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★ To find analytic function whose Real & Imaginary part is given.

We'll solve these problems using Milne-Thompson's Method:-

1) Consider $f(z)$ be analytic function whose real part is u is given to us, then to find $f(z)$.

As u is given then we can find u_x & u_y as $f(z)$ is analytic then it satisfies C.R eqn i.e.

$u_x = v_y$ & $u_y = -v_x$

Also we know that $f'(z) = u_x + iv_x = u_x - iu_y$
— as we know u_y.

Then, $f'(z) = u_x(x,y) - i\, u_y(x,y)$

Then by Milne-Thompson's method put $x=z$ & $y=0$.

$\Rightarrow f'(z) = u_x(z,0) - i\, u_y(z,0)$

Then on integrating we get

$$f(z) = \int u_x(z,0)\,dz - i\int u_y(z,0)\,dz + C.$$

2) Similarly, if $f(z)$ which is analytic function whose imaginary part v is given, one can find $f(z)$.

As v is given then we can find v_x & v_y

& as $f(z)$ is analytic then $u_x = v_y$ & $u_y = -v_x$

Also, $f'(z) = u_x + iv_x = v_y + iv_x$.

Then $f'(z) = v_y(x,y) + i\,v_x(x,y)$.

By Milne-Thompson's method put $x=z$ & $y=0$.

$\Rightarrow f'(z) = v_y(z,0) + i\cdot v_x(z,0)$.

Then on integration we get

$$f(z) = \int v_y(z,0)\,dz + i\int v_x(z,0)\,dz + C.$$

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1) Find analytic function whose real part is

i) $u = e^x(x\cos y - y\sin y)$.

Let $f(z) = u + iv$ be analytic function $\Rightarrow u_x = v_y$ & $u_y = -v_x$

Here, $u_x = e^x(\cos y) + (x\cos y - y\sin y)e^x$

$= e^x[\cos y + x\cos y - y\sin y]$

$u_y = e^x(-x\sin y - (y\cos y + \sin y))$.

$= e^x(-x\sin y - y\cos y - \sin y)$.

As $f(z)$ is analytic $\Rightarrow f'(z) = u_x + iv_x = u_x - iu_y$.

$\therefore f'(z) = e^x[\cos y + x\cos y - y\sin y] + ie^x[-x\sin y - y\cos y - \sin y]$

put $x = z$ & $y = 0$. By Milne-Thompson's method

$f'(z) = e^z[1 + z] + ie^z[0 - 0 - 0]$

$= e^z[1 + z]$.

On integrating we get

$f(z) = \int e^z(1+z)\,dz + C$

$= (1+z)e^z - \int e^z dz + C$

$= (e^z + ze^z) - e^z + C = ze^z + C$

Note: Suppose they ask to find img. part v then sub. $z = x + iy$ in $f(z)$. we'll get img. part.

In above eg. $f(z) = (x+iy)\,e^{x+iy}$

$= (x+iy)\,e^x e^{iy}$

$= (x+iy)\,e^x(\cos y + i\sin y)$

$= e^x[x\cos y + ix\sin y + iy\cos y - y\sin y]$

$= e^x[x\cos y - y\sin y] + ie^x[x\sin y + y\cos y]$

$\therefore v = e^x[x\sin y + y\cos y]$.

2) $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$.

Let $f(z) = u + iv$ is analytic funct^n $\Rightarrow u_x = v_y$ & $u_y = -v_x$

$u_x = \dfrac{(\cosh 2y - \cos 2x)\,2\cos 2x - \sin 2x(0 + 2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$

$= \dfrac{2\cosh 2y\cos 2x - 2\cos^2 2x - 2\sin^2 2x}{(\cosh 2y + \cos 2x)^2}$

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$$\therefore u_x = \frac{2\cosh 2y \cos 2x - 2(\cos^2 2x + \sin^2 2x)}{(\cosh 2y - \cos 2x)^2}$$

$$= \frac{2\cosh 2y \cos 2x - 2}{(\cosh 2y - \cos 2x)^2}$$

$$u_y = \frac{-2\sin 2x \ \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

As $f(z)$ is analytic.

