Week 06 IVLE Quiz

1. Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be vectors in $\mathbb{R}^3$. Which of the following statements is/are definitely true?

   (I) If $\boldsymbol{x}$ and $\boldsymbol{y}$ are both linear combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$, then $\boldsymbol{x}+\boldsymbol{2y}$ is also a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

   (II) The zero vector $(0,0,0)$ is a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

   (III) If $\boldsymbol{x}$ is a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$, then $\boldsymbol{x}$ belongs to $\mathrm{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$.

   (A) (I) and (III) only.

   (B) (II) and (III) only.

   (C) All 3 statements.

   (D) None of the given combinations is correct.

   Answer: (C). (I) is correct. For instance, if $\boldsymbol{x} = a\boldsymbol{u}+b\boldsymbol{v}+c\boldsymbol{w}$ and $\boldsymbol{y} = e\boldsymbol{u}+f\boldsymbol{v}+g\boldsymbol{w}$, then $\boldsymbol{x}+2\boldsymbol{y} = (a+2e)\boldsymbol{u}+(b+2f)\boldsymbol{v}+(c+2g)\boldsymbol{w}$ which is indeed a linear combination of $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$. (II) is correct since $(0,0,0) = 0\boldsymbol{u} + 0\boldsymbol{v} + 0\boldsymbol{w}$. (III) is correct since $\mathrm{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ contains all linear combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

2. Suppose you are given the following, which of the statements below is/are true?

$$\left(\begin{array}{ccc|c} 1 & -1 & -3 & x \\ 1 & 2 & 9 & y \\ -1 & 7 & 27 & z \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 1 & -1 & -3 & x \\ 0 & 3 & 12 & y-x \\ 0 & 0 & 0 & z+3x-2y \end{array}\right).$$

   (I) There are vectors in $\mathbb{R}^3$ that **cannot** be written as linear combinations of $(1,1,-1)^T$, $(-1,2,7)^T$ and $(-3,9,27)^T$.

   (II) There are vectors in $\mathbb{R}^3$ that can be written as linear combinations of $(1,1,-1)^T$, $(-1,2,7)^T$ and $(-3,9,27)^T$ in more than one way.

   (III) $(1,1,-1)^T$, $(-1,2,7)^T$ and $(-3,9,27)^T$ spans $\mathbb{R}^3$.

   (A) (I) only.

   (B) (III) only.

   (C) (II) and (III) only

   (D) (I) and (II) only

   Answer: (D). Consider the augmented matrix below and its row-echelon form

$$\left(\begin{array}{ccc|c} 1 & -1 & -3 & x \\ 1 & 2 & 9 & y \\ -1 & 7 & 27 & z \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & -3 & x \\ 0 & 3 & 12 & y-x \\ 0 & 0 & 0 & z-2y+3x \end{array}\right)$$

We see that (I) is correct since, for example, $(x, y, z)^T = (1, 0, 0)$ is not a linear combination of $(1, 1, 1)^T$, $(-1, 2, 7)^T$ and $(-3, 9, 27)^T$ (since $z - 2y + 3z \neq 0$ when $x = 1, y = 0, z = 0$). (II) is correct since, for example, $(x, y, z)^T = (1, 1, -1)^T$ can be written as linear combinations of the three vectors in more than one way (since $z - 2y + 3z = 0$ when we let $x = 1, y = 1, z = -1$). (III) is incorrect since there are vectors (e.g. $(1, 0, 0)^T$) in $\mathbb{R}^3$ that are not linear combinations of the three vectors.

