$f'(a) = 2\pi i\,(8a+1)$

$f''(a) = 2\pi i\,[8]$

$f''(-i) = 2\pi i\,(8)$

$f''(-i) = 16\pi i$

17/2/15
Tuesday

Taylor's Series & Laurent's Series.

Taylor's series ~~Expansion~~ :-

If $f(z)$ is analytic in a circle C with centre z_0 then for all z inside C $f(z)$ can be expanded as Taylor's series as follows.

$$f(z) = f(z_0) + (z-z_0)f'(z_0) + \frac{(z-z_0)^2}{2!}f''(z_0) + \cdots$$

Laurent's Series:- Expansion

If $f(z)$ is not analytic then we can ~~find~~ write Laurent's series expansion of $f(z)$ as follows,

$$f(z) = \underbrace{\sum_{n=0} a_n (z-z_0)^n}_{\text{analytic part or Regular part}} + \underbrace{\sum_{n=1} b_n (z-z_0)^{-n}}_{\text{principal part.}}$$

NOTE :-

1) for analytic function we can always find the Taylor's series which contains only positive powers of $(z-z_0)$

2) For non-analytic function we can find Laurent's series expansion which contains positive & negative powers of $(z-z_0)$.

$\cancel{S.T.\ e^{z} = e \cdot e \sum_{n=0} f(z-1)^n}$

eg. 1 find Taylor's series of $f(z) = e^{z}$ at $z=1$

→ By Taylor's series we have,

$$f(z) = f(z_0) + (z-z_0) f'(z_0) + \frac{(z-z_0)}{2!} f''(z_0) + \dots$$

$z_0 = 1$

$$f(z) = f(1) + (z-1) f'(1) + \frac{(z-1)^2}{2!} f''(1) + \dots \quad \text{--- (1)}$$

$f(z) = e^{z}$ at $z=1$

$f(z) = e^{z}$ $\qquad f(1) = e$

$f'(z) = e^{z}$ $\qquad f'(1) = e$

$f''(z) = e^{z}$ $\qquad f''(1) = e$

from ①,

$$f(z) = e + (z-1)\,e + \frac{(z-1)^2}{2!}\,e + \cdots$$

$$f(z) = e\left[1 + (z-1) + \frac{(z-1)^2}{2!} + \cdots\right]$$

Important formulaes :- (For Laurent's series expansion)

1) $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

2) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$

3) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$

4) $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$ — $(|z| < 1)$

5) $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$ — $(|z| < 1)$

6) $\frac{1}{(1+z)^2} = 1 - 2z + 3z^2 - \cdots$

7) $\frac{1}{(1-z)^2} = 1 + 2z + 3z^2 + \cdots$

$\Big\}\ |z| < 1$

2) Find Taylor's series of $\cos z$ at $z = \frac{\pi}{4}$.

→ By Taylor's series we have,

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \cdots$$

Here, $z_0 = \frac{\pi}{4}$

$$f(z) = f\left(\frac{\pi}{4}\right) + \left(z - \frac{\pi}{4}\right) f'\left(\frac{\pi}{4}\right) + \frac{\left(z-\frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right) + \cdots \quad \text{—— (i)}$$

$\cancel{f(z) = \cos z}$

$\cancel{f\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4}}$

$f(z) = \cos z \qquad f\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f'(z) = -\sin z \qquad f'\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

$f''(z) = -\cos z \qquad f''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

$f'''(z) = \sin z \qquad f'''\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$f(z) = \frac{1}{\sqrt{2}} + \left(z - \frac{\pi}{4}\right)\left(-\frac{1}{\sqrt{2}}\right) + \frac{\left(z-\frac{\pi}{4}\right)^2}{2!}\left(-\frac{1}{\sqrt{2}}\right) + \frac{\left(z-\frac{\pi}{4}\right)^3}{3!}\left(\frac{1}{\sqrt{2}}\right) + \cdots$$

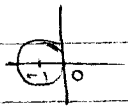
$$\therefore f(z) = \frac{1}{\sqrt{2}}\left[1 - \left(z - \pi/4\right) - \frac{\left(z-\frac{\pi}{4}\right)^2}{2!} + \frac{\left(z-\frac{\pi}{4}\right)^3}{3!} + \cdots\right]$$

6M (3) Show that $\frac{1}{z^2} = 1 + \sum_{n=1} \frac{(n+1)!}{n!}(z+1)^n$

where $|z+1| < 1$.