$$f'(z) = u_x + i v_x = u_x - i u_y$$

$$f'(z) = \frac{(2\cosh 2y \cos 2x - 2) + i 2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

By M-T method put $x = z$ & $y = 0$

$$f'(z) = \frac{2\cos 2z - 2 + i2(0)}{(1 - \cos 2z)^2} \qquad \sinh(0) = 0$$

$$= \frac{-2(1-\cos 2z)}{(1-\cos 2z)^{\cancel{2}}} = \frac{-2}{1-\cos 2z} = \frac{-\cancel{2}}{\cancel{2}\sin^2 z}$$

$$\therefore f'(z) = -\operatorname{cosec}^2 z$$

On integrating, $f(z) = -\int \operatorname{cosec}^2 z \, dz + C$

$$= \cot z + C$$

3) $u = \frac{x}{2}\log(x^2+y^2) - y\tan^{-1}\left(\frac{y}{x}\right) + \sin x \cosh y$.

$\Rightarrow$ Let $f(z) = u + iv$ be analytic funⁿ. where $u = \frac{x}{2}\log(x^2+y^2) - y\tan^{-1}(y/x) + \sin x \cosh y$

Then, $u_x = \frac{x}{2}\frac{2x}{x^2+y^2} + \frac{\log(x^2+y^2)}{2} - y\frac{1}{\left(1+\frac{y^2}{x^2}\right)}\left(\frac{-y}{x^2}\right) + \cos x \cosh y$

$$= \frac{x^2}{x^2+y^2} + \frac{\log(x^2+y^2)}{2} + \frac{y^2}{x^2\left(\frac{x^2+y^2}{x^2}\right)} + \cos x \cosh y.$$

$$= \frac{x^2+y^2}{x^2+y^2} + \frac{\log(x^2+y^2)}{2} + \cos x \cosh y = 1 + \frac{\log(x^2+y^2)}{2} + \cos x \cosh y$$

$$u_y = \frac{x}{2}\frac{2y}{x^2+y^2} - \frac{y}{\left(1+\frac{y^2}{x^2}\right)}\frac{1}{x} - \tan^{-1}\left(\frac{y}{x}\right) + \sin x \sinh y$$

$$= \frac{xy}{x^2+y^2} - \frac{xy}{x^2+y^2} - \tan^{-1}\left(\frac{y}{x}\right) + \sin x \sinh y = -\tan^{-1}\left(\frac{y}{x}\right) + \sin x \sinh y$$

As $f(z)$ is analytic $\Rightarrow u_x = v_y$ & $u_y = -v_x$

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Also $f'(z) = u_x + iv_x = u_x - iu_y$

$= 1 + \log\frac{(x^2+y^2)}{2} + \cos x \cosh y - i\left[\tan^{-1}\left(\frac{y}{x}\right) + \sin x \sinh y\right]$

By M.T. method put $x = z$ & $y = 0$.

$f'(z) = 1 + \frac{\log(z^2)}{2} + \cos z(1) - i(-\tan^{-1}(0) + 0)$

$= 1 + \frac{\cancel{2}\log z}{\cancel{2}} + \cos z + i(0)$

$f'(z) = 1 + \log z + \cos z$

On integrating we get.

$f(z) = z + \int \log z (1) dz + \sin z + C$

$= z + z\log z - \int z \frac{1}{z} dz + \sin z + C$

$= \cancel{z} + z\log z - \cancel{z} + \sin z + C$

$= z\log z + \sin z + C$.

4) $x^4 - 6x^2y + y^4$

Let $f(z) = u + iv$ be analytic where $u = x^4 - 6x^2y + y^4$.

