Unit 063 Complext vectors - an introduction

Slide 01: In this unit, we will see a brief recap on complex numbers and introduce vectors with complex numbers as its components. Some of concepts discussed earlier for vectors in $\mathbb{R}^n$ will now be extended to vectors in $\mathbb{C}^n$.

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Slide 02: You probably recall that a complex number z is normally written as $a+bi$. Now

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a is known as the real part of z while b is the imaginary part.

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The modulus or absolute value of z is defined to be the square root of the sum a^2+b^2.

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The complex number $z = a + bi$ has its complex conjugate $\bar{z}$ which is $a - bi$.

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A simple algebraic simplification will reveal that the product of z with its conjugate will give us the square of the modulus of z.

Slide 03: When we represent the complex number $z = a + bi$ like in the figure shown here, the angle θ is known as the argument of z. You can also think of this as decomposing z into its real part and imaginary part.

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From the figure it is easy to see that the real part of z is the modulus of z multiplied by $\cos(\theta)$ while the imaginary part of z is the modulus of z multiplied by $\sin(\theta)$.

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Lastly, you may be familiar with the polar form of a complex number as shown here.

Slide 04: In all the previous units, we have discussed vectors in $\mathbb{R}^n$, where the components of a vector are all real numbers.

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For example, the vector $\boldsymbol{u}$ is a vector in $\mathbb{R}^3$ since each of its 3 components are real numbers.

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Since complex numbers can be represented as $a + bi$, where i is the square root of -1,

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it is a natural extension to define the set $\mathbb{C}^n$ to be the set of all vectors $\boldsymbol{v}$ with n components and each component is a complex number.

Slide 05: For example, the vector $\boldsymbol{v}$ shown here is a vector in $\mathbb{C}^3$. Note that the second component of $\boldsymbol{v}$ is 3, which is real, but which can also be considered as a complex number with no imaginary part.

Slide 06: As seen previously, we can split a vector in $\mathbb{C}^n$ into real and imaginary parts. More precisely,

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if $\boldsymbol{v}$ is the vector in $\mathbb{C}^n$ shown here,

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we can rewrite $\boldsymbol{v}$ as the vector with components a_1, a_2 and so on plus i times the vector with components b_1, b_2 and so on.

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The first vector with components a_1, a_2 and so on is called the real part of $\boldsymbol{v}$ while the second vector with components b_1, b_2 and so on is called the imaginary part of $\boldsymbol{v}$. So we have rewritten $\boldsymbol{v}$ as $\mathrm{Re}(\boldsymbol{v})$ plus i times $\mathrm{Im}(\boldsymbol{v})$.

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By definition of conjugates, the conjugate of $\boldsymbol{v}$ would be the following vector

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which can also be splitted into the real and imaginary parts.

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Thus the conjugate of $\boldsymbol{v}$ is $\mathrm{Re}(\boldsymbol{v})$ minus i times $\mathrm{Im}(\boldsymbol{v})$.

Slide 07: Let us see some algebraic properties of complex conjugates. Suppose $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{C}^n$ and k is a scalar. Then

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it is clear that the conjugate of $\boldsymbol{u}$ conjugate is just $\boldsymbol{u}$ itself.

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The conjugate of the vector $k\boldsymbol{u}$ is the conjugate of k times the conjugate of $\boldsymbol{u}$.

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The conjugate of the sum $\boldsymbol{u} + \boldsymbol{v}$ is the sum of the two conjugate vectors.

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And similarly, the conjugate of the difference $\boldsymbol{u} - \boldsymbol{v}$ is the difference of the two conjugate vectors.

Slide 08: Analogous to the dot product between vectors in $\mathbb{R}^n$, we also have the complex dot product between vectors in $\mathbb{C}^n$. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in $\mathbb{C}^n$ with components as shown.

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The complex dot product of $\boldsymbol{u}$ with $\boldsymbol{v}$ is defined as follows, where we multiply each component of $\boldsymbol{u}$ with the conjugate of its corresponding component of $\boldsymbol{v}$ and then sum up the n terms.

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The Euclidean norm of a vector $\boldsymbol{v}$ in $\mathbb{C}^n$ is similarly defined like its real vector counterparts. More precisely, it is the square root of the vector $\boldsymbol{v}$ dot with itself.

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A unit vector is a vector with norm equals to 1

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and two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{C}^n$ are orthogonal if their complex dot product is 0.

Slide 09: Let us consider an example. Here we have two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{C}^3$.

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First, the complex dot product of $\boldsymbol{u}$ with $\boldsymbol{v}$ can be computed as follows. Recall that we multiply each component of $\boldsymbol{u}$ with the conjugate of its corresponding component of $\boldsymbol{v}$ and then sum up the three terms.

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A little simplification
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reveals the answer to be $-2-10i$.
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The complex dot product of $\boldsymbol{v}$ with $\boldsymbol{u}$ can be computed similarly.
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Again upon simplifying,
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we have the answer as $-2+10i$
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Notice that the two answers are in fact complex conjugates of each other. We will soon learn that this is by no means a coincidence.

