Q. 2

a) Find Singular value of Decomposition of matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

Solution: Given that $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$

· Let $A = UDV^T$ be SVD of A.

Step 1: Find V :

Consider the matrix $A^TA = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$.

eigenvalues of A^TA are 2, 3.

Let $\lambda_1 = 3$, $\lambda_2 = 2$

For $\lambda_1 = 3$,

we get corresponding eigenvector. $x_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

For $\lambda_2 = 2$,

we get, corresponding eigenvector. $x_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$\therefore V = \begin{bmatrix} \frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$V^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ ——— } (*).$$

Step 2: Find D: order of D = order of A.

$\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$, $\sigma_2 = \sqrt{\lambda_2} = \sqrt{2}$.

No. of non zero eigenvalues = Rank = 2.

$$D = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 3: Find U:

Consider the matrix $AA^T = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$\therefore$ eigenvalues of AA^T are $3, 2, 0$.

Let $\lambda_1 = 3$, $\lambda_2 = 2$, $\lambda_3 = 0$

For $\lambda_1 = 3$, we get corresponding eigenvector $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda_2 = 2$, we get $x_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

For $\lambda_3 = 0$, we get $x_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$$U = \begin{bmatrix} | & | & | \\ \frac{x_1}{\|x_1\|} & \frac{x_2}{\|x_2\|} & \frac{x_3}{\|x_3\|} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

$\therefore$ $A = UDV^T$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}_{3\times 3} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{2\times 2}$$

random variable X has following probability distribu...

Q1 (a) $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$

The char. eqn. is $|A - \lambda I| = 0$

(1)

$$\begin{vmatrix} -5-\lambda & 2 \\ -7 & 4-\lambda \end{vmatrix} = 0$$

$$(-5-\lambda)(4-\lambda) + 14 = 0$$

$$-20 + 5\lambda - 4\lambda + \lambda^2 + 14 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$\lambda = 2, -3$$

(a) When $\lambda = 2$

$$[A - \lambda I]X = 0$$

$$[A - 2I]X = 0$$

$$\begin{bmatrix} -7 & 2 \\ -7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 - R_1$

$$\begin{bmatrix} -7 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-7x_1 + 2x_2 = 0$$

$$x_1 = 2, \quad x_2 = 7$$

$\therefore$ for $\lambda = 2$, $X_1 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$

(b) When $\lambda = -3$

$$[A - \lambda I]X = 0$$

$$[A + 3I]X = 0$$

$$\begin{bmatrix} -2 & 2 \\ -7 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2/2$, $R_3/7$

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 - R_1$

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 = 0$

Let $x_1 = 1$, $x_2 = 1$

$\therefore$ for $\lambda = -3$, $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Q1 (b)

A random variable X has following probability distribution

X	0	1	2	3	4	5	6
P(X=x)	K	3K	5K	7K	9K	11K	13K

Find (i) Value of K and Mean of X

(ii)Find Cumulative Distribution function of X

Sol: (i) $\Sigma p_i = 1$

$$k+3k+5k+7k+9k+11k+13k = 1$$

$$49k = 1$$

$$k = \frac{1}{49}$$

$$E(X) = \Sigma x_i p_i$$

$$= (0)(k)+(1)(3k)+(2)(5k)+(3)(7k)+(4)(9k)$$
$$+(5)(11k)+(6)(13k)$$

$$= 3k+10k+21k+36k+55k+78k$$

$$= 203k$$

$$= 203 \times \frac{1}{49} = \frac{203}{49} = 4.1428$$

(2)

X	0	1	2	3	4	5	6
P(X=x)	$\frac{1}{49}$	$\frac{3}{49}$	$\frac{5}{49}$	$\frac{7}{49}$	$\frac{9}{49}$	$\frac{11}{49}$	$\frac{13}{49}$
F(X=x)	$\frac{1}{49}$	$\frac{4}{49}$	$\frac{9}{49}$	$\frac{16}{49}$	$\frac{25}{49}$	$\frac{36}{49}$	1

Q1 (d)

Obtain the Hessian Matrix for the function

$$Z = x_1x_2 + 9x_1 + 6x_3 - x_1^2 - x_2^2 - x_3^2$$

Sol:

$$\frac{\partial z}{\partial x_1} = x_2 + 9 - 2x_1$$

$$\frac{\partial z}{\partial x_2} = x_1 + 0 - 2x_2$$

$$\frac{\partial z}{\partial x_3} = 6 - 2x_3$$

$$\frac{\partial^2 z}{\partial x_1^2} = -2, \quad \frac{\partial^2 z}{\partial x_2^2} = -2, \quad \frac{\partial^2 z}{\partial x_3^2} = -2$$

