* Let z be a complex no.
i.e. $z = x+iy$
where x & y are real numbers.
& $i = \sqrt{-1}$ is imaginary number.
then,
$$|z| = \sqrt{x^2+y^2}$$

* Evaluation of line integrals i.e.
$$\int_C f(z)\,dz$$

where C is a curve
as $z = x+iy$
$dz = dx + i\,dy$
& one can separate $f(z)$ in form of real & imaginary parts.
i.e. $f(z) = u+iv$.

$$\int_C f(z)\,dz$$
$$= \int_C (u+iv)(dx+i\,dy)$$

The function under integral sign is a functn of x & y. Then using curve C. will convert the function under integral sign in terms of one variable either x or y.

17) Evaluate :

$$\int_0^{1+i} (x-y+ix^2)\,dz$$

i) along the line $z=0$ to $z=1+i$

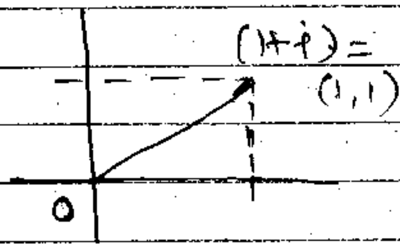
ii) along the real axis from $z=0$ to $z=1$ when the line parallel to imaginary axis from $z=1$ to $z=1+i$

iii) along imaginary axis from $z=0$ to $z=i$ when along the line parallel to real axis from $z=i$ to $z=1+i$

iv) along the parabola $y^2=x$

Ans:-

i) along $z=0$ to $z=1+i$



$$I=\int_0^{1+i}(x-y+ix^2)\,dz$$

as $z=x+iy$

$dz=dx+i\,dy$

$$=\int_0^{1+i}(x-y+ix^2)(dx+i\,dy)$$

C is $y=x$

$dy=dx$

limit $x=0$, $y=1$

$$I=\int_0^1 (x-x+ix^2)(dx+i\,dx)$$

$$=\int_0^1 ix^2(1+i)\,dx$$

$$= i(1+i)\int_0^1 x^2\,dx$$

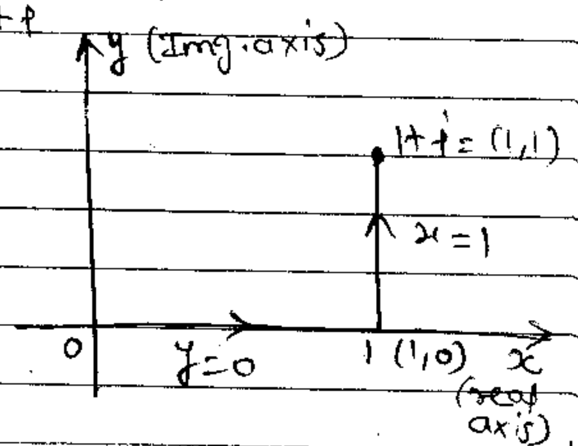
$$I = (i-1)\int_0^1 x^2\,dx$$

$$= (i-1)\left[\frac{x^3}{3}\right]_0^1$$

$$= (i-1)\left[\frac{1}{3} - \frac{0}{3}\right]$$

$$I = (i-1)\left(\frac{1}{3}\right)$$

ii> along the real axis from $z=0$ to $z=1$ along the line parallel the imaginary axis from $z=1$ to $z=1+i$



Here, C_1 is $y=0$

C_2 is $x=1$

$$I = \int_0^{1+i} (x-y+ix^2)\,dz$$

$$I = \int_{C_1} (x-y+ix^2)\,dz + \int_{C_2} (x-y+ix^2)\,dz \quad \text{--- (1)}$$

