Engineering Mathematics 3 Solved Question Paper from Dec 2017 to May 2019

COMPUTER ENGINEERING

APPLIED MATHS – 3

(CBCGS DEC 2017)

Q1.a) Find the Laplace transform of $\frac{1}{t}e^{-t}\sin t$. **(5)**

Sol: To find : $L\left[\frac{1}{t}e^{-t}\sin t\right]$

$\Rightarrow L[\sin t] = \frac{1}{s^2+1}$ $\quad$ [Since $L\{\sin at\} = \frac{1}{s^2+a^2}$]

By First Shifting Theorem,

$\Rightarrow L[e^{-t}\sin t] = \frac{1}{(s+1)^2+1}$ $\quad$ [Since $L\{e^{at}f(t)\} = \Phi(s-a)$]

$\Rightarrow L\left[\frac{1}{t}e^{-t}\sin t\right] = \int_s^\infty \frac{1}{(s+1)^2+1}ds$ $\quad$ [Effect of division by t]

$\Rightarrow \left[\tan^{-1}(s+1)\right]_s^\infty$ $\quad$ [$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$]

$\Rightarrow \left[\tan^{-1}(\infty) - \tan^{-1}(s+1)\right]$

$\Rightarrow \frac{\pi}{2} - \tan^{-1}(s+1)$

$\Rightarrow \cot^{-1}(s+1)$ $\quad$ [$\frac{\pi}{2} - \tan^{-1}x = \cot^{-1}x$]

Ans : $L\left[\frac{1}{t}e^{-t}\sin t\right] = \cot^{-1}(s+1)$

Q1.b) Find the inverse Laplace transform of $\frac{1}{\sqrt{2s+1}}$. **(5)**

Sol: To find : $L^{-1}\left[\frac{1}{\sqrt{2s+1}}\right]$

$\Rightarrow L^{-1}\left[\frac{1}{\sqrt{2s+1}}\right] = L^{-1}\left[\frac{1}{2\sqrt{s+\frac{1}{2}}}\right]$

$$\Rightarrow \frac{e^{-\frac{t}{2}}}{\sqrt{2}} L^{-1}\left[\frac{1}{\sqrt{s}}\right] \qquad [L^{-1}\{\phi(s+a)\} = e^{-at}L^{-1}[\phi(s)]]$$

$$\Rightarrow \frac{e^{-\frac{t}{2}}}{\sqrt{2}} \left[\frac{t^{-\frac{1}{2}}}{\Gamma\frac{1}{2}}\right] \qquad [L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{\Gamma n}]$$

$$\Rightarrow \frac{e^{-\frac{t}{2}}}{\sqrt{2\pi}} \left[t^{-\frac{1}{2}}\right]$$

Ans : $L^{-1}\left[\frac{1}{\sqrt{2s+1}}\right] = \frac{e^{-\frac{t}{2}}}{\sqrt{2\pi}} t^{-\frac{1}{2}}$

Q1.c) Show that the function, f(z) = $\sinh(z)$ is analytic and find $f'(z)$ in terms of z (5)

Sol: Given : $f(z) = \sinh(z)$

$\Rightarrow \sinh(x+iy) = \sinh(x)\cosh(iy) + \cosh(x)\sinh(iy)$

$\Rightarrow \sinh(x)\cos(y) + i\cosh(x)\sin(y)$ [cosh(iy)=cos y, sinh(iy)=isin(y)]

Comparing real and imaginary parts,

$u=\sinh(x)\cos(y); v = \cosh(x)\sin(y)$

Differentiating u and v partially with respect to x and y,

$u_x = \cosh(x)\cos(y);\ u_y = -\sinh(x)\sin(y)$

$v_x = \sinh(x)\sin(y);\ v_y = \cosh(x)\cos(y)$

From above equations clearly, we can see that : $u_x = v_y$ & $u_y = -v_x$

Thus CR equations are satisfied and thus the function is analytic.

Therefore; $f'(z) = u_x + iv_x$

$f'(z) = \cosh(x)\cos(y) + i\sinh(x)\sin(y)$

$f'(z) = \cos h(x+iy)$

$f'(z) = \cosh(z)$

Ans : $f'(z)$ **= cosh(z)**

Q1.d) Find the Fourier series for $f(x) = x$ in $(0, 2\pi)$. (5)

Sol: $f(x) = x$

Fourier series is given by : $f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos n\pi x + b_n \sin n\pi x\right)$

Calculating a_0,

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx \quad => \quad a_0 = \frac{1}{2\pi}\int_0^{2\pi} x dx$$

$$a_0 = \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} dx \quad => \quad a_0 = \frac{1}{2\pi}\left[\frac{4\pi^2}{2} - 0\right] dx$$

$$a_0 = \pi \qquad \text{-----------1}$$

Calculating a_n,

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos(nx)dx$$

$$=> \quad a_n = \frac{1}{\pi}\left[\frac{x\sin(nx)}{n} - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi} dx$$

$$=> \quad a_n = \frac{1}{\pi}\left[\left\{\frac{2\pi \sin(2n\pi)}{n} + \frac{\cos(2n\pi)}{n^2}\right\} - \left\{0 + \frac{\cos 0}{n^2}\right\}\right]$$

$$=> \quad a_n = \frac{1}{\pi}\left[\left\{0 + \frac{1}{n^2}\right\} - \left\{0 + \frac{1}{n^2}\right\}\right] \quad => a_n = 0 \qquad \text{-----------2}$$

Calculating b_n,

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin(nx)dx$$

$$=> \quad b_n = \frac{1}{\pi}\left[\frac{x(-\cos nx)}{n} - \left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi} dx$$

$$=> \quad b_n = \frac{1}{\pi}\left[\left\{\frac{2\pi(-\cos 2n\pi)}{n} + \frac{\sin(2n\pi)}{n^2}\right\} - \left\{0 + \frac{\sin 0}{n^2}\right\}\right]$$

$$=> \quad b_n = \frac{1}{\pi}\left[\left\{\frac{-2\pi}{n} + 0\right\} - \{0+0\}\right] \quad => b_n = \frac{-2}{n} \qquad \text{-----------3}$$

Substituting in the Fourier Series, we get;

$$x = \pi + 0 + \sum_{n=1}^{\infty}\left(\frac{-2}{n}\sin n\pi x\right)$$

Ans : $x = \pi - 2\sum_{n=1}^{\infty}\left(\frac{\sin n\pi x}{n}\right)$

Q2.a) Use Laplace transform to prove : $\int_0^{\infty} e^{-t}\frac{\sin^2 t}{t}dt= \frac{1}{4}\log 5.$ **(6)**

Sol: To prove $\int_0^{\infty} e^{-t}\frac{\sin^2 t}{t}dt= \frac{1}{4}\log 5$

LHS : $\int_0^{\infty} e^{-t}\frac{\sin^2 t}{t}dt$

$\Rightarrow L[\sin^2 t] = L\left[\frac{1-\cos(2t)}{2}\right]$

$\Rightarrow \frac{1}{2}L[1 - \cos(2t)]$

$\Rightarrow \frac{1}{2}L\left[\frac{1}{s}-\frac{2}{s^2+4}\right]$ $\left[L\{\cos at\} = \frac{s}{s^2+a^2} ; L[1] = \frac{1}{s} \right]$

$L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\left[\int_s^{\infty}\frac{1}{s}-\frac{s}{s^2+4}\right]ds$

$= \frac{1}{2}\left[\log s-\frac{1}{2}\log(s^2+4)\right]$

$= -\frac{1}{4}[\log(s^2+4)-\log s^2]$

$= -\frac{1}{4}\left[\log\left(\frac{s^2+4}{s^2}\right)\right]_s^{\infty}$

$= \frac{1}{4}\log\left[\frac{s^2+4}{s^2}\right]$

Therefore , $\int_0^{\infty} e^{-st}\frac{\sin^2 t}{t}dt= \frac{1}{4}\log\left[\frac{s^2+4}{s^2}\right]$

Putting s =1;

$$\int_0^\infty e^{-t}\frac{\sin^2 t}{t}dt = \frac{1}{4}\log\left[\frac{1^2+4}{1}\right]$$

$$\int_0^\infty e^{-st}\frac{\sin^2 t}{t}dt = \frac{1}{4}\log[5]$$

Ans : $\int_0^\infty e^{-st}\frac{\sin^2 t}{t}dt = \frac{1}{4}\log[5]$

Q2.b) If $\{f(k)\} = \begin{cases} 4^k, k<0 \\ 3^k, k\geq 0 \end{cases}$ **, find Z{f(k)}. (6)**

Sol: By definition,

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)\,.\,z^{-k}$$

$$Z\{f(k)\} = \sum_{k=-\infty}^{1} 5^k\,.\,z^{-k} + \sum_{k=0}^{\infty} 3^k\,.\,z^{-k}$$

Put k= -n in 1^{st} series,

$$\Rightarrow Z\{f(k)\} = \sum_{n=1}^{\infty} 5^{-n}\,.\,z^{n} + \sum_{k=0}^{\infty} 3^k\,.\,z^{-k}$$

$$\Rightarrow Z\{f(k)\} = \left[\left(\frac{z}{5}\right)+\left(\frac{z}{5}\right)^2+\left(\frac{z}{5}\right)^3+\ldots\right] + \left[1+\left(\frac{3}{z}\right)+\left(\frac{3}{z}\right)^2+\left(\frac{3}{z}\right)^3+\ldots\right]$$

The above two series are sum of infinite GP terms whose summation is given by,

$$S = \frac{a}{1-r}, \text{ where a is 1st term and r is the common ratio between the terms}$$

$$\Rightarrow Z\{f(k)\} = \frac{z}{5}\left[\frac{1}{1-\left(z/5\right)}\right] + \left[\frac{1}{1-\left(3/z\right)}\right]$$

$$\Rightarrow Z\{f(k)\} = \frac{z}{5}\left[\frac{5}{5-z}\right] + \left[\frac{z}{z-3}\right]$$

$$\Rightarrow Z\{f(k)\} = \left[\frac{z}{5-z}\right] + \left[\frac{z}{z-3}\right]$$

$$\Rightarrow Z\{f(k)\} = \frac{z(z-3)+z(5-z)}{(5-z)(z-3)}$$

$$\Rightarrow Z\{f(k)\} = \frac{2z}{(5-z)(z-3)}$$

Ans : $Z\{f(k)\} = \dfrac{2z}{(5-z)(z-3)}$

Q2.c) Show that the function u = $\cos x\cosh y$ **is a harmonic function . Find its harmonic conjugate and corresponding analytic function** **(8)**

Sol : Given : $u = \cos x\cosh y$

$u_x = -\sin x\cosh y \quad ; u_y = \cos y\sinh y$

$u_x^2 = -\cos x\cosh y \quad ; u_y^2 = \cos x\cosh y$

From the above equations,

$u_x^2 + u_y^2 = 0$ Thus the Laplace equation is satisfied.

Therefore, u is harmonic

Let $u_x = \Psi_1(x,y)$ and $u_y = \Psi_2(x,y)$

$\Psi_1(z,0) = -\sin z$ and $\Psi_2(z,0)=0$

By Milne Thompson method,

$f(z) = \int \Psi_1(z,0)dz - \int \Psi_2(z,0)dz$

$f(z) = \int -\sin z dz - \int 0 dz$

$f(z) = \cos z+c$ This is the required analytic function.

Separating real and imaginary parts, putting z=x+iy,

$f(z) = \cos(x+iy)$

$f(z) = \cos x\cos iy - \sin x\sin iy$

$f(z) = \cos x\cos hy - i\sin x\sinh y$ [cos(iy)=cosh(y) and sin(iy)=isinh(y)]

Therefore, $v = -\sin x\sinh y$

Ans : Required analytic function is $f(z) = \cos z+c$

Harmonic conjugate of u = $v = -\sin x\sinh y$

Q3.a) Find the equation of the line of regression of Y on X for the following data. (6)

X	5	6	7	8	9	10	11
Y	11	14	14	15	12	17	16

Sol. The Line of regression Y on X is given as y=a + bx.

x	x^2	y	y^2	xy
5	25	11	121	55
6	36	14	196	84
7	49	14	196	98
8	64	15	225	120
9	81	12	144	108
10	100	17	324	170
11	121	16	256	176
$\Sigma = 56$	$\Sigma = 476$	$\Sigma = 99$	$\Sigma = 1427$	$\Sigma = 811$

Here N=7,

The normal equation are given as follows;

$\Sigma y = Na + b\Sigma x$

$\Sigma xy = a\Sigma x + b\Sigma x^2$

Substituting the values from the above table;

$7a + 56b = 99$

$56a + 476b = 811$

Solving the above two equations simultaneously, we get; a=8.714 and b=0.6786

Thus, the equation of line of regression is : 8.714 $+ \ 0.6786x$

Q3.b) Find the bilinear transformation which maps the points 1, -i, 2 on z plane onto 0, 2, -i respectively of w-plane. (6)

Sol: Let the transformation be $w = \frac{az+b}{cz+d}$ ----- i

Putting the given values, $0 = \frac{a+b}{c+d}$; $2 = \frac{-ai+b}{-ci+d}$; $-i = \frac{2a+b}{2c+d}$

From these equations we get, $a + b = 0$ ----- ii

$(a-2c)i + (2d-b) = 0$ ----- iii

$(2c+d)i + (2a+b) = 0$ ----- iv

From ii we get b = -a.

Putting this value of b in iii and iv, we get

$(a-2c)i + (2d+a) = 0$ ----- v

$(2c+d)i + (a) = 0$ ----- vi

Adding v and vi we get

$(a+d)i + 2(a+d) = 0$ Therefore, $(a+d)(i + 2) = 0$

Thus , d = -a [Since, i ≠ -2]

Putting these values of d and b in $2 = \frac{-ai+b}{-ci+d}$, we get $2 = \frac{-ai-a}{-ci-a} = \frac{a(1+i)}{ci+a}$

Therefore, $2ci + 2a = a + ai$ $\Rightarrow 2ci = -a + ai$

$\Rightarrow 2ci = ai^2 + ai$ $\Rightarrow 2ci = ai(i + 1)$

$2c = a(1 + i)$ $\Rightarrow c = \left(\frac{1+i}{2}\right)a$

Putting these values o b, c, d in (i),

$$w = \frac{az-a}{\left(\frac{1+i}{2}\right)az-a}$$

$$w = \frac{z-1}{\left(\frac{1+i}{2}\right)z-1}$$

$$w = \frac{2(z-1)}{(1+i)z-2}$$

Ans : $w = \dfrac{2(z-1)}{(1+i)z-2}$

Q3.c) Find half range sine series for $f(x) = \begin{cases} x & ,0<x<\frac{\pi}{2} \\ \pi-x & ,\frac{\pi}{2}<x<\pi \end{cases}$. **(8)**

Hence find the sum of $\sum_{(2n-1)}^{\infty}\frac{1}{n^4}$.

Sol: Half range sine series is given by:

$$f(x) = \sum b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$\Rightarrow \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx\, dx + \int_{\pi/2}^{2\pi}(\pi-x)\sin nx\, dx\right]$$

$$\Rightarrow \frac{2}{\pi}\left[\left\{x\left(\frac{-\cos nx}{n}\right)-\left(\frac{-\sin nx}{n^2}\right)(1)\right\}_0^{\pi/2}+\left\{(\pi-x)\left(\frac{-\cos nx}{n}\right)-\left(\frac{-\sin nx}{n^2}\right)(-1)\right\}_{\pi/2}^{\pi}\right]$$

$$\Rightarrow \frac{2}{\pi}\left[\left\{\frac{\pi}{2}\frac{\cos(n\pi/2)}{n}+\frac{\sin(n\pi/2)}{n^2}-0-0\right\}+\left\{0-0+\frac{\pi}{2}\frac{\cos(n\pi/2)}{n}+\frac{\sin(n\pi/2)}{n^2}\right\}\right]$$

$$\Rightarrow \frac{4}{\pi}\frac{\sin(n\pi/2)}{n^2}$$

$$b_1 = \frac{4}{\pi}\frac{1}{1^2}; b_2 = 0; b_3 = -\frac{4}{\pi}\frac{1}{3^2}; b_4 = 0; \ldots.$$

$$f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1^2}-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\frac{\sin 7x}{7^2}+\ldots\right]$$

By Parseval's identity;

$$\frac{1}{\pi}\int_0^{\pi}[f(x)]^2dx = \frac{1}{2}\left[b_1^{\,2}+b_2^{\,2}+b_3^{\,3}+b_4^{\,2}+\ldots+\infty\right] \qquad \text{----- i}$$

$$\frac{1}{\pi}\left[\int_0^{\pi/2} x^2dx+\int_{\pi/2}^{\pi}(\pi-x)^2dx\right] = \frac{1}{2}\left[b_1^{\,2}+b_2^{\,2}+b_3^{\,3}+b_4^{\,2}+\ldots+\infty\right]$$

LHS : $\frac{1}{\pi}\left[\int_0^{\pi/2}x^2dx+\int_{\pi/2}^{\pi}(\pi^2-2\pi x+x^2)dx\right]$

$$=\frac{1}{\pi}\left[\left\{\frac{x^3}{3}\right\}_0^{\pi/2}+\left\{\pi^2x-\pi x^2+\frac{x^3}{3}\right\}_{\pi/2}^{\pi}\right]$$

$$=\frac{1}{\pi}\left[\left\{\frac{x^3}{24}-0\right\}+\left\{\pi^3-\pi^3+\frac{\pi^3}{3}\right\}-\left\{\frac{\pi^3}{2}-\frac{\pi^3}{4}+\frac{\pi^3}{24}\right\}\right]$$

$$=\frac{\pi^2}{12}$$

$$\frac{\pi^2}{12}=\frac{1}{2}\left[\frac{16}{\pi^2}.\frac{1}{1^4}+\frac{16}{\pi^2}.\frac{1}{3^4}+\frac{16}{\pi^2}.\frac{1}{5^4}+\ldots\right]$$

$$\Rightarrow\frac{\pi^2}{96}=\left[\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots\right]$$

$$\Rightarrow\sum_{(2n-1)}^{\infty}\frac{1}{n^4}=\frac{\pi^2}{96},\ n=1,2,3,\ldots$$

Ans : $\sum_{(2n-1)}^{\infty}\frac{1}{n^4}=\frac{\pi^2}{96}$, $n = 1,2,3,\ldots$

Q4.a) Find the inverse Laplace Transform using convolution theorem

$\frac{1}{(s-a)(s+a)^2}$ **(6)**

Sol: $\phi_1(s)=\frac{1}{s-a}$; $\phi_2(s)=\frac{1}{(s+a)^2}$

$$L^{-1}[\phi_1(s)]=L^{-1}\left[\frac{1}{s-a}\right]=e^{at}$$

$$L^{-1}[\phi_2(s)]=L^{-1}\left[\frac{1}{(s+a)^2}\right]=e^{-at}t$$

$$L^{-1}[\phi(s)]=\int_0^t e^{au}.e^{-a(t-u)}(t-u)du$$

$$=\int_0^t e^{au}.e^{-a(t-u)}(t-u)du$$

$$=e^{-at}\int_0^t e^{2au}(t-u)du$$

$$=e^{-at}\left[(t-u)\frac{e^{2au}}{2a}-\frac{e^{2au}}{4a^2}(-1)\right]_0^t$$

$$= e^{-at}\left[0+\frac{e^{2at}}{4a^2}-\left\{\frac{t}{2a}+\frac{1}{4a^2}\right\}\right]$$

$$= \frac{1}{4a^2}[e^{at}-2ate^{-at}+e^{-at}]$$

Ans : $L^{-1}\left[\frac{1}{(s-a)(s+a)^2}\right] = \frac{1}{4a^2}[e^{at}-2ate^{-at}+e^{-at}]$

Q4.b) Calculate the coefficient of correlation between X and Y from the following data (6)

X	8	8	7	5	6	2
Y	**3**	**4**	**10**	**13**	**22**	**8**

Sol:

x	x^2	y	y^2	xy
8	64	3	9	24
8	64	4	16	32
7	49	10	100	70
5	25	12	144	65
6	36	22	484	132
2	4	8	64	16
Σ 36	Σ 242	Σ 60	Σ 842	Σ 339

