W04-07

Slide 01: In this unit, we will wrap up the discussion on the concept of linear span.

Slide 02: By now, you should have the understanding that a linear span of non zero vectors is a set with infinitely many vectors. The following theorem gives a necessary and sufficient condition on when one linear span will be entirely contained inside another linear span. Let S_1 and S_2 be two subsets of $\mathbb{R}^n$. More precisely, let S_1 consists of vectors $\boldsymbol{u_1}, \boldsymbol{u_2}$ and so on till $\boldsymbol{u_k}$ while S_2 contains vectors $\boldsymbol{v_1}, \boldsymbol{v_2}$ till $\boldsymbol{v_m}$. Here you can think of the vectors in S_1 being represented by the orange dots while the vectors in S_2 are represented by the red dots.

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The linear span of S_1 is the set of all linear combination of the $\boldsymbol{u}$'s while the linear span of S_2 is the set of all linear combination of the $\boldsymbol{v}$'s. Then the linear span of S_1 is a subset of the linear span of S_2

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if and only if each $\boldsymbol{u_i}$ is a linear combination of the vectors in S_2. In other words, each of the $\boldsymbol{u}$'s can be written in terms of the $\boldsymbol{v}$'s. We will omit the proof of this result and proceed with some illustrative examples.

Slide 03: In this example, we have $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ and also vectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$. We would like to show that the linear span of $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ is the same as the linear span of $\boldsymbol{v_1}, \boldsymbol{v_2}$.

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The idea is really to use the previous theorem which gave us a necessary and sufficient condition for one linear span to be contained inside another linear span. We will first show the first subset inclusion.

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In order to do so using the theorem, we need to show that each of $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ and $\boldsymbol{u_3}$ can be written as linear combination of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$.

Slide 04: How do we check that each of the $\boldsymbol{u}$ is a linear combination of the $\boldsymbol{v}$'s?

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This is nothing new as writing a vector as a linear combination of other vectors has been described in a previous unit. We start off with $\boldsymbol{u_1}$ and set up a vector equation with unknowns a and b which we will try to solve for.

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We do the same for $\boldsymbol{u_2}$. Note that even though we use the same symbols of a and b to represent the unknowns, the ones for $\boldsymbol{u_1}$ and those for $\boldsymbol{u_2}$ are clearly different.

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We will do the same for $\boldsymbol{u_3}$.

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Arising from the first vector equation for $\boldsymbol{u_1}$, we have the following linear system.

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From the second equation for $\boldsymbol{u_2}$, we have the following,

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and the last system is for $\boldsymbol{u_3}$.

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You should notice that even though these are three different linear systems, they have something in common. Namely, the left hand side of the three systems are exactly identical. This is because the three vector equations for $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ and $\boldsymbol{u_3}$ has the same right hand side. Because of this similarity, we can in fact attempt to solve the three linear systems together by using a special augmented matrix that is modified from the one we are familiar with.

Slide 05: Consider the augmented matrix on the left. Here you see the first two columns which is the identical left hand side of the three linear systems. Following that, we have three columns, each representing the right hand side of the three linear systems. There is one column for $\boldsymbol{u_1}$, the next one for $\boldsymbol{u_2}$ and the last one for $\boldsymbol{u_3}$.

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We will perform Gauss-Jordan elimination as per normal on this modified augmented matrix. The resulting matrix in reduced row-echelon form is shown here. How can we interpret this matrix?

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Remember that the first column on the right corresponds to the $\boldsymbol{u_1}$ linear system. So the two numbers $\frac{1}{5}$ and $\frac{2}{5}$ seen here will give us the unique solution to writing $\boldsymbol{u_1}$ in terms of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$. Thus we have $\boldsymbol{u_1} = \frac{1}{5}\boldsymbol{v_1} + \frac{2}{5}\boldsymbol{v_2}$.

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The next column in the reduced row-echelon form gives us the solution to the $\boldsymbol{u_2}$ linear system. Once again, the solution is unique and we have $\boldsymbol{u_2} = \frac{3}{5}\boldsymbol{v_1} + \frac{1}{5}\boldsymbol{v_2}$.

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Finally, the last column gives us the unique solution to write $\boldsymbol{u_3}$ as a linear combination of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$. More precisely, we have $\boldsymbol{u_3} = \frac{3}{5}\boldsymbol{v_1} - \frac{4}{5}\boldsymbol{v_2}$.

