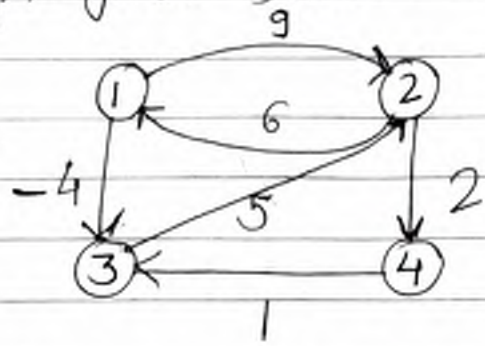
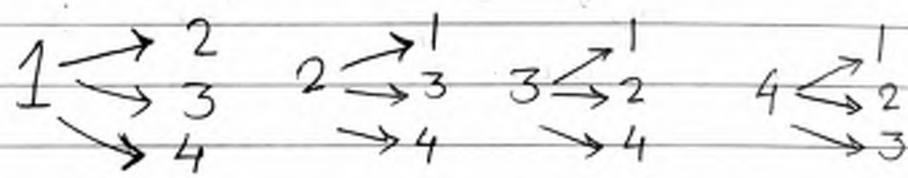
All pair shortest path (Floyd Warshall Algo)

Suppose we have a directed ~~graph~~ weighted graph given below.



Now we need to find all pair shortest path.

If we consider 1 as source we need to find shortest path from 1 to 2, 1 to 3, 1 to 4



If we consider 2 as source then we need to find shortest path from 2 to 1, 2 to 3, 2 to 4.

And similarly for vertex 3 & vertex 4.

This is the problem statement of all pair shortest path.

Advantages of floyd Warshall Algorithm :-

1) It works on the principle of dynamic programming.

2) It works with the graphs having weight in negative ~~values~~ weights.

3) single execution of the algo will give shortest path for all pairs

Step 1 Let's consider 1 as middle element ~~in~~
Now we need to find shortest distance between 1 to 2, 1 to 3 & 1 to 4.

For finding shortest distance between 1 to 2 there are 3 possibilites with minimum weighted path

$1 \longrightarrow 2$

$1 \longrightarrow 3 \longrightarrow 2$

$1 \longrightarrow 4 \longrightarrow 2$

Draw distance matrix

Initial Distance Matrix

$$D^0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 9 & -4 & \infty \\ 6 & 0 & \infty & 2 \\ \infty & 5 & 0 & \infty \\ \infty & \infty & 1 & 0 \end{bmatrix} \end{array}$$

Step 2 Now let's cosider 1 as middle element.

So we need to find the shortest distance for each ~~matrix~~ vertexe via 1. i.e. 1 is middle element.

Now let's calculate distance matrix D^1 by consider D^0 as base matrix.

$$D^0 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 9 & -4 & \infty \\ 2 & 6 & 0 & \infty & 2 \\ 3 & \infty & 5 & 0 & \infty \\ 4 & \infty & \infty & 1 & 0 \end{array}$$

working row

working col^m

Copy working row & working colum as it is D^1

$$D^1 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 9 & -4 & \infty \\ 2 & 6 & 0 & & \\ 3 & \infty & & 0 & \\ 4 & \infty & & & 0 \end{array}$$

To findout distance between 2 to 3

$D^0[2,3]$ $\qquad$ $D^0[2,1] + D^0[1,3]$

$= \infty$ $\qquad$ $= 6 + (-4)$

$= \infty \quad > \quad = 2$

As 2 is minimum distance compared to ∞

we will consider distance between 2 to 3 as 2 instead of ∞.

$$D^1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 0 & 9 & -4 & \infty \\ 6 & 0 & \boxed{2} & \\ \infty & & 0 & \\ \infty & & & 0 \end{array}\right] \end{array}$$

Now calculate distance between 2 to 4

$$\begin{array}{lcl} D^0[2,4] & & D^0[2,1] + D^0[1,4] \\ = 2 & & = 6 + \infty \\ = 2 & \boxed{<} & = \infty \end{array}$$

As 2 is less than ∞ so we will not update distance between 2 to 4 from D^0 we will keep it as it is.

$$D^1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 0 & 9 & -4 & \infty \\ 6 & 0 & 2 & \boxed{2} \\ \infty & & 0 & \\ \infty & & & 0 \end{array}\right] \end{array}$$

Now calculate distance between 3 to 2

$$\begin{array}{lcl} D^0[3,2] & & D^0[3,1] + D^0[1,2] \\ = 5 & & = \infty + 9 \\ = 5 & \boxed{<} & = \infty \end{array}$$

So we will not update value for 3 to 2.