3. How many of the following sets are subspaces?

$$S_1 = \{(x, y, z) \mid x = y = 0\}$$

$$S_2 = \{(a, b, c, d, e) \mid a + b + c + d + e = 1 \quad \text{and} \quad a - b + c - d - e = 0\}$$

$$S_3 = \{(x_1, x_2, x_3, x_4) \mid x_1^2 = x_2^2 \quad \text{and} \quad x_3 = x_4\}$$

(A) None

(B) Exactly one

(C) Excatly two

(D) All three

Answer: (B). S_1 is a subspace since $S_1 = \{(0, 0, z) \mid z \in \mathbb{R}\} = \text{span}\{(0, 0, 1)\}$. S_2 is not a subspace since it does not contain the zero vector (the zero vector does not satisfy the equation $a+b+c+d+e = 1$). S_3 is not a subspace since $(1, 1, 0, 0)$ and $(-1, 1, 0, 0)$ belongs to S_3 but $(1, 1, 0, 0) + (-1, 1, 0, 0) = (0, 2, 0, 0)$ does not.

4. Suppose $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are vectors in $\mathbb{R}^3$ such that

$$\boldsymbol{u} = 3\boldsymbol{v} - 2\boldsymbol{w}.$$

Which of the following statements is/are definitely true?

(I) $\text{span}\{\boldsymbol{v}, \boldsymbol{w}\} = \text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$

(II) $\text{span}\{\boldsymbol{u}, \boldsymbol{v}\} = \text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$

(III) $\text{span}\{\boldsymbol{u}\} = \text{span}\{\boldsymbol{v}, \boldsymbol{w}\}$

(IV) $\text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a line in $\mathbb{R}^3$

(A) All four statements are true.

(B) Only (I) and (IV) are true.

(C) Only (I) and (II) are true.

(D) Only (II), (III) and (IV) are true.

Answer: (C). (I) is correct since $\boldsymbol{u}$ is a linear combination of $\boldsymbol{v}$ and $\boldsymbol{w}$, so the linear span of $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is the same as the linear span of $\{\boldsymbol{u}, \boldsymbol{w}\}$. (II) is also correct since $\boldsymbol{w}$ is a linear combination of $\boldsymbol{u}$ and $\boldsymbol{v}$ ($\boldsymbol{w} = \frac{1}{2}(3\boldsymbol{v} - \boldsymbol{u})$). (III) is not necessarily correct since, for example, when $\boldsymbol{v} = (1, 0, 0), \boldsymbol{w} = (0, 1, 0)$ and $\boldsymbol{u} = (3, -2, 0)$, then $\boldsymbol{v} = (1, 0, 0)$ belongs to $\text{span}\{\boldsymbol{v}, \boldsymbol{w}\}$ but does not belong to $\text{span}\{\boldsymbol{u}\}$. (IV) is not necessarily correct, for example, if $\boldsymbol{v} = (1, 0, 0), \boldsymbol{w} = (0, 1, 0)$ and $\boldsymbol{u} = (3, -2, 0)$, then $\text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a plane in $\mathbb{R}^3$.

5. Let S_1 and S_2 be subsets of $\mathbb{R}^4$. Furthermore, we know that $S_1 \subseteq S_2$. Which of the following statements is/are definitely true?

(I) If $\boldsymbol{u} \in S_1$ (that is, $\boldsymbol{u}$ is a vector in S_1), then $\boldsymbol{u} \in S_2$.

(II) If $\boldsymbol{u} \in S_1$, then $\boldsymbol{u} \in \text{span}(S_2)$.

(III) $\text{span}(S_1) \subseteq \text{span}(S_2)$.

(A) All three statements are true.

(B) Only (I) and (II) are true.

(C) Only (II) and (III) are true.

(D) None of the given combination given is correct.

Answer: (A). (I) is correct since S_1 is a subset of S_2 (so any element in S_1 is definitely found in S_2). (II) is correct since $\boldsymbol{u}$ is in S_2 (since it is in S_1 and S_1 is a subset of S_2), $\boldsymbol{u}$ will be in $\text{span}(S_2)$ ($\boldsymbol{u}$ is a linear combination of itself and other vectors). (III) is correct since every vector in S_1 is a linear combination of vectors in S_2 (by part (II) argument).