Unit 009 Homogeneous linear systems

Slide 01: In this unit, we will discuss a special type of linear system called homogeneous linear systems.

Slide 02: A homogeneous linear system is one that has the following form. Note that it is similar to the general linear system we have seen earlier where there is a collection of linear equations involving some variables.

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The special condition that a linear system must have is that for each of the linear equations in the system, the right hand side constant must be zero.

Slide 03: Recall that for any linear system, there are only three possibilities in terms of how many solutions the linear system can have. The system could be inconsistent, consistent with exactly one unique solution, or consistent with infinitely many solutions.

Slide 04: However for a homogeneous linear system, it is easy to see that if we choose all the variables to take on the value of 0, every equation in the linear system would be satisfied. In other words, we have already found a solution to the system.

Slide 05: Thus a homogeneous linear system can never be inconsistent, since we have already found a solution to such a system.

Slide 06: The solution to a homogeneous linear system where we let all the variables take on the value of 0 is called the trivial solution of the homogeneous linear system.

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While it may not exist, any other solution to a homogeneous linear system is called a non trivial solution.

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Thus, for this special class of linear systems, there are only two possibilties in terms of how many solutions it can have. If it has a unique solution, then it must be the trivial one. On the other hand, if it has infinitely many solutions, then we will have the trivial solution as well as non trivial ones.

Slide 07: Consider the following simple example of a homogeneous linear system with two equations involving two variables x and y. We know that geometrically, this system represents two straight lines l_1 and l_2 in the xy plane that passes through the origin.

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Since this is a homogeneous linear system, it has only two possibilities in terms of how many solutions it can have. In the first scenario, where the system has only the trivial solution, this corresponds to the case where the two straight lines l_1 and l_2 are different lines and intersect only at the origin. The origin here, is essentially, the trivial solution $x = 0$, $y = 0$.

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The second scenario is when the two lines are actually the same line. In this case, every point on the line is a solution to the system and therefore there would be infinitely many solutions. Note that we have both the trivial as well as non trivial solutions in this case.

Slide 08: We will now look at a few augmented matrices. For each of them consider what is the maximum number of leading entries the augmented matrix can have if it is reduced to row-echelon form.

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This augemented matrix has two rows and three variable columns. Therefore the corresponding linear system is one that has two equations and three variables.

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Since at row-echelon form, there is at most one leading entry in every row, we know that there is at most 2 leading entries in a row-echelon form of the augmented matrix. The fact that there are less rows than variable columns restricts the maximum number of leading entries in row-echelon form to 2.

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Similarly, this augmented matrix has fewer rows compared to variable columns. To be precise, the homogeneous linear system here has three equations and five unknowns. Thus, the number of leading entries in a row-echelon form of the matrix is at most 3.

Slide 09: Let us make a general statement that reflects our observation. If a homogeneous linear system has more unknowns than equations, in other words, more variable columns than rows,

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this would imply that at row-echelon form, the augmented matrix will always have some variable column to be non pivot.

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As we have discussed in an earlier unit, this implies that the system will have infinitely many solutions. As we are dealing with homogeneous linear systems, this would mean

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that we will have non trivial solutions in addition to the trivial one.

Slide 10: Such systems are commonly known as underdetermined systems. For the case of underdetermined homogeneous linear systems, we are now able to conclude that they will always have infinitely many solutions.

Slide 11: In summary, this unit

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introduces a special class of linear systems that are homogeneous.

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We also defined, for homogeneous linear systems, what is meant by the trivial solution and what are called non trivial solutions.

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Remember that a special property that homogeneous systems have is that they can never be inconsistent.

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Lastly, we saw that underdetermined homogeneous linear systems are guranteed to have infinitely many solutions.