W03-08

Slide 01: In this unit, we will discuss the interplay between elementary row operations and the determinant of a matrix.

Slide 02: Let us consider the first type of elementary row operation. For the 3×3 matrix $\boldsymbol{A}$ here, we are now able to write down its determinant.

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Suppose we perform kR_3 on $\boldsymbol{A}$

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and obtain the matrix $\boldsymbol{B_1}$.

(#)

Recall that the elementary matrix that represents this elementary row operation is the matrix $\boldsymbol{E_1}$ shown here and pre-multiplying $\boldsymbol{E_1}$ to $\boldsymbol{A}$ gives us $\boldsymbol{B_1}$.

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The determinant of $\boldsymbol{B_1}$ can also be computed

(#)

and subsequently simplified to give $k\det(\boldsymbol{A})$.

Slide 03: Since the determinant of the elementary matrix $\boldsymbol{E_1}$ is k and we have observed that the determinant of $\boldsymbol{B_1}$ is k times the determinant of $\boldsymbol{A}$,

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we now see that corresponding to $\boldsymbol{B_1}$ equals to $\boldsymbol{E_1 A}$,

(#)

we have determinant of $\boldsymbol{B_1}$ equals to determinant of $\boldsymbol{E_1}$ multiplied to determinant of $\boldsymbol{A}$.

(#)

We make two observations here. The first is that elementary row operations of the first type, when performed on $\boldsymbol{A}$ seems to change the determinant of $\boldsymbol{A}$ by a factor of k.

(#)

Secondly, we have the determinant of a matrix product is equal to the product of the determinant of the matrices.

Slide 04: Consider the second type of elementary row operation. Once again, we have the determinant of the 3×3 matrix $\boldsymbol{A}$.

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Suppose we perform a row swap between rows 1 and 3

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and obtain the matrix $\boldsymbol{B_2}$.

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Recall that the elementary matrix that represents this elementary row operation is the matrix $\boldsymbol{E_2}$ shown here and pre-multiplying $\boldsymbol{E_2}$ to $\boldsymbol{A}$ gives us $\boldsymbol{B_2}$.

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The determinant of $\boldsymbol{B_2}$ can also be computed

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and subsequently simplified to give $-\det(\boldsymbol{A})$.

Slide 05: Since the determinant of the elementary matrix $\boldsymbol{E_2}$ is -1 and we have observed that the determinant of $\boldsymbol{B_2}$ is -1 times the determinant of $\boldsymbol{A}$,

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we now see that corresponding to $\boldsymbol{B_2}$ equals to $\boldsymbol{E_2 A}$,

(#)

we have determinant of $\boldsymbol{B_2}$ equals to determinant of $\boldsymbol{E_2}$ multiplied to determinant of $\boldsymbol{A}$.

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We again make two observations here. The first is that elementary row operations of the second type, when performed on $\boldsymbol{A}$ seems to change the determinant of $\boldsymbol{A}$ by a factor of -1.

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Secondly, we again have the determinant of a matrix product being equal to the product of the determinant of the matrices.

Slide 06: Let us now consider the third type of elementary row operation. Again the determinant of matrix $\boldsymbol{A}$ is noted.

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Suppose we perform $R_2 + kR_3$

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and obtain the matrix $\boldsymbol{B_3}$.

(#)

Recall that the elementary matrix that represents this elementary row operation is the matrix $\boldsymbol{E_3}$ shown here and pre-multiplying $\boldsymbol{E_3}$ to $\boldsymbol{A}$ gives us $\boldsymbol{B_3}$.

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The determinant of $\boldsymbol{B_3}$ can also be computed

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and subsequently simplified to give the same determinant as $\boldsymbol{A}$.

Slide 07: Since the determinant of the elementary matrix $\boldsymbol{E_3}$ is 1 and we have observed that the determinant of $\boldsymbol{B_3}$ is 1 times the determinant of $\boldsymbol{A}$,

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we now see that corresponding to $\boldsymbol{B_3}$ equals to $\boldsymbol{E_3 A}$,

(#)

we have determinant of $\boldsymbol{B_3}$ equals to determinant of $\boldsymbol{E_3}$ multiplied to determinant of $\boldsymbol{A}$.

(#)

We again make two observations. The first is that elementary row operations of the third type, when performed on $\boldsymbol{A}$ does not change the determinant of $\boldsymbol{A}$.

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Secondly, we once again have the determinant of a matrix product being equal to the product of the determinant of the matrices.

