NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

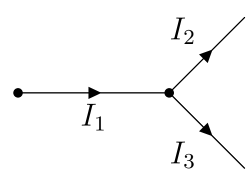
Module: MA1508E Linear Algebra for Engineering
Year/Semester: 2018-2019 (Semester 2)
Tutorial: 2

1. **(Application)** Electrical networks provides information about power sources, such as batteries, and devices powered by these sources, such as light bulbs or motors. A power source 'forces' a current of electrons to flow through the network, where it encounters various resistors, each of which requires that a certain amount of force be applied in order for the current to flow through it.

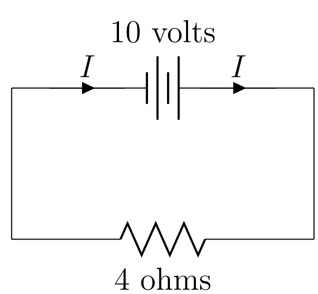
   The fundamental law of electricity is Ohm's law, which states exactly how much force E is needed to drive a current I through a resistor with resistance R. Ohm's law states $E = IR$, in other words, force = current × resistance. Here, force is measured in volts, resistance in ohms and current in amperes.

   The following two laws (discovery due to Kirchhoff), govern electrical networks. The first is a 'conservation of flow' law at each node; the second is a 'balancing of votage' law around each circuit.

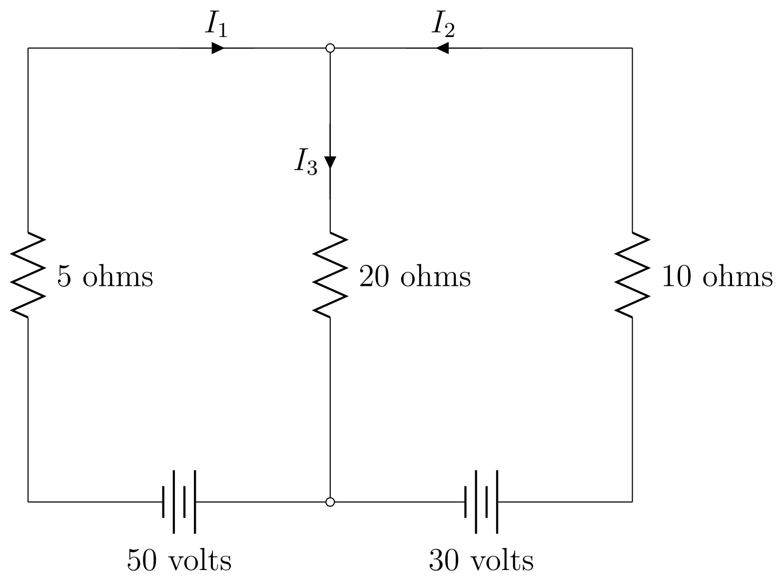
   (Kirchoff's Current Law (KCL)) At each node, the sum of the currents flowing into any node is equal to the sum of the currents flowing out of that node. For example, in the diagram below, by KCL, we have $I_1 = I_2 + I_3$.



   (Kirchoff's Voltage Law (KVL)) For every circuit, the sum of the voltage drops around the circuit is equal to the total voltage around the circuit (provided by the batteries). For example in the diagram below, by KVL, we have $4I = 10$.



   Consider the following electrical network. Using both KCL and KVL, form a linear system involving I_1, I_2 and I_3. Solve the system to determine the currents in the network.



By KCL,

$$I_1 + I_2 = I_3 \Leftrightarrow I_1 + I_2 - I_3 = 0.$$

By KVL, consider the left loop,

$$-5I_1 - 20I_3 = 50.$$

Consider the right loop,

$$20I_3 + 10I_2 = 30 \Leftrightarrow 10I_2 + 20I_3 = 30.$$

Consider the outside loop,

$$-5I_1 + 10I_2 = 80.$$

However, we can ignore this last equation as it is the sum of the previous two. We thus have the linear system

$$\begin{cases} I_1 & + & I_2 & - & I_3 & = & 0 \\ -5I_1 & & & - & 20I_3 & = & 50 \\ & & 10I_2 & + & 20I_3 & = & 30 \end{cases}$$

Solving the system

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -5 & 0 & -20 & 50 \\ 0 & 10 & 20 & 30 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

So $I_1 = -6$, $I_2 = 5$, $I_3 = -1$. The fact that I_1 and I_3 are negative means that the direction of this current is opposite of what is shown in the figure.

2. Let $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$.

(i) Compute each of the following

(a) $\boldsymbol{A}^3$; (b) $\boldsymbol{B}^2$; (c) $(\boldsymbol{AB})^3$; (d) $\boldsymbol{A}^3\boldsymbol{B}^3$.

