**Matriculation Number:** 

| A | | | | | | | | |
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NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2018-2019

**MA1512**

**DIFFERENTIAL EQUATIONS FOR ENGINEERING**

November 2018 Time allowed: 1 hour 30 minutes

# INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** Do not write your name anywhere in this booklet. This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.

2. This examination paper consists of **FOUR (4)** questions and comprises **NINE (9)** printed pages.

3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question. The marks for each question are indicated at the beginning of the question. The maximum possible total score for this examination paper is 40 marks.

4. This is a **closed book (with authorized material)** examination. Students are only allowed to bring into the examination hall **ONE** piece A4 size help-sheet which can be used on both sides.

5. Candidates may use any calculators that satisfy MOE A-Level examination guidelines. However, they should lay out systematically the various steps in the calculations.

**For official use only. Do not write below this line.**

| Question | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Marks | | | | |

## Question 1 [10 marks]

(a) A fossilized bone is found to contain 82% of the original amount of Carbon-14. We know that the half-life of Carbon-14 is 5600 years. Find the age of this fossilized bone. Give your answer in years correct to the nearest integer.

(b) Let $y(x)$ denote the solution of the differential equation $2x\frac{dy}{dx} = xy + y - 6y^3$ with $x > 0$, $y > 0$ and $y(1) = \frac{1}{3}$. Find the value of $y(2)$. Give your answer correct to two decimal places.

| Answer 1(a) | 1603 | Answer 1(b) | 0.53 |
|---|---|---|---|

*(Show your working below and on the next page.)*

(a) $\frac{dy}{dt} = -ky$

$y = Ae^{-kt}$

$k = \frac{\ln 2}{5600}$

$0.82 = e^{-kt}$

$\Rightarrow t = -\frac{\ln 0.82}{k}$

$= -\frac{5600 \ln 0.82}{\ln 2}$

$= 1603.3\ldots$

$\approx \underline{1603}$

(b) $\frac{dy}{dx} - \frac{x+1}{2x} y = -\frac{3}{x} y^3$

let $z = y^{1-3} = y^{-2} \Rightarrow \frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}$

$\therefore \frac{dz}{dx} + \frac{x+1}{x} z = \frac{6}{x}$

$R = e^{\int (1+\frac{1}{x})dx} = e^{x+\ln x} = xe^x$

$\frac{1}{y^2} = z = \frac{1}{xe^x}\int 6e^x dx = \frac{1}{xe^x}(6e^x + c)$

$\therefore y = \sqrt{\frac{xe^x}{6e^x+c}}$ $(\because y > 0)$

$y(1) = \frac{1}{3} \Rightarrow c = 3e \Rightarrow y = \sqrt{\frac{xe^x}{6e^x+3e}}$

$\therefore y(2) = \sqrt{\frac{2e^2}{6e^2+3e}} = \sqrt{\frac{2e}{6e+3}}$

$= 0.530\ldots$

$\approx \underline{0.53}$

## Question 2 [10 marks]

(a) Let $y(x)$ denote the solution of the differential equation $y'' - 3y' - 4y = 5e^{4x}$ with $y(0) = 3$ and $y(1) = \frac{2(1+e^5)}{e}$. Find the value of $y\left(\frac{1}{4}\right)$. Give your answer correct to two decimal places.

(b) A certain bacterial population follows a logistic growth model with initial population 1000 and reaches a logistic equilibrium population of 10000. It is known that at time $t = 1$ hour there are 2000 bacteria. Find the number of bacteria at time $t = 3$ hour. Give your answer correct to the nearest integer.

| Answer 2(a) | 4.96 | Answer 2(b) | 5586 |
|---|---|---|---|

*(Show your working below and on the next page.)*

(a) $\lambda^2 - 3\lambda - 4 = 0$

$(\lambda-4)(\lambda+1) = 0$

Try $y = Axe^{4x}$

$y' = Ae^{4x} + 4Axe^{4x}$

$y'' = 8Ae^{4x} + 16Axe^{4x}$

$y'' - 3y' - 4y = 5Ae^{4x}$

$\therefore A = 1$

$\therefore y = C_1e^{4x} + C_2e^{-x} + xe^{4x}$

$y(0) = 3 \Rightarrow C_1 + C_2 = 3$

$y(1) = \frac{2(1+e^5)}{e} \Rightarrow C_1e^4 + C_2e^{-1} + e^4 = \frac{2(1+e^5)}{e}$

$\therefore C_1 = 1,\ C_2 = 2$

$\therefore y = e^{4x} + 2e^{-x} + xe^{4x}$

$y(\frac{1}{4}) = e + 2e^{-\frac{1}{4}} + \frac{1}{4}e$

$= 4.955\ldots$

$\approx \underline{\underline{4.96}}$

(b) $N = \dfrac{10000}{1 + \left(\frac{10000}{1000} - 1\right)e^{-Bt}} = \dfrac{10000}{1 + 9e^{-Bt}}$

$\therefore 2000 = \dfrac{10000}{1 + 9e^{-B}}$

$\therefore 9e^{-B} = 4 \Rightarrow B = -\ln\frac{4}{9}$

$\therefore t = 3$

$\Rightarrow N = \dfrac{10000}{1 + 9e^{(\ln\frac{4}{9})\times 3}}$

$= 5586.2\ldots$

$\approx \underline{\underline{5586}}$

## Question 3 [10 marks]

(a) Let $a$ and $b$ denote two positive constants with $a < 2$. A particle is moving along the positive part of the $x$-axis away from the origin with an acceleration equal to $\frac{497}{x^2}$ metre/second$^2$ when it is at a distance $x$ metre from the origin. It is known that initially at time $t = 0$ second the particle is at rest at a distance $a$ metre from the origin. It is observed that at time $t = b$ second the particle is at a distance 2 metre from the origin and its velocity at that moment is 39 metre per second. Find the value of $a$. Give your answer correct to two decimal places.