Here N=6,

$\dot{X} = \frac{36}{6} = 6$ and $\dot{Y} = \frac{60}{6} = 10$

Coefficient of correlation ,

$$r = \frac{\Sigma xy-\frac{\Sigma x\Sigma y}{n}}{\sqrt{\Sigma x^2-\frac{(\Sigma x)^2}{n}}\sqrt{y^2-\frac{(\Sigma y)^2}{n}}}$$

Substituing the values , we get :

$$r = \frac{339 - \frac{36x60}{6}}{\sqrt{242 - \frac{(36)^2}{6}}\sqrt{842 - \frac{(60)^2}{6}}}$$

$r = -0.2647$

Ans : Coefficient of correlation, r = -0.2647

Q4.c) Find the inverse Z-transform of : **(8)**

i) $\frac{1}{(z-a)^2}$**, $|z|$ < a**

ii) $\frac{1}{(z-3)(z-2)}$**, $|z|$ > 3**

Sol: i) $F(z) = \frac{1}{(z-a)^2}, |z| < a$

$$\frac{1}{a^2[1-\left(z/_a\right)]^2} \quad \Rightarrow \frac{1}{a^2}\left(1-\frac{z}{a}\right)^{-2}$$

$$\Rightarrow \frac{1}{a^2}[1 + 2\left(\frac{z}{a}\right)^1 + 3\left(\frac{z}{a}\right)^2 + 4\left(\frac{z}{a}\right)^3 + \dots + (n+1)\left(\frac{z}{a}\right)^n]$$

Coefficient of $z^n = \frac{n+1}{a^{n+2}}$; $n \geq 0$

Put $n = -k$, $z^{-k} = \frac{-k+1}{a^{-k+2}}$; $k \leq 0$

ii) $F(z) = \frac{1}{(z-3)(z-2)}, |z| > 3$

$$\frac{1}{(z-3)(z-2)} = \frac{A}{z-3} + \frac{B}{z-2}$$

$1 = A(z-2) + B(z-3)$

Putting z=2; 1 = -B => B = -1

Putting z=3; 1 = A => A = 1

$$\frac{1}{(z-3)(z-2)} = \frac{1}{z-3} - \frac{1}{z-2}$$

RHS

$$\Rightarrow \frac{1}{z(1-\frac{3}{z})} - \frac{1}{z(1-\frac{2}{z})}$$

$$\Rightarrow \frac{1}{z}\left[1-\frac{3}{z}\right]^{-1} - \frac{1}{z}\left[1-\frac{2}{z}\right]^{-1}$$

$$\Rightarrow \frac{1}{z}[1 + \frac{3}{z} + \left(\frac{3}{z}\right) + \left(\frac{3}{z}\right) + \ldots + \left(\frac{3}{z}\right)^{k-1}] - \frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)+\ldots+\left(\frac{2}{z}\right)^{k-1}\right]$$

Coefficient of z^{-k} = $3^{k-1} - 2^{k-1}$; $k \geq 1$

$$Z^{-1}[F(z)] = 3^{k-1} - 2^{k-1}$$

Q5.a) Using Laplace transform evaluate $\int_0^\infty e^{-t}(1+2t-t^2+t^3)H(t-1)dt$ (6)

Sol : To evaluate $\int_0^\infty e^{-t}(1+2t-t^2+t^3)H(t-1)dt$

$\Rightarrow f(t) = 1 + 2t - t^2 + t^3$; a=1

$\Rightarrow f(t + 1) = 1 + 2(t + 1) - (t+1)^2 + (t+1)^3$

$= 1 + 2t + 2 - (t^2 + 2t + 1) + t^3 + 3t^2 + 3t + 1$

$= t^3 + 2t^2 + 3t + 3$

$L[f(t+1)] = L[t^3 + 2t^2 + 3t + 3]$

$$= \frac{3!}{s^4} + 2\frac{2!}{s^3} + \frac{3}{s^2} + \frac{3}{s} \qquad \text{----- i}$$

We know, $L[f(t)H(t-a)] = e^{-as}L[f(t + a)]$

Substituting the value of L[f(t+a)] in above equation, we get

$$L[(1+2t-t^2+t^3)H(t-1)] = e^{-as}\left[\frac{3!}{s^4}+2\frac{2!}{s^3}+\frac{3}{s^2}+\frac{3}{s}\right]$$

$$\int_0^\infty e^{-st}(1+2t-t^2+t^3)H(t-1)dt = e^{-s}\left[\frac{3!}{s^4}+2\frac{2!}{s^3}+\frac{3}{s^2}+\frac{3}{s}\right]$$

Putting s=1 in the above equation;

$$\int_0^\infty e^{-t}\left(1+2t-t^2+t^3\right)H(t-1)dt = e^{-1}\left[\frac{3!}{1^4}+2\frac{2!}{1^3}+\frac{3}{1^2}+\frac{3}{1}\right]$$

$$=e^{-1}[6+4+3+3] = \frac{16}{e}$$

Ans :

$$\int_0^\infty e^{-t}\left(1+2t-t^2+t^3\right)H(t-1)dt = \left[\frac{16}{e}\right]$$

Q5.b) Show that the set of functions cos x, cos 2x, cos 3x, … is a set of orthogonal functions over [-π,π]. Hence construct set of orthonormal functions. (6)

Sol : We have $f_n(x) = \cos nx$;n=1, 2, 3, …

Therefore, $\int_{-\pi}^{\pi} f_m(x) \,.\, f_m(x)dx \Rightarrow \int_{-\pi}^{\pi} \cos mx.\cos nxdx$

$$\Rightarrow \frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)x+\cos(m-n)xdx$$

$$\Rightarrow \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n}+\frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi}$$

Now two cases arise:

i. When m ≠ n :

$$= \frac{1}{2}\left[\left\{\frac{\sin(m+n)\pi}{m+n}+\frac{\sin(m-n)\pi}{m-n}\right\}-\left\{\frac{-\sin(m+n)\pi}{m+n}-\frac{\sin(m-n)\pi}{m-n}\right\}\right]$$

$$= \left[\left\{\frac{\sin(m+n)\pi}{m+n}+\frac{\sin(m-n)\pi}{m-n}\right\}\right]$$

$= 0$

ii. When m=n:

$$\int_{-\pi}^{\pi} \cos^2 nxdx = \int_{-\pi}^{\pi} \frac{1+\cos 2x}{2}dx$$

$$\Rightarrow \frac{1}{2}\left[x+\frac{\sin 2x}{2}\right]_{-\pi}^{\pi}$$

$$\Rightarrow \frac{1}{2}[\pi+0-(-\pi+0)] \Rightarrow \pi \neq 0$$

Therefore the functions are orthogonal in [-π,π]

$$\int_{-\pi}^{\pi}[f(x)]^2dx = \pi$$

dividing the above equation by π;

$$\Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2dx = 1$$

$$\Rightarrow \int_{-\pi}^{\pi}\frac{1}{\sqrt{\pi}}f(x).\frac{1}{\sqrt{\pi}}f(x)dx = 1$$

This is obviously an orthonormal set where $\phi(x) = \frac{1}{\sqrt{\pi}}\cos nx$

Thus the required orthonormal set is $\frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\cos 2x, \frac{1}{\sqrt{\pi}}\cos 3x, \ldots$

Q5.c) Solve using Laplace transform: (8)

$(D^3 - 2D^3 + 5D)y = 0$ with $y(0) = 0; y'(0) = 0; y''(0) = 1$

Sol: Let $L(y) = \dot{y}$

Taking Laplace transform on both sides of the given equation ;

$$L(y''') - 2L(y'') + 5L(y') = 0$$

$$\Rightarrow L(y') = s(\dot{y}) - y(0) \,;\, L(y'') = s^2\dot{y} - sy(0) - y'(0) \,;\, L(y''') = s^3\dot{y} - s^2y(0) - sy'(0) - y''(0)$$

From the given conditions;

$$L(y') = s(\dot{y}) \,;\,;\, L(y'') = s^2\dot{y} \,;\,;\, L(y''') = s^3\dot{y} - 1$$

Therefore the equation becomes;

$$\Rightarrow s^3\dot{y} - 1 - 2s^2\dot{y} + 5s(\dot{y}) = 0$$

$$\Rightarrow \dot{y} = \frac{1}{s^3-2s^2+5s}$$

Taking inverse Laplace transform,

$$\Rightarrow y = L^{-1}\left[\frac{1}{s^3-2s^2+5s}\right]$$

$$\Rightarrow y = L^{-1}\left[\frac{1}{s(s^2-2s+5)}\right] \Rightarrow L^{-1}\left[\frac{1}{s[(s-1)^2+2^2]}\right]$$

We obtain the inverse by convolution theorem,

$\phi_1(s) = \frac{1}{(s-1)^2+2^2}$; $\phi_2(s) = \frac{1}{s}$

$f_1(t) = L^{-1}[\phi_1(s)] \Rightarrow L^{-1}\left(\frac{1}{(s-1)^2+2^2}\right) \Rightarrow e^t L^{-1}\left(\frac{1}{(s)^2+2^2}\right) = \frac{1}{2}.e^t.\sin 2t$

$f_2(t) = L^{-1}[\phi_2(s)] \Rightarrow L^{-1}[\frac{1}{s}] \Rightarrow 1$

$\Rightarrow f_1(u) = \frac{1}{2}.e^u.\sin 2u$

$\Rightarrow L^{-1}[\phi(s)] = \frac{1}{2}\left[\frac{1}{1+2^2}[e^u(\sin 2u - 2\cos 2u)]\right]_0^t$

*The above integral is of this format : $[\int e^{ax}\sin bx = \frac{1}{a^2+b^2}(e^{ax}\{\sin ax - b\cos bx\})]$

$\Rightarrow L^{-1}[\phi(s)] = \frac{1}{2}\left[\frac{1}{5}[e^t(\sin 2t - 2\cos 2t) + 2]\right]$

$\Rightarrow L^{-1}[\phi(s)] = \left[\frac{1}{10}[e^t(\sin 2t - 2\cos 2t) + 2]\right]$

The solution is : $\left[\frac{1}{10}[e^t(\sin 2t - 2\cos 2t) + 2]\right]$

Q6.a) Find the Complex Form of the Fourier Series for f(x) = 2x in (0,2π) (6)

Sol : f(x)=2x, range (0, 2π);

$\Rightarrow \sum_{-\infty}^{\infty} C_n e^{inx}$; where $C_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$

Hence, $C_n = \frac{1}{2\pi}\int_0^{2\pi} 2x.e^{-inx}dx$ ----- i

$\Rightarrow \frac{1}{\pi}\int_0^{2\pi} x.e^{-inx}dx$

$\Rightarrow \frac{1}{\pi}\left[x.\frac{e^{-inx}}{-in} - \frac{e^{-inx}}{(in)^2}\cdot 1\right]_0^{2\pi}$

$\Rightarrow \frac{1}{\pi}\left[-\frac{2\pi e^{-i2n\pi}}{in} + \frac{e^{i2n\pi}}{n^2} - 0 - \frac{1}{n^2}\right]$

$\Rightarrow \frac{1}{\pi}\left[-\frac{2\pi}{in} + \frac{1}{n^2} - 0 - \frac{1}{n^2}\right] \Rightarrow \frac{1}{\pi}(-\frac{2\pi}{in})$

$$\Rightarrow \left(-\frac{2}{in}\right)\left(\frac{i}{i}\right) \Rightarrow \frac{2i}{n} \qquad \{n \neq 0\}$$

For n=0, substitute it in (i);

$$C_0 = \frac{1}{2\pi}\int_0^{2\pi} 2x\,dx \Rightarrow \frac{1}{\pi}\left(\frac{x^2}{2}\right)_0^{2\pi} = \frac{1}{2\pi}(4\pi^2) = 2\pi$$

Therefore, $f(x) = 2\pi + \sum_{-\infty}^{\infty} \frac{2i}{n} e^{inx}$

$$\boldsymbol{\Rightarrow f(x) = 2\pi + 2i\sum_{-\infty}^{\infty} \frac{e^{inx}}{n}}$$

Q6.b) If $f(z)$ and $f(z)$ are both analytic, prove that f(z) is constant (6)

Sol: $f(z) = u + iv$

$f(z) = u + i(-v)$

For $f(z)$:

$u_x = v_y$ ----- i

$u_y = -v_x$ ----- ii

For $f(z)$:

$u_x = -v_y$ ----- iii

$u_y = -(-v_x)$ ----- iv

From i and iii;

$$v_y = -v_y \Rightarrow 2v_y = 0 \Rightarrow v_y = 0$$

From ii and iv;

$$v_x = -v_x \Rightarrow 2v_x = 0 \Rightarrow v_x = 0$$

Substituting in i and ii,

$$u_x = u_y = 0$$

Therefore u=k and v=k [partial derivatives of constant are zero]

Hence u+iv is constant

⇨ f(z) is constant.

Q6.c) Fit a curve of the form y = ab^x to the following data. (8)

X	1	2	3	4	5	6
Y	151	100	61	50	20	8

Sol: $y = ab^x$

Taking log on both sides,

$\log y = \log a + x \log b$

Let log y = Y, log a= A, x=X and log b= B

=>Y=A+X(B)

x	y	X	Y	X^2	XY
1	151	1	2.1789	1	2.1789
2	100	2	2	4	4
3	61	3	1.7853	9	5.3559
4	50	4	1.6989	16	6.7956
5	20	5	1.3010	25	6.5050
6	8	6	0.9031	36	5.4186
		∑21	∑9.8672	∑91	∑30.254

Here , N=6

$\Sigma Y = NA + B\Sigma X$

$\Sigma XY = A\Sigma X + B\Sigma X^2$

Substituting the values from the above table;

$6A + 21B = 9.8672$

$21A+91B = 30.254$

On solving simultaneously ;

A=2.5 and B=-0.2446

Hence, b= antilog(-0.2446) =>0.5668 $[10^{-0.2446} = 0.56676]$

a=antilog(2.5) =>316.2278 $[10^{2.5} = 0.56676]$

Therefore , $y = (316.2278)(0.5668)^x$

Ans : y = $(316.2278)(0.5668)^x$

COMPUTER ENGINEERING

APPLIED MATHEMATICS -3

(CBCGS - MAY 2018)

Q1.a) Find the Laplace transform of $e^{-2t}t\cos t$ **[5]**

Sol : $L[\cos t] = \frac{s}{s^2+1}$ $\left\{\because L[\cos at]=\frac{s}{s^2+a^2}\right\}$

$$\Rightarrow L[t\cos t] = (-1)\left[\frac{(s^2+1)-s(2s)}{(s^2+1)^2}\right] \quad \left\{\text{By }\frac{u}{v}\text{ rule of differentiation}\right\}$$

$$\Rightarrow L[t\cos t] = -\left[\frac{(s^2+1)-2s^2}{(s^2+1)^2}\right] \Rightarrow \left[\frac{(s^2-1)}{(s^2+1)^2}\right]$$

$$\Rightarrow L[e^{-2t}t\cos t] = \left[\frac{(s+2)^2-1}{((s+2)^2+1)^2}\right] \quad \{L[e^{-at}f(t)] = \Phi(s+a)\}$$

$$\Rightarrow L[e^{-2t}t\cos t] = \left[\frac{s^2+4s-3}{(s^2+4s+5)^2}\right]$$

Ans : $L[e^{-2t}t\cos t]$ **=** $\left[\frac{s^2+4s-3}{(s^2+4s+5)^2}\right]$

Q1.b) Find the inverse Laplace transform of $\frac{3s+7}{s^2-2s-3}$ **[5]**

Sol : Adjusting the numerator and denominator

$$\Rightarrow \frac{3(s-1)+10}{(s-1)^2-2^2}$$

Splitting the terms;

$$\Rightarrow 3L^{-1}\left[\frac{(s-1)}{(s-1)^2-2^2}\right] + 10L^{-1}\left[\frac{1}{(s-1)^2-2^2}\right]$$

$$\Rightarrow 3e^tL^{-1}\left[\frac{s}{s^2-2^2}\right] + 10e^tL^{-1}\left[\frac{1}{s^2-2^2}\right] \quad \{\because \Phi(s+a) = e^{-at}L[f(t)]\}$$

$$\Rightarrow 3e^t\cosh 2t + \frac{10}{2}e^t\sinh 2t \quad \left\{\because L\left[\frac{s}{s^2-a^2}\right]=\cosh at,\ L\left[\frac{1}{s^2-a^2}\right]=\frac{1}{a}\sinh at\right\}$$

$$\Rightarrow e^t(3\cosh 2t+5\sinh 2t)$$

Ans : $L^{-1}\left[\frac{3s+7}{s^2-2s-3}\right] = e^t(3\cosh 2t+5\sinh 2t)$

Q1.c) Determine whether the function $f(z) = (x^3+3xy^2-3x) + i(3x^2y - y^3 + 3y)$ is analytic and if so, find its derivative. [5]

Sol : Given $f(z) = (x^3+3xy^2-3x) + i(3x^2y - y^3 + 3y)$

Comparing real and imaginary parts, we get

$u = (x^3+3xy^2-3x)$; $v = (3x^2y - y^3 + 3y)$

Differentiating u partially w.r.t x and y,

$u_x = 3x^2+3y^2-3$; $u_y = 6xy$

Differentiating v partially w.r.t x and y,

$v_x = 6xy$; $v_y = 3x^2 - 3y^2 + 3$

$\because$ CR equations are not satisfied $\{u_x \neq v_y ; u_y \neq -v_x\}$

Therefore the function is not analytic and thus its derivative does not exists.

Q1.d) Find the Fourier series for $f(x) = x^2$ in the interval $(-\pi,\pi)$ [5]

Sol : $f(x) = x^2$ is an even function as $f(-x) = (-x)^2 = x^2 = f(x)$

Fourier transform for even function is given by :

$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ -----(i)

$\Rightarrow a_0 = \frac{1}{\pi}\int_0^\pi f(x)dx$

$\Rightarrow a_0 = \frac{1}{\pi}\int_0^\pi x^2 dx \Rightarrow \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^\pi \Rightarrow \frac{1}{3\pi}(\pi^3 - 0)$

$\Rightarrow a_0 = \frac{\pi^2}{3}$

$\Rightarrow a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx dx$

$\Rightarrow a_n = \frac{2}{\pi}\int_0^\pi x^2 \cos nx dx$

$$\Rightarrow a_n = \frac{2}{\pi}\left[x^2\left\{\frac{\sin nx}{n}\right\}-2x\left\{\frac{-\cos nx}{n^2}\right\}+2\left\{\frac{-\sin nx}{n^3}\right\}\right]_0^{\pi}$$

$$\Rightarrow a_n = \frac{2}{\pi}\left[\left\{0-2\pi\left(\frac{-\cos n\pi}{n^2}\right)+0\right\}-\{0-0+0\}\right]$$

$$\Rightarrow a_n = \frac{2}{\pi}\left[\left\{2\pi\left(\frac{\cos n\pi}{n^2}\right)\right\}\right] \Rightarrow a_n = \frac{4}{n^2}(-1)^n \qquad \{\because \cos n\pi=(-1)^n\}$$

Resubstituting the values in (i)

$$\textbf{Ans}: x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$$

Q2.a) Evaluate $\int_0^{\infty}\left(\frac{\sin 2t+\sin 3t}{te^t}\right)dt = \frac{3\pi}{4}$ **[6]**

Sol : LHS :

$$L(\sin 2t+\sin 3t) = \frac{2}{s^2+4} + \frac{3}{s^2+9} \qquad \left\{\because L[\sin at]=\frac{a}{s^2+a^2}\right\}$$

$$\Rightarrow L\left(\frac{\sin 2t+\sin 3t}{t}\right) = \int_s^{\infty}\frac{2}{s^2+4}ds + \int_s^{\infty}\frac{3}{s^2+9}ds$$

$$\Rightarrow L\left(\frac{\sin 2t+\sin 3t}{t}\right) = \left[\tan^{-1}\left(\frac{s}{2}\right)+\tan^{-1}\left(\frac{s}{3}\right)\right]_s^{\infty}$$

$$\Rightarrow L\left(\frac{\sin 2t+\sin 3t}{t}\right) = \left[\{\tan^{-1}(\infty)+\tan^{-1}(\infty)\}-\left\{\tan^{-1}\left(\frac{s}{2}\right)+\tan^{-1}\left(\frac{s}{3}\right)\right\}\right]$$

$$\Rightarrow L\left(\frac{\sin 2t+\sin 3t}{t}\right) = \left[\left\{\frac{\pi}{2}+\frac{\pi}{2}\right\}-\left\{\tan^{-1}\left(\frac{s}{2}\right)+\tan^{-1}\left(\frac{s}{3}\right)\right\}\right]$$

$$\Rightarrow L\left(\frac{\sin 2t+\sin 3t}{t}\right) = \left[\pi-\left\{\tan^{-1}\left(\frac{s}{2}\right)+\tan^{-1}\left(\frac{s}{3}\right)\right\}\right]$$

$$\int_0^{\infty}e^{-st}L\left(\frac{\sin 2t+\sin 3t}{t}\right) = \left[\pi-\left\{\tan^{-1}\left(\frac{s}{2}\right)+\tan^{-1}\left(\frac{s}{3}\right)\right\}\right]$$

On Putting s=1,

$$\int_0^{\infty}e^{-t}L\left(\frac{\sin 2t+\sin 3t}{t}\right) = \left[\pi-\left\{\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)\right\}\right]$$

$$\int_0^{\infty} e^{-t} L\left(\frac{\sin 2t+\sin 3t}{t}\right) = \left[\pi - \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}.\frac{1}{3}}\right)\right] \qquad \left\{\because \tan^{-1} A+\tan^{-1} B=\tan^{-1}\left(\frac{A+B}{1-AB}\right)\right\}$$

$$\Rightarrow\left[\pi - \tan^{-1}\left(\frac{5}{5}\right)\right]$$

$$\Rightarrow\left[\pi-\frac{\pi}{4}\right]$$

$$\Rightarrow\left[\frac{3\pi}{4}\right]$$

=RHS

Hence proved.