Slide 10: The norm of $\boldsymbol{u}$ is the square root of $\boldsymbol{u}$ dot $\boldsymbol{u}$
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which gives us
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$\sqrt{13}$ as the norm of $\boldsymbol{u}$.
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Similarly, we can now compute the norm of $\boldsymbol{v}$
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which gives us $\sqrt{22}$ as the norm of $\boldsymbol{v}$.

Slide 11: The following theorem gives us some fundamental results involving complex dot products. Let $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ be vectors in $\mathbb{C}^n$ and let k be a scalar. Now
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The complex dot product of $\boldsymbol{u}$ with $\boldsymbol{v}$ is the complex conjugate of the complex dot product of $\boldsymbol{v}$ with $\boldsymbol{u}$. This is known as the antisymmetry property of complex dot product. Recall that if $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{R}^n$, then $\boldsymbol{u}\cdot\boldsymbol{v}$ will always be equals to $\boldsymbol{v}\cdot\boldsymbol{u}$. This is obviously not the case for complex dot products.
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Similar to real vectors dot products, we also have the distributive property as shown here.
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The scalar k can be multiplied to the result $\boldsymbol{u}\cdot\boldsymbol{v}$ or multiplied to $\boldsymbol{u}$ first then compute the dot product with $\boldsymbol{v}$. The two numerical results will be the same.
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The complex dot product of $\boldsymbol{u}$ with $k\boldsymbol{v}$ is the conjugate of k multiplied with the complex dot product of $\boldsymbol{u}$ with $\boldsymbol{v}$.
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Lastly, we have the analogous result from real vectors dot product in the sense that even if the vector $\boldsymbol{v}$ is in $\mathbb{C}^n$, the dot product of $\boldsymbol{v}$ with itself is still non negative. In fact, the only time where $\boldsymbol{v}\cdot\boldsymbol{v}$ is zero is when $\boldsymbol{v}$ is the zero vector.

Slide 12: Let us turn our attention to a recent topic of eigenvalues and eigenvectors of a square matrix. This theorem states that if $\boldsymbol{A}$ is a square matrix of order n where all its entries are real numbers, then whenever λ is an eigenvalue of $\boldsymbol{A}$, the conjugate of λ will also be an eigenvalue of $\boldsymbol{A}$. Furthermore, if $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ associated with λ, then the conjugate of $\boldsymbol{x}$ will also be an eigenvector of $\boldsymbol{A}$ associated with the conjugate of λ.

Slide 13: To best understand this result, we will look at an example. In this example, we would like to show that $\boldsymbol{A}$ is diagonalizable by finding an invertible matrix $\boldsymbol{P}$ with complex entries that will diagonalize $\boldsymbol{A}$.

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We start off in the same way as before, which is to compute the characteristic polynomial of $\boldsymbol{A}$.

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This simplifies to $\lambda^2 - 4\lambda + 5$.

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The roots of the characteristic equation, in this case

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are the conjugate pair $2+i$ and $2-i$. Thus $\boldsymbol{A}$ has two eigenvalues $2+i$ and $2-i$ which are conjugates of each other.

Slide 14: Let λ_1 be the eigenvalue $2+i$. We will proceed to investigate the eigenspace E_{λ_1} by solving the homogeneous linear system as shown.

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While we perform elementary row operations, note that we are able to take a complex scalar multiple of one row to add to another row, as shown here.

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We arrive at the row-echelon form of the augmented matrix as shown here

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and this allows us to write a general solution for the system

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which leads us to the conclusion that the eigenspace E_{λ_1} is spanned by a single vector in $\mathbb{C}^2$, namely the vector $\begin{pmatrix} 1-i \\ 2 \end{pmatrix}$.

Slide 15: Consider the second eigenspace E_{λ_2}. We will again solve the homogeneous linear system as shown

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and by performing the elementary row operation as shown, we have

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the following row-echelon form of the augmented matrix.

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This allows us to write down a general solution for the system

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which leads us to the conclusion that the eigenspace E_{λ_2} is spanned by a single vector in $\mathbb{C}^2$, namely the vector $\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$.

Slide 16: We now see the results of the previous theorem.

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The two eigenvalues of $\boldsymbol{A}$ are indeed complex conjugates of each other.

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While their corresponding eigenvectors $\begin{pmatrix} 1-i \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$ are also complex conjugates of each other.

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If we let $\boldsymbol{P}$ be the matrix as shown, where the columns of $\boldsymbol{P}$ are once again the linearly independent eigenvectors of $\boldsymbol{A}$, we see that $\boldsymbol{A}$ can now be written as $\boldsymbol{PDP}^{-1}$

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where the diagonal matrix $\boldsymbol{D}$ would contain the two eigenvalues along the diagonal.

Slide 17: Let us summarise the main points in this unit.

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We started off with a quick review of complex numbers, giving definitions for terms like modulus and conjugate.

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We extended the defintion of $\mathbb{R}^n$ to $\mathbb{C}^n$. Remember that a vector in $\mathbb{C}^n$ is one that has n componenets, each of which is a complex number.

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We also introdued complex dot product and saw some of its properties. Notice the difference between real vectors dot product and complex dot products.

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Lastly, we presented a theorem that states that complex eigenvalues always happen in pairs. In fact, so do their corresponding eigenvectors of $\boldsymbol{A}$.