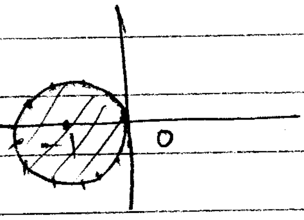
→

$f(z) = f(z_0) + (z-z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \cdots$



~~f(z)~~

Given $|z+1| < 1$ which is interior part of circle having centre have $(-1, 0)$, & radius is 1.



Clearly $z = 0$ is on the boundry of the circle.

$\therefore f(z) = \frac{1}{z^2}$ is analytic on $|z+1| < 1$

By Taylors series, we have

$f(z) = f(z_0) + (z-z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \cdots$

Here $z_0 = -1$

5M

$f(z) = f(-1) + (z+1) f'(-1) + \frac{(z+1)^2}{2!} f''(-1) + \cdots$

$f(z) = \frac{1}{z^2}$, $f(-1) = \frac{1}{(-1)^2} = 1$

$\cancel{f'(z)} = \cancel{\frac{-1}{z^2}}$ $\cancel{f'(-1)} =$

z^2

$f'(z) = (-2) z^{-3} \quad f'(-1) = (-2)(-1)^{-3}$
$= \frac{(-2)}{(-1)^3} = 2!$

$f''(z) = (-2)(-3) z^{-4} \quad f''(-1) = (-2)(-3)(-1)^{-4}$
$= (2)(3) = 3!$

$f'''(z) = (-2)(-3)(-4) z^{-5} \quad f'''(-1) = (-2)(-3)(-4) \frac{1}{(-1)^5}$
$= 2 \cdot 3 \cdot 4$
$= 4!$

$$f(z) = 1 + (z+1) 2! + \frac{(z+1)^2}{2!} 3! + \frac{(z+1)^3}{3!} 4! + \ldots$$

$$f(z) = 1 + \sum_{n=1}^{\infty} \frac{(n+1)!}{n!} (z+1)^n.$$
//

5m (4) find series of $f(z) = z^3 e^{1/z}$ at $z=0$.

$\rightarrow$ Given function $f(z) = z^3 e^{1/z}$ is not analytic at point $z=0$.
Hence we will find. Laurent's series expansion.

$$e^z = e^{1/z} = 1 + \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^3}{3!} + \ldots$$

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$$e^{z} = e^{1/z} = 1 + \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^3}{3!} + \cdots$$

$$f(z) = z^3\left[1 + \frac{1}{z} + \frac{1}{z^2 \cdot 2!} + \frac{1}{z^3 \cdot 3!} + \cdots\right]$$

$$\left[f(z) = z^3 + z^2 + \frac{z}{2!} + \frac{1}{3!} + \cdots\right] //$$

(5) $f(z) = \dfrac{e^{3z}}{(z-1)^3}$ at $z=1$

$\rightarrow$ The function $f(z)$ is not analytic at pt $z=1$

Hence we find Laurent's series expn,

$$f(z) = \frac{e^{3z}}{(z-1)^3}$$

$$= \frac{1}{(z-1)^3}\left[e^{3z}\right]$$

$$= \frac{\cancel{1}}{\cancel{(z-1)^3}} \cancel{\{e^{3z+3}}$$

$$= \frac{1}{(z-1)^3}\left[e^{3z-3+3}\right]$$

$$= \frac{1}{(z-1)^3} e^{3z-3} \cdot e^{3}$$

$$f(z) = \frac{e^3}{(z-1)^3} \quad e^{3(z-1)}$$

$$= \frac{e^3}{(z-1)^3}\left[1 + 3(z-1) + \frac{3^2(z-1)^2}{2!} + \cdots\right]$$

6 $f(z) = (z-3)\sin\left(\frac{1}{z+2}\right)$ at $z = -2$

→ The function $f(z)$ is not analytic at pt $z = -2$.

