CHAPTER 2. MORE APPLICATIONS OF ODEs

2.1. THE HARMONIC OSCILLATOR

Consider the pendulum shown.

The small object, mass m,

at the end of the pendulum,

is moving on a circle of radius

L, so the component of its velocity

tangential to the circle is $L\dot{\theta}$

Hence its tangential acceleration is

$L\ddot{\theta}$ and so by $\overrightarrow{F} = M\overrightarrow{a}$ we have

$$mL\ddot{\theta} = -mg\sin\theta.$$

An obvious solution is $\theta = 0$. This is called an EQUILIBRIUM solution, meaning that θ is a CONSTANT function. This means that if you set $\theta = 0$ initially, then θ will remain at 0 and the pendulum will not move — which of course we know is correct. There is ANOTHER equilibrium solution, $\theta = \pi$. Again, IN THEORY, if you set the pendulum EXACTLY at $\theta = \pi$, then it will remain in that position forever. IN REALITY, of course, it won't! Because the slightest puff of air will knock it over! So this equilibrium is very different from the one at $\theta = 0$. This is a very important distinction!

Equilibrium is said to be STABLE if a SMALL push away from equilibrium REMAINS small. If the small push tends to grow large, then the equilibrium is UNSTABLE. Obviously this is important for engineers! Especially you want vibrations of structures, engines, etc to remain small.

Let's look at $\theta = \pi$. By Taylor's theorem, near $\theta = \pi$, we have

$$f(\theta) = f(\pi) + f'(\pi)(\theta - \pi) + \frac{1}{2}f''(\pi)(\theta - \pi)^2 + \ldots$$

Now $\sin(\pi) = 0$, $\sin'(\pi) = \cos(\pi) = -1$, $\sin''(\pi) = -\sin(\pi) = 0$ etc so

$$\sin(\theta) = 0 - (\theta - \pi) - 0 + \frac{1}{6}(\theta - \pi)^3 \quad \text{etc}$$

For small deviations away from π, $\theta - \pi$ is small, $(\theta - \pi)^3$ is much smaller, etc, so we can approximate

$$\sin(\theta) \approx -(\theta - \pi)$$

so our equation is approximately

$$ML\ddot{\theta} = -mg\sin\theta = mg(\theta - \pi).$$

Let $\phi = \theta - \pi$, so $\ddot{\phi} = \ddot{\theta}$, and now

$$\ddot{\phi} = \frac{g}{L}\phi.$$

The general solution is

$$\phi = Ae^{(\sqrt{g/L})t} + Be^{-(\sqrt{g/L})t}$$

so $\theta = \phi + \pi = Ae^{(\sqrt{g/L})t} + Be^{-(\sqrt{g/L})t} + \pi$.

As you know, the exponential function grows very quickly; so even if θ is close to π initially, it won't stay near to it very long! Very soon, θ will arrive either at $\theta = 0$ or 2π, far away from $\theta = \pi$. The equilibrium is UNSTABLE!

Now what about $\theta = 0$? Here of course we use Taylor's theorem around zero,

$$f(\theta) = f(0) + f'(0)\theta + \frac{1}{2}f''(0)\theta^2 + ...$$

$$\sin(\theta) = 0 + \theta - 0 - \frac{1}{6}\theta^3 + ...$$

so $\sin(\theta) \approx \theta$ and we have approximately

$$mL\ddot{\theta} = -mg\theta \quad \text{or}$$

$$\ddot{\theta} = -\frac{g}{L}\theta = -\omega^2\theta$$

with $\omega^2 = g/L$. That minus sign is crucial!

General solution is $C\cos(\omega\, t) + D\sin(\omega\, t)$ where C and D are arbitrary constants.

Now using trigonometric identities you can show that ANY expression of the form $C\cos(x) + D\sin(x)$ can be written as

$$C\cos(x) + D\sin(x) = \sqrt{C^2 + D^2}\cos(x - \gamma)$$

where $\tan(\gamma) = D/C$. [You can see this easily by taking the scalar product of the vectors $\begin{bmatrix} C \\ D \end{bmatrix}$ and $\begin{bmatrix} \cos(x) \\ \sin(x) \end{bmatrix}$.]

So now we can write our general solution as

$$\theta = A\cos(\omega t - \delta)$$

[**Check**: this does satisfy $\ddot{\theta} = -\omega^2\theta$ and it does contain TWO arbitrary constants, A and δ]. In this case, θ is never larger than A, never smaller than $-A$, so IF θ WAS SMALL INITIALLY, it REMAINS SMALL! [We call A the AMPLITUDE.] So the equilibrium in this case is STABLE. This is called SIMPLE HARMONIC MOTION. Clearly θ repeats its values every time ωt increases by 2π [since cos is periodic with period 2π]. Now

$$\omega t \to \omega t + 2\pi$$

means

$$t \to t + \frac{2\pi}{\omega}$$

So $\frac{2\pi}{\omega} = 2\pi\sqrt{L/g}$ is the time taken for θ to return to its initial value, the PERIOD. The number ω is called the ANGULAR FREQUENCY.

