Half Range Sine & Cosine Series.

Note:- Half range series we have to find on interval $(0, l)$. Hence no need to find l separately. It is already given in interval.

Problems:-

1) Find Half range sine & cosine series for $f(x) = lx - x^2 \quad 0 < x < l$

From sine series Prove that;

i) $\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots$

ii) $\frac{\pi^6}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \cdots$

iii) $\frac{\pi^6}{945} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \cdots$

Sol^n Cosine Series:- $b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad \text{—— (1)}$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\int_0^l (lx - x^2)\,dx$$

$$= \frac{2}{l}\left[\frac{lx^2}{2} - \frac{x^3}{3}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l^3}{2} - \frac{l^3}{3}\right]$$

$$= \frac{2}{l}\left[\frac{l^3}{6}\right] = \frac{l^2}{3}$$

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$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l (lx - x^2)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[(lx-x^2)\left[\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right] - (l-2x)\left[\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right] + (-2)\left[\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right]\right]_0^l$$

$$= \frac{2}{l}\left[0 - (-l)\left[-\cos n\pi\left(\frac{l}{n\pi}\right)^2\right] + 0 - 0 + l\left[-\left(\frac{l}{n\pi}\right)^2\right] - 0\right]$$

$$= \frac{2}{l}\left[-(-1)^n\frac{l^3}{(n\pi)^2} - \frac{l^3}{(n\pi)^2}\right]$$

$$= \frac{-2}{\cancel{l}}\frac{l^{\cancel{3}}}{n^2\pi^2}\left[1+(-1)^n\right]$$

$$= \frac{-2l^2}{n^2\pi^2}\left[1+(-1)^n\right]$$

From ① we get,

$$f(x) = \frac{l^2}{3\times 2} + \sum_{n=1}^{\infty}\left(\frac{-2l^2}{n^2\pi^2}\right)\left(1+(-1)^n\right)\cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{l^2}{6} - \frac{2l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{\left[1+(-1)^n\right]}{n^2}\cos\left(\frac{n\pi x}{l}\right).$$

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Sine series :- $a_0 = 0$, $a_n = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{---} \; (2)$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l (lx - x^2)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[(lx-x^2)\left[\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)}\right] - (l-2x)\left[\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}\right] + (-2)\left[\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right]\right]_0^l$$

$$= \frac{2}{l}\left[0 - 0 - 2\cos n\pi\left(\frac{l}{n\pi}\right)^3 - 0 + 0 - (-2)(1)\left(\frac{l}{n\pi}\right)^3\right]$$

$$= \frac{2}{l}\left[-2(-1)^n\frac{l^3}{n^3\pi^3} + 2\frac{l^3}{n^3\pi^3}\right]$$

$$= \frac{2}{\cancel{l}}\,\frac{2l^3}{n^3\pi^3}\left[1-(-1)^n\right]$$

$$= \frac{4l^2}{n^3\pi^3}\left[1-(-1)^n\right]$$

from (2) we get

$$f(x) = \sum_{n=1}^{\infty}\frac{4l^2}{n^3\pi^3}\left[1-(-1)^n\right]\sin\left(\frac{n\pi x}{l}\right).$$

$$= \frac{4l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^n\right]}{n^3}\sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{4l^2}{\pi^3}\left[\frac{2}{1^3}\sin\left(\frac{\pi x}{l}\right) + 0 + \frac{2}{3^3}\sin\left(\frac{3\pi x}{l}\right) + 0 + \cdots\right]$$

$$= \frac{4l^2 \times 2}{\pi^3}\left[\frac{1}{1^3}\sin\left(\frac{\pi x}{l}\right) + \frac{1}{3^3}\sin\left(\frac{3\pi x}{l}\right) + \cdots\right]$$

$$lx-x^2 = \frac{8l^2}{\pi^3}\left[\frac{1}{1^3}\sin\left(\frac{\pi x}{l}\right)+\frac{1}{3^3}\sin\left(\frac{3\pi x}{l}\right)+\cdots\right]$$

put $x=\frac{l}{2}$, we get

$$\frac{l^2}{2}-\frac{l^2}{4} = \frac{8l^2}{\pi^3}\left[\frac{1}{1^3}(1)+\frac{1}{3^3}(-1)+\cdots\right]$$

$$\frac{2l^2-l^2}{4} = \frac{8l^2}{\pi^3}\left[\frac{1}{1^3}-\frac{1}{3^3}+\cdots\right]$$

$$\frac{\cancel{l^2}}{4}\times\frac{\pi^3}{8\cancel{l^2}} = \frac{1}{1^3}-\frac{1}{3^3}+\cdots$$

$$\frac{\pi^3}{32} = \frac{1}{1^3}-\frac{1}{3^3}+\cdots$$ (proved)

