MA1512 TUTORIAL 2

KEY CONCEPTS – CHAPTER 1 DIFFERENTIAL EQUATIONS

1st order DE (continued)

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Technique 3: Integrating Factor

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Step 1 Define **Integrating Factor**:

$R(x) = e^{\int P(x)dx}$, where $R' = RP$ (chain rule)

Step 2 We use $(Ry)' = RQ$ and we get

$y = \frac{1}{R}\left(\int RQ \; dx + C\right)$.

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Technique 4: Reduction to Linear Form

Bernoulli's Equation: $y' + p(x)y = q(x)y^n$

Step 1 Let $z = y^{1-n}$ to get $\frac{dz}{dx} = \frac{(1-n)}{y^n}\frac{dy}{dx}$. Then, the DE is transformed into:

$$z' + (1-n)p(x)z = (1-n)q(x).$$

Step 2 Solve the first order linear differential equation using the integrating factor method.

- In this case, $n \neq 0, 1$. (Why?)
- Remember to fit the DE into the standard form i.e. find the value of n, $p(x)$ and $q(x)$.

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TUTORIAL PROBLEMS

Question 1

Solve the following differential equations:

Solutions

(a) $xy' + (1+x)y = e^{-x}$, $x > 0$

There is only 1 term in terms of y, thus we guess that we can use the integrating factor method. In order to use the integrating factor formula, we need to change the DE into the standard form:

$$y' + \left(\frac{1}{x} + 1\right)y = x^{-1}e^{-x}$$

Step 1 With $P(x) = \frac{1}{x} + 1$ and $Q(x) = x^{-1}e^{-x}$, the integrating factor is given by

$$R(x) = e^{\int \left(\frac{1}{x}+1\right) dx} = e^{\ln|x|+x} = e^{\ln|x|}e^x = xe^x$$

Step 2 The solution is given by

$$\begin{aligned} y &= \frac{1}{R}\left(\int RQ \; dx + C\right) \\ &= \frac{1}{xe^x}\left[\int (xe^x \cdot x^{-1}e^{-x})\, dx + C\right] \\ &= \frac{1}{xe^x}\left(\int dx + C\right) \end{aligned}$$

$$\boxed{y = \frac{1}{e^x} + \frac{C}{xe^x}}$$

The integration constant is inside the bracket, hence needs to be multiplied by $\frac{1}{R}$ if the brackets were to be removed.

Why can the absolute sign on x in $\ln|x|$ be dropped?

(b) $y' - \left(1 + \frac{3}{x}\right) y = x + 2 \qquad x > 0 \qquad y(1) = e - 1$

The power of y is 1. We can guess that we can use the integrating factor method. The question is already in the standard form where with $P(x) = -\left(1 + \frac{3}{x}\right)$ and $Q(x) = x + 2$.

Step 1 Hence, the integrating factor is given by

$$R(x) = e^{-\int \left(1+\frac{3}{x}\right) dx} = e^{-x-3\ln|x|} = e^{-x} e^{\ln|x^{-3}|} = x^{-3} e^{-x}$$

Step 2 Thus, the solution is given by

$$\begin{aligned} y &= x^3 e^x \left[\int x^{-3} e^{-x} \cdot (x+2)\, dx + C\right] \\ &= x^3 e^x \left[\left(\int \frac{1}{x^2 e^x}\, dx\right) + \int \frac{2}{x^3 e^x} dx + C\right] \\ &= x^3 e^x \left[\left(-x^{-2} e^{-x} - 2\int x^{-3} e^{-x}\, dx\right) + \int \frac{2}{x^3 e^x} dx + C\right] \\ &= -x + C x^3 e^x \end{aligned}$$

Integration by Parts:

$$\boxed{\int u\, dv = uv - \int v\, du}$$

Step 3 With condition $y(1) = e - 1$, we get $C = 1$. Thus the particular solution to the DE is:

$$\boxed{y = x^3 e^x - x}$$

(c) $y' + y + \frac{x}{y} = 0$ is a Bernoulli's equation, with $n = -1$, $P(x) = 1$ and $Q(x) = -x$.

