W05-06

Slide 01: In this unit, we continue our discussion on bases and introduce the notion of coordinate vectors.

Slide 02: Recall that we have shown in a previous unit that if S is a basis for a vector space V, then every vector $\boldsymbol{v}$ in the vector space can be expressed in terms of the basis vectors in exactly one way.

Slide 03: This result gives rise to the definition of coordinate vectors. So for a set of basis vectors $\boldsymbol{u_1}, \boldsymbol{u_2}$ to $\boldsymbol{u_k}$ for a vector space V, suppose a vector $\boldsymbol{v}$ from the vector space is written as a linear combination of the basis vectors in the following way,

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then the coefficients from the linear combination, namely, c_1, c_2 to c_k are called the coordinates of $\boldsymbol{v}$ relative to the basis S.

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Furthermore, we can write a vector $(\boldsymbol{v})_S$ using these coordinates by arranging them from c_1 to c_k as shown. This vector is called the coordinate vector of $\boldsymbol{v}$ relative to the basis S. Note that this vector is a vector in $\mathbb{R}^k$ since it contains k components.

Slide 04: Several remarks are to be noted. Firstly, in order to discuss the notion of coordinate vectors meaningfully, we need to make sure that the basis vectors in S are ordered, meaning we need to be clear which vector is $\boldsymbol{u_1}$, which is $\boldsymbol{u_2}$ and so on. This is because eventually, the coefficients in the linear combination expression will be arranged in that order when we write down the coordinate vector.

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Once we have a fixed basis S, the coordinate vector $(\boldsymbol{v})_S$ is unique and well-defined for each vector $\boldsymbol{v}$ in the vector space.

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For two different bases of the same vector space, a vector $\boldsymbol{v}$ from the vector space will be expressed differently in terms of the two different sets of basis vectors. Thus, corresponding to different bases, we will have different coordinate vectors for the same $\boldsymbol{v}$. Thus it is always important to specify a coordinate vector is with respect to which basis.

Slide 05: Let us look at a few examples. Consider the following set S.

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In fact we have already shown, from a previous unit that S is a basis for $\mathbb{R}^3$. Thus we will not repeat the working here.

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Next we would like to find the coordinate vector of the vector $(5, -1, 9)$ relative to the basis S.

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To do this, we just need to write $\boldsymbol{v}$ as a linear combination of the basis vectors and find the three coefficients a, b and c.

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From the vector equation, we have the following equivalent linear system which we will solve.

Slide 06: From the starting augmented matrix of the system,

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we perform Gauss-Jordan elimination and arrive at the following row-echelon form, which tells us that the linear system and thus also the vector equation has a unique solution

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of $a = 1$, $b = -1$ and $c = 2$. Thus the coordinate vector of $\boldsymbol{v}$ relative of S is the vector $(1, -1, 2)$.

Slide 07: What if we are now asked which vector $\boldsymbol{w}$ in $\mathbb{R}^3$ would have the coordinate vector $(-1, 3, 2)$ relative to the basis S? This question is in fact much easier to answer.

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Since we are told $(\boldsymbol{w})_S$ is $(-1, 3, 2)$, we simply recover the vector $\boldsymbol{w}$ by linearly combining the three basis vectors using the coefficients -1, 3 and 2.

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Thus, $\boldsymbol{w}$ is found to be the vector $(11, 31, 7)$.

Slide 08: Consider the following example where $\boldsymbol{v}$ is a vector in $\mathbb{R}^2$ and S_1, S_2 and S_3 are three different bases for $\mathbb{R}^2$.

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Consider the basis S_1. To write $\boldsymbol{v}$ as a linear combination of the two basis vectors in S_1 is easy.

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Thus we have $(\boldsymbol{v})_{S_1}$ to be $(2, 3)$.

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To visualise this, we have the following diagram. The two red arrows represent the two basis vectors in S_1. They are in the conventional direction of the x and y axes that we are familiar with. The black arrow is $(\boldsymbol{v})_{S_1}$ and as you can see, it is obtained by adding 2 units of $(1, 0)$ to 3 units of $(0, 1)$. So to measure $\boldsymbol{v}$ with respect to the basis S_1, we have the coordinate vector $(2, 3)$.

