ELEMENTARY MATRICES (PART I)

Discussion

Recall that there are three types of elementary row operations.

(1) Multiplying the i^{th} row by a non zero constant c. (cR_i)

(2) Interchanging the i^{th} and j^{th} row. ($R_i \leftrightarrow R_j$)

(3) Adding k times the i^{th} row to the j^{th} row. ($R_j + kR_i$)

What happens when these elementary row operations were performed on an identity matrix $\boldsymbol{I}_m$?

First type of E.R.O.

(1) Multiplying the i^{th} row by a non zero constant c. (cR_i)

$$\boldsymbol{I}_m \xrightarrow{\quad cR_i \quad} \begin{array}{cl} \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & \cdots & \cdots & 0 \\ \vdots & & c & & & \vdots \\ \vdots & & & 1 & & \vdots \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix} & \begin{array}{l} \\ \\ \leftarrow i^{\text{th}} \text{ row} \\ \\ \\ \\ \end{array} \end{array} = \boldsymbol{E}_1$$

What if we pre-multiply $\boldsymbol{E}_1$ to a $m \times n$ matrix $\boldsymbol{B}$?

First type of E.R.O.

(1) Multiplying the i^{th} row by a non zero constant c. (cR_i)

i^{th} row

$$\overbrace{\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & \cdots & \cdots & 0 \\ \vdots & & c & & & \vdots \\ \vdots & & & 1 & & \vdots \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix}}^{m \text{ columns}} \quad m\overset{n}{\begin{pmatrix} \boxed{\text{row } i \text{ of } \boldsymbol{B}} \\ \boldsymbol{B} \\ \\ \\ \end{pmatrix}} = \begin{pmatrix} \boxed{c \times \text{row } i} \\ \\ \\ \\ \end{pmatrix}$$

$$\xrightarrow{cR_i}$$

First type of E.R.O.

It seems like we can 'represent' performing the elementary row operation cR_i on $\boldsymbol{B}$ by pre-multiplying a suitable matrix to $\boldsymbol{B}$.

$$\boldsymbol{B} \xrightarrow{cR_i} \boldsymbol{C}_1$$

$$\boldsymbol{I}_m \xrightarrow{cR_i} \boldsymbol{E}_1$$

Then we have $\boldsymbol{E}_1\boldsymbol{B} = \boldsymbol{C}_1$.

Second type of E.R.O.

(2) Interchanging the i^{th} and j^{th} row. ($R_i \leftrightarrow R_j$)

$$
\boldsymbol{I}_m \xrightarrow{R_i \leftrightarrow R_j}
\begin{array}{rc}
 & \begin{array}{ccccccc} \quad & \quad & i^{\text{th}}\text{ col} & \quad & j^{\text{th}}\text{ col} & \quad & \quad \end{array} \\
\begin{array}{r} \\ \\ i^{\text{th}}\text{ row} \\ \\ j^{\text{th}}\text{ row} \\ \\ \\ \end{array} &
\begin{pmatrix}
1 & & & & & & \\
 & \ddots & & & & & \\
 & & 0 & & 1 & & \\
 & & & \ddots & & & \\
 & & 1 & & 0 & & \\
 & & & & & \ddots & \\
 & & & & & & 1
\end{pmatrix}
\end{array}
= \boldsymbol{E}_2
$$

What if we pre-multiply $\boldsymbol{E}_2$ to a $m \times n$ matrix $\boldsymbol{B}$?

Second type of E.R.O.

(2) Interchanging the i^{th} and j^{th} row. $(R_i \leftrightarrow R_j)$

$$\begin{pmatrix}1 &&&&&&\\ &\ddots&&&&&\\ &&0&&1&&\\ &&&\ddots&&&\\ &&1&&0&&\\ &&&&&\ddots&\\ &&&&&&1\end{pmatrix}\begin{array}{l} \\ \\ \leftarrow i^{\text{th}}\text{ row}\\ \\ \leftarrow j^{\text{th}}\text{ row}\\ \\ \\ \end{array}
\begin{array}{cc} & n \\ m & \begin{pmatrix}\text{row } i \text{ of } \boldsymbol{B}\\ \boldsymbol{B}\\ \text{row } j \text{ of } \boldsymbol{B}\end{pmatrix}\end{array} = \begin{pmatrix}\text{row } j \text{ of } \boldsymbol{B}\\ \\ \text{row } i \text{ of } \boldsymbol{B}\end{pmatrix}$$

$$R_i \leftrightarrow R_j$$

Second type of E.R.O.

It seems like we can 'represent' performing the elementary row operation $R_i \leftrightarrow R_j$ on $\boldsymbol{B}$ by pre-multiplying a suitable matrix to $\boldsymbol{B}$.

$$\boldsymbol{B} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{C}_2$$

$$\boldsymbol{I}_m \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E}_2$$

Then we have $\boldsymbol{E}_2\boldsymbol{B} = \boldsymbol{C}_2$.

