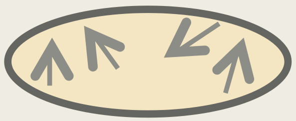
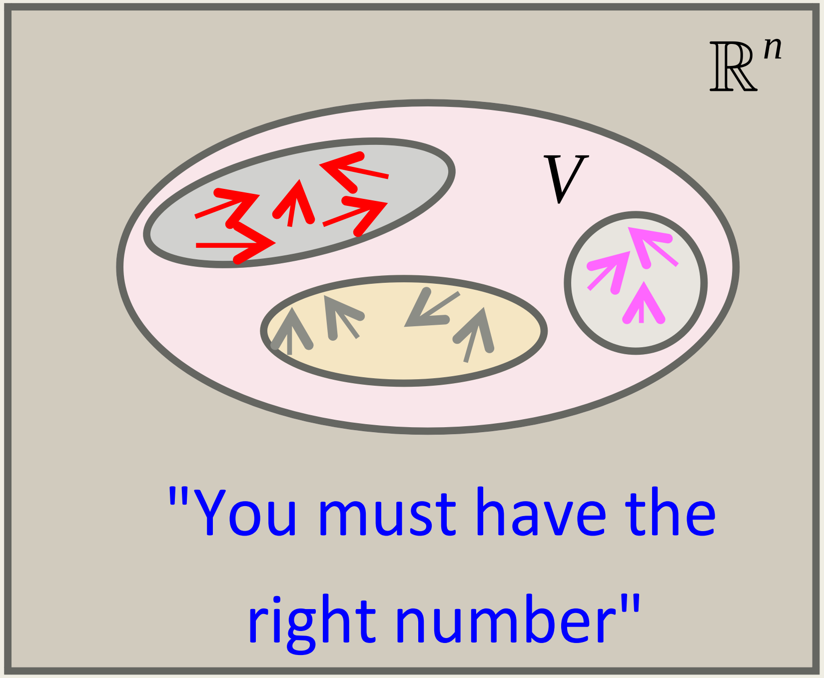
DIMENSIONS PART I

Theorem

Let V be a vector space which has a basis with k vectors.

 $=$ basis for V



Then

(1) any subset of V with more than k vectors is always linearly dependent (so cannot be a basis);

(2) any subset of V with less than k vectors cannot span V (so also cannot be a basis);

Definition

The dimension of a vector space V denoted by $\dim(V)$, is defined to be the number of vectors in a basis for V.

The dimension of the zero space is defined to be zero.

Example

1) $\dim(\mathbb{R}^n) = n$

(recall a basis for $\mathbb{R}^n$ can be $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$).

2) Subspaces of $\mathbb{R}^2$:

$\{\mathbf{0}\}$: dimension $\boxed{0}$

lines through the origin: dimension $\boxed{1}$

$\mathbb{R}^2$: dimension $\boxed{2}$

3) Subspaces of $\mathbb{R}^3$:

$\{\mathbf{0}\}$: dimension $\boxed{0}$

lines through the origin: dimension $\boxed{1}$

planes containing the origin: dimension $\boxed{2}$

$\mathbb{R}^3$: dimension $\boxed{3}$

Example

Find a basis for and determine the dimension of the subspace $W = \{(x, y, z) \mid y = 2z\}$.

$$= \{(x, 2z, z) \mid x, z \in \mathbb{R}\}$$

$$= \{x(1,0,0) + z(0,2,1) \mid x, z \in \mathbb{R}\}$$

$$= \text{span}\{(1,0,0),(0,2,1)\}$$

$\{(1,0,0),(0,2,1)\}$ spans W

$\{(1,0,0),(0,2,1)\}$ is linearly independent (why?)

$\{(1,0,0),(0,2,1)\}$ is a basis for W and $\dim(W) = 2$.

Example

Find a basis for and determine the dimension of the solution space of the homogeneous system

$$\begin{cases} 2v + 2w - x + z = 0 \\ -v - w + 2x - 3y + z = 0 \\ x + y + z = 0 \\ v + w - 2x - z = 0 \end{cases}$$

Example

$$
\left(\begin{array}{ccccc|c}
2 & 2 & -1 & 0 & 1 & 0 \\
-1 & -1 & 2 & -3 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & -2 & 0 & -1 & 0
\end{array}\right)
\longrightarrow
\left(\begin{array}{ccccc|c}
1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

$$
\begin{cases}
v & = & -s-t \\
w & = & s \\
x & = & -t \\
y & = & 0 \\
z & = & t, \qquad s,t \in \mathbb{R}
\end{cases}
\qquad
\begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix}
=
\begin{pmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{pmatrix}
= s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
+ t\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}
$$

Example

$$\begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} \text{ belongs to the solution space } \Leftrightarrow \begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix} = s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{Solution space} = \text{span}\boxed{\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}\right\}}$$

A basis for the solution space

Linearly independent

Dimension of the solution space is 2.

Remark

$$\text{Solution space} = \text{span}\left\{ \boxed{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}} \right\}$$

A basis for the solution space

Linearly independent

A set of vectors that spans the solution space for any homogeneous linear system found using the above method will always be linearly independent.

Summary

1) All bases for the same vector space have the same number of vectors.

2) Definition of dimension (of a vector space).

3) Finding a basis for the solution space of a homogeneous linear system.