Matriculation Number: MA1512

A								

NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2017-2018

MA1512

DIFFERENTIAL EQUATIONS FOR ENGINEERING

May 2018 Time allowed: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** Do not write your name anywhere in this booklet. This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **FIVE (5)** questions and comprises **ELEVEN (11)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question. The marks for each question are indicated at the beginning of the question. The maximum possible total score for this examination paper is 50 marks.
4. This is a **closed book (with authorized material)** examination. Students are only allowed to bring into the examination hall **ONE** piece A4 size help-sheet which can be used on both sides.
5. Candidates may use any calculators that satisfy MOE A-Level examination guidelines. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5
Marks					

Question 1 [10 marks]

(a) Let $y(x)$ denote the solution of the differential equation

$$\frac{dy}{dx} = \frac{y}{x},$$

with $y(1) = 2$. Find the **exact value** of $y(756)$.

(b) Let a denote a positive constant. There are two tanks. Tank A is initially filled with 100 gallons of water in which a lbs of salt are dissolved. Tank B is initially filled with 100 gallons of water in which 8 lbs of salt are dissolved. Starting at time $t = 0$ water with a salt concentration of 0.2 lb/gal is being pumped into tank A at a constant rate of 5 gal/min while the well stirred solution in tank A is being pumped into tank B at a constant rate of 5 gal/min and the well stirred solution in tank B is being pumped out and discarded at a constant rate of 5 gal/min. If there are 13 lbs of salt in tank B at time $t = 20$ minutes, find the value of a. Give your answer correct to two decimal places.

Answer 1(a)	1512	**Answer 1(b)**	12.97

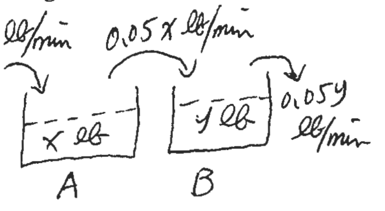
(Show your working below and on the next page.)

(a) $\begin{cases} \frac{dy}{dx} = \frac{y}{x} \\ y(1) = 2 \end{cases}$

$\Rightarrow y = 2x$

$\Rightarrow y(756) = \underline{\underline{1512}}$

(b)



$\frac{dx}{dt} = 1 - 0.05x$

$= -0.05(x-20)$

$\Rightarrow x - 20 = Ae^{-0.05t}$

$\Rightarrow A = a - 20$

$\Rightarrow x = 20 + (a-20)e^{-0.05t}$

$\frac{dy}{dt} = 0.05x - 0.05y \Rightarrow \frac{dy}{dt} + 0.05y = 0.05x$

$\Rightarrow \frac{dy}{dt} + 0.05y = 1 + 0.05(a-20)e^{-0.05t}$

$R = e^{\int 0.05\,dt} = e^{0.05t}$

$y = e^{-0.05t}\int \{e^{0.05t} + 0.05(a-20)\}\,dt$

$= 20 + 0.05(a-20)te^{-0.05t} + Ce^{-0.05t}$

$y(0) = 8 \Rightarrow C = -12$

$y(20) = 13 \Rightarrow 13 = 20 + (a-20)e^{-1} - 12e^{-1}$

$\Rightarrow a = 12.972\ldots$

$\approx \underline{\underline{12.97}}$

Question 2 [10 marks]

(a) Let a and b denote two positive integers. It is known that a is a two digit prime number. If $y = 2018e^{ax}$ is a solution of the differential equation $\frac{d^2y}{dx^2} - b\frac{dy}{dx} + 847y = 0$,
find the exact value of b.

(b) Let $y(x)$ denote the solution of the differential equation $y'' + 4y = \tan 2x$ with $-\frac{\pi}{4} < x < \frac{\pi}{4}$, $y(0) = 0$ and $y'(0) = \frac{3}{2}$. Find the value of $y\left(\frac{\pi}{8}\right)$. Give your answer correct to two decimal places.

Answer 2(a)	88	Answer 2(b)	0.55

(Show your working below and on the next page.)

(a) a is a root of

$\lambda^2 - b\lambda + 847 = 0.$

Let β = the other root.

