Parshvanath Charitable Trust's
A. P. SHAH INSTITUTE OF TECHNOLOGY
(Approved by AICTE New Delhi & Govt. of Maharashtra, Affiliated to University of Mumbai)
(Religious Jain Minority)

Subject: Applied Mathematics III

SEM: III

INVERSE LAPLACE TRANSFORM.

- Inverse Laplace Transform:

If $L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\,dt = \phi(s)$ then $f(t)$ is called as the inverse Laplace transform of $\phi(s)$ and it is denoted by $L^{-1}[\phi(s)] = f(t)$.

- Formulae :

Laplace Transform	Inverse Laplace Transform
1) $L(1) = \frac{1}{s}$	1) $L^{-1}\left(\frac{1}{s}\right) = 1$
2) $L[e^{at}] = \frac{1}{s-a}$	2) $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$.
3) $L[e^{-at}] = \frac{1}{s+a}$	3) $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$.
4) $L[\sin at] = \frac{a}{s^2+a^2}$.	4) $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$.
5) $L[\cos at] = \frac{s}{s^2+a^2}$.	5) $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$
6) $L[\sinh at] = \frac{a}{s^2-a^2}$.	6) $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at$.
7) $L[\cosh at] = \frac{s}{s^2-a^2}$.	7) $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$.
8) $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$.	8) $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{\Gamma(n)}$.

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Examples:

1) Find $L^{-1}\left[\frac{3+2s+s^2}{s^3}\right]$

Sol^n: $L^{-1}\left[\frac{3+2s+s^2}{s^3}\right] = L^{-1}\left[\frac{3}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right]$

$$= 3\cdot L^{-1}\left[\frac{1}{s^3}\right] + 2\cdot L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{1}{s}\right]$$

$$= 3\cdot\frac{t^2}{\lceil 3} + 2\cdot\frac{t}{\lceil 2} + 1$$

$$= \frac{3t^2}{2} + 2t + 1$$

2) Find $L^{-1}\left[\left(\frac{1-\sqrt{s}}{s^2}\right)^2\right]$

Sol^n: $L^{-1}\left[\left(\frac{1-\sqrt{s}}{s^2}\right)^2\right] = L^{-1}\left[\frac{1-2\sqrt{s}+s}{s^4}\right]$

$$= L^{-1}\left[\frac{1}{s^4}\right] - 2L^{-1}\left[\frac{s^{1/2}}{s^4}\right] + L^{-1}\left[\frac{1}{s^3}\right]$$

$$= \frac{t^3}{\lceil 4} - 2\cdot\frac{t^{5/2}}{\lceil 5/2} + \frac{t^2}{\lceil 3}$$

$$= \frac{t^3}{3!} - \frac{2\cdot t^{5/2}}{\lceil 5/2} + \frac{t^2}{2}$$

$$= \frac{t^3}{6} - \frac{2\cdot t^{5/2}}{\lceil 5/2} + \frac{t^2}{2}.$$

- Examples for practice:

*) Find Inverse Laplace transform of following

1) $\frac{2s+3}{s^2+9}$ 2) $\left(\frac{s^2-1}{s^5}\right)^2$ 3) $\frac{s+3}{s^2+4}$ 4) $\frac{4s+15}{16s^2-25}$

5) $\frac{1}{4s-5}$ 6) $\frac{1}{s^{3/2}}$ 7) $\frac{1}{s^2+25}$.

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- Using first shifting theorem,

We know that,

If $L[f(t)] = \phi(s)$, then $L[e^{-at} f(t)] = \phi(s+a)$

Hence, $L^{-1}[\phi(s+a)] = e^{-at} f(t) = e^{-at} L^{-1}[\phi(s)]$

Therefore,

$$L^{-1}[\phi(s+a)] = e^{-at} L^{-1}[\phi(s)]$$

$$L^{-1}[\phi(s-a)] = e^{at} L^{-1}[\phi(s)].$$

- Examples :

1) Find $L^{-1}\left[\frac{s+2}{(s+2)^2-1}\right]$

Sol^n: $L^{-1}\left[\frac{s+2}{(s+2)^2-1}\right] = e^{-2t} L^{-1}\left[\frac{s}{s^2-1}\right]$ --- (We can use above formula since, every s is the form of (s+2))

