First Week

Example 2. Experiments show that a radioactive substance decomposes at a rate proportional to the amount present. Starting with 2 mg at certain time, say $t = 0$, what can be said about the amount available at a later time?

First order linear equation

To solve $y' + Py = Q$:

First: find $R = e^{\int P\,dx}$.

Second: Write down the answer

$$y = \frac{1}{R}\int RQ\,dx$$

Bernoulli equations

$$\frac{dy}{dx} + Py = Qy^n, \quad n \neq 0, 1.$$

Let $\boxed{z = y^{1-n}}$

- Example

$$y' + y = x^2y^2. \ [y(Ae^x + x^2 + 2x + 2) = 1]$$

$$y' + y = x^2 y^2$$

Let $z = y^{1-2} = y^{-1}$

$$\therefore z' = -y^{-2} y'$$

$$\therefore -y^2 z' + y = x^2 y^2$$

$$z' - y^{-1} = -x^2$$

$$z' - z = -x^2$$

$$R = e^{\int -dx} = e^{-x}$$

$$z = e^x \int e^{-x}(-x^2)dx$$

$$= e^x \left\{ \int x^2 d(e^{-x}) \right\}$$

$$= e^{x}\left\{ x^{2}e^{-x} - \int 2xe^{-x}dx \right\}$$

$$= e^{x}\left\{ x^{2}e^{-x} - \int (-2x)\, d(e^{-x}) \right\}$$

$$= e^{x}\left\{ x^{2}e^{-x} + 2xe^{-x} - 2\int e^{-x}dx \right\}$$

$$= e^{x}\left\{ x^{2}e^{-x} + 2xe^{-x} + 2e^{-x} + c \right\}$$

$$\therefore \frac{1}{y} = ce^x + x^2 + 2x + 2$$

$$\therefore y = \frac{1}{ce^x + x^2 + 2x + 2}$$

Tutorial 1

4. One theory about the behaviour of moths states that they navigate at night by keeping a fixed angle between their velocity vector and the direction of the Moon. A certain moth flies near to a candle and mistakes it for the Moon.

(i) Prove that in polar coordinates (r, θ) with the candle at the origin, the formula for the angle ψ between the radius vector and the velocity vector is given by $\tan(\psi) = \mathrm{r}\frac{d\theta}{dr}$.

(ii)Use this formula to solve for r as a function of θ and discuss what will happen to the moth.

3. In very dry regions, the phenomenon called **Virga** is very important because it can endanger aeroplanes. [See http://en.wikipedia.org/wiki/Virga]. Virga is rain in air that is so dry that the raindrops evaporate before they can reach the ground. Suppose that the volume of a raindrop is proportional to the $3/2$ power of its surface area. [Why is this reasonable? Note: raindrops are not spherical, but let's assume that they always have the same shape, no matter what their size may be.] Suppose that the rate of reduction of the volume of a raindrop is proportional to its surface area. [Why is this reasonable?] Find a formula for the amount of time it takes for a virga raindrop to evaporate completely, expressed in terms of the constants you introduced and the initial surface area of a raindrop. Check that the units of your formula are correct. Suppose somebody suggests that the rate of reduction of the volume of a raindrop is proportional to the **square** of the surface area. Argue that this cannot be correct.