$u_x = 4x^3 - 12xy^2$, $u_y = -12x^2y + 4y^3$.

As $f(z)$ is analytic $\Rightarrow u_x = v_y$ & $u_y = -v_x$

Also $f'(z) = u_x + iv_x = u_x - iu_y$

$\therefore f'(z) = (4x^3 - 12xy^2) - i(-12x^2y + 4y^3)$

By M.T. method put $x = z$ & $y = 0$.

$\therefore f'(z) = 4z^3 - 0 - i(0 + 4z(0))$

$f'(z) = 4z^3$

On integrating we get, $f(z) = \frac{\cancel{4}z^4}{\cancel{4}} + C \Rightarrow f(z) = z^4 + C$.

5) $\frac{\sin 2x}{\cosh 2y + \cos 2x}$.

Let $f(z) = u + iv$ be analytic. s.t. $u = \frac{\sin 2x}{\cosh 2y + \cos 2x}$.

$u_x = \frac{(\cosh 2y + \cos 2x)2\cos 2x + \sin 2x(2\sin 2x)}{(\cosh 2y + \cos 2x)^2} = \frac{\cosh 2y \cdot 2\cos 2x + 2\cos^2 2x + 2\sin^2 2x}{(\cosh 2y + \cos 2x)^2}$

$= \frac{2\cosh 2y \cos 2x + 2}{(\cosh 2y + \cos 2x)^2}$

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$$u_y = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y + \cos 2x)^2}$$

As $f(z)$ is analytic $\Rightarrow u_x = v_y$ & $u_y = -v_x$

& $f'(z) = u_x + i v_x = u_x - i u_y$

$$\Rightarrow f'(z) = \frac{2\cosh 2y \cos 2x + 2}{(\cosh 2y + \cos 2x)^2} + i \frac{2\sin 2x \sinh 2y}{(\cosh 2y + \cos 2x)^2}.$$

By M.T. method put $x = z$ & $y = 0$.

$$\Rightarrow f'(z) = \frac{2(1)\cos 2z + 2}{(1+\cos 2z)^2} + i\,(0) \quad \text{--- } \because \sinh(0) = 0.$$

$$= \frac{2(1+\cos 2z)}{(1+\cos 2z)^2} = \frac{2}{1+\cos 2z} = \frac{2}{2\cos^2 z} = \sec^2 z$$

$\therefore f'(z) = \sec^2 z$

On integrating we get, $f(z) = \tan z + C$

HW 2) Find the imaginary part of the analytic function whose real part is $e^{2x}(x\cos 2y - y\sin 2y)$. Also verify v is harmonic

→ Let, $f(z) = u + iv$ be analytic function. s.t $u = e^{2x}(x\cos 2y - y\sin 2y)$

$u_x = e^{2x}(\cos 2y) + (x\cos 2y - y\sin 2y)\, 2e^{2x}$.

$= e^{2x}(\cos 2y + 2x\cos 2y - 2y\sin 2y)$

$u_y = e^{2x}(-2x\sin 2y - \sin 2y - 2y\cos 2y)$

As $f(z)$ is analytic $\Rightarrow u_x = v_y$ & $u_y = -v_x$

Also, $f'(z) = u_x + i v_x = u_x - i u_y$

$f'(z) = e^{2x}(\cos 2y + 2x\cos 2y - 2y\sin 2y)$
$\quad + i\, e^{2x}(2x\sin 2y + \sin 2y + 2y\cos 2y)$

By M.T. method put $x = z$ & $y = 0$.

$f'(z) = e^{2z}(1+2z) + i\,e^{2z}(0) = e^{2z}(1+2z)$

On integrating.