$$\int_{C_1} (x-y+ix^2)\,dz$$

$$= \int_{C_1} (x-y+ix^2)\,(dx+i\,dy)$$

C_1 is $y=0 \Rightarrow dy=0$

$$= \int_0^1 (x+ix^2)\,(dx)$$

$$= \int_0^1 (x + i x^2)\, dx$$

$$= \left[\frac{x^2}{2} + i\frac{x^3}{3}\right]_0^1$$

$$= \left[\frac{1}{2} + \frac{i}{3}\right] //$$

$$\int_{C_2} (x - y + i x^2)\, dz$$

$$= \int_{C_2} (x - y + i x^2)(dx + i dy)$$

C_2, $x = 1 \Rightarrow dx = 0$

$$= \int_0^1 (1 - y + i)(0 + i\, dy)$$

$$= i \int_0^1 (1 - y + i)\, dy$$

$$= i\left[y - \frac{y^2}{2} + iy\right]_0^1$$

$$= i\left[1 - \frac{1}{2} + i\right]$$

$$= i\left[\frac{1}{2} + i\right] //$$

$$I = C_1 + C_2$$

$$= \left[\frac{1}{2} + \frac{i}{3}\right] + i\left[\frac{1}{2} + i\right]$$

$$= \frac{1}{2} + \frac{i}{3} + \frac{i}{2} + i^2$$

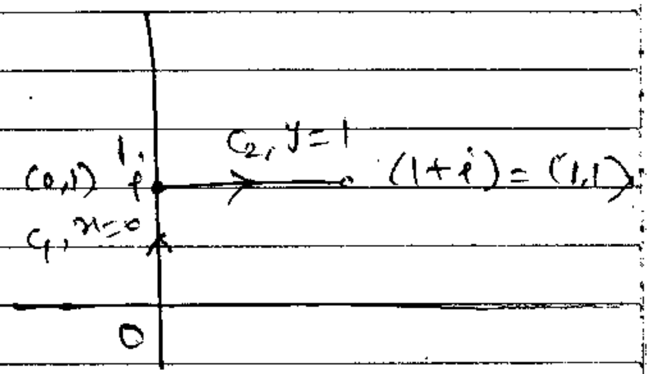
$$= \frac{1}{2} + \frac{5i}{6} - 1$$

$$\therefore \boxed{I = \frac{-1}{2} + \frac{5i}{6}} \;//$$

iii) along img axis, $z = 0$ to $z = i$
parallel to real $z = i$ to $z = 1+i$

Here C_1 is $x = 0$
C_2 is $y = 1$

$$I = \int_0^{1+i} (x - y + ix^2)\,dz$$



$$I = \int_{C_1} (x - y + ix^2)\,dz + \int_{C_2} (x - y + ix^2)\,dz \quad \text{--- (2)}$$

$$\int_{C_1} (x - y + ix^2)\,dz$$

C_1 is $x = 0 \Rightarrow dx = 0$

$$= \int_{C_1} (-y)\, i\, dy$$

$$= -i \int_0^1 (y)\, dy$$

$$= -i \left[\frac{y^2}{2} \right]_0^1$$

$$= -i \left[\frac{1}{2} \right]$$

$$= \frac{-i}{2}$$

$$\int_{C_2} (x - y + i x^2)\, dz$$

C_2 is $y = 1 \Rightarrow dy = 0$

$$= \int_0^1 (x - 1 + i x^2)\, dx$$

$$= \left[\frac{x^2}{2} - x + i \frac{x^3}{3} \right]_0^1$$

$$= \left[\frac{1}{2} - 1 + \frac{i}{3} \right]$$

$$= \frac{-1}{2} + \frac{i}{3}$$

from eqn (11)

$$I = C_1 + C_2$$

$$= -\frac{i}{2} - \frac{1}{2} + \frac{i}{3}$$

$$I = -\frac{1}{2} - \frac{i}{6}$$

$$\therefore \boxed{I = -\frac{1}{2} - \frac{i}{6}}$$

iv) along the parabola $y^2 = x$

$$= \int_0^{1+i} (x - y + ix^2)\, dz$$

$$= \int_0^{1+i} (x - y + ix^2)(dx + i\,dy)$$

$$\therefore y^2 = x$$

$$x = y^2$$

$$\therefore dx = 2y\,dy$$

$$= \int_0^1 (y^2 - y + iy^4)(2y\,dy + i\,dy)$$

$$= \int_0^1 (y^2 - y + iy^4)(2y + i)\,dy$$

$$= \int_0^1 \left[2y(y^2 - y + iy^4) + i(y^2 - y + iy^4)\right] dy$$

$$= \int_0^1 \left[2y^3 - 2y^2 + 2iy^5 + iy^2 - iy - y^4\right] dy$$

$$= \int_0^1 \left[2y^3 - 2y^2 + 2iy^5 + iy^2 - iy - y^4\right] dy$$

$$= \left[\frac{2y^4}{4} - \frac{2y^3}{3} + \frac{2iy^6}{6} + \frac{iy^3}{3} - \frac{iy^2}{2} - \frac{y^5}{5}\right]_0^1$$