$z_0 = -2$ Hence we find Laurent's series expansion,

$z - z_0$

$z + 2$

$$f(z) = (z-3)\sin\left(\frac{1}{z+2}\right) \text{ at } z = -2$$

$$f(z) = (z+2-2-3)\sin\left(\frac{1}{z+2}\right)$$

$$= [(z+2)-5]\sin\left(\frac{1}{z+2}\right)$$

$$= (z+2)\sin\left(\frac{1}{z+2}\right) - 5\sin\left(\frac{1}{z+2}\right)$$

$$= (z+2)\left[\frac{1}{(z+2)} - \frac{\left(\frac{1}{z+2}\right)^3}{3!} + \frac{\left(\frac{1}{z+2}\right)^5}{5!} - \cdots\right]$$

$$-5\left[\frac{1}{(z+2)} - \frac{\left(\frac{1}{z+2}\right)^3}{3!} + \frac{\left(\frac{1}{z+2}\right)^5}{5!} - \cdots\right]$$

$$f(z) = (z+2)\left[\frac{1}{z+2} - \frac{1}{3!(z+2)^3} + \frac{1}{5!(z+2)^5} - \cdots\right]$$

$$-5\left[\frac{1}{(z+2)} - \frac{1}{3!(z+2)^3} + \frac{1}{5!(z+2)^5}\right]$$

19/2/15

① find Laurent's series of $f(z) = \frac{\sin z}{z-\pi}$ at $z=\pi$

→

$$f(z) = \frac{\sin z}{z-\pi}$$

$$= \frac{\sin[(z-\pi)+\pi]}{(z-\pi)}$$

$$= \frac{1}{(z-\pi)}\left[\sin(z-\pi)\cos\pi + \cos(z-\pi)\sin\pi\right]$$

$$= \frac{1}{(z-\pi)}\left[-\sin(z-\pi) + 0\right]$$

$$f(z) = \frac{-1}{z-\pi}\left[\sin(z-\pi)\right]$$

$$f(z) = \frac{-1}{(z-\pi)}\left[(z-\pi) - \frac{(z-\pi)^3}{3!} + \frac{(z-\pi)^5}{5!} \cdots\right]$$

(2) $f(z) = \dfrac{1}{z^2 \sinh z}$ at $z=0$

→

$$= \frac{1}{z^2}\left[\frac{1}{z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots}\right]$$

$$= \frac{1}{z^2 z}\left[\frac{1}{1 + \left(\frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right)}\right]$$

$$= \frac{1}{z^3}\left[1 - \left(\frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right) + \left(\frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right)^2 - \cdots\right]$$

$$= \frac{1}{z^3}\left[1 - \frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^4}{(3!)^2} \cdots\right]$$

(3) Find all possible Laurent's series expansion of the function.

$$f(z) = \frac{2 - z^2}{z(1-z)(2-z)} \text{ about } z=0$$

indicating region of convergens in each case.

→ $\cancel{f(z) = \dfrac{2+z^2}{z(z-1)(z-2)}}$

$$f(z) = \frac{2 - z^2}{z(z-1)(z-2)}$$

$$\frac{2 - z^2}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{(1-z)} + \frac{C}{(2-z)}$$

$$z = 0$$

$$2 \cancel{- z^2 = A(1-z}$$

$$2 - z^2 = A(1-z)(2-z) + B(z)(2-z) + C(z)(1-z)$$

$$z = 0$$

$$2 = A(1)(2) + 0 + 0$$

$$\boxed{A = 1}$$

$$z = 1$$

$$2 - (1)^2 = A(0) + B(1)(2-1)$$

$$2 = B(1)$$

$$\boxed{B = 2}$$

$$z = 2.$$

$$2 - (2)^2 = A(0) + B(0) + C(2)(1-2)$$

$$-2 = C(2)(-1)$$

$$\boxed{C = 1}$$

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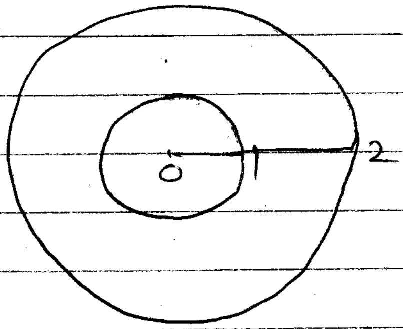
$$f(z) = \frac{2-z^2}{z(1-z)(2-z)}$$

