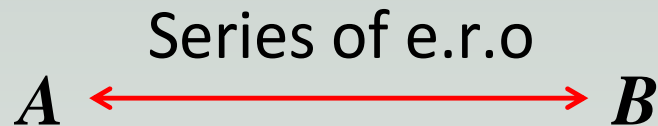


FINDING A BASIS FOR ROW SPACE

DISCUSSION

Recall the definition of row equivalent.

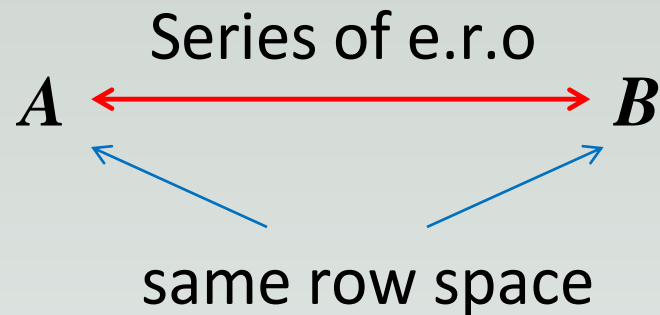


A and B are
row equivalent

Two matrices A and B (of the same size) are row equivalent if and only if they have a similar row-echelon form (or they have the same unique reduced row-echelon form).

THEOREM

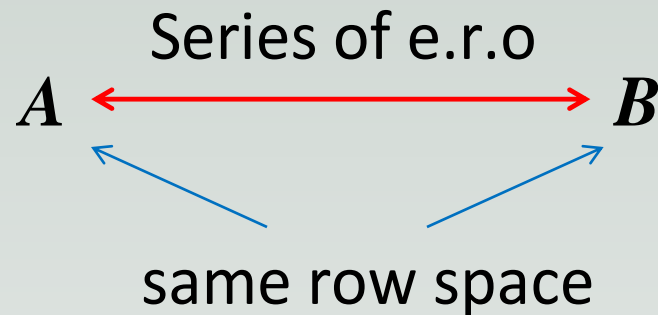
Let A and B be row equivalent matrices. Then the row space of A and the row space of B are identical.



That is to say, performing elementary row operations on A does not change its row space.

THEOREM

Let A and B be row equivalent matrices. Then the row space of A and the row space of B are identical.



Proof: If we can show that the row space of a matrix does not change after a single elementary row operation (any one of the three types), then it follows that the row space will always be the same after a series of elementary row operations.

THEOREM

Proof: We will use this theorem introduced in an earlier unit on linear spans.

Let $S_1 = \{u_1, u_2, \dots, u_k\}$ and $S_2 = \{v_1, v_2, \dots, v_m\}$ be subsets of \mathbb{R}^n .

Then $\text{span}(S_1) \subseteq \text{span}(S_2) \iff$ each u_i is a linear combination of v_1, v_2, \dots, v_m .

Consider the first type of elementary row operation:

$$A \xrightarrow{cR_i \ (c \neq 0)} B$$

THEOREM

Let $S_1 = \{u_1, u_2, \dots, u_k\}$ and $S_2 = \{v_1, v_2, \dots, v_m\}$ be subsets of \mathbb{R}^n .

Then $\text{span}(S_1) \subseteq \text{span}(S_2) \iff$ each u_i is a linear combination of v_1, v_2, \dots, v_m .

Rows of $A = \{r_1, r_2, \dots, r_i, \dots, r_n\}$



Rows of $B = \{r_1, r_2, \dots, cr_i, \dots, r_n\}$

Row space of A

Row space of B

$= \text{span}\{r_1, r_2, \dots, r_i, \dots, r_n\} \subseteq ?$

$= \text{span}\{r_1, r_2, \dots, cr_i, \dots, r_n\}$

Is every vector from  a linear combination of vectors from  ?

THEOREM

Let $S_1 = \{u_1, u_2, \dots, u_k\}$ and $S_2 = \{v_1, v_2, \dots, v_m\}$ be subsets of \mathbb{R}^n .