$\beta = b - a \Rightarrow \beta = \text{integer}$

$\therefore a\beta = 847 = 11 \times 11 \times 7$

$\Rightarrow a = 11$ ($\because a = 2$ digit prime)

$\therefore \beta = 77$

$a + \beta = b = \underline{\underline{88}}$

(b) $\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$

Let $y_1 = \cos 2x$, $y_2 = \sin 2x$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$u = -\frac{1}{2}\int \sin 2x \tan 2x \, dx$

$= -\frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x} dx$

$= \frac{1}{4}\int (\cos 2x - \sec 2x)\, d(2x)$

$= \frac{1}{4}\sin 2x - \frac{1}{4}\ln|\sec 2x + \tan 2x|$

$v = \frac{1}{2}\int \cos 2x \tan 2x dx = -\frac{1}{4}\cos 2x$

$uy_1 + vy_2 = -\frac{1}{4}\cos 2x \ln|\sec 2x + \tan 2x|$

$\therefore y = C_1\cos 2x + C_2 \sin 2x - \frac{1}{4}\cos 2x \ln|\sec 2x + \tan 2x|$

$y(0) = 0 \Rightarrow C_1 = 0$

$\Rightarrow y = C_2 \sin 2x - \frac{1}{4}\cos 2x \ln|\sec 2x + \tan 2x|$

$y' = 2C_2\cos 2x + \frac{1}{2}\sin 2x \ln|\sec 2x + \tan 2x|$
$- \frac{1}{2}\cos 2x\left(\frac{\sec 2x\tan 2x + \sec^2 2x}{\sec 2x + \tan 2x}\right)$

$y'(0) = \frac{3}{2} \Rightarrow \frac{3}{2} = 2C_2 - \frac{1}{2} \Rightarrow C_2 = 1$

$\therefore y = \sin 2x - \frac{1}{4}\cos 2x \ln|\sec 2x + \tan 2x|$

$y\left(\frac{\pi}{8}\right) = \sin\frac{\pi}{4} - \frac{1}{4}\cos\frac{\pi}{4}\ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right|$

$= 0.551\ldots$

$\approx \underline{\underline{0.55}}$

Question 3 [10 marks]

(a) The Martian population on the planet Mars follows a Malthus model with a birth rate per capita of 10% per year and a death rate per capita of 5% per year. How long does it take for the Martian population to double itself? Give your answer in years correct to two decimal places.

(b) The monkey population at the Bukit Timah Nature Reserve follows a logistic model with a birth rate per capita of 10% per year. After a very long time, the population settled down to the carrying capacity of 2300 monkeys. The AVA then decided to allow E monkeys to be captured and put to rest per year so that the monkey population will eventually settle down to a new stable equilibrium value of 1500 monkeys. Find the value of E. Give your answer correct to the nearest integer.

Answer 3(a)	13.86	**Answer 3(b)**	52

(Show your working below and on the next page.)

(a) $\frac{dN}{dt} = (0.1 - 0.05)N$

$\Rightarrow N = N_0 e^{0.05t}$

$2N_0 = N_0 e^{0.05t}$

$\Rightarrow t = \frac{\ln 2}{0.05}$

$= 13.862\ldots$

$\approx \underline{13.86}$

(b) $B = 0.1$, $S = \frac{0.1}{2300} = \frac{1}{23000}$

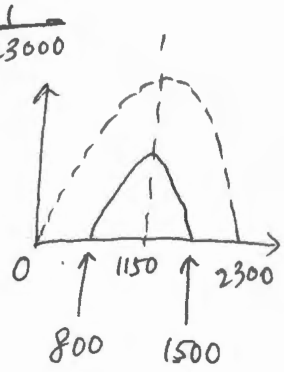
$BN - SN^2 - E = 0$

$\frac{E}{S} = 800 \times 1500$

$E = \frac{800 \times 1500}{23000}$

$= 52.17\ldots$

$\approx \underline{52}$



Question 4 [10 marks]

(a) Let $y(t)$ denote the solution to the differential equation $y'' - 2y' + y = e^t + t\,(u\,(t-2))$ with $y(0) = y'(0) = 0$, where u denotes the unit step function. Find the value of $y(3)$. Give your answer correct to two decimal places.

(b) Let a, b, c denote three positive constants. It is known that $y = 13e^{-t}\cos t - 13e^{-t}\sin t$ is the solution to the differential equation $ay + b\frac{dy}{dt} + \int_0^t y(u)\,du = c\delta(t)$ with $y(0) = 0$, where δ denotes the Dirac delta function. Find the **exact value** of $a + b + c$.

