FOURIER SERIES

1) Even Function

If $f(-x) = f(x)$ then $f(x)$ is said to be even function.

eg. $\cos x$, x^2, $|x|$, $|\sin x|$

2) Odd Function

If $f(-x) = -f(x)$ then $f(x)$ is said to be odd function.

eg. $\sin x$, x, x^3

Note: $f(x) = e^x$ is neither even nor odd function.
(as $f(-x) = e^{-x} \neq f(x)$ & $\neq -f(x)$)

3) $$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \quad \text{— if } f(x) \text{ is even}$$
$$= 0 \quad \text{— if } f(x) \text{ is odd}$$

4) $f(x) = f_1(x) \cdot f_2(x)$

$f_1(x)$	E	O	E	O
$f_2(x)$	E	O	O	E
$f(x) = f_1(x) \cdot f_2(x)$	E	E	O	O

5) If n is integer

$\sin n\pi = 0 \quad \therefore \sin 2n\pi = 0.$

$\cos n\pi = (-1)^n \quad \therefore \cos 2n\pi = 1$

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- Dirichlet's Conditions :

If $f(x)$ is defined in the interval $c_1 \le x \le c_2$ can be expressed as fourier series if in the interval

1) $f(x)$ and its integrals are finite and single valued.
2) $f(x)$ has finite number of discontinuties.
3) $f(x)$ has finite number of maxima and minima.

These conditions are known as Dirichlet's Conditions.

- Determination of Fourier Co-efficients (Euler's Formulae)

1) If $f(x)$ is defined in interval $(c, c+2l)$, then Fourier Series of $f(x)$ is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right) dx$$

- Parseval's Identity

If $f(x)$ is defined in interval $(c, c+2l)$ then by Parseval's Identity

$$\frac{1}{l}\int_c^{c+2l} [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

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2) If $f(x)$ is defined in interval $(-l, l)$, then we can check $f(x)$ is even or odd.

Case 1:-

If $f(x)$ is even $\Rightarrow b_n = 0$,
then Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

Hence,
Parseval's Identity is

$$\frac{2}{l}\int_0^l [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

Case 2:-

If $f(x)$ is odd $\Rightarrow a_0 = 0, a_n = 0$
then Fourier Series of $f(x)$ is given by.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Hence,
Parseval's Identity is

$$\frac{2}{l}\int_0^l [f(x)]^2 dx = \sum_{n=1}^{\infty} b_n^2$$

Note:-
If $f(x)$ is neither even nor odd then we have to use formulae written in 1)

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3) Half Range Series

Half Range series of $f(x)$ is defined on interval $(0, l)$

1) Half Range Cosine Series

Here, $b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$\therefore$ Hence, Parseval's Identity

$$\frac{2}{l}\int_0^l [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

2) Half Range Sine Series.

Here, $a_0 = 0$ & $a_n = 0$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx.$$

Hence, Parseval's Identity

$$\frac{2}{l}\int_0^l [f(x)]^2\,dx = \sum_{n=1}^{\infty} b_n^2$$

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Problems :-

1) Find Fourier series of $f(x)=x^2$ in $(0,2\pi)$.
Hence find $\frac{\pi^2}{12}=\frac{1}{1^2}-\frac{1}{2^2}+\cdots$

Sol^n Comparing with $(c, c+2l)$

$\therefore (c, c+2l) = (0, 2\pi)$

$\Rightarrow c=0$ & $c+2l=2\pi \Rightarrow 0+2l=2\pi \Rightarrow \boxed{l=\pi}$

As,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{l}\right)+\sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{l}\right)$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx+\sum_{n=1}^{\infty} b_n\sin nx \quad \text{—— 1)}$$

$$a_0=\frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$=\frac{1}{\pi}\int_0^{2\pi} x^2\,dx$$

$$=\frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left(\frac{8\pi^3}{3}\right)=\frac{8\pi^2}{3}$$

$$a_n=\frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$=\frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx$$

$$=\frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)-(2x)\left(\frac{-\cos nx}{n^2}\right)+2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[0-4\pi\left(\frac{-\cos 2n\pi}{n^2}\right)+0-0+0-0\right]$$

—— $\because \sin 2n\pi=0$

$$= \frac{1}{\pi}\left[\frac{4\pi}{n^2}\right]$$

$$a_n = \frac{4}{n^2}$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx \, dx$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) - (2x)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{+\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[4\pi^2\left(\frac{-\cos 2n\pi}{n}\right) - 0 + 2\frac{\cos 2n\pi}{n^3} - 0 + 0 - 2\left(\frac{1}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[-4\pi^2\left(\frac{1}{n}\right) + 2\left(\cancel{\frac{1}{n^3}}\right) - 2\left(\cancel{\frac{1}{n^3}}\right)\right]$$

$$= \frac{1}{\cancel{\pi}} \times \frac{-4\pi^{\cancel{2}}}{n}$$

$$= \frac{-4\pi}{n}$$

$\therefore$ from (1)

