Strategies in Game Theory

Pure and Mixed Strategies

Definition and Examples

- **Pure Strategies**:
  - A pure strategy is a specific predetermined action that a player will follow. In each situation of the game, the player chooses the same action.
  - **Example**: In the game of rock-paper-scissors, choosing "rock" every time is a pure strategy.
- **Mixed Strategies**:
  - A mixed strategy is a probability distribution over possible pure strategies. Instead of choosing one specific action, a player randomly selects among available actions according to assigned probabilities.
  - **Example**: In rock-paper-scissors, choosing rock with probability $\frac{1}{3}$, paper with probability $\frac{1}{3}$, and scissors with probability $\frac{1}{3}$ is a mixed strategy.

How to Represent Strategies

- **Pure Strategy Representation**:
  - Each pure strategy can be represented as a specific action or a tuple in a matrix.
  - **Example**: In a 2x2 matrix game, pure strategies for Player 1 could be the rows (A1, A2) and for Player 2, the columns (B1, B2).
- **Mixed Strategy Representation**:
  - A mixed strategy is represented as a vector of probabilities.
  - **Example**: If Player 1 has two strategies A1 and A2, a mixed strategy might be represented as $(p, 1-p)$ where p is the probability of choosing A1 and $1-p$ the probability of choosing A2.

Example:

- Consider a simple game with two players. Player 1 can choose A or B, and Player 2 can choose X or Y. The payoff matrix is:

	X	Y
A	(3,2)	(1,4)
B	(2,3)	(4,1)

  - Pure strategy for Player 1: A or B.
  - Pure strategy for Player 2: X or Y.
  - Mixed strategy for Player 1: (p_A, p_B).
  - Mixed strategy for Player 2: (p_X, p_Y).

Dominant Strategies and Nash Equilibrium

Finding Dominant Strategies

- **Dominant Strategy**: A strategy is dominant if it yields a higher payoff than any other strategy, regardless of what the other players do.
  - **Example**: In the Prisoner's Dilemma, each player has two strategies: confess (C) or remain silent (S). The payoff matrix might look like:

	C	S
C	(-1,-1)	(0,-3)
S	(-3,0)	(1,1)

    - Here, "C" is a dominant strategy for both players because confessing always results in a better or equal payoff than remaining silent, regardless of the other player's choice.

Definition and Significance of Nash Equilibrium

- **Nash Equilibrium**:
  - A set of strategies, one for each player, such that no player has an incentive to unilaterally change their strategy. In other words, given the strategies of the other players, each player's strategy is the best response.
  - **Significance**: Nash equilibrium represents a state of mutual best responses, providing a stable outcome where players do not benefit from deviating.
- **Finding Nash Equilibrium**:
  - **Best Response Analysis**: Identify the best response for each player to every possible strategy of the opponents.

- **Intersection of Best Responses**: The strategies that are mutual best responses form a Nash equilibrium.

Examples and Case Studies

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- **Example 1: Battle of the Sexes**:
  - Players: Husband (H) and Wife (W).
  - Strategies: H can choose (Opera, Football), W can choose (Opera, Football).
  - Payoff matrix:

	Opera	**Football**
Opera	(2,1)	(0,0)
Football	(0,0)	(1,2)

    - Nash Equilibria: (Opera, Opera) and (Football, Football), as both are stable outcomes where neither player wants to deviate unilaterally.

- **Example 2: Cournot Duopoly**:
  - Firms choose quantities q_1 and q_2 to produce.
  - Demand function: $P = 100 - (q_1 + q_2)$.
  - Cost functions: $C(q_1) = 20q_1, C(q_2) = 20q_2$.
  - Profit functions: $\pi_1 = q_1(100 - q_1 - q_2) - 20q_1, \pi_2 = q_2(100 - q_1 - q_2) - 20q_2$.
  - Best response functions:

$$q_1 = \frac{80 - q_2}{2}, \quad q_2 = \frac{80 - q_1}{2}$$

  - Nash Equilibrium: Solve the system of equations to get $q_1^* = q_2^* = 20$.

Examples

- **Prisoner's Dilemma** revisited:
  - **Strategies**: Confess (C), Silent (S).
  - **Nash Equilibrium**: Both prisoners choose to confess (C, C), as it is the dominant strategy.
- **Matching Pennies**:
  - Players simultaneously choose either heads (H) or tails (T).
  - Payoffs: If both choose the same, Player 1 wins; otherwise, Player 2 wins.
  - **Mixed Strategy Nash Equilibrium**: Each player randomizes with equal probability $\frac{1}{2}$ between H and T.

Summary

- **Pure Strategies**: Fixed actions chosen by players.
- **Mixed Strategies**: Probabilistic combination of pure strategies.
- **Dominant Strategies**: Best strategy regardless of opponents' actions.
- **Nash Equilibrium**: Stable outcome where no player benefits from deviating unilaterally