$f(z) = \int (1+2z)\,e^{2z} + c.$

$$= (1+2z)\frac{e^{2z}}{2} - \int \frac{e^{2z}}{\cancel{2}}(\cancel{2})\,dz + C$$

$$= (1+2z)\frac{e^{2z}}{2} - \frac{e^{2z}}{2} + C = \cancel{\frac{e^{2z}}{2}} + ze^{2z} - \cancel{\frac{e^{2z}}{2}} + C$$

$= ze^{2z} + C$

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as $f(z) = ze^{2z} = (x+iy)e^{2(x+iy)}$
$= (x+iy)\, e^{2x} e^{i2y} = (x+iy)\, e^{2x}(\cos 2y + i\sin 2y)$.
$= e^{2x}(x\cos 2y + i x\sin 2y + iy\cos 2y - y\sin 2y)$
$= e^{2x}[x\cos 2y - y\sin 2y) + i e^{2x}(x\sin 2y + y\cos 2y)$
$V = e^{2x}(x\sin 2y + y\cos 2y)$
$V_x = e^{2x}(\sin 2y) + (x\sin 2y + y\cos 2y)\, 2e^{2x}$
$= e^{2x}(\sin 2y + 2x\sin 2y + 2y\cos 2y)$.
$V_{xx} = e^{2x}(2\sin 2y) + (\sin 2y + 2x\sin 2y + 2y\cos 2y)\, 2e^{2x}$
$= e^{2x}[4\sin 2y + 4x\sin 2y + 4y\cos 2y]$ —— 1>
$V_y = e^{2x}(2x\cos 2y + y(-2\sin 2y) + \cos 2y)$
$= e^{2x}(2x\cos 2y - 2y\sin 2y + \cos 2y)$
$V_{yy} = e^{2x}(-4x\sin 2y - (4y\cos 2y + 2\sin 2y) - 2\sin 2y)$
$= e^{2x}(-4x\sin 2y - 4y\cos 2y - 4\sin 2y)$
$= -e^{2x}(4x\sin 2y + 4y\cos 2y + 4\sin 2y)$ —— 2>
$\therefore V_{xx} + V_{yy} = 0$ —— from 1> & 2>
$\Rightarrow$ V satisfies Laplace's eq^n $\Rightarrow$ V is harmonic.

3> Find analytic function whose img. part is.
1> $e^{-x}[2xy\cos y + (y^2 - x^2)\sin y]$
Let $f(z) = u + iv$ be analytic fun^n, s.t.
$v = e^{-x}[2xy\cos y + (y^2 - x^2)\sin y]$.
$\therefore V_x = e^{-x}[2y\cos y + \sin y(-2x)] + [2xy\cos y + (y^2 - x^2)\sin y](-e^{-x})$
$= e^{-x}[2y\cos y - 2x\sin y - 2xy\cos y - (y^2 - x^2)\sin y]$
$V_y = e^{-x}[2x(-y\sin y + \cos y) + (y^2 - x^2)\cos y + \sin y(2y)]$
$= e^{-x}[-2xy\sin y + 2x\cos y + (y^2 - x^2)\cos y + 2y\sin y]$
As $f(z)$ is analytic fun^n $\Rightarrow$ $u_x = v_y$ & $u_y = -v_x$.
also $f'(z) = u_x + iv_x = v_y + iv_x$.
$\therefore f'(z) = e^{-x}[-2xy\sin y + 2x\cos y + (y^2 - x^2)\cos y + 2y\sin y]$
$+ i e^{-x}[2y\cos y - 2x\sin y - 2xy\cos y - (y^2 - x^2)\sin y]$
By M.T. method put $x = z$ & $y = 0$.
$f'(z) = e^{-z}[2z - z^2] + ie^{-z}[0] = e^{-z}[2z - z^2]$
$\therefore f'(z) = 2ze^{-z} - z^2 e^{-z}$

On integrating,

$f(z) = 2\int z e^{-z} dz - \int z^2 e^{-z} dz + C$

$= 2\left[z \frac{e^{-z}}{-1} + \int e^{-z} dz\right] - \left[z^2 \frac{e^{-z}}{-1} + \int e^{-z} 2z\, dz\right] + C.$