By Parseval's Identity for sine series

$$\frac{2}{l}\int_0^l [f(x)]^2\,dx = \sum_{n=1}^{\infty} b_n^2$$

$$\Rightarrow \frac{2}{l}\int_0^l [lx-x^2]^2\,dx = \sum_{n=1}^{\infty}\left[\frac{4l^2}{n^3\pi^3}(1-(-1)^n)\right]^2$$

$$\Rightarrow \frac{2}{l}\int_0^l (l^2x^2-2lx^3+x^4)\,dx = \sum_{n=1}^{\infty}\frac{16l^4}{n^6\pi^6}(1-(-1)^n)^2$$

$$\Rightarrow \frac{2}{l}\left[\frac{l^2x^3}{3}-\frac{2lx^4}{4}+\frac{x^5}{5}\right]_0^l = \frac{16l^4}{\pi^6}\sum_{n=1}^{\infty}\frac{(1-(-1)^n)^2}{n^6}$$

$$\Rightarrow \frac{2}{l}\left[\frac{l^5}{3}-\frac{\cancel{2}l^5}{\cancel{4}_2}+\frac{l^5}{5}\right] = \frac{16l^4}{\pi^6}\left[\frac{2^2}{1^6}+0+\frac{2^2}{3^6}+0+\cdots\right]$$

$$\Rightarrow \frac{\cancel{2}}{l}\left[\frac{10l^5-15l^5+6l^5}{30}\right]\times\frac{\pi^6}{8\,\cancel{16}\,l^4} = 2^2\left[\frac{1}{1^6}+\frac{1}{3^6}+\cdots\right]$$

$$\Rightarrow \frac{\cancel{l^5}\times\pi^6}{\cancel{l}\times 30\times 8\cancel{l^4}} = 4\left[\frac{1}{1^6}+\frac{1}{3^6}+\cdots\right]$$

$$\Rightarrow \frac{\pi^6}{30\times8\times4} = \frac{1}{1^6}+\frac{1}{3^6}+\cdots$$

$$\Rightarrow \frac{\pi^6}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \cdots \quad \text{---} \quad (3)$$

How To prove,

$$\frac{\pi^6}{945} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \cdots$$

Let,

$$S = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \cdots$$

$$= \left(\frac{1}{1^6} + \frac{1}{3^6} + \cdots\right) + \left(\frac{1}{2^6} + \frac{1}{4^6} + \cdots\right)$$

$$= \frac{\pi^6}{960} + \left[\frac{1}{(2\cdot1)^6} + \frac{1}{(2\cdot2)^6} + \frac{1}{(2\cdot3)^6} + \cdots\right]$$

$$S = \frac{\pi^6}{960} + \frac{1}{2^6}\left[\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \cdots\right]$$

$$S = \frac{\pi^6}{960} + \frac{1}{64}(S)$$

$$\Rightarrow S - \frac{S}{64} = \frac{\pi^6}{960}$$

$$\Rightarrow \frac{63S}{64} = \frac{\pi^6}{960}$$

$$\Rightarrow S = \frac{\pi^6}{\cancel{960}_{\cancel{60}\ 15}} \times \frac{\cancel{64}^{\cancel{4}\ 1}}{63}$$

$$= \frac{\pi^6}{15 \times 63}$$

$$\Rightarrow S = \frac{\pi^6}{945} \Rightarrow \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \cdots = \frac{\pi^6}{945}$$

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2) Obtain half range sine series for $f(x) = x^2$ in $0 < x < 3$

Soln Here, $l = 3$.