Step 1 Let $z = y^2 \implies \frac{dz}{dx} = 2y\frac{dy}{dx}$. DE becomes

$$\begin{aligned} y' + y &= -xy^{-1} \\ yy' + yy &= -x \\ 2yy' + 2y^2 &= -2x \\ z' + 2z &= -2x \end{aligned}$$

Step 2A Find the integrating factor, $R(x)$.

$$R(x) = e^{\int 2\, dx} = e^{2x}$$

Step 2B The solution is found to be

$$\begin{aligned} z &= e^{-2x}\left[\int e^{2x} \cdot (-2x)\, dx + C\right] \\ &= e^{-2x}\left[-xe^{2x} - \int -e^{2x}\, dx + C\right] \end{aligned}$$

$$\boxed{y^2 = -x + \frac{1}{2} + Ce^{-2x}}$$

(d) $2xyy' + (x-1)y^2 = x^2 e^x,\ x > 0$

Putting into standard form, we get

$$y' + \left(\frac{1}{2} - \frac{1}{2x}\right) y = \frac{1}{2} x e^x y^{-1}$$

This is the standard Bernoulli equation form, with $n = -1$, $P(x) = \frac{1}{2} - \frac{1}{2x}$ and $Q(x) = \frac{1}{2} x e^x$.

Step 1 Let $z = y^2 \implies \frac{dz}{dx} = 2y\frac{dy}{dx}$. DE becomes

$$\begin{aligned} z' + 2\left(\frac{1}{2} - \frac{1}{2x}\right) z &= 2 \cdot \frac{1}{2} x e^x \\ z' + \left(1 - \frac{1}{x}\right) z &= x e^x \end{aligned}$$

Step 2A Find the integrating factor, $R(x)$.

$$R(x) = e^{\int \left(1-\frac{1}{x}\right) dx} = e^{x - \ln|x|} = \frac{1}{x} e^x$$

Step 2B The solution is found to be

$$\begin{aligned} z &= \frac{x}{e^x}\left[\int \left(\frac{e^x}{x} \cdot x e^x\right) dx + C\right] \\ &= \frac{x}{e^x}\left[\frac{1}{2} e^{2x} + C\right] \end{aligned}$$

$$\boxed{y^2 = \frac{1}{2} x e^x + \frac{Cx}{e^x}}$$

Question 2

If a cable is held up at two ends at the same height, then it will sag in the middle, making a U-shaped curve called a **catenary**. This is the shape seen in electricity cables suspended between poles, in countries less advanced than Singapore, such as Japan and the US. It can be shown using simple physics that if the shape is given by a function $y(x)$, then this function satisfies

$$\frac{dy}{dx} = \frac{\mu}{T}\int_0^x \sqrt{\left(\frac{dy}{dt}\right)^2 + 1}\, dt,$$

where $\boldsymbol{x = 0}$ **at the lowest point** of the catenary and $y(0) = 0$, where μ is the weight per unit length of the cable, and where T is the horizontal component of its tension; this horizontal component is a constant along the cable. Find a formula for the shape of the cable. [Hint: Use the Fundamental Theorem of Calculus, and think of the resulting equation as a **first-order** ODE.]

Solutions

- (Extra) The DE given above is known as an implicit first order DE. This means that the term $y'(x)$ cannot be explicitly expressed as a function of the other variables (such as x, y). In general, the method used to solve this kind of DE is to use the substitution $v = y'$. After which, the methods will vary, depending on the type of DE given.

$$\frac{dy}{dx} = \frac{\mu}{T}\int_0^x \sqrt{\left(\frac{dy}{dt}\right)^2 + 1}\, dt$$

The key to solving the DE is to first differentiate both sides with respect to x. This step might not be entirely obvious. We are differentiating both sides so that the integral will disappear by the use of the Fundamental Theorem of Calculus.

