# Least squares solution to a linear system

# Definition (Least squares solution)

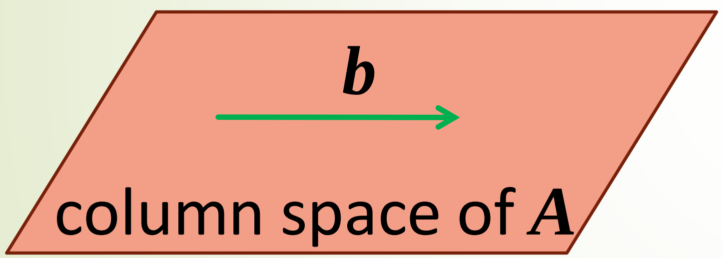
Recall:

Let $\boldsymbol{Ax} = \boldsymbol{b}$ be a linear system where $\boldsymbol{A}$ is a $m \times n$ matrix.

A vector $\boldsymbol{u} \in \mathbb{R}^n$ is called a least squares solution to the linear system if $\|\boldsymbol{b} - \boldsymbol{Au}\| \leq \|\boldsymbol{b} - \boldsymbol{Av}\|$ for all $\boldsymbol{v} \in \mathbb{R}^n$.

# When is a linear system consistent?

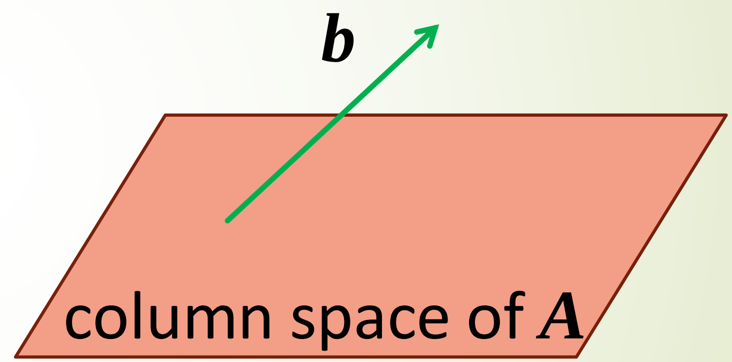
Recall that $\boldsymbol{Ax} = \boldsymbol{b}$ is consisent if and only if $\boldsymbol{b}$ belongs to the column space of $\boldsymbol{A}$.




$\boldsymbol{Ax} = \boldsymbol{b}$ is consistent.

Least squares solution

$=$ Exact solution.

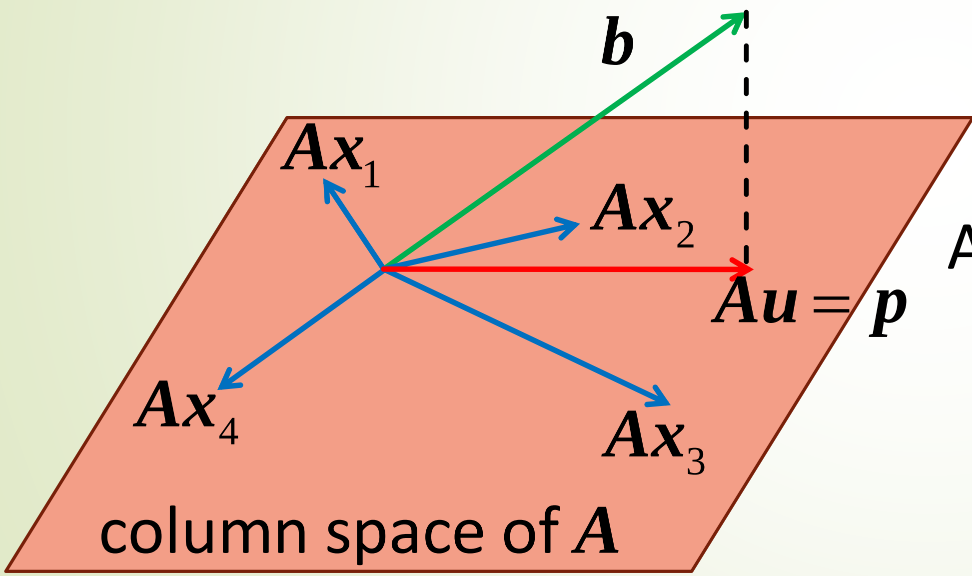



$\boldsymbol{Ax} = \boldsymbol{b}$ is inconsistent.