Q2.b) Find the Z-transform of $\left\{\left(\frac{1}{4}\right)^{|k|}\right\}$ **[6]**

Sol : $F(Z) = \begin{cases}\left(\frac{1}{4}\right)^{k} ;k\geq 0\\ \left(\frac{1}{4}\right)^{-k} ;k<0\end{cases}$

The equation can be expressed as : $\sum_{-\infty}^{\infty} F(z).z^{-k}$

$$\Rightarrow \sum_{-\infty}^{-1}\left(\frac{1}{4}\right)^{-k} z^{-k}+\sum_{0}^{\infty}\left(\frac{1}{4}\right)^{k} z^{-k}$$

$$\Rightarrow\left[...+\left(\frac{z}{4}\right)^3+\left(\frac{z}{4}\right)^2+\left(\frac{z}{4}\right)^1\right] + \left[1+\left(\frac{1}{4z}\right)^1+\left(\frac{1}{4z}\right)^2+\left(\frac{1}{4z}\right)^3+...\right]$$

The above two series are sum of infinite GP whose sum is given as: $\frac{a}{1-r}$

Where a = 1^{st} term, r is the common ratio between the terms.

$$\Rightarrow\left(\frac{z}{4}\right)\left[...+\left(\frac{z}{4}\right)^2+\left(\frac{z}{4}\right)^1+1\right] + \left[1+\left(\frac{1}{4z}\right)^1+\left(\frac{1}{4z}\right)^2+\left(\frac{1}{4z}\right)^3+...\right]$$

$$\Rightarrow\left(\frac{z}{4}\right)\left[\frac{1}{1-\frac{z}{4}}\right] + \left[\frac{1}{1-\left(\frac{1}{4z}\right)}\right] \qquad , \qquad \left|\frac{z}{4}\right| < 1 \text{ and } \left|\frac{1}{4z}\right| < 1$$

Ans : $Z\{f(k)\} = \left(\frac{z}{4}\right)\left[\frac{1}{1-\frac{z}{4}}\right] + \left[\frac{1}{1-\left(\frac{1}{4z}\right)}\right]; \left|\frac{1}{4}\right| < z < 4$

Q2.c) Show that the function v = $e^x(x \sin y + y \cos y)$ **is harmonic function. Find its harmonic conjugate and corresponding analytic function. [8]**

Sol : $v = e^x(x \sin y + y \cos y)$

$v = e^x x \sin y + e^x y \cos y$

Differentiating partially wrt. x and y twice,

$v_x = e^x(x\sin y + y\cos y) + e^x \sin y$

$v_y = e^x(x\cos y + \cos y - y \sin y)$

$v_x^{\ 2} = e^x(x\sin y + y\cos y) + e^x \sin y + e^x \sin y$ -----(i)

$v_y^{\ 2} = e^x(-x\sin y - \sin y - \sin y - y\cos y)$ -----(ii)

Adding equations i and ii ;

$$v_x^{\ 2} + v_y^{\ 2} = 0$$

Therefore, v satisfies Laplace equation and thus v is harmonic.

$v_x = e^x(x\sin y + y\cos y) + e^x \sin y$

$\Psi_1(z,0) = 0$

$v_y = e^x(x\cos y + \cos y - y \sin y)$

$\Psi_2(z,0) = e^z(z + 1)$

$\Rightarrow f(z) = \Psi_1(z,0) + i\Psi_2(z,0)$

$f(z) = \int e^z(z+1)\, dz$

$= ze^z$

Ans : $f(z) = ze^z$

Q3.a) From 8 observations the following results were obtained : **[6]**

$\Sigma x = 59; \Sigma y = 40; \Sigma x^2 = 524; \Sigma y^2 = 256; \Sigma xy = 364$

Find the equation of line of regression of x on y and the coefficient of correlation.

Sol : $\bar{X} = \frac{59}{8} = 7.375$; $\bar{Y} = \frac{40}{8} = 5$

Coefficient of regression of y on x :

$$\Rightarrow b_{yx} = \frac{\Sigma xy - \frac{\Sigma x \Sigma y}{N}}{\Sigma x^2 - \frac{(\Sigma x)^2}{N}}$$

$$\Rightarrow b_{yx} = \frac{364 - \frac{(59)(40)}{8}}{524 - \frac{(59)^2}{8}}$$

$\therefore b_{yx} = 0.7764$

Coefficient of regression of x on y :

$$\Rightarrow b_{xy} = \frac{\Sigma xy - \frac{\Sigma x \Sigma y}{N}}{\Sigma y^2 - \frac{(\Sigma y)^2}{N}}$$

$$\Rightarrow b_{xy} = \frac{364 - \frac{(59)(40)}{8}}{256 - \frac{(40)^2}{8}}$$

$\therefore b_{xy} = 1.2321$

Equation of line of regression of x on y is given by

$X - \bar{X} = b_{xy}(Y - \bar{Y})$

$\Rightarrow X - 7.375 = 1.2321(Y - 5)$

$\Rightarrow X = 1.2321(Y - 5) + 7.375$

$\Rightarrow X = 1.2321Y + 1.2145$

$r = \sqrt{b_{xy} . b_{yx}}$

$r = \sqrt{(1.2321)(0.7764)}$

r = 0.9781

Ans : X = 1.2321Y + 1.2145

r = 0.9781

Q3.b) Find the bilinear transformation which maps the points z=-1, 0, 1 onto the plane w=-1, -i, 1 **[6]**

Sol : Let z=-1, 0, 1 be the points in the z-plane with the images w=-1, -I, 1 in the w plane.

The bilinear transformation is given by,

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\Rightarrow \frac{(w+1)(-i-1)}{(-1+i)(1-w)} = \frac{(z+1)(0-1)}{(-1-0)(1-z)}$$

$$\Rightarrow \frac{(w+1)(-i-1)}{(1-w)(-1+i)} = \frac{(z+1)}{(1-z)}$$

$$\Rightarrow \frac{-(w+1)(i+1)}{-(w-1)(-1+i)} = \frac{(z+1)}{-(z-1)} \qquad \Rightarrow \frac{(w+1)(i+1)}{-(w-1)(1-i)} = \frac{(z+1)}{-(z-1)}$$

$$\Rightarrow \frac{(w+1)(i+1)}{(w-1)(1-i)} = \frac{(z+1)}{(z-1)}$$

$$\Rightarrow \frac{(w+1)(i+1)}{(w-1)(1-i)}\frac{(1+i)}{(1+i)} = \frac{(z+1)}{(z-1)} \qquad \text{----- (i)} \qquad \text{(Rationalising)}$$

$$\Rightarrow \frac{(w+1)(i+1)^2}{(w-1)\left(1^2-i^2\right)} = \frac{(z+1)}{(z-1)}$$

$$\Rightarrow \frac{(w+1)(-1+2i+1)}{(w-1)(2)} = \frac{(z+1)}{(z-1)}$$

$$\Rightarrow \frac{(w+1)i}{(w-1)} = \frac{(z+1)}{(z-1)}$$

$$\Rightarrow \frac{(w+1)}{(w-1)} = \frac{(z+1)}{(iz-i)}$$

Applying componendo – dividendo;

$$\Rightarrow \frac{(w+1)+(w-1)}{(w+1)-(w-1)} = \frac{(z+1)+(iz-i)}{(z+1)-(iz-i)}$$

$$\Rightarrow \frac{2w}{2} = \frac{z+1+iz-i}{z+1-iz+i}$$

$$\Rightarrow w = \frac{z(1+i)+(1-i)}{z(1-i)+(1+i)}$$

$$\Rightarrow w = \frac{z\frac{(1+i)}{(1-i)}+1}{z+\frac{(1+i)}{(1-i)}}$$

From above steps(rationalising eqn i we know $(1 + i)/(1 - i) = i$)

$$\Rightarrow w = \frac{zi+1}{z+i}$$

Ans : Therefore, the required transformation, w = $\frac{zi+1}{z+i}$

Q3.c) Obtain half – range cosine series for f(x) **=** $(x-1)^2$ in $0 < x < 1$.

Hence find $\sum_{n=1}^{\infty}\frac{1}{n^2}$ **[8]**

Sol : $f(x) = (x-1)^2$ in $0 < x < 1$

$\therefore$The half range cosine series or $f(x)$ is given as :

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{1}\int_0^1 (x-1)^2 dx$$

$$\Rightarrow a_0 = 1\left[\frac{(x-1)^3}{3}\right]_0^1 \qquad \Rightarrow a_0 = 0 - \left(-\frac{1}{3}\right) \qquad \Rightarrow a_0 = \frac{1}{3}$$

$$a_n = \frac{2}{1}\int_0^1 (x-1)^2 \cos n\pi x dx$$

$$a_n = 2\left[(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) - 2(x-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + 2\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$a_n = 2\left[(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) + 2(x-1)\left(\frac{\cos n\pi x}{n^2\pi^2}\right) - 2\left(\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$a_n = 2\left[0+0-0-\left\{0-\frac{2}{n^2\pi^2}-0\right\}\right]$$

$$a_n = \frac{4}{n^2\pi^2}$$

$$\therefore f(x) = \frac{1}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\cos n\pi x$$

$$\therefore (x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$$

Put x=0;

$$\Rightarrow 1 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \frac{2}{3} = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

Ans : $(x-1)^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Q4.a) Find the inverse Laplace transform by using convolution theorem $\frac{1}{(s^2+a^2)(s^2+b^2)}$ **[6]**

Sol : $L^{-1}[\phi_1(s)] = L^{-1}\left[\frac{1}{(s^2+a^2)}\right] = \frac{1}{a}\sin at$

$$L^{-1}[\phi_2(s)]=L^{-1}\left[\frac{1}{(s^2+b^2)}\right] = \frac{1}{b}\sin bt$$

$$L^{-1}[\phi(s)] = L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right] = \int_0^t \frac{1}{a}\sin au.\frac{1}{b}\sin b(t-u)du$$

$$\Rightarrow \frac{1}{ab}\int_0^t \sin au.\sin b(t-u)du$$

$$\Rightarrow \frac{-1}{2ab}\int_0^t \{\cos([a-b]u+bt)-\cos[a+b]u-bt)\}du$$

$$\left\{\because \sin A \sin B = -\frac{1}{2}[\cos(A+B)-\cos(A-B)]\right\}$$

$$\Rightarrow \frac{-1}{2ab}\left[\frac{\sin\{(a-b)u+bt\}}{a-b}-\frac{\sin\{(a+b)u-bt\}}{a+b}\right]_0^t$$

$$\Rightarrow \frac{-1}{2ab}\left[\frac{\sin at}{a-b}-\frac{\sin at}{a+b}-\frac{\sin bt}{a-b}-\frac{\sin bt}{a+b}\right]$$

$$\Rightarrow \frac{-1}{2ab}\left[2b.\frac{\sin at}{a^2-b^2}-2a.\frac{\sin bt}{a^2-b^2}\right]$$

$$\Rightarrow \left[\frac{a.\sin bt}{a^2-b^2}-\frac{b.\sin at}{a^2-b^2}\right]$$

Ans : $L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right] = \left[\frac{a.\sin bt}{a^2-b^2}-\frac{b.\sin at}{a^2-b^2}\right]$

Q4.b) Compute Spearman's Rank correlation coefficient for the following data: [6]

X	85	74	85	50	65	78	74	60	74	90
Y	78	91	78	58	60	72	80	55	68	70

Sol :

X	R1	Y	R2	D	$D^2 = (R1-R2)^2$
85	8.5	78	7.5	1	1
74	5	91	10	-5	25
85	8.5	78	7.5	1	1
50	1	58	2	-1	1
65	3	60	3	0	0
78	7	72	6	1	1
74	5	80	9	-4	16
60	2	55	1	1	1

74	5	68	4	1	1
90	10	70	5	5	25
N=10					∑=72

Therefore, $R = 1 - \dfrac{6\{\Sigma D^2+\frac{1}{12}(m_1^3-m_1)+\frac{1}{12}(m_1^3-m_1)+\frac{1}{12}(m_1^3-m_1)+...\}}{N^3-N}$

Here m1=2, m2=2, m3=3,

$$R = 1 - \frac{6\{72+\frac{1}{12}(2^3-2)+\frac{1}{12}(2^3-2)+\frac{1}{12}(3^3-3)+...\}}{10^3-10}$$

On solving, R=0.5454

Ans : R=0.5454

Q4.c) Find the inverse Z-transform for the following: [8]

i) $\dfrac{1}{(z-5)^2}$ **, $|z|$ < 5**

ii) $\dfrac{z}{(z-2)(z-3)}$**, $|z|$ > 3**

Sol :

i) $\dfrac{1}{(z-5)^2}$, $|z| < 5$

$$\Rightarrow \frac{1}{5^2\left(1-\left(\frac{5}{z}\right)\right)^2}$$

$$\Rightarrow \frac{1}{5^2}\left[1-\left(\frac{z}{5}\right)\right]^{-2}$$

$$\Rightarrow \frac{1}{5^2}\left[1+2\left(\frac{z}{5}\right)+3\left(\frac{z}{a}\right)^2+...+(n+1)\left(\frac{z}{5}\right)^n\right]^1$$

$$\Rightarrow \left[\frac{1}{5^2}+2\left(\frac{z}{5^3}\right)+3\left(\frac{z^2}{5^4}\right)^2+...+(n+1)\left(\frac{z^n}{5^{n+2}}\right)\right]^1$$

Coefficient of $z^n = \dfrac{n+1}{5^{n+2}}, n \geq 0$

Put n = -k ;

Coefficient of $z^{-k} = \frac{-k+1}{5^{-k+2}}, k \leq 0$

Ans : $Z^{-1}[F(z)] = \frac{-k+1}{5^{-k+2}}, \mathbf{k \leq 0}$

ii) $\frac{z}{(z-2)(z-3)}, |z| > 3$

Applying Partial Fractions;

$$\frac{z}{(z-2)(z-3)} = \frac{A}{z-3} + \frac{B}{z-2} \qquad \text{------ (i)}$$

$\Rightarrow z = A(z-2) + B(z-3)$

Put z=2 Put z=3

$\Rightarrow 2 = B(-1) \Rightarrow \mathbf{B = -2}$ $\Rightarrow 3 = A(1) \Rightarrow \mathbf{A = 3}$

Resubstituting in (i);

$$\frac{z}{(z-2)(z-3)} = \frac{3}{z-3} - \frac{2}{z-2}$$

RHS:

$$\Rightarrow -\frac{3}{3\left[1-\left(\frac{z}{3}\right)\right]} + \frac{2}{2\left[\left(1-\left(\frac{z}{2}\right)\right)\right]}$$

$$\Rightarrow \left(1-\frac{z}{2}\right)^{-1} - \left(1-\frac{z}{3}\right)^{-1}$$

$$\Rightarrow \left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^{2}+\ldots+\left(\frac{z}{2}\right)^{n}\right] - \left(1+\frac{z}{3}+\left(\frac{z}{3}\right)^{2}+\ldots+\left(\frac{z}{3}\right)^{n}\right)$$

The coefficient of $z^{n} = 2^{-n} - 3^{-n}; n \geq 0$

Put n=-k;

$z^{-k} = 2^{k} - 3^{k}; k \leq 0$

Ans : $Z^{-1}[F(z)] = 2^{k} - 3^{k}, \mathbf{k \leq 0}$

Q5.a) Using Laplace Transform evaluate $\int_0^\infty e^{-t}(1+3t+t^2)H(t-2)dt$ **[6]**

Sol : To evaluate $\int_0^\infty e^{-t}(1+3t+t^2)H(t-2)dt$

$$\Rightarrow f(t) = 1 + 3t + t^2 \quad ; \ a=2$$

$$\Rightarrow f(t+1) = 1 + 3(t+2) + (t+2)^2$$

$$= 1 + 3t + 6 + (t^2 + 4t + 4)$$

$$= t^2 + 7t + 11$$

$$L[f(t+2)] = L[t^2 + 7t + 11]$$

$$= \frac{2!}{s^3} + 7\frac{1!}{s^2} + \frac{11}{s} \qquad \text{----- i}$$

We know, $L[f(t)H(t-a)] = e^{-as}L[f(t + a)]$

Substituting the value of L[f(t+a)] in above equation, we get

$$L[(1+3t+t^2)H(t-2)] = e^{-2s}\left[\frac{2!}{s^3}+7\frac{1!}{s^2}+\frac{11}{s}\right]$$

$$\int_0^\infty e^{-st}(1+3t+t^2)H(t-2)dt = e^{-2s}\left[\frac{2!}{s^3}+7\frac{1!}{s^2}+\frac{11}{s}\right]$$

Putting s=1 in the above equation;

$$\int_0^\infty e^{-t}(1+3t+t^2)H(t-2)dt = e^{-2}\left[\frac{2!}{1}+7\frac{1!}{1}+\frac{11}{1}\right]$$

$$=e^{-2}[2+7+11] = \frac{20}{e^2}$$

Ans : $\int_0^\infty e^{-t}(1+3t+t^2)H(t-2)dt = \frac{20}{e^2}$

Q5.b) Prove that $f_1(x) = 1; f_2(x) = x; f_3(x) = \frac{3x^2-1}{2}$ are orthogonal over (-1,1).