Answer 4(a)	92.67	**Answer 4(b)**	8

(Show your working below and on the next page.)

(a) $y'' - 2y' + y = e^t + (t-2)u(t-2) + 2u(t-2)$

$s^2Y - 2sY + Y = \frac{1}{s-1} + \frac{1}{s^2}e^{-2s} + \frac{2}{s}e^{-2s}$

$Y = \frac{1}{(s-1)^3} + \frac{1+2s}{s^2(s-1)^2}e^{-2s}$

$= \frac{1}{(s-1)^3} + \left\{\frac{1}{s^2} + \frac{4}{s} + \frac{3}{(s-1)^2} - \frac{4}{s-1}\right\}e^{-2s}$

$y = \frac{1}{2}t^2e^t$
$+ \left\{(t-2) + 4 + 3(t-2)e^{(t-2)} - 4e^{(t-2)}\right\}u(t-2)$

$y(3) = \frac{9}{2}e^3 + 5 - e$
$= 92.666\ldots$
$\approx \underline{\underline{92.67}}$

(b) $aY + bsY + \frac{1}{s}Y = c$

$Y = \frac{cs}{bs^2 + as + 1}$

$L(13e^{-t}\cos t - 13e^{-t}\sin t)$
$= \frac{13(s+1)}{(s+1)^2+1} - \frac{13}{(s+1)^2+1}$
$= \frac{13s}{s^2+2s+2}$

$\therefore \frac{2cs}{2bs^2 + 2as + 2} = \frac{13s}{s^2+2s+2}$

$\Rightarrow a = 1,\ b = 0.5,\ c = 6.5$

$a + b + c = \underline{\underline{8}}$

Question 5 [10 marks]

(a) Let $u(x,y)$ denote the solution to the PDE $yu_{xy} - 2xu = 0$, with $y > 0$ and $u(0,y) = 5y$, found by using the separation of variable method. Find the value of $u(2,2)$. Give your answer correct to two decimal places.

(b) Let $u(x,t)$ denote the solution to the Heat Equation $u_t = u_{xx}$, with $0 \le x \le \pi$, $t \ge 0$, $u(0,t) = u(\pi,t) = 0$ and initial condition $u(x,0) = 2\sin 3x$. Find the value of $u\left(\frac{\pi}{5}, 0.1\right)$. Give your answer correct to two decimal places.

(Hint: you may use the fact that if $X(x)$ is a solution to the ordinary differential equation $X'' - kX = 0$, $X(x)$ is not identically equal to zero and $X(0) = X(\pi) = 0$, where k denotes a constant, then $k = -n^2$, where n denotes an integer.)

Answer 5(a)	545.98	**Answer 5(b)**	0.77

(Show your working below and on the next page.)

(a) Let $u = XY$

$yX'Y' = 2xXY$

$\frac{X'}{xX} = \frac{2Y}{yY'} = k$

$\frac{X'}{X} = kx \Rightarrow X = Ae^{\frac{1}{2}kx^2}$

$\frac{Y'}{Y} = \frac{2}{ky} \Rightarrow Y = By^{2/k}$

$\therefore u = Cy^{2/k}e^{\frac{1}{2}kx^2}$

$u(0,y) = 5y \Rightarrow u = 5ye^{x^2}$

$u(2,2) = 10e^4$

$= 545.981\cdots$

$\approx \underline{545.98}$

(b) Let $u = XT \Rightarrow XT' = X''T \Rightarrow \frac{T'}{T} = \frac{X''}{X} = k$

$\therefore X'' - kX = 0,\ X(0) = X(\pi) = 0$

$\Rightarrow k = -n^2$

$\therefore X'' + n^2X = 0 \Rightarrow X = C_1\cos nx + C_2\sin nx$

$X(0) = 0 \Rightarrow C_1 = 0 \Rightarrow X = C_n\sin nx$

$\frac{T'}{T} = k = -n^2 \Rightarrow T = C_3e^{-n^2t}$

$\therefore u_n = a_n\sin nx\, e^{-n^2t}$

$u(x,0) = 2\sin 3x \Rightarrow u = 2\sin 3x\, e^{-9t}$

$u(\frac{\pi}{5}, 0.1) = 2\sin\frac{3\pi}{5}e^{-0.9}$

$= 0.773\cdots$

$\approx \underline{0.77}$