Third type of E.R.O.

(3) Adding k times the i^{th} row to the j^{th} row. $(R_j + kR_i)$

$$
\boldsymbol{I}_m \xrightarrow{R_j + kR_i}
\begin{array}{r c}
 & \begin{array}{ccccccc} & & i^{\text{th}}\text{ col} & & j^{\text{th}}\text{ col} & & \end{array} \\
\begin{array}{r} \\ \\ i^{\text{th}}\text{ row} \\ \\ j^{\text{th}}\text{ row} \\ \\ \\ \end{array} &
\begin{pmatrix}
1 & & & & & & \\
 & \ddots & & & & & \\
 & & 1 & & 0 & & \\
 & & & \ddots & & & \\
 & & \boxed{k} & & 1 & & \\
 & & & & & \ddots & \\
 & & & & & & 1
\end{pmatrix}
\end{array}
= \boldsymbol{E}_3
$$

$(i < j)$

What if we pre-multiply $\boldsymbol{E}_3$ to a $m \times n$ matrix $\boldsymbol{B}$?

Third type of E.R.O.

(3) Adding k times the i^{th} row to the j^{th} row. $(R_j + kR_i)$

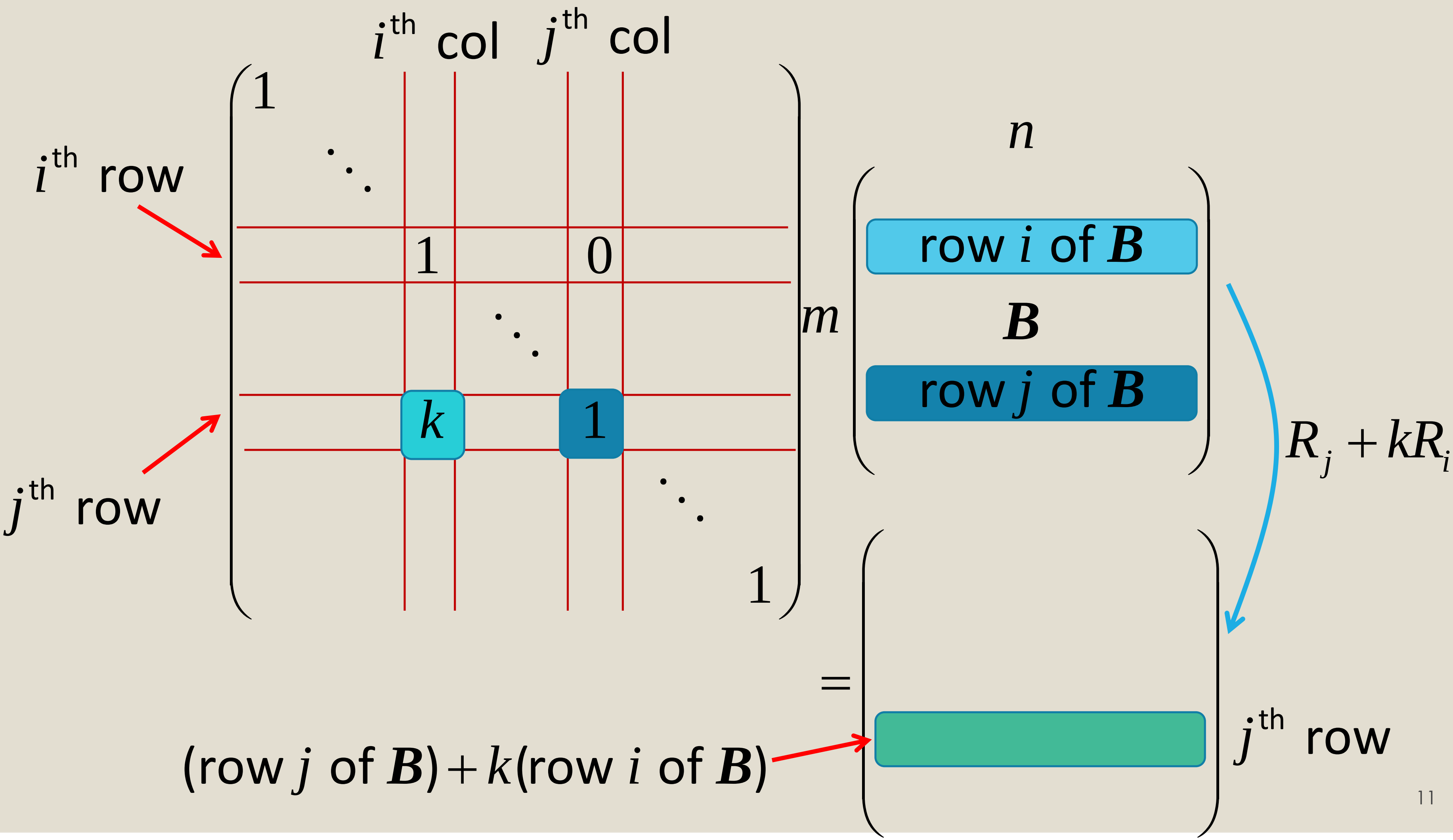
$$\boldsymbol{I}_m \xrightarrow{R_j + kR_i} \begin{array}{r c}
 & \begin{array}{ccccc} & j^{\text{th}}\text{ col} & & i^{\text{th}}\text{ col} & \end{array} \\
\begin{array}{r} \\ \\ j^{\text{th}}\text{ row} \\ \\ i^{\text{th}}\text{ row} \\ \\ \\ \end{array} &
\begin{pmatrix}
1 & & & & \\
 & \ddots & & & \\
 & 1 & & \boxed{k} & \\
 & & \ddots & & \\
 & 0 & & 1 & \\
 & & & & \ddots \\
 & & & & 1
\end{pmatrix}
\end{array} = \boldsymbol{E}_3$$