[6]

Sol : Conditions for functions to be orthogonal are

i) $\int_a^b f_m(x).f_n(x)dx=0 \ ;m \neq n$

ii) $\int_a^b [f_n(x)]^2 dx \neq 0 \ ;m=n$

i) Proving 1st condition is true,

We have, $\int_{-1}^{1} f_1(x).f_2(x)dx = \int_{-1}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1}$

$\Rightarrow \frac{1}{2}(1^2 - (-1)^2) = 0$

$\int_{-1}^{1} f_1(x).f_3(x)dx = \int_{-1}^{1} \frac{3x^2-1}{2}\,dx = \frac{1}{2}\left[x^3-x\right]_{-1}^{1}$

$\Rightarrow \frac{1}{2}\left[\left(1^3-1\right)-\{(-1)^3-(-1)\}\right] \Rightarrow \frac{1}{2}[(0)-(0)] = 0$

$\int_{-1}^{1} f_2(x).f_3(x)dx = \int_{-1}^{1} \frac{x}{2}\left(3x^2-1\right)dx = \frac{1}{2}\int_{-1}^{1}\left(3x^3-x\right)dx$

$\Rightarrow \frac{1}{2}\left[\frac{3x^4}{4} - \frac{x^2}{2}\right]_{-1}^{1} \Rightarrow \frac{1}{2}\left[\left(\frac{3}{4}-\frac{1}{2}\right) - \left(\frac{3}{4} - \frac{1}{2}\right)\right] = 0$

ii) Proving 2^{nd} condition in true;

$\int_{-1}^{1}\left[f_1(x)\right]^2dx = \int_{-1}^{1} 1^2dx = [x]_{-1}^{1} = [1-(-1)] = 2 \neq 0$

$\int_{-1}^{1}\left[f_2(x)\right]^2dx = \int_{-1}^{1} x^2dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \left[\frac{1}{3}-\left(-\frac{1}{3}\right)\right] = \frac{2}{3} \neq 0$

$\int_{-1}^{1}\left[f_3(x)\right]^2dx = \int_{-1}^{1}\left(\frac{3x^2-1}{2}\right)^2dx$

$\Rightarrow \frac{1}{4}\int_{-1}^{1}\left(9x^4-6x^2+1\right)dx \Rightarrow \frac{1}{4}\left[\frac{9x^5}{5}-2x^3+x\right]_{-1}^{1}$

$\Rightarrow \frac{1}{4}\left[\frac{9}{5}-2+1-\left\{-\frac{9}{5}-2(-1)^3-1\right\}\right]$

$\Rightarrow \frac{1}{4}\left[\frac{18}{5}-4+2\right] \Rightarrow \frac{2}{5} \neq 0$

Hence, the given set is orthogonal on [-1,1]

Q5.c) Solve using Laplace transform

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{3x} \text{ ; } y = 2 \text{ and } y' = 3 \text{ at } x=0 \qquad [8]$$

Sol : $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^{3x}$

$\therefore (D^2 - 3D + 2)y = 2e^{3x}$

$\therefore y'' - 3y' + 2y = 2e^{3x}$

Taking Laplace transform on both sides, we get

$L[y''] - 3L[y'] + 2L[y] = \frac{2}{s-3}$ $\qquad \{\because L[e^{at}] = \frac{1}{s-a}\}$

$L[y''] = s^2\bar{y} - sy(0) - y'(0)$

$L[y'] = s\bar{y} - y(0)$

Substituting the values in the equation,

$$s^2\bar{y} - 2s - 3 - 3\left(s\bar{y} - 2\right) + 2\bar{y} = \frac{2}{s-3}$$

$$\Rightarrow \bar{y}\left(s^2-3s+2\right) - 2s + 3 = \frac{2}{s-3}$$

$$\Rightarrow \bar{y}\left(s^2-3s+2\right) = \frac{2}{s-3} + (2s-3)$$

$$\Rightarrow \bar{y}\left(s^2-3s+2\right) = \frac{2+(2s-3)(s-3)}{s-3}$$

$$\Rightarrow \bar{y}\left(s^2-3s+2\right) = \frac{2s^2-9s+11}{s-3}$$

$$\Rightarrow \bar{y} = \frac{2s^2-9s+11}{\left(s^2-3s+2\right)(s-3)}$$

$$\Rightarrow \bar{y} = \frac{2s^2-9s+11}{(s-1)(s-2)(s-3)} \qquad \left[\left(x^2-3x+3\right)=(x-1)(x-2)\right]$$

Applying partial fractions;

$$\frac{2s^2-9s+11}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$\Rightarrow 2s^2 - 9s + 11 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$

Put s=1 | Put s=2 | Put s=3

$4 = 2A$ | $1 = -B$ | $2 = 2C$

A=2 B=-1 C=1

$$\therefore \frac{2s^2-9s+11}{(s-1)(s-2)(s-3)} = \frac{2}{s-1} - \frac{1}{s-2} + \frac{1}{s-3}$$

$$\bar{y} = \frac{2}{s-1} - \frac{1}{s-2} + \frac{1}{s-3}$$

Taking inverse Laplace on both sides,

$$L^{-1}\left[\bar{y}\right] = L^{-1}\left[\frac{2}{s-1} - \frac{1}{s-2} + \frac{1}{s-3}\right]$$

$$y = 2e^{t} - e^{2t} + e^{3t}$$

Ans : $y = 2e^{t} - e^{2t} + e^{3t}$

Q6.a) Find the complex form of the Fourier series for $f(x) = e^{x}, (-\pi, \pi)$ [6]

Sol : The complex form of the Fourier series for $f(x) = e^{x}$ is given by

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{inx} \text{ where } C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

$$\Rightarrow C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x}.e^{-inx}dx$$

$$\Rightarrow C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x}dx$$

$$\Rightarrow C_n = \frac{1}{2\pi}\left[\frac{e^{(1-in)x}}{(1-in)}\right]_{-\pi}^{\pi}$$

$$\Rightarrow C_n = \frac{1}{2\pi}\left[\left\{\frac{e^{(1-in)\pi}}{(1-in)}\right\} - \left\{\frac{e^{(1-in)(-\pi)}}{(1-in)}\right\}\right]$$

$$\Rightarrow C_n = \frac{1}{2\pi(1-in)}\left[e^{\pi}.e^{-in\pi} - e^{-\pi}e^{in\pi}\right]$$

But $e^{\pm(in\pi)} = \cos(\pm n\pi) + i\sin(\pm n\pi)$

$$\therefore C_n = \frac{1}{2\pi(1-in)}\left[e^{\pi}.(-1)^{n} - e^{-\pi}(-1)^{n}\right]$$

$$\Rightarrow C_n = \frac{(-1)^n}{\pi(1-in)}\left[\frac{e^{\pi}-e^{-\pi}}{2}\right]$$

$$\Rightarrow C_n = \frac{(-1)^n}{\pi(1-in)}\sinh\pi \qquad \left\{\because \frac{e^{x}-e^{-x}}{2} = \sinh(x)\right\}$$

Rationalising the denominator, multiply divide by (1+in);

$$\Rightarrow C_n = \frac{(-1)^n}{\pi(1-in)}\sinh\pi \cdot \frac{1+in}{1+in}$$

$$\Rightarrow C_n = \frac{(-1)^n(1+in)}{\pi(1^2-(in)^2)}\sinh\pi \Rightarrow \frac{(-1)^n(1+in)}{\pi(1+n^2)}\sinh\pi$$

Substituting the value in f(x)

$$f(x) = \sum_{-\infty}^{\infty}\frac{(-1)^n(1+in)}{\pi(1+n^2)}\sinh\pi . e^{inx}$$

Ans : $e^x = \sum_{-\infty}^{\infty}\frac{(-1)^n(1+in)}{\pi(1+n^2)}\sinh\pi . e^{inx}$

Q6.b) If u, v are harmonic conjugate functions, show that uv is a harmonic function [6]

Sol : Let f(z) = u + iv be the analytic function;

$\therefore u_x = v_y$ and $u_y = -v_x$

And u, v are harmonic therefore $u_x{}^2 + u_y{}^2 = 0$ and $v_x{}^2 + v_y{}^2 = 0$ -----(i)

Now, $uv_x = uv_x + vu_x$

$(uv)_x{}^2 = u_xv_x + u(v_x)^2 + v_xu_x + v(u_x)^2$

$(uv)_x{}^2 = 2u_xv_x + u(v_x)^2 + v(u_x)^2$ -----(ii)

Similarly, we can prove that,

$(uv)_y{}^2 = 2u_yv_y + u(v_y)^2 + v(u_y)^2$

But $u_x = v_y$ and $u_y = -v_x$

$\therefore (uv)_y{}^2 = -2u_xv_x + u(v_y)^2 + v(u_y)^2$ -----(iii)

Adding (ii) and (iii), we get;

$(uv)_x{}^2 + (uv)_y{}^2 = u\left(v_x{}^2+v_y{}^2\right) + v(u_x{}^2 + u_y{}^2)$

=0 {from i}

Therefore, uv is harmonic

Q6.c) Fit a straight line of the form, y = a + bx to the following data and estimate the value of y for x = 3.5 [8]

X	0	1	2	3	4
Y	1	1.8	3.3	4.5	6.3

Solution:-

x	y	x^2	xy
0	1.0	0	0.0
1	1.8	1	1.8
2	3.3	4	6.6
3	4.5	9	13.5
4	6.3	16	25.2
$\sum$=10	$\sum$=16.9	$\sum$=30	$\sum$=47.1

Here N=5.

Let the equation of the line be $y = a + bx$

Then the normal equations are :

$\sum y = Na + b\sum x$

$\sum xy = N\sum x + b\sum x^2$

Substituting the values in the above equation,

$\therefore 16.9=5a+10b$

$\therefore 47.1=10a+30b$

Solving the above equations simultaneously,

$a = 0.72$ and $b= 1.33$

$y = 0.72 + 1.33x$

At $x=3.5$; substituting the value in above equation,

$y = 0.72 + 1.33(3.5)$

$y = 5.375$

Ans : $y = 0.72 + 1.33(x)$

y at $x = 3.5$:5.375

COMPUTER ENGINEERING

APPLIED MATHEMATICS – 3

(CBCGS – DEC 2018)

Q1] a) If Laplace Transform of erf($\sqrt{t}$) = $\frac{1}{s\sqrt{s+1}}$ then find L$\{e^{t}.erf(2\sqrt{t})\}$ (5)

Solution:-

Given :- $L[erf(\sqrt{t})] = \frac{1}{s\sqrt{s+1}}$

$L[erf(2\sqrt{t})] = L[erf(\sqrt{4t})]$

By change of scale property ; $\left\{L[f(at)]= \frac{1}{a}\varphi(\frac{s}{a})\right\}$

$$L[erf(2\sqrt{t})] = \frac{1}{4} \times \frac{1}{\left(\frac{s}{4}\right)\sqrt{\left(\frac{s}{4}\right)+1}}$$

$$= \frac{2}{s\sqrt{s+4}} = \varphi(-s) \quad \text{................ (1)}$$

$L[e^{f}. erf(2\sqrt{t})]$ can be found by first shifting theorem,

$\{L[e^{at}f(t)]= \varphi(s-a)\}$

$L[e^{t} erf(2\sqrt{t})] = \varphi(s-1)$

From I; replace s by s-1

We get $L[e^{t} erf(2\sqrt{t})] = \frac{2}{(s-1)\sqrt{(s-1)+4}} = \frac{2}{(s-1)\sqrt{s+3}}$

$$\mathbf{L[e^{t}\, erf(2\sqrt{t})] = \frac{2}{(s-1)\sqrt{s+3}}}$$

Q1] b) Find the orthogonal trajectory of the family of curves given by $e^{-x}\cos y + xy = C$ **(5)**

Solution:-

Let $u = e^{-x}\cos y + xy$;

To find orthogonal trajectory of u = C

i.e. find v(hormonal conjugate of u)

$u_x = -e^{-x}\cos y + y$ ……………… [differentiating partially wrt x]

$u_y = -e^{-x}\sin y + x$ ………………. [differentiating partially wrt y]

$f'(z) = u_x + iv_x = u_x - iu_y$ ……………[by CR eqn ; $v_x = -u_y$

By Milne-Thompson's method ; replace x = z; y = 0

$f'(z) = -e^{-z}\cos(0) + (0) - i[-e^{-z}\sin(0) + z] = -e^{-z} - iz$

By integrating both sides;

$$f(z) = \frac{-e^{-z}}{-1} - \frac{iz^2}{2} + c = e^{-z} - \frac{iz^2}{2} + c$$

put z = x + iy

$$f(z) = e^{-(x+iy)} - \frac{i(x+iy)^2}{2} + C$$

$$f(z) = e^{-x}.e^{-iy} - \frac{i}{2}[x^2-y^2+2xiy] + C$$

$$f(z) = e^{-x}(\cos y - i\sin y) - \frac{i}{2}[x^2-y^2+2xiy] + C$$

Imaginary part ; $v = -e^{-x}\sin y - \frac{1}{2}[x^2-y^2]$

Hence required orthogonal trajectory = $-e^{-x}\sin y - \frac{1}{2}[x^2-y^2]$

Q1] c) Find Complex form of Fourier Series for e^{2x}; 0 < x < 2 **(5)**

Solution:-

In interval (0,2l); $f(x) = e^{2x}$

$$F(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/l} \quad \text{where } C_n = \frac{1}{2l}\int_0^{2l} f(x)e^{in\pi x/l}dx$$

Put l = 1 therefore in interval (0 < x < 2)

$$\text{We get } f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x} \quad ; \quad C_n = \frac{1}{2}\int_0^2 f(x)e^{in\pi x}dx$$

$$C_n = \frac{1}{2}\int_0^2 e^{2x}.e^{in\pi x}dx \quad = \quad \frac{1}{2}\int_0^2 e^{(2-in\pi)x}dx$$

$$C_n = \frac{1}{2}\left[\frac{e^{(2-in\pi)x}}{(2-in\pi)x}\right]_0^2 \quad = \quad \frac{1}{2}\left[\left[\frac{e^{(2-in\pi)x}}{(2-in\pi)x}\right]-\frac{1}{(2-in\pi)}\right] \quad = \quad \frac{1}{2}\left[\frac{(e^4 \times e^{i2n\pi})-1}{2-in\pi}\right] \quad = \quad \frac{1}{2}\left[\frac{e^4-1}{2-in\pi}\right]$$

$$C_n = \frac{e^4-1}{4-2in\pi}$$

$$e^{2x} = \sum_{-\infty}^{\infty} C_n e^{in\pi x} \quad = \quad \sum_{-\infty}^{\infty}\left[\frac{e^4-1}{4-2in\pi}\right].e^{in\pi x}$$

$$e^{2x} = (e^4-1)\sum_{-\infty}^{\infty}\frac{e^{in\pi x}}{4-2in\pi}$$

Q1] d) If the regression equations are x - 6y + 90 = 0 ; 15x - 8y - 180 = 0. Find the means of x and y, correlation coefficients and standard derivation of x if variance of y = 1 **(5)**

Solution:-

Given equation:- $5x - 6y + 90 = 0$; $\quad 15x - 8y - 180 = 0$

(1) Means of x,y:

Solving the equation n simultaneously,

$5x - 6y = -90$

$15x - 8y = 180$

We get, $X = 36$; $Y = 45$

(2) Correlation coefficients

Suppose the first equation represents the lines of regression of X on Y

Writing it as $X = \frac{6Y}{5} - \frac{90}{5} = b_{xy} = \frac{6}{5}$

Suppose the second equation represents the lines of regression of Y on X

Writing it as $Y = \frac{15X}{8} - \frac{180}{8} = b_{yx} = \frac{15}{8}$

$$r = \sqrt{b_{yx}.\, b_{xy}} = \sqrt{\frac{6}{5} \times \frac{15}{8}} = 1.5$$

But r cannot be greater than 1.

Hence our assumption is wrong;

Treating equation 1 as line of regression of Y on X and equation 2 as line of regression of X on Y .

$$Y = \frac{5X}{6} + \frac{90}{6} = b_{yx} = \frac{5}{6}$$

$$X = \frac{8Y}{15} + \frac{180}{15} = b_{xy} = \frac{8}{15}$$

$$r = \sqrt{b_{yx}.\, b_{xy}} = \sqrt{\frac{5}{6} \times \frac{8}{15}} = 0.6667$$

(3) To find σ_x; given $\sigma_y^2 = 1$

$\sigma_y^2 = 1$

$\sigma_y = 1$

$b_{yx} = r \times \frac{\sigma_y}{\sigma_x}$

$\frac{5}{6} = \frac{2}{3} \times \frac{\sigma_y}{\sigma_x}$

$\frac{15}{12} = \frac{1}{\sigma_x}$

$\sigma_x = \frac{12}{15} = 0.8$

$\bar{X} = 36$; $\bar{Y} = 45$

r = 0.6667

$\sigma_x = 0.8$

Q2] a) Show that the function is Harmonic and find the Harmonic conjugate v = $e^x\cos y + x^3 - 3xy^2$ **(6)**

Solution:-

Given:-

$\frac{\partial v}{\partial x} = e^x\cos y + 3x^2 - 3y^2$

$\frac{\partial^2 v}{\partial x^2} = e^x\cos y + 6x$ ………………… (1)

$\frac{\partial v}{\partial y} = -e^x\sin y - 6yx$

$\frac{\partial^2 v}{\partial y^2} = -e^x\cos y - 6x$ ……………………. (2)

$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = e^x\cos y + 6x - e^x\cos y - 6x = 0$

Therefore, v satisfies Laplace's equation

v is harmonic

Finding harmonic conjugate , u;

$V_y = \psi_1(x,y)$ and $V_x = \psi_2(x,y)$

$\psi_1(z,0) = -e^z.0 - 6(z).0 = 0$

$\psi_2(z,0) = e^z + 3z^2$

$f'(z) = \psi_1(z,0) + i\psi_2(z,0) = i(e^z + 3z^2)$

On integrating; $f(z) = i[e^z+z^3] = i[e^{(x+iy)} + (x+iy)^3] = i[e^x.e^{iy} + (x+iy)^3]$

$f(z) = i[e^x\{\cos y+i\sin y\} + x^3 + 3x^2iy - 3xy^2 - iy^3]$

Real part; $u = -e^x\sin y - 3x^2y + y^3$

Harmonic conjugate = $-e^x\sin y - 3x^2y + y^3$

Q2] b) Find Laplace Transform of:-

$f(t) = \begin{cases} t, & 0<t<1, \\ 0, & 1<t<2 \end{cases}$ $\quad f(t+2) = f(t)$ (6)

Solution:-

f(t) is periodic with period a = 2; we have

$$L[f(t)] = \frac{1}{1-e^{-as}}\int_0^a e^{-st}f(t)dt = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st}f(t)dt$$

$$L[f(t)] = \frac{1}{1-e^{-2s}}\left[\int_2^0 e^{-st}tdt + \int_1^2 0dt\right] = \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}tdt\right]$$

$$L[f(t)] = \frac{1}{1-e^{-2s}}\left[t\left(\frac{-e^{-st}}{s}\right) - \left(\frac{e^{-st}}{s^2}\right)\right]_0^1 = \frac{1}{1-e^{-2s}}\left[\left(\frac{-e^{-s}}{s}\right) - \left(\frac{-e^{-s}}{s^2}\right) + \frac{e^{-s}}{s^2}\right]$$

$$L[f(t)] = \frac{1}{1-e^{-2s}}\left[\left(\frac{-e^{-s}}{s}\right) - \frac{-e^{-s}+1}{s^2}\right]$$

$$L[f(t)] = \frac{1}{s^2(1-e^{-2s})}[-se^{-s}+1-e^{-s}]$$

$$\mathbf{L[f(t)]} = \frac{1}{s^2(1-e^{-2s})}[1-e^{-s}-se^{-s}]$$

Q2] c) Find Fourier expansion of $f(x) = -x^2$; $-1<x<1$ (8)

Solution:-

$f(x) = x - x^2 \quad ; -1<x<1$

Given function is difference b/w odd and even function

$f(x) = f_1(x) - f_2(x)$

Here , $l = 1$

For $f_1(x) = x$ which is odd ; $a_n = 0$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx \quad = \quad 2\int_0^1 x\sin(n\pi x)dx$$

$$b_n = 2\left[x\left\{\frac{-\cos n\pi x}{n\pi}\right\}-(1)\left\{\frac{-\sin(n\pi x)}{n^2\pi^2}\right\}\right]_0^1 = 2\left[1\left\{\frac{-\cos n\pi}{n\pi}\right\}-1\left\{\frac{-\sin(n\pi)}{n^2\pi^2}\right\}-\left\{\frac{\sin 0}{n^2\pi^2}\right\}\right]$$

$$b_n = 2\left[\frac{-(1)^n}{n\pi}\right]$$

For $f_2(x) = x^2$ which is even ; $b_n = 0$

$$a_0 = \frac{1}{l}\int_0^l f(x)dx = \int_0^1 x^2dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} \quad \text{................ (2)}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx = 2\int_0^1 x^2\cos(n\pi x)dx$$

$$a_n = 2\left[x^2\left(\frac{\sin(n\pi x)}{n\pi}\right)-2x\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)+2\left(\frac{-\sin(n\pi x)}{n^3\pi^3}\right)\right]_0^1$$