$$= \left[\frac{2}{4} - \frac{2}{3} + \frac{2i}{6} + \frac{i}{3} - \frac{i}{2} - \frac{1}{5}\right]$$

$$= \left[\frac{2}{4} - \frac{2}{3} - \frac{1}{5} + \frac{2i}{6} + \frac{i}{3} - \frac{i}{2}\right]$$

$$= \left[\frac{6-8}{12} - \frac{1}{5} + \frac{i}{3} + \frac{i}{3} - \frac{i}{2}\right]$$

$$= \left[-\frac{1}{6} - \frac{1}{5} + \frac{2i}{3} - \frac{i}{2}\right]$$

$$= \left[\frac{-5-6}{30} + \frac{4i-3i}{6}\right]$$

$$I = \left[-\frac{11}{30} + \frac{i}{6}\right]$$

NOTE:

i) for above problem line integral is different along different paths i.e. depending upon path.

ii) Result :

If $f(z)$ is analytic function then line integral is path independant.

[If $f(z)$ satisfies C.R. equations $u_x = v_y$ & $u_y = -v_x$]

iii) For above problem.

- $f(z) = x - y + ix^2$
$= u + iv$

$u = x - y$ $\quad\quad v = x^2$

$u_x = 1$ $\quad\quad v_x = 2x$

$u_y = -1$ $\quad\quad v_y = 0$

$\therefore u_x \neq v_y$

$\Rightarrow f(z)$ is not analytic ~~does not satisfy~~. because C-R equations are not ~~satisfi~~ equal.

$\therefore$ Above line integral depending on path.

2nd lect $\Longrightarrow$

① $\displaystyle\int_{1-i}^{2+i} (2x + iy + 1)\,dz$

i) along straight line $1-i$ to $2+i$

ii) $x = t+1$, $y = 2t^2 - 1$.

Ans:- i) along straight line $1-i$ to $2+i$

i.e. $(1,-1)$ to $(2,1)$

line eqn,

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow \frac{y - (-1)}{x - 1} = \frac{1 - (-1)}{2 - 1}$$

$$\Rightarrow \frac{y+1}{x-1} = \frac{2}{1}$$

$$\frac{y+1}{x-1} = \frac{2}{1}$$

$$\frac{y+1}{x-1} = \frac{2}{1}$$

$$y+1 = 2x-2$$
$$y = 2x-2-1$$
$$y = 2x-3 \;//$$

$$\int_{1-i}^{2+i} (2x+iy+1)\,dz$$

$$z = x+iy$$
$$dz = dx + i\,dy$$

$$\int_{1-i}^{2+i} (2x+iy+1)(dx+i\,dy)$$

$$y = 2x-3 \Rightarrow dy = 2dx.$$

$$I = \int_{1}^{2} [2x+i(2x-3)+1]\,(dx+i2dx)$$

$$I = \int_{1}^{2} [2x+i2x-3i+1][1+2i]\,dx$$

$$I = (1+2i)\int_{1}^{2} [2x+i2x-3i+1]\,dx.$$

$$\therefore I = (1+2i)\int_1^2 [2x + i2x - 3i + 1]\,dx$$

$$I = (1+2i)\left[\frac{2x^2}{2} + \frac{i2x^2}{2} - 3ix + x\right]_1^2$$

$$I = (1+2i)\left[x^2 + ix^2 - 3ix + x\right]_1^2$$

$$I = (1+2i)\left\{\left[(2)^2 + i(2)^2 - 3i(2) + 2\right] - \left[(1)^2 + i(1)^2 - 3i(1) + 1\right]\right\}$$

