Taylor's and Laurent's Series

Taylor's series

It is a representation of a function as an infinite sum of terms that are calculated from the values of the functions derivatives at a single point

Applications:-

* Used to find the sum of series
* to Evaluate the limits.
* We can use Taylor's polynomials to approximate the functions.

Formula

$$f(z) = f(z_0) + (z-z_0)f'(z_0) + \frac{(z-z_0)^2}{2!}f''(z_0) + \cdots + \infty.$$

If we put $z_0 = 0$, then

$$f(z) = f(0) + z f'(0) + \frac{z^2}{2!}f''(0) + \cdots + \infty.$$

Laurent's Series:-

The Laurent's series of a complex function $f(z)$ is a representation of that function as a power series which includes terms of negative degree.

Laurent's series expansion

If C_1 and C_2 are two concentric circles of radii r_1 & r_2 with centre at z_0 and if $f(z)$ is analytic on C_1 & C_2 and in the annular region R between the two circles, then for any point z in R

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

where $a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-z_0)^{n+1}} dw.$

$$b_n = \frac{1}{2\pi i} \oint_{C} \frac{f(w)}{(w-z_0)^{-n+1}} dw.$$

Note:-

i) $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

ii) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} \mp \cdots$

iii) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} \mp \cdots$

iv) $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$

v) $\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$

vi) $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} \cdots$

vii) $(1+z)^{-1} = 1 - z + z^2 - z^3 + z^4 - \cdots$

viii) $(1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots$

Note:- The series (vii) & (viii) are convergent only if $|z| < 1$. Hence in $\frac{1}{a-z}$

i) if $a > z$

$$\frac{1}{a-z} = \frac{1}{a(1-z/a)} = \frac{1}{a}(1-z/a)^{-1}$$

ii) If $z > a$

$$\frac{1}{a-z} = \frac{1}{z(1-a/z)} = \frac{-1}{z}(1-a/z)^{-1}.$$

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1) Find the Laurent's series which represents the function $f(z) = \dfrac{2}{(z-1)(z-2)}$ when i) $|z|<1$ ii) $1<|z|<2$ iii) $|z|>2$.

Soln:-

Let $\dfrac{2}{(z-1)(z-2)} = \dfrac{A}{z-1} + \dfrac{B}{z-2}$

$$2 = A(z-2) + B(z-1).$$

$z=2$ $\boxed{2 = B}$

$z=1$ $2 = -A \Rightarrow \boxed{A = -2}$

$$\therefore \frac{2}{(z-1)(z-2)} = \frac{-2}{z-1} + \frac{2}{z-2}.$$

i) $|z| < 1$

$\Rightarrow |z| < 2$

$$\therefore f(z) = \frac{-2}{(1-z)} + \frac{2}{2(z/2-1)}.$$

$$= \frac{-2}{1-z} - \frac{1}{(1-z/2)}.$$

$$= -2(1-z)^{-1} - (1-z/2)^{-1}$$

$$= -2(1+z+z^2+\dots) - (1+z/2+(z/2)^2+(z/2)^3+\dots)$$

ii) $1<|z|<2$.

$\Rightarrow |z| > 1$; $|z| < 2$.

$$\therefore f(z) = \frac{-2}{z(1-1/z)} + \frac{2}{2(z/2-1)}.$$

$$= \frac{-2}{z}(1-1/z)^{-1} - (1-z/2)^{-1}$$

$$= \frac{-2}{z}\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\dots\right) - \left(1-(z/2)+(z/2)^2+\dots\right)$$

iii) $|z| > 2$

$\Rightarrow |z| > 1$

$$\therefore f(z) = \frac{-2}{z(1-1/z)} + \frac{2}{z(1-2/z)}$$

$$= \frac{-2}{z}(1-1/z)^{-1} + \frac{2}{z}(1-2/z)^{-1}$$

$$= \frac{-2}{z}\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right) + \frac{2}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right)$$

2) Obtain Laurent's series expansion of

$f(z) = \frac{1}{z^2+4z+3}$ when i) $1<|z|<3$ ii) $|z|>3$.

Soln:-

Let $f(z) = \frac{1}{z^2+4z+3} = \frac{1}{(z+1)(z+3)}$

$$= \frac{A}{z+1} + \frac{B}{z+3}.$$

i.e.) $1 = A(z+3) + B(z+1)$.

$z = -3$

$1 = -2B$

$\Rightarrow \boxed{B = -1/2}$

$\underline{z = -1}$

$$1 = 2A$$

$$\Rightarrow \boxed{A = 1/2}$$

$$\therefore f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

i) $1 < |z| < 3$

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

$$1 < |z| < 3 \Rightarrow 1 < |z| \; ; \; |z| < 3.$$

$$\therefore |z| > 1$$

$$\therefore f(z) = \frac{1}{2z(1 + 1/z)} - \frac{1}{2 \times 3(z/3 + 1)}$$

$$= \frac{1}{2z}(1 + 1/z)^{-1} - \frac{1}{6}(1 + z/3)^{-1}$$

$$= \frac{1}{2z}\left(1 - 1/z + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \cdots\right)$$

$$- 1/6\left(1 - z/3 + (z/3)^2 - (z/3)^3 + \cdots\right)$$

ii) $|z| > 3$

$\Rightarrow |z| > 1$

$$f(z) = \frac{1}{2z(1+1/z)} - \frac{1}{2z(1+3/z)}$$

$$= \frac{1}{2z}(1+1/z)^{-1} - \frac{1}{2z}(1+3/z)^{-1}$$

$$= \frac{1}{2z}\left(1 - 1/z + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \cdots\right)$$

$$- \frac{1}{2z}\left(1 - 3/z + (3/z)^2 - (3/z)^3 + \cdots\right)$$

3) Obtain Laurent's expansions of $f(z) = \dfrac{z-1}{z^2-2z-3}$ indicating regions of convergence.

Soln:-

Let $\dfrac{z-1}{z^2-2z-3} = \dfrac{z-1}{(z+1)(z-3)} = \dfrac{A}{z+1} + \dfrac{B}{z-3}$

$$\Rightarrow z-1 = A(z-3) + B(z+1)$$