ROW SPACE AND COLUMN SPACE

DEFINITION

Given any $m \times n$ matrix $\boldsymbol{A}$,

$$\boldsymbol{A} = \begin{pmatrix} \overbrace{a_{11} & a_{12} & \cdots & a_{1n}}^{n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

$$\boldsymbol{r}_1 = (a_{11} \quad a_{12} \quad \cdots \quad a_{1n})$$
$$\boldsymbol{r}_2 = (a_{21} \quad a_{22} \quad \cdots \quad a_{2n})$$
$$\vdots$$
$$\boldsymbol{r}_m = (a_{m1} \quad a_{m2} \quad \cdots \quad a_{mn})$$

The rows of $\boldsymbol{A}$ can be considered as vectors in $\mathbb{R}^n$.

$\Rightarrow$ span$\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_m\}$ is a subspace of $\mathbb{R}^n$,

This subspace is called the row space of $\boldsymbol{A}$.

DEFINITION

Given any $m \times n$ matrix $\boldsymbol{A}$,

$$\boldsymbol{A} = \left.\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}\right\} m$$

The columns of $\boldsymbol{A}$ can be considered as vectors in $\mathbb{R}^m$.

$\Rightarrow$ span$\{\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_n\}$ is a subspace of $\mathbb{R}^m$,

$$\boldsymbol{c}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \quad \boldsymbol{c}_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \cdots \quad \boldsymbol{c}_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

This subspace is called the column space of $\boldsymbol{A}$.

REMARK

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 4 & -2 \\ 1 & 2 & 0 \\ 4 & 1 & 1 \end{pmatrix}$$

The row space of A is the column space of A^T

The column space of A is the row space of A^T

EXAMPLE

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

The row space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^3$.

The column space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^4$.

Note that if $\boldsymbol{A}$ is not a square matrix, then the row space and column space of $\boldsymbol{A}$ contains totally 'different type' of vectors.

EXAMPLE

$$\boldsymbol{A} = \begin{array}{c} \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{l} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \boldsymbol{r}_3 \\ \boldsymbol{r}_4 \end{array}$$

We write $\boldsymbol{r}_1 = (2,-1,0)$ (as a vector) rather than a row matrix $\begin{pmatrix} 2 & 1 & 0 \end{pmatrix}$.

The row space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^3$.

$$= \text{span}\{\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_4\}$$

$$= \{a(2,1,0) + b(1,-1,3) + c(-5,1,0) + d(1,0,1) \mid a,b,c,d \in \mathbb{R}\}$$

$$= \{(2a + b - 5c + d, -a - b + c, 3b + d) \mid a,b,c,d \in \mathbb{R}\}$$

EXAMPLE

$$\boldsymbol{A} = \begin{array}{c} \begin{pmatrix} \boxed{\begin{matrix} 2 \\ 1 \\ -5 \\ 1 \end{matrix}} & \boxed{\begin{matrix} -1 \\ -1 \\ 1 \\ 0 \end{matrix}} & \boxed{\begin{matrix} 0 \\ 3 \\ 0 \\ 1 \end{matrix}} \end{pmatrix} \\ \begin{matrix} \quad \boldsymbol{c}_1 & \quad \boldsymbol{c}_2 & \quad \boldsymbol{c}_3 \end{matrix} \end{array}$$

The column space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^4$.

$= \text{span}\{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3\}$

$$= \left\{ a\begin{pmatrix} 2 \\ 1 \\ -5 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 2a - b \\ a - b + 3c \\ -5a + b \\ a + c \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}$$

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

How to find a basis for the row space of $\boldsymbol{A}$?

The row space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^5$.

The column space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^3$.

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

How to find a basis for the row space of $\boldsymbol{A}$?

row space of $\boldsymbol{A} = \text{span}\{(1,0,-1,1,4),(0,1,4,2,1),(0,0,-2,0,1)\}$

If $(1,0,-1,1,4),(0,1,4,2,1),(0,0,-2,0,1)$ (that is, the rows of $\boldsymbol{A}$) are linearly independent, then obviously they will form a basis for the row space of $\boldsymbol{A}$.

EXAMPLE

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

How to find a basis for the row space of A?

row space of $A = \text{span}\{(1,0,-1,1,4),(0,1,4,2,1),(0,0,-2,0,1)\}$

$$(0,0,0,0,0) = a(1,0,-1,1,4) + b(0,1,4,2,1) + c(0,0,-2,0,1)$$

$$\Rightarrow a = 0, b = 0, c = 0$$

So the three rows of A are linearly independent and thus form a basis for the row space of A.

EXAMPLE

$$A=\begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

Note that $\boldsymbol{A}$ is in row echelon form.

What if we want to find a basis for the row space of a matrix $\boldsymbol{R}$ that is in row echelon form?

$$\boldsymbol{R}=\begin{pmatrix} & \otimes & * & & & & * \\ & & \otimes & * & & & * \\ & \mathbf{0} & & \ddots & & & * \\ & & & & \otimes & * & * \\ 0 & \cdots & \cdots & & \cdots & 0 & 0 \end{pmatrix}\begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \\ \end{array}\right\} \text{non zero rows} \\ \left.\begin{array}{l} \\ \end{array}\right\} \text{zero rows (if any)} \end{array}$$

EXAMPLE

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

Note that $\boldsymbol{A}$ is in row echelon form.

The non zero rows of $\boldsymbol{R}$ are always linearly independent and thus forms a basis for the row space of $\boldsymbol{R}$.

$$\boldsymbol{R} = \begin{pmatrix} \otimes & * & & & & * \\ & & \otimes & * & & * \\ & \mathbf{0} & & \ddots & & * \\ & & & \otimes & * & * \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix}$$

non zero rows

zero rows (if any)

SUMMARY

1) Definition of the row space and column space of a $m \times n$ matrix $\boldsymbol{A}$.

2) If the rows of $\boldsymbol{A}$ are linearly independent, then they form a basis for the row space of $\boldsymbol{A}$.

3) If $\boldsymbol{R}$ is a matrix in row-echelon form, then the non-zero rows of $\boldsymbol{R}$ form a basis for the row space of $\boldsymbol{R}$.