$$\frac{\partial^2 z}{\partial x_1 \partial x_2} = 1, \quad \frac{\partial^2 z}{\partial x_1 \partial x_3} = 0, \quad \frac{\partial^2 z}{\partial x_2 \partial x_3} = 0$$

$$\therefore \quad H = \begin{bmatrix} \frac{\partial^2 z}{\partial x_1^2} & \frac{\partial^2 z}{\partial x_1 \partial x_2} & \frac{\partial^2 z}{\partial x_1 \partial x_3} \\ \frac{\partial^2 z}{\partial x_2 \partial x_1} & \frac{\partial^2 z}{\partial x_2^2} & \frac{\partial^2 z}{\partial x_2 \partial x_3} \\ \frac{\partial^2 z}{\partial x_3 \partial x_1} & \frac{\partial^2 z}{\partial x_3 \partial x_2} & \frac{\partial^2 z}{\partial x_3^2} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Q 2 (b)

Two samples of sizes 9 and 8 give the sum of squares of deviations from their respective means as 160 inches square and 91 inches square respectively. Test the hypothesis that the variances of the two populations from which the samples are drawn are equal at 10% level of significance.

(Given $f_{((8,7),0.05)}=3.73$, $f_{((8,7),0.95)}=0.286$)

Sol.

Step 1: $H_0: \sigma_x^2 = \sigma_y^2$

$H_a: \sigma_x^2 \neq \sigma_y^2$

Step 2: Data

$m=9, n=8$

$\sum_{i=1}^{9} (x-\bar{x})^2 = 160, \quad \sum_{j=1}^{8} (y-\bar{y})^2 = 91$

Step 3: Level of significance $\alpha = 10\%$

Step 4: Test statistics:

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} = \frac{S_1^2}{S_2^2}$$

$$S_x^2 = \frac{1}{m-1} \sum_{i=1}^{m} (x_i-\bar{x})^2, \quad S_y^2 = \frac{1}{n-1} \sum_{j=1}^{n} (y_i-\bar{y})^2$$

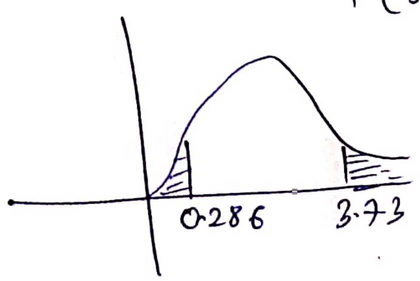
$$S_x^2 = \frac{160}{8} = 20, \quad S_y^2 = \frac{91}{7} = 13$$

$$F = \frac{S_x^2}{S_y^2} = \frac{20}{13} = 1.54$$

Step 5: critical value:

The critical value for two tailed test at (8,7) with 10% L.O.S. is

$f_{(8,7)\,0.05} = 3.73$ & $f_{(8,7)\,0.95} = 0.286$



Step 6: Decision

$\because$ F (calculated) = 1.54 lies in the region of acceptance of H_0

$\therefore$ H_0 is accepted

Q 3 (a)

The following table gives the random sample of marks obtained by students in two schools, A and B

School A	63	72	80	60	85	83	70	72	81
School B	86	93	64	82	81	75	86	63	63

Is the variance of Marks of the students in School A is less than that of those in School B? Test at 5% level of significance.

(Given $F_{((8,8),0.95)=0.291}$)

Sol:

Step 1: Null Hypothesis $H_0: \sigma_x^2 = \sigma_y^2$

Alternate Hypothesis $H_a: \sigma_x^2 < \sigma_y^2$

Step 2:

$$S_x^2 = \frac{1}{m-1} \sum_{i=1}^{m} (x_i - \bar{x})^2 = 78.5$$

$$S_y^2 = \frac{1}{n-1} \sum_{i=1}^{m} (y_i - \bar{y})^2 = 128$$

$\bar{x} = 74$ & $\bar{y} = 77$

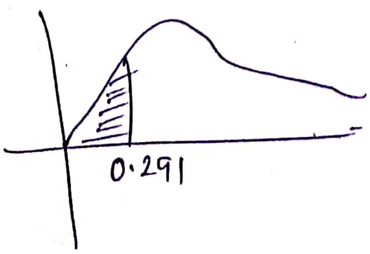
$$F = \frac{S_x^2}{S_y^2} = \frac{78.5}{128} = 0.613$$

Step 3: Level of significance $\alpha = 5\%$

Step 4: Critical value

$f_{(8,8), 0.95} = 0.291$

Step 5: Decision



$\because$ f(calculated) = 0.613 lies in the region of acceptance of Ho

$\therefore$ Ho is accepted

Q 5(a)

