W06-06

Slide 01: In this unit, we introduce another subspace associated with a matrix.

Slide 02: We have already discussed, in some detail, the row space and column space of a matrix. We will introduce another subspace associated with a matrix in this unit, but it turns out that this subspace is something we have already seen previously.

Slide 03: Let $\boldsymbol{A}$ be a $m \times n$ matrix. Consider the homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$. Since $\boldsymbol{A}$ has n columns, the variable $\boldsymbol{x}$ will have a total of n unknowns.

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We have already seen that the solution set of this homogeneous linear system is a subspace, and thus a subspace of $\mathbb{R}^n$. We also call this the solution space of $\boldsymbol{Ax} = \boldsymbol{0}$.

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We can also call this subspace the nullspace of the coefficient matrix $\boldsymbol{A}$.

Slide 04: Since the nullspace of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^n$, the dimension of the nullspace does not exceed n.

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The dimension of the nullspace of $\boldsymbol{A}$ is called the nullity of $\boldsymbol{A}$.

Slide 05: Let us find a basis for, and also determine the dimension of the nullspace of the following matrix $\boldsymbol{A}$.

Slide 06: The procedure of doing so is nothing new. We simply proceed to solve the homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$. By performing Gauss-Jordan elimination, we obtain the reduced row-echelon form of the augmented matrix as shown. With that, we can easily write down a general solution for $\boldsymbol{Ax} = \boldsymbol{0}$ as shown here. Note that the general solution involves one arbitrary parameter s.

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Thus, an arbitrary vector in the solution space can be written as follows, which is simply a scalar multiple of the vector $(-4, 0, 0, 1)$.

Slide 07: So the nullspace of $\boldsymbol{A}$ is just the set of all scalar multiples of the vector $(-4, 0, 0, 1)$, or in other words, the linear span of $(-4, 0, 0, 1)$. A basis for the nullspace can be the set with just this one vector and the nullity of $\boldsymbol{A}$ would be 1.

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From the working shown on this slide, can we determine what is the rank of $\boldsymbol{A}$?

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Clearly, the left hand side of the reduced row-echelon form is the reduced row-echelon form of the matrix $\boldsymbol{A}$. Since there are three pivot columns, we see that the rank of $\boldsymbol{A}$ is 3.

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For this example, we notice that the three pivot columns meant that the rank of $\boldsymbol{A}$ is 3, while the 1 non pivot column meant that our general solution had one arbitrary parameter. This would imply that the basis for the nullspace of $\boldsymbol{A}$ has only one vector and thus the nullity of $\boldsymbol{A}$ is 1.

Slide 08: We will consider the same question, now with a different matrix $\boldsymbol{B}$.

Slide 09: From the reduced row-echelon form of the augmented matrix, we can write down a general solution for the homogeneous linear system. Notice that the general solution involves two arbitrary parameters s and t.

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An arbitrary vector in the solution space can be written as a linear combination of $(-1, 1, 0, 0, 0)$ and $(-1, 0, -1, 0, 1)$.

Slide 10: These two vectors would form a basis for the nullspace of $\boldsymbol{B}$ and thus the nullity of $\boldsymbol{B}$ is 2.

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Can you tell what is the rank of $\boldsymbol{B}$ from the information on this slide?

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Indeed we see that the rank of $\boldsymbol{B}$ is 3 since there are three pivot columns at row-echelon form as highlighted in pink. There are two non-pivot columns on the left hand side at row-echelon form and these two non-pivot columns led to the two arbitrary parameters in a general solution to $\boldsymbol{Bx} = \mathbf{0}$, which eventually led to the conclusion that nullity of $\boldsymbol{B}$ is 2.

Slide 11: The observations in the two preceeding examples give rise to the following theorem, known as the Dimension Theorem for matrices. Let $\boldsymbol{A}$ be a matrix with n columns, then the rank of $\boldsymbol{A}$ plus the nullity of $\boldsymbol{A}$ will be equals to n.

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To prove this, let $\boldsymbol{R}$ be the reduced row-echelon form of $\boldsymbol{A}$.

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The n columns in $\boldsymbol{R}$ can be classified into two groups, namely those that are pivot columns and those that are non-pivot columns.

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Thus, the total number of pivot columns in $\boldsymbol{R}$ plus the total number of non-pivot columns in $\boldsymbol{R}$ must be equal to n.

Slide 12: We will now see how these two groups of columns in $\boldsymbol{R}$ is related to rank and nullity.

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Firstly, the number of pivot columns in $\boldsymbol{R}$ is equal to the number of leading entries in $\boldsymbol{R}$.

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While the number of non-pivot columns in $\boldsymbol{R}$ will determine how many arbitrary parameters will there be when we write down a general solution to $\boldsymbol{Ax} = \mathbf{0}$.

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Returning to the number of leading entries in $\boldsymbol{R}$, this is simply the rank of $\boldsymbol{A}$.

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Whereas the number of arbitrary parameters in a general solution to $\boldsymbol{Ax} = \mathbf{0}$ is equal to the number of vectors in a basis for the nullspace of $\boldsymbol{A}$

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which is just the nullity of $\boldsymbol{A}$.

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We have now proven the Dimension Theorem for matrices, which states that the rank of $\boldsymbol{A}$ plus the nullity of $\boldsymbol{A}$, will be equal to the number of columns in $\boldsymbol{A}$, which is n.

Slide 13: In summary, for this unit,

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we define what is known as the nullspace and nullity of a matrix. Remember that this is not a new subspace, but in fact something we have already learnt about in an earlier unit.

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We then stated and proved the Dimension Theorem for matrices.