$= 2\left[-z e^{-z} - e^{-z}\right] + z^2 e^{-z} - 2\left[-z e^{-z} - e^{-z}\right] + C$

$= -2z e^{-z} - 2e^{-z} + z^2 e^{-z} + 2z e^{-z} + 2e^{-z} + C$

$= z^2 e^{-z} + C$

2) $(x^4 - 6x^2y^2 + y^4) + (x^2 - y^2) + 2xy$

Let $f(z) = u + iv$ be analytic s.t. $v = (x^4 - 6x^2y^2 + y^4) + (x^2 - y^2) + 2xy$

$v_x = 4x^3 - 12xy^2 + 2x + 2y$, $v_y = -12x^2y + 4y^3 - 2y + 2x$

As $f(z)$ is analytic $\Rightarrow u_x = v_y$ & $u_y = -v_x$.

Also $f'(z) = u_x + iv_x = v_y + iv_x$

$\therefore f'(z) = (-12x^2y + 4y^3 - 2y + 2x) + i(4x^3 - 12xy^2 + 2x + 2y)$

By M.T. method put $x = z$ & $y = 0$

$\Rightarrow f'(z) = 2z + i(4z^3 + 2z)$

On integ. we get, $f(z) = \int 2z\, dz + i\int (4z^3 + 2z) dz + C$

$\Rightarrow f(z) = z^2 + i(z^4 + z^2) + C$

H.W 3) $e^x(x\sin y + y\cos y)$.

Let $f(z) = u + iv$ be analytic s.t. $v = e^x(x\sin y + y\cos y)$

$v_x = e^x(\sin y) + (x\sin y + y\cos y)e^x = e^x(\sin y + x\sin y + y\cos y)$

$v_y = e^x(x\cos y + y(-\sin y) + \cos y)$.

As $f(z)$ is analytic $\Rightarrow u_x = v_y$ & $u_y = -v_x$.

Also $f'(z) = u_x + iv_x = v_y + iv_x$.

$\therefore f'(z) = e^x(x\cos y - y\sin y + \cos y) + i(e^x(\sin y + x\sin y + y\cos y))$.

By M.T. method put $x = z$ & $y = 0$

$\therefore f'(z) = e^z(z+1) + ie^z(0) = e^z(z+1)$.

On integrating, $f(z) = \int (z+1)e^z + C$.

$= (z+1)e^z - e^z + C = ze^z + e^z - e^z + C$

$= ze^z$

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H.W 4) $\frac{x}{x^2+y^2} + \cosh x \cos y$

Let $f(z) = u + iv$ be analytic funⁿ s.t. $v = \frac{x}{x^2+y^2} + \cosh x \cos y$

$V_x = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} + \sinh x \cos y = \frac{y^2-x^2}{(x^2+y^2)^2} + \sinh x \cos y$.

$V_y = \frac{-2xy}{(x^2+y^2)^2} + \cosh x(-\sin y)$.

As $f(z)$ is analytic $\Rightarrow u_x = v_y$ & $u_y = -v_x$.

Also, $f'(z) = u_x + iv_x = v_y + iv_x$

$\therefore f'(z) = \frac{-2xy}{(x^2+y^2)^2} + \cosh x(-\sin y) + i\left[\frac{(y^2-x^2)}{(x^2+y^2)^2} + \sinh x \cos y\right]$

By M.T. method put $x = z$, $y = 0$.

$f'(z) = 0 + 0 + i\left[\frac{-z^2}{(z^2)^2} + \sinh z\right] = i\left[-\frac{1}{z^2} + \sinh z\right]$

On integrative, $f(z) = i\int\left(-\frac{1}{z^2} + \sinh z\right)dz + C$.

$\therefore f(z) = i\left[\frac{1}{z} + \cosh z\right] + C$.

