Parshvanath Charitable Trust's
A. P. SHAH INSTITUTE OF TECHNOLOGY
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(Religious Jain Minority)

**Subject: Applied Mathematics III** **SEM: III**

# Use of Differentiation of $\phi(s)$ to I.L.T.

$$L^{-1}[\phi(s)] = \frac{-1}{t} L^{-1}[\phi'(s)]$$

$\because$ $L[t \cdot f(t)] = -\phi'(s)$

$\therefore$ $L^{-1}[\phi'(s)] = \frac{-1}{t} f(t) = \frac{-1}{t} \cdot L^{-1}[\phi(s)]$.

Note : 1] Use this property only for these functions where inverse Laplace transform of $\phi(s)$ is not easy to find but inverse Laplace transform of $\phi'(s)$ is easy to find.

2] Generally we will use above property for logarithmic and Inverse function (e.g. log, $\tan^{-1}$, $\cot^{-1}$, etc.).

Examples :

1] Find $L^{-1}\left[\log\left(\frac{s+a}{s+b}\right)\right]$

sol$^n$ : consider, $L^{-1}\left[\log\left(\frac{s+a}{s+b}\right)\right] = L^{-1}[\log(s+a) - \log(s+b)]$

$$= \frac{-1}{t} \cdot L^{-1}\left[\frac{d}{ds}(\log(s+a) - \log(s+b))\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$$

$$= \frac{-1}{t}\left[e^{-at} - e^{-bt}\right]$$

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2] $L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$

Soln: consider, $L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = L^{-1}\left[\log(s^2+a^2)-\log(s^2+b^2)\right]$

$$= -\frac{1}{t}L^{-1}\left[\frac{2s}{s^2+a^2}-\frac{2s}{s^2+b^2}\right]$$

$$= -\frac{2}{t}L^{-1}\left[\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right]$$

$$= -\frac{2}{t}\left[\cos at-\cos bt\right]$$

3] $L^{-1}\left[\log\left(\frac{s^2+a^2}{\sqrt{s+a}}\right)\right] = L^{-1}\left[\log(s^2+a^2)-\log(\sqrt{s+a})\right]$

$$= L^{-1}\left[\log(s^2+a^2)-\frac{1}{2}\log(s+a)\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left(\log(s^2+a^2)-\frac{1}{2}\log(s+a)\right)\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{2s}{s^2+a^2}-\frac{1}{2}\cdot\frac{1}{s+a}\right]$$

$$= -\frac{1}{t}\cdot\left[2\cos at-\frac{e^{-at}}{2}\right]$$

4] $\log\left(1+\frac{1}{s^2}\right)$ 5] $\frac{1}{2}\log\left(1-a^2/s^2\right)$

6] $\log\left(\frac{s^2-4}{(s-3)^2}\right)$ 7) $\log\left(\frac{s^2+1}{s(s+1)}\right)$

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8) $L^{-1}\left[\tan^{-1}(2/s^2)\right]$

Soln: $L^{-1}\left[\tan^{-1}(2/s^2)\right] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\left(\tan^{-1}(2/s^2)\right)\right]$

$$= \frac{-1}{t} L^{-1}\left[\frac{1}{1+4/s^4} \cdot -4/s^3\right]$$

$$= \frac{-1}{t} \cdot L^{-1}\left[\frac{-4/s^3}{\frac{s^4+4}{s^4}}\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{-4s}{s^4+4}\right]$$

$$= \frac{4}{t} \cdot L^{-1}\left[\frac{s}{(s^2)^2+2^2}\right]$$

$$= \frac{4}{t} \cdot L^{-1}\left[\frac{s}{(s^2)^2+2^2+2s^2(2)-2s^2(2)}\right]$$

$$= \frac{4}{t} \cdot L^{-1}\left[\frac{s}{(s^2+2)^2-4s^2}\right]$$

$$= \frac{4}{t} \cdot L^{-1}\left[\frac{s}{(s^2+2-2s)(s^2+2+2s)}\right]$$

$$= \frac{4}{t} L^{-1}\left[\frac{1}{4}\left[\frac{1}{s^2+2-2s} - \frac{1}{s^2+2+2s}\right]\right]$$

$$= \frac{4}{t \cdot 4} L^{-1}\left[\frac{1}{s^2-2s+1+1} - \frac{1}{s^2+2s+1+1}\right]$$

$$= \frac{1}{t} L^{-1}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right].$$

