NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

Module: MA1508E Linear Algebra for Engineering
Year/Semester: 2018-2019 (Semester 2)
Tutorial: 1

1. Solve the following linear systems by Gaussian Elimination or Gauss-Jordan Elimination.

(a)

$$\left\{\begin{array}{rcrcrcr} x_1 & + & 3x_2 & + & x_3 & = & 4 \\ 2x_1 & + & 2x_2 & + & x_3 & = & -1 \\ 2x_1 & + & 3x_2 & + & x_3 & = & 3 \end{array}\right.$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 4 \\ 2 & 2 & 1 & -1 \\ 2 & 3 & 1 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -7 \end{array}\right).$$

So the unique solution is $x_1 = -1, x_2 = 4, x_3 = -7$.

(b)

$$\left\{\begin{array}{rcrcrcrcr} & - & b & - & 2c & - & 3d & = & 0 \\ a & + & b & + & 4c & + & 4d & = & 7 \\ a & + & 3b & + & 7c & + & 9d & = & 4 \\ -a & - & 2b & - & 4c & - & 6d & = & 6 \end{array}\right.$$

$$\left(\begin{array}{cccc|c} 0 & -1 & -2 & -3 & 0 \\ 1 & 1 & 4 & 4 & 7 \\ 1 & 3 & 7 & 9 & 4 \\ -1 & -2 & -4 & -6 & 6 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -6 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 & -7 \end{array}\right)$$

So the unique solution is $a = -6, b = 1, c = 10, d = -7$.

(c)

$$\left\{\begin{array}{rcrcrcrcrcr} & & 2x_2 & + & x_3 & + & 2x_4 & - & x_5 & = & 4 \\ & & x_2 & & & + & x_4 & - & x_5 & = & 3 \\ 4x_1 & + & 6x_2 & + & x_3 & + & 4x_4 & - & 3x_5 & = & 8 \\ 2x_1 & + & 2x_2 & & & + & x_4 & - & x_5 & = & 2 \end{array}\right.$$

$$\left(\begin{array}{ccccc|c} 0 & 2 & 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 1 & -1 & 3 \\ 4 & 6 & 1 & 4 & -3 & 8 \\ 2 & 2 & 0 & 1 & -1 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & -2 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

So a general solution is

$$\left\{\begin{array}{rcl} x_1 & = & -2 + \frac{s}{2} - \frac{t}{2} \\ x_2 & = & 3 - s + t \\ x_3 & = & -2 - t \\ x_4 & = & s \\ x_5 & = & t, \qquad s, t \in \mathbb{R}. \end{array}\right.$$

(d)

$$\left\{\begin{array}{rcrcrcl} x & - & y & & & = & 1 \\ & & y & - & z & = & 1 \\ -x & & & + & z & = & 3 \\ x & - & 2y & + & 2z & = & 1 \end{array}\right.$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 3 \\ 1 & -2 & 2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

So the linear system is inconsistent.

2. For each of the following, a general solution to a linear system is given.

   (a) Write down the reduced row-echelon form (of appropriate size) of the augemented matrix representing the system that will give rise to the general solution.

   (b) Hence write down a linear system with the required number of equations such that the coefficients of the variables in each equation are all non zero. You may assume that the variables are $x_1, x_2, \cdots$.

   (i) Linear system has 2 equations.

$$\left\{\begin{array}{lcl} x_1 & = & 4 + 5s - t \\ x_2 & = & s \\ x_3 & = & t, \qquad s, t \in \mathbb{R}. \end{array}\right.$$

$$\left(\begin{array}{ccc|c} 1 & -5 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left\{\begin{array}{rcrcrcl} x_1 & - & 5x_2 & + & x_3 & = & 4 \\ 2x_1 & - & 10x_2 & + & 2x_3 & = & 8 \end{array}\right.$$

   (ii) Linear system has 3 equations.

$$\left\{\begin{array}{lcl} x_1 & = & 2 + 20s - 29t \\ x_2 & = & -7s + 8t \\ x_3 & = & s \\ x_4 & = & t, \qquad s, t \in \mathbb{R}. \end{array}\right.$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -20 & 29 & 2 \\ 0 & 1 & 7 & -8 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \left\{\begin{array}{rcrcrcrcl} -x_1 & + & 2x_2 & + & 34x_3 & - & 45x_4 & = & -2 \\ -x_1 & + & x_2 & + & 27x_3 & - & 37x_4 & = & -2 \\ -2x_1 & + & 2x_2 & + & 54x_3 & - & 74x_4 & = & -4 \end{array}\right.$$

