• Properties of Laplace Transform

(We will see proofs of only first three properties then we will list all the properties together.)

1) Change of Scale property :

If $L[f(t)] = \phi(s)$, then $L[f(at)] = \frac{1}{a}\phi\left(\frac{s}{a}\right)$

proof Given $L[f(t)] = \phi(s)$

$$\Rightarrow \int_0^\infty e^{-st} f(t)\,dt = \phi(s) \quad \text{— by def}^n \quad \text{— 1)}$$

$$L[f(at)] = \int_0^\infty e^{-st} f(at)\,dt$$

put $at = u \Rightarrow t = \frac{u}{a} \Rightarrow dt = \frac{du}{a}$

as $t: 0 \to \infty \Rightarrow u: 0 \to \infty$

$$= \int_0^\infty e^{-s\left(\frac{u}{a}\right)} f(u) \frac{du}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)u} f(u)\,du$$

$$L[f(at)] = \frac{1}{a}\phi\left(\frac{s}{a}\right) \quad \text{— from 1)}$$

2) First Shifting Theorem :

If $L[f(t)] = \phi(s)$, then $L[e^{-at} f(t)] = \phi(s+a)$

proof Given $L[f(t)] = \phi(s)$

$$\Rightarrow \int_0^\infty e^{-st} f(t)\,dt = \phi(s) \quad \text{— by def}^n \quad \text{— 1)}$$

$$L[e^{-at} f(t)] = \int_0^\infty e^{-st} e^{-at} f(t)\,dt$$

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$$= \int_0^\infty e^{-st-at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\,dt$$

$L[e^{-at} f(t)] = \phi(s+a)$ —— from 1>

similarly $L[e^{at} f(t)] = \phi(s-a)$ one can prove

3> Second Shifting Theorem:

If $L[g(t)] = \phi(s)$ and $f(t) = g(t-a)$, $t>a$
$= 0$, $t<a$

then prove that $L[f(t)] = e^{-as}\phi(s)$

Soln $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^a e^{-st} f(t)\,dt + \int_a^\infty e^{-st} f(t)\,dt$$

$$= \int_0^a e^{-st} (0)\,dt + \int_a^\infty e^{-st} g(t-a)\,dt$$

put $t-a = u \Rightarrow t = a+u \Rightarrow dt = du$

as $t: a \to \infty$ $u: 0 \to \infty$

$$= 0 + \int_0^\infty e^{-s(a+u)} g(u)\,du$$

$$= \int_0^\infty e^{-sa} e^{-su} g(u)\,du$$

$$= e^{-as} \int_0^\infty e^{-su} g(u)\,du$$

$$= e^{-as} L[g(u)]$$

$= e^{-as}\phi(s)$ —— as $L[g(u)] = L[g(t)] = \phi(s)$

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- List of all properties of Laplace Transform

(If $L[f(t)] = \phi(s)$ then

1) Change of Scale:-

$$L[f(at)] = \frac{1}{a}\,\phi\left(\frac{s}{a}\right)$$

2) First Shifting Theorem:-

$$L[e^{at} f(t)] = \phi(s-a)$$

$$L[e^{-at} f(t)] = \phi(s+a)$$

3) Multiplication by t

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\,\phi(s)$$

4) Division by t

$$L\left[\frac{1}{t} f(t)\right] = \int_s^{\infty} \phi(s)\, ds$$

5) Laplace Transform of Derivative.

$$L\left[\frac{d}{dt}(f(t))\right] = L[f'(t)] = -f(0) + sL[f(t)]$$

$$L\left[\frac{d^2}{dt^2}(f(t))\right] = L[f''(t)] = -f'(0) - sf(0) + s^2 L[f(t)]$$

6) Laplace Transform of Integration

$$L\left[\int_0^t f(u)\, du\right] = \frac{1}{s}\,\phi(s) \qquad \text{where, } L[f(t)] = L[f(u)] = \phi(s)$$

$$L\left[\underbrace{\int_0^t \int_0^t \cdots \int_0^t}_{n\,\text{times}} f(u)\, \underbrace{du\, du \cdots du}_{n\,\text{times}}\right] = \frac{1}{s^n}\,\phi(s)$$

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- **Problems :-**

Find Laplace Transform

1) If $L[f(t)] = \frac{2}{s^3} e^{-s}$ Find $L[f(2t)]$

Soln Given, $L[f(t)] = \frac{2}{s^3} e^{-s} = \phi(s)$

By Change of Scale property,
$$L[f(at)] = \frac{1}{a}\phi\left(\frac{s}{a}\right).$$

$$\therefore L[f(2t)] = \frac{1}{2}\phi\left(\frac{s}{2}\right) \quad \text{— here } a=2$$

$$= \frac{1}{\cancel{2}} \frac{\cancel{2} e^{-s/2}}{(s/2)^3} = \frac{8 e^{-s/2}}{s^3}$$

2) If $L[f(t)] = \frac{20-4s}{s^2-4s+20}$ find $L[f(3t)]$

Soln Given, $L[f(t)] = \frac{20-4s}{s^2-4s+20} = \phi(s)$

By Change of Scale property, $L[f(at)] = \frac{1}{a}\phi\left(\frac{s}{a}\right)$

$$\therefore L[f(3t)] = \frac{1}{3}\phi\left(\frac{s}{3}\right) \quad \text{— here } a=3$$

$$= \frac{1}{3} \frac{20 - 4\frac{s}{3}}{\left(\frac{s}{3}\right)^2 - 4\left(\frac{s}{3}\right) + 20}$$

$$= \frac{1}{3} \frac{60-4s}{3\left(\frac{s^2}{9} - \frac{4s}{3} + 20\right)} = \frac{1}{\cancel{9}} \frac{(60-4s)}{\frac{s^2-12s+180}{\cancel{9}}}$$

$$= \frac{60-4s}{s^2-12s+180}$$

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