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$$= \frac{1}{t}\cdot\left[e^{t} L^{-1}\left(\frac{1}{s^2+1}\right) - e^{-t} L^{-1}\left(\frac{1}{s^2+1}\right)\right]$$

$$L^{-1}\left[\tan^{-1}(2/s^2)\right] = \frac{1}{t}\left[e^{t}\cdot \sin t - e^{-t}\cdot \sin t\right]$$

9] show that $L^{-1}\left[\frac{1}{s}\tan^{-1}(2/s)\right] = \int_0^t \frac{1}{u}\cdot \sin 2u\, du$

⇒ Sol$^n$:

let, $\phi(s) = \tan^{-1}(2/s)$ & $\psi(s) = 1/s$.

$$\therefore L^{-1}[\phi(s)] = L^{-1}\left[\tan^{-1}(2/s)\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{d}{ds}(\tan^{-1}(2/s))\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{1}{1+4/s^2}\cdot (-2/s^2)\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{s^2}{s^2+4}\cdot \frac{-1(2)}{s^2}\right]$$

$$= \frac{2}{t} L^{-1}\left[\frac{1}{s^2+2^2}\right]$$

$$= \frac{2}{t}\cdot\frac{1}{2}\cdot \sin 2t$$

$$\therefore f(t) = \frac{\sin 2t}{t}$$

& $L^{-1}[\psi(s)] = L^{-1}[1/s] = 1 = g(t)$

By convolution theorem,

$$L^{-1}[\phi(s)\cdot\psi(s)] = \int_0^t f(u)\cdot g(t-u)\, du.$$

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$$\therefore L^{-1}\left[\tan^{-1}(2/s)\cdot 1/s\right] = \int_0^t \frac{\sin 2u}{u}(1)\,du$$

$$= \int_0^t \frac{\sin 2u}{u}\,du$$

Hence proved

10] show that $L^{-1}\left[\frac{1}{s}\cdot\log\left(a^2+\frac{b^2}{s^2}\right)\right] = \int_0^t \frac{2}{u}\cdot\left(1-\cos\left(\frac{b}{a}\right)u\right)du$

⇒ Sol$^n$: let,

$\phi(s) = \log(a^2+b^2/s^2)$ & $\psi(s) = 1/s$.

$$\therefore L^{-1}[\phi(s)] = L^{-1}[\log(s^2a^2+b^2)-\log s^2)]$$

$$= L^{-1}[\log(a^2s^2+b^2)-\log s^2]$$

using $L^{-1}[\phi(s)] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\phi(s)\right]$

$$\therefore L^{-1}[\log(a^2+b^2/s^2)] = \frac{-1}{t}L^{-1}\left[\frac{2a^2s}{a^2s^2+b^2} - \frac{1}{s^2}2s\right]$$

$$= \frac{-2}{t}L^{-1}\left[\frac{\cancel{a^2}s}{\cancel{a^2}(s^2+b^2/a^2)} - \frac{1}{s}\right]$$

$$= \frac{-2}{t}\left[\cos\left(\frac{b}{a}\right)t - 1\right]$$

$$= \frac{2}{t}\left[1-\cos\left(\frac{b}{a}\right)t\right] = f(t)$$

& $L^{-1}[\psi(s)] = 1 = g(t)$.

By convolution theorem.

$$L^{-1}[\phi(s)\cdot\psi(s)] = \int_0^t f(u)\cdot g(t-u)\,du = \int_0^t \frac{2}{u}\cdot(1-\cos(b/a)u)du$$