2.2. FORCED OSCILLATIONS

Suppose you have a mass m which can move in a horizontal line. It is attached to the end of a spring which exerts a force

$$F_{\text{spring}} = -kx$$

where x is the extension of the spring and k is a constant (called the spring constant). This is Hooke's Law. Now we attach an external MOTOR to the mass m. This motor exerts a force $F_0 \cos(\alpha t)$, where

F_0 is the amplitude of the external force and α is the frequency. If $F_0 = 0$ we just have, from Newton,

$$m\ddot{x} = -kx,$$

so we get $\ddot{x} = -\omega^2 x, \ \ \omega = \sqrt{k/m}$. Here ω is the frequency that the system has if we leave it alone that is, it is the NATURAL frequency. It has NOTHING TO DO with α of course – we can choose α to suit ourselves.

If $F_0 \neq 0$, then we have

$$m\ddot{x} + kx = F_0 \cos \alpha t.$$

Let z be a complex function satisfying

$$m\ddot{z} + kz = F_0 \ e^{i\alpha t}.$$

Clearly the real part, $Re\ z$, satisfies the above equation, so we can solve for z and them take the real part. We try

$$z = C\ e^{i\alpha t}$$

and get

$$mC(i\alpha)^2 e^{i\alpha t} + C\ k\ e^{i\alpha t} = F_0 e^{i\alpha t}$$

$$\Rightarrow C = \frac{F_0}{k - m\alpha^2} = \frac{F_0/m}{\omega^2 - \alpha^2}$$

So $$Re\ z = \tfrac{F_0/m}{\omega^2-\alpha^2} \cos(\alpha t)$$

and the general solution is

$$x = A\cos(\omega t - \delta) + \frac{F_0/m}{\omega^2 - \alpha^2} \cos(\alpha t).$$

Note that

$$\dot{x} = -A\omega \sin(\omega t - \delta) - \frac{\alpha F_0/m}{\omega^2 - \alpha^2} \sin(\alpha t).$$

The arbitrary constants A and δ are fixed by giving $x(0)$ and $\dot{x}(0)$ as usual. For example, suppose $x(0) = \dot{x}(0) = 0$, then

$$0 = A\cos(\delta) + \frac{F_0/m}{\omega^2 - \alpha^2}$$

$$0 = A\omega\sin(\delta).$$

Assuming $F_0 \neq 0$, we cannot have $A = 0, \Rightarrow \delta = 0$. So $A = -\frac{F_0/m}{\omega^2-\alpha^2}$,

$$x = \frac{F_0/m}{\omega^2 - \alpha^2}\left[\cos(\alpha t) - \cos(\omega t)\right].$$

Using the trigonometric identity

$$\cos A - \cos B = -2\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right)$$

we find

$$x = \frac{2F_0/m}{\alpha^2 - \omega^2} \sin\left[\left(\frac{\alpha - \omega}{2}\right) t\right] \sin\left[\left(\frac{\alpha + \omega}{2}\right) t\right]$$

What happens if we let $\alpha \to \omega$? We have

$$\begin{aligned} A(t) &= \frac{2F_0/m}{\alpha + \omega} \times \frac{\sin\left[\frac{\alpha-\omega}{2} t\right]}{\alpha - \omega} \\ &\to \frac{F_0}{m\omega} \times \frac{t}{2} = \frac{F_0 t}{2m\omega} \end{aligned}$$

by L'Hopital's rule. So in this limit

$$x = \frac{F_0 t}{2m\omega} \sin(\omega t)$$

and we see that the oscillations

go completely out of control.

This situation is called

RESONANCE. We see that IF A

SYSTEM IS FORCED IN A WAY

THAT AGREES WITH ITS OWN

NATURAL FREQUENCY, IT CAN OSCILLATE

UNCONTROLLABLY.

This can be very dangerous!

2.3. CONSERVATION.

Newton's 2nd law involves TIME derivatives, but sometimes it can be expressed in terms of SPATIAL derivatives, by means of the following trick:

$$\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) = \dot{x}\frac{d\dot{x}}{dx} = \frac{dx}{dt}\,\frac{d\dot{x}}{dx} = \ddot{x} \quad \text{(chain rule)}.$$

For SHM we have

$$m\ddot{x} = -kx$$

so

$$m\frac{d}{dx}(\frac{1}{2}\dot{x}^2) = -kx.$$

But now we can integrate both sides:

$$\frac{1}{2}m\dot{x}^2 = -\frac{1}{2}kx^2 + E$$

where E is a constant of integration. So we have

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2.$$

As you know, $\frac{1}{2}m\dot{x}^2$ is called the KINETIC ENERGY of the oscillator, and $\frac{1}{2}kx^2$ is called the POTENTIAL ENERGY. We call E the TOTAL ENERGY. The fact that E is CONSTANT is called the conservation of energy.