Solving equation we get,

$$a_n = \frac{4(-1)^n}{n^2\pi^2} \quad \text{...................... (3)}$$

$f(x) = f_1(x) - f_2(x)$

$$f(x) = \sum b_n\sin\left(\frac{n\pi x}{l}\right)-\left\{a_0+\sum a_n\cos\left(\frac{n\pi x}{l}\right)\right\}$$

$$\mathbf{f(x) = \frac{-2}{\pi}\sum\frac{(-1)^n}{n}\sin(n\pi x)-\frac{1}{3}+\frac{4}{\pi^2}\sum\frac{(-1)^n}{n^2}\cos(n\pi x)}$$

Q3] a) Find the analytic function f(z) = u + iv if $v = \log(x^2+y^2) + x-2y$ (6)

Solution:-

$v = \log(x^2+y^2) + x-2y$

Differentiating partially with respect to x and y

$$\frac{\partial v}{\partial x} = \frac{1}{(x^2+y^2)} \times 2x + 1 \quad \ldots\ldots\ldots\ldots (1)$$

$$\frac{\partial v}{\partial y} = \frac{1}{(x^2+y^2)} \times 2y - 2 \quad \ldots\ldots\ldots\ldots (2)$$

$$\frac{\partial v}{\partial y} = \psi_1(x,y) \quad \text{and} \quad \frac{\partial v}{\partial x} = \psi_2(x,y)$$

$$\psi_1(z,0) = \frac{0}{(Z^2)} - 2 \quad ; \qquad \psi_2(z,0) = \frac{2z}{(Z^2)} + 1$$

$$f'(z) = \psi_1(z,0) + i\,\psi_2(z,0)$$

Integrating both sides;

$$f(z) = -\int 2dz + i\int \frac{2z}{Z^2} + 1dz \;=\; -2z + i\int \frac{2}{z} + 1dz$$

$$f(z) = -2z + i(2\log z + z) \;=\; -2z + 2i\log z + iz$$

$$\mathbf{f(z) = z(i-2) + 2i\log z}$$

Q3] b) Find inverse z transform of

$$\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)} \quad ; \quad 3< |z| <4 \qquad \textbf{(6)}$$

Solution:-

By partial fraction:-

$$\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)} = \frac{A}{(z-2)} + \frac{B}{(z-3)} + \frac{C}{(z-4)}$$

$$3z^2 - 18z + 26 = A(z-3)(z-4) + B(z-2)(z-4) + C(z-2)(z-3)$$

Put $z = 4$

$3(4)^2 + 26 - 18(4) = 0(A) + 0(B) + C(4-2)(4-3)$

$2 = C(2)(1)$

$C = 1$

Put $Z = 3$

$3(2)^2 - 18(2) + 26 = A(2-3)(2-4) + 0(B) + 0(C)$

$2 = A(-1)(-2)$

$A = 1$

$$\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)} = \frac{1}{(z-2)} + \frac{1}{(z-3)} + \frac{1}{(z-4)}$$

Since $|z| > 3$ we take common z from first two terms and $4 > |z|$ we take 4 common from last term.

$$\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)} = \frac{1}{z(1-\frac{2}{z})} + \frac{1}{z(1-\frac{3}{z})} + \frac{1}{4(\frac{z}{4}-1)}$$

$$\frac{3z^2-18z+26}{(z-2)(z-3)(z-4)} = \frac{1}{z(1-\frac{2}{z})} + \frac{1}{z(1-\frac{3}{z})} - \frac{1}{4(1-\frac{z}{4})}$$

RHS:-

$$= \frac{1}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+ \text{---}\right] + \frac{1}{z}\left[1+\left(\frac{3}{z}\right)+\left(\frac{3}{z}\right)^2+ \text{---}\right] - \frac{1}{4}\left[1+\left(\frac{z}{4}\right)+\left(\frac{z}{4}\right)^2+ \text{--}\right]$$

$$= \frac{1}{z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+ \text{---}\left(\frac{2}{z}\right)^{k-1}+ \text{---}\right] + \frac{1}{z}\left[1+\left(\frac{3}{z}\right)+\left(\frac{3}{z}\right)^2+ \text{---}\left(\frac{3}{z}\right)^{k-1}+\right] - \frac{1}{4}\left[1+\left(\frac{z}{4}\right)+\left(\frac{z}{4}\right)^2+ \text{---}\left(\frac{z}{4}\right)^{k-1}+ \text{--}\right]$$

Coefficient of z^{-k} in 1st term $= 2^{k-1}$; $k \geq 1$

Coefficient of z^{-k} in 2nd term $= 3^{k-1}$; $k \geq 1$

Coefficient of z^{k} in 3rd term $= \frac{-1}{4^{k+1}}$; $k \geq -1$

Coefficient of z^k in 3^{rd} term = $\frac{-1}{4^{-k+1}}$; $k \le 0$

Hence $z^{-1}[f(z)] = 2^{k-1} + 3^{k-1}$; $k \ge 1$

$$= \frac{-1}{4^{-k+1}} \;;\; k \le 0$$

Q3] c) Solve the differential equation:-

$$\frac{d^2y}{dt^2} + 4y = f(t) \;;\; f(t) = H(t-2) \;;\; y(0) = 0;\; y'(0) = 1 \qquad \textbf{(8)}$$

Using Laplace transform

Solution:-

Let y be the Laplace transform of y L[y] = y

Taking Laplace on both sides

$L[y''] + L[4y] = L[f(t)]$

$S^2 y + Sy(0) - y'(0) + 4y = L[f(t)]$

$S^2 y + 0 - 1 + 4y = L[f(t)]$

$S^2 y - 1 + 4y = L[f(t)]$

$(S^2 + 4) y = 1 + L[f(t)]$

$(S^2 + 4) y = 1 + L[H(t-2)]$

$$y = \frac{1}{S^2+4} + \frac{e^{-2s}}{s(S^2+4)}$$

$$y = \frac{1}{S^2+4} + \left[\frac{1}{s} - \frac{1}{S^2+4}\right]\frac{e^{-2s}}{4}$$

Taking inverse on both sides

$$y = L^{-1}\left(\frac{1}{S^2+4}\right) + L^{-1}\left[\frac{e^{-2s}}{4}\left(\frac{1}{s}\right)\right] - L^{-1}\left[\frac{e^{-2s}}{4(S^2+4)}\right]$$

$$\mathbf{y = \frac{\sin 2t}{2} + \frac{1}{4}H(t\text{-}2) - \frac{1}{4}\cos 2(t-2)H(t-2)}$$

Q4] a) Find $Z\{f(k)\times g(k)\}$ if $(k) = \left(\frac{1}{2}\right)^k$; $g(k) = \cos\pi k$ (6)

Solution:-

$$Z\left\{\left(\frac{1}{2}\right)^k \times \cos\pi k\right\}$$

$$Z\left\{\left(\frac{1}{2}\right)^k\right\} = \sum_{k=0}^{\infty}\frac{1}{2^k}\times Z^{-k} = \sum_{k=0}^{\infty}\frac{1}{2Z^k}$$

$$Z\left\{\left(\frac{1}{2}\right)^k\right\} = 1 + \frac{1}{2Z} + \frac{1}{(2Z)^2} + \frac{1}{(2Z)^3} + \text{- - -}$$

$$Z\left\{\left(\frac{1}{2}\right)^k\right\} = \frac{2Z}{2Z-1}$$

$$Z\{\cos\pi k\} = \sum_{k=0}^{\infty}\cos\pi k\times Z^{-k}$$

$$Z\{\cos\pi k\} = \frac{Z(Z-\cos\pi)}{Z^2-2Z\cos\pi+1} = \frac{Z(Z-(-1))}{Z^2-2Z(-1)+1} = \frac{Z(Z+1)}{Z^2+2Z+1} = \frac{Z}{Z+1}$$

$$\mathbf{Z}\{\cos\pi k\} = \frac{Z}{Z+1}$$

By convolution Theorem; $Z\{f(k)\times g(k)\} = \left(\frac{2Z}{2Z-1}\right)\left(\frac{Z}{Z+1}\right)$

Q4] b) Find the Sperman's Rank Correlation Coefficient b/w X and Y (6)

X	**60**	**30**	**37**	**30**	**42**	**37**	**55**	**45**
Y	**50**	**25**	**33**	**27**	**40**	**33**	**50**	**42**

Solution:-

X	R_1	Y	R_2	$R_1 - R_2$	D^2 $(R_1-R_2)^2$
60	8	50	7.5	-0.5	0.25
30	1.5	25	1	0.5	0.25
37	3.5	33	3.5	0	0
30	1.5	27	2	-0.5	0.25
42	5	40	5	0	0
37	3.5	33	3.5	0	0
55	7	50	7.5	-0.5	0.25
45	6	42	6	0	0
					$\sum = 1$

For repeated ranks;

$$R = 1 - \frac{6\left\{\sum D^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \text{------} \ \frac{1}{12}(m_4^3 - m_4)\right\}}{8^3 - 8}$$

$$R = 1 - \frac{6\left\{1 + \frac{1}{12}(8-2) + \frac{1}{12}(8-2) + \frac{1}{12}(8-2) + \frac{1}{12}(8-2)\right\}}{8^3 - 8}$$

R = 0.9643

Q4] c) Find inverse Laplace transform of **(8)**

1) $\dfrac{3S+1}{(s+1)^4}$ 2) $\dfrac{e^{4-3s}}{(s+4)^{5/2}}$

Solution:-

1) $\dfrac{3S+1}{(s+1)^4}$

By first shifting theorem of replace inverse;

$$L^{-1}\left[\frac{1}{(s+a)^n}\right] = e^{-at}L^{-1}\left[\frac{1}{(s)^n}\right]$$

$$L^{-1}\left[\frac{3S+1}{(s+1)^4}\right] = e^{-t}L^{-1}\left[\frac{3(s-1)+1}{(s+1-1)^n}\right] = e^{-t}L^{-1}\left[\frac{3s-2}{(s)^n}\right] = e^{-t}L^{-1}\left[\frac{3s}{(s)^3}-\frac{2}{(s)^4}\right]$$

$$L^{-1}\left[\frac{3S+1}{(s+1)^4}\right] = e^{-t}\left[\frac{3t^2}{2!}-\frac{2t^3}{3!}\right]$$

$$L^{-1}\left[\frac{3S+1}{(s+1)^4}\right] = e^{-t}\left[\frac{3t^2}{2!}-\frac{2t^3}{3!}\right]$$

2) $\dfrac{e^{4-3s}}{(s+4)^{\frac{5}{2}}}$

$$\frac{e^4.e^{-3s}}{(s+4)^{5/2}} = e^4L^{-1}\left[\frac{e^{-3s}}{(s+4)^{5/2}}\right]$$

Here $\varphi(s) = \dfrac{1}{(s+4)^{5/2}}$ and $a = 3$

$$L^{-1}[\varphi(s)] = L^{-1}\left[\frac{1}{(s+4)^{5/2}}\right] = e^{-4t}L^{-1}\left[\frac{1}{(s)^{5/2}}\right] = e^{-4t}\frac{t^{3/2}}{\sqrt{5/2}} = \frac{e^{-4t}.\ t^{3/2}}{3/2\times1/2\times\sqrt{1/2}}$$

$$L^{-1}[\varphi(s)] = \frac{e^{-4t}.\ t^{3/2}.4}{3\sqrt{\pi}}$$

$$L^{-1}\left[\frac{e^{-as}}{(s+4)^{5/2}}\right] = f(t-a)H(t-a) = \frac{4}{3\sqrt{\pi}}\times e^{-4(t-3)}(t-3)^{\frac{3}{2}}H(t-3)$$

$$L^{-1}\left[\frac{e^4.e^{-3s}}{(s+4)^{5/2}}\right] = e^4\times\frac{4}{3\sqrt{\pi}}\times e^{-4(t-3)}\mathbf{(t-3)}^{\frac{3}{2}}\mathbf{H(t-3)}$$

Q5] a) Find inverse Laplace Transform using convolution theorem; (6)

$$\frac{1}{(s-4)^2(s+3)}$$

Solution:-

Let $\varphi_1(s) = \dfrac{1}{s+3}$ and $\varphi_2(s) = \dfrac{1}{(s-4)^2}$

$L^{-1}[\varphi_1(s)] = e^{-3t}$ and $L^{-1}[\varphi_2(s)] = e^{4t}L^{-1}\left[\dfrac{1}{s^2}\right] = e^{4t}t$

$$L^{-1}[\varphi_1(s)] = \int_0^t e^{-3u}.e^{4(t-u)}(t-u)du = \int_0^t e^{(4t-7u)}(t-u)du$$

$$L^{-1}[\varphi_1(s)] = e^{4t}\int_0^t e^{-7u}(t-u)du = e^{4t}\left[(t-u)\frac{e^{-7u}}{-7}-\frac{(-1)e^{-7u}}{49}\right]_0^t$$

$$L^{-1}[\varphi_1(s)] = e^{4t}\left[\frac{e^{-7t}}{49}+\frac{t}{7}+\frac{1}{49}\right] = e^{4t}\left[\frac{t}{7}+\frac{e^{-7t}-1}{49}\right]$$

$$\mathbf{L^{-1}[\varphi_1(s)] = e^{4t}\left[\frac{t}{7}+\frac{e^{-7t}-1}{49}\right]}$$

Q5] b) Show that the functions $f_1(x) = 1$; $f_2(x) = x$ are orthogonal on (-1,1); determine the constant a,b such that the function $f(x) = -1 + ax + bx^2$ is orthogonal to both $f_1(x)$,$f_2(x)$ on the (-1,1). (6)

Solution:-

$f_1(x) = 1$; $f_2(x) = x$; $f_3(x) = 1 + ax + bx^2$

Case 1:- $m \neq n$

$$\int_{-1}^{1}[f_1(x)]^2dx = \int_{-1}^{1}1dx = [x]_{-1}^{1} = 1-(-1) = 2 \neq 0$$

$$\int_{-1}^{1}[f_2(x)]^2dx = \int_{-1}^{1}x^2dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3}-\left(\frac{-1}{3}\right) = \frac{2}{3} \neq 0$$

$f_1(x)$ & $f_2(x)$ are orthogonal in [-1,1]

$f_3(x)$ is orthogonal with $f_1(x)$

$\int_{-1}^{1}[f_1(x)\times f_3(x)]dx=0$

$\int_{-1}^{1}[-1+ax+bx^2]dx=0$

$\int_{-1}^{1}[-1]dx+\int_{-1}^{1}[ax]dx+\int_{-1}^{1}[bx^2]dx=0$

$-(1-(-1)) + a\left[\frac{x^2}{2}\right]_{-1}^{1} + b\left[\frac{x^3}{3}\right]_{-1}^{1} = 0$

$-2 + 0 + b\left[\frac{1}{3}+\frac{1}{3}\right] = 0$

$-2 + \frac{2b}{3} = 0$

$b = 3$

Also $f_3(x)$ is orthogonal with $f_2(x)$

$\int_{-1}^{1}[f_2(x)\times f_3(x)]dx=0$

$\int_{-1}^{1}x[-1+ax+bx^2]dx=0$

$\int_{-1}^{1}[-x+ax^2+bx^3]dx=0$

$\left[\frac{-x^2}{2}+\frac{ax^3}{3}+\frac{bx^4}{4}\right]_{-1}^{1} = 0$

$\left(\frac{-1}{2}+\frac{a}{3}+\frac{b}{4}\right)-\left(\frac{-1}{2}-\frac{a}{3}+\frac{b}{4}\right) = 0$

$\frac{2a}{3} = 0$

$a = 0$

Ans :- a = 0 and b = 3

Q5] c) Find the Laplace transform of:-

1) $e^{-3t}\int_0^t t\sin 4t\,dt$ **2)** $\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}dt$ **(8)**

Solution:-

1) $e^{-3t}\int_0^t t\sin 4t\,dt$

$$L[\sin 4t] = \frac{4}{s^2+16} = \varphi(s)$$

$$L[\sin 4t] = \frac{(-1)d[\ \varphi(s)]}{ds} = \frac{(-1)d\left[\frac{4}{s^2+16}\right]}{ds}$$

$$-4\frac{d}{ds}\left[\frac{1}{s^2+16}\right] = -4\left[\frac{(3^2+16)0-1(2s)}{(s^2+16)^2}\right]$$

$$-4\left[\frac{-2s}{(s^2+16)^2}\right] = \frac{8s}{(s^2+16)^2}$$

$$L\left[\int_0^t t\sin 4t\,dt\right] = \frac{1}{s}\times\frac{8s}{(s^2+16)^2} = \frac{8}{(s^2+16)^2}$$

$$L\left[e^{-3t}\int_0^t t\sin 4t\,dt\right] = \frac{8}{[(s+3)^2+16]^2}$$ ....(by first shifting method)

$$L\left[e^{-3t}\int_0^t t\sin 4t\,dt\right] = \frac{8}{[s^2+6s+9+16]^2} = \frac{8}{[s^2+6s+25]^2}$$

$$\mathbf{L}\left[e^{-3t}\int_0^t t\sin 4t\,dt\right] = \frac{8}{[s^2+6s+25]^2}$$

2) $\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}dt$

$$L[e^{-t}-e^{-2t}] = \frac{1}{s+1}-\frac{1}{s+2} = \varphi(s)$$

$$L\left[\frac{e^{-t}-e^{-2t}}{t}\right] = \int_s^\infty \varphi(s)$$ .... [division by t]

$$\int_s^\infty \frac{1}{s+1}-\frac{1}{s+2}$$

$$[\ln(s+1)-\ln(s+2)]_s^\infty = \left[\ln\left(\frac{s+1}{s+2}\right)\right]_s^\infty$$

$$\left[\ln\left(\frac{\frac{1}{s}+1}{\frac{2}{s}+1}\right)\right]_s^{s\,\infty} = \left[\ln(0)-\ln\left(\frac{s+1}{s+2}\right)\right] = -\ln\left(\frac{s+1}{s+2}\right) = \ln\left(\frac{s+2}{s+1}\right)\int_0^\infty e^{-st}\times\frac{e^{-t}-e^{-2t}}{t}dt = \ln\left(\frac{s+2}{s+1}\right)$$

Put s = 0

$$\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}dt = \ln(2)$$

Q6] a) Fit a second degree parabola to the given data (6)

X	1	1.5	2	2.5	3	3.5	4
Y	1.1	1.3	1.6	2	2.7	3.4	4.1

Solution:-

Sr	x	y	x^2	x^3	x^4	xy	x^2y
1	1	1.1	1	1	1	1.1	1.1
2	1.5	1.3	2.25	3.375	5.0625	1.95	2.925
3	2	1.6	4	8	16	3.2	6.4
4	2.5	2	6.25	15.625	39.062	5	12.5
5	3	2.7	9	27	81	8.1	24.3
6	3.5	3.4	12.25	42.875	150.06	11.9	41.65
7	4	4.1	16	64	256	16.4	65.6
$\sum$	17.5	16.2	50.75	161.875	548.1845	47.65	154.475

The normal equation are:

$\sum y = Na + b\sum x + c\sum x^2$

$\sum xy = a\sum x + b\sum x^2 + c\sum x^3$

$\sum x^2y = a\sum x^2 + b\sum x^3 + c\sum x^4$

16.2 = 7a + b(17.5) + c(50.75)

$47.65 = 17.5a + b(50.75) + c(161.875)$

$154.475 = 50.75a + b(161.875) + c(548.1845)$

Solving simultaneously;

$a = 0.8329 \quad b = 2.4091 \times 10^{-4} \quad c = 0.2042$

$$\mathbf{y = 0.8329 + 2.4091 \times 10^{-4}x + 0.2042x^2}$$

Q6] b) Find the image of $\left|z-\frac{5}{2}\right| = \frac{1}{2}$ under the transformation $\omega = \frac{3-z}{z-2}$ (6)

