BASES II AND COORDINATE VECTORS

Recall from a previous unit

If $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}$ is a basis for a vector space V, then every vector $\boldsymbol{v}\in V$ can be expressed in the form (as a linear combination of $\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k$)

$$\boldsymbol{v}=c_1\boldsymbol{u}_1+c_2\boldsymbol{u}_2+\ldots+c_k\boldsymbol{u}_k$$

in exactly one way, where $c_1,c_2,\ldots,c_k\in\mathbb{R}$.

Coordinate vectors

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a basis for a vector space V and $\boldsymbol{v}$ be a vector in V. If

$$\boldsymbol{v} = \boxed{c_1}\boldsymbol{u}_1 + \boxed{c_2}\boldsymbol{u}_2 + \ldots + \boxed{c_k}\boldsymbol{u}_k$$

then the coefficients $\boxed{c_1, c_2, \ldots, c_k}$ are called the **coordinates** of $\boldsymbol{v}$ relative to the basis S.

The vector

$$(\boldsymbol{v})_S = (c_1, c_2, \ldots, c_k) \text{ (belonging to } \mathbb{R}^k\text{)}$$

is called the **coordinate vector** of $\boldsymbol{v}$ relative to the basis S.

Remarks

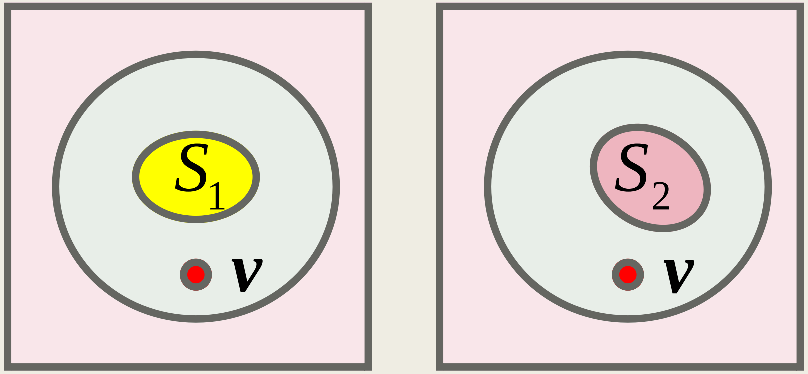
1) In order to discuss coordinate vectors meaningfully, the vectors in $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}$ must be ordered.

$$\boldsymbol{v}=c_1\boldsymbol{u}_1+c_2\boldsymbol{u}_2+\ldots+c_k\boldsymbol{u}_k \qquad (\boldsymbol{v})_S=(c_1,c_2,\ldots,c_k)$$

2) Once $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}$ is fixed, $(\boldsymbol{v})_S$ is unique and well-defined for each $\boldsymbol{v}\in V$.

3) Different basis, different coordinate vectors.

$$\boxed{(\boldsymbol{v})_{S_1}\neq(\boldsymbol{v})_{S_2}}$$



Example

$$S=\{(1,2,1),(2,9,0),(3,3,4)\}$$

1) Prove that S is a basis for $\mathbb{R}^3$.

2) Find the coordinate vector of $\boldsymbol{v}=(5,-1,9)$ relative to S.

Solution: Solve for the coefficients a,b,c in the equation

$$\boldsymbol{v}=(5,-1,9)=a(1,2,1)+b(2,9,0)+c(3,3,4)$$

$$\begin{cases} a & + & 2b & + & 3c & = & 5 \\ 2a & + & 9b & + & 3c & = & -1 \\ a & & & + & 4c & = & 9 \end{cases}$$

Example

$$S=\{(1,2,1),(2,9,0),(3,3,4)\}$$

$$\boldsymbol{v}=(5,-1,9)=a(1,2,1)+b(2,9,0)+c(3,3,4)$$

$$\begin{cases} a + 2b + 3c = 5 \\ 2a + 9b + 3c = -1 \\ a + 4c = 9 \end{cases}$$

$$a=1, b=-1, c=2$$

$$\boxed{\text{So } (\boldsymbol{v})_S=(1,-1,2).}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 2 & 9 & 3 & -1 \\ 1 & 0 & 4 & 9 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

unique solution

Example

$S = \{(1,2,1),(2,9,0),(3,3,4)\}$

1) Prove that S is a basis for $\mathbb{R}^3$.