(a) $\begin{pmatrix} 3 & 0 & 1 \\ 5 & 8 & -1 \\ 2 & 4 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} -36 & 12 & 8 \\ 4 & 20 & 104 \\ -16 & 24 & 96 \end{pmatrix}$

(d) $\boldsymbol{A}^3\boldsymbol{B}^3 = \boldsymbol{A}^3\boldsymbol{B}^2\boldsymbol{B} = \begin{pmatrix} 3 & 0 & 1 \\ 5 & 8 & -1 \\ 2 & 4 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 0 & 14 \\ 12 & 8 & 34 \\ 20 & 4 & 28 \end{pmatrix}$.

(ii) If $\boldsymbol{C} = (c_{ij})$ is a 3×3 matrix, write down the expression for the following:

(a) $(1, 3)$ entry of $(\boldsymbol{AB})\boldsymbol{C}$.

(b) $(2, 3)$ entry of $\boldsymbol{A}(\boldsymbol{CB})$.

(c) $(3, 2)$ entry of $(\boldsymbol{BC})\boldsymbol{A}$.

(a) $-3c_{13} + c_{23}$;

(b) $2(c_{11} + c_{12} + c_{13}) + (c_{21} + c_{22} + c_{23}) + (c_{31} + c_{32} + c_{33})$;

(c) $(2c_{11} + c_{31}) + (2c_{12} + c_{32}) + (2c_{13} + c_{33})$

3. Let $\boldsymbol{A}$ be an $m \times n$ matrix.

(a) Let $\boldsymbol{B_1}$ and $\boldsymbol{B_2}$ be $n \times p$ and $n \times q$ matrices respectively. Show that

$$\boldsymbol{A}\begin{pmatrix} \boldsymbol{B_1} & \boldsymbol{B_2} \end{pmatrix} = \begin{pmatrix} \boldsymbol{AB_1} & \boldsymbol{AB_2} \end{pmatrix}.$$

(In here, $\begin{pmatrix} \boldsymbol{B_1} & \boldsymbol{B_2} \end{pmatrix}$ is an $n \times (p+q)$ matrix such that its jth column is equal to the jth column of $\boldsymbol{B_1}$ if $j \leq p$ and equal to the $(j-p)$th column of $\boldsymbol{B_2}$ if $j > p$.)

(b) Let $\boldsymbol{C_1}$ and $\boldsymbol{C_2}$ be $r \times m$ matrices. Is it true that $\begin{pmatrix} \boldsymbol{C_1} & \boldsymbol{C_2} \end{pmatrix}\boldsymbol{A} = \begin{pmatrix} \boldsymbol{C_1A} & \boldsymbol{C_2A} \end{pmatrix}$?

(c) Let $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ be $s \times m$ and $t \times m$ matrices respectively. Show that

$$\begin{pmatrix} \boldsymbol{D_1} \\ \boldsymbol{D_2} \end{pmatrix}\boldsymbol{A} = \begin{pmatrix} \boldsymbol{D_1A} \\ \boldsymbol{D_2A} \end{pmatrix}.$$

(In here, $\begin{pmatrix} \boldsymbol{D_1} \\ \boldsymbol{D_2} \end{pmatrix}$ is an $(s+t) \times m$ matrix such that its ith row is equal to the ith row of $\boldsymbol{D_1}$ if $i \leq s$ and equal to the $(i-s)$th row of $\boldsymbol{D_2}$ if $i > s$.)

(a) Let $\boldsymbol{B_1} = \begin{pmatrix} \boldsymbol{b_1} & \cdots & \boldsymbol{b_p} \end{pmatrix}$ and $\boldsymbol{B_2} = \begin{pmatrix} \boldsymbol{c_1} & \cdots & \boldsymbol{c_q} \end{pmatrix}$ where $\boldsymbol{b_1}, \ldots, \boldsymbol{b_p}$ are columns of $\boldsymbol{B_1}$ and $\boldsymbol{c_1}, \ldots, \boldsymbol{c_p}$ are columns of $\boldsymbol{B_2}$. Then

$$\begin{pmatrix} \boldsymbol{B_1} & \boldsymbol{B_2} \end{pmatrix} = \begin{pmatrix} \boldsymbol{b_1} & \cdots & \boldsymbol{b_p} & \boldsymbol{c_1} & \cdots & \boldsymbol{c_q} \end{pmatrix}.$$