$$\frac{d^2y}{dx^2} = \frac{\mu}{T}\frac{d}{dx}\int_0^x \sqrt{\left(\frac{dy}{dt}\right)^2 + 1}\, dt$$
$$= \frac{\mu}{T}\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}$$

Fundamental Theorem of Calculus:

$$\boxed{\frac{d}{dx}\int_a^x f(t)\, dt = f(x)}$$

We have just converted the original DE into a 2nd order non-linear implicit ODE. We can do a reduction to 1st order ODE by a substitution $v = \frac{dy}{dx}$.

$$\frac{1}{\sqrt{(v)^2 + 1}}\frac{dv}{dx} = \frac{\mu}{T}$$
$$\int \frac{1}{\sqrt{v^2 + 1}}\, dv = \frac{\mu}{T}\int dx$$
$$\operatorname{arcsinh} v = \frac{\mu}{T}x + C$$

$$\int \frac{1}{\sqrt{v^2 + 1}}\, dv = \operatorname{arcsinh} v + C$$

With initial condition: lowest point at $x = 0 \Longrightarrow \frac{dy}{dx}(0) = 0 \Longrightarrow v(0) = 0$, we get $C = 0$.

$$\operatorname{arcsinh} v = \frac{\mu}{T}x$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\frac{dy}{dx} = v = \sinh\left(\frac{\mu x}{T}\right)$$

Integrating both sides with respect to x:

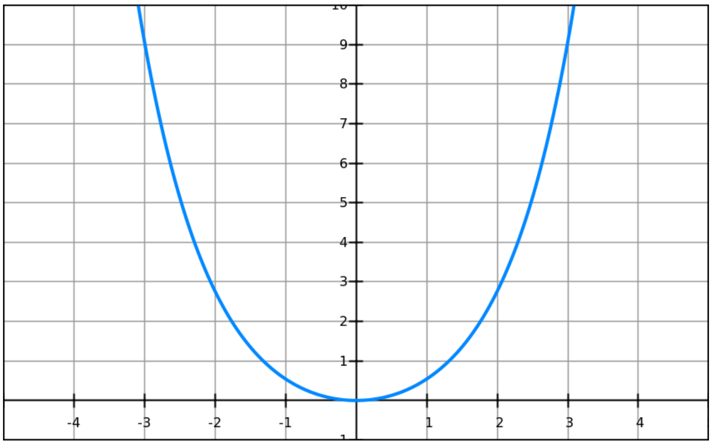
$$y = \frac{T}{\mu}\cosh\left(\frac{\mu x}{T}\right) + D$$

Using initial condition $y(0) = 0$, we get $D = -\frac{T}{\mu}$. Note that $\cosh(0) = 1$. The final equation that describes the shape of the cable is given by

$$\boxed{y = \frac{T}{\mu}\left[\cosh\left(\frac{\mu x}{T}\right) - 1\right]}$$

$$(\sinh x)' = \cosh x$$
$$(\cosh x)' = \sinh x$$

The graph of the solution is given with parameters $T = \mu = 1$. Note that the graph looks like a quadratic equation. Typically, $\cosh x$ is not found in graphing software, so one needs to convert it back to its definition: $\cosh x = \frac{1}{2}(e^x + e^{-x})$. Also, by defining the reference frame, we get the minimum point to be at $x = 0$. There is also a need to specify the domain of x values, as the cable cannot go infinitely high when x goes too big or too small.



Question 3

Psychologists talk about something called a **Performance Curve**. Suppose a MA1512 student is solving mathematics problems. She starts with ordinary differential equations. Let $P(t)$ be a non-negative function that measures her performance, that is, her success rate at solving DEs. Her performance increases rapidly at first, but then the rate of increase slows down as she becomes more expert. Let M, a positive constant, be the best possible performance; then one can suppose that P satisfies

$$\frac{dP}{dt} = C(M - P),$$

where C is a constant. What are the units of this constant? What does this constant measure? Solve this equation assuming that she is completely incompetent at $t = 0$ [that is, $P(0) = 0$].