$= e^{-2t} \cosh t.$

2) Find $L^{-1}\left[\frac{s}{(s+1)^2+2}\right]$

Sol^n: $L^{-1}\left[\frac{s}{(s+1)^2+2}\right] = L^{-1}\left[\frac{(s+1)-1}{(s+1)^2+(\sqrt{2})^2}\right]$

$= e^{-t} L^{-1}\left[\frac{s-1}{s^2+(\sqrt{2})^2}\right]$

$= e^{-t} L^{-1}\left[\frac{s}{s^2+(\sqrt{2})^2} - \frac{1}{s^2+(\sqrt{2})^2}\right]$

$= e^{-t}\left[\cos\sqrt{2}t - \frac{1}{\sqrt{2}} \cdot \sin\sqrt{2}t\right].$

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3] Evaluate $L^{-1}\left[\frac{1}{(s+3)^{3/2}}\right]$

Sol^n: $L^{-1}\left[\frac{1}{(s+3)^{3/2}}\right] = e^{-3t} \cdot L^{-1}\left[\frac{1}{s^{3/2}}\right]$

$$= e^{-3t} \cdot \frac{t^{3/2-1}}{\Gamma 3/2}$$

$$= e^{-3t} \cdot \frac{t^{1/2}}{\frac{1}{2}\Gamma\frac{1}{2}}$$

$$= \frac{2 \cdot e^{-3t} \cdot t^{1/2}}{\sqrt{\pi}}.$$

4] Evaluate $L^{-1}\left[\frac{s}{(s-2)^6}\right]$

Sol^n: $L^{-1}\left[\frac{s}{(s-2)^6}\right] = L^{-1}\left[\frac{(s-2)+2}{(s-2)^6}\right]$

$$= e^{2t} \cdot L^{-1}\left[\frac{s+2}{s^6}\right]$$

$$= e^{2t} \cdot L^{-1}\left[\frac{1}{s^5} + \frac{2}{s^6}\right]$$

$$= e^{2t} \cdot \left[\frac{t^4}{\Gamma 5} + \frac{2t^5}{\Gamma 6}\right]$$

$$= e^{2t} \cdot \left[\frac{t^4}{4!} + \frac{2t^5}{5!}\right]$$

5] Evaluate $L^{-1}\left[\frac{6s-4}{s^2-4s+20}\right]$

Sol^n: $L^{-1}\left[\frac{6s-4}{s^2-4s+20}\right] = L^{-1}\left[\frac{6s-4}{s^2-4s+4-4+20}\right]$

$$= L^{-1}\left[\frac{6s-4}{(s-2)^2+16}\right]$$

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Department of Humanities & Applied Sciences

$$= L^{-1}\left[\frac{6s-12+12-4}{(s-2)^2+4^2}\right]$$

$$= L^{-1}\left[\frac{6(s-2)+8}{(s-2)^2+4^2}\right]$$

$$= e^{2t}\cdot L^{-1}\left[\frac{6s+8}{s^2+4^2}\right]$$

$$= e^{2t}\left[L^{-1}\left[\frac{6s}{s^2+4^2}\right] + 8\cdot L^{-1}\left[\frac{1}{s^2+4^2}\right]\right]$$

$$= e^{2t}\left[6\cdot\cos 4t + 8\cdot\frac{1}{4}\sin 4t\right]$$

$$= e^{2t}\left[6\cos 4t + 2\sin 4t\right].$$

- We can use above method only when, we have factors of denominator as not in integer form. otherwise we will use method of partial fraction.

Method of Partial Fractions:

one can use method of partial fraction directly when degree of polynomial in numerator is less than the degree of polynomial in denominator.

How to apply partial fraction:

IF $\phi(s)$ has following form then we can express as below,

1) $\phi(s) = \dfrac{F(s)}{(s+a)(s+b)} = \dfrac{A}{s+a} + \dfrac{B}{s+b}$.

2) $\phi(s) = \dfrac{F(s)}{(s+a)(s+b)^2} = \dfrac{A}{s+a} + \dfrac{B}{s+b} + \dfrac{C}{(s+b)^2}$.

3) $\phi(s) = \dfrac{F(s)}{(s+a)(s^2+b^2)} = \dfrac{A}{s+a} + \dfrac{Bs+C}{s^2+b^2}$.

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