Week 03 F2F Example Solutions

1. **Example 3.1** Let $\boldsymbol{X} = \begin{pmatrix} \boldsymbol{x_1} & \boldsymbol{x_2} & \boldsymbol{x_3} \end{pmatrix}$, then we may choose

$$\boldsymbol{x_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{x_2} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{x_3} = 3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$$

So $\boldsymbol{X} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

2. **Example 3.2**

(a) $\boldsymbol{A}^2 = \begin{pmatrix} 4 & -6 & -6 \\ 0 & 10 & 6 \\ 0 & 6 & 10 \end{pmatrix}$, $-6\boldsymbol{A} = \begin{pmatrix} -12 & 6 & 6 \\ 0 & -18 & -6 \\ 0 & -6 & -18 \end{pmatrix}$, $8\boldsymbol{I} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$.

It is easy to be checked that $\boldsymbol{A}^2 - 6\boldsymbol{A} + 8\boldsymbol{I} = \boldsymbol{0}$.

(b) By (a), $\boldsymbol{A}^2 = 6\boldsymbol{A} - 8\boldsymbol{I}$. Since

$$\boldsymbol{A}\left[\tfrac{1}{8}(6\boldsymbol{I} - \boldsymbol{A})\right] = \tfrac{1}{8}\boldsymbol{A}(6\boldsymbol{I} - \boldsymbol{A}) = \tfrac{1}{8}(6\boldsymbol{A} - \boldsymbol{A}^2) = \tfrac{1}{8}(6\boldsymbol{A} - 6\boldsymbol{A} + 8\boldsymbol{I}) = \boldsymbol{I},$$

$\boldsymbol{A}^{-1} = \frac{1}{8}(6\boldsymbol{I} - \boldsymbol{A})$.

3. **Example 3.3**

(a) Since $(\boldsymbol{I}-\boldsymbol{A})(\boldsymbol{I}+\boldsymbol{A}) = \boldsymbol{I}-\boldsymbol{A}^2 = \boldsymbol{I}$, $\boldsymbol{I}-\boldsymbol{A}$ is invertible and $(\boldsymbol{I}-\boldsymbol{A})^{-1} = \boldsymbol{I}+\boldsymbol{A}$.

(b) Since $(\boldsymbol{I}-\boldsymbol{A})(\boldsymbol{I}+\boldsymbol{A}+\boldsymbol{A}^2) = \boldsymbol{I}-\boldsymbol{A}^3 = \boldsymbol{I}$, $\boldsymbol{I}-\boldsymbol{A}$ is invertible and $(\boldsymbol{I}-\boldsymbol{A})^{-1} = \boldsymbol{I}+\boldsymbol{A}+\boldsymbol{A}^2$. In general, we have $(\boldsymbol{I}-\boldsymbol{A})(\boldsymbol{I}+\boldsymbol{A}+\cdots+\boldsymbol{A}^{n-1}) = \boldsymbol{I}-\boldsymbol{A}^n$. So if $\boldsymbol{A}^n = \boldsymbol{0}$, then $\boldsymbol{I}-\boldsymbol{A}$ is invertible and its inverse is $\boldsymbol{I}+\boldsymbol{A}+\cdots+\boldsymbol{A}^{n-1}$.

(c) Yes, $\boldsymbol{A}$ is invertible. $(\boldsymbol{A}-k\boldsymbol{I})(\boldsymbol{A}+k\boldsymbol{I}) = \boldsymbol{0} \Leftrightarrow \boldsymbol{A}^2 - k^2\boldsymbol{I}^2 = \boldsymbol{0} \Leftrightarrow \boldsymbol{A}^2 = k^2\boldsymbol{I} \Leftrightarrow (\frac{1}{k}\boldsymbol{A})(\frac{1}{k}\boldsymbol{A}) = \boldsymbol{I}$. Thus $\frac{1}{k}\boldsymbol{A}$ is invertible and since k is nonzero, $k(\frac{1}{k}\boldsymbol{A}) = \boldsymbol{A}$ is also invertible.

4. **Example 3.4** It is clear that the sizes of $(\boldsymbol{AB})^T$ and $\boldsymbol{B}^T\boldsymbol{A}^T$ are the same. Let $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$. The (i,j)-entry of $(\boldsymbol{AB})^T$ is the $/(j,i)$-entry of $\boldsymbol{AB}$ which is

$$a_{j1}b_{1i} + a_{j2}b_{2i} + \cdots + a_{jn}b_{ni}.$$

On the other hand, let $\boldsymbol{A}^T = (a'_{ij})$ and $\boldsymbol{B}^T = (b'_{ij})$. By definition of transpose, $a'ij = a_{ji}$ and $b'_{ij} = b_{ji}$. So the (i,j)-entry of $\boldsymbol{B}^T\boldsymbol{A}^T$ is

$$b'_{i1}a'_{1j} + b'_{i2}a'_{2j} + \cdots + b'_{in}a'_{nj} = b_{1i}a_{j1} + b_{2i}a_{j2} + \cdots + b_{ni}a_{jn}.$$

Thus the (i,j)-entry of $(\boldsymbol{AB})^T$ is equal to the (i,j)-entry of $\boldsymbol{B}^T\boldsymbol{A}^T$.

5. **Example 3.5**

$$\boldsymbol{E_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{E_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \quad \boldsymbol{E_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{E_4} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$\boldsymbol{F_1} = \boldsymbol{E_4}^{-1} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{F_2} = \boldsymbol{E_3}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\boldsymbol{F_3} = \boldsymbol{E_2}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \boldsymbol{F_4} = \boldsymbol{E_1}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

6. **Example 3.6**

(a)

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ -1 & 2 & -2 & 3 \end{array}\right) \xrightarrow{R_3 + R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 5 \end{array}\right) \xrightarrow{R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 3 \end{array}\right) \xrightarrow{-R_3}$$
$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right) \xrightarrow{R_1 - R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

Thus $\boldsymbol{x} = \begin{pmatrix} 5 \\ 1 \\ -3 \end{pmatrix}$.

(b) From (a), we know that $\boldsymbol{E_4 E_3 E_2 E_1 A} = \boldsymbol{I}$ where

$$\boldsymbol{E_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \boldsymbol{E_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \quad \boldsymbol{E_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \boldsymbol{E_4} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So $\boldsymbol{Ax} = \boldsymbol{b} \Rightarrow \boldsymbol{E_4 E_3 E_2 E_1 Ax} = \boldsymbol{E_4 E_3 E_2 E_1 b}$, we have $\boldsymbol{x} = \boldsymbol{E_4 E_3 E_2 E_1 b}$.