Now the student turns to another kind of problem, say in partial differential equations. Again, her performance is low at first but gets better in accordance with this equation. Now as the years go by, her overall ability to solve mathematics problems gradually gets better, so C, instead of being a constant, is really a slowly increasing function of time. Suppose that $C(t) = K\tanh\left(\frac{t}{T}\right)$, $t \geq 0$, where K and T are positive constants. [Is this reasonable? Why? What are the meanings of the constants K and T?] Replacing C with $C(t)$, solve for P, again assuming that $P(0) = 0$.

<u>Solutions</u>

Part 1 What are the units of this constant C? What does this constant measure?

By dimensional analysis, C has the units of 1/time, $[T^{-1}]$. The constant C measures the ability of the student to learn, how fast the student can absorb the content. Initially, when the performance P is low, the rate of change of P is large (meaning that the student learns a lot very quickly), because of the large value of $M - P$. Subsequently as time passes by and P starts to approach the maximum value of M, the value of $M - P$ becomes small, hence the rate of change of P decreases, indicating that the rate of learning decreases.

Part 2 Solve the differential equation. The DE is separable:

$$\frac{1}{M-P}\frac{dP}{dt} = C$$

$$\int \frac{1}{M-P}\, dP = \int C\, dt$$

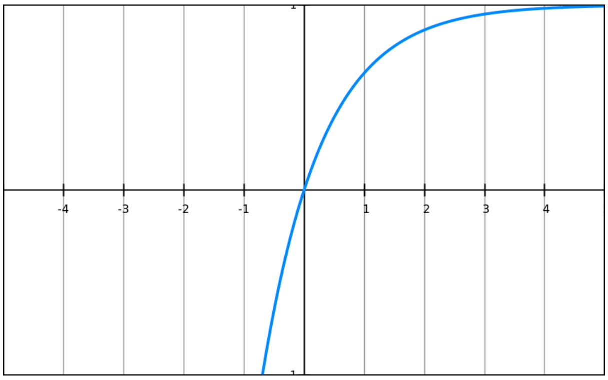
$$-\ln|M-P| = Ct + D$$

$$P = M - Ae^{-Ct}$$

With the initial condition $\mathrm{P}(0) = 0$, we get $A = M$, hence the DE becomes

$$\boxed{P = M(1 - e^{-Ct})}$$

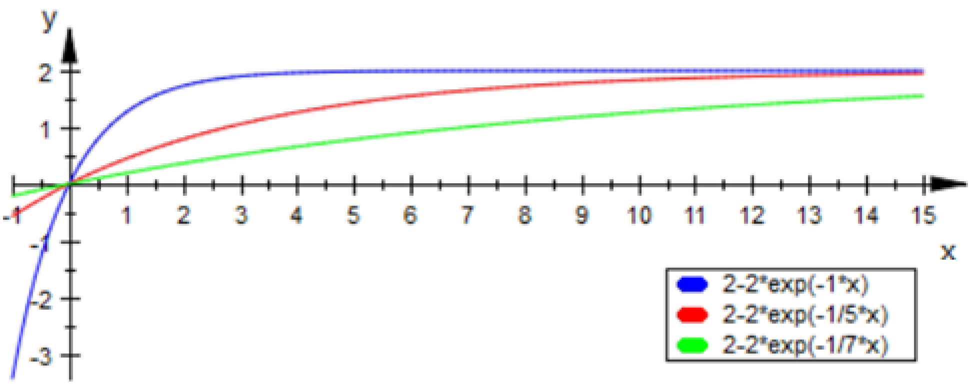
For simplicity's sake, we will take $M = C = 1$, to produce the performance-time graph.



Question The graph will tend towards an asymptote. What is the asymptote? What is the physical interpretation of this asymptote?

Further Analysis

We can provide a deeper analysis by looking at how the values of C affects the graph of the performance curve.

