NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

Module: MA1508E Linear Algebra for Engineering
Year/Semester: 2018-2019 (Semester 2)
Practice Problem Set: 3 (Solutions)

Name:

Matriculation Number:

Tutorial Group:

Write down the solutions to the problems below, showing all working involved (imagine it is an examination question). Hand in your answers **together with this question paper** before you leave the classroom. You may refer to any materials while answering the questions. You may also discuss with your friends but do not **copy blindly**.

1. Let $\boldsymbol{u} = (a, b, c, d)$ be a vector in $\mathbb{R}^4$. Find condition(s) on a, b, c, d such that $\boldsymbol{u}$ is orthogonal to $\boldsymbol{v_1} = (3, -1, 3, 14)$, $\boldsymbol{v_2} = (6, -2, 3, 1)$ and $\boldsymbol{v_3} = (9, -3, 5, 6)$.

   Solution: For $\boldsymbol{u}$ to be orthogonal to $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$, we must have

$$\left\{\begin{array}{rcl} \boldsymbol{u}\cdot\boldsymbol{v_1} & = & 0 \\ \boldsymbol{u}\cdot\boldsymbol{v_2} & = & 0 \\ \boldsymbol{u}\cdot\boldsymbol{v_3} & = & 0 \end{array}\right. \Rightarrow \left\{\begin{array}{rcrcrcrcl} 3a & - & b & + & 3c & + & 14d & = & 0 \\ 6a & - & 2b & + & 3c & + & d & = & 0 \\ 9a & - & 3b & + & 5c & + & 6d & = & 0 \end{array}\right.$$

   Solving the above linear system

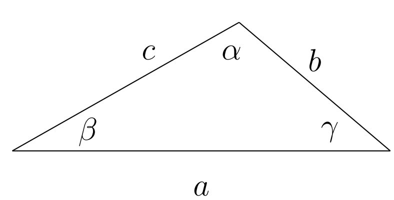
$$\left(\begin{array}{cccc|c} 3 & -1 & 3 & 14 & 0 \\ 6 & -2 & 3 & 1 & 0 \\ 9 & -3 & 5 & 6 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & -\frac{1}{3} & 0 & -\frac{13}{3} & 0 \\ 0 & 0 & 1 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

   So a general solution is

$$\left\{\begin{array}{rcl} a & = & \frac{s}{3} + \frac{13t}{3} \\ b & = & s \\ c & = & -9t \\ d & = & t, \qquad s, t \in \mathbb{R} \end{array}\right.$$

   So the conditions are $c = -9d$ and $a = \frac{1}{3}b + \frac{13}{3}d$.

2. Consider the triangle shown below, whose side lengths are given by a, b, c and whose interior angles are respectively given by α, β, γ.



(a) Using simple trigonometry consideration, prove that

$$b\cos\gamma + c\cos\beta = a.$$

Once proven, by symmetry, you would also have the following:

$$c\cos\alpha + a\cos\gamma = b \qquad \text{and} \qquad a\cos\beta + b\cos\alpha = c.$$

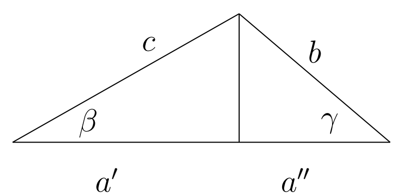
(b) Write down a linear system using the above and find an expression for $\cos\gamma$ in terms of a, b and c. (**Hint:** Cramer's Rule)

(c) Hence prove the Law of Cosines:

$$a^2 + b^2 - 2ab\cos\gamma = c^2.$$

Solution:

(a) We drop a perpendicular from the vertex with angle α to the side with length a, as shown below. This would divide the side into two segments with lengths a' and a'' such that $a = a' + a''$.



Now $\cos\beta = \frac{a'}{c}$ and $\cos\gamma = \frac{a''}{b}$. This implies $c\cos\beta = a'$ and $b\cos\gamma = a''$. Since $a = a' + a''$, we have $a = c\cos\beta + b\cos\gamma$.

(b) Let $X = \cos\alpha$, $Y = \cos\beta$, $Z = \cos\gamma$. Then the linear system is

$$\begin{cases} \quad cY + bZ = a \\ cX + aZ = b \\ bX \quad aY = c \end{cases} \Rightarrow \begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Let the coefficient matrix of the linear system be $\boldsymbol{A}$. Then

$$\det(\boldsymbol{A}) = -c\begin{vmatrix} c & a \\ b & 0 \end{vmatrix} + b\begin{vmatrix} c & 0 \\ b & a \end{vmatrix} = cba + bca = 2abc.$$

Since a, b, c are all positive real numbers, $\det(\boldsymbol{A}) \neq 0$ which implies that $\boldsymbol{A}$ is invertible. By Cramer's Rule,

$$\cos\gamma = Z = \frac{\begin{vmatrix} 0 & c & a \\ c & 0 & b \\ b & a & c \end{vmatrix}}{2abc} = \frac{0 - c(c^2 - b^2) + a(ca)}{2abc} = \frac{a^2 + b^2 - c^2}{2ab}.$$

(c) It follows immediately that

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab} \Rightarrow a^2 + b^2 - 2ab\cos\gamma = c^2.$$