$$= (1+2i)\left[4 + 4i - 6i + 2 - 1 - i + 3i - 1\right]$$

$$= (1+2i)\left[4\right]$$

$$\boxed{I = 4 + 8i}$$

ii) $x = t+1,\ y = 2t^2 - 1$

$dx = dt \quad , \quad dy = 4t\,dt.$

$$I = \int_{1-i}^{2+i} (2x + iy + 1)\,dz$$

$$I = \int_{1-i}^{2+i} (2x + iy + 1)\,(dx + i\,dy)$$

$$I = \int_{1-i} (2x + iy + 1)(dx + i\,dy)$$

$$I = \int_0^1 \left[2(t+1) + i(2t^2-1) + 1\right]\left[dt + i4t\,dt\right]$$

$$\therefore I = \int_0^1 \left[2(t+1) + i(2t^2-1) + 1\right](1 + i4t)\,dt$$

$$= \int_0^1 \left[2t + 2 + i2t^2 - i + 1\right]\left[1 + i4t\right] dt$$

$$= \int_0^1 \left\{\left[i2t^2 + 2t - i + 3\right]\left[1 + i4t\right]\right\} dt$$

$$= \int_0^1 \left\{1\left(i2t^2 + 2t - i + 3\right) + i4t\left(i2t^2 + 2t - i + 3\right)\right\} dt$$

$$= \int_0^1 \left\{\left(i2t^2 + 2t - i + 3\right) + i^2 8t^3 + i8t^2 - i^2 4t + 12it\right\} dt$$

$$= \int_0^1 \left[i2t^2 + 2t - i + 3 - 8t^3 + i8t^2 + 4t + 12it\right] dt$$

$$= \int_0^1 \left[10it^2 - 8t^3 + 6t - i + 3 + 12it\right] dt$$

$$= \int_0^1 \left[10it^2 - 8t^3 + 12it + 6t - i + 3\right] dt$$

$$= \left[\frac{10it^3}{3} - \frac{8t^4}{4} + \frac{12it^2}{2} + \frac{6t^2}{2} - it + 3t\right]_0^1$$

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$$I = \left[\frac{10it^3}{3} - \frac{8t^4}{4} + \frac{12it^2}{2} + \frac{6t^2}{2} - it + 3t\right]_0^1$$

$$= \left[\frac{10i}{3} - \frac{8}{4} + \frac{12i}{2} + \frac{6}{2} - i + 3\right]$$

$$= \left[\frac{10i}{3} - 2 + 6i + 3 - i + 3\right]$$

$$= \left[\frac{10i}{3} + 6i - i - 2 + 3 + 3\right]$$

$$= \left[\frac{10i}{3} + 5i + 4\right]$$

$$= \left[\frac{10i + 15i}{3} + 4\right]$$

$$= \left[\frac{25i}{3} + 4\right]$$

$$I = \left[\frac{25i + 12}{3}\right]$$ //

(2) $\int_0^{1+i} (x^2 + iy)\, dz$

along i) $y = x$, ii) $y = x^2$

Is the line integral is indepedent of the path if yes then why?

