$\underline{z = x + iy}$

$\bar{z} = x - iy$

$(\bar{\bar{z}}) = x + iy = z$

$|z| = \sqrt{x^2 + y^2}$

$|z|^2 = x^2 + y^2$

$z\bar{z} = (x+iy)(x-iy)$

$= x^2 + y^2 = |z|^2$

$|z_1 + z_2| \le |z_1| + |z_2|$

$\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

$z + \bar{z} = x + iy + x - iy = 2x$

$= 2\,Re(z)$

$z - \bar{z} = 2iy = 2i\,Im(z)$

Inner Product Space:-

$\underline{V_n(F)} \longrightarrow$ Finite Dim. V.S. on Field F. — R, C

$V_n(C)$

$x = (x_1, x_2, \ldots, x_n)$

$y = (y_1, y_2, \ldots, y_n)$

Inner product

$\underline{(\,,\,)} : V_n \times V_n \longrightarrow C$ by

defined by $(x, y) = x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n$

= complex no.

$F = R$ $\quad (x, y) = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$ (Real no.)

$(y, x) = y_1\bar{x}_1 + y_2\bar{x}_2 + \cdots + y_n\bar{x}_n$

$x = (2, 3, 5)$, $y = (-1, 0, 3)$

$(x, y) = 2(-1) + 3(0) + 5(3) = -2 + 0 + 15 = 13$

$x = (2, 3i, 5)$, $y = (-1, 0, 3i)$

$(x, y) = 2(-1) + 3i(0) + 5(\overline{3i}) = -2 + 5(-3i)$

$= -2 - 15i$

$\underline{z = x + iy}$

$\bar{z} = x - iy$

$(\bar{\bar{z}}) = x + iy = z$

$|z| = \sqrt{x^2 + y^2}$

$|z|^2 = x^2 + y^2$

$z\bar{z} = (x+iy)(x-iy)$

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Inner Product Space:

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Inner product $(\ ,\ ): V_n \times V_n \rightarrow C$ by

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= complex no

$F = R$: $(x, y) = x_1y_1 + x_2y_2 + \dots + x_ny_n$ (Real no)

$(y, x) = y_1\bar{x}_1 + y_2\bar{x}_2 + \dots + y_n\bar{x}_n$

$x = (2, 3, 5)$, $y = (-1, 0, 3)$

$(x, y) = 2(-1) + 3(0) + 5(3) = -2 + 0 + 15 = 13$

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$z + \bar{z} = x + iy + x - iy = 2x$

$= 2 R_e(z)$

$z - \bar{z} = 2iy = 2i\, Im(z)$

Inner Product (Properties): Let $x, y, z \in V_n(C)$ and any $a \in C$ then:

i) $(x, x) \geq 0$ and $(x, x) = 0 \Leftrightarrow x = 0$.

ii) $(x, y) = \overline{(y, x)}$

iii) $(x, y + z) = (x, y) + (x, z)$

iv) $(ax, y) = a(x, y)$

v) $(x, ay) = \bar{a}(x, y)$

Let $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$

$z = (z_1, z_2, \dots, z_n)$

by defn

$(x, x) = x_1\bar{x}_1 + x_2\bar{x}_2 + \dots + x_n\bar{x}_n$

$= |x_1|^2 + |x_2|^2 + \dots + |x_n|^2$

since each $|x_1|^2 \geq 0, |x_2|^2 \geq 0 \dots |x_n|^2 \geq 0$

$\Rightarrow (x, x) \geq 0$

IF $(x, x) = 0 \Leftrightarrow |x_1|^2 + |x_2|^2 + \dots + |x_n|^2 = 0$

$\Leftrightarrow |x_1|^2 = 0, |x_2|^2, \dots |x_n|^2 = 0$

$\Leftrightarrow x_1 = 0, x_2 = 0, \dots x_n = 0$

$\Leftrightarrow x = 0$ hence $(x, x) = 0 \Leftrightarrow x = 0$

(ii) $(y, x) = y_1\bar{x}_1 + y_2\bar{x}_2 + \dots + y_n\bar{x}_n$

$\Rightarrow \overline{(y, x)} = \overline{y_1\bar{x}_1 + y_2\bar{x}_2 + \dots + y_n\bar{x}_n}$

$= (\overline{y_1\bar{x}_1}) + (\overline{y_2\bar{x}_2}) + \dots + \overline{y_n\bar{x}_n}$

$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$= \bar{y}_1\bar{\bar{x}}_1 + \bar{y}_2\bar{\bar{x}}_2 + \dots + \bar{y}_n\bar{\bar{x}}_n$