$$f(z) = \frac{1}{z} + \frac{1}{1-z} + \frac{1}{2-z}$$

$$z=0,\ z=1,\ z=2$$



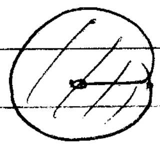
case i> $0<|z|<1$

case ii> $1<|z|<2$

case iii> $|z|>2$

case i> $0<|z|<1$, $|z|<2$

$$\frac{|z|}{2} < 1$$



$$f(z) = \frac{1}{z} + \frac{1}{1-z} + \frac{1}{2-z}$$

$$= \frac{1}{z} + \frac{1}{1-z} + \frac{1}{2\left(1-\frac{z}{2}\right)}$$

$$f(z) = \frac{1}{z} + \left(1+z+z^2+\cdots\right) + \frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right)$$

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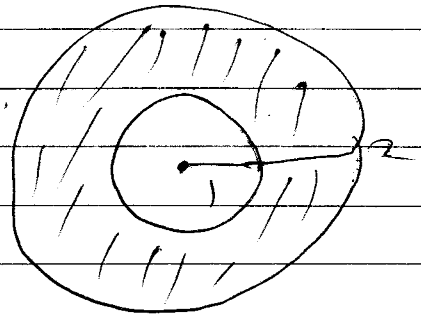
case ii> $1 < |z| < 2$

$1 < |z|$, $\frac{1}{|z|} < 1$



$|z| < 2$

$\frac{|z|}{2} < 1$

$$f(z) = \frac{1}{z} + \frac{1}{1-z} + \frac{1}{2-z}$$

$$= \frac{1}{z} + \frac{1}{z\left(\frac{1}{z} - 1\right)} + \frac{1}{2\left(1 - \frac{z}{2}\right)}$$

$$= \frac{1}{z} - \frac{1}{z\left(1 - \frac{1}{z}\right)} + \frac{1}{2\left(1 - \frac{z}{2}\right)}$$

$$= \frac{1}{z} - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right) + \frac{1}{2}\left(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots\right)$$

case iii>

$|z| > 2$

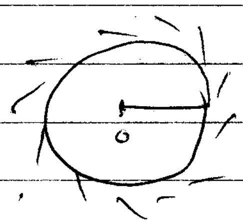
$2 < |z|$

$\frac{2}{|z|} < 1$

$|z| > 1$

$1 < |z|$

$\frac{1}{|z|} < 1$



$$f(z) = \frac{1}{z} + \frac{1}{1-z} + \frac{1}{2-z}$$

$$= \frac{1}{z} + \frac{1}{z\left(\frac{1}{z}-1\right)} + \frac{1}{z\left(\frac{2}{z}-1\right)}$$

$$= \frac{1}{z} - \frac{1}{z\left(1-\frac{1}{z}\right)} - \frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$= \frac{1}{z} - \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$$

$$- \frac{1}{z}\left(1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\cdots\right)$$

④. Find Laurent's series.

