Examples

1) a) Find the expectation of the number on a die when thrown.

b) Two unbiased dice are thrown. Find the expected values of the sums of numbers of points on them.

$\rightarrow$ a) Let X be the random variable representing the number on a die when thrown.

x	1	2	3	4	5	6
$P(X=x)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

$$\therefore E(X) = \sum_x x \cdot P_x$$

$$= 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6}$$

$$= \frac{1}{6}(1+2+\cdots+6) \quad = \frac{1}{6}\cdot\frac{6\times 7}{2}$$

$$\therefore \boxed{E(X) = \frac{7}{2}}$$

b) The probability function of X (the sum of numbers obtained on two dice) is

x	2	3	4	5	6	7	.....	11	12
$P(X=x)$	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$	....	$2/36$	$1/36$

$$\therefore E(X) = \sum_x x \cdot P_x$$

$$= 2\times\frac{1}{36} + 3\times\frac{2}{36} + 4\times\frac{3}{36} + 5\times\frac{4}{36} + 6\times\frac{5}{36} + 7\times\frac{6}{36} +$$

$$8\times\frac{5}{36} + 9\times\frac{4}{36} + 10\times\frac{3}{36} + 11\times\frac{2}{36} + 12\times\frac{1}{36}$$

$$= \frac{1}{36}\times 252$$

$$\therefore \boxed{E(X) = 7}$$

Prof. Nancy Sinollin

2) Let the r.v. X have the distribution :

$P(X=0) = P(X=2) = p$; $P(X=1) = 1-2p$, $0 \le p \le 1/2$

For what p is the $Var(x)$ a maximum?

→

X	0	1	2
$P(X=x)$	p	$1-2p$	p

$\therefore \quad E(x) = 0 \times p + 1 * (1-2p) + 2 \times p = 1$

& $E(x^2) = 0 \times p + 1^2 \times (1-2p) + 2^2 \times p = 1+2p$

$\therefore \quad Var(x) = E(x^2) - (E(x))^2 = 2p \quad , \quad 0 \le p \le 1/2$

$\therefore$ For $0 \le p \le \frac{1}{2}$, $Var(x)$ is maximum when $p=\frac{1}{2}$

$\therefore \quad [Var(x)]_{max} = 2 \times \frac{1}{2} = \underline{1}$

3) Two cards are drawn at random with replacement from a box which contains 4 cards numbered 1, 1, 2 & 2. If X denotes the sum of the numbers shown on the two cards, find the mean & variance of X.

→ X can takes values 2, 3, 4

$\therefore P(X=2) = P(\text{two 1's are drawn})$

$= \frac{^2C_2}{^4C_2} = \frac{1}{6}$

$P(X=3) = P(\text{one 1 \& one 2 are drawn})$

$= \frac{^2C_1 \cdot {^2C_1}}{^4C_2} = \frac{4}{6}$

$P(X=4) = P(\text{two 2's are drawn})$

$= \frac{^2C_2}{^4C_2} = \frac{1}{6}$

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The Probability distribution of X is

X	2	3	4
$P(X=x)$	$1/6$	$4/6$	$1/6$

$$\therefore E(X) = \sum_x X \cdot P_x = \frac{2}{6} + 3\times\frac{4}{6} + 4\times\frac{1}{6} = 3$$

$$E(X^2) = \sum_x X^2 P_x = 2^2\cdot\frac{1}{6} + 3^2\times\frac{4}{6} + 4^2\times\frac{1}{6} = \frac{28}{3}$$

$$\therefore Var(X) = E(X^2) - [E(X)]^2 = \frac{28}{3} - 9 = \frac{1}{3}$$

4) A box contains 2^n tickets of which ${}^nC_r$ tickets bear the number r ($r = 0, 1, \cdots n$). Two tickets are drawn from the box. Find the expectation of the sum of their numbers.

→ Total number of tickets in the box,

$$\sum_{r=0}^{n} {}^nC_r = {}^nC_0 + {}^nC_1 + \cdots + {}^nC_n = \overset{=(1+1)^n}{2^n} \text{ (given)}$$

Let X: number on the first ticket

Y: number on the second ticket

Then $E(X+Y) = E(X) + E(Y)$

X	0	1	2	$\cdots$	n
$P(X=x)$	$\frac{{}^nC_0}{2^n}$	$\frac{{}^nC_1}{2^n}$	$\frac{{}^nC_2}{2^n}$	$\cdots$	$\frac{{}^nC_n}{2^n}$

$$\therefore E(X) = 1\times\frac{{}^nC_1}{2^n} + 2\times\frac{{}^nC_2}{2^n} + \cdots + n\times\frac{{}^nC_n}{2^n}$$

$$= \frac{n}{2^n}\left\{{}^{(n-1)}C_0 + {}^{(n-1)}C_1 + \cdots + {}^{(n-1)}C_{n-1}\right\}$$

$$= \frac{n}{2^n}(1+1)^{n-1} = \frac{n}{2}$$

Similarly, $E(Y) = \frac{n}{2}$

$$\therefore E(X+Y) = n$$

Prof. Nancy Sinollin