$\overline{z_1z_2} = \bar{z}_1\bar{z}_2$

$= \bar{y}_1x_1 + \bar{y}_2x_2 + \dots + \bar{y}_nx_n$

$\bar{\bar{z}} = z$

$\overline{(y, x)} = x_1\bar{y}_1 + x_2\bar{y}_2 + \dots + x_n\bar{y}_n$

$= (x, y)$

Inner Product (Properties): Let $x, y, z \in V_n(C)$ and any $a \in C$ then:

i) $(x, x) \geq 0$ and $(x, x) = 0 \Leftrightarrow x = 0$

ii) $(x, y) = \overline{(y, x)}$

iii) $(x, y+z) = (x, y) + (x, z)$

iv) $(ax, y) = a(x, y)$ ✓

v) $(x, ay) = \bar{a}(x, y)$ ✓

Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$

$z = (z_1, z_2, \ldots, z_n)$

$y+z = (y_1+z_1, y_2+z_2, \ldots, y_n+z_n)$

Now $(x, y+z) = x_1\overline{(y_1+z_1)} + x_2\overline{(y_2+z_2)} + \cdots + x_n\overline{(y_n+z_n)}$

$= x_1(\bar{y}_1+\bar{z}_1) + x_2(\bar{y}_2+\bar{z}_2) + \cdots + x_n(\bar{y}_n+\bar{z}_n)$

$= (x_1\bar{y}_1 + x_1\bar{z}_1) + (x_2\bar{y}_2 + x_2\bar{z}_2) + \cdots + x_n\bar{y}_n + x_n\bar{z}_n$

$= (x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n) + (x_1\bar{z}_1 + x_2\bar{z}_2 + \cdots + x_n\bar{z}_n)$

$= (x, y) + (x, z)$

iv) $ax = (ax_1, ax_2, \ldots, ax_n)$

$\Rightarrow (ax, y) = ax_1\bar{y}_1 + ax_2\bar{y}_2 + \cdots + ax_n\bar{y}_n$

$= a(x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n)$

$= a(x, y)$

v) $ay = (ay_1, ay_2, \ldots, ay_n)$

$(x, ay) = x_1\overline{ay_1} + x_2\overline{ay_2} + \cdots + x_n\overline{ay_n}$

$= x_1\bar{a}\bar{y}_1 + x_2\bar{a}\bar{y}_2 + \cdots + x_n\bar{a}\bar{y}_n$

$= \bar{a}(x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n)$

$= \bar{a}(x, y)$

Inner Product Space:- Let $V(F)$ be a vector space on field F (F is either R or C) and $(,) : V \times V \to F$ be the inner product defined on V. Then $V(F)$ is called inner product space if inner product on V satisfy following condition/axioms:

1) Non-Negativity: $\forall \alpha \in V \Rightarrow (\alpha, \alpha) \geq 0$ and $(\alpha, \alpha) = 0 \Leftrightarrow \alpha = 0$

2) Conjugate Symmetry: $\forall \alpha, \beta \in V \Rightarrow (\alpha, \beta) = \overline{(\beta, \alpha)}$

3) Linearity: $\forall \alpha, \beta, \gamma \in V$ and $a, b \in F \Rightarrow (a\alpha + b\beta, \gamma) = a(\alpha, \gamma) + b(\beta, \gamma)$

\# (α, β) $F = R$ $(\alpha, \beta) = (\alpha, \beta)$ $\Downarrow$ $F = R$

\# $V(F) \to F = R$ $\Downarrow$ IPS $\searrow$ Euclidean space

$F = C$ unitary space

Q. Show that $V_n(C)$ is an inner product space with inner product define on $\alpha = (a_1, a_2, \ldots, a_n)$, $\beta = (b_1, b_2, \ldots, b_n) \in V_n(C)$ by $(\alpha, \beta) = a_1\bar{b}_1 + a_2\bar{b}_2 + \cdots + a_n\bar{b}_n$ (1) } — standard inner prod in $V_n(F)$

Sol: Let $\alpha = (a_1, a_2, \ldots, a_n)$, $\beta = (b_1, b_2, \ldots, b_n)$ & $\gamma = (c_1, c_2, \ldots, c_n) \in V_n(F)$ and $a, b \in C$

(i) Non-Negativity: $(\alpha, \alpha) = |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2 \geq 0$ since $|a_i|^2 \geq 0$

$(x, y) = \overline{(y, x)}$ $\quad (\alpha, \alpha) = 0 \Leftrightarrow |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2 = 0$

$\Leftrightarrow$ each $a_i = 0 \Rightarrow \boxed{\alpha = 0}$

(ii) Conjugate Symmetry: $(\alpha, \beta) = \overline{(\beta, \alpha)}$

(iii) Linearity: $a\alpha + b\beta = a(a_1, a_2, \ldots, a_n) + b(b_1, b_2, \ldots, b_n)$

$= (aa_1 + bb_1, aa_2 + bb_2, \ldots, aa_n + bb_n)$

Now $(a\alpha + b\beta, \gamma) = (aa_1 + bb_1)\bar{c}_1 + (aa_2 + bb_2)\bar{c}_2 + \cdots + (aa_n + bb_n)\bar{c}_n$

$= a(a_1\bar{c}_1 + a_2\bar{c}_2 + \cdots + a_n\bar{c}_n) + b(b_1\bar{c}_1 + b_2\bar{c}_2 + \cdots + b_n\bar{c}_n)$

$= a(\alpha, \gamma) + b(\beta, \gamma)$

Here inner product define by (1) satisfy all 3 condition so $V_n(C)$ is an inner product space