Lectures 02 **(Bayes' Theorem)**

Conditional Probability:

The Conditional Probability of an event A assuming or given that another event M has occurred, is denoted by $P(A \mid M)$ (read as *probability of A given M*) and defined as:

$$P(A \mid M) = \frac{P(A \cap M)}{P(M)},\ P(M) > 0$$

Remark: The above definition gives,

$$P(A \cap M) = P(M)\ P(A \mid M) \quad or \quad P(A \cap M) = P(A)\ P(M \mid A)$$

Examples:

1. Suppose a fair die is tossed. Let events A, B and C be respectively defined as follows:
   A: The outcome is even; B: The outcome is a prime number and C: The outcome is greater than 2. Then
   $A = \{2,4,6\};\ B = \{2,3,5\};\ C = \{3,4,5,6\}$
   P(A) = 1/2 ; P(B)= 1/2
   Now, $A \cap B = \{2\};\ A \cap C = \{4,6\};\ B \cap C = \{3,5\}$
   $$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{3/6} = \frac{1}{3} \neq \text{P(A)}$$
   $$P(A \mid C) = \frac{P(A \cap C)}{P(C)} = \frac{2/6}{4/6} = \frac{1}{2} = \text{P(A)}$$
   $$P(B \mid C) = \frac{P(B \cap C)}{P(C)} = \frac{2/6}{4/6} = \frac{1}{2} = \text{P(B)}$$

Independence

Two events A and B are said to be independent if the occurrence of one does not affect the probability of occurrence of the other. That is, if A and B are independent, then we should have,

$P(A \mid B) = P(A)$ (and $P(B \mid A) = P(B)$)

[In the above example, events A and C are independent, events B and C are independent. Events A and B are **not** independent.]

From the definition of conditional probability, this means

$$\frac{P(A \cap B)}{P(B)} = P(A)$$

$$\Rightarrow P(A \cap B) = P(A)P(B)$$

Note: $P(A \cap B)$ is often written as $P(AB)$

Examples:

1. A missile can be accidentally launched if two relays A and B both have failed. The probabilities of A and B failing are known to be 0.01 and 0.03 respectively. It is also known that B is more likely to fail (probability = 0.06) if A has failed.
(a) What is the probability of an accidental missile launch?
(b) What is the probability that A will fail, if B has failed?
(c) Are the events ‘A fails’ and ‘B fails’ statistically independent?
Solution:
Let *A*: relay A fails and *B*: relay B fails
(This notation means that *A* denotes the event that relay A fails and *B* denotes the event that relay B fails).

$$
\begin{aligned}
\text{(a) } \mathrm{P}(\text{accidental missile launch}) &= P(A \cap B) \\
&= P(B \mid A)P(A) \\
&= .06 * .01 \\
&= .006
\end{aligned}
$$

$$
\begin{aligned}
\text{(b) } \mathrm{P}(\text{A will fail if B has failed}) &= P(A \mid B) \\
&= \frac{P(AB)}{P(B)} \\
&= \frac{.006}{.03} \\
&= .02
\end{aligned}
$$

$$
\begin{aligned}
&\text{(c) } P(AB) = .006 \ (from\,(a)) \\
&\quad Also,\ P(A)P(B) = 0.01 * 0.03 \\
&\qquad\qquad\qquad = 0.003 \\
&\qquad\qquad\qquad \neq P(AB) \\
&\therefore\ events\ A\ and\ B\ are\ not\ independent
\end{aligned}
$$

<u>Bayes’ Theorem</u>: Suppose S is the sample space of a random experiment and $A_1, A_2, \ldots, A_n$ is a ***partition*** of S. If B is any event, then

$$
P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + \ldots + P(B \mid A_n)P(A_n)}, \quad i = 1, 2, \ldots n
$$

Proof: Since $A_1, A_2, \ldots, A_n$ is a *partition* of *S*, we have

$$(i)\ A_i \cap A_j = \varphi,\ i \neq j$$

and

$$(ii)\ S = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\begin{aligned} Now,\ B &= B \cap S \\ &= B \cap (A_1 \cup A_2 \cup \ldots \cup A_n) \\ &= (B \cap A_1) \cup (B \cap A_2) \cup \ldots (B \cap A_n) \end{aligned}$$

$$\Rightarrow P(B) = P(B \cap A_1) + P(B \cap A_2) + \ldots + P(B \cap A_n)$$

$$(By\ Axiom\ 3, \because A_i \cap A_j = \varphi \Rightarrow (B \cap A_i) \cap (B \cap A_j) = \varphi)$$

$$i.e\ P(B) = P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + \ldots + P(B \mid A_n)P(A_n), \quad i = 1, 2, \ldots n \quad \ldots(i)$$

$$Now\ by\ definition,\ P(A_i \mid B) = \frac{P(A_i B)}{P(B)}$$

$$= \frac{P(A_i)P(B \mid A_i)}{P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + \ldots + P(B \mid A_n)P(A_n)}, \quad i = 1, 2, \ldots n$$

(using (i), *the total probability theorem*)

Thus Bayes' theorem is proved.

Examples:

1. There are 6 true coins and 1 false coin with 'head' on both sides. A coin is chosen at random and tossed 4 times. If 'head' occurs all the 4 times, what is the probability that the false coin has been chosen and used?

Solution: Let A_1 and A_2 respectively be the events that the coin chosen is a true one and the coin chosen is the false one.