$(j < i)$

What if we pre-multiply $\boldsymbol{E}_3$ to a $m \times n$ matrix $\boldsymbol{B}$?

Third type of E.R.O.

(3) Adding k times the i^{th} row to the j^{th} row. ($R_j + kR_i$)



Third type of E.R.O.

(3) Adding k times the i^{th} row to the j^{th} row. $(R_j + kR_i)$

$$\begin{array}{cc}
 & \begin{array}{ccccc} \quad & j^{\text{th}}\text{ col} & \quad & i^{\text{th}}\text{ col} & \quad \end{array} \\
\begin{array}{c} \\ \\ j^{\text{th}}\text{ row} \\ \\ i^{\text{th}}\text{ row} \\ \\ \\ \end{array} &
\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & \boxed{1} & & \boxed{k} & \\ & & \ddots & & \\ & 0 & & 1 & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}
\end{array}
\;
m\overset{n}{\begin{pmatrix} \boxed{\text{row } j \text{ of } \boldsymbol{B}} \\ \boldsymbol{B} \\ \boxed{\text{row } i \text{ of } \boldsymbol{B}} \end{pmatrix}}
\xrightarrow{R_j + kR_i}$$

$$= \begin{array}{cc} \begin{pmatrix} \boxed{\quad (\text{row } j \text{ of } \boldsymbol{B}) + k(\text{row } i \text{ of } \boldsymbol{B}) \quad} \\ \\ \\ \end{pmatrix} & \begin{array}{c} j^{\text{th}}\text{ row} \\ \\ \\ \end{array} \end{array}$$

Third type of E.R.O.

It seems like we can 'represent' performing the elementary row operation $R_j + kR_i$ on $\boldsymbol{B}$ by pre-multiplying a suitable matrix to $\boldsymbol{B}$.

$$\boldsymbol{B} \xrightarrow{R_j + kR_i} \boldsymbol{C}_3$$

$$\boldsymbol{I}_m \xrightarrow{R_j + kR_i} \boldsymbol{E}_3$$

Then we have $\boldsymbol{E}_3\boldsymbol{B} = \boldsymbol{C}_3$.

In summary

Let $\boldsymbol{B}$ be a $m \times n$ matrix. For each of the three types of elementary row operations we can perform on $\boldsymbol{B}$:

$$\begin{tikzcd} & \boldsymbol{C}_1 \\ \boldsymbol{B} \arrow[ur, "cR_i"] \arrow[r, "R_i \leftrightarrow R_j"] \arrow[dr, "R_j + kR_i"'] & \boldsymbol{C}_2 \\ & \boldsymbol{C}_3 \end{tikzcd} \qquad \begin{tikzcd} & \boldsymbol{E}_1 \\ \boldsymbol{I}_m \arrow[ur, "cR_i"] \arrow[r, "R_i \leftrightarrow R_j"] \arrow[dr, "R_j + kR_i"'] & \boldsymbol{E}_2 \\ & \boldsymbol{E}_3 \end{tikzcd}$$

there are matrices $\boldsymbol{E}_i$ $(i = 1, 2, 3)$ such that

$\boldsymbol{E}_i \boldsymbol{B} = \boldsymbol{C}_i$ for $i = 1, 2, 3$.

Definition

A square matrix is an elementary matrix if it can be obtained from an identity matrix by performing a single elementary row operation.

Summary

1) For each elementary row operation X there is a corresponding square matrix $\boldsymbol{E}$ (of order m) such that performing X on a $m \times n$ matrix $\boldsymbol{B}$ produces the same effect as pre-multiplying $\boldsymbol{E}$ to $\boldsymbol{B}$.

In other words,

$$\boldsymbol{B} \xrightarrow{X} \boldsymbol{EB}$$

2) The matrix $\boldsymbol{E}$, defined as an elementary matrix is obtained by performing the corresponding elementary row operation X on $\boldsymbol{I}_m$.

$$\boldsymbol{I}_m \xrightarrow{X} \boldsymbol{E}$$