Subject: Applied Mathematics-IV **SEM:IV**

(5) Find the eigen values and eigen vectors of the following matrix $A = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{pmatrix}$

Soln:-

The characteristic eqn is

$$\lambda^3 - (4+3-2)\lambda^2 + \big((12-6-8)-(6-6-10)\big)\lambda - [4(-6+10)-6(-2+2)+6(-5+3)] = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\therefore \lambda = 1, 2, 2$$

$\lambda = 1$

$$(A - \lambda I)X = 0$$

$$\begin{pmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -5 & -2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

sub $\lambda = 1$

$$\begin{pmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad \xrightarrow{R_1 \times 1/3} \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -5 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

Solving by Cramer's rule,

$$\frac{x_1}{-6+10} = \frac{-x_2}{-3+2} = \frac{x_3}{-5+2}$$ (By considering second and third row).

$$\frac{x_1}{4} = \frac{+x_2}{+1} = \frac{x_3}{-3} = t$$

$x_1 = 4t,\ x_2 = t,\ x_3 = -3t$

$$X = \begin{pmatrix} 4t \\ t \\ -3t \end{pmatrix} = t\begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}$$

$\therefore \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}$ is the eigen vector corresponding to $\lambda = 1$.

$\underline{\lambda = 2}$ $(A - \lambda I)X = 0$

$$\begin{pmatrix} 2 & 6 & 6 \\ 1 & 1 & 2 \\ -1 & -5 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

By using cramer's rule,

$$\frac{x_1}{12-6} = \frac{-x_2}{4-6} = \frac{x_3}{2-6} = t$$

$$\frac{x_1}{6} = \frac{-x_2}{-2} = \frac{x_3}{-4} = t$$

$$\frac{x_1}{3} = \frac{x_2}{1} = \frac{x_3}{-2} = t$$

$x_1 = 3t$, $x_2 = t$, $x_3 = -2t$.

$$X = \begin{pmatrix} 3t \\ t \\ -2t \end{pmatrix} = t\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$$

$\therefore \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ is the eigenvector corresponding to $\lambda = 2$.

Exercise

i) Find the eigen values & eigen vectors of

a) $\begin{pmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{pmatrix}$ b) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}$

c) $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

① Two of the eigen values of a 3x3 matrix are $-1, 2$. If the determinant of the matrix is 4, find its third eigen value.

Soln:-

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.

WKT Product of eigen values $= |A|$

$$-1 \times 2 \times \lambda_3 = 4.$$

$$\lambda_3 = \frac{4}{-2}$$

$$\boxed{\lambda_3 = -2}$$

$\therefore$ The third eigen value is -2.

Eigen values of a Hermitian matrix are real.

Proof:-

Let A be a Hermitian matrix, λ the eigen value & X the eigen vector.

$$\therefore AX = \lambda X$$

Premultiply by X^{θ} we get

$$X^{\theta}AX = X^{\theta}\lambda X = \lambda X^{\theta}X \longrightarrow ①.$$

By taking complex conjugate transpose of both sides,

$$(X^\theta A X)^\theta = (\lambda X^\theta X)^\theta$$

$$X^\theta A^\theta (X^\theta)^\theta = \bar{\lambda} X^\theta (X^\theta)^\theta$$

Since A is Hermitian $A^\theta = A$ & also $(X^\theta)^\theta = X$.

$$\therefore X^\theta A X = \bar{\lambda} X^\theta X . \longrightarrow (2).$$

$\therefore$ From (1) & (2)

$$\lambda X^\theta X = \bar{\lambda} X^\theta X$$

$$(\lambda - \bar{\lambda}) X^\theta X = 0 .$$

Since X is a non-zero vector $X^\theta X \neq 0$.

$$\therefore \lambda - \bar{\lambda} = 0$$

$$\therefore \lambda = \bar{\lambda} .$$

Hence λ is real.

Remark * Eigen values of a real symmetric matrix are all real

* Eigen values of a skew-Hermitian matrix are either purely imaginary or zero.

* The eigen values of a real skew-symmetric matrix are purely imaginary or zero.

* The eigen values of a unitary matrix are of unit modulus.

* Eigen values of an orthogonal matrix are of unit modulus.

* If λ is an eigen value of A then $\bar{\lambda}$ is an eigen value of A^{θ}.

* If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A then $k\lambda_1, k\lambda_2, \ldots k\lambda_n$ are the eigen values of kA.

* If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots \frac{1}{\lambda_n}$ are the eigen values of A^{-1}.

* If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A then $\lambda_1^n, \lambda_2^n, \ldots \lambda_n^n$ are the eigen values of A^n.

* If λ is an eigen value of a non-singular

matrix A, then $\frac{|A|}{\lambda}$ is an eigen value of adj A.

Examples:

If $A = \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}$ then find the eigen values of $4A^{-1}+3A+2I$.

Soln:-

$$\lambda^2 - 5\lambda + 4 = 0$$

$$\lambda = 1, 4.$$

$\therefore$ The eigen values of A are 1, 4.

The eigen values of A^{-1} are $1, \frac{1}{4}$

The eigen values of I are 1, 1

For $\lambda = 1$

Eigen value of $4A^{-1}+3A+2I = 4(1)+3(1)+2$
$= 9.$

For $\lambda = 4$

Eigen value of $4A^{-1}+3A+2I = 4\left(\frac{1}{4}\right)+3(4)+2$
$= 15$

$\therefore$ The eigen values are 9, 15.