For half range sine series $a_0 = 0$, $a_n = 0$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{3}\right) \longrightarrow (1)$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{3}\int_0^3 x^2 \sin\left(\frac{n\pi x}{3}\right) dx$$

$$= \frac{2}{3}\left[x^2\left(\frac{-\cos\left(\frac{n\pi x}{3}\right)}{\left(\frac{n\pi}{3}\right)}\right) - (2x)\left(\frac{-\sin\left(\frac{n\pi x}{3}\right)}{\left(\frac{n\pi}{3}\right)^2}\right) + 2\left(\frac{\cos\left(\frac{n\pi x}{3}\right)}{\left(\frac{n\pi}{3}\right)^3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[9(-\cos n\pi)\left(\frac{3}{n\pi}\right) - 0 + 2\cos(n\pi)\frac{3^3}{n^3\pi^3} - 0 + 0 - 2(1)\frac{3^3}{n^3\pi^3}\right]$$

$$= \frac{2}{3}\left[\frac{27}{n\pi}(-(-1)^n) + \frac{2\times 27}{n^3\pi^3}(-1)^n - \frac{2\times 27}{n^3\pi^3}\right]$$

$$= \frac{2\times \cancel{27}^{9}}{\cancel{3}_1}\left[\frac{-(-1)^n}{n\pi} + \frac{2(-1)^n - 2}{n^3\pi^3}\right]$$

$$= 18\left[\frac{2(-1)^n - 2}{n^3\pi^3} - \frac{(-1)^n}{n\pi}\right]$$

$$\therefore f(x) = \sum_{n=1}^{\infty} 18\left[\frac{2(-1)^n - 2}{n^3\pi^3} - \frac{(-1)^n}{n\pi}\right]\sin\left(\frac{n\pi x}{3}\right)$$

—— From (1)

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3) Find Half range sine series for

$f(x) = x \quad 0 \le x \le 2$

$= 4-x, \quad 2 \le x \le 4.$

Solⁿ Given interval in $(0,4)$ $\therefore l = 4$.

For Half range sine series $a_0 = 0$ & $a_n = 0$.

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{4}\right) \quad \text{——— (1)}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{4}\int_0^4 f(x)\sin\left(\frac{n\pi x}{4}\right)dx$$

$$= \frac{1}{2}\left[\int_0^2 x\sin\left(\frac{n\pi x}{4}\right)dx + \int_2^4 (4-x)\sin\left(\frac{n\pi x}{4}\right)dx\right]$$

$$= \frac{1}{2}\left\{\left[x\left(\frac{-\cos\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)}\right) - (1)\left(\frac{-\sin\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)^2}\right)\right]_0^2 + \left[(4-x)\left(\frac{-\cos\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)}\right) - (-1)\left(\frac{-\sin\left(\frac{n\pi x}{4}\right)}{\left(\frac{n\pi}{4}\right)^2}\right)\right]_2^4\right\}$$

$$= \frac{1}{2}\left[2\left(-\cos\left(\frac{n\pi}{2}\right)\right)\frac{4}{n\pi} + \sin\left(\frac{n\pi}{2}\right)\left(\frac{4}{n\pi}\right)^2 - 0 + 0 + 0 - 0 - (2)\left(-\cos\left(\frac{n\pi}{2}\right)\right)\left(\frac{4}{n\pi}\right) + (-1)\left(-\sin\left(\frac{n\pi}{2}\right)\right)\left(\frac{4}{n\pi}\right)^2\right]$$

$$= \frac{1}{2}\left[\cancel{\frac{-8}{n\pi}\cos\left(\frac{n\pi}{2}\right)} + \frac{16}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \cancel{\frac{8}{n\pi}\cos\left(\frac{n\pi}{2}\right)} + \frac{16}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{32}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{16}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

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From (1)

$$f(x) = \sum \frac{16}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{4}\right)$$

4) Ex:-

Obtain half-range cosine series for $f(x) = x(2-x)$ in $0 < x < 2$.

5) Ex:-

Find half range cosine series for the function $f(x) = \frac{\pi}{4} - x$; $0 < x < \pi$

Hence deduce that, $\frac{\pi}{4}\left(\frac{\pi}{2} - x\right) = \frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots$

$$\frac{\pi}{8} x(\pi - x) = \frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots$$

6) Ex:-

Find a cosine series of period 2π to represent $\sin x$ in $0 \le x \le \pi$

7) Ex:-

Obtain half range cosine series $f(x) = x$ $(0, 2)$ & prove that $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$

$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

8) Ex:-

Obtain half range sine series of $f(x) = \begin{cases} x & 0 < x < \pi/2 \\ \pi - x & \pi/2 < x < \pi \end{cases}$

and deduce $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \cdots$

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9) Ex :-

Obtain half range cosine series of $f(x) = x(\pi - x)$ in $(0, \pi)$. Hence deduce that,

i) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \Rightarrow$ i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

ii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12} \Rightarrow$ ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \cdots = \frac{\pi^2}{12}$

iii) $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$

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