SUBSPACES

DEFINITION (SUBSPACES)

Let V be a subset of $\mathbb{R}^n$.

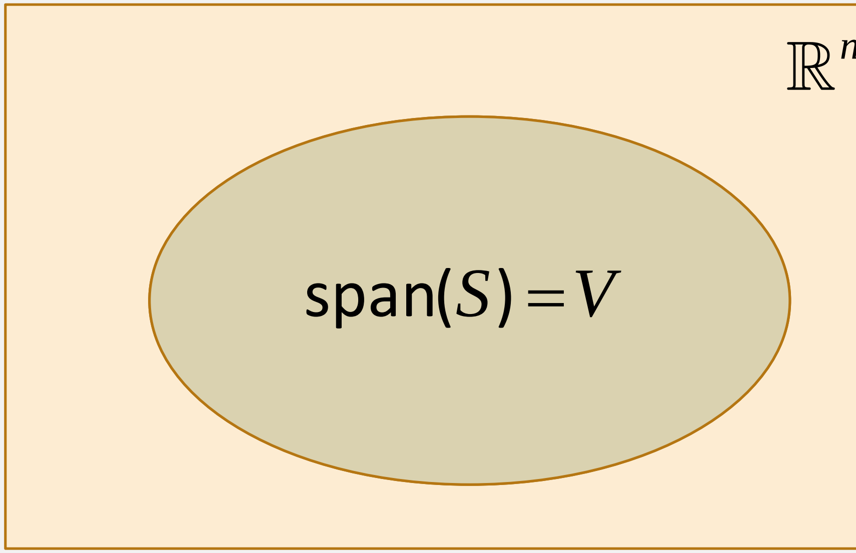
If there exists a set of vectors

$$S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} \text{ in } \mathbb{R}^n$$

such that $\text{span}(S) = V$,

then V is said to be

a subspace of $\mathbb{R}^n$.



Remember: $\text{span}(S) =$ set of all linear combinations of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$.

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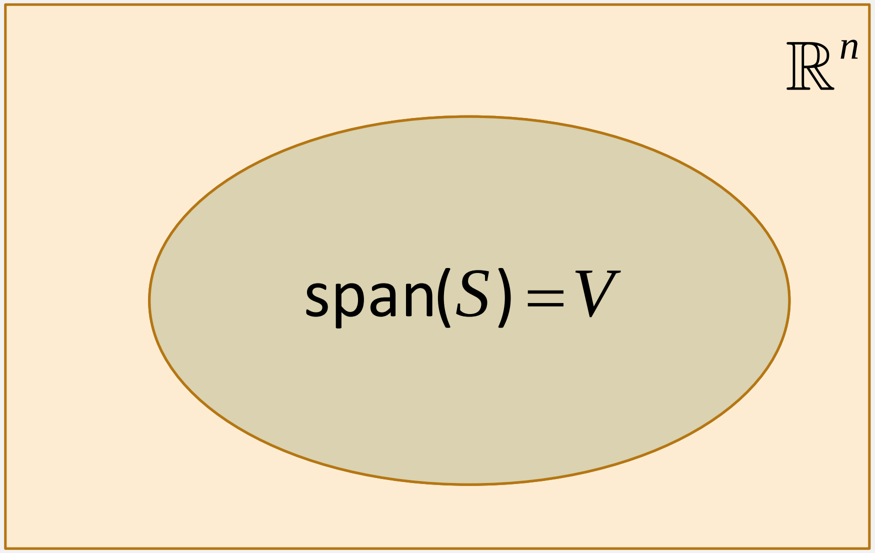


More precisely, we say:

1) V is the subspace spanned by S.

2) V is the subspace spanned by $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k$.

3) S spans V.

THE ZERO SPACE

$\{\mathbf{0}\} = \text{span}\{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$.

(Here, $\mathbf{0}$ is the zero vector of $\mathbb{R}^n$.)

It is also called the zero space of $\mathbb{R}^n$.

It is also the only subspace of $\mathbb{R}^n$ with a finite number (in this case, one), and thus the least, of vectors.

THE "ENTIRE UNIVERSE"

Consider the vectors in $\mathbb{R}^n$:

$$\boldsymbol{e}_1 = (1,0,\ldots.,0) \qquad \boldsymbol{e}_2 = (0,1,\ldots.,0) \qquad \cdots\cdot \qquad \boldsymbol{e}_n = (0,0,\ldots.,1)$$

$$\begin{aligned}
\mathbb{R}^n &= \{(u_1, u_2, \ldots, u_n) \,|\, u_1, u_2, \ldots, u_n \in \mathbb{R}\} \\
&= \{u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + \ldots + u_n\boldsymbol{e}_n \,|\, u_1, u_2, \ldots, u_n \in \mathbb{R}\} \\
&= \text{the set of all linear combinations of } \boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n \\
&= \text{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\} = \text{a subspace of } \mathbb{R}^n
\end{aligned}$$

So $\mathbb{R}^n$ is a subspace of itself and to some extent, it can be thought of as the subspace of $\mathbb{R}^n$ with the 'largest' number of vectors.