Least squares solution $= ?$

# An inconsistent linear system

Recall that $\boldsymbol{Ax} = \boldsymbol{b}$ is consisent if and only if $\boldsymbol{b}$ belongs to the column space of $\boldsymbol{A}$.




Which $\boldsymbol{u}$ will be such that $\boldsymbol{Au}$ is 'closest' to $\boldsymbol{b}$?

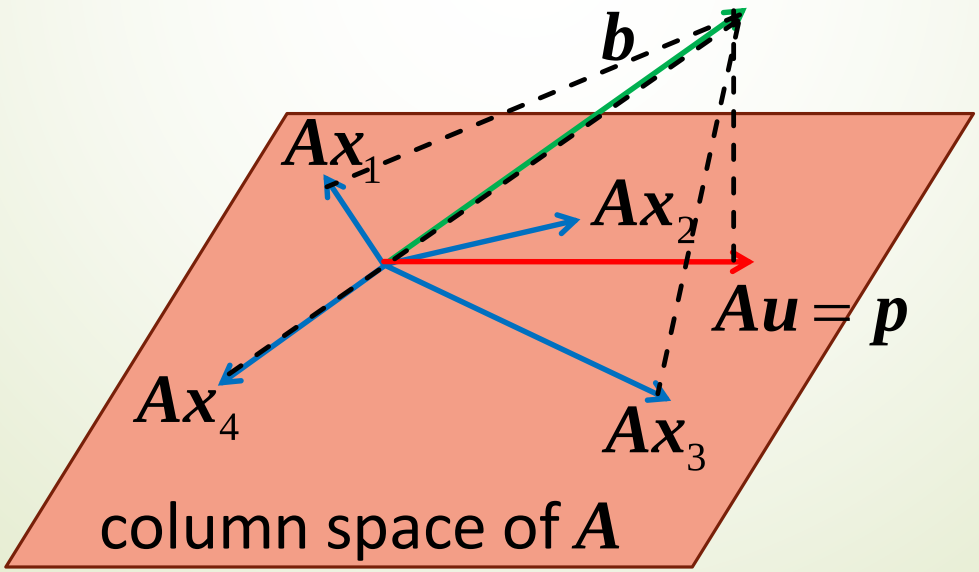
Answer: An $\boxed{\boldsymbol{u} \text{ such that } \boldsymbol{Au} = \boldsymbol{p}}$ where $\boldsymbol{p}$ is the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$.

To find least squares solution $\boldsymbol{u}$, we solve $\boldsymbol{Ax} = \boldsymbol{p}$.

# Theorem

Let $\boldsymbol{Ax} = \boldsymbol{b}$ be a linear system, where $\boldsymbol{A}$ is an $m \times n$ matrix, and let $\boldsymbol{p}$ be the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$. Then

$$\boxed{\left\|(\boldsymbol{b}-\boldsymbol{p})\right\| \le \left\|(\boldsymbol{b}-\boldsymbol{Av})\right\| \text{ for all } \boldsymbol{v} \in \mathbb{R}^n}$$



# Example

$$\boldsymbol{A}=\begin{pmatrix}1&1\\0&1\\1&1\end{pmatrix},\boldsymbol{b}=\begin{pmatrix}1\\2\\3\end{pmatrix}\text{ and } V=\text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$$

(column space of $\boldsymbol{A}$)

$V=\text{span}\{(1,0,1),(1,1,1)\}$ (a plane in $\mathbb{R}^3$ containing origin).

Find the (shortest) distance from $\boldsymbol{u}=(1,2,3)$ to $V$.