Solution:-

$\omega = \frac{3-z}{z-2}$

$\omega(z-2) = (3-z)$

$\omega z - 2\omega = 3 - z$

$\omega z + z = 3 + 2\omega$

$z(1+\omega) = 3 + 2\omega$

$z = \frac{3+2\omega}{(1+\omega)}$

$\left|\frac{3+2\omega}{(1+\omega)} - \frac{5}{2}\right| = \frac{1}{2}$ i.e. $\left|\frac{6+4\omega-5-5\omega}{2(1+\omega)}\right| = \frac{1}{2}$

$\left|\frac{1-\omega}{2+2\omega}\right| = \frac{1}{2}$ i.e. $\left|\frac{1-(u-iv)}{2+2(u+iv)}\right| = \frac{1}{2}$

$\left|\frac{(1-u)-iv)}{(2+2u)+2iv}\right| = \frac{1}{2}$

$\frac{(1-u)^2+v^2}{(2+2u)^2+4v^2} = \frac{1}{4}$

$4\left[(1-u)^2+v^2\right] = (2+2u)^2 + 4v^2$

$u = 0$

Imaginary axis

Q6] c) Find half range cosine series for f(x) = xsinx and hence find (8)

$$\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\dots\dots = \frac{\pi-2}{4}$$

Solution:-

F(x) = xsinx

$$a_0 = \frac{1}{\pi}\int_0^x f(x)dx \quad = \frac{1}{\pi}\int_0^x xsinxdx$$

$$a_0 = \frac{1}{\pi}[x(-cosx)-(-sinx)]_0^\pi$$

$$a_0 = \frac{1}{\pi}[\pi(-(-1))] = 1 \quad \dots\dots\dots\dots(1)$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)cosnxdx \quad = \frac{2}{\pi}\int_0^\pi xsinxcosnxdx$$

$$a_n = \frac{2}{\pi}\int_0^\pi x\left\{\frac{1}{2}[sin(n+1)x+sin(n-1)x]\right\}dx$$

$$a_n = \frac{1}{\pi}\int_0^\pi x\{sin(n+1)x+sin(n-1)x\}dx$$

$$a_n = \frac{1}{\pi}\left[x\left\{\frac{-cos(n+1)x}{n+1}-\frac{cos(n-1)x}{n-1}\right\}-\left\{\frac{-sin(n+1)x}{(n+1)^2}-\frac{sin(n-1)}{(n-1)^2}\right\}\right]_0^\pi$$

$$a_n = \frac{1}{\pi}\left[\pi\left\{\frac{cosn\pi}{n+1}-\frac{cosn\pi}{n-1}\right\}\right] = \frac{1}{\pi}\left[\pi\left\{\frac{-1^n}{n+1}-\frac{-1^n}{n-1}\right\}\right]$$

$$a_n = \frac{-2(-1)^n}{n^2-1} = \frac{2(-1)^{n+1}}{n^2-1} \qquad \text{for } n \neq 1$$

For n=1 put n=1 in equation (1)

$$a_1 = \frac{2}{\pi}\int_0^\pi xsinxcosnxdx$$

$$a_1 = \frac{1}{\pi}\int_0^\pi 2xsinxcosnxdx$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$a_1 = \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right)-\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[\pi\left(\frac{-1}{2}\right)-0\right] = \frac{-1}{2}$$

$$x\sin x = a_0 + \sum_{n=0}^{\infty} a_n\cos nx = 1 + a_1\cos x + \sum_{n=2}^{\infty} a_n\cos nx$$

$$x\sin x = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx$$

Put $x = \pi/2$

$$\frac{\pi}{2} = 1 - 0 + 2\left[\sum\frac{(-1)^3}{3}\cos 2\left(\frac{\pi}{2}\right)+\frac{(-1)^4}{8}\cos\frac{3\pi}{2}\dots\dots\dots\right]$$

$$\frac{\pi}{2} - 1 = 2\left[\frac{1}{3}-\frac{1}{15}+\frac{1}{35}\dots\dots\dots\right]$$

$$\frac{\pi-2}{4} = \left[\frac{1}{3}-\frac{1}{15}+\frac{1}{35}\dots\dots\dots\right]$$

COMPUTER ENGINEERING

APPLIED MATHEMATICS – 3

SEM – 3(CBCGS MAY – 2019)

Q1] a) Find the Laplace transform of te^t sin2t cost **(5)**

Solution:-

$$L[\sin 2t\,.\,\cos t] = L\left[\frac{1}{2}\{\sin 3t+\sin t\}\right] = \frac{1}{2}L[\sin 3t+\sin t]$$

$$L[\sin 2t\,.\,\cos t] = \frac{1}{2}\left[\frac{3}{s^2+9}+\frac{1}{s^2+1}\right]$$

$$L[\sin 2t\,.\,\cos t] = (-1)^n\frac{1}{2}\frac{d}{ds}\left[\frac{3}{s^2+9}+\frac{1}{s^2+1}\right] = \frac{-1}{2}\left[3\left\{\frac{-2s}{(s^2+9)^2}\right\}-\left\{\frac{2s}{(s^2+1)^2}\right\}\right]$$

$$L[\sin 2t\,.\,\cos t] = \frac{1}{2}\left[\frac{6s}{(s^2+9)^2}+\frac{2s}{(s^2+1)^2}\right] = \frac{3s}{(s^2+9)^2}+\frac{s}{(s^2+1)^2} = \varphi(s)$$

$$L\left[te^t\sin 2t\,.\,\cos t\right] = \varphi(s-a)$$

By first shifting theorem;

$$L\left[te^t\sin 2t\,.\,\cos t\right] = \frac{3(s-1)}{[(s-1)^2+9]^2}+\frac{(s-1)}{[(s-1)^2+1]^2} = \frac{3(s-1)}{[s^2-2s+10]^2}+\frac{(s-1)}{[s^2-2s+2]^2}$$

$$L\left[te^t\sin 2t\,.\,\cos t\right] = \frac{3(s-1)}{[s^2-2s+10]^2}+\frac{(s-1)}{[s^2-2s+2]^2}$$

Q1] b) Find the inverse Laplace transform of $\frac{s+2}{s^2(s+3)}$ **(5)**

Solution:-

$$\frac{s+2}{s^2(s+3)} = \frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+3}$$

$$s+2 = As^2(s+3)+B(s+3)+Cs^2$$

Put s = 0;

$2 = B(3) \Rightarrow \quad B = \frac{2}{3}$

Put s = -3;

$-1 = C(-3)^2 \quad \Rightarrow \quad C = -\frac{1}{9}$

Comparing coefficient of s^2 on both sides;

$0 = A + C$

$A = \frac{1}{9}$

$$L^{-1}\left[\frac{s+2}{s^2(s+3)}\right] = \frac{1}{9}L^{-1}\left[\frac{1}{s}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{9}L^{-1}\left[\frac{1}{s+3}\right] = \frac{1}{9}(1) + \frac{2}{3}t - \frac{1}{9}e^{-3t}$$

$$L^{-1}\left[\frac{s+2}{s^2(s+3)}\right] = \frac{1}{9} + \frac{2}{3}t - \frac{1}{9}e^{-3t}$$

Q1] c) Determine whether the function $f(z) = x^2 - y^2 + 2ixy$ is analytic and if so find its derivative (5)

Solution:-

$f(z) = x^2 - y^2 + 2ixy$

$u + iv = (x^2-y^2) + i(2xy)$

$u = (x^2-y^2)$ and $v = 2xy$

Differentiating u & v partially wrt x & y

$u_x = 2x; \quad u_y = -2y$

$v_x = 2y; \quad v_y = 2x$

$u_x = v_y \quad \& \quad u_y = -v_x$

CR equation are satisfied

Hence f(z) is analytic.

$f'(z) = u_x + iv_x = 2x + i2y$

Derivative of $f(z) = 2(x+iy) = 2z$

Q1] d) Find the Fourier series for $f(x) = e^{-|x|}$ in the interval $(-\pi,\pi)$ (5)

Solution:-

Fourier series for $f(x) = e^{-|x|}$ in $(-\pi,\pi)$

$f(x) = e^{-|x|}$

$f(-x) = e^{-|-x|} = e^{-|x|} = f(x)$

Hence $f(x)$ is an even function and hence $b_n = 0$

Also, $f(x) = \begin{cases} e^{x}, & -\pi<x<0 \\ e^{-x}, & 0<x<\pi \end{cases}$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} e^{-x}dx = \frac{1}{\pi}[-e^{-x}]_0^{\pi} = \frac{1}{\pi}[-e^{-\pi}-(-1)] = \frac{1}{\pi}[1-e^{-\pi}]$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} e^{-x}\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\cos nx+n\sin nx)\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{e^{-x}}{1+n^2}\{-\cos nx+n\sin nx\}-\frac{1}{1+n^2}(-\cos 0+n\sin 0)\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[\frac{-e^{-x}\cos n\pi+1}{(1+n^2)}\right] = \frac{2(-e^{-x}(-1)^n+1)}{\pi(1+n^2)}$$

$$f(x) = \frac{1}{\pi}(1-e^{-\pi}) + \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-e^{-x}(-1)^n}{(1+n^2)}\right)\cos nx$$

Q2] a) Evaluate $\int_0^{\infty}\frac{e^{-t}-\cos t}{te^{4t}}\,dt$ (6)

Solution:-

$$L[e^{-t}-\cos t] = L\left[\frac{1}{s+1} - \frac{s}{s^2+1}\right]$$

$$L\left[\frac{e^{-t}-\cos t}{te^{4t}}\right] = \int_s^\infty \frac{1}{s+1}-\frac{s}{s^2+1}\,ds = [\ln(s+1)-\frac{1}{2}\ln(s^2+1)]_s^\infty$$

$$L\left[\frac{e^{-t}-\cos t}{te^{4t}}\right] = [\ln(s+1)-\frac{1}{2}\ln\sqrt{(s^2+1)}]_s^\infty = [\ln\left(\frac{(s+1)}{\sqrt{(s^2+1)}}\right)]_s^\infty$$

$$L\left[\frac{e^{-t}-\cos t}{te^{4t}}\right] = [\ln\left(\frac{\left(1+\frac{1}{s}\right)}{\sqrt{(1+\frac{1}{s^2})}}\right)]_s^\infty = \ln(1)-\ln\left(\frac{(s+1)}{\sqrt{(s^2+1)}}\right) = -\ln\left(\frac{(s+1)}{\sqrt{(s^2+1)}}\right)$$

$$L\left[\frac{e^{-t}-\cos t}{te^{4t}}\right] = \ln\left(\frac{(s+1)}{\sqrt{(s^2+1)}}\right)$$

$$\int_0^\infty e^{-st}\left[\frac{e^{-t}-\cos t}{t}\right]dt=\ln\left(\frac{\sqrt{(s^2+1)}}{(s+1)}\right)$$

Put s = 4

$$\int_0^\infty \frac{e^{-t}-\cos t}{te^{4t}} =\ln\left(\frac{\sqrt{(16+1)}}{(4+1)}\right) = \ln\left(\frac{\sqrt{(17)}}{(5)}\right)$$

Q2] b) Find the Z – transform of $f(k) = f(x) = \begin{cases} 3^k, & k<0 \\ 2^k, & k\geq 0 \end{cases}$ **(6)**

Solution:-

$$Z\{f(k)\} = \sum_{k=-\infty}^{-1} 3^k z^{-k} + \sum_{k=-0}^{\infty} 2^k z^{-k}$$

$$Z\{f(k)\} = \sum_{n=1}^{\infty} 3^{-n} z^{n} + \left\{1+\frac{2}{z}+(\frac{2}{z})^2+.........\right\}$$ [put –k = n in 1st series]

$$Z\{f(k)\} = \left\{\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+...\right\} + \left\{1+\frac{2}{z}+(\frac{2}{z})^2+.........\right\}$$

$$Z\{f(k)\} = \frac{z}{3}\left(1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+.....\right) + \frac{1}{1-\frac{2}{z}} = \frac{z}{3}\left(\frac{1}{1-\frac{z}{3}}\right) + \frac{1}{z-2}z$$

$$Z\{f(k)\} = \frac{z}{3-z} + \frac{z}{z-2} = \frac{z(z-2)+z(3-z)}{(3-z)(z-2)} = \frac{z^2-2z+3z-z^2}{(3-z)(z-2)}$$

$$Z\{f(k)\} = \frac{z}{(3-z)(z-2)}$$

Q2] c) Show that the function $u = 2x(1 - y)$ is a harmonic function . Find its harmonic conjugate and corresponding analytic function (8)

Solution:-

$u = 2x - 2xy$

$u_x = 2 - 2y \qquad u_y = -2x$

$u_x^2 = 0 \qquad u_y^2 = 0$

$u_x^2 + u_y^2 = 0$ laplace equation is satisfied

U is harmonic

$u_x = \varphi_1(x,y) = 2 - 2y$

$u_y = \varphi_1(x,y) = -2x$

$\varphi_1(z,0) = 2 - 2(0) = 2$

$\varphi_2(z,0) = -2z$

By Milne Thompsons method;

$f(z) = \int u_x - iu_z dz = \int 2 - i(-2z)dz = 2\int(1+z)dz = 2\left[z + \frac{z^2}{2}\right]$

$f(z) = 2z + z^2$

$u + iv = 2(x+iy) + (x+iy)^2$

$u + iv = 2x + 2iy + x^2 - y^2 + 2ixy$

On comparing imaginary part,

$v = 2y + 2xy$

$v = 2y(1+x)$

Q3] a) Find the equation of the line of regression of y on x for the following data **(6)**

X	**10**	**12**	**13**	**16**	**17**	**20**	**25**
Y	**19**	**22**	**24**	**27**	**29**	**33**	**37**

Solution:-

X	Y	X^2	Y^2	XY
10	19	100	361	190
12	22	144	484	264
13	24	169	576	312
16	27	256	729	432
17	29	289	841	493
20	33	400	1089	660
25	37	625	1369	925
$\sum X = 113$	181	1983	5449	3276

$\sum X = 113$; $\sum Y = 181$; $\sum X^2 = 1983$; $\sum Y^2 = 5449$; $\sum XY = 3276$

Here N = 7

Line of regression y on x => $y = a + bx$

The normal equation are:

$\sum y = Na + b\sum x$; $181 = 7a + 113b$

$\sum xy = \sum x + b\sum x^2$; $3276 = 113a + 1983b$

On solving simultaneously;

$a = -10.1304$ and $b = 2.2293$

line of regression ; y on x => $y = -10.1304 + 2.2293b$

Q3] b) Find the bilinear transformation which maps z = 2, $1, 0$ onto w = 1, $0, i$ (6)

Solution:-

Let the transformation be $\omega = \dfrac{az+b}{cz+d}$ ....(1)

Putting the given values of z & ω we get $1 = \dfrac{2a+b}{2c+a}$; $0 = \dfrac{a+b}{c+d}$; $i = \dfrac{b}{d}$

From third we get b = di

From second we get a = - b = di

$2c + 2d = 2a + b$

We get $2c + d = -2di + di$

$2c = -di - d$

$c = -\dfrac{(i+1)d}{2}$

Hence from 1; we get

$$\omega = \frac{-diz+di}{\left\{-\dfrac{(i+1)dz}{2}\right\}+d} = \frac{2(-iz+i)}{-(i+1)+2}$$

$$\omega = \frac{2(z-1)}{(1-i)z+2i}$$

Q3] c) Obtain the expansion of f(x) = x$(\pi-x)$, $0 < x < \pi$ as a half range cosine series. Hence show that $\sum_1^{\infty} \dfrac{(-1)^{n+1}}{n^2} = \dfrac{\pi^2}{12}$ (8)

Solution:-

Cosine series

$f(x) = a_0 + \sum a_n \cos nx$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \pi x - x^2 dx = \frac{1}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^3}{3}\right] = \frac{\pi^2}{6} \quad ...... (1)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi x-x^2)\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\left[(\pi x-x^2)\frac{\sin(nx)}{n}-(\pi-2x)\left(\frac{-\cos nx}{n^2}\right)+(-2)\left(\frac{-\sin(nx)}{n^3}\right)\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[\left\{0-\frac{\pi\cos\pi}{n^2}+0\right\}-\left\{0+\frac{\pi\cos n(0)}{n^2}\right\}+0\right] = \frac{2}{\pi}\left[\frac{-\pi(-1)^n}{n^2}-\frac{\pi}{n^2}\right] = \frac{-2}{n^2}[(-1)^n+1]$$

$$f(x) = \frac{\pi^2}{6} - 2\sum_{n=1}^{\infty}\frac{(-1)^n+1}{n^2}\cos nx$$

$$x(\pi-x) = \frac{\pi^2}{6} - 2\sum_{n=1}^{\infty}\frac{(-1)^n+1}{n^2}\cos nx$$

Put x = 0

$$0 = \frac{\pi^2}{6} - 2\sum_{n=1}^{\infty}\frac{(-1)^n+1}{n^2}\cos nx$$

$$-\frac{\pi^2}{6} = -2\sum_{n=1}^{\infty}\frac{(-1)^n+1}{n^2}$$

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{(-1)^n+1}{n^2}$$

Q4] a) Find the inverse Laplace Transform by using convolution theorem

$$\frac{1}{(s^2+1)(s^2+9)}$$ **(6)**

Solution:-

Let $\varphi_1(s) = \frac{1}{(s^2+1)}$ and $\varphi_2(s) = \frac{1}{(s^2+9)}$

$$L^{-1}[\varphi_1(s)] = L^{-1}\left[\frac{1}{(s^2+1)}\right] = \sin t$$

$$L^{-1}[\varphi_2(s)] = L^{-1}\left[\frac{1}{(s^2+9)}\right] = \frac{1}{3}\sin 3t$$

$$L^{-1}\left[\frac{1}{(s^2+1)}\cdot\frac{1}{(s^2+9)}\right] = \int_0^{l}\sin u.\frac{1}{3}\sin 3(l-u)du$$

$$L^{-1}\left[\frac{1}{(s^2+1)}\cdot\frac{1}{(s^2+9)}\right] = -\frac{1}{6}\int_0^t \cos[(1-3)u+3t]-\cos[(4u-3t)]du$$

$$L^{-1}\left[\frac{1}{(s^2+1)}\cdot\frac{1}{(s^2+9)}\right] = -\frac{1}{6}\int_0^t \cos(3t-2u)-\cos(4u-3t)du$$

$$L^{-1}\left[\frac{1}{(s^2+1)}\cdot\frac{1}{(s^2+9)}\right] = -\frac{1}{6}\left[\frac{\sin(3t-2u)}{-2}-\frac{\sin(4u-3t)}{4}\right]_0^t = -\frac{1}{6}\left[\frac{-\sin t}{2}-\frac{\sin t}{4}-\left(\frac{-\sin 3t}{2}-\frac{\sin(-3t)}{4}\right)\right]$$

$$L^{-1}\left[\frac{1}{(s^2+1)}\cdot\frac{1}{(s^2+9)}\right] = -\frac{1}{6}\left[\frac{\sin 3t}{2}-\frac{\sin 3t}{4}-\frac{\sin t}{2}-\frac{\sin t}{4}\right]$$

$$L^{-1}\left[\frac{1}{(s^2+1)}\cdot\frac{1}{(s^2+9)}\right] = -\frac{1}{6}\left[\frac{2\sin 3t-\sin 3t}{4}-\frac{2\sin t-\sin t}{4}\right]$$

$$L^{-1}\left[\frac{1}{(s^2+1)}\cdot\frac{1}{(s^2+9)}\right] = \frac{1}{24}[3\sin t-\sin 3t]$$

Q4] b) Calculate the coefficient of correlation between Price and Demand (6)

Price : 2, 3, 4, 7, 4

Demand : 8, 7, 3, 1, 1.