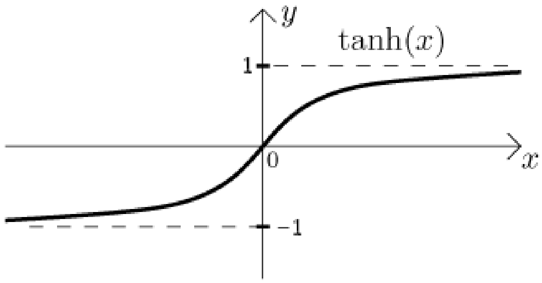


A larger value of C causes the performance curve to reach values close to 2 in a shorter period of time. A larger C value indicates better learning ability of the student, and is thus reflected as higher performance on the performance time chart.

Part 3 Improving the Model: $C(t) = K\tanh\left(\frac{t}{T}\right),\ t \geq 0$

How is this new model reasonable?

- The **rate of learning**, C should not just be a constant. It should be increasing slowly with time.
- There should also be a limit to the **maximum learning ability**, hence the function should also be bounded above.
- The function $\tanh$ is a suitable function and its graph is given below.



What do the constants K and T mean?

The constant K (units: 1/time) should be the maximum rate of learning, while T (units: time) is the amount of time for the student to reach her maximum potential for rate of learning.

- Note that $y = 1$ is a asymptote for $t \geq 0$ when $t \to \infty$. It means that one can never attain his/her maximum potential for learning. Is this reasonable in physical sense? (Also, we will never know what our maximum potential is.)

The new DE is given by

$$\frac{dP}{dt} = K\tanh\left(\frac{t}{T}\right) \cdot (M - P)$$

Part 4 Solving the new DE

$$\int \frac{1}{M-P}\,dP = \int K\tanh\left(\frac{t}{T}\right)dt$$

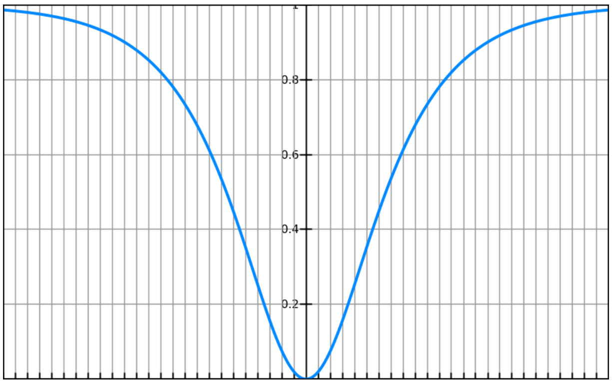
$$-\ln|M-P| = KT\ln\left|\cosh\left(\frac{t}{T}\right)\right| + E$$

$$P = M - F\,\text{sech}^{KT}\left(\frac{t}{T}\right)$$

With the initial condition $P(0) = 0$, we get $F = M$, hence the DE becomes

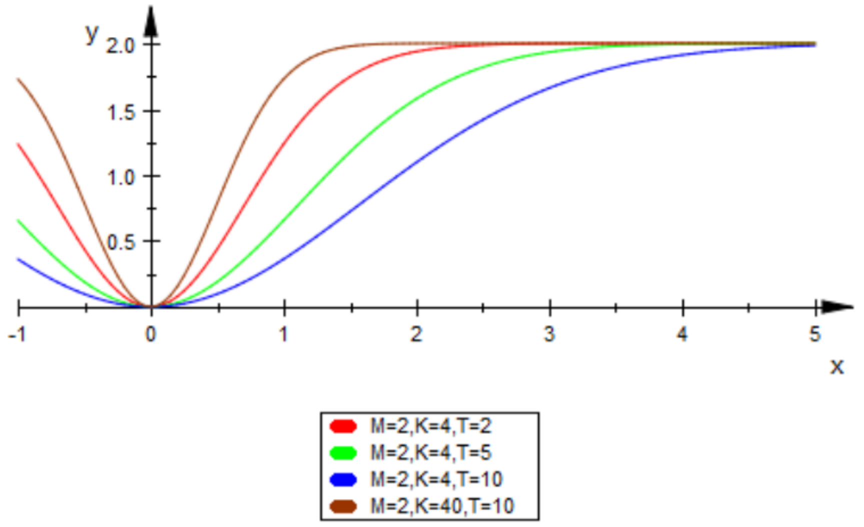
$$\boxed{P = M\left[1 - \text{sech}^{KT}\left(\frac{t}{T}\right)\right]}$$

Once again, for sake of simplicity, we will take constants $M = T = K = 1$.