By Gram-Schmidt Process, $(1,0,1)$ and $(0,1,0)$ forms an orthogonal basis for $V$.

$$\boldsymbol{p}=\frac{(1,2,3)\cdot(1,0,1)}{(1,0,1)\cdot(1,0,1)}(1,0,1)+\frac{(1,2,3)\cdot(0,1,0)}{(0,1,0)\cdot(0,1,0)}(0,1,0)\quad=(2,2,2)$$

# Example

$$\boldsymbol{A}=\begin{pmatrix}1 & 1\\ 0 & 1\\ 1 & 1\end{pmatrix},\boldsymbol{b}=\begin{pmatrix}1\\ 2\\ 3\end{pmatrix} \text{ and } V=\text{span}\left\{\begin{pmatrix}1\\ 0\\ 1\end{pmatrix},\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}\right\}$$

(column space of $\boldsymbol{A}$)

What is a least squares solution to $\boldsymbol{Ax}=\boldsymbol{b}$?

Projection of $\boldsymbol{b}$ onto $V$ is $\boldsymbol{p}=\begin{pmatrix}2\\ 2\\ 2\end{pmatrix}$.

$\begin{pmatrix}x\\ y\end{pmatrix}$ is a least squares solution if and only if

$$\boldsymbol{A}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}2\\ 2\\ 2\end{pmatrix}\Leftrightarrow\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 2\end{pmatrix}.$$

# Theorem (Least squares solution)

Let $\boldsymbol{Ax} = \boldsymbol{b}$ be a linear system. Then $\boldsymbol{x}$ is a least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$ if and only if $\boldsymbol{x}$ is a solution to

$$\boxed{\boldsymbol{A}^T \boldsymbol{Ax} = \boldsymbol{A}^T \boldsymbol{b}} \quad \text{(Normal equation)}$$

**Proof:** Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_n \end{pmatrix}$ $\quad \boldsymbol{u}_i = i$th column of $\boldsymbol{A}$

Let $V = \text{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\} =$ column space of $\boldsymbol{A}$.

$\boldsymbol{x}$ is a least squares solution

$\Leftrightarrow$ $\boldsymbol{x}$ is a solution to $\boldsymbol{Ax} = \boldsymbol{p}$ $\quad$ ($\boldsymbol{p}$ is the projection of $\boldsymbol{b}$ onto $V$)

$\Leftrightarrow$ $\boldsymbol{Ax}$ is the projection of $\boldsymbol{b}$ onto $V$

# Theorem (Least squares solution)

**Proof:** Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \dots & \boldsymbol{u}_n \end{pmatrix}$ $\quad \boldsymbol{u}_i = i$th column of $\boldsymbol{A}$

Let $\boxed{V = \text{span}\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_n\}}$ = column space of $\boldsymbol{A}$.

$\boldsymbol{x}$ is a least squares solution

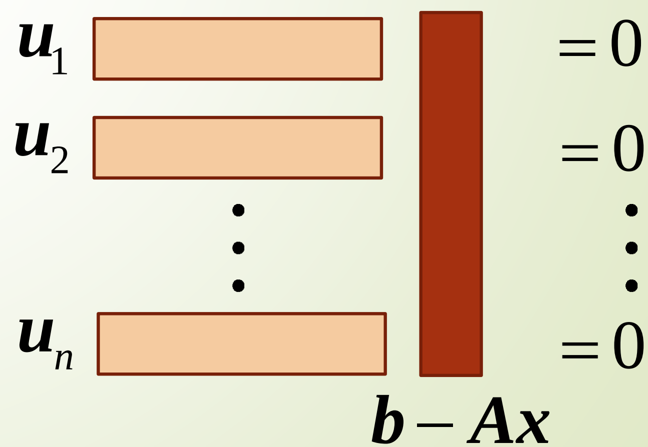
$\Leftrightarrow$ $\boldsymbol{x}$ is a solution to $\boldsymbol{Ax} = \boldsymbol{p}$ $\quad$ ($\boldsymbol{p}$ is the projection of $\boldsymbol{b}$ onto $V$)

$\Leftrightarrow$ $\boldsymbol{Ax}$ is the projection of $\boldsymbol{b}$ onto $V$

$\Leftrightarrow$ $\boldsymbol{b} - \boldsymbol{Ax}$ is orthogonal to $V$

$\Leftrightarrow$ $\boldsymbol{u}_i \cdot (\boldsymbol{b} - \boldsymbol{Ax}) = 0$ for all $i = 1, \dots, n$.