EXAMPLE

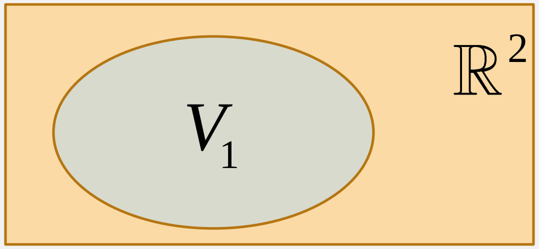
$V_1 = \{(a-2b, 3b) \mid a, b \in \mathbb{R}\}$. V_1 is a subset of $\mathbb{R}^2$.

Is V_1 a subspace of $\mathbb{R}^2$?

Idea: Can you try to write V_1 as a linear span?

$$\begin{aligned} V_1 &= \{(a-2b, 3b) \mid a, b \in \mathbb{R}\} \\ &= \{a(1,0) + b(-2,3) \mid a, b \in \mathbb{R}\} \\ &= \operatorname{span}\{(1,0), (-2,3)\} \end{aligned}$$

Is $V_1 = \mathbb{R}^2$?



So V_1 is a subspace of $\mathbb{R}^2$.

Is there a vector in $\mathbb{R}^2$ that is not in V_1?

EXAMPLE

$V_1 = \{(a-2b, 3b) \mid a, b \in \mathbb{R}\}$. V_1 is a subspace of $\mathbb{R}^2$.

Is $V_1 = \mathbb{R}^2$?

Idea: Is $\text{span}\{(1,0),(-2,3)\} = \mathbb{R}^2$?

$$a(1,0) + b(-2,3) = (x, y)$$

$$\begin{cases} a - 2b = x \\ \quad\;\; 3b = y \end{cases} \qquad \left(\begin{array}{cc|c} 1 & -2 & x \\ 0 & 3 & y \end{array}\right)$$

Row-echelon form

No zero rows

$$V_1 = \text{span}\{(1,0),(2,-3)\} = \mathbb{R}^2$$

EXAMPLE

$V_2 = \{(x,y,z) \mid x - 3y + 2z = 0\}$. V_2 is a subset of $\mathbb{R}^3$.

Describe V_2 geometrically. Is V_2 a subspace of $\mathbb{R}^3$? YES!

Idea: Can we express V_2 in another form?

$$(x,y,z) \in V_2 \Leftrightarrow x - 3y + 2z = 0 \Leftrightarrow \begin{cases} x &= 3s - 2t \\ y &= s \\ z &= t, \quad s,t \in \mathbb{R} \end{cases}$$

$$\begin{aligned} V_2 &= \{(x,y,z) \mid x - 3y + 2z = 0\} \\ &= \{(3s - 2t, s, t) \mid s,t \in \mathbb{R}\} \\ &= \{s(3,1,0) + t(-2,0,1) \mid s,t \in \mathbb{R}\} = \text{span}\{(3,1,0),(-2,0,1)\} \end{aligned}$$

EXAMPLE

$V_3 = \{(x, y, z) \mid x - 3y + 2z = 1\}$. V_3 is a subset of $\mathbb{R}^3$.

Describe V_3 geometrically. Is V_3 a subspace of $\mathbb{R}^3$?

$(0,0,0) \notin V_3$ since $0 - 3(0) + 2(0) \neq 1$.

We have already shown that any linear span must contain the zero vector. Since V_3 does not contain the zero vector, it cannot be a linear span.

So V_3 is not a subspace of $\mathbb{R}^3$.

EXAMPLE

$V_4 = \{(x,y,z) \mid x \le y \le z\}$. V_4 is a subset of $\mathbb{R}^3$.

Is V_4 a subspace of $\mathbb{R}^3$? No!

If V_4 is a subspace, then it can be expressed as a linear span, that is, $V_4 = \text{span}(T)$ for some finite set T.

By the closure property of linear spans,

For any $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r \in \text{span}(T) = V_4$ and $c_1, c_2, \ldots, c_r \in \mathbb{R}$,

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_r\boldsymbol{v}_r \in \text{span}(T) = V_4.$$

$(1,1,2),(0,2,4) \in V_4$ but $(1,1,2) - 2(0,2,4) = (1,-3,-6) \notin V_4$.

So V_4 is not a subspace of $\mathbb{R}^3$.

SUMMARY

1) Definition of a subspace.

2) Zero space is a subspace of $\mathbb{R}^n$ and $\mathbb{R}^n$ is a subspace of itself.

3) How we can show that a given subset is NOT a subspace.