We have

$$\boldsymbol{AB_1} = \begin{pmatrix} \boldsymbol{Ab_1} & \cdots & \boldsymbol{Ab_p} \end{pmatrix},$$

$$\boldsymbol{AB_2} = \begin{pmatrix} \boldsymbol{Ac_1} & \cdots & \boldsymbol{Ac_q} \end{pmatrix},$$
$$\boldsymbol{A}\begin{pmatrix} \boldsymbol{B_1} & \boldsymbol{B_2} \end{pmatrix} = \begin{pmatrix} \boldsymbol{Ab_1} & \cdots & \boldsymbol{Ab_p} & \boldsymbol{Ac_1} & \cdots & \boldsymbol{Ac_q} \end{pmatrix}.$$

Hence $\boldsymbol{A}\begin{pmatrix} \boldsymbol{B_1} & \boldsymbol{B_2} \end{pmatrix} = \begin{pmatrix} \boldsymbol{AB_1} & \boldsymbol{AB_2} \end{pmatrix}$.

(b) No. The size of $\begin{pmatrix} \boldsymbol{C_1} & \boldsymbol{C_2} \end{pmatrix}$ is $r \times 2m$ and hence we cannot pre-multiply the matrix to $\boldsymbol{A}$.

(c) Let $\boldsymbol{D_1} = \begin{pmatrix} \boldsymbol{d_1} \\ \vdots \\ \boldsymbol{d_s} \end{pmatrix}$ and $\boldsymbol{D_2} = \begin{pmatrix} \boldsymbol{f_1} \\ \vdots \\ \boldsymbol{f_t} \end{pmatrix}$ where $\boldsymbol{d_1}, \ldots, \boldsymbol{d_s}$ are rows of $\boldsymbol{D_1}$ and $\boldsymbol{f_1}, \ldots, \boldsymbol{f_t}$ are rows of $\boldsymbol{D_2}$. Then

$$\begin{pmatrix} \boldsymbol{D_1} \\ \boldsymbol{D_2} \end{pmatrix} = \begin{pmatrix} \boldsymbol{d_1} \\ \vdots \\ \boldsymbol{d_s} \\ \boldsymbol{f_1} \\ \vdots \\ \boldsymbol{f_t} \end{pmatrix}.$$

We have

$$\boldsymbol{D_1A} = \begin{pmatrix} \boldsymbol{d_1A} \\ \vdots \\ \boldsymbol{d_sA} \end{pmatrix}, \quad \boldsymbol{D_2A} = \begin{pmatrix} \boldsymbol{f_1A} \\ \vdots \\ \boldsymbol{f_tA} \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{D_1} \\ \boldsymbol{D_2} \end{pmatrix}\boldsymbol{A} = \begin{pmatrix} \boldsymbol{d_1A} \\ \vdots \\ \boldsymbol{d_sA} \\ \boldsymbol{f_1A} \\ \vdots \\ \boldsymbol{f_tA} \end{pmatrix}.$$

Hence $\begin{pmatrix} \boldsymbol{D_1} \\ \boldsymbol{D_2} \end{pmatrix}\boldsymbol{A} = \begin{pmatrix} \boldsymbol{D_1A} \\ \boldsymbol{D_2A} \end{pmatrix}$.

4. (a) Show that if $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{A}+\boldsymbol{B}$ are invertible matrices of the same size, then

$$\boldsymbol{A}(\boldsymbol{A}^{-1}+\boldsymbol{B}^{-1})\boldsymbol{B}(\boldsymbol{A}+\boldsymbol{B})^{-1} = \boldsymbol{I}.$$

$$\begin{aligned} & \boldsymbol{A}(\boldsymbol{A}^{-1}+\boldsymbol{B}^{-1})\boldsymbol{B}(\boldsymbol{A}+\boldsymbol{B})^{-1} \\ = \; & (\boldsymbol{AA}^{-1}+\boldsymbol{AB}^{-1})\boldsymbol{B}(\boldsymbol{A}+\boldsymbol{B})^{-1} \\ = \; & (\boldsymbol{I}+\boldsymbol{AB}^{-1})\boldsymbol{B}(\boldsymbol{A}+\boldsymbol{B})^{-1} \\ = \; & (\boldsymbol{B}+\boldsymbol{AB}^{-1}\boldsymbol{B})(\boldsymbol{A}+\boldsymbol{B})^{-1} \\ = \; & (\boldsymbol{B}+\boldsymbol{A})(\boldsymbol{A}+\boldsymbol{B})^{-1} = \boldsymbol{I} \end{aligned}$$

(b) What does this tell you about the invertibility of $\boldsymbol{A}^{-1}+\boldsymbol{B}^{-1}$?

