Bezier Curves

- Bezier curve section can be fitted to any number of control points.
- Number of control points and their relative position gives degree of the Bezier polynomials.
- With the interpolation spline Bezier curve can be specified with boundary condition or blending function.
- Most convenient method is to specify Bezier curve with blending function.
- Consider we are given n+1 control point position from p_0 to p_n where $p_k = (x_k, y_k, z_k)$.
- This is blended to gives position vector p(u) which gives path of the approximate Bezier curve is:

  $$p(u) = \sum_{k=0}^{n} p_k BEZ_{k,n}(u) \qquad 0 \le u \le 1$$

  Where $BEZ_{k,n}(u) = C(n,k)u^k(1-u)^{n-k}$

  And $C(n,k) = {n!}/{k!\,(n-k)!}$
- We can also solve Bezier blending function by recursion as follow:

  $BEZ_{k,n}(u) = (1-u)BEZ_{k,n-1}(u) + uBEZ_{k-1,n-1}(u) \qquad n > k \ge 1$

  Here $BEZ_{k,k}(u) = u^k$ and $BEZ_{0,k}(u) = (1-u)^k$
- Parametric equation from vector equation can be obtain as follows.

  $$x(u) = \sum_{k=0}^{n} x_k BEZ_{k,n}(u)$$

  $$y(u) = \sum_{k=0}^{n} y_k BEZ_{k,n}(u)$$

  $$z(u) = \sum_{k=0}^{n} z_k BEZ_{k,n}(u)$$
- Bezier curve is a polynomial of degree one less than the number of control points.
- Below figure shows some possible curve shapes by selecting various control point.

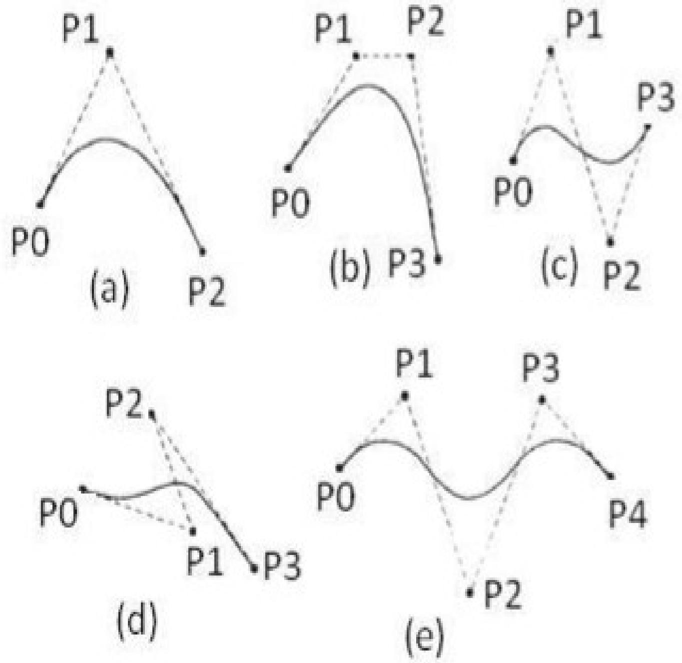


Fig. 4.20: -Example of 2D Bezier curves generated by different number of control points.

- Efficient method for determining coordinate positions along a Bezier curve can be set up using recursive calculation
- For example successive binomial coefficients can be calculated as

$$C(n,k) = \frac{n-k+1}{k} C(n,k-1) \qquad n \geq k$$

Properties of Bezier curves

- It always passes through first control point i.e. $p(0) = p_0$
- It always passes through last control point i.e. $p(1) = p_n$
- Parametric first order derivatives of a Bezier curve at the endpoints can be obtain from control point coordinates as:

$$p'(0) = -np_0 + np_1$$
$$p'(1) = -np_{n-1} + np_n$$

- Parametric second order derivatives of endpoints are also obtained by control point coordinates as:

$$p''(0) = n(n-1)[(p_2 - p_1) - (p_1 - p_0)]$$
$$p''(1) = n(n-1)[(p_{n-2} - p_{n-1}) - (p_{n-1} - p_n)]$$

- Bezier curve always lies within the convex hull of the control points.
- Bezier blending function is always positive.
- Sum of all Bezier blending function is always 1.

$$\sum_{k=0}^{n} BEZ_{k,n}(u) = 1$$

- So any curve position is simply the weighted sum of the control point positions.
- Bezier curve smoothly follows the control points without erratic oscillations.

Cubic Bezier Curves

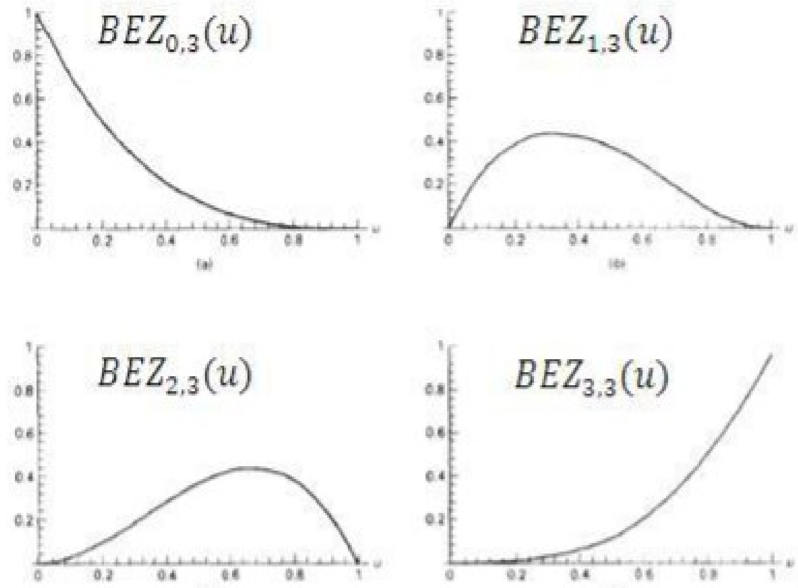
- Many graphics package provides only cubic spline function because this gives reasonable design flexibility in average calculation.
- Cubic Bezier curves are generated using 4 control points.
- 4 blending function obtained by substituting n=3