Further Analysis

The performance-time graph is reproduced below using different set of M, K, T values.



Question Try to analyse why the graphs behave this way, using the interpretation of M, K, T values as described above.

Question 4

A student starts a rumour in school. The number of students who have heard the rumour, $R(t)$, is given by

$$\frac{dR}{dt} = KR(1400 - R),$$

where K is a positive constant, and 1400 is the number of students in that school. What is the meaning of K? Is this equation reasonable? By regarding this equation as a Bernoulli equation, find $R(t)$.

<u>Solutions</u>

Part I Interpreting the DE

- Rate of rumour spread is dependent on K (some scaling factor), the number of students who have heard the rumour, R, and the number of students who have not heard the rumour, $1400 - R$.
- The constant K is probably a measure of **how fast the rumour spreads**. It is dependent on the tendency of students to spread the rumour, how interesting the rumour is, how gossipy students are, how gullible students are etc.

Question What is the sign of $\frac{dR}{dt}$? Is this reasonable?

The equation is reasonable – (1) it is dependent on the number of people who knows the rumour. The more the people who know the rumour, the faster the rumour can potentially spread; (2) However, the rumour cannot spread forever, since there is a maximum number of people who can know this rumour, which is the size of the class (1400). Thus, the rate of spread of rumour should depend on the number of people who do not know of the rumour. This is because if there are lesser people who do not know of the rumour after an extended period of time, the rumour will spread less quickly.

Part II Solving the DE

The equation is simplified to the Bernoulli's form:

$$\frac{dR}{dt} - 1400KR = -KR^2$$

Can you find another method to solve the DE stated above?

Using the substitution $z = R^{-1}$, we obtain the new DE:

$$\frac{dz}{dt} + 1400Kz = K$$

Integrating factor, $R_1(t)$ is given by

$$R_1(t) = e^{\int 1400K\,dt} = e^{1400Kt}$$

The general solution is given by

$$z = e^{-1400Kt}\left[\int e^{1400Kt} \cdot K\,dt + C\right]$$

$$= e^{-1400Kt}\left(\frac{e^{1400Kt}}{1400} + C\right)$$

$$\frac{1}{R(t)} = \frac{1}{1400} + Ce^{-1400Kt}$$

Since at the start, only one student know of the rumour, the initial condition can be $R(0) = 1$.

- Note: We are setting the initial conditions by ourselves again.

Thus, $C = \frac{1399}{1400}$. The particular solution is given by

$$\boxed{\frac{1}{R(t)} = \frac{1}{1400} + \frac{1399}{1400}e^{-1400Kt}}$$

Note that as $t \to \infty$, $R(t) \to 1400$. It makes sense, since at extended periods of time, everyone will know the rumour eventually.

Question 5

The half-life of Thorium 230 is about 75 000 years, while that of Uranium 234 is about 245 000 years. A certain sample of ancient coral has a Thorium/Uranium ratio of 10 percent. How old is the coral?

Solutions

Firstly, Uranium will decay into Thorium, thus each decay of one Uranium atom produces exactly one Thorium atom. The two differential equations can be formulated:

$$\frac{dU}{dt} = -k_U U -- (5.1)$$

$$\frac{dT}{dt} = k_U U - k_T T -- (5.2)$$

where k_U and k_T are constants related to their half-lives with $k_U \neq k_T$ and the initial conditions are

$$U(0) = U_0 \qquad\qquad T(0) = 0$$

Solving equation (5.1) gives us $\boxed{U(t) = U_0 e^{-k_U t}}$.