2) Find the coordinate vector of $\boldsymbol{v} = (5,-1,9)$ relative to S.

3) Find a vector $\boldsymbol{w}$ in $\mathbb{R}^3$ such that $(\boldsymbol{w})_s = (-1,3,2)$.

Answer: If $(\boldsymbol{w})_S = (-1,3,2)$, then

$$\boldsymbol{w} = -(1,2,1) + 3(2,9,0) + 2(3,3,4) = (11,31,7)$$

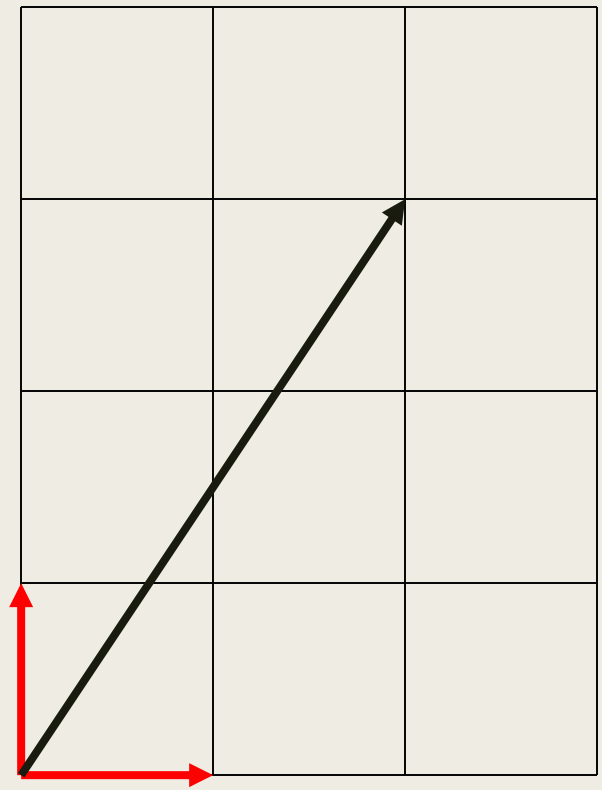
Example

$\boldsymbol{v} = (2,3) \in \mathbb{R}^2$