H.W 5) $e^{-x}(y\sin y + x\cos y)$

Let $f(z) = u + iv$ be analytic funⁿ s.t. $v = e^{-x}(y\sin y + x\cos y)$

$V_x = e^{-x}(\cos y) + (y\sin y + x\cos y)(-e^{-x}) = e^{-x}(\cos y - y\sin y - x\cos y)$

$V_y = e^{-x}(y\cos y + \sin y - x\sin y)$.

As $f(z)$ is analytic $\Rightarrow u_x = v_y$ & $u_y = -v_x$.

Also $f'(z) = u_x + iv_x = v_y + iv_x$.

$\therefore f'(z) = e^{-x}(y\cos y + \sin y - x\sin y) + i\, e^{-x}(\cos y - y\sin y - x\cos y)$

By M.T. method put $x = z$ & $y = 0$

$f'(z) = e^{-z}(0) + i e^{-z}(1-z) = i\, e^{-z}(1-z)$

$\therefore$ On integrating $f(z) = i\int(1-z)e^{-z}\,dz + C$

$f(z) = i\left[-(1-z)e^{-z} + \int e^{-z}(-1)\,dz\right] + C$

$= i\left[-(1-z)e^{-z} + e^{-z}\right] + C = i\left[\cancel{-e^{-z}} + ze^{-z} + \cancel{e^{-z}}\right] + C$

$= i z e^{-z} + C$

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HW 6) $\tan^{-1} y/x$

Let $f(z) = u + iv$ be analytic s.t. $v = \tan^{-1} y/x$

$$V_x = \frac{1}{1+(y^2/x^2)}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$V_y = \frac{1}{1+(y^2/x^2)} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

As $f(z)$ be analytic $\Rightarrow u_x = V_y$ & $u_y = -V_x$

Also, $f'(z) = u_x + iV_x = V_y + iV_x$.

$$f'(z) = \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right).$$

By M.T. method, put $x = z$ & $y = 0$

$$f'(z) = \frac{z}{z^2} + i(0) = \frac{1}{z} \Rightarrow f'(z) = \frac{1}{z}$$

On integrating, $f(z) = \int \frac{1}{z} + C = \log z + C$.

4) If the img. part of the analytic fun^n $\omega = f(z)$ is $v = x^2 - y^2 + \frac{x}{x^2+y^2}$, show that the real part

$$u = -2xy + \frac{y}{x^2+y^2} + c.$$

Let, $f(z) = u + iv$ be analytic s.t. $v = x^2 - y^2 + \frac{x}{x^2+y^2}$

$$V_x = 2x - \frac{(x^2-y^2)}{(x^2+y^2)^2}, \quad V_y = -2y - \frac{2xy}{(x^2+y^2)^2}.$$

As $f(z)$ is analytic $\Rightarrow u_x = V_y$ & $u_y = -V_x$

Also, $f'(z) = u_x + iV_x = V_y + iV_x$

$$f'(z) = \left(-2y - \frac{2xy}{(x^2+y^2)^2}\right) + i\left(2x - \frac{(x^2-y^2)}{(x^2+y^2)^2}\right).$$

By M.T. method, put $x = z$ & $y = 0$.

$$f'(z) = 0 + i\left(2z - \frac{z^2}{z^4}\right) = i\left(2z - \frac{1}{z^2}\right).$$

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On integrating, $f(z) = i\int\left(2z - \frac{1}{z^2}\right) dz + c$

$\therefore f(z) = i\left(z^2 + \frac{1}{z}\right) + c$

put $z = x + iy$.

$f(z) = i\left[(x+iy)^2 + \frac{1}{x+iy}\right] + c$

$= i\left[x^2 + i2xy - y^2 + \left(\frac{x-iy}{x^2+y^2}\right)\right] + c$

$= i\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right) + i\left(i2xy - \frac{iy}{x^2+y^2}\right) + c$

$= \left(-2xy + \frac{y^2}{x^2+y^2}\right) + i\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right) + c$

$u = \left(-2xy + \frac{y}{x^2+y^2}\right) + c$

★ To find analytic funⁿ when u+v or u−v is given

1) Find analytic function $f(z) = u + iv$ in terms of z if

1) $u + v = \frac{x}{x^2+y^2}$.