$$f(z) = \frac{z^2-1}{z^2+5z+6} \quad \text{at } z=0 \ \& \ z=1$$

→ Degree of N^R = Degree of D^R

$$z^2+5z+6 \overline{\big)\, z^2-1 \,}\ 1$$

$$-\ z^2+5z+6$$

$$-5z-7$$

$$z^2-1 = (z^2+5z+6)(1) + (-5z-7)$$

$$f(z) = \frac{z^2-1}{(z^2+5z+6)} = 1 + \frac{(-5z-7)}{(z^2+5z+6)}$$

$$f(z) = 1 + \frac{(-5z-7)}{(z+3)(z+2)}$$

$$\frac{(-5z-7)}{(z+3)(z+2)} = \frac{A}{(z+3)} + \frac{B}{(z+2)}$$

$$(-5z-7) = A(z+2) + B(z+3)$$

Put $z = -2$,

$$-5z-7 = B(-2+3)$$

$$-5z-7 = B(1)$$

$$-5(-2)-7 = B$$

$$10-7 = B$$

$$\boxed{B = 3}$$

Put $z = -3$

$$(-5z-7) = A(z+2) + B(z+3)$$

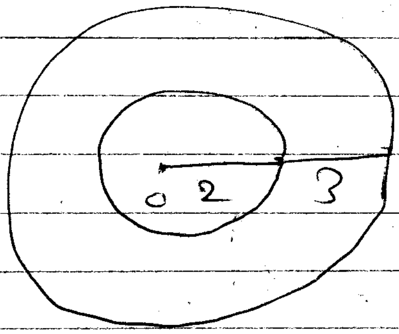
$$-5(-3)-7 = A(-3+2) + 0$$

$$15-7 = A(-1)$$

$$\boxed{-8 = A}$$

$$\frac{(-5z-7)}{(z+3)(z+2)} = \frac{-8}{(z+3)} + \frac{3}{(z+2)}$$

$z=0,\ z=-2,\ z=-3$



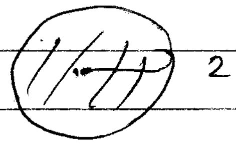
$f(z) =$

$$= 1 - \frac{8}{z+3} + \frac{3}{z+2}$$

case i> $|z|<2$
case ii> $2<|z|<3$
case iii> $|z|>3$

case i> $|z|<2$, $|z|<3$

$\frac{|z|}{2}<1$ $\quad \frac{|z|}{3}<1$



$$f(z) = 1 - \frac{8}{z+3} + \frac{3}{z+2}$$

$$= 1 - \frac{8}{3\left(\frac{z}{3}+1\right)} + \frac{3}{2\left(\frac{z}{2}+1\right)}$$

$$= 1 - \frac{8}{3}\left(\frac{1}{1+\frac{z}{3}}\right) + \frac{3}{2}\left(\frac{1}{1+\frac{z}{2}}\right)$$

$$= 1 - \frac{8}{3}\left[1 - \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 - \cdots\right]$$

$$+ \frac{3}{2}\left[1 - \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 - \cdots\right]$$

case ii) $2<|z|<3$

$2<|z| \qquad |z|<3$

$\frac{2}{|z|}<1 \qquad \frac{|z|}{3}<1$

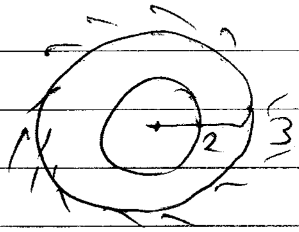
$$f(z)=1-\frac{8}{z+3}+\frac{3}{z+2}$$

$$=1-\frac{8}{3\left(\frac{z}{3}+1\right)}+\frac{3}{z\left(1+\frac{2}{z}\right)}$$

$$=1-\frac{8}{3}\left(\frac{1}{1+\frac{z}{3}}\right)+\frac{3}{z}\left(\frac{1}{1+\frac{2}{z}}\right)$$

$$=1-\frac{8}{3}\left(1-\frac{z}{3}+\left(\frac{z}{3}\right)^2+\cdots\right)+\frac{3}{z}\left(1-\frac{2}{z}+\left(\frac{2}{z}\right)^2-\cdots\right)$$

case iii)



$|z|>3$

$3<|z|$

$\frac{3}{|z|}<1$

$\Rightarrow |z|>2$

$2<|z|$

$\frac{2}{|z|}<1$

$$f(z)=1-\frac{8}{z+3}+\frac{3}{z+2}$$

$$f(z) = 1 - \frac{8}{z\left(1+\frac{3}{z}\right)} + \frac{3}{z\left(1+\frac{2}{z}\right)}$$

$$= 1 - \frac{8}{z}\left(1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - \dots\right) + \frac{3}{z}\left(1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \dots\right)$$