$\boxed{S_1 = \{(1,0),(0,1)\}}$

$S_2 = \{(1,-1),(1,1)\}$

$S_3 = \{(1,0),(1,1)\}$



$(2,3) = 2(1,0) + 3(0,1)$

$\Rightarrow (\boldsymbol{v})_{S_1} = (2,3)$

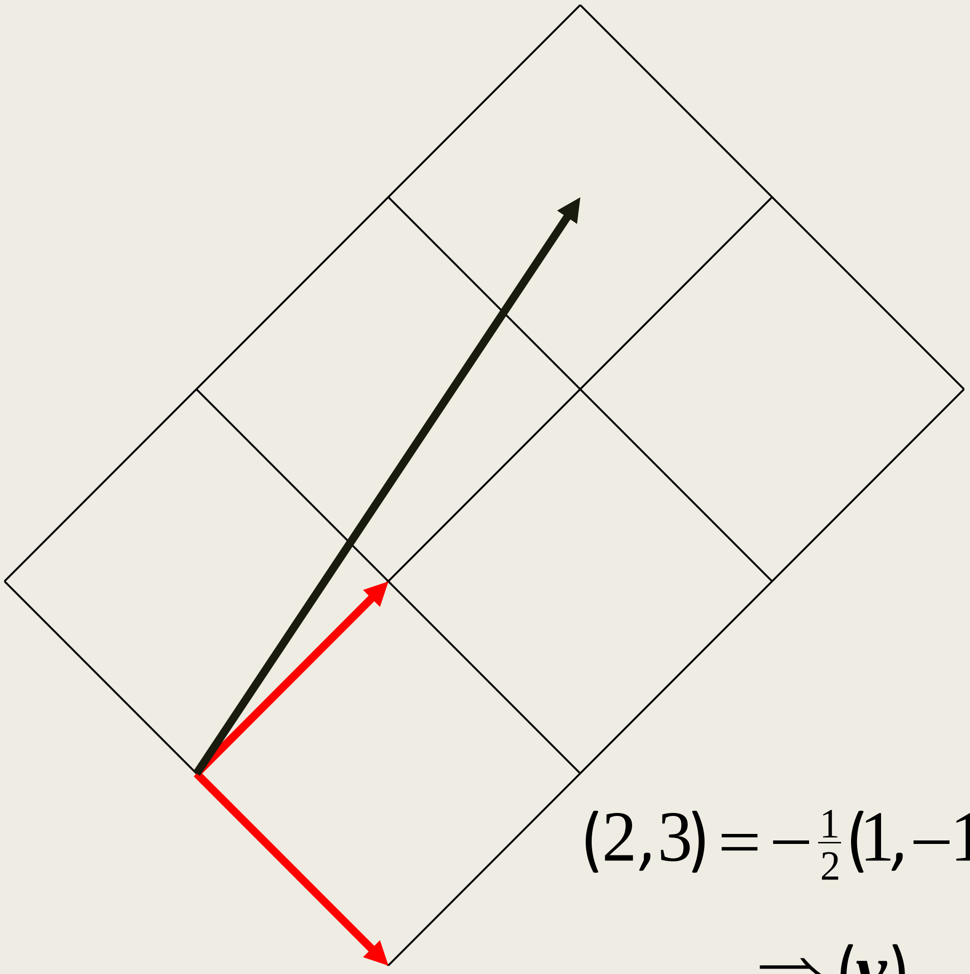
Example

$\boldsymbol{v} = (2,3) \in \mathbb{R}^2$

$S_1 = \{(1,0),(0,1)\}$

$S_2 = \{(1,-1),(1,1)\}$

$S_3 = \{(1,0),(1,1)\}$



$(2,3) = -\tfrac{1}{2}(1,-1) + \tfrac{5}{2}(1,1)$