Minimize the function $f(x_1, x_2) = 4x_1 + 8x_2 - x_1^2 - x_2^2$

subject to $x_1 + x_2 = 4, \quad x_1, x_2 \geq 0$

Sol: The Langrange's function is given by

$$L(x, \lambda) = 4x_1 + 8x_2 - x_1^2 - x_2^2 - \lambda(x_1 + x_2 - 4)$$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 4 - 2x_1 - \lambda = 0 \quad \text{———} (1)$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 8 - 2x_2 - \lambda = 0 \quad \text{———} (2)$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x_1 + x_2 = 4 \quad \text{———} (3)$$

solving (1), (2) & (3), we get

$$x_1 = 1, \quad x_2 = 3, \quad \lambda = 2$$

$$\therefore \; p \equiv (1, 3)$$

$$Q = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$O =$ Null matrix of size $1 \times 1 = [0]$

$$P = [\nabla g_1(x)] = \left[\frac{\partial}{\partial x_1} g_1(x) \;\; \frac{\partial}{\partial x_2} g_2(x)\right] = [1 \;\; 1]$$

$$\therefore \; H_B = \left[\begin{array}{c|c} O & P \\ \hline P^t & Q \end{array}\right] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

starting order $= 2m+1 = 2 \times 1 + 1 = 3$

No. of principle minor determinants $= n - m$
$= 2 - 1 = 1$

$$\Delta = \begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{vmatrix} = 4 > 0$$

$\because \; \Delta > 0$

$\therefore \; f(x_1, x_2)$ has maxima at $p \equiv (1, 3)$

$\therefore \; Z_{max} = 18$ at $x_1 = 1, \; x_2 = 3$

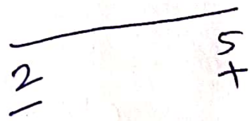
Q5 (b)

Find the minimizer of $f(x) = x^2 + \frac{54}{x}$ using bisection method in (2,5) within a range of 0.3

Sol:

$$f(x) = x^2 + \frac{54}{x}$$

$$f'(x) = 2x - \frac{54}{x^2}$$



Iteration 1

Let $a = x_1 = 2$

$b = x_2 = 5$

$$f'(2) = 4 - \frac{54}{4} = -9.5 < 0$$

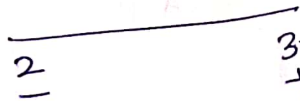
$$f'(5) = 10 - \frac{54}{\cancel{100}\ 25} = 7.84 > 0$$

$$\therefore z = \frac{x_1 + x_2}{2} = \frac{2+5}{2} = 3.5$$

$$f'(3.5) = 2(3.5) - \frac{54}{(3.5)^2} = 2.5918$$

$$|f'(3.5)| = 2.5918 \neq 0.3$$

Iteration 2



$\because f'(3.5) = 2.5918 > 0$

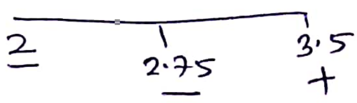
Let $x_1 = 2$ & $x_2 = 3.5$

$$\therefore z = \frac{x_1 + x_2}{2} = \frac{2+3.5}{2} = 2.75$$

$$f'(2.75) = 2(2.75) - \frac{54}{(2.75)^2} = -1.6405$$

$$|f'(2.75)| = |-1.6405| \neq 0.3.$$

Iterations 3



$\because f'(2.75) < 0$

Let $x_1 = 2.75$ & $x_2 = 3.5$

$$\therefore z = \frac{x_1 + x_2}{2} = \frac{2.75 + 3.5}{2} = 3.125$$

$$f'(3.125) = 2(3.125) - \frac{54}{(3.125)^2} = 0.7204$$

$$|f'(3.125)| = |0.7204| \nleq 0.3$$

Iteration 4,

2.75 (−) 3.125 (+) 3.5 (+)

$\because$ $f'(3.125) > 0$

Let $x_1 = 2.75$, $x_2 = 3.125$

$$z = \frac{x_1 + x_2}{2} = \frac{2.75 + 3.125}{2} = 2.9375$$

$$f'(z) = f'(2.9375) = 2(2.9375) - \frac{54}{(2.9375)^2} = -0.3830$$

$\therefore$ $|f'(2.9375)| = |-0.3830| \nleq 0.3$

Iteration 5

2.75 (−) 2.9375 (−) 3.125 (+)

$\because$ $f'(2.9375) < 0$

Let $x_1 = 2.9375$, $x_2 = 3.125$

$$z = \frac{x_1 + x_2}{2} = \frac{2.9375 + 3.125}{2} = 3.03125$$

$$f'(z) = f'(3.03125) = 2(3.03125) - \frac{54}{(3.03125)^2}$$

$$= 0.1856$$

$$|f'(3.03125)| = |0.1856| \leq 0.3$$

$\therefore$ The ~~root~~ minimium is $z = 3.03125$