DIAGONALIZATION PART II

RECALL - THEOREM

Let $\boldsymbol{A}$ be a square matrix of order n. Then $\boldsymbol{A}$ is diagonalizable if and only if $\boldsymbol{A}$ has n linearly independent eigenvectors.

It is important to emphasize the n eigenvectors have to be linearly independent since

E_λ contains ALL the eigenvectors of $\boldsymbol{A}$ associated with λ.

that is, $\boldsymbol{A}$ already has infinitely eigenvectors associated with a particular eigenvalue λ.

EXAMPLE

$\boldsymbol{M} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ det$(\lambda \boldsymbol{I} - \boldsymbol{M}) = (\lambda - 2)^2$

$\boldsymbol{M}$ is a 2×2 matrix and has one eigenvalue 2.

$E_2 =$ span$\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ dim$(E_2) = 1 < 2$

$\boldsymbol{M}$ has only 1 linearly independent eigenvector and so it is not diagonalizable.

EXAMPLE

Is $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{pmatrix}$ diagonalizable? If so, find an invertible matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{A}$.

Answer: We know immediately that the eigenvalues of $\boldsymbol{A}$ are 1 and 2.

We need to check if we can find 3 linearly independent eigenvectors of $\boldsymbol{A}$.

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{pmatrix}$$

Consider E_1: Solving $(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 3 & -5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 3 & -5 & -1 & 0 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{-1}{8} & 0 \\ 0 & 1 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x = \frac{t}{8} \\ y = \frac{-t}{8} \\ z = t, \quad t \in \mathbb{R} \end{cases}$$

So $E_1 = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 8 \end{pmatrix}\right\}$

and $\dim(E_1) = 1$.

$\boldsymbol{A}$ has only two linearly independent eigenvectors

EXAMPLE

$\boldsymbol{A}$ is not diagonalizable

$$\boldsymbol{A}=\begin{pmatrix}1 & 0 & 0\\ 1 & 2 & 0\\ -3 & 5 & 2\end{pmatrix}$$

$\det(\lambda \boldsymbol{I}-\boldsymbol{A})$

$=(\lambda-2)^2(\lambda-1)$

Consider E_2: Solving $(2\boldsymbol{I}-\boldsymbol{A})\boldsymbol{x}=\boldsymbol{0}$

$$\begin{pmatrix}1 & 0 & 0\\ -1 & 0 & 0\\ 3 & -5 & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix} \Leftrightarrow \left(\begin{array}{ccc|c}1 & 0 & 0 & 0\\ -1 & 0 & 0 & 0\\ 3 & -5 & 0 & 0\end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$$

$$\begin{cases}x = 0\\ y = 0\\ z = t, \quad t\in\mathbb{R}\end{cases}$$

So $E_2=\text{span}\left\{\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\right\}$

and $\dim(E_2)=1.$

REMARK

Step 1: Solve $\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$ to find all eigenvalues of $\boldsymbol{A}$

$\lambda_1, \lambda_2, \ldots, \lambda_k$ (suppose $\boldsymbol{A}$ has k distinct eigenvalues, $k \le n$)

In general, when solving for the roots of

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$$

it is possible that some λ_i is a complex number.

Complex vector spaces $\mathbb{C}^n$ (instead of $\mathbb{R}^n$) needs to be introduced.

A SUFFICIENT CONDITION FOR DIAGONALIZABILITY

Let $\boldsymbol{A}$ be a square matrix of order n. If $\boldsymbol{A}$ has n distinct eigenvalues, then $\boldsymbol{A}$ is diagonalizable.

EXAMPLE

Is $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 3 \end{pmatrix}$ diagonalizable?

Answer: Yes. Since $\boldsymbol{A}$ is a 3×3 matrix that has 3 distinct eigenvalues.

APPLICATION OF DIAGONALIZATION

Once a square matrix $\boldsymbol{A}$ can be diagonalized, we can compute $\boldsymbol{A}^n$ easily.

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}=\boldsymbol{D}=\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & \mathbf{0} & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n\end{pmatrix}$$

$$\Leftrightarrow \boldsymbol{A}=\boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$$

$$\Leftrightarrow \boldsymbol{A}^k=\boldsymbol{P}\boldsymbol{D}^k\boldsymbol{P}^{-1}$$

$$=\boldsymbol{P}\begin{pmatrix}\lambda_1{}^k & & & \\ & \lambda_2{}^k & \mathbf{0} & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n{}^k\end{pmatrix}\boldsymbol{P}^{-1}$$

EXAMPLE

Let $\boldsymbol{A}=\begin{pmatrix}-4 & 0 & -6\\ 2 & 1 & 2\\ 3 & 0 & 5\end{pmatrix}$. Compute $\boldsymbol{A}^{10}$.

$$\det(\lambda \boldsymbol{I}-\boldsymbol{A})=\begin{vmatrix}\lambda+4 & 0 & 6\\ -2 & \lambda-1 & -2\\ -3 & 0 & \lambda-5\end{vmatrix}=(\lambda+4)\begin{vmatrix}\lambda-1 & -2\\ 0 & \lambda-5\end{vmatrix}+6\begin{vmatrix}-2 & \lambda-1\\ -3 & 0\end{vmatrix}$$

$$=(\lambda+4)(\lambda-1)(\lambda+5)+18(\lambda-1)$$

$$=(\lambda-1)[(\lambda^2-\lambda-20+18)]$$

$$=(\lambda-1)(\lambda^2-\lambda-2)$$

$$=(\lambda-1)(\lambda+1)(\lambda-2)$$

So the eigenvalues of $\boldsymbol{A}$ are $-1,1$ and 2.

EXAMPLE

Using the method described previously, we find:

$$E_{-1} = \text{span}\left\{\begin{pmatrix}-2\\1\\1\end{pmatrix}\right\} \qquad E_{1} = \text{span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}\right\} \qquad E_{2} = \text{span}\left\{\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$$

(verify it yourself!)

So if we let

$$\boldsymbol{P} = \begin{pmatrix}-2 & 0 & -1\\1 & 1 & 0\\1 & 0 & 1\end{pmatrix}$$

So the eigenvalues of $\boldsymbol{A}$ are $-1, 1$ and 2.

EXAMPLE

We will have

$$P^{-1}AP = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ or equivalently } A = P\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}P^{-1}$$

$$\Rightarrow A^{10} = P\begin{pmatrix} (-1)^{10} & 0 & 0 \\ 0 & 1^{10} & 0 \\ 0 & 0 & 2^{10} \end{pmatrix}P^{-1}$$

$$= \begin{pmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1024 \end{pmatrix}\begin{pmatrix} -1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -1022 & 0 & -2046 \\ 0 & 1 & 0 \\ 1023 & 0 & 2047 \end{pmatrix}$$

SUMMARY

1) A sufficient condition for a $n \times n$ matrix to be diagonalizable.

2) Computing powers of $\boldsymbol{A}$ when $\boldsymbol{A}$ is diagonalizable.