$$f(x) = \frac{\overset{4}{\cancel{8}}\pi^2}{3\times\cancel{2}} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx + \sum_{n=1}^{\infty}\left(-\frac{4\pi}{n}\right)\sin nx$$

$$x^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} + (-4\pi)\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

$$\therefore x^2 = \frac{4\pi^2}{3} + 4\left(\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \cdots\right) - 4\pi\left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \cdots\right).$$

put $x = \pi$, we get

$$\pi^2 = \frac{4\pi^2}{3} + 4\left(\frac{\cos\pi}{1^2} + \frac{\cos 2\pi}{2^2} + \cdots\right) - 4\pi(0) \quad \text{—— } \because \sin n\pi = 0$$

$$\pi^2 - \frac{4\pi^2}{3} = 4\left(\frac{-1}{1^2} + \frac{1}{2^2} + \frac{-1}{3^2} \cdots\right)$$

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$$\therefore -\frac{\pi^2}{3} = -4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right)$$

$$\Rightarrow \frac{\pi^2}{3\times4} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

$$\Rightarrow \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

2) Find Fourier Series of $f(x) = \left(\frac{\pi - x}{2}\right)^2 \quad 0 \le x \le 2\pi$

Hence prove that,

i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

2) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

3) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

4) $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$

Soln $\quad (c, c+2l) = (0, 2\pi)$

$\therefore c = 0 \;\&\; c+2l = 2\pi \Rightarrow 2l = 2\pi \Rightarrow \boxed{l = \pi}$

As

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 dx$$

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$$= \frac{1}{\pi \times 4} \int_0^{2\pi} (\pi - x)^2 \, dx$$

$$= \frac{1}{4\pi} \left[\frac{(\pi - x)^3}{3(-1)} \right]_0^{2\pi}$$

$$= \frac{1}{4\pi} \left[\frac{(-\pi)^3}{-3} - \frac{\pi^3}{-3} \right]$$

$$= \frac{1}{4\pi} \left[\frac{+\pi^3}{+3} + \frac{\pi^3}{3} \right]$$

$$= \frac{1}{4\pi} \left[\frac{2\pi^3}{3} \right]$$

$$= \frac{\pi^2}{6}$$

$$a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 \cos nx \, dx$$

$$= \frac{1}{4\pi} \int_0^{2\pi} (\pi - x)^2 \cos nx \, dx$$

$$= \frac{1}{4\pi} \left[(\pi - x)^2 \left(\frac{\sin nx}{n}\right) - [2(\pi - x)(-1)] \left(-\frac{\cos nx}{n^2}\right) + [2(-1)(-1)] \left(-\frac{\sin nx}{n^3}\right) \right]_0^{2\pi}$$

$$= \frac{1}{4\pi} \left[(-\pi)^2 (0) - [2(-\pi)(-1)] \left(-\frac{\cos 2n\pi}{n^2}\right) + 0 - 0 + 2(\pi)(-1) \left(\frac{-\cos(0)}{n^2}\right) - 0 \right]$$

$$= \frac{1}{4\pi} \left[2\pi \left(\frac{1}{n^2}\right) + 2\pi \left(\frac{1}{n^2}\right) \right] = \frac{1}{4\pi} \frac{4\pi}{n^2}$$

$$a_n = \frac{1}{n^2}$$

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$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{2}\right)^2 \sin nx\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2 \sin nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left(\frac{-\cos nx}{n}\right) - [2(\pi-x)(-1)]\left(\frac{-\sin nx}{n^2}\right) + [2(-1)(-1)]\left(\frac{+\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(-\pi)^2\left(\frac{-\cos 2n\pi}{n}\right) - 0 + 2\left(\frac{\cos 2n\pi}{n^3}\right) - \pi^2\left(\frac{-1}{n}\right) + 0 - 2\left(\frac{1}{n^3}\right)\right]$$

$$= \frac{1}{4\pi}\left[\pi^2\left(\frac{-1}{n}\right) + \frac{2}{n^3} + \pi^2\left(\frac{1}{n}\right) - \frac{2}{n^3}\right]$$

$$= \frac{1}{4\pi}\left[-\frac{\pi^2}{n} + \frac{\pi^2}{n}\right]$$

$$b_n = 0$$

$\therefore$ from (1)

$$f(x) = \frac{\pi^2}{6\times 2} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx + 0$$

$$= \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$$

$$\left(\frac{\pi-x}{2}\right)^2 = \frac{\pi^2}{12} + \left(\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots\right) \quad \text{--- (2)}$$

put $x=0$ in (2)

$$\left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{12} + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

$$\frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

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$$\frac{3\pi^2-\pi^2}{12} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+ \cdots$$

$$\frac{\cancel{2}\pi^2}{\cancel{12}\,6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+ \cdots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+ \cdots \quad \text{—— (3)}$$

put $x=\pi$ in (2) we get,

$$0 = \frac{\pi^2}{12}+\frac{\cos\pi}{1^2}+\frac{\cos 2\pi}{2^2}+\frac{\cos 3\pi}{3^2}+ \cdots$$

$$-\frac{\pi^2}{12} = \frac{(-1)}{1^2}+\frac{1}{2^2}+\frac{(-1)}{3^2}+ \cdots$$

$$-\frac{\pi^2}{12} = -\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2} \cdots\right)$$

$$\Rightarrow \frac{\pi^2}{12} = \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2} \cdots \quad \text{—— (4)}$$