Slide 09: Let's move on to S_2. A little effort is required to write $\boldsymbol{v}$ as a linear combination of the two vectors in S_2. The two coefficients obtained are $-\frac{1}{2}$ and $\frac{5}{2}$, which gives us the coordinate vector $(\boldsymbol{v})_{S_2}$.

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To visualise this, we now see the two red arrows to be in the direction of $(1, -1)$ and $(1, 1)$, which are the two vectors in S_2. The black arrow is $(\boldsymbol{v})_{S_2}$ and as you can see, it is obtained by adding $-\frac{1}{2}$ units of $(1, -1)$ to $\frac{5}{2}$ units of $(1, 1)$. So to measure $\boldsymbol{v}$ with respect to the basis S_2, we have the coordinate vector $(-\frac{1}{2}, \frac{5}{2})$.

Slide 10: Lastly, consider the basis S_3. Once again, we can find that the coordinate vector of $\boldsymbol{v}$ relative to S_3 is $(-1, 3)$.

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The two red arrows are now in the direction of $(1, 0)$ and $(1, 1)$ and the black arrow is $(\boldsymbol{v})_{S_3}$ which is obtained by adding -1 times $(1, 0)$ to 3 times of $(1, 1)$. So to measure $\boldsymbol{v}$ with respect to the basis S_3, we have the coordinate vector $(-1, 3)$.

Slide 11: The significance of this discussion is to illustrate the following. Regardless of whether a vector is measured with respect to which basis, whether it is S_1,

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or S_2,

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or S_3, while the red vectors, which are your respective basis vectors are different,

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it is essentially still the same $\boldsymbol{v}$ that we are talking about. The different coordinate vectors only tells us the different ways of measuring the same vector.

Slide 12: Let us revisit the previous example again. The basis S_1 was actually a very convenient basis to use, as we see that $\boldsymbol{v}$ and $(\boldsymbol{v})_{S_1}$ actually have the same representation of $(2, 3)$.

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In fact, for any vector in $\mathbb{R}^2$, say (x, y), the coordinate vector of $\boldsymbol{v}$ relative to S_1 is the same as the vector $\boldsymbol{v}$ itself.

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Such a basis is convenient to use and in fact we have all been using them to build our $\mathbb{R}^2$, similar to what you see in the graph paper that you have been using since your earlier school days. Those square grids on the graph paper was drawn with this basis S_1 in mind, for otherwise, your graph paper would contain slanted squares instead.

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Let us define this type of bases formally. Let E be the set containing vectors $\boldsymbol{e_1}$, $\boldsymbol{e_2}$ to $\boldsymbol{e_n}$ where the vectors $\boldsymbol{e_1}$ to $\boldsymbol{e_n}$ are as shown here. Notice that each $\boldsymbol{e_i}$ has only one non zero component , which is 1, at the i-th position.

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For such a basis, it is clear that for any vector $\boldsymbol{v}$ in $\mathbb{R}^n$ the coordinate vector of $\boldsymbol{v}$ with respect to the basis E is the same as the vector $\boldsymbol{v}$ itself.

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E is what we call the standard basis for $\mathbb{R}^n$.

Slide 13: Remember that the standard bases for different $\mathbb{R}^n$ contains entirely different vectors, even though they are represented by the same notation $\boldsymbol{e_i}$.

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For example, here we have the standard basis for $\mathbb{R}^3$, comprising of vectors $\boldsymbol{e_1}$, $\boldsymbol{e_2}$ and $\boldsymbol{e_3}$.

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While the standard basis vectors for $\mathbb{R}^4$, namely vectors $\boldsymbol{e_1}$ to $\boldsymbol{e_4}$ looks entirely different, since they are vectors with 4 components and not 3.

Slide 14: Before we end this unit, think about the following question.

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While we have seen that for a vector space V, V can have many different bases. While these different bases may contain very different vectors, would all of them have the **same number** of vectors? This important question, will be answered in a subsequent unit.

Slide 15: To summarise the points in this unit.
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We defined what is a coordinate vector relative to a fixed basis.
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We also introduced a very convenient basis to use, known as the standard basis for $\mathbb{R}^n$.