Solution:-

x	y	x^2	y^2	xy
2	8	4	64	16
3	7	9	49	21
4	3	16	9	12
7	1	49	1	7
4	1	16	1	4
∑N =5 ; 20	20	94	124	60

$$X = \frac{\sum x}{N} = \frac{20}{5} = 4$$

$$Y = \frac{\sum y}{N} = \frac{20}{5} = 4$$

$$r = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sqrt{\left(\sum x^2 - \frac{(\sum x)^2}{n}\right)\left(\sum y^2 - \frac{(\sum y)^2}{n}\right)}} = \frac{60 - \frac{400}{5}}{\sqrt{\left(94 - \frac{400}{5}\right)\left(124 - \frac{400}{5}\right)}} = -0.8058$$

Q4] c) Find the inverse Z- transform for the following: (8)

1. $\frac{z}{z-5}$ **, $|z| < 5$** **2.** $\frac{1}{(z-1)^2}$ **, $|z| > 1$**

Solution:-

1. $\frac{z}{z-5}$, $|z| < 5$

$$\frac{-z}{5(1-\frac{z}{5})} = -\frac{z}{5}\left[1-\frac{z}{5}\right]^{-1} = -\frac{z}{5}\left[1+\left(\frac{z}{5}\right)+\left(\frac{z}{5}\right)^2+\ldots\ldots\right]$$

$$= -\left[\left(\frac{z}{5}\right)+\left(\frac{z}{5}\right)^2+\left(\frac{z}{5}\right)^3\ldots\ldots\right]$$

Coefficient of $z^n = -5^n$

Put $n = -k$; $n \geq 1$

$z^{-k} = -5^{-k}$; $k \leq -1$

$Z^{-1}[F(z)] = -5^{-k}$; $k \leq -1$

2. $\frac{1}{(z-1)^2}$, $|z| > 1$

$$\frac{1}{z^2(1-\frac{1}{z})^2} = \frac{1}{z^2}\left[1-\frac{1}{z}\right]^{-2} = \frac{1}{z^2}\left[1+2\left(\frac{1}{z}\right)+3\left(\frac{1}{z}\right)^2+\ldots\right]$$

$$= \left[\frac{1}{z^2}+\frac{2}{z^3}+\frac{3}{z^4}+\ldots\frac{(n-1)}{z^n}\right]$$

Coefficient of $z^{-n} = n - 1$; $n \geq 2$

Put n = k

$z^{-k} = k - 1 \;\; ; \; k \geq 2$

$z^{-1}[f(z)] = k - 1 \; ; k \geq 2$

Q5] a) Find the Laplace transform of e^{-t}sint $H(t - \pi)$ (6)

Solution:-

Here $f(t) = e^{-t}\sin t$ and $a = \pi$

$f(t + \pi) = e^{-(t+\pi)}\sin(t+\pi) = -e^{-(t+\pi)}\sin t = -e^{-(\pi)}.e^{-(t)}\sin t$

$$L[f(t+\pi)] = L\left[-e^{-\pi}.e^{-(t)}.\sin t\right] = -e^{-\pi}L\left[e^{-(t)}\sin t\right] = -e^{-\pi}\left[\frac{1}{(s+1)^2+1}\right]$$

$$L[f(t+\pi)H(t-\pi)] = -e^{-\pi s}.e^{-\pi}.\frac{1}{s^2+2s+2} = -e^{-\pi(s+1)}.\frac{1}{(s+1)^2+1}$$

$$L\left[e^{-t}\sin t\, H(t-\pi)\right] = \frac{-e^{-\pi(s+1)}}{s^2+2s+2}$$

Q5] b) Show that the set of functions {sinx, $\sin 3x$, $\sin 5x$, ……} is orthogonal over [0,π/2] . Hence construct orthogonal set of functions. (6)

Solution:-

We have $f_n(x) = \sin(nx)$

$$\int_0^{\pi/2} f_m(x).f_n(x)dx = \int_0^{\pi/2} \sin mx.\sin nx\, dx$$

$$\int_0^{\pi/2} f_m(x).f_n(x)dx = -\frac{1}{2}\int_0^{\pi/2} \sin(m+n)x - \sin(m-n)x\, dx$$

$$\int_0^{\pi/2} f_m(x).f_n(x)dx = -\frac{1}{2}\left[\frac{-\cos(m+n)x}{(m+n)} - \left\{\frac{-\cos(m-n)x}{(m-n)}\right\}\right]_0^{\pi/2}$$

$$\int_0^{\pi/2} f_m(x).f_n(x)dx = \frac{1}{2}\left[\frac{\cos(m+n)x}{(m+n)} - \frac{\cos(m-n)x}{(m-n)}\right]_0^{\pi/2}$$

Now two cases arises;

1. m $\neq$ n

$$= \frac{1}{2}\left[\frac{\cos(m+n)\pi/2}{(m+n)} - \frac{\frac{\cos(m-n)\pi}{2}}{(m-n)} - \{1-1\}\right] = 0$$

for m = n

$$\int_0^{\pi/2}\sin^2(2n+1)dx = \int_0^{\pi/2}\frac{1-\cos2(2n+1)x}{2} = \frac{1}{2}\left[x - \frac{\sin2(2n+1)x}{2(2n+1)}\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 0 - \{0-0\}\right]_0^{\pi/2} = \frac{\pi}{4} \neq 0$$

$$\int_0^{\pi/2} \begin{matrix} f_m(x)f_n(x)dx = 0 & \text{if } m \neq n \\ f_m(x)f_n(x)dx \neq 0 & \text{if } m = n \end{matrix}$$

given set of functions is orthogonal in $[0,\frac{\pi}{2}]$

$$\int_0^{\pi/2}[f_n(x)]^2dx = \frac{\pi}{4}$$

Divide by $\frac{\pi}{4}$ on both sides;

$$\int_0^{\pi/2}\frac{4}{\pi}[f_n(x)]^2dx = \frac{4}{\pi}.\frac{\pi}{4}$$

i.e. $\int_0^{\frac{\pi}{2}}\frac{2}{\sqrt{\pi}}f_n(x).\frac{2}{\sqrt{\pi}}f_n(x)dx = 1$

this is orthonormal set, $\varphi_n(x) = \frac{2}{\sqrt{\pi}}\sin(2n+1)x$

Q5] c) Solve using Laplace transform $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 3te^t$, given that y(0) = 4 and y'(0) = 2 **(8)**

Solution:-

Let $L(y) = \bar{y}$

Taking laplace transform on both sides

$$L(y'') + 2L(y') + L(y) = L(3te^{-t})$$

But , $L(y') = S\,y - y(0) = S\,y - 4$

$$L(y'') = S^2y - Sy(0) - y'(0) = S^2y - 4s - 2$$

$$L(e^{-t}) - \frac{1}{s+1}$$

$$L[te^{-t}] = \frac{-d}{ds}\left(\frac{1}{s+1}\right) = \frac{1}{(s+1)^2}$$

The equation becomes,

$$\left(s^2\,y - 4s - 2\right) + 2\left(s\,y - 4\right) + y = 3.\frac{1}{(s+1)^2}$$

$$\left(s^2 - 4s - 2\right)y - 4s - 10 = \frac{3}{(s+1)^2}$$

$$(s+1)^2\,y = 4s + 10 + \frac{3}{(s+1)^2}$$

$$y = \frac{3}{(s+1)^4} + \frac{4s+10}{(s+1)^2}$$

$$y = \frac{3}{(s+1)^4} + \frac{4s+10}{(s+1)^2} + \frac{6}{(s+1)^2}$$

$$y = \frac{3}{(s+1)^4} + \frac{4}{(s+1)} + \frac{6}{(s+1)^2}$$

Take inverse on both sides

$$y = L^{-1}\left[\frac{3}{(s+1)^4}\right] + 4L^{-1}\left[\frac{1}{s+1}\right] + 6L^{-1}\left[\frac{1}{(s+1)^2}\right]$$

$$y = 3e^{-t}.\frac{t^3}{3!} + 4e^{-t} + 6e^{-t}t$$

Q6] a) Find the complex form of Fourier series for $f(x) = 3x$ in $(0, 2\pi)$ (6)

Solution:-

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}$$

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{inx}dx \;=\; \frac{1}{2\pi}\int_0^{2\pi} 3xe^{inx}dx \;=\; \frac{3}{2\pi}\int_0^{2\pi} xe^{inx}dx$$

$$C_n = \frac{3}{2\pi}\left[\frac{xe^{-inx}}{-in} - \frac{e^{-inx}}{(-in)^2}\right]_0^{2\pi} = \frac{3}{2\pi}\left[\frac{-2\pi e^{-inx}}{in} + \frac{e^{-in2\pi}}{n^2} - \left\{0 + \frac{1}{n^2}\right\}\right] = -\frac{3}{in}$$

$$C_n = -\frac{3}{in}\cdot\frac{i}{i} = \frac{3i}{n}; \quad n \neq 0$$

$$C_0 = 3\pi$$

$$f(x) = C_0 + \sum_{-\infty}^{\infty} C_n e^{inx}$$

$$f(x) = 3\pi + 3i\sum_{-\infty}^{\infty} e^{inx}/n$$

Q6] b) If f(z) is an analytic function with constant modulus then, prove that f(z) is constant **(6)**

Solution:-

Let $f(z) = u + iv$

But $|f(z)| = C$

$|u + iv| = C$

$u^2 + v^2 = C^2$

Differentiate it partially wrt x and y

$uu_x + vv_x = 0$

$uu_y + vv_y = 0$

f(z) is analytic , $u_x = u_y$ and $u_y = -v_x$

$uu_x - vv_y = 0$ and $uu_y + vv_x = 0$

$uu_x = vu_y \quad \Rightarrow \quad u_y = \frac{uu_x}{v}$

$u\left(\frac{uu_x}{v}\right) + vu_x = 0 \quad \Rightarrow \quad (u^2+v^2)u_x = 0$

Eliminating u_y; $(u^2 + v^2)\, u_x = 0$

$C^2 u_x = 0 \quad => \quad u_x = 0$

Similarly we can prove that

$u_y = v_x = v_y = 0$

f(z) is analytic

$f'(z) = U_x + iV_x = 0$

As derivative of constant function is 0

Hence f(z) is constant.

Q6] c) Fit a circle of the form y = ax^b to the following data (8)

X	**1**	**2**	**3**	**4**
y	**2.5**	**8**	**19**	**50**

Solution:-

Taking log on both sides of $y = ax^b$

$\log y = \log a + b\log x$

let $\log y = Y, \log a = A; b = B; \log x = X$

x	y	X = log x	Y = log y	XY	X^2
1	2.5	0	0.3979	0	0
2	8	0.3010	0.9031	0.2718	0.0906
3	19	0.4771	1.2788	0.6101	0.2276
4	50	0.6020	1.6990	1.0228	0.3624
∑N = 4		1.3801	4.2788	1.9047	0.6806

Putting values in above equations:

4.2788 = 4A + B(1.3801)

1.9047 = (1.3801)A + B(0.6806)

Solving simultaneously;

A = 0.3466;   B = 2.09

A = log a => a = 2.22

$y = 2.22x^{2.09}$

MATHEMATICS SOLUTION

CBCGS (DEC – 2019) SEM – 3

BRANCH – COMPUTER ENGINEERING

Q1. a) If $L\{t\sin\omega t\} = \frac{2\omega s}{(s^2+\omega^2)^2}$. Find $L\{\omega t\cos t\omega t + \sin\omega t\}$ **[5]**

Soln.: $L\{\omega t\cos t\omega t + \sin\omega t\}$

$L\{\omega t\cos\omega t\} + L\{\sin\omega t\}$

$\omega\, L\{t\cos\omega t\} + L\{\sin\omega t\}$ -----(i)

Finding $L\{t\cos\omega t\}$;

$L[\cos\omega t] = \frac{s}{s^2+\omega^2}$

$L[t\cos\omega t] = -\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right) = \frac{s^2-\omega^2}{s^2+\omega^2}$

$L[\sin\omega t] = \frac{1}{s^2+\omega^2}$

Substituting in (i), we get;

$$\mathbf{L\{\omega t\cos t\omega t + \sin\omega t\} = \omega\frac{s^2-\omega^2}{s^2+\omega^2} + \frac{1}{s^2+\omega^2}}$$

Q1. b) If $f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$ is analytic, find a, b, c and d. **[5]**

Soln.: We have $f(z) = u + iv$ and $u = (x^2 + axy + by^2)$; $v = (cx^2 + dxy + y^2)$

$\therefore u_x = 2x + ay$ $\qquad u_y = ax + 2by$

$v_x = 2cx + dy$ $\qquad v_y = dx + 2y$

By CR equation,

$u_x = v_y$

$2x + ay = dx + 2y$

On comparing the coefficients,

$a = 2$ and $d = 2$

Also, $u_y = -v_x$

$ax + 2by = -(2cx + dy)$

$2x + 2by = -2cx - 4y$

On comparing the coefficients,

$c = -1$ and $b = -2$

Ans: $\mathbf{a = 2, b = -2, c = -1 \text{ and } d = 2}$

Q1. c) Find the Fourier series of expansion of $\mathbf{f(x) = x^3\ (-\pi,\ \pi)}$ **[5]**

Soln. : $f(x) = x^3$ is an odd function as $f(x) = f(-x) = -f(x)$

Therefore in the range $(-\pi,\ \pi)$, $a_0 = a_1 = 0$

$\therefore\ b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$

$b_n = \frac{2}{\pi}\int_0^{\pi} x^3 \sin(nx)\,dx$

$b_n = \frac{2}{\pi}\left[x^3\left(-\frac{\cos nx}{n}\right) - \left\{3x^2\left(-\frac{\sin nx}{n^2}\right)\right\} + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right]_0^{\pi}$

$b_n = \frac{2}{\pi}\left[\pi^3\left(-\frac{\cos n\pi}{n}\right) - \left\{3\pi^2\left(-\frac{\sin n\pi}{n^2}\right)\right\} + 6\pi\left(\frac{\cos n\pi}{n^3}\right) - 6\left(\frac{\sin n\pi}{n^4}\right) - \{0 - 0 + 0 - 0\}\right]$

$b_n = \frac{2}{\pi}\left[\pi^3\left(-\frac{(-1)^n}{n}\right) + 6\pi\left(\frac{(-1)^n}{n^3}\right)\right]$

$b_n = 2\left[-\pi^2\left(\frac{-1^n}{n}\right) + 6\left(\frac{-1^n}{n^3}\right)\right]$

$b_n = (-1)^n\left[\frac{12}{n^3} - \frac{2\pi^2}{n}\right]$

Fourier series for the given function is given as: $f(x) = \sum_{x=1}^{x=\infty} b_n \sin nx$

$$\mathbf{f(x) = \sum_{x=1}^{x=\infty}(-1)^n\left[\frac{12}{n^3} - \frac{2\pi^2}{n}\right]\sin nx}$$

Q1. d) If the two regression equations are $4x - 5y + 33 = 0, 20x - 9y - 107 = 0$. Find:

i) The mean values of x and y
ii) The Correlation Coefficient
iii) Standard Deviation of y if variance of x is 9 **[5]**

Soln.:

i) Solving the equations simultaneously,

$$4x - 5y = -33$$
$$20x - 9y = 107$$

We get $\bar{x}$= 13 and $\bar{y}$=17

ii) Suppose the second equation represents the line of regression of X on Y

$$20x = 9y + 107$$

$\therefore b_{xy} = \frac{9}{20}$

Suppose the first equation represents the line of regression of X on Y

$$5y = 4x + 33$$

$\therefore b_{yx} = \frac{4}{5}$

$$r = \sqrt{b_{xy}.b_{yx}} = \sqrt{\frac{9}{20}.\frac{4}{5}} = \mathbf{0.6}$$

iii) $b_{yx} = r\frac{\sigma_y}{\sigma_x}$

$\frac{4}{5} = 0.6(\frac{\sigma_y}{3})$ [Standard devn = $\sqrt{\text{variance}}$]

$\boldsymbol{\sigma_y = 4}$

Q2. a) Show that the function is harmonic and find the harmonic conjugate. **[6]**

$$\mathbf{u = \cos x \ \cosh y - 2xy}$$

Soln.: Given: $u = \cos x \ \cosh y - 2xy$

Partially double differentiating wrt x and y.

$u_x = -\cosh y \sin x - 2y$

$u_x^2 = -\cosh y \cos x$

$u_y = \cos x \sinh y - 2x$

$u_y^2 = \cos x \cosh y$

By Laplace's equation,

$u_x^2 + u_y^2 = 0$

$-\cosh y \cos x + \cos x \cosh y = 0 = \text{RHS}$

Thus, the function is harmonic

$-\int u_y dx = -\int (\cos x \sinh y - 2x) dx$

$= -\sin x \sinh y + x^2$

Integrating terms in u_x free from x

$\int -2y\, dy = -y^2$

$\therefore \quad \mathbf{v = \sin x \sinh y + x^2 - y^2 + c}$

Q2. b) Evaluate $\int_0^\infty e^{-t}(\int_0^t u^2 \sinh u \cosh u\, du)dt$ using Laplace Transform **[6]**

Soln.: $L[\sinh u \cosh u] = \frac{1}{2}[2 \sinh u \cosh u] = \frac{1}{2} L[\sinh 2u] = \frac{1}{2}[\frac{2}{s^2-2^2}]$

$L[u^2 \sinh u \cosh u] = \frac{d^2}{ds^2}(\frac{1}{s^2-4})$

$\frac{d}{ds}\left(-\frac{2s}{s^2-4}\right) = \frac{2(3s^2+4)}{(s^2-4)^3}$

$L\left[\int_0^t u^2 \sinh u \cosh u\, du\right] = \frac{2(s^2+4)}{s(s^2-4)^3}$

$\therefore \int_0^\infty e^{-st}(\int_0^t u^2 \sinh u \cosh u\, du)dt = \frac{2(3s^2+4)}{s(s^2-4)^3}$

Put s=1, we get

$$\int_0^\infty \mathbf{e^{-t}}(\int_0^t \mathbf{u^2 \sinh u \cosh u\, du})\mathbf{dt} = \frac{\mathbf{2(3+4)}}{\mathbf{1(1-4)^3}} = -\frac{\mathbf{14}}{\mathbf{27}}$$

Q2. c) Find the Fourier Series expansion of $f(x) = \begin{cases} x\,;\ -1 < x < 0 \\ x+2\,;\ 0 < x < 1 \end{cases}$ **[8]**

Soln.: Fourier series for f(x) is given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$a_0 = \frac{1}{2l}\int_{-1}^{1} f(x)dx = \frac{1}{2}[\int_{-1}^{0}(x+2)dx + \int_{0}^{1} xdx]$

$a_0 = \frac{1}{2}\left[\left(\frac{x^2+4x}{2}\right)\Big|_{-1}^{0} + \left(\frac{x^2}{2}\right)\Big|_{0}^{1}\right]$

$a_0 = \frac{1}{2}\left[\left(0+\frac{3}{2}\right) + \left(\frac{1}{2}\right)\right] = 1$

$a_n = \frac{1}{l}\int_{-1}^{1} f(x)\cos(\frac{n\pi x}{l})dx = [\int_{-1}^{0}(x+2)\cos(n\pi x)dx + \int_{0}^{1} x\cos(n\pi x)dx]$

$a_n = 1\left[\left(\frac{(x+2)}{n\pi}\sin n\pi x + \frac{2(\cos n\pi x)}{n^2\pi^2}\right)\Big|_{-1}^{0} + \left(\frac{x}{n\pi}\sin n\pi x + \frac{1(\cos n\pi x)}{n^2\pi^2}\right)\Big|_{0}^{1}\right]$

$a_n = 1\left[\left(0+\frac{2}{n^2\pi^2} - 0 - \frac{2}{n^2\pi^2}\right) + \left(0+\frac{(-1)^n}{n^2\pi^2} - 0 - \frac{(-1)^n}{n^2\pi^2}\right)\right] = 0$

$b_n = \frac{1}{l}\int_{-1}^{1} f(x)\sin(\frac{n\pi x}{l})dx = [\int_{-1}^{0}(x+2)\sin(n\pi x)dx + \int_{0}^{1} x\sin(n\pi x)dx]$

$b_n = 1\left[\left(-\frac{(x+2)}{n\pi}\cos n\pi x + \frac{2(\sin n\pi x)}{n^2\pi^2}\right)\Big|_{-1}^{0} + \left(-\frac{x}{n\pi}\cos n\pi x + \frac{1(\sin n\pi x)}{n^2\pi^2}\right)\Big|_{0}^{1}\right]$