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Rows of $B = \{r_1, r_2, \dots, cr_i, \dots, r_n\}$

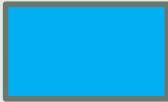

Row space of A

Row space of B

$= \text{span}\{r_1, r_2, \dots, r_i, \dots, r_n\}$

$\supseteq ?$

$= \text{span}\{r_1, r_2, \dots, cr_i, \dots, r_n\}$

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Shown: Row space of A = Row space of B

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Then $\text{span}(S_1) \subseteq \text{span}(S_2) \iff$ each u_i is a linear combination of v_1, v_2, \dots, v_m .

Consider the second type of elementary row operation:

$$A \xrightarrow{R_i \leftrightarrow R_j} B$$

THEOREM

Let $S_1 = \{u_1, u_2, \dots, u_k\}$ and $S_2 = \{v_1, v_2, \dots, v_m\}$ be subsets of \mathbb{R}^n .

Then $\text{span}(S_1) \subseteq \text{span}(S_2) \iff$ each u_i is a linear combination of v_1, v_2, \dots, v_m .

Rows of $A = \{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_i, \dots, \mathbf{r}_j, \dots, \mathbf{r}_n\}$

Rows of $B = \{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_j, \dots, \mathbf{r}_i, \dots, \mathbf{r}_n\}$

Row space of $A = \text{span}\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_i, \dots, \mathbf{r}_j, \dots, \mathbf{r}_n\}$



Row space of $B = \text{span}\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_j, \dots, \mathbf{r}_i, \dots, \mathbf{r}_n\}$

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Shown: Row space of A = Row space of B

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Then $\text{span}(S_1) \subseteq \text{span}(S_2) \iff$ each u_i is a linear combination of v_1, v_2, \dots, v_m .

Consider the third type of elementary row operation:

$$A \xrightarrow{R_i + cR_j} B$$

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Let $S_1 = \{u_1, u_2, \dots, u_k\}$ and $S_2 = \{v_1, v_2, \dots, v_m\}$ be subsets of \mathbb{R}^n .

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Rows of $A = \{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_i, \dots, \mathbf{r}_j, \dots, \mathbf{r}_n\}$

Rows of $B = \{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_i + c\mathbf{r}_j, \dots, \mathbf{r}_j, \dots, \mathbf{r}_n\}$

Row space of $A = \text{span}\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_i, \dots, \mathbf{r}_j, \dots, \mathbf{r}_n\}$

$\subseteq ?$

Row space of $B = \text{span}\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_i + c\mathbf{r}_j, \dots, \mathbf{r}_j, \dots, \mathbf{r}_n\}$

Every vector

from a

linear combination

of vectors

from ?

THEOREM

Let $S_1 = \{u_1, u_2, \dots, u_k\}$ and $S_2 = \{v_1, v_2, \dots, v_m\}$ be subsets of \mathbb{R}^n .

Then $\text{span}(S_1) \subseteq \text{span}(S_2) \iff$ each u_i is a linear combination of v_1, v_2, \dots, v_m .

YES!

$$\mathbf{r}_i = (\mathbf{r}_i + c\mathbf{r}_j) - c\mathbf{r}_j$$

Every vector

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linear combination

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Row space of $A = \text{span}\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_i, \dots, \mathbf{r}_j, \dots, \mathbf{r}_n\}$

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Row space of $A = \text{span}\{r_1, r_2, \dots, r_i, \dots, r_j, \dots, r_n\}$

$\supseteq ?$

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Then $\text{span}(S_1) \subseteq \text{span}(S_2) \iff$ each u_i is a linear combination of v_1, v_2, \dots, v_m .

YES!

$$\mathbf{r}_i + c\mathbf{r}_j = \mathbf{r}_i + c\mathbf{r}_j$$

Every vector

from a

linear combination

of vectors

from ?

Row space of $A = \text{span}\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_i, \dots, \mathbf{r}_j, \dots, \mathbf{r}_n\}$

\supseteq ?

