INVERSE OF A MATRIX

DISCUSSION

If x is a real number, it is easy to solve

$$2x = 5,$$

since corresponding to '2', there is another number '$\frac{1}{2}$' such that $2 \times \frac{1}{2} = 1$, allowing us to have

$$\frac{1}{2} \times 2x = \frac{1}{2} \times 5$$

$$\Rightarrow 1 \times x = \frac{5}{2}$$

$$\Rightarrow x = \frac{5}{2}$$

DISCUSSION

If $\boldsymbol{X}$ is a matrix, how can we solve

$$\begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 3 \\ 3 & 1 & 2 \end{pmatrix} \boldsymbol{X} = \begin{pmatrix} 2 & 1 & -1 \\ 3 & -5 & 0 \\ 0 & 1 & 4 \end{pmatrix} ?$$

Is there a matrix $\begin{pmatrix} \boxed{?} \end{pmatrix}$ such that

$$\begin{pmatrix} \boxed{?} \end{pmatrix} \begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 3 \\ 3 & 1 & 2 \end{pmatrix} \boldsymbol{X} = \begin{pmatrix} \boxed{?} \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 3 & -5 & 0 \\ 0 & 1 & 4 \end{pmatrix}$$

DISCUSSION

$\Big($  $\Big)\begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 3 \\ 3 & 1 & 2 \end{pmatrix} \boldsymbol{X} = \Big($  $\Big)\begin{pmatrix} 2 & 1 & -1 \\ 3 & -5 & 0 \\ 0 & 1 & 4 \end{pmatrix}$

$\Rightarrow \Big($  $\Big) \boldsymbol{X} = \Big($  $\Big)\begin{pmatrix} 2 & 1 & -1 \\ 3 & -5 & 0 \\ 0 & 1 & 4 \end{pmatrix}$

 $= \boldsymbol{I}$ (identity matrix)

$\Rightarrow \boldsymbol{X} = \Big($  $\Big)\begin{pmatrix} 2 & 1 & -1 \\ 3 & -5 & 0 \\ 0 & 1 & 4 \end{pmatrix}$

DEFINITION

Let A be a square matrix of order n.

A is said to be an invertible matrix if there exists another square matrix B of the same order such that

$$\boxed{AB = BA = I_n}$$

If such a B exists, it is called an inverse of A.

A is said to be singular if it has no inverse.

REMARK

The definition of an invertible matrix is an existential one.

"I tried very hard to find $\boldsymbol{B}$ but I could not..."

...does not mean $\boldsymbol{B}$ does not exists!

Could there be more than one such $\boldsymbol{B}$?

EXAMPLE

Is $\begin{pmatrix} -2 & 0 & -1 \\ 2 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix}$ an inverse of $\begin{pmatrix} 2 & 1 & 1 \\ -6 & -3 & -2 \\ -5 & -2 & -2 \end{pmatrix}$?

Check $\begin{pmatrix} -2 & 0 & -1 \\ 2 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ -6 & -3 & -2 \\ -5 & -2 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Check $\begin{pmatrix} 2 & 1 & 1 \\ -6 & -3 & -2 \\ -5 & -2 & -2 \end{pmatrix}\begin{pmatrix} -2 & 0 & -1 \\ 2 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

EXAMPLE

Find $\boldsymbol{X}$ if $\begin{pmatrix} 2 & 1 & 1 \\ -6 & -3 & -2 \\ -5 & -2 & -2 \end{pmatrix} \boldsymbol{X} = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}$.

$$\begin{pmatrix} -2 & 0 & -1 \\ 2 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ -6 & -3 & -2 \\ -5 & -2 & -2 \end{pmatrix} \boldsymbol{X} = \begin{pmatrix} -2 & 0 & -1 \\ 2 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{IX} = \begin{pmatrix} -7 & 2 & -3 \\ 8 & -2 & 2 \\ 9 & -3 & 5 \end{pmatrix} \quad \Rightarrow \boldsymbol{X} = \begin{pmatrix} -7 & 2 & -3 \\ 8 & -2 & 2 \\ 9 & -3 & 5 \end{pmatrix}.$$

EXAMPLE

Show that $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is singular.

Suppose $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is invertible, then there must be a square matrix of order 2, say $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

(By definition of inverse)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

EXAMPLE

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a+b & 0 \\ c+d & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow$ a contradiction, by looking at the $(2,2)$-entry

Thus it is impossible for $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ to have an inverse,

and so the matrix has to be singular.

MORE EFFICIENT WAY?

We knew $\boldsymbol{A} = \begin{pmatrix} 2 & 1 & 1 \\ -6 & -3 & -2 \\ -5 & -2 & -2 \end{pmatrix}$ is invertible because

we were 'given' $\boldsymbol{B} = \begin{pmatrix} -2 & 0 & -1 \\ 2 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix}$ to 'test' whether

$\boldsymbol{AB}$ and $\boldsymbol{BA}$ are both equal to $\boldsymbol{I}$.

We had to use contradiction to show that $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ has no inverse. What if the matrix was bigger and more complicated?

UNIQUENESS OF INVERSE

If $\boldsymbol{B}$ and $\boldsymbol{C}$ are inverses of a square matrix $\boldsymbol{A}$, then

$$\boldsymbol{B} = \boldsymbol{C}.$$

That is, if $\boldsymbol{A}$ is an invertible square matrix, then it has one and only one inverse.

Inverses are unique!

Since inverses are unique, we will write $\boxed{\boldsymbol{A}^{-1}}$ as the inverse of $\boldsymbol{A}$ if $\boldsymbol{A}$ is invertible.

UNIQUENESS OF INVERSE

If $\boldsymbol{B}$ and $\boldsymbol{C}$ are inverses of a square matrix $\boldsymbol{A}$, then

$$\boldsymbol{B} = \boldsymbol{C}.$$

Proof:

$\boldsymbol{B}$ is an inverse of $\boldsymbol{A}$, then

$$\boldsymbol{BA} = \boldsymbol{I}$$

$$\boldsymbol{BAC} = \boldsymbol{IC}$$

$\boldsymbol{C}$ is an inverse of $\boldsymbol{A}$, then

$$\boxed{\boldsymbol{AC}} = \boldsymbol{I}$$

$$\boldsymbol{B}(\boxed{\boldsymbol{AC}}) = \boldsymbol{C}$$

$$\boldsymbol{B}(\boldsymbol{I}) = \boldsymbol{C}$$

$$\boldsymbol{B} = \boldsymbol{C}$$

REMARK

It turns out that to check whether a given square matrix $\boldsymbol{B}$ is the inverse of $\boldsymbol{A}$, we only need to check either

$$\boldsymbol{AB} = \boldsymbol{I} \qquad \text{OR} \qquad \boldsymbol{BA} = \boldsymbol{I}$$

The reason will be explained in a later unit.

SUMMARY

1) Definition for the inverse of a square matrix.

2) Not so easy to determine (at least for now) whether a matrix is invertible or singular.

3) Uniqueness of inverse.