$$z = 1$$

$$f(z) = \frac{z^2-1}{z^2+5z+6}$$

$$f(z) = 1 - \frac{8}{z+3} + \frac{3}{z+2}$$

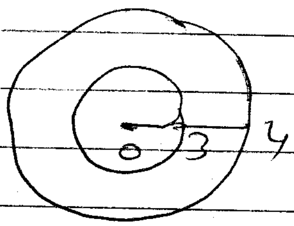
$$= 1 - \frac{8}{z-1+1+3} + \frac{3}{z-1+1+2}$$

$$f(z) = 1 - \frac{8}{(z-1)+4} + \frac{3}{(z-1)+3}$$

$$z-1 = u$$

$$f(z) = 1 - \frac{8}{u+4} + \frac{3}{u+3}$$

1. $|u| < 3$
2. $3 < |u| < 4$
3. $|u| > 4$



Prof. Nancy Sinollin

$$f(z) = 1 - \frac{8}{u+4} + \frac{3}{u+3}$$

$$= 1 - \frac{8}{4\left(\frac{u}{4}+1\right)} + \frac{3}{3\left(\frac{u}{3}+1\right)}$$

$$= 1 - \frac{8}{4\left(\frac{u}{4}+1\right)} + \frac{3}{3\left(\frac{u}{3}+1\right)}$$

$$= 1 - 2\left(1 - \frac{u}{4} + \left(\frac{u}{4}\right)^2 - \cdots\right)$$
$$+ \left(1 - \frac{u}{3} + \left(\frac{u}{3}\right)^2 - \cdots\right)$$

$$= 1 - 2\left[1 - \frac{(z-1)}{4} + \frac{(z-1)^2}{4} - \cdots\right]$$
$$+ \left[1 - \frac{(z-1)}{3} + \frac{(z-1)^2}{3} - \cdots\right]$$

case ii> $3 < |u| < 4$

$3 < |u|$ $\qquad$ $|u| < 4$

$\frac{3}{|u|} < 1$ $\qquad$ $\frac{|u|}{4} < 1$

$$f(z) = 1 - \frac{8}{u+4} + \frac{3}{u+3}$$

$$= 1 - \frac{8}{4\left(\frac{u}{4}+1\right)} + \frac{3}{u\left(1+\frac{3}{u}\right)}$$

$$\cancel{= 1 - \frac{8}{4}}$$

$$= 1 - 2\left[1 - \frac{u}{4} + \left(\frac{u}{4}\right)^2 - \cdots\right]$$

$$+ \frac{3}{u}\left[1 - \frac{3}{u} + \left(\frac{3}{u}\right)^2 - \cdots\right]$$

$$u = z-1$$

$$f(z) = 1 - 2\left[1 - \frac{(z-1)}{4} + \left(\frac{z-1}{4}\right)^2 - \cdots\right]$$

$$+ \frac{3}{(z-1)}\left[1 - \frac{3}{(z-1)} + \left[\frac{3}{(z-1)}\right]^2 - \cdots\right]$$

Case iii>

$$|u| > 4 \qquad\qquad |u| > 3$$

$$4 < |u| \qquad\qquad 3 < |u|$$

$$\frac{4}{|u|} < 1 \qquad\qquad \frac{3}{|u|} < 1$$

$$= 1 - \frac{8}{u+4} + \frac{3}{u+3}$$

$$= 1 - \frac{8}{u\left(1+\frac{4}{u}\right)} + \frac{3}{u\left(1+\frac{3}{u}\right)}$$

$$= 1 - \frac{8}{u}\left(\frac{1}{1+\frac{4}{u}}\right) + \frac{3}{u}\left(\frac{1}{1+\frac{3}{u}}\right)$$

$$= 1 - \frac{8}{u}\left[1 - \frac{4}{u} + \left(\frac{4}{u}\right)^2 - \cdots\right]$$

$$u = z-1$$

$$= 1 - \frac{8}{(z-1)}\left[1 - \frac{4}{(z-1)} + \left(\frac{4}{z-1}\right)^2 - \cdots\right]$$

$$+ \frac{3}{(z-1)}\left[1 - \frac{3}{z-1} + \left(\frac{3}{z-1}\right)^2 - \cdots\right]$$