The half-lives of Uranium (245000 years) and Thorium (75000 years) will give us conditions to find k_U and k_T.

Solving

$$\frac{1}{2}U_0 = U(245000) = U_0 e^{-k_U 245000}$$

and an analogous equation for Thorium will give us:

$$k_U = \frac{\ln 2}{245000} \qquad\qquad k_T = \frac{\ln 2}{75000}$$

- **Question** How is the expression of k_T derived?

Equation (5.2) becomes

$$\frac{dT}{dt} + k_T T = k_U U_0 e^{-k_U t}$$

Using the integrating factor method and together with initial conditions $T(0) = 0$ (Try working this out for yourself), we get

$$\boxed{T(t) = \frac{k_U}{k_T - k_U} U_0 (e^{-k_U t} - e^{-k_T t})}.$$

To get rid of unknown U_0, we take the ratio of T/U.

$$\boxed{\frac{T(t)}{U(t)} = \frac{k_U}{k_T - k_U}\left[1 - e^{(k_U - k_T)t}\right]}.$$

Since the question gives us $\frac{T}{U} = 0.1$, we are able to find the age of the coral. The age of the coral is about **40083 years**.

Question 6

A reaction sequence $A \to B \to C$ takes place in wastewater treatment plants as well as in natural aquatic ecosystems in the breakdown of dead or decaying organic matter. In the first step, A (organic nitrogen) is transformed to B (ammonia) through the degradation of organic nitrogen by heterotrophic bacteria. In the second step, B (ammonia) is transformed into C (nitrate) through the nitrification process by nitrifying bacteria. The overall treatment results in the transformation of nitrogen from complex organic forms (organic nitrogen) to simple inorganic forms, which can then either be taken up by plants as nutrients (in aquatic ecosystem), or broken down even further into nitrogen gas by denitrifying bacteria in wastewater treatment plants.

The reactions then satisfy the following system of ODE, where k_1 and k_2 are two positive constants:

$$\begin{cases} \dfrac{\mathrm{d}A}{\mathrm{d}t} = -k_1 A \\ \dfrac{\mathrm{d}B}{\mathrm{d}t} = k_1 A - k_2 B \\ \dfrac{\mathrm{d}C}{\mathrm{d}t} = k_2 B \end{cases}$$

Suppose at $t = 0$, we have $A = A_0$ and $B = C = 0$. Find a formula for C in terms of A_0, k_1 and k_2.

Solutions

From Question 5, we have

$$B(t) = \frac{k_1}{k_2 - k_1} A_0 (e^{-k_1 t} - e^{-k_2 t}) \text{ and } A(t) = A_0 e^{-k_1 t}.$$

Then,

$$\frac{dC}{dt} = \frac{k_1 k_2}{k_2 - k_1} A_0 (e^{-k_1 t} - e^{-k_2 t})$$

$$C(t) = \frac{A_0 k_1 k_2}{k_2 - k_1}\left(\frac{1}{k_2} e^{-k_2 t} - \frac{1}{k_1} e^{-k_1 t}\right) + c$$

Using initial conditions $C(0) = 0$, we have

$$0 = \frac{A_0 k_1 k_2}{k_2 - k_1}\left(\frac{1}{k_2} - \frac{1}{k_1}\right) + c$$

$$c = -\frac{A_0 k_1 k_2}{k_2 - k_1}\left(\frac{k_1 - k_2}{k_1 k_2}\right)$$

$$= A_0$$

Thus,

$$C(t) = \frac{A_0 k_1 k_2}{k_2 - k_1}\left(\frac{1}{k_2} e^{-k_2 t} - \frac{1}{k_1} e^{-k_1 t}\right) + A_0$$

$$\boxed{C(t) = \frac{A_0}{k_2 - k_1}(k_1 e^{-k_2 t} - k_2 e^{-k_1 t}) + A_0}.$$