(b) Let $f(t) = 1000(\cos t)(u(t-1))$ where $u$ denotes the unit step function. Let $F(s) = L(f(t))$ be the Laplace transform of $f(t)$. Find the value of $F(2)$. Give your answer correct to two decimal places.

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| Answer 3(a) | 0.49 | Answer 3(b) | 6.47 |
|---|---|---|---|

*(Show your working below and on the next page.)*

(a) $\frac{d^2x}{dt^2} = \frac{497}{x^2} \Rightarrow \frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) = \frac{497}{x^2}$

$\int_0^{39} d\left(\frac{1}{2}\dot{x}^2\right) = \int_a^2 \frac{497}{x^2}dx$

$\frac{1}{2}(39)^2 = -\frac{497}{x}\Big|_a^2$

$= -\frac{497}{2} + \frac{497}{a}$

$\frac{497}{a} = \frac{39^2+497}{2} = \frac{2018}{2}$

$a = \frac{497}{1009} = 0.492\ldots$

$\approx \underline{\underline{0.49}}$

(b) $f(t) = 1000\cos[(t-1)+1]u(t-1)$

$= 1000\{\cos 1 \cos(t-1)u(t-1) - \sin 1 \sin(t-1)u(t-1)\}$

$L(f(t)) = 1000\left\{(\cos 1)\frac{s}{s^2+1}e^{-s} - (\sin 1)\frac{1}{s^2+1}e^{-s}\right\}$

$F(2) = 1000\left\{(\cos 1)\frac{2}{5}e^{-2} - (\sin 1)\frac{1}{5}e^{-2}\right\}$

$= 6.472\ldots$

$\approx \underline{\underline{6.47}}$

## Question 4 [10 marks]

(a) Let $y(t)$ be the solution to the differential equation $y'' + 2y' + 2y = (\delta(t-\pi))\, u(t-\pi)$ with $y(0) = 1$ and $y'(0) = 0$, where $\delta$ denotes the delta function and $u$ denotes the unit step function. Find the value of $y\left(\frac{18}{5}\right)$. Give your answer correct to two decimal places.

(b) Let $u(x, y)$ denote a solution to the PDE $u_{xy} = xyu$, with $u(1,1) = 3e^{\frac{5}{4}}$ and $u(2,2) = 3e^5$, found by using the separation of variable method. Find the value of $\ln(u(3,3))$. Give your answer correct to two decimal places.

| Answer 4(a) | 0.24 | Answer 4(b) | 12.35 |
|---|---|---|---|

*(Show your working below and on the next page.)*

(a) Let $Y = L(y)$.

$s^2Y - s + 2sY - 2 + 2Y = e^{-\pi s}$

$Y = \dfrac{s+2+e^{-\pi s}}{s^2+2s+2}$

$= \dfrac{s+1}{(s+1)^2+1} + \dfrac{1}{(s+1)^2+1} + \dfrac{e^{-\pi s}}{(s+1)^2+1}$

$\therefore y = e^{-t}\cos t + e^{-t}\sin t + \{e^{-(t-\pi)}\sin(t-\pi)\}u(t-\pi)$

$y\left(\frac{18}{5}\right) = e^{-\frac{18}{5}}\cos\frac{18}{5} + e^{-\frac{18}{5}}\sin\frac{18}{5} + e^{-(\frac{18}{5}-\pi)}\sin\left(\frac{18}{5}-\pi\right)$

$= 0.243\ldots$

$\approx \underline{\underline{0.24}}$

(b) Let $u = XY$

$X'Y' = xyXY$

$\dfrac{X'}{xX} = \dfrac{yY}{Y'} = k$

$\dfrac{X'}{X} = kx \Rightarrow \ln|X| = \frac{1}{2}kx^2 + C \Rightarrow X = Ae^{\frac{1}{2}kx^2}$

$\dfrac{Y'}{Y} = \dfrac{y}{k} \Rightarrow Y = Be^{\frac{1}{2k}y^2}$

$\therefore u = Ce^{\frac{1}{2}kx^2 + \frac{1}{2k}y^2}$

$\therefore Ce^{\frac{1}{2}k + \frac{1}{2k}} = 3e^{\frac{5}{4}}$

$Ce^{2k + \frac{2}{k}} = 3e^5$

$\therefore e^{2k + \frac{2}{k} - \frac{1}{2}k - \frac{1}{2k}} = e^{5 - \frac{5}{4}}$

$\therefore 2k^2 - 5k + 2 = 0 \Rightarrow k = \frac{1}{2} \text{ or } 2 \Rightarrow C = 3$

$\therefore u = 3e^{\frac{1}{4}x^2 + y^2}$ or $3e^{x^2 + \frac{1}{4}y^2}$

$u(3,3) = 3e^{\frac{45}{4}}$

$\ln u(3,3) = \frac{45}{4} + \ln 3 = 12.348\ldots$

$\approx \underline{\underline{12.35}}$