$\Leftrightarrow$ $\boldsymbol{A}^T(\boldsymbol{b} - \boldsymbol{Ax}) = \boldsymbol{0}$ $\Leftrightarrow$ $\boxed{\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}}$



# Back to our experiments!

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $r_i$ | 0 | 0 | 1 | 1 | 2 | 2 |
| $s_i$ | 0 | 1 | 2 | 0 | 1 | 2 |
| $t_i$ | 0.5 | 1.6 | 2.8 | 0.8 | 5.1 | 5.9 |

$$\begin{pmatrix} r_1^2 & s_1 & 1 \\ r_2^2 & s_2 & 1 \\ \vdots & \vdots & \vdots \\ r_6^2 & s_6 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \\ e \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_6 \end{pmatrix} \Leftrightarrow \boldsymbol{Ax} = \boldsymbol{b}$$

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \\ 4 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} 0.5 \\ 1.6 \\ 2.8 \\ 0.8 \\ 5.1 \\ 5.9 \end{pmatrix}$$

# Back to our experiments!

$$\boldsymbol{A}^T\boldsymbol{A} = \begin{pmatrix} 34 & 14 & 10 \\ 14 & 10 & 6 \\ 10 & 6 & 6 \end{pmatrix} \qquad \boldsymbol{A}^T\boldsymbol{b} = \begin{pmatrix} 47.6 \\ 24.1 \\ 16.7 \end{pmatrix}$$

Solving

$$\begin{pmatrix} 34 & 14 & 10 \\ 14 & 10 & 6 \\ 10 & 6 & 6 \end{pmatrix}\begin{pmatrix} c \\ d \\ e \end{pmatrix} = \begin{pmatrix} 47.6 \\ 24.1 \\ 16.7 \end{pmatrix}$$

we have $c = 0.9275$, $d = 0.9225$, $e = 0.3150$.

# Example

Let $V = \text{span}\{(1,-1,1,-1),(1,2,0,1),(2,1,1,0)\}$. Find the projection of $(1,1,1,1)$ onto $V$.

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

Note that $V$ is the column space of $\boldsymbol{A}$.

We first obtain a least squares solution of $\boldsymbol{Ax} = \boldsymbol{b}$.

# Example

Let $V = \text{span}\{(1,-1,1,-1),(1,2,0,1),(2,1,1,0)\}$. Find the projection of $(1,1,1,1)$ onto $V$.

We first obtain a least squares solution of $\boldsymbol{Ax} = \boldsymbol{b}$.

Solving $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$:

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 6 & 4 \\ 2 & 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}$$

solving...

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -t+\frac{2}{5} \\ -t+\frac{4}{5} \\ t \end{pmatrix}, t \in \mathbb{R}$$

Infinitely many least squares solutions

choose one: $\begin{pmatrix} \frac{2}{5} \\ \frac{4}{5} \\ 0 \end{pmatrix}$

# Example

Let $V = \text{span}\{(1,-1,1,-1),(1,2,0,1),(2,1,1,0)\}$. Find the projection of $(1,1,1,1)$ onto $V$.

$$\boldsymbol{A}\boxed{\begin{pmatrix} \frac{2}{5} \\ \frac{4}{5} \\ 0 \end{pmatrix}} = \begin{pmatrix} \frac{6}{5} \\ \frac{6}{5} \\ \frac{2}{5} \\ \frac{2}{5} \end{pmatrix}$$ is the projection of $(1,1,1,1)$ onto $V$.

choose one: $\boxed{\begin{pmatrix} \frac{2}{5} \\ \frac{4}{5} \\ 0 \end{pmatrix}}$

# Summary

1) A least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$ is given by the vector $\boldsymbol{u}$ that satisfies $\boldsymbol{Au} = \boldsymbol{p}$ where $\boldsymbol{p}$ is the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$.

2) Finding a least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$ by solving the normal equation $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$.