W03-02

Slide 01: We continue our discussion on elementary matrices.

Slide 02: Are elementary matrices invertible? The matrix shown here is recognised as an elementary matrix. It is the 4×4 matrix that results when $2R_3$ is performed on $\boldsymbol{I}_4$. In other words, this elementary matrix represents the elementary row operation $2R_3$.

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What happens when we pre-multiply with this other 4×4 matrix?

(#)

What about if we post-multiply instead? A simple verification will show that in both cases, we have the identity matrix of order 4 as the result.

(#)

Let us look at another elementary matrix representing the first type of elementary row operation. Here you see a 3×3 matrix that represents $-3R_1$.

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Let us pre-multiply the matrix with another 3×3 matrix as follows.

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Once again, both the pre- and post-multiplication results in the identity matrix. For these two examples, we have indeed found the respective inverses of the elementary matrices.

Slide 03: It is indeed true that every elementary matrix $\boldsymbol{E}$, representing elementary row operation of the first type is invertible. The general form of $\boldsymbol{E}$, as well as $\boldsymbol{E}^{-1}$ is shown here.

Slide 04: This is another elementary matrix. You should recognise it as one representing a row swap, in particular, the swap between rows 1 and 3.

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Pre- or post-multiplying the matrix with itself results in $\boldsymbol{I}_4$.

(#)

This is another elementary matrix representing a row swap operation.

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Once again, pre- or post-multiplying the matrix with itself results in the identity matrix. For these two examples, we have again shown that the elementary matrices are invertible. In fact, these elementary matrices are inverses of themselves.

Slide 05: Every elementary matrix $\boldsymbol{E}$, representing elementary row operation of the second type is invertible. For such elementary matrices, their inverse is basically the matrix itself.

Slide 06: This is an elementary matrix representing an elementary row operation of the third type. More precisely, it represents $R_1 - 3R_4$.

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Let us pre-multiply with this other matrix.

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We also post-multiply to the same matrix and again in both cases, we obtain the identity matrix $\boldsymbol{I}_4$.

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This is another example of an elementary matrix. This matrix would represent the operation $R_2 - \frac{1}{2}R_1$.

(#)

Let us pre-multiply with this other matrix.

(#)

We also post-multiply to the same matrix and once again, in both cases, we obtain the idenity matrix.

Slide 07: Every elementary matrix $\boldsymbol{E}$, representing elementary row operations of the third type is invertible. The general form of $\boldsymbol{E}$, as well as $\boldsymbol{E}^{-1}$ is shown here. Notice that while the matrix $\boldsymbol{E}$ represents $R_j + kR_i$, depending on whether $j < i$ or $i < j$, the position of the entry with the k could either be in the top right portion of the matrix,

Slide 08: or it could happen in the bottom left portion.

Slide 09: We are now ready to state a theorem on the invertibility of elementary matrices. All elementary matrices are indeed invertible and in fact, if you observe the format of their inverses, as shown in the previous slides, you would