   (iii) Linear system has 4 equations.

$$\left\{\begin{array}{lcl} x_1 & = & -3r - 4s - 2t \\ x_2 & = & r \\ x_3 & = & -2s \\ x_4 & = & s \\ x_5 & = & t \\ x_6 & = & 0, \qquad r, s, t \in \mathbb{R}. \end{array}\right.$$

$$\left(\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left\{\begin{array}{rcrcrcrcrcrcr} x_1 & + & 3x_2 & - & 4x_3 & - & 4x_4 & + & 2x_5 & - & 2x_6 & = & 0 \\ 2x_1 & + & 6x_2 & + & 4x_3 & + & 16x_4 & + & 4x_5 & + & 3x_6 & = & 0 \\ x_1 & + & 3x_2 & + & x_3 & + & 6x_4 & + & 2x_5 & + & x_6 & = & 0 \\ x_1 & + & 3x_2 & + & x_3 & + & 6x_4 & + & 2x_5 & + & x_6 & = & 0 \end{array}\right.$$

3. For each of the following, determine the values of $b_1, b_2, \cdots$ such that the linear system is

   (i) inconsistent;

   (ii) consistent with a unique solution;

   (iii) consistent with infinitely many solutions.

   (a)

$$\left\{\begin{array}{rcrcrcr} 2x_1 & - & b_1x_2 & + & 3x_3 & = & 0 \\ 4x_1 & - & 2x_2 & + & 5x_3 & = & -b_1 \\ -2x_1 & + & b_1x_2 & - & 2x_3 & = & b_1 \end{array}\right.$$

   Linear system has a unique solution if $b_1 \neq 1$. Linear system has infinitely many solutions if $b_1 = 1$.

   (b)

$$\left\{\begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2x_3 & = & b_1 \\ x_1 & & & + & x_3 & = & b_2 \\ 2x_1 & + & x_2 & + & 3x_3 & = & b_3 \end{array}\right.$$

   Linear system is inconsistent if $b_3 - b_2 - b_1 \neq 0$. Linear system has infinitely many solutions if $b_3 - b_2 - b_1 = 0$.

   (c)

$$\left\{\begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & 3x_3 & = & 0 \\ 4x_1 & - & b_1x_2 & + & 5x_3 & = & -1 \\ -2x_1 & + & x_2 & - & b_2x_3 & = & b_3 \end{array}\right.$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 4 & -b_1 & 5 & -1 \\ -2 & 1 & -b_2 & b_3 \end{array}\right) \xrightarrow[R_3+R_1]{R_2-2R_1} \left(\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 0 & -b_1+2 & -1 & -1 \\ 0 & 0 & 3-b_2 & b_3 \end{array}\right) = \boldsymbol{R}$$

   (i) If $b_1 = 2, b_2 = 3$, then linear system has infinitely many solutions if $b_3 = 0$ and no solutions when $b_3 \neq 0$.

   (ii) If $b_1 = 2$ and $b_2 \neq 3$, then

$$\boldsymbol{R} = \left(\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 3-b_2 & b_3 \end{array}\right)$$

and the linear system is consistent and has infinitely many solutions if $3 - b_2 = b_3$ which is equivalent to $b_2 + b_3 = 3$. If $b_2 + b_3 \neq 3$, then the linear system is inconsistent.

(iii) If $b_1 \neq 2$ and $b_2 \neq 3$, then the linear system has a unique solution for all $b_3 \in \mathbb{R}$.

(iv) If $b_1 \neq 2$ and $b_2 = 3$, then the linear system has infinitely many solutions if $b_3 = 0$ and no solutions if $b_3 \neq 0$.

So in summary, the linear system

(i) is inconsistent if
  (a) $b_1 = 2, b_2 = 3, b_3 \neq 0$; or
  (b) $b_1 = 2, b_2 \neq 3, b_2 + b_3 \neq 3$; or
  (c) $b_1 \neq 2, b_2 = 3, b_3 \neq 0$.

(ii) is consistent with a unique solution if $b_1 \neq 2$, $b_2 \neq 3$, $b_3 \in \mathbb{R}$.

(iii) is consistent with infinitely many solutions if
  (a) $b_1 = 2, b_2 = 3, b_3 = 0$; or
  (b) $b_1 = 2, b_2 \neq 3, b_2 + b_3 = 3$; or
  (c) $b_1 \neq 2, b_2 = 3, b_3 = 0$.