(3) + (4) gives us,

$$\frac{\pi^2}{6}+\frac{\pi^2}{12} = \left(\frac{1}{1^2}+\cancel{\frac{1}{2^2}}+\frac{1}{3^2}+\cdots\right)+\left(\frac{1}{1^2}\cancel{-\frac{1}{2^2}}+\frac{1}{3^2} \cdots\right)$$

$$\frac{2\pi^2+\pi^2}{12} = \frac{2}{1^2}+\frac{2}{3^2}+ \cdots$$

$$\frac{\cancel{3}\pi^2}{\cancel{12}\,4} = 2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+ \cdots\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+ \cdots \quad \text{—— (5)}$$

To prove last series we will use Parseval's Identity which is as follows

$$\frac{1}{l}\int_c^{c+2l}[f(x)]^2\,dx = \frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2).$$

$$\therefore \frac{1}{\pi}\int_0^{2\pi}\left[\left(\frac{\pi-x}{2}\right)^2\right]^2 dx = \frac{\left(\frac{\pi^2}{6}\right)^2}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{1}{n^2}\right)^2 + 0\right]$$

$$\Rightarrow \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{2}\right)^4 dx = \frac{\pi^4}{36\times 2} + \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow \frac{1}{2^4\pi}\int_0^{2\pi}(\pi-x)^4 dx = \frac{\pi^4}{72} + \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow \frac{1}{16\pi}\left[\frac{(\pi-x)^5}{5(-1)}\right]_0^{2\pi} = \frac{\pi^4}{72} + \sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow \frac{1}{16\pi}\left[\frac{(-\pi)^5}{-5} - \frac{\pi^5}{-5}\right] = \frac{\pi^4}{72} + \left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+ \cdots\right)$$

$$\Rightarrow \frac{1}{16\pi}\left[\frac{+\pi^5}{+5} + \frac{\pi^5}{5}\right] = \frac{\pi^4}{72} + \left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+ \cdots\right)$$

$$\Rightarrow \frac{1}{\cancel{16}_8\pi}\left(\frac{\cancel{2}\pi^5}{5}\right) - \frac{\pi^4}{72} = \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+ \cdots$$

$$\Rightarrow \frac{\pi^4}{40} - \frac{\pi^4}{72} = \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+ \cdots$$

$$\Rightarrow \frac{\cancel{32}^{8}\pi^4}{\cancel{40}_{10}\times \cancel{72}_9} = \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+ \cdots$$

$$\Rightarrow \frac{\pi^4}{90} = \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+ \cdots$$

8) Find Fourier Series of $f(x) = \sin ax$ in $(-\pi, \pi)$, a is not an integer.

Solⁿ Given interval is $(-\pi, \pi)$

($\therefore$ we can check $f(x)$ is even or odd.)

$f(-x) = \sin(-ax) = -\sin ax = -f(x)$

$\Rightarrow f(x)$ is odd function

$\Rightarrow a_0 = 0,\ a_n = 0$

$l = \pi$

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$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{——— } (1)$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sin ax \sin nx \, dx$$

$$= \frac{2}{\pi} \times \frac{1}{2}\int_0^{\pi} [\cos(ax-nx) - \cos(ax+nx)] dx$$

$$= \frac{1}{\pi}\int_0^{\pi} [\cos(a-n)x - \cos(a+n)x] dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(a-n)x}{(a-n)} - \frac{\sin(a+n)x}{(a+n)}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin(a-n)\pi}{(a-n)} - \frac{\sin(a+n)\pi}{a+n} - 0 + 0\right] \quad n \neq a$$

$$= \frac{1}{\pi}\left[\frac{\sin(a\pi - n\pi)}{(a-n)} - \frac{\sin(a\pi + n\pi)}{(a+n)}\right]$$

$$\sin(a\pi + n\pi) = \sin a\pi \cos n\pi \pm \cos a\pi \sin n\pi$$

$$= \sin a\pi (-1)^n \pm 0 = (-1)^n \sin a\pi$$

$$\therefore b_n = \frac{1}{\pi}\left[\frac{(-1)^n \sin a\pi}{a-n} - \frac{(-1)^n \sin a\pi}{a+n}\right]$$

$$= \frac{(-1)^n \sin a\pi}{\pi}\left[\frac{1}{a-n} - \frac{1}{a+n}\right]$$

$$= \frac{(-1)^n \sin a\pi}{\pi}\left[\frac{a+n-(a-n)}{a^2-n^2}\right]$$

$$= \frac{(-1)^n \sin a\pi}{\pi} \frac{2n}{(a^2-n^2)} \quad \text{——— } n \neq a$$

$$\therefore \text{from } (1) \quad f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^n (\sin a\pi) n}{(a^2-n^2)} \sin nx$$

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