$b_n = 1\left[\left(-\frac{2}{n\pi} - 0 + \frac{(-1)^n}{n\pi} - 0\right) + \left(-\frac{(-1)^n}{n\pi} - 0\right)\right] = -\frac{2}{n\pi}$

Substituting the values in expansion,

$$f(x) = 1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{l}\right)$$

Q3.a) Find the Analytic function f(z)=u+iv if $u - v = e^x(\cos y - \sin y)$ **[6]**

Soln.: Let $U = u - v = e^x(\cos y - \sin y)$

$U_x = e^x(\cos y - \sin y) = \phi 1(x)$

$U_y = e^x(-\sin y - \cos y) = -e^x(\sin y + \cos y) = \phi 2(x)$

$\therefore\ (1+i)f'(z) = U_x - iU_y = \phi 1(z,0) - i\phi 2(z,0)$

$\therefore\ (1+i)f(z) = \int[e^z + ie^z]dz = (1+i)\int e^z dz = (1+i)e^z + C$

$\therefore\ \mathbf{f(z) = e^z + C}$

Q3.b) Find Inverse Z transform of $\frac{5z}{(2z-1)(z-3)}$ $\quad \frac{1}{2} < |z| < 3$ **[6]**

Soln.: We have, $F(Z) = \frac{5z}{(2z-1)(z-3)}$

Applying Partial fractions;

$$\frac{z}{(2z-1)(z-3)} = \frac{A}{2z-1} + \frac{B}{z-3}$$

$$\frac{z}{(2z-1)(z-3)} = \frac{A(z-3)+B(2z-1)}{(2z-1)(z-3)}$$

Comparing the coefficients on both the sides,

$$1 = A + 2B \text{ and } 0 = 3A + B$$

Solving the equations simultaneously,

$$A = -\frac{1}{5} \text{ and } B = \frac{3}{5}$$

$$\frac{5z}{(2z-1)(z-3)} = \left[\frac{3}{z-3} - \frac{1}{2z-1}\right]$$

$$\frac{5z}{(2z-1)(z-3)} = \left[\frac{3}{z-3} - \frac{1}{2(z-\frac{1}{2})}\right]$$

$$\frac{5z}{(2z-1)(z-3)} = \left[-\frac{3}{3(1-\frac{z}{3})} - \frac{1}{2z(1-\frac{1}{2z})}\right]$$

$$\frac{5z}{(2z-1)(z-3)} = \left[-\frac{1}{1(1-\frac{z}{3})} - \frac{1}{2z(1-\frac{1}{2z})}\right]$$

$$\frac{5z}{(2z-1)(z-3)} = \left[-\left(1-\frac{z}{3}\right)^{-1} - \frac{1}{2z}\left(1-\frac{1}{2z}\right)^{-1}\right]$$

$$\frac{5z}{(2z-1)(z-3)} = \left[-\left(1+\frac{z}{3}+\frac{z^2}{9}+\cdots+\left(\frac{z}{3}\right)^n\right)-\frac{1}{2z}\left(1+\frac{1}{2z}+\frac{1}{4z^2}+\cdots+\frac{1}{(2z)^n}\right)\right]$$

Coefficient of z^n in first series $= -3$

Put n=-k

$z^{-k} = -3$ k > =0

Coefficient of z^{-n} in second series $= \frac{1}{2^n}$

Put n=k

$z^{-k} = \frac{1}{2^k}$ k > = 0

$$\mathbf{Z^{-1}[F(Z)] = -3 + \frac{1}{2^k}; k >= 0}$$

Q3. c) Solve the differential equation using Laplace Transform: **[8]**

$$(D^2 - 2D + 1)y = e^t, y(0) = 2 \text{ and } y'(0) = -1$$

Soln.: Let $L(y) = \bar{y}$, then taking Laplace transform on both sides,

$L(y'') - 2L(y') + L(y) = L(e^t)$

But $L(y') = s\bar{y} - y(0) = s\bar{y} - 2$

and $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - 2s + 1$

and $L(e^t) = \frac{1}{s-1}$

$\therefore$ the equation becomes,

$s^2\bar{y} - 2s + 1 - 2(s\bar{y} - 2) + \bar{y} = \frac{1}{s-1}$

$\Rightarrow s^2\bar{y} - 2s + 1 - 2s\bar{y} + 4 + \bar{y} = \frac{1}{s-1}$

$\Rightarrow \bar{y}(s^2 - 2s + 1) = \frac{1}{s-1} + 2s - 5 \quad \Rightarrow \quad \bar{y}(s-1)^2 = \frac{1}{s-1} + 2s - 5$

$\Rightarrow \bar{y} = \frac{1}{(s-1)(s-1)^2} + \frac{2s}{(s-1)^2} - \frac{5}{(s-1)^2}$

$\Rightarrow \bar{y} = \frac{1}{(s-1)^3} + \frac{2s}{(s-1)^2} - \frac{5}{(s-1)^2}$

$\Rightarrow \bar{y} = \frac{e^t t^2}{2} + 2\left[\frac{(s-1)}{(s-1)^2} + \frac{1}{(s-1)^2}\right] - \frac{5}{(s-1)^2} \Rightarrow \bar{y} = \frac{e^t t^2}{2} + 2\left[\frac{1}{(s-1)} + \frac{1}{(s-1)^2}\right] - \frac{5}{(s-1)^2}$

$\Rightarrow \bar{y} = \frac{e^t t^2}{2} + 2[e^t] - \frac{3}{(s-1)^2}$

$\Rightarrow \frac{e^t t^2}{2} + 2[e^t] - \frac{3e^t}{(s)^2} \Rightarrow \frac{e^t t^2}{2} + 2[e^t] - 3te^t$

Ans : $\boldsymbol{\frac{e^t t^2}{2} + 2[e^t] - \frac{3e^t}{(s)^2} \Rightarrow \frac{e^t t^2}{2} + 2[e^t] - 3te^t}$

Q4. a) Find the Complex form of Fourier Series for f(x)= cos ax ; (-π,π) **[6]**

Soln.: We have $\cos ax = (e^{aix} + e^{-aix}) / 2$

Complex form of f(x) is given by $f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}$

For e^{aix}:

$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{aix}e^{-inx}\,dx$

$$= \frac{1}{2\pi}\left[\frac{e^{(ai-in)x}}{a-in}\right]_{-\pi}^{\pi} = \frac{1}{2\pi(ai-in)}\left(e^{ai\pi}e^{-inx} - e^{-ai\pi}e^{in\pi}\right)$$

We know $e^{in\pi} = (-1)^n$

$\therefore$

$$C_n = \frac{1}{2\pi(ai-in)}\left(e^{ai\pi}(-1)^n - e^{-ai\pi}(-1)^n\right) = \frac{(-1)^n}{2\pi(ai-in)}\left(e^{ai\pi} - e^{-ai\pi}\right)$$

Multiply and divide by 2,

$C_n = \frac{(-1)^n}{\pi(ai-in)}\left(\frac{e^{ai\pi}-e^{-ai\pi}}{2}\right) = \frac{(-1)^n}{\pi(ai-in)}(\sinh ai\pi) = \frac{i(-1)^n}{\pi(ai-in)}\sin a\pi$

$C_n = \frac{i(-1)^n}{\pi(ai-in)}\sin a\pi \,.\, \frac{ai+in}{ai+in} = \frac{i(-1)^n(ai+in)}{\pi(-a^2+n^2)}\sin a\pi = \frac{(-1)^n(a+n)}{\pi(a^2-n^2)}\sin a\pi$

$\therefore$

$$f(x) = \frac{\sin a\pi}{\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n(a+n)}{(a^2-n^2)}e^{inx}$$

Similarly for e^{-aix}, we get

$$f(x) = \frac{\sin a\pi}{\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n(a-n)}{(a^2-n^2)}e^{inx}$$

$\therefore$

$$\cos ax = (e^{aix} + e^{-aix}) / 2$$

$$\cos ax = \frac{\sin a\pi}{2\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n(a+n)}{(a^2-n^2)}e^{inx} + \frac{\sin a\pi}{2\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n(a-n)}{(a^2-n^2)}e^{inx}$$

$$\therefore \boldsymbol{\cos ax = \frac{a\sin a\pi}{\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n}{(a^2-n^2)}e^{inx}}$$

Q4. b) Find the Spearman's Rank correlation coefficient between X and Y. [6]

X	**68**	**64**	**75**	**50**	**64**	**80**	**75**	**40**	**55**	**64**
Y	**62**	**58**	**68**	**45**	**81**	**60**	**68**	**48**	**50**	**70**

Soln.:

Sr No.	X	R1	Y	R2	$D = (R1 - R2)^2$
1	68	7	62	6	1
2	64	5	58	4	1
3	75	8.5	68	7.5	1
4	50	2	45	1	1
5	64	5	81	10	25
6	80	8.5	60	5	12.25
7	75	10	68	7.5	6.25
8	40	1	48	2	1
9	55	3	50	3	0
10	64	5	70	9	16
N=10					∑=64.5

m1=3, m2=2 and m3=3

$$R = 1 - \frac{6\left[\Sigma D^2 + \frac{1}{12}(m1^3 - m1) + \frac{1}{12}(m2^3 - m2) + \frac{1}{12}(m3^3 - m3)\right]}{N^3 - N}$$

$$R = 1 - \frac{6\left[64.5 + \frac{1}{12}(24) + \frac{1}{12}(6) + \frac{1}{12}(6)\right]}{990}$$

$$\mathbf{Ans : R = 0.9327}$$

Q4.c) Find the inverse Laplace transform of

i) $\frac{s-1}{s^2+2s+2}$ **ii)** $\frac{e^{-\pi s}}{s^2(s^2+1)}$ **[8]**

Soln.:

i) $L^{-1}\left[\frac{s-1}{s^2+2s+2}\right] = L^{-1}\left[\frac{(s+1)-1}{(s+1)^2+1}\right]$

$= L^{-1}\left[\frac{(s+1)}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right]$

$= e^{-t}L^{-1}\left[\frac{(s)}{(s)^2+1} - \frac{1}{(s)^2+1}\right]$

$== e^{-t}[\cos t - \sin t]$

ii) $L^{-1}\left[\frac{e^{-\pi s}}{s^2(s^2+1)}\right]$

Here $\phi(s) = \frac{1}{s^2(s^2+1)}$ and $a = \pi$

$\therefore L^{-1}[\phi(s)] = L^{-1}[\frac{1}{s^2(s^2+1)}]$

Applying convolution theorem,

Let $\Phi_1(s) = \frac{1}{s^2}$; $\Phi_2(s) = \frac{1}{s^2+1}$

$\therefore$ $L^{-1}[\Phi_1(s)] = t$; $L^{-1}[\Phi_2(s)] = \sin t$

$\therefore L^{-1}[\phi(s)] = \int_0^t \sin t \,.(t-u)du$

$=\sin t \left[tu - \frac{u^2}{2}\right]_0^t = \sin t(t^2 - \frac{t^2}{2})$

$\therefore$ $L^{-1}\left[\frac{e^{-\pi s}}{s^2(s^2+1)}\right] = \mathbf{f(t-a)H(t-a)}$

$\therefore$

$$\text{Ans: } L^{-1}\left[\frac{e^{-\pi s}}{s^2(s^2+1)}\right] = \sin(t-\pi)\left[(t-\pi)^2 - \frac{(t-\pi)^2}{2}\right]H(t-\pi)$$

Q5. a) Find the $Z\{f(k)\} = 4^k, k < 0$ [6]

$= 3^k, k \geq 0$

Soln.: By definition $Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k)z^{-k}$

$\therefore Z\{f(k)\} = \sum_{k=-\infty}^{-1} 4^k z^{-k} + \sum_{k=0}^{\infty} 3^k z^{-k}$

Putting k = -n in the first series, we get

$$Z\{f(k)\} = \sum_{k=-\infty}^{-1} 4^{-n}z^{n} + \sum_{k=0}^{\infty} 3^k z^{-k}$$

$$Z\{f(k)\} = \left[\frac{z}{4} + \frac{z^2}{4^2} + \cdots\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \cdots\right]$$

$$Z\{f(k)\} = \frac{z}{4}\left[1 + \frac{z}{4} + \frac{z^2}{4^2} + \cdots\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \cdots\right]$$

$$Z\{f(k)\} = \left[\frac{4}{4-z}\right] + \left[\frac{z}{z-3}\right]$$

ROC is $3 < |z| < 4$

Q5.b) Show that {cos x, cos 2x, cos 3x, … } is orthogonal set over the interval [0,2π]. Construct the corresponding orthonormal set. [6]

Soln.: We have $f_n(x) = \cos nx; n = 1,2,3$

Therefore, $\int_{-\pi}^{\pi} f_m(x)f_n(x)dx \Rightarrow \int_{-\pi}^{\pi} \cos m x . \cos nxdx$

$\Rightarrow \frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)x + \cos(m-n)xdx \Rightarrow \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)}{m-n}\right]_{-\pi}^{\pi}$

Now two cases arises:

i) When m≠n: $\frac{1}{2}\left[\left\{\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)}{m-n}\right\} - \left\{-\frac{\sin(m+n)\pi}{m+n} - \frac{\sin(m-n)\pi}{m-n}\right\}\right] = 0$

ii) When m=n: $\int_{-\pi}^{\pi} \cos^2 nx\, dx = \int_{-\pi}^{\pi} \frac{1+\cos 2nx}{2} dx$

$\frac{1}{2}\left[x + \frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi} = \frac{1}{2}[\pi + 0 + \pi - 0] = \pi \neq 0$

Therefore the functions are orthogonal in [-π,π]

$$\int_{-\pi}^{\pi}[f(x)]^2 dx = \pi$$

Dividing the equation by π,

$\Rightarrow \int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}} f(x).\frac{1}{\sqrt{\pi}} f(x) dx = 1$

This is obviously an orthonormal set where $\phi(x) = \frac{1}{\sqrt{\pi}} \cos nx$

Thus the required orthonormal set is $\frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\cos 2x, \frac{1}{\sqrt{\pi}}\cos 3x, \ldots$

Q5. c) Find the bilinear transformation which maps the points z= 1, i, -1 into the points w = i, 0, -i. Hence find the image of |z|<1 [8]

Soln.: Let the transformation be $w = \frac{az+b}{cz+d}$ -----(1)

Putting the given values of z and w, we get,

$i = \frac{a+b}{c+d}$; $0 = \frac{ai+b}{ci+d}$; $-1 = \frac{-a+b}{-c+d}$

From these equalities, we get,

(a+b) - i(c+d) = 0 -----(2) b+ia=0 -----(3)

(-a-b) + i(-c+d) = 0 -----(4)

From 2 and 4 we get c=b/i

Subtracting 4 from 2, we get 2a – 2id =0. ∴d=-ia

Putting the values b=-ia, c=-a and d=-ia in (1) we get,

$w = \frac{az-ia}{-az-ia} = \frac{z-i}{-z-i}$

∴ **$w = \frac{i-z}{i+z}$ is the required bilinear transformation.**

∴ wi + wz = i - z

w – i = -z(1 + w)

Further, |z| < 1 is mapped onto the region

$$\left|\frac{i(1-w)}{1+w}\right| < 1$$

∴ |1-w| < |1+w| [|i|=1]

|(1-u)-iv| < |(1+u)+iv|

$\therefore (1-u)^2 + v^2 < (1+u)^2 + v^2$

-4u<0 $\Rightarrow$ **u>0**

Q6. a) Fit a straight line to the given data **[6]**

X	**10**	**12**	**15**	**23**	**20**
Y	**14**	**17**	**23**	**25**	**21**

Soln.:

x	y	x^2	xy
10	14	100	140
12	17	144	204
15	23	225	345
23	25	529	575
20	31	400	620
$\sum$x=80	$\sum$y=110	$\sum x^2 = 1398$	$\sum$xy=1884

Let the equation be y=a+bx

The normal equations are

$\Sigma y = Na + b\Sigma x$ $\therefore$110=5a+80b

And $\Sigma xy = a\Sigma x + b\Sigma x^2$ $\therefore$1884=80a+1398b

Solving the equations simultaneously,

a=306/59 and b=62/59

Q6.b) Find the Inverse Laplace Transform using convolution theorem **[6]**

$$\mathbf{\frac{1}{(s-2)^3(s+3)}}$$

Soln.: Let $\Phi_1(s) = \frac{1}{s+3}$; $\Phi_2(s) = \frac{1}{(s-2)^3}$

$\therefore\ L^{-1}[\Phi_1(s)] = e^{-3t}$; $L^{-1}[\Phi_2(s)] = e^{2t}L^{-1}\left[\frac{1}{s^4}\right] = e^{2t}.\frac{t^2}{2}$

$$\therefore L^{-1}[\phi(s)] = \int_0^t e^{-3u}.e^{2(t-u)}.\frac{(t-u)^2}{2}du$$

$$= \int_0^t e^{(2t-5u)}.\frac{(t-u)^2}{2}du$$

$$= e^{2t}\left[\frac{(t-u)^2}{2}\left(\frac{-e^{-5u}}{5}\right) - (t-u)\left(\frac{e^{-5u}}{25}\right) + \left(\frac{e^{-5u}}{125}\right)\right]_0^t$$

$$= e^{2t}\left[\left\{0 - 0 - \left(\frac{e^{-5t}}{125}\right)\right\} - \left\{\frac{(t)^2}{2}\left(\frac{-1}{5}\right) - (t)\left(\frac{1}{25}\right) + \left(\frac{1}{125}\right)\right\}\right] = e^{2t}[-\left(\frac{e^{-5t}}{125}\right) + \frac{t^2}{10} + \frac{t}{25} + \frac{1}{125}]$$

$$\textbf{Ans}: e^{2t}\left[\left(\frac{t^2}{10} + \frac{t}{25} + \frac{1}{125}\right) - \frac{e^{-5t}}{125}\right]$$

Q6.c) Find Half Range Cosine Series for f(x)=sin x in (0,π) and hence deduce that [8]

$$\frac{\pi^2 - 8}{16} = \frac{1}{1^2.3^2} + \frac{1}{3^2.5^2} + \frac{1}{5^2.7^2} + \cdots$$

Soln.: Let $f(x) = a_0 + \sum a_n \cos nx$

$$a_0 = \frac{1}{\pi}\int_0^\pi f(x)dx = \frac{1}{\pi}\int_0^\pi \sin x\, dx = \frac{1}{\pi}[-\cos x]_0^\pi$$

$$\therefore a_0 = -\frac{1}{\pi}[-1-1] = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\left[\int_0^\pi f(x)\cos nx\, dx\right] = \frac{2}{\pi}\left[\int_0^\pi \sin x \cos nx\, dx\right]$$

$$a_n = \frac{2}{2\pi}\left[\int_0^\pi \sin(1+n)x + \sin(1-n)x\, \right]dx$$

$$a_n = \frac{1}{\pi}\left[\frac{-\cos(1+n)\pi}{(1+n)} - \frac{\cos(1-n)\pi}{1-n} - \left(-\frac{1}{1+n} - \frac{1}{1-n}\right)\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{\cos n\pi}{(1+n)} - \frac{\cos n\pi}{n-1} + \left(\frac{1}{1+n} + \frac{1}{n-1}\right)\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{(-1)^n}{(1+n)}\left(\frac{2}{n^2-1}\right) - \frac{2}{n^2-1}\right] = -\frac{2}{\pi(n^2-1)}[(-1)^n + 1]$$

= 0 if n is odd and n is not = 1

$\therefore a_n = -\frac{4}{\pi(n^2-1)}$ if n is even

$\therefore$ If n =1 , we get

$$a_1 = \frac{2}{\pi}\int_0^\pi \sin x \cos x\, dx = \frac{1}{\pi}\int_0^\pi \sin 2x dx = \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^\pi$$

$a_1 = \frac{1}{\pi}\left[-\frac{1}{2} + \frac{1}{2}\right] = 0$

$\therefore$

$$\textbf{Ans: } f(x) = \sin\ x = \frac{2}{\pi} - \sum \frac{4}{\pi(n^2-1)} \cos nx$$