4. **(Application)** Chemical reactions can be described by equations. The expression on the left side are called the reactants, and those on the right side are the products, which are produced from the reaction of the chemicals on the left. The two sides are separated by an arrow $\rightarrow$, which indicates that the reactants form the products. A chemical equation is balanced, provided that the number of atoms of each type on the left is the same as the number of atoms of the corresponding type on the right. An example of such a **balanced** chemical equation is shown below, when propane (C_3H_8) gas burns, combining with oxygen (O_2) to form carbon dioxide (CO_2) and water (H_2O).

$$C_3H_8 + 5O_2 \rightarrow 3CO_2 + 4H_2O.$$

Consider another chemical equation where, in the process of photosynthesis, plants use radiant energy from sunlight to convert carbon dioxide (CO_2) and water (H_2O) into glucose ($C_6H_{12}O_6$) and oxygen (O_2). The **unbalanced** chemical equation is

$$CO_2 + H_2O \rightarrow O_2 + C_6H_{12}O_6.$$

(a) Let x and y be the number of carbon dioxide and water molecules, respectively, required for the reaction; and z and w are the number of oxygen and glucose, respectively, produced. By equating the number of carbon (C), oxygen (O) and hydrogen (H) atoms on both sides of the chemical equations, write down a homogeneous linear system with 3 equations and 4 unknowns.

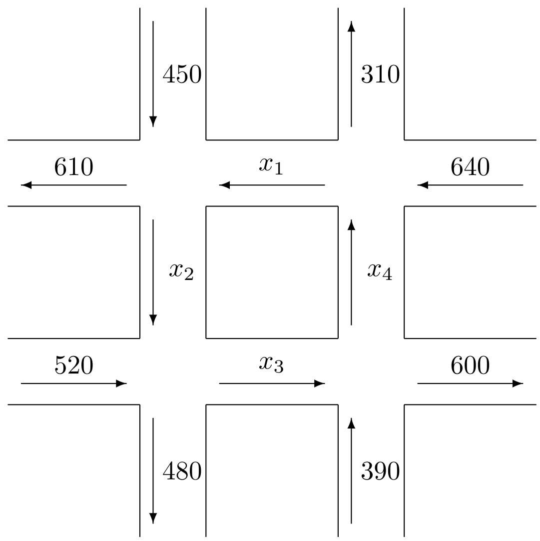
$$\left\{\begin{array}{rcrcrcrcl} x & & & - & 6w & & & = & 0 \\ 2x & + & y & - & 6w & - & 2z & = & 0 \\ & & 2y & - & 12w & & & = & 0 \end{array}\right.$$

(b) Find a general solution for the homogeneous linear sytem obtained in Part (a).

$$\begin{cases} x &= s \\ y &= s \\ w &= \frac{s}{6} \\ z &= s, \qquad s \in \mathbb{R}. \end{cases}$$

(c) Find the (non-trivial) solution of x, y, z, w with the smallest values. (Note that x, y, z, w are positive integers.)
$x = 6, y = 6, w = 1, z = 6.$

5. **(Application)** In a downtown section of a certain city, two sets of one-way streets intersect as shown below. The average hourly volume of traffic entering and leaving this section during rush hour is given in the diagram.



(a) Do we have enough information to find the traffic volumes x_1, x_2, x_3 and x_4? Explain your answer.

$$\begin{cases} x_1 + 450 = x_2 + 610 \\ x_2 + 520 = x_3 + 480 \\ x_3 + 390 = x_4 + 600 \\ x_4 + 640 = x_1 + 310. \end{cases} \Leftrightarrow \begin{cases} x_1 - x_2 = 160 \\ x_2 - x_3 = -40 \\ x_3 - x_4 = 210 \\ -x_1 + x_4 = -330. \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & 1 & -1 & 210 \\ -1 & 0 & 0 & 1 & -330 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 330 \\ 0 & 1 & 0 & -1 & 170 \\ 0 & 0 & 1 & -1 & 210 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The system has infinitely many solutions. We cannot determine the values of x_1, x_2, x_3, x_4 uniquely.

(b) Given that $x_4 = 200$, find x_1, x_2 and x_3. If $x_4 = 200$, then $x_1 = 330 + x_4 = 530$, $x_2 = 170 + x_4 = 370$ and $x_3 = 210 + x_4 = 410$.