Let, $f(z) = u + iv \Rightarrow if(z) = i(u+iv) = iu - v = -v + iu$.

$\therefore f(z) + if(z) = (u-v) + i(u+v)$

$\Rightarrow (1+i) f(z) = U + iV = F(z)$.

$F(z)$ where $U = u - v$ & $V = u + v$

as $f(z)$ is analytic $\Rightarrow (1+i)f(z)$ is also analytic.

$\Rightarrow F(z)$ is analytic. $\Rightarrow U_x = V_y$ & $U_y = -V_x$

Also $F'(z) = U_x + iV_x$ — 1)

Here, $V = u + v = \frac{x}{x^2+y^2} \Rightarrow V_x = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$

$V_y = \frac{-2yx}{(x^2+y^2)^2}$

$F(z) = (1+i) f(z)$

from 1) $f'(z) = U_x + iV_x = V_y + iV_x$.

$\Rightarrow (1+i) f'(z) = \frac{-2xy}{(x^2+y^2)^2} + i\left(\frac{y^2-x^2}{(x^2+y^2)^2}\right)$

By M.T. method put $x = z, y = 0$

$(1+i) f'(z) = 0 + i\frac{(-z^2)}{(z^2)^2} = -\frac{i}{z^2}$

On Integrating, $(1+i)f(z) = -i\int \frac{1}{z^2} dz + C$.

$\Rightarrow (1+i) f(z) = \frac{i}{z} + C \Rightarrow f(z) = \frac{1}{(1+i)}\left(\frac{i}{z} + C\right)$

2) $u - v = (x-y)(x^2+4xy+y^2)$.

Let $f(z) = u + iv \Rightarrow if(z) = i(u+iv) = iu - v = -v + iu$.

$f(z) + if(z) = (u-v) + i(u+v) \Rightarrow (1+i)f(z) = (u-v) + i(u+v)$

$\therefore (1+i) f(z) = U + iV = F(z)$

where, $U = (u-v) = (x-y)(x^2+4xy+y^2)$

$\therefore U_x = (x-y)(2x+4y) + (x^2+4xy+y^2)$

$= 2x^2 + 4xy - 2xy - 4y^2 + x^2 + 4xy + y^2$

$= 3x^2 + 6xy - 3y^2$

$U_y = (x-y)(4x+2y) + (x^2+4xy+y^2)(-1)$

$= 4x^2 - 4xy + 2xy - 2y^2 - x^2 - 4xy - y^2$

$= 3x^2 - 6xy - 3y^2$.

As $f(z)$ is analytic $\Rightarrow (1+i)f(z)$ is analytic.

$\Rightarrow F(z)$ is analytic $\Rightarrow U_x = V_y$ & $U_y = -V_x$

Also $f'(z) = U_x + iV_x = U_x - iU_y$

$\Rightarrow (1+i) f'(z) = 3x^2 + 6xy - 3y^2 - i(3x^2 - 6xy - 3y^2)$

By M.T. method put $x = z$ & $y = 0$.

$\Rightarrow (1+i) f'(z) = 3z^2 - i\,3z^2 = 3z^2(1-i)$

On Integ. we get, $(1+i) f(z) = (1-i)\int 3z^2 + C$.

$\Rightarrow (1+i) f(z) = (1-i) z^3 + C$.

$\Rightarrow f(z) = \frac{(1-i)}{(1+i)} z^3 + \frac{C}{(1+i)} = \frac{(1-i)^2}{1+1} z^3 + C' = \frac{(1-2i-1)}{2} z^3 + C'$

$\Rightarrow f(z) = -iz^3 + C'$.

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