$$

$$\tilde{w} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

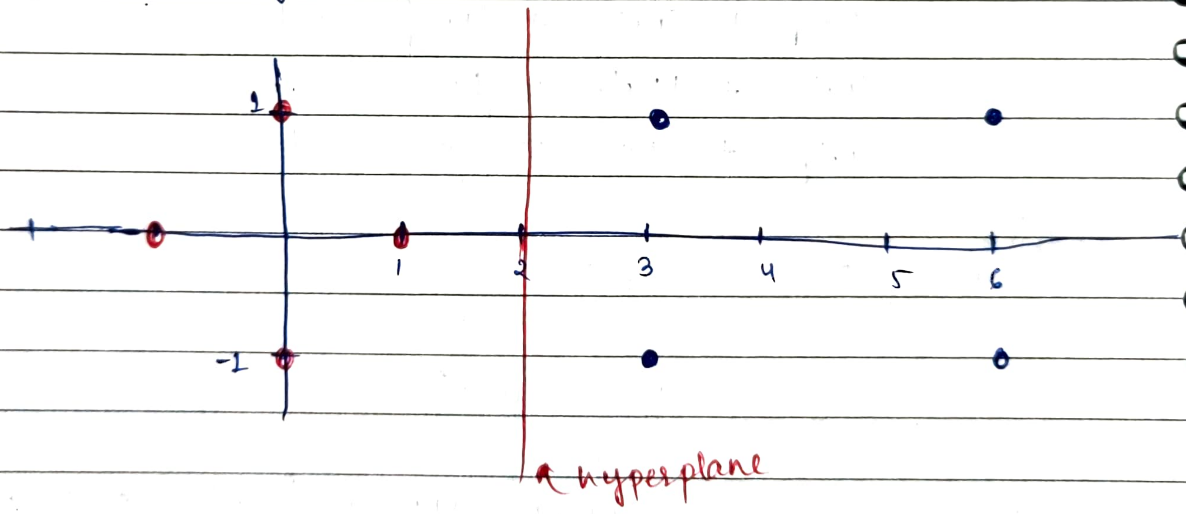
→ Finally, remembering that our vectors are augmented with a bias. Hence we can equate the last entry in $\tilde{w}$ as the hyperplane offset b.

Therefor the seperating hyperplane equation

$$y = wx + b$$

with $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $b = (-2)$

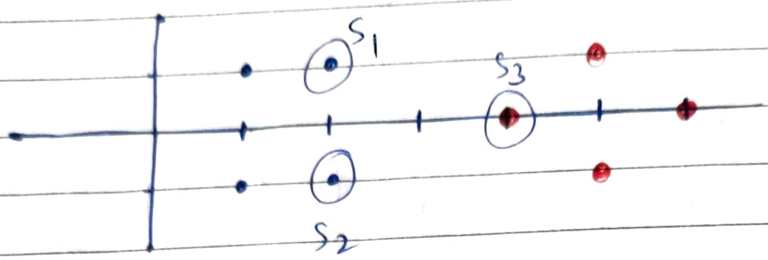
$w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ means $x = 1$, $y = 0$ → the line is parallel to y axis.



Ex: Find optimal hyperplane for the set of data points:

class 1 : {(1, 1) (2, 1) (1, -1) (2, -1)} (Blue)

class 2 : {(4, 0) (5, 1) (5, -1) (6, 0)} (Red)



$S_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad S_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $\quad S_3 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$

we use vector augmented with 1 as a bias input, se

$\tilde{S}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ $\quad \tilde{S}_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ $\quad \tilde{S}_3 = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$

we need to find $\alpha_1, \alpha_2, \alpha_3$ based on following eq^n

$$\alpha_1 \tilde{S}_1 \cdot \tilde{S}_1 + \alpha_2 \tilde{S}_2 \cdot \tilde{S}_1 + \alpha_3 \tilde{S}_3 \cdot \tilde{S}_1 = -1$$
$$\alpha_1 \tilde{S}_1 \cdot \tilde{S}_2 + \alpha_2 \tilde{S}_2 \cdot \tilde{S}_2 + \alpha_3 \tilde{S}_3 \cdot \tilde{S}_2 = -1$$
$$\alpha_1 \tilde{S}_1 \cdot \tilde{S}_3 + \alpha_2 \tilde{S}_2 \cdot \tilde{S}_3 + \alpha_3 \tilde{S}_3 \cdot \tilde{S}_3 = 1$$

Solving we get

$$6\alpha_1 + 4\alpha_2 + 9\alpha_3 = -1$$
$$4\alpha_1 + 6\alpha_2 + 9\alpha_3 = -1$$
$$9\alpha_1 + 9\alpha_2 + 17\alpha_3 = 1$$

$\Rightarrow \alpha_1 = -3.25, \quad \alpha_2 = -3.25, \quad \alpha_3 = 3.5$

hyperplane $\tilde{w} = \sum \alpha_i \tilde{s}_i$

$$= -3.25\begin{pmatrix}2\\1\\1\end{pmatrix} - 3.25\begin{pmatrix}2\\-1\\1\end{pmatrix} + 3.5\begin{pmatrix}4\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\-3\end{pmatrix}$$

so, Hyperplane eq^n

$y = wx + b$ with $w = \begin{pmatrix}1\\0\end{pmatrix}$ and offset $b = -3$