→ $I = \int_0^{1+i} (x^2 + iy)(dx + i\,dy)$

$$\therefore I = \int_0^{1+i} (x^2+iy)(dx+idy)$$

$$= \int_0^1 (x^2+ix)(dx+idx)$$

$$= \int_0^1 (x^2+ix)(1+i)\,dx$$

$$= (1+i)\int_0^1 (x^2+ix)\,dx$$

$$= (1+i)\left[\frac{x^3}{3}+\frac{ix^2}{2}\right]_0^1$$

$$= (1+i)\left[\frac{1}{3}+\frac{i}{2}\right]$$

$$I = (1+i)\left[\frac{1}{3}+i\frac{1}{2}\right] //$$

ii) $y = x^2$

$dy = 2x\,dx$

$$\int_0^{1+i} (x^2+iy)(dx+idy)$$

$$I = \int_0^1 (x^2+ix^2)(dx+i\,2x\,dx)$$

$$= \int_0^1 (1+i)x^2\cdot(1+i2x)\,dx$$

$$\therefore I = (1+i)\int_0^1 x^2(1+i2x)\,dx$$

$$= (1+i)\int_0^1 (x^2 + i2x^3)\,dx$$

$$= (1+i)\left[\frac{x^3}{3} + i\frac{2x^4}{4}\right]_0^1$$

$$= (1+i)\left[\frac{x^3}{3} + \frac{ix^4}{2}\right]_0^1$$

$$= (1+i)\left[\frac{1}{3} + i\frac{1}{2}\right]$$

Now, to check that the line integral path is independent or not.
We have to check that the f(z) is analytic or not.

$$f(z) = x^2 + iy$$
$$= u + iv$$

$u = x^2$ $\quad$ $v = y$
$u_x = 2x$ $\quad$ $v_x = 0$
$u_y = 0$ $\quad$ $v_y = 1$

$$\therefore u_x \neq v_y$$

$\therefore$ The CR eqn s are not equal.
$\therefore$ The f(z) is not analytic.
Hence, the line integral is path dependant.

NOTE :

It doesn't mean that if along the different paths line integral is same then path indepedent. Hence always verify C-R equations to check the function is analytic or not as we know that every analytic function is path indepedent.

(3) find $\int_C f(z)\,dz$

where $f(z) = \begin{cases} 4y & y>0 \\ 1 & y<0 \end{cases}$

C is arc from $z = -1-i$ to $1+i$ of the cubical curve $y = x^3$

$\rightarrow$

$$I = \int_C f(z)\,dz$$

$$= \int_C f(z)\,(dx + i\,dy)$$

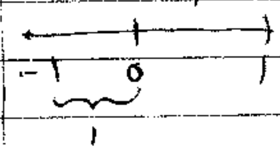
$y = x^3$

$dy = 3x^2\,dx$

$$f(z) = \begin{cases} 4y, & y>0 \\ 1, & y<0 \end{cases} = \begin{cases} 4x^3 & x^3>0 \quad x>0 \\ 1 & x^3<0 \quad x<0 \end{cases}$$

$$I = \int_{-1}^{1} f(z)\,(dx + i\,3x^2\,dx)$$

4·3



$$= \int_{-1}^{0} f(z)\,(dx + i\,3x^2\,dx) + \int_{0}^{1} f(z)\,(dx + i\,3x^2\,dx)$$

$$= \int_{-1}^{0} (1+i3x^2)\,dx + \int_{0}^{1} 4x^3(1+i3x^2)\,dx$$

$$\int_{-1}^{0} (1+i3x^2)\,dx$$

$$= \left[x + \frac{i3x^3}{3}\right]_{-1}^{0}$$

$$= \left[x + ix^3\right]_{-1}^{0}$$

$$= -\left[-1 + i(-1)^3\right]$$

$$= -\left[-1 - i\right]$$

$$\boxed{I_1 = 1+i}$$ //

$$I_2 = 4\int_{0}^{1} x^3(1+i3x^2)\,dx$$

$$I_2 = 4\int_{0}^{1} (x^3 + i3x^5)\,dx$$

$$= 4\left[\frac{x^4}{4} + i3\frac{x^6}{6}\right]_{0}^{1}$$

$$= 4\left[\frac{1}{4} + i3\frac{1}{6}\right]$$

$$= 4\left[\frac{1}{4} + i\frac{1}{2}\right]$$

$$= \boxed{1 + i2}$$ //

$$\therefore I = I_1 + I_2$$

$$I = (1+i) + (1+2i)$$

$$\boxed{I = 2+3i}$$ //

(4) $\int_0^{2+i} (\bar{z})^2 \, dz$

$\longrightarrow$

along i) line $x = 2y$
ii) Real axis from 0 to 2 &
vertically from 2 to $2+i$

• $z = x + iy$
$\bar{z} = x - iy$

i) line $x = 2y$

$$\int_0^{2+i} (x - iy)^2 (dx + i\,dy)$$

$x = 2y$
$dx = 2dy$

ii) $\int_0^{2+i} (x - iy)^2 (dx + i\,dy)$