Row space of $B = \text{span}\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_i + c\mathbf{r}_j, \dots, \mathbf{r}_j, \dots, \mathbf{r}_n\}$

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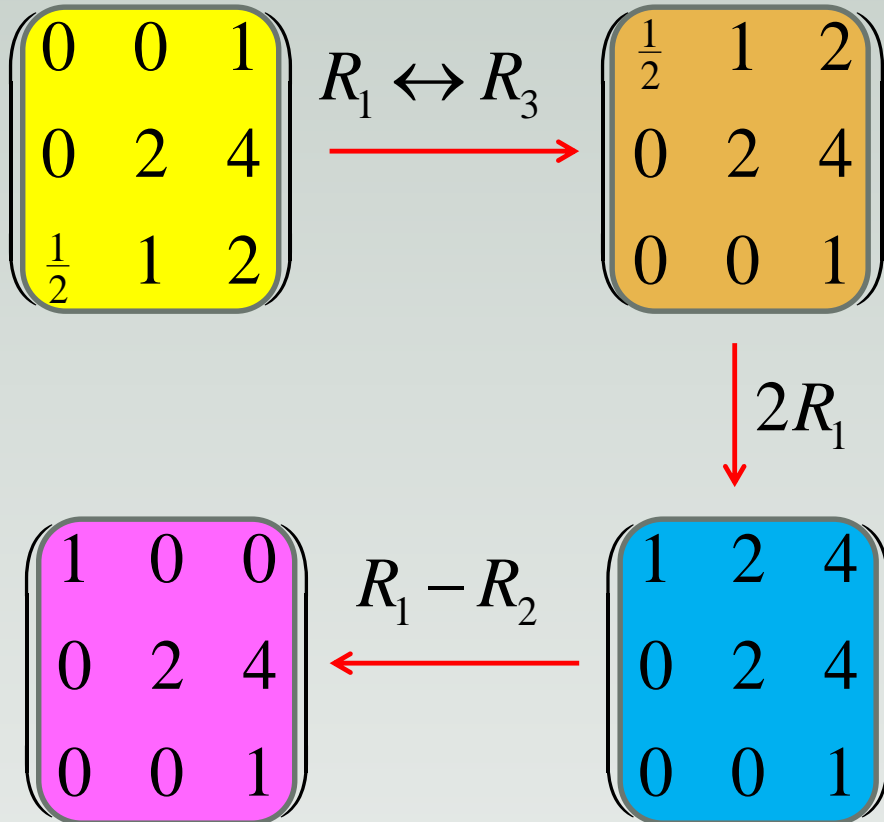
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Shown: Row space of A = Row space of B

EXAMPLE



$$\begin{aligned} & \text{span}\{(0,0,1), (0,2,4), (\tfrac{1}{2},1,2)\} \\ &= \text{span}\{(\tfrac{1}{2},1,2), (0,2,4), (0,0,1)\} \\ &= \text{span}\{(1,2,4), (0,2,4), (0,0,1)\} \\ &= \text{span}\{(1,0,0), (0,2,4), (0,0,1)\} \end{aligned}$$

BACK TO THIS QUESTION

Question: How to find a basis for the row space of a matrix A ?

$$R = \begin{pmatrix} \text{⊗} & * & & * \\ & \text{⊗} & * & * \\ & & \ddots & * \\ 0 & & & \text{⊗} & * & * \\ 0 & \dots & \dots & \dots & 0 & 0 \end{pmatrix}$$

The non zero rows of R are always linearly independent and thus forms a basis for the row space of R .

Answer:

Find a row-echelon form R of A .

A basis for the row space of R is also a basis for the row space of A .

Let A and B be row equivalent matrices. Then the row space of A and the row space of B are identical.

EXAMPLE

Find a basis for the row space of the following matrix.

$$A = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & \frac{3}{2} & -3 & \frac{3}{2} \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Performing Gaussian Elimination on A :

A basis for the row space of A is

$$\{(2, 2, -1, 0, 1), (0, 0, \frac{3}{2}, -3, \frac{3}{2}), (0, 0, 0, 3, 0)\}$$

SUMMARY

- 1) Row equivalent matrices have the same row space.
- 2) A method to find a basis for the row space of a matrix.