Week 06

MA1508E LINEAR ALGEBRA FOR ENGINEERING

Week 5

IVLE Quiz Discussion

Review of Week 05 (Units 023-028) content

- How do elementary row operations affect the determinant of a matrix?
  - First type (multiplying a row by *k*) → determinant changes by a factor of *k*
  - Second type (row swap) → determinant changes by a factor of -1
  - Third type (adding a multiple of one row to another row) → determinant not changed.
- Let ***A*** be a square matrix. Then

$$\begin{tikzcd}
 & \boldsymbol{B}_1 \\
\boldsymbol{A} \arrow[ur, "kR_i"] \arrow[r, "R_i \leftrightarrow R_j"] \arrow[dr, "R_j + kR_i"'] & \boldsymbol{B}_2 \\
 & \boldsymbol{B}_3
\end{tikzcd}$$

$$\det(\boldsymbol{B}_1) = k\det(\boldsymbol{A})$$

$$\det(\boldsymbol{B}_2) = -\det(\boldsymbol{A})$$

$$\det(\boldsymbol{B}_3) = \det(\boldsymbol{A})$$

Furthermore, if ***E*** is an elementary matrix of the same size as ***A***, then det(***EA***) = det(***E***)det(***A***)

Review of Week 05 (Units 023-028) content

- Finding the determinant of a matrix using elementary row operations.
- ***A*** is invertible if and only if det(***A***) is not zero.
- Some properties on determinants
  - Multiplying a scalar *c* to ***A***
  - Determinant of product = product of determinants (that is, det(***AB***)=det(***A***)det(***B***))
  - Determinant of the inverse of an invertible matrix.
- What are Euclidean vectors? When are two vectors equal?
- Addition, subtraction, scalar multiple of (geometric) vectors.
- Components of a vector.
- Generalisation of geometric vectors (R^2 and R^3) to *n*-vectors.
- Definition of Euclidean *n*-space. Subsets of R^n.

Review of Week 05 (Units 023-028) content

- Length of a vector; distance between two vectors; angle between two vectors.
- Dot product of two vectors.
- Identifying dot product with matrix product.
- Some laws involving dot product; dot product of any vector with itself is non-negative…
- When are two vectors orthogonal?
- When is a set of vectors orthogonal? When is a set of vectors orthonormal?
- The concept of orthogonality as an extension to the concept of two perpendicular vectors.
- Converting an orthogonal set into an orthonormal set.

Week 06 (units 029-034) overview

029 Linear combinations

- What is a linear combination of vectors?
- How to check whether a given vector is a linear combination of some other vectors?
- Vector equation → linear system → check consistency
- How to check whether every vector in $\boldsymbol{R}^n$ is a linear combination of some (collection of) vectors?

030 Linear span Part I

- The linear span of a set of vectors
- How to write a set as a linear span (when possible)
- How to check whether a vector belongs to span(S)
- How to check whether span(S) = $\boldsymbol{R}^n$ (or not)

Week 06 (units 029-034) overview

031 Linear span Part II

- Detailed discussion on checking if span(S) = $\boldsymbol{R}^n$
- We can never span $\boldsymbol{R}^n$ with less than n vectors
- Two properties of span(S) (contain zero vector and closure)

032 Linear span Part III

- Necessary and sufficient condition for span(S) to be a subset of span(T)
- Using the necessary and sufficient condition above to check whether one linear span is contained inside another linear span
- When a vector does not add ‘value’ to a linear span

Week 06 (units 029-034) overview

033 Subspaces

- Definition of a subspace
- Zero space is a subspace of $\boldsymbol{R}^n$ and $\boldsymbol{R}^n$ is a subspace of itself
- How we can show that a given subset is not a subspace

034 Subspaces in $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$

- Linear span of one vector (geometrical)
- Linear span of two vectors (geometrical)
- Characterisation of all subspaces of $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$

Theorem

The solution set of a homogeneous system of linear equations in n variables is a subspace of $\mathbb{R}^n$.

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & 0 \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & 0 \end{cases}$$

Proof of Theorem

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & 0 \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & 0 \end{cases}$$

If the homogeneous linear system has only the trivial solution, then the solution set is $\{\mathbf{0}\}$, which is the zero subspace.

Suppose the homogeneous linear system has infinitely many solutions. We will solve the system using Gaussian elimination.

Proof of Theorem

Suppose a general solution for the system involves a total of k arbitrary parameters $t_1, t_2, \cdots, t_k$. The general solution can be written as follows.

$$\begin{cases} x_1 & = & r_{11}t_1 & + & r_{12}t_2 & + & \cdots & + & r_{1k}t_k \\ x_2 & = & r_{21}t_1 & + & r_{22}t_2 & + & \cdots & + & r_{2k}t_k \\ & \vdots & & & & & & & \\ x_n & + & r_{n1}t_1 & + & r_{n2}t_2 & + & \cdots & + & r_{nk}t_k \end{cases}$$

Here, $r_{11}, r_{12}, \cdots, r_{nk}$ are real numbers.