$\Rightarrow (\boldsymbol{v})_{S_2} = (-\tfrac{1}{2}, \tfrac{5}{2})$

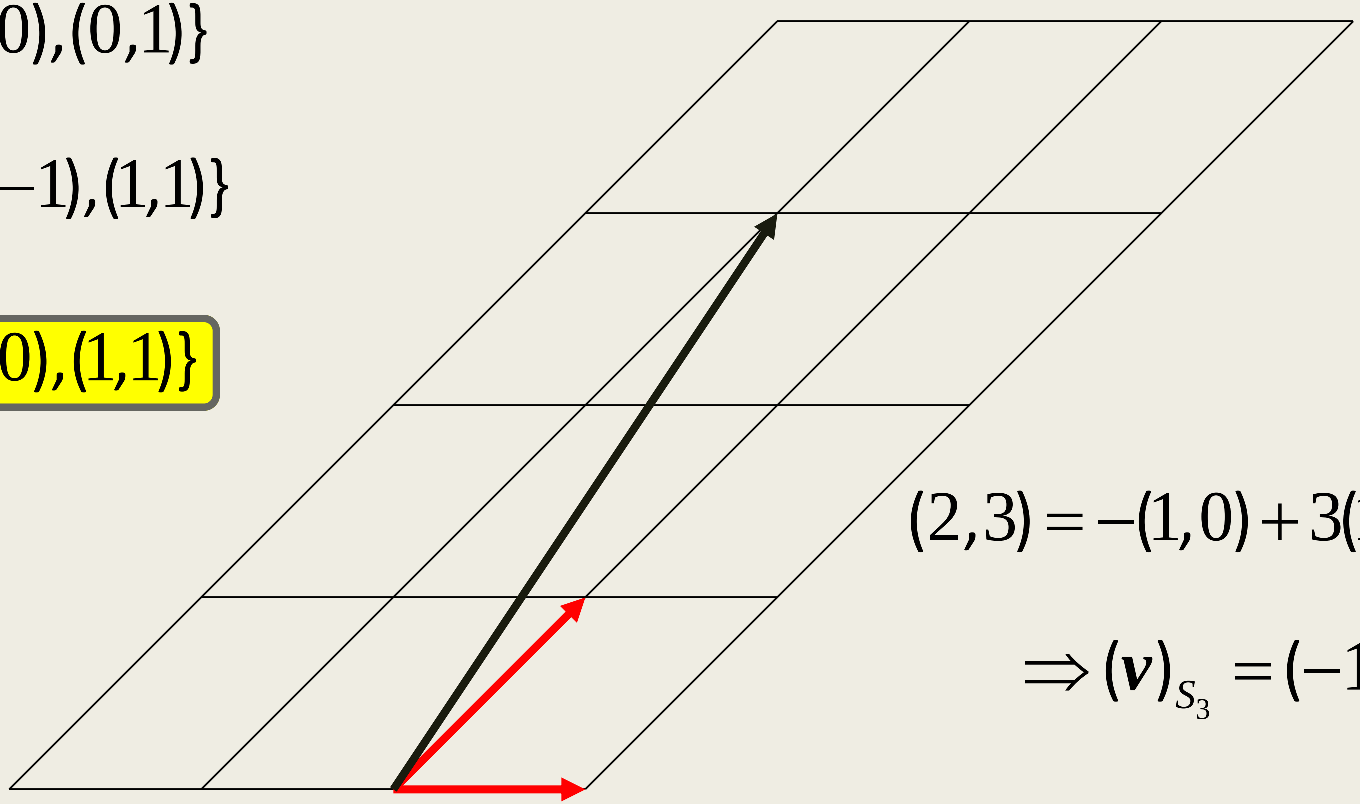
Example

$\boldsymbol{v} = (2,3) \in \mathbb{R}^2$

$S_1 = \{(1,0),(0,1)\}$

$S_2 = \{(1,-1),(1,1)\}$

$\boxed{S_3 = \{(1,0),(1,1)\}}$



$(2,3) = -(1,0) + 3(1,1)$

$\Rightarrow (\boldsymbol{v})_{S_3} = (-1,3)$

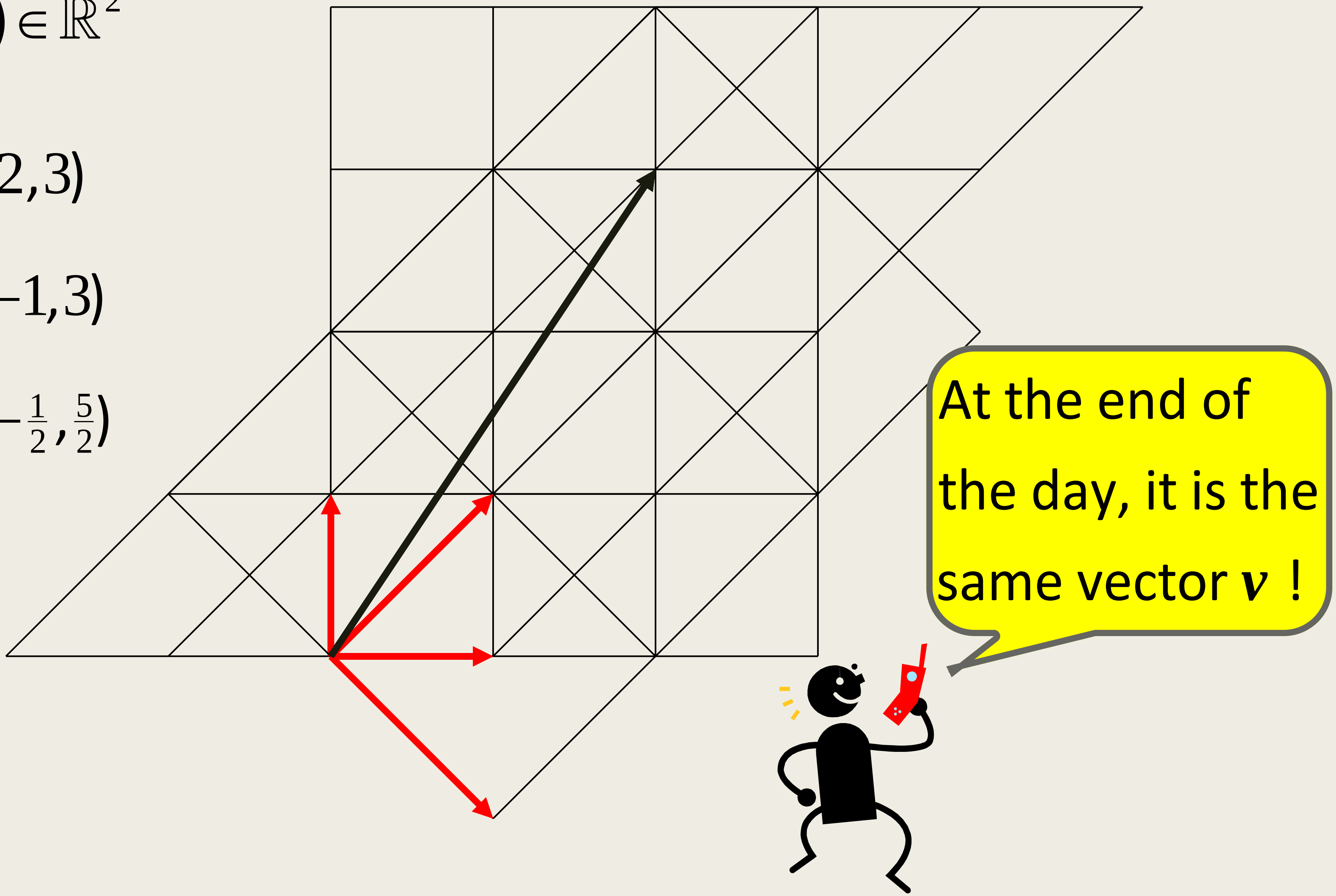
What is the significance?

