W05-02

Slide 01: In this unit, we introduce the concept of linear independence.

Slide 02: Recall that from a previous unit we saw that in a set of k vectors, if the last vector $\boldsymbol{u_k}$ is a linear combination of the first $k-1$ vectors, then the linear span of all the k vectors is the same as the linear span of just the first $k-1$ vectors. In this case, we say that the last vector $\boldsymbol{u_k}$ is redundant in the linear span of $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$.

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Intuitively, we understand that the vector $\boldsymbol{u_k}$ does not add value to the linear span in the sense that having this additional vector to take linear span with does not generate additional vectors compared to what the first $k-1$ vectors can generate.

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This notion of redundancy is going to be developed over the next few units.

Slide 03: Let us define formally the notion of linear independence. Let the set S, consisting of vectors $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$ be a subset of $\mathbb{R}^n$. Consider the following vector equation (*). In particular, we are interested in finding the solutions to this equation. In other words, we wolud like to find out what real values of c_1 to c_k can be used to satisfy the equation, meaning to make the left hand side equal to the zero vector.

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Clearly if we let all of c_1 to c_k to take on the value 0, then equation (*) would be satisfied as the left hand side will be the zero vector. This solution to equation (*) is known as the trivial solution.

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If it turns out that vector equation (*) has only this trivial solution, then we say that the set S is a linearly independent set. We also say, in this case, that $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$ are linearly independent vectors.

Slide 04: If vector equation (*) can be satisfied by another set of numbers c_1 to c_k that are not all zero, then it has non trivial solutions and we will say that S is a linearly dependent set. We also say that the vectors $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$ are linearly dependent vectors.

Slide 05: Let us determine if the following three vectors are linearly independent vectors in $\mathbb{R}^3$.

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We first set up the vector equation with a linear combination of the 3 vectors on the left and the zero vector on the right. Remember that we wish to find solutions to the vector equation in the coefficients a, b and c.

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Solving such a vector equation is nothing new. We write down the corresponding linear system by comparing components on both sides of the vector equation. Note that the resulting linear system is a homogeneous linear system since the right hand side vector in the vector equation is the zero vector. We already know that homogeneous linear systems always have the trivial solution. What is interesting for us now is to find out if it has non trivial solutions as well.

Slide 06: Performing Gaussian elimination on the augmented matrix, we arrive at the following row-echelon form.

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Can you tell from the row-echelon form, how many solutions does the linear system have?

Slide 07: Since the linear system is just another representation of the vector equation, we should also be able to find out how many solutions does the vector equation have.

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Notice that there is a column on the left hand side that is a non-pivot column. This indicates that this homogeneous linear system and thus the vector equation (*) has infinitely many solutions. In other words, there are non trivial solutions to vector equation (*) in addition to the trivial one.

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Thus our conclusion is that the vectors are linearly dependent vectors.

Slide 08: We will look at another similar example. Here, we have three vectors from $\mathbb{R}^4$ and we wish to determine if they are linearly independent vectors.

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Again, we start off by setting up the vector equation, with unknowns a, b and c.

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We then write down the corresponding homogeneous linear system by comparing components on both sides of the vector equation.

Slide 09: We solve the homogeneous linear system by performing Gaussian elimination and arrive at the following row-echelon form.

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The row-echelon form should once again tell us how many solutions does the homogeneous linear system have.

Slide 10: And equivalently, how many solutions does the vector equation have.

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In this case, we observe that the columns on the left hand side are all pivot columns, this means that the homogeneous linear system has only the trivial solution and equivalently, the vector equation will have only the trivial solution $a = b = c = 0$.

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This allows us to conclude that the vectors are linearly independent vectors.

Slide 11: Now that we have seen how to determine algebraically if a set is linearly independent or not, let us get some intuitive understanding on how what linear independence means for sets with only one or two vectors. Consider a set S with only one vector $\boldsymbol{u}$. When can we say that such a set S is linearly independent? To answer this question, we need to find our when does the equation $c\boldsymbol{u} = \mathbf{0}$ have only the trivial solution $c = 0$.

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Now if $c\boldsymbol{u} = \mathbf{0}$ can only be satisfied by $c = 0$, this means that the vector $\boldsymbol{u}$ must be a non zero vector, for otherwise, any non zero c would satisfy the equation.

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Thus, for a set S containing only one vector $\boldsymbol{u}$, S is linearly independent if and only if $\boldsymbol{u}$ is not the zero vector.

Slide 12: What about a set S with exactly two vectors? When is S a linearly independent set? We would need to know when does the equation $c_1\boldsymbol{u} + c_2\boldsymbol{v} = \mathbf{0}$ have non trivial solutions for c_1 and c_2.

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For non trivial solutions c_1 and c_2, at least one of them is non zero. Suppose c_1 is not zero. Then the vector equation can be rewritten as $\boldsymbol{u}$ equals to $-\frac{c_2}{c_1}\boldsymbol{v}$

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which means that $\boldsymbol{u}$ is in fact a scalar multiple of $\boldsymbol{v}$.

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Therefore, a set S with exactly two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is a linearly dependent set if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are scalar multiples of each other. This result gives us a very easy and convenient way of determining whether a set of two vectors is a linearly independent set.

Slide 13: In summary, in this unit,

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We gave the formal definition of what is meant by a linearly independent set.

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The procedure to check whether a set of vectors are linearly independent or not

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involves the setting up of a vector equation and then its corresponding homogeneous linear system. Upon solving the system

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if we find that the linear system has only the trivial solution, then the vectors are linearly independent.

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On the other hand, if non trivial solutions exists, then the vectors will be linearly dependent.

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In this unit, we also discussed linear independence for sets with one or two vectors.