Slide 08: Let us put all that we have discussed so far into a Theorem. Let $\boldsymbol{A}$ be a square matrix. If we perform the elementary row operation kR_i on $\boldsymbol{A}$ to obtain matrix $\boldsymbol{B_1}$, then the determinant of $\boldsymbol{B_1}$ is the determinant of $\boldsymbol{A}$ multiplied by k.

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If we perform any row swap on $\boldsymbol{A}$ to obtain matrix $\boldsymbol{B_2}$, then the determinant of $\boldsymbol{B_2}$ is the determinant of $\boldsymbol{A}$ multiplied by -1.

(#)

If we perform $R_j + kR_i$ on $\boldsymbol{A}$ to obtain matrix $\boldsymbol{B_3}$, then the determinant of $\boldsymbol{B_3}$ is the same as the determinant of $\boldsymbol{A}$.

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Furthermore, if $\boldsymbol{E}$ is also a square matrix of the same size as $\boldsymbol{A}$, then the determinant of $\boldsymbol{EA}$ is just the product of the determinant of $\boldsymbol{E}$ with the determinant of $\boldsymbol{A}$.

Slide 09: How can we make use of such a result? This result actually provides us with an alternative way to compute the determinant of a square matrix $\boldsymbol{A}$, if we do not wish to perform cofactor expansion. We first perform elementary row operations to reduce $\boldsymbol{A}$ to a row-echelon form $\boldsymbol{R}$ and then compute the determinant of $\boldsymbol{R}$.

Slide 10: Notice that $\boldsymbol{R}$ is always a triangular matrix whose determinant is easy to compute. If we keep track of the elementary row operations that have been performed on $\boldsymbol{A}$, we can now backtrack from the determinant of $\boldsymbol{R}$ to find the determinant of $\boldsymbol{A}$. We will now proceed to illustrate this method with examples.

Slide 11: For this 3×3 matrix, we wish to evaluate its determinant using elementary row operations instead of cofactor expansion.

Slide 12: We begin our Gaussian elimination process by first performing swapping rows 1 and 3 of $\boldsymbol{A}$.

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Note that if the matrix obtained after the row swap is denoted by $\boldsymbol{A_1}$, then we know that the determinant of $\boldsymbol{A_1}$ is -1 times the determinant of $\boldsymbol{A}$, since this is an elementary row operation of the second type.

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Proceeding with the next elementary row operation, we add -2 times of row 1 to row 2. Since this is the third type of elementary row operation, which does not affect determinants,

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we know that the current matrix $\boldsymbol{A_2}$ will have the same determinant as $\boldsymbol{A_1}$.

(#)

We perform one more elementary row operation of the third type to arrive at this matrix, which is already in row-echelon form.

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Once again, the determinant is not affected and thus the determinant of $\boldsymbol{A_3}$ is the same as the determinant of $\boldsymbol{A_2}$.

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The determinant of $\boldsymbol{A_3}$ is computed by multiplying its diagonal entries, which gives -5.

(#)

Since the determinant of $\boldsymbol{A_3}$ is the same as that of $\boldsymbol{A_2}$,

(#)

which is in turn the same as that of $\boldsymbol{A_1}$,

(#)

which is the negative of the determinant of $\boldsymbol{A}$,

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we now obtain that the determinant of $\boldsymbol{A}$ must be equal to 5 and the problem is solved.

Slide 13: In this next example, we do not have information on the matrix $\boldsymbol{A}$ other than the fact that it underwent four elementary row operations as shown here before becoming matrix $\boldsymbol{B}$ as shown. We would like to find the determinat of $\boldsymbol{A}$.

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As long as we are able to identify the effect of each elementary row operation has on the determinant of the matrix, we should be able to write down the following equation. For this example, we see that starting with the determinant of $\boldsymbol{A}$, it undergoes the multiplication of three factors, namely, 1, -1 and 4 before arriving at the determinant of $\boldsymbol{B}$.

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The determinant of $\boldsymbol{B}$ is simply the product of its diagonal entries,

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so the determinant of $\boldsymbol{A}$ can now be easily found to be $\frac{5}{6}$.

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For this question, you may wish to consider for a while and see if you can recover the matrix $\boldsymbol{A}$.

Slide 14: To summarise this unit,

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we discussed how the three types of elementary row operations changes the determinant of a square matrix.

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We now have an alternative method to cofactor expansion for the computation of the determinant of a matrix.