$$\boldsymbol{v} = (2,3) \in \mathbb{R}^2$$

$$(\boldsymbol{v})_{S_1} = (2,3)$$

$$(\boldsymbol{v})_{S_3} = (-1,3)$$

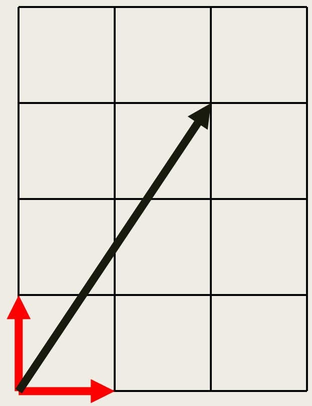
$$(\boldsymbol{v})_{S_2} = (-\tfrac{1}{2}, \tfrac{5}{2})$$



Example

$\boldsymbol{v} = (2,3) \in \mathbb{R}^2$ $\qquad S_1 = \{(1,0),(0,1)\}$ $\qquad (\boldsymbol{v})_{S_1} = (2,3) = \boldsymbol{v}$

In fact, for any $\boldsymbol{v} = (x, y) \in \mathbb{R}^2$,



$$(\boldsymbol{v})_{S_1} = (x, y) = \boldsymbol{v}$$

Such a basis (like S_1) is convenient to use.

Let $E = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ where

For any $\boldsymbol{v} \in \mathbb{R}^n$, $(\boldsymbol{v})_E = \boldsymbol{v}$

$$\boldsymbol{e}_1 = (1,0,\ldots,0) \quad \boldsymbol{e}_2 = (0,1,\ldots,0) \quad \cdots\cdots\cdots \quad \boldsymbol{e}_n = (0,0,\ldots,1)$$

E is called the standard basis for $\mathbb{R}^n$.

Remark

Remember the standard basis for $\mathbb{R}^3$ and the standrd basis for $\mathbb{R}^4$ contains entirely different vectors.

Do not be confused!

Standard basis for $\mathbb{R}^3$

$$\boldsymbol{e}_1 = (1,0,0),\ \boldsymbol{e}_2 = (0,1,0),\ \boldsymbol{e}_3 = (0,0,1)$$

Standard basis for $\mathbb{R}^4$

$$\boldsymbol{e}_1 = (1,0,0,0),\ \boldsymbol{e}_2 = (0,1,0,0),\ \boldsymbol{e}_3 = (0,0,1,0),\ \boldsymbol{e}_4 = (0,0,0,1)$$

A question to ponder

For a vector space V, we know that V can have many different bases. But do all these bases have the <u>same number</u> of vectors?

Summary

1) Definition of coordinate vectors (relative to a basis).

2) The standard basis for $\mathbb{R}^n$.