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We have now managed to verify that each of the $\boldsymbol{u}$'s is a linear combination of the $\boldsymbol{v}$'s, so we have the linear span of $\boldsymbol{u_1}, \boldsymbol{u_2}$ and $\boldsymbol{u_3}$ to be entirely contained inside the linear span of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$

Slide 06: This is only half the battle won. In order to show that the two linear spans are equal, we need to show the other subset inclusion. In other words, we need to show that the linear span of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ is contained inside the linear span of $\boldsymbol{u_1}, \boldsymbol{u_2}$ and $\boldsymbol{u_3}$.

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The method to do so is the same as before. We need to show that each of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ is a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}$ and $\boldsymbol{u_3}$.

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We write down the $\boldsymbol{v_1}$ vector equation, where a, b, c are the unknowns.

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Likewise the $\boldsymbol{v_2}$ vector equation as shown.

Slide 07: Similar to the previous case, we write down the two linear systems,

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one for $\boldsymbol{v_1}$

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and the other for $\boldsymbol{v_2}$.

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Again, the two linear systems has the same left hand side which means that we can once again solve both systems together with a modified augmented matrix.

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Notice that the two columns on the right is such that one represents the $\boldsymbol{v_1}$ linear system while the other is for $\boldsymbol{v_2}$.

Slide 08: When we perform Gauss-Jordan elimination on the augmented matrix, we obtain the following reduced row-echelon form.

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The $\boldsymbol{v_1}$ linear system can be solved by considering the highlighted portion of the reduced row-echelon form.

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From the reduced row-echelon form, it is clear that the linear system has inifintely many solutions and we can apply knowledge from a previous unit to write down a general solution for this system.

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By letting the arbitrary parameter take on the value of 0, we have a solution to the system as $a = -1$, $b = 2$ and $c = 0$.

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This means that we can write $\boldsymbol{v_1}$ as $-\boldsymbol{u_1} + 2\boldsymbol{u_2} + 0\boldsymbol{u_3}$.

Slide 09: The $\boldsymbol{v_2}$ linear system can be solved by considering the highlighted portion of the reduced row-echelon form.

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Once again, we observe that this linear system has infinitely many solutions and we can write down a general solution to the system as follows.

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By letting the arbitrary parameter take on the value of 0, we have a solution to the system as $a = 3$, $b = -1$ and $c = 0$.

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This means that we can write $\boldsymbol{v_2}$ as $3\boldsymbol{u_1} - \boldsymbol{u_2} + 0\boldsymbol{u_3}$.

Slide 10: Now that we have shown that each $\boldsymbol{v}$ is a linear comvbination of the $\boldsymbol{u}$'s, we have established that the linear span of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ is contained inside the linear span of $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ and $\boldsymbol{u_3}$.

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Together with what we have shown earlier, we have completed the proof that the two linear spans are identical.

Slide 11: Let us consider another example. Here we have three vectors $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, $\boldsymbol{u_3}$ and three other vectors $\boldsymbol{v_1}$, $\boldsymbol{v_2}$ and $\boldsymbol{v_3}$. In this case, we would like to show that the linear span of the $\boldsymbol{v}$'s is not contained inside the linear span of $\boldsymbol{u}$'s.

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We will still try to write each of the $\boldsymbol{v}$'s as a linear combination of the $\boldsymbol{u}$'s and see what happens.

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As before, we set up the $\boldsymbol{v_1}$ vector equation and then subsequently write down the associated linear system. Note that there are 4 equations involving 3 unknowns.

Slide 12: Likewise we write down the $\boldsymbol{v_2}$ vector equation and the associated linear system.

Slide 13: We do the same for the $\boldsymbol{v_3}$ vector equation.

Slide 14: Once again, the three linear systems share the same left hand side and thus we can try to solve them together. The modified augmented matrix is shown here and

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upon performing Gaussian elimination, we arrive at the following row-echelon form. Unlike the examples we have seen previously, this row-echelon form actually indicates that some of the vector equations are inconsistent.

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In other words, there are some $\boldsymbol{v_i}$ that is not a linear combination of $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ and $\boldsymbol{u_3}$. Can you identify which are these $\boldsymbol{v_i}$'s?