Using same formula we need find distance between 3 to 4, 4 to 2 & 4 to 3.

$$D^1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 9 & -4 & \infty \\ 6 & 0 & 2 & 2 \\ \infty & 5 & 0 & \infty \\ \infty & \infty & 1 & 0 \end{bmatrix} \end{array}$$

<u>Step 3</u> Now let's consider 2 as middle element

We need to calculate shortest path for all vertices via 2 by considering D^1 as base matrix

$$D^1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & \boxed{9} & -4 & \infty \\ \boxed{6} & \boxed{0} & \boxed{2} & \boxed{2} \\ \infty & \boxed{5} & 0 & \infty \\ \infty & \boxed{\infty} & 1 & 0 \end{bmatrix} \end{array}$$

← working row

↑ working colm

$$D^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 9 & & \\ 6 & 0 & 2 & 2 \\ & 5 & 0 & \\ & \infty & & 0 \end{bmatrix} \end{array}$$

Now find out value for 1 to 3

$D^1[1,3]$ $\quad$ $D^1[1,2] + D^1[2,3]$
$= -4$ $\quad$ $= 9 + 2$
$= -4$ $\boxed{<}$ $= 11$

So we will not update ~~di~~ value for 1 to 3 in D^2

Now find out value for 1 to 4 ~~in~~ D^2

$D^1[1,4]$ $\quad$ $D^1[1,2] + D^1[2,4]$
$= \infty$ $\quad$ $= 9 + 2$
$= \infty$ $\boxed{>}$ $= 11$

So we will update value for 1 to 4 in D^2

Now find out value for 3 to 1

$D^1[3,1]$ $\quad$ $D^1[3,2] + D^1[2,1]$
$= \infty$ $\quad$ $= 5 + 6$
$= \infty$ $\boxed{>}$ $= 11$

So we need to update value for 3 to 1 in D^2

Similarly we need to calculate value for 3 to 4, 4 to 1, 4 to 3.

$$D^2 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 9 & -4 & 11 \\ 2 & 6 & 0 & 2 & 2 \\ 3 & 11 & 5 & 0 & 7 \\ 4 & \infty & \infty & 1 & 0 \end{array}$$

Step 4 Now let's consider 3 as middle element

So we need to calculate shortest path for all vertices via 3 by considering D^2 as base matrix

$$D^3 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & & -4 & \\ 2 & & 0 & 2 & \\ 3 & 11 & 5 & 0 & 7 \\ 4 & & & 1 & 0 \end{array}$$

Find out distance between 1 to 2

$$\begin{array}{lcl} D^2[1,2] & & D^2[1,3] + D^2[3,2] \\ = 9 & & = -4 + 5 \\ = 9 & \boxed{>} & = 1 \end{array}$$

So we need to update value for 1 to 2.

$$\begin{array}{lcl} D^2[1,4] & & D^2[1,3] + D^2[3,4] \\ = 11 & & = -4 + 7 \\ = 11 & \boxed{>} & = 3 \end{array}$$

So we need to update value for 1 to 4

$$D^3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & -4 & 3 \\ 6 & 0 & 2 & 2 \\ 11 & 5 & 0 & 7 \\ 12 & 6 & 1 & 0 \end{bmatrix} \end{array}$$

Step 5 Now let's consider 4 as middle element by considering D^3 as base matrix

$$D^4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & & & 3 \\ & 0 & & 2 \\ & & 0 & 7 \\ 12 & 6 & 1 & 0 \end{bmatrix} \end{array}$$

$D^3[1,2]$ $D^3[1,4] + D^3[4,2]$

$= 1$ $= 3 + 6$

$= 1$ $\boxed{<}$ $= 9$

So we will not update value of 1 to 2.

$$D^4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & -4 & 3 \\ 6 & 0 & 2 & 2 \\ 11 & 5 & 0 & 7 \\ 12 & 6 & 1 & 0 \end{bmatrix} \end{array}$$

Matrix D^4 will give the solution which is all pair shortest path

Formula based on working principle

$$D^k[i,j] = \min\{D^{k-1}[i,j],\ D^{k-1}[i,k] + D^{k-1}[k,j]\}$$

```
for (k=1 to n)
{
    for (i= 1 to n)      // row
    {
        for (j=1 to n)   // column
        {
            Formula
        }
    }
}
```