W05-09

Slide 01: In this unit, we will add two more equivalent statements to our collection of statements that are equivalent to $\boldsymbol{A}$ is an invertible square matrix.

Slide 02: Let us recap what we have obtained so far. A total of 5 logically equivalent statements, including the statement that $\boldsymbol{A}$ is an invertible square matrix of order n. The equivalence of these statements have been establised in two seprate units you have seen earlier.

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The sixth statement to be added to this collection is that the rows of $\boldsymbol{A}$ can be taken to form a basis for $\mathbb{R}^n$. Since $\boldsymbol{A}$ is a square matrix of order n, here we will n rows, each representing a vector from $\mathbb{R}^n$.

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The seventh statement is that the columns of $\boldsymbol{A}$ can also be taken to form a basis for $\mathbb{R}^n$. Similar to the sixth statement, here we are taking the n columns of the matrix to represent n vectors from $\mathbb{R}^n$. Let us see how we can establish the equivalence of these two statements with the first 5 we have discussed earlier.

Slide 03: We will first prove that the statement $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution, which is one of the five statements we have so far,

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is in fact equivalent to the statement that the columns of $\boldsymbol{A}$ can be taken to form a basis for $\mathbb{R}^n$.

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To do this, we let $\boldsymbol{c_1}, \boldsymbol{c_2}$ and so on till $\boldsymbol{c_n}$ be the columns of $\boldsymbol{A}$. So $\boldsymbol{c_i}$ is the i-th column of $\boldsymbol{A}$.

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Since we already know that the dimension of $\mathbb{R}^n$ is n, the n columns of $\boldsymbol{A}$ is the correct number of vectors we need to have, in order to form a basis for $\mathbb{R}^n$. Thus, to show that these n columns are indeed a basis for $\mathbb{R}^n$, it suffices to show that the columns are linearly independent.

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The matrix expression $\boldsymbol{Ax} = \boldsymbol{0}$ can be rewritten as follows, where we write $\boldsymbol{A}$ in terms of its columns, and the unknown $\boldsymbol{x}$ be explicitly written down in terms of its components x_1 to x_n.

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We can further rewrite this as $x_1\boldsymbol{c_1} + x_2\boldsymbol{c_2} + \cdots x_n\boldsymbol{c_n}$ on the left hand side while on the right hand side, we have the zero vector.

Slide 04: Now it is clear that $\boldsymbol{Ax} = \boldsymbol{0}$ can be represented as a vector equation as shown on the right hand side.

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The statement that $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution

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is thus equivalent to the statement that the vector equation, as highlighted here, has only the trivial solution. However, you now realise that the vector equation is precisely the one we write down to check whether $\boldsymbol{c_1}, \boldsymbol{c_2}$ until $\boldsymbol{c_n}$ are linearly independent or not.

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Thus, the vector equation having only the trivial solution is equivalent to the statement that $\boldsymbol{c_1}, \boldsymbol{c_2}$ until $\boldsymbol{c_n}$ are linearly independent

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which in turn is equivalent to the statement that the columns of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$.

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We have thus shown that the seventh statement is equivalent to the first 5.

Slide 05: We now turn our attention to the rows of $\boldsymbol{A}$.

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We will show that $\det(\boldsymbol{A}) \neq 0$ is equivalent to the statement that the rows of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$.

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Since we have already established that the seventh statement, stated in terms of the columns of $\boldsymbol{A}$, is equivalent to the first 5, we can say that the determinant of $\boldsymbol{A}$ is non zero is equivalent to the statement that the columns of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$.

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By simply replacing the matrix $\boldsymbol{A}$ by $\boldsymbol{A}^T$ in the previous equivalence statement we have the following.

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However, now note that the columns of $\boldsymbol{A}^T$ are in fact the rows of $\boldsymbol{A}$. Thus we now have $\det(\boldsymbol{A}^T) \neq 0$ is equivalent to the statement that the rows of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$.

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But now recall that the determinant of $\boldsymbol{A}$ and $\boldsymbol{A}^T$ is actually the same, thus we have now obtained the equivalence as desired.

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We have successfully included the sixth statement into the collection.

Slide 06: In summary, we now see that $\boldsymbol{A}$ is invertible is logically equivalent to 6 other statements that relates to various concepts we have covered so far, for example, in terms elementary matrices, determinants, and now bases.

Slide 07: We will now use the theorem to answer the following question. Is this set of 3 vectors a basis for $\mathbb{R}^3$? Once again,we could have answered the problem using first principles, meaning we will go through checking for linear independence or linear span.

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However, with the preceding theorem on equivalent statements, we can approach the problem differently. For example, if we construct a 3×3 matrix where the rows of the matrix are precisely the 3 vectors we are looking at, followed by computing the determinant of this 3×3 matrix.

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Upon cofactor expansion,
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we find that the determinant of the matrix is non zero.
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Thus the three vectors does indeed form a basis for $\mathbb{R}^3$.
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We will look at a similar problem, now with 4 vectors from $\mathbb{R}^4$. Will these 4 vectors be a basis for $\mathbb{R}^4$?
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We can once again use the vectors to construct a matrix. This time, we will put the 4 vectors as colmuns of the matrix as shown and then proceed to compute the determinant of this 4×4 matrix.
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We find that the determinant of the matrix is 0.
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This would the be equivalent to saying that the columns of the matrix,namely the 4 vectors in question, **does not** form a basis for $\mathbb{R}^4$.

Slide 08: As a summary,
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in this unit, we have added two more equivalent statements to the statement that $\boldsymbol{A}$ is invertible.