Then $P(A_1) = \frac{6}{7}$ and $P(A_2) = \frac{1}{7}$

Let B be the event of getting 4 heads in 4 tosses of the coin.

Then $P(B \mid A_1) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$ since the coin is true (fair).

Also $P(B \mid A_2) = 1$ since the false coin has 'head' on both sides.

Now we have to find $P(A_2 \mid B)$.

By Bayes' theorem,

$$P(A_2 \mid B) = \frac{P(A_2)P(B \mid A_2)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)}$$

$$= \frac{\frac{1}{7} \times 1}{\frac{6}{7} \times \frac{1}{16} + \frac{1}{7} \times 1}$$

$$= \frac{\frac{1}{7}}{\frac{23}{112}} = \frac{16}{23}$$

Therefore the probability that the false coin has been chosen and used is $\frac{16}{23}$.

2. A manufacturing plant makes radios that each contain an integrated circuit (IC), supplied by three sources A, B and C. The probability that the IC in a radio came from one of the sources is 1/3, same for all sources. IC's are known to be defective with probabilities 0.001, 0.003 and 0.002 for sources A, B and C respectively.

(a) What is the probability that any given radio will contain a defective IC?

(b) If a radio contains a defective IC, find the probability that it came from source A. Repeat for sources B and C.

Solution: Let A: Source A contains a defective IC

B: Source B contains a defective IC

and C: Source C contains a defective IC

Let E: A radio contains a defective IC

We have

$$P(A)=P(B)=P(C)=\frac{1}{3}$$

Also $P(E\mid A)=0.001,\ P(E\mid B)=0.003,\ P(E\mid C)=0.002$

(a) $\text{P(a radio contains a defective IC)} = \text{P(E)}$

$$=P(E\mid A)P(A)+P(E\mid B)P(B)+P(E\mid C)P(C)$$

$$(By\, total\ prob\, thm)$$

$$=0.001*\frac{1}{3}+0.003*\frac{1}{3}+0.002*\frac{1}{3}$$

$$=0.002$$

(b) $\text{P(the defective came from source A| the radio contains a defective IC)}$

$$= \text{P(A|E)}$$

- $$=\frac{P(E\mid A)P(A)}{P(E)}$$

$$=\frac{(0.001)(1/3)}{0.002}$$

$$=\frac{1}{6}$$

Similarly, we can get

$\text{P(the defective came from source B| the radio contains a defective IC)}$

$$= \text{P(B|E)}$$

$$=\frac{1}{2}$$

And

$\text{P(the defective came from source C| the radio contains a defective IC)}$

$$= \text{P(C|E)}$$

$$=\frac{1}{3}$$

3. A mechanism consists of three paths A,B,C and probabilities of their failure are p, q, r respectively. The mechanism works if there is no failure in any of these parts. Find the probability that (i) the mechanism is working and (ii) the mechanism is not working.

Solution: Let A: Path A is working

B: Path B is working

and C: Path C is working

Let E: The mechanism is working

We have

$$P(A)=1-p; P(B)=1-q; P(C)=1-r$$

$$(i)\, P(E)=P(Mechanism\ is\ working)$$

$$=P(all\ the\ paths\ A, B\ and\ C\ are\ working)$$

$$=P(A\cap B\cap C)$$

$$=P(A)P(B)P(C) \quad (\because A, B\ and\ C\ are\ indep\ events)$$

$$=(1-p)(1-q)(1-r)$$

$$(ii)\ Now,\ P(Mechanism\ is\ not\ working)$$

$$=1-P(Mechanism\ is\ working)$$

$$=1-(1-p)(1-q)(1-r)$$

4. In a certain binary communication channel, the probability a transmitted zero is received as a zero is 0.90 and the probability a transmitted one is received as a one is 0.85. Assuming that the probability a zero is transmitted is 0.3, find the probability that (i) a one is received (ii) a one was transmitted given that a one was received

probability that an error is committed.

Solution: Let A_0: A zero is transmitted

A_1: A one is transmitted

B_0: A zero is received

B_1: A one is received

$$Given:\ P(A_0)=0.3; P(A_1)=0.7$$

$$P(B_0 | A_0)=0.90 \Rightarrow P(B_1 | A_0)=0.1;$$

$$P(B_1 | A_1)=0.85 \Rightarrow P(B_0 | A_1)=0.15$$

(i) Probability that a one is received $= \mathrm{P(B_1)}$

$$=P(\mathrm{B}_1 | A_0)P(A_0)+P(\mathrm{B}_1 | A_1)P(A_1)$$

$$(By\ total\ prob\ thm)$$

$$=0.1*0.3+0.85*0.7$$

$$=0.625$$

(i) Probability that a one is transmitted | one is received $= P(A_1 | B_1)$

$$= \frac{P(B_1 | A_1) P(A_1)}{P(B_1)}$$

$$= \frac{0.85 * 0.7}{0.625}$$

$$= 0.952$$

(iii) Probability an error is committed $= P(0 \textit{ is transmitted } \& \ 1 \textit{ is received or } 1 \textit{ is transmitted } \& \ 0 \textit{ is received})$

$$= P(A_0 \cap B_1) + P(A_1 \cap B_0)$$

$$= P(B_1 | A_0) P(A_0) + P(B_0 | A_1) P(A_1)$$

$$= 0.1 * 0.3 + 0.15 * 0.7$$

$$= 0.135$$