Proof of Theorem

$$\begin{cases} x_1 & = & r_{11}t_1 & + & r_{12}t_2 & + & \cdots & + & r_{1k}t_k \\ x_2 & = & r_{21}t_1 & + & r_{22}t_2 & + & \cdots & + & r_{2k}t_k \\ & \vdots & & & & & & & \\ x_n & + & r_{n1}t_1 & + & r_{n2}t_2 & + & \cdots & + & r_{nk}t_k \end{cases} \qquad t_1, t_2, \cdots, t_k \in \mathbb{R}$$

We can rewrite the general solution as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = t_1 \begin{pmatrix} r_{11} \\ r_{21} \\ \vdots \\ r_{n1} \end{pmatrix} + t_2 \begin{pmatrix} r_{12} \\ r_{22} \\ \vdots \\ r_{n2} \end{pmatrix} + \cdots + t_k \begin{pmatrix} r_{1k} \\ r_{2k} \\ \vdots \\ r_{nk} \end{pmatrix}. \qquad t_1, t_2, \cdots, t_k \in \mathbb{R}$$

Proof of Theorem

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So the solution set of the linear system is

$$\left\{ t_1 \begin{pmatrix} r_{11} \\ r_{21} \\ \vdots \\ r_{n1} \end{pmatrix} + t_2 \begin{pmatrix} r_{12} \\ r_{22} \\ \vdots \\ r_{n2} \end{pmatrix} + \cdots + t_k \begin{pmatrix} r_{1k} \\ r_{2k} \\ \vdots \\ r_{nk} \end{pmatrix} \middle| \, t_1, t_2, \cdots, t_k \in \mathbb{R} \right\}$$

Proof of Theorem

So the solution set of the linear system is

$$\left\{ t_1 \begin{pmatrix} r_{11} \\ r_{21} \\ \vdots \\ r_{n1} \end{pmatrix} + t_2 \begin{pmatrix} r_{12} \\ r_{22} \\ \vdots \\ r_{n2} \end{pmatrix} + \cdots + t_k \begin{pmatrix} r_{1k} \\ r_{2k} \\ \vdots \\ r_{nk} \end{pmatrix} \;\middle|\; t_1, t_2, \cdots, t_k \in \mathbb{R} \right\}$$

$$= \operatorname{span} \left\{ \begin{pmatrix} r_{11} \\ r_{21} \\ \vdots \\ r_{n1} \end{pmatrix}, \begin{pmatrix} r_{12} \\ r_{22} \\ \vdots \\ r_{n2} \end{pmatrix}, \cdots, \begin{pmatrix} r_{1k} \\ r_{2k} \\ \vdots \\ r_{nk} \end{pmatrix} \right\}$$

Since the solution set can be written as a linear span, it is a subspace of $\mathbb{R}^n$.

Example 5.1

Let $\boldsymbol{u}_1 = (2,1,0,3)$, $\boldsymbol{u}_2 = (3,-1,5,2)$, $\boldsymbol{u}_3 = (-1,0,2,1)$. Determine which of the following vectors are linear combinations of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$.

(a) $(2,3,-7,3)$ (b) $(1,1,1,1)$

Example 5.2

For each of the following sets, determine whether the set spans $\mathbb{R}^3$.

(a) $\{(1,1,-1),(-2,2,1)\}$

(b) $\{(1,1,-1),(-2,2,1),(1,5,-2)\}$

(c) $\{(1,1,-1),(-2,2,1),(4,0,3)\}$

(d) $\{(1,1,-1),(-2,2,1),(-1,7,-1),(0,8,-2)\}$

Example 5.3

Determine whether span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ = span$\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ if

$$\boldsymbol{u}_1 = (2,-2,0),\ \ \boldsymbol{u}_2 = (-1,1,-1),\ \ \boldsymbol{u}_3 = (0,0,9),\ \ \boldsymbol{v}_1 = (1,-1,-5),\ \ \boldsymbol{v}_2 = (0,1,1).$$

Example 5.4

Determine which of the following are subspaces of $\mathbb{R}^4$.

(a) $\{(w,x,y,z)\,|\,w+x=y+z\}$

(b) $\{(w,x,y,z)\,|\,wx=yz\}$

(c) $\{(w,x,y,z)\,|\,w+x+y=z^2\}$

(d) $\{(w,x,y,z)\,|\,w=0$ and $z=0\}$

(e) $\{(w,x,y,z)\,|\,w=0$ or $z=0\}$

(f) $\{(w,x,y,z)\,|\,w=1$ and $z=0\}$

(g) $\{(w,x,y,z)\,|\,w+z=0$ and $x+4y-4z=0\}$

Finally…

THE END