Since $\boldsymbol{A}(\boldsymbol{A}^{-1}+\boldsymbol{B}^{-1})$ and $\boldsymbol{B}(\boldsymbol{A}+\boldsymbol{B})^{-1}$ are both square matrices of the same size and their product is $\boldsymbol{I}$, then $\boldsymbol{A}(\boldsymbol{A}^{-1}+\boldsymbol{B}^{-1})$ is invertible. Again, since $\boldsymbol{A}$ and $(\boldsymbol{A}^{-1}+\boldsymbol{B}^{-1})$ are square matrices of the same size and their product is invertible, then $(\boldsymbol{A}^{-1}+\boldsymbol{B}^{-1})$ is invertible.

5. Consider the matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}, \boldsymbol{F}$ shown below.

$$\boldsymbol{A}=\begin{pmatrix}3&4&1\\2&-7&-1\\8&1&5\end{pmatrix},\quad \boldsymbol{B}=\begin{pmatrix}8&1&5\\2&-7&-1\\3&4&1\end{pmatrix},\quad \boldsymbol{C}=\begin{pmatrix}3&4&1\\2&-7&-1\\2&-7&3\end{pmatrix},$$

$$\boldsymbol{D}=\begin{pmatrix}8&1&5\\-6&-8&-6\\3&4&1\end{pmatrix},\quad \boldsymbol{F}=\begin{pmatrix}8&1&5\\8&1&1\\3&4&1\end{pmatrix}.$$

For each of the following, find an elementary matrix $\boldsymbol{E}$ that satisfies the given equation.

(a) $\boldsymbol{EA}=\boldsymbol{B}$; (b) $\boldsymbol{EB}=\boldsymbol{A}$; (c) $\boldsymbol{EA}=\boldsymbol{C}$ (d) $\boldsymbol{EC}=\boldsymbol{A}$.
(e) $\boldsymbol{EB}=\boldsymbol{D}$; (f) $\boldsymbol{ED}=\boldsymbol{B}$; (g) $\boldsymbol{EB}=\boldsymbol{F}$; (h) $\boldsymbol{EF}=\boldsymbol{B}$.

(a) $\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$ (b) Same as (a) (c) $\begin{pmatrix}1&0&0\\0&1&0\\-2&0&1\end{pmatrix}$.

(d) $\begin{pmatrix}1&0&0\\0&1&0\\2&0&1\end{pmatrix}$ (e) $\begin{pmatrix}1&0&0\\-1&1&0\\0&0&1\end{pmatrix}$ (f) $\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}$.

(g) $\begin{pmatrix}1&0&0\\0&1&2\\0&0&1\end{pmatrix}$ (h) $\begin{pmatrix}1&0&0\\0&1&-2\\0&0&1\end{pmatrix}$.

Are all the 5 given matrices row equivalent?

Yes they are all row equivalent.

6. For the given matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, show that $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent by finding a sequence of elementary row operations that produces $\boldsymbol{B}$ from $\boldsymbol{A}$, and then use that result to find a matrix $\boldsymbol{C}$ such that $\boldsymbol{CA}=\boldsymbol{B}$.

$$\boldsymbol{A}=\begin{pmatrix}1&2&3\\1&4&1\\2&1&9\end{pmatrix},\quad \boldsymbol{B}=\begin{pmatrix}1&0&5\\0&2&-2\\1&1&4\end{pmatrix}.$$

Note that

$$\boldsymbol{A}\ \underset{\boldsymbol{E_1}}{\xrightarrow{R_2-R_1}}\ \underset{\boldsymbol{E_2}}{\xrightarrow{R_3-2R_1}}\ \underset{\boldsymbol{E_3}}{\xrightarrow{-\frac{1}{3}R_3}}\ \begin{pmatrix}1&2&3\\0&2&-2\\0&1&-1\end{pmatrix}=\boldsymbol{X}.$$

Also

$$\boldsymbol{B} \underset{\boldsymbol{E_4}}{\overset{R_3 - R_1}{\longrightarrow}} \underset{\boldsymbol{E_5}}{\overset{R_1 + 2R_3}{\longrightarrow}} \boldsymbol{X}.$$

So $\boldsymbol{A}$ and $\boldsymbol{B}$ are indeed row equivalent. We may let $\boldsymbol{C} = \boldsymbol{E_4}^{-1}\boldsymbol{E_5}^{-1}\boldsymbol{E_3}\boldsymbol{E_2}\boldsymbol{E_1}$ and thus

$$\boldsymbol{CA} = \begin{pmatrix} -\frac{1}{3} & 0 & \frac{2}{3} \\ -1 & 1 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 1 \\ 2 & 1 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 2 & -2 \\ 1 & 1 & 4 \end{pmatrix} = \boldsymbol{B}.$$