5) $f(z) = \dfrac{1}{(z-1)(z-2)}$ in the regions

i) $|z-1| < 1$

ii) $1 < |z-3| < 2$

iii) $|z| < 1$

→ $f(z) = \dfrac{1}{(z-1)(z-2)}$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$1 = A(z-2) + B(z-1)$$

$$z = 2$$

$$1 = A(0) + B(2-1)$$

$$1 = B(1)$$

$$\boxed{B = 1}$$

$$z = 1$$

$$1 = A(1-2) + B(0)$$

$$\cancel{1 = A(1-1}$$

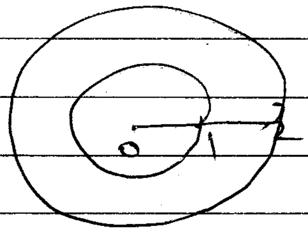
$$1 = A(-1)$$

$$\boxed{A = -1}$$

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

$$f(z) = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

$$z = 1, \quad z = 2$$



case i) $|z-1| < 1$ $\qquad$ $z - 1 = u$

$$|u| < 1$$

$$f(z) = \frac{-1}{z-1} + \frac{1}{z-1+1-2}$$

$$= \frac{-1}{z-1} + \frac{1}{(z-1)-1}$$

$$= \frac{-1}{u} + \frac{1}{u-1}$$

$$= -\frac{1}{u} - \frac{1}{1-u}$$

$$= -\frac{1}{u} - \left[1 + u + \frac{u^2}{2} + \cdots\right]$$

$$f(z) = \frac{-1}{(z-1)} - \left[1 + (z-1) + (z-1)^2 + \cdots\right]$$

case ii) $1 < |z-3| < 2$

$1 < |z-3| \qquad |z-3| < 2$

$$z - 3 = u$$

$1 < |u| \qquad |u| < 2$

$\frac{1}{|u|} < 1 \qquad \frac{|u|}{2} < 1$

$$f(z) = \frac{-1}{(z-1)} + \frac{1}{z-2}$$

$$= \frac{-1}{(z-3)+3-1} + \frac{1}{(z-3)+3-2}$$

$$f(z) = \frac{-1}{(z-3)+2} + \frac{1}{(z-3)+1}$$

$$= \frac{-1}{u+2} + \frac{1}{u+1}$$

$$= \frac{-1}{2\left(\frac{u}{2}+1\right)} + \frac{1}{u\left(1+\frac{1}{u}\right)}$$

$$= \frac{-1}{2}\left(\frac{1}{1+\frac{u}{2}}\right) + \frac{1}{u}\left(\frac{1}{1+\frac{1}{u}}\right)$$

$$= \frac{-1}{2}\left[1 - \frac{u}{2} + \left(\frac{u}{2}\right)^2 - \cdots\right]$$

$$+ \frac{1}{u}\left[1 - \frac{1}{u} + \left(\frac{1}{u}\right)^2 - \cdots\right]$$

$$u = z-3$$

$$f(z) = \frac{-1}{2}\left[1 - \frac{(z-3)}{2} + \left(\frac{z-3}{2}\right)^2 - \cdots\right]$$

$$\cancel{u = z-3} \quad + \frac{1}{u}\left[1 - \frac{1}{u} + \left(\frac{1}{u}\right)^2 - \cdots\right]$$

$$u = z-3$$

$$f(z) = \frac{-1}{2}\left[1 - \frac{(z-3)}{2} + \left(\frac{z-3}{2}\right)^2 - \cdots\right]$$

$$+ \frac{1}{(z-3)}\left[1 - \frac{1}{z-3} + \left(\frac{1}{z-3}\right)^2 - \cdots\right]$$

case iii>

$|z| < 1$ $\quad\quad |z| < 2$

$\frac{|z|}{2} < 1$

$$f(z) = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$= \frac{1}{1-z} + \frac{1}{2\left(\frac{z}{2}-1\right)}$$

$$= \frac{1}{1-z} + \frac{1}{2\left(1-\frac{z}{2}\right)}$$

$$= \frac{1}{1-z} - \frac{1}{2}\left(\frac{1}{1-\frac{z}{2}}\right)$$

$$f(z) = \left[1+z+z^2 - \dots\right] - \frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\dots\right]$$

