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* Recurrence relations -

In discrete computation problems, it is easier to obtain the numeric function in terms of or in the form of a relation between its terms. The recursive formula for defining the numeric function is called a recurrence relation.

If $a = \{a_0, a_1, a_2, \ldots, a_r, \ldots\}$ is a numeric function, then the recurrence relation for a is an equation relating a_r, for any r, to one or more a_i $(i < r)$.

A recurrence relation is also called as difference ~~ne~~ equation. To define the numeric function completely using the recurrence relation, the values of the numeric function at one or more points are required to initiate the computation.

$$a_r = a_{r-1} + 3, \quad r \geq 1 \text{ with } a_0 = 2$$

Here $a_1 = a_0 + 3 = 2 + 3 = 5$

$a_2 = a_1 + 3 = 5 + 3 = 8$

$a_3 = a_2 + 3 = 8 + 3 = 11$

$\vdots$

Hence the given recurrence relation

recursively defines the numeric function

$a = \{2, 5, 8, 11, \ldots\}$

The condition $a_0 = 2$ is the initial condition

ex. of recurrence relation
⇒ Fibonacci sequence of numbers.
It is defined by the recurrence relation.

$a_r = a_{r-2} + a_{r-1}, \quad r \geq 2.$

with the initial conditions

$a_0 = 1$ and $a_1 = 1$

Here, $a_2 = a_0 + a_1 = 2$
$a_3 = a_1 + a_2 = 3$
$a_4 = a_2 + a_3 = 5$

Thus, the fibonacci series is given by
1, 1, 2, 3, 5, 8, 13, ...

The numeric function which is computed using recurrence relation is known as the solution of the recurrence relation.
A sequence is called a solution of a recurrence relation of its terms satisfy the recurrence relation.

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ex (2) Let $\{a_n\}$ be a sequence that satisfies that recurrence relation $a_n = a_{n-1} - a_{n-2}$, for $n = 2, 3, \ldots$ and suppose that $a_0 = 3$ and $a_1 = 5$, what are a_2 and a_3?

$\Rightarrow$ Given that,

$$a_n = a_{n-1} - a_{n-2}$$

$\therefore$

$$a_0 = 3$$
$$a_1 = 5$$
$$a_2 = a_1 - a_0 = 5 - 3 = 2$$
$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

$\therefore a_2 = 2, \quad a_3 = -3$

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* Linear Recurrence relation with constant coefficients -

⇒

A recurrence relation of the form,

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + C_3 a_{r-3} + \dots + C_k a_{r-k} = f(r) \quad (1)$$

where C_i are constant, is called a linear recurrence relation with constant coefficients.

The recurrence relation in above eq^n is known as k^{th} order recurrence relation provided that both $C_0 \neq 0$ and $C_k \neq 0$.

Order = ~~lower~~ higher order - lower order

e.g. second order recurrence relation is

$$= C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} = f(r)$$

Third order recurrence relation

$$= C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + C_3 a_{r-3} = f(r)$$

Solution for eq^n (1) is,

$a_r = a_r^{(h)} + a_r^{(p)}$ where $a_r^{(h)}$ = homogenous solution, $a_r^{(P)}$ = particular solution

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* Homogeneous recurrence relation -

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + \dots + c_k a_{r-k} = 0$$

when $eq^n = 0$ (at RHS) or $f(r) = 0$ then eq^n is called as homogeneous recurrence relation.

* Non homogeneous recurrence relation -

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + \dots + c_k a_{r-k} = f(r)$$

where $f(r) \neq 0$.

examples -

1) suppose the second order homogeneous linear recurrence relation is

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} = 0$$

The characteristic equation (Auxiliary Equation) is

$$c_0 m^2 + c_1 m + c_2 = 0$$

case 1 : If roots of Auxiliary eq^n are real and unequal

say, $m_1 \neq m_2$

the general solution is

$$a_r = C_1 m_1^r + C_2 m_2^r$$

case 2 : If the roots of Auxiliary Equation are real and equal

say, $m_1 = m_2 = m$

The general solution is

$$a_r = (C_1 + r C_2) m^r$$

case 3 : If roots of Auxiliary eq^n are in complex numbers.

say,
$$m = \alpha \pm i\beta$$

The general solution is

$$a_r = (C_1 \cos r\theta + C_2 \sin r\theta) R^r$$

where
$$R = \sqrt{\alpha^2 + \beta^2}$$

and
$$\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

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example ①

solve the recurrence relation

$$a_r + 5a_{r-1} + 6a_{r-2} = 0$$

⇒

Here, we have second order linear recurrent relⁿ

$$a_r + 5a_{r-1} + 6a_{r-2} = 0 \quad -(1)$$

The characteristic equation is

$$m^2 + 5m + 6 = 0$$

$$(m+2)(m+3) = 0$$

$$m = -2, -3$$

$a_r = m^2$

$a_{r-1} = m$

② Solve the recurrence relation

$$a_r - 7a_{r-1} + 10a_{r-2} = 0 \text{ given that}$$

$$a_0 = 0, a_1 = 3$$

⇒ Given,

$$a_r - 7a_{r-1} + 10a_{r-2} = 0 \quad -(1)$$

Given that $a_0 = 0, a_1 = 3$

This is second order recurrence relation

The characteristic eqⁿ

$$m^2 - 7m + 10 = 0$$

$$(m-2)(m-5) = 0$$

$$m = 2, 5$$

The general solⁿ is

$$a_r = C_1\{(2)^r\} + C_2(5)^r \quad -(2)$$

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putting in eq^n 2

$a_0 = 0$. i.e. $a_r = 0$ & $r = 0$

$$C_1(2)^0 + C_2(5)^0 = 0$$

$$C_1 + C_2 = 0 \quad -(3)$$

again put in eq^n (2) $a_1 = 3$ i.e. $a_r = 3$, r

$$C_1(2)^1 + C_2(5)^1 = 3$$

$$2C_1 + 5C_2 = 3 \quad -(4)$$

Solving eq^n 3 & 4 we get

$C_1 = -1$ & $C2 = +1$ [by solving using calc]

The required general solution is

$$\boxed{a_r = 5^r - 2^r.}$$