  $BEZ_{0,3}(u) = (1-u)^3$

  $BEZ_{1,3}(u) = 3u(1-u)^2$

  $BEZ_{2,3}(u) = 3u^2(1-u)$

  $BEZ_{3,3}(u) = u^3$
- Plots of this Bezier blending function are shown in figure below



- The form of blending functions determines how control points affect the shape of the curve for values of parameter u over the range from 0 to 1.

  At u = 0 $BEZ_{0,3}(u)$ is only nonzero blending function with values 1.

  At u = 1 $BEZ_{3,3}(u)$ is only nonzero blending function with values 1.
- So the cubic Bezier curve is always pass through p_0 and p_3.
- Other blending function is affecting the shape of the curve in intermediate values of parameter u.
- $BEZ_{1,3}(u)$ is maximum at $u = {}^1/_3$and $BEZ_{2,3}(u)$ is maximum at $u = {}^2/_3$
- Blending function is always nonzero over the entire range of u so it is not allowed for local control of the curve shape.
- At end point positions parametric first order derivatives are :

  $p'(0) = 3(p_1 - p_0)$

  $p'(1) = 3(p_3 - p_2)$
- And second order parametric derivatives are.

  $p''(0) = 6(p_0 - 2p_1 + p_2)$

  $p''(1) = 6(p_1 - 2p_2 + p_3)$
- This expression can be used to construct piecewise curve with C^1 and C^2 continuity.
- Now we represent polynomial expression for blending function in matrix form:

$$p(u) = [u^3 \quad u^2 \quad u \quad 1] \cdot M_{BEZ} \cdot \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$M_{BEZ} = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- We can add additional parameter like tension and bias as we did with the interpolating spline.

B-Spline Curves

- General expression for B-Spline curve in terms of blending function is given by:

$$p(u) = \sum_{k=0}^{n} p_k B_{k,d}(u) \qquad u_{min} \leq u \leq u_{max}, 2 \leq d \leq n+1$$

  Where p_k is input set of control points.
- The range of parameter u is now depends on how we choose the B-Spline parameters.
- B-Spline blending function $B_{k,d}$ are polynomials of degree d-1 , where d can be any value in between 2 to n+1.
- We can set d=1 but then curve is only point plot.
- By defining blending function for subintervals of whole range we can achieve local control.
- Blending function of B-Spline is solved by Cox-deBoor recursion formulas as follows.

$$B_{k,1}(u) = \begin{cases} 1 & if\ u_k \leq u \leq u_{k+1} \\ 0 & otherwise \end{cases}$$

$$B_{k,d}(u) = \frac{u - u_k}{u_{k+d-1} - u_k} B_{k,d-1}(u) + \frac{u_{k+d} - u}{u_{k+d} - u_{k+1}} B_{k+1,d-1}(u)$$

- The selected set of subinterval endpoints u_j is reffered to as a **knot vector**.
- We can set any value as a subinterval end point but it must follow $u_j \leq u_{j+1}$
- Values of u_{min} and u_{max} depends on number of control points, degree d, and knot vector.
- Figure below shows local control

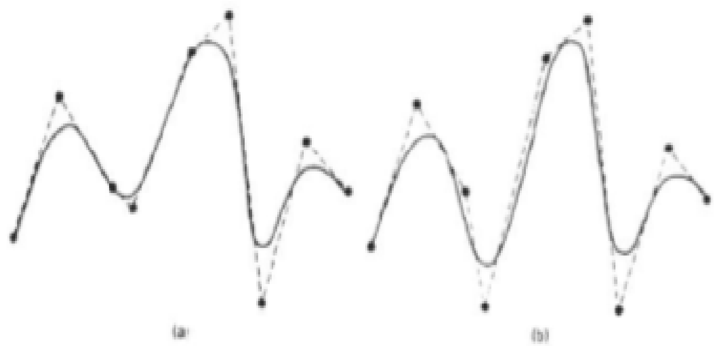


Fig. 4.26: -Local modification of B-Spline curve.

- B-Spline allows adding or removing control points in the curve without changing the degree of curve.
- B-Spline curve lies within the convex hull of at most d+1 control points so that B-Spline is tightly bound to input positions.

- For any u in between u_{d-1} to u_{n+1}, sum of all blending function is 1 i.e. $\sum_{k=0}^{n} B_{k,d}(u) = 1$
- There are three general classification for knot vectors:
  - Uniform
  - Open uniform
  - Non uniform

Properties of B-Spline Curves

- It has degree d-1 and continuity C^{d-2} over range of u.
- For n+1 control point we have n+1 blending function.
- Each blending function $B_{k,d}(u)$ is defined over d subintervals of the total range of u, starting at knot value u_k.
- The range of u is divided into n+d subintervals by the n+d+1 values specified in the knot vector.
- With knot values labeled as $\{u_0, u_1, \ldots, u_{n+d}\}$ the resulting B-Spline curve is defined only in interval from knot values u_{d-1} up to knot values u_{n+1}
- Each spline section is influenced by d control points.
- Any one control point can affect at most d curve section.