(1) $f(z) = \dfrac{1}{(1-z)^2(2-z)}$ at $z=0$

$\rightarrow$

$$\frac{1}{(1-z)^2(2-z)} = \frac{A}{(1-z)} + \frac{B}{(1-z)^2} + \frac{C}{(2-z)}$$

$$\frac{1}{(1-z)^2(2-z)} = A(1-z)(2-z) + B\cancel{(1-z)}(2-z) + C\cancel{(1-z)}(1-z)^2$$

$$1 = A(1-z)(2-z) + B(2-z) + C(1-z)^2$$

Put $z=1$

$$1 = A(0) + B(2-1) + C(0)$$

$$1 = B(1)$$

$$\boxed{B=1}$$

put $z=2$

$$1 = A(1-2)(0) + B(0) + C(1-2)^2$$

$$1 = C(-1)^2$$

$$1 = C(1)$$

$$\boxed{C=1}$$

Put $z = 0$

$$1 = A(1-0)(2-0) + B(2-0) + C(1-0)^2$$

$$1 = 2A + 2B + C$$

$$1 = 2A + 2(1) + 1$$

$$1 = 2A + 3$$

$$\boxed{A = -1}$$

$$f(z) = \frac{-1}{(1-z)} + \frac{1}{(1-z)^2} + \frac{1}{(2-z)}$$

i) $|z| < 1$, $|z| < 2$

$\frac{|z|}{2} < 1$

$$f(z) = \frac{-1}{(1-z)} + \frac{1}{(1-z)^2} + \frac{1}{(2-z)}$$

$$= \frac{-1}{(1-z)} + \frac{1}{(1-z)^2} + \frac{1}{2}\left(\frac{1}{1-\frac{z}{2}}\right)$$

$$= -(1+z+z^2+\cdots) + (1+2z+3z^2+\cdots) + \frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right).$$

case ii> $1 < |z| < 2$

$1 < |z|$ $\quad\quad$ $|z| < 2$

$\frac{1}{|z|} < 1$ $\quad\quad$ $\frac{|z|}{2} < 1$

$$f(z) = \frac{-1}{(1-z)} + \frac{1}{(1-z)^2} + \frac{1}{(2-z)}$$

$$= \frac{-1}{z\left(\frac{1}{z} - 1\right)} + \frac{1}{\left[z\left(\frac{1}{z} - 1\right)\right]^2} + \frac{1}{2\left(1 - \frac{z}{2}\right)}$$

$$= \frac{1}{z\left(1 - \frac{1}{z}\right)} + \frac{1}{\left[z\left(1 - \frac{1}{z}\right)\right]^2} + \frac{1}{2\left(1 - \frac{z}{2}\right)}$$

$$= \frac{1}{z}\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \cdots\right] + \frac{1}{z^2}\left[1 + \left(\frac{2}{z}\right) + 3\left(\frac{1}{z}\right)^2 + \cdots\right] + \frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots\right]$$

case iii> $|z| > 2$ $\quad\quad$ $|z| > 1$

$2 < |z|$ $\quad\quad$ $1 < |z|$

$\frac{2}{|z|} < 1$ $\quad\quad$ $\frac{1}{|z|} < 1$

~~case~~

$$f(z) = \frac{-1}{(1-z)} + \frac{1}{(1-z)^2} + \frac{1}{(2-z)}$$

$$= \frac{-1}{z\left(\frac{1}{z}-1\right)} + \frac{1}{\left[z\left(\frac{1}{z}-1\right)\right]^2} + \frac{1}{z\left(\frac{2}{z}-1\right)}$$

$$= \frac{-1}{z\left(\frac{1}{z}-1\right)} + \frac{1}{z^2\left(\frac{1}{z}-1\right)^2} + \frac{1}{z\left(\frac{2}{z}-1\right)}$$

$$= \frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{z^2\left(1-\frac{1}{z}\right)^2} - \frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$= \frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right] + \frac{1}{z^2}\left[1+\frac{2}{z}+3\left(\frac{1}{z}\right)^2+\cdots\right]$$

$$- \frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right]$$