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If we look at the part of the row-echelon form corresponding to the $\boldsymbol{v_1}$ vector equation, we see that there is a row where the right hand side column is a pivot column. This means that the linear system is inconsistent and thus $\boldsymbol{v_1}$ is not a linear combination of the $\boldsymbol{u}$'s. There is another $\boldsymbol{v_i}$ that is also not a linear combination of the $\boldsymbol{u}$'s. Can you identify which $\boldsymbol{v_i}$ is it?

Slide 15: We will conclude this unit with a theorem. Suppose $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ until $\boldsymbol{u_k}$ are vectors taken from $\mathbb{R}^n$.

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If the last vector $\boldsymbol{u_k}$ is a linear combination of the other $k-1$ vectors, namely, $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, until $\boldsymbol{u_{k-1}}$, then

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the linear span of first $k-1$ vectors is the same as the linear span of all the k vectors.

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What this means is that the set of vectors that can be generated by taking linear combinations using the first $k-1$ vectors is the sames as the set of vectors that can be generated by taking linear combinations using all the k vectors.

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In other words, in this case, it turns out that having one more vector, namely $\boldsymbol{u_k}$ to take linear combinations with, did not generate any additional vectors.

Slide 16: To establish this result we will show that the two linear span are equal by showing two subset inclusions. First, we show that that the linear span of the $k-1$ vectors is contained inside the linear span of the k vectors.

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We will do this by using the theorem presented earlier in this unit. Starting with the first vector $\boldsymbol{u_1}$, is $\boldsymbol{u_1}$ a linear combination of the k vectors $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$?

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Certainly it is, in fact writing $\boldsymbol{u_1}$ in terms of the k vectors is trivial.

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Next the second vector $\boldsymbol{u_2}$

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can also be easily written as a linear combination of the k vectors.

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We can continue doing this

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until $\boldsymbol{u_{k-1}}$, which is also easily written as a linear combination of the k vectors. Thus the first subset inclusion has been established.

Slide 17: We proceed with the second subset inclusion which is to show that the linear span of the k vectors is contained inside the linear span of the first $k-1$ vectors.

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We go through the same considerations, starting with $\boldsymbol{u_1}$,

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which is clearly a linear combination of the $k-1$ vectors.

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Likewise for $\boldsymbol{u_2}$

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which can be easily written in terms of $\boldsymbol{u_1}$ to $\boldsymbol{u_{k-1}}$.

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we will continue with this and all will be fine

Slide 18: until we reach the last vector $\boldsymbol{u_k}$. Since $\boldsymbol{u_k}$ is not one of the first $k-1$ vectors, you may wonder if $\boldsymbol{u_k}$ is indeed a linear combination of $\boldsymbol{u_1}$ to $\boldsymbol{u_{k-1}}$.

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Here is where the assumption comes in, the fact that $\boldsymbol{u_k}$ is actually a linear combination of $\boldsymbol{u_1}$ to $\boldsymbol{u_{k-1}}$ is something that we have not used up till now. Thus we have now established that each of the k vectors is a linear combination of the first $k-1$ vectors

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and this establishes the second subset inclusion. Together with the first case we have proven, we have shown that the two linear spans are the same.

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Intuitively, as we have noted at the beginning before we started on the proof, if $\boldsymbol{u_k}$ is already a linear combination of the first $k-1$ vectors, then having $\boldsymbol{u_k}$ actually does not add any value or allows additional vectors to be generated in the linear span.

Slide 19: Consider the following example, with the three vectors as shown. It is easily verified that $\boldsymbol{u_3}$ is a linear combination of $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$.

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By the theorem we have just proven, since $\boldsymbol{u_3}$ is a linear combination of $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$, the linear span of all three vectors is the same as the linear span of just the first two.

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Are you able to describe the linear span of $\boldsymbol{u_1}, \boldsymbol{u_2}$ and $\boldsymbol{u_3}$ geometrically? Since we now know that it is the same as the linear span of just the first two vectors,

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then the linear span of $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ is the span of $(1, 0, 0)$, which is the x-axis and $(0, -1, 0)$, which is the opposite direction of the y-axis. Thus the linear span of $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ is simply the xy-plane, or what is also known as the plane $z = 0$.

Slide 20: To summarise this unit,

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We presented a necessary and sufficient condition for one linear span to be entirely contained inside another linear span.

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We saw a few examples on how this necessary and sufficient condition can be used.

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Lastly, we proved a theorem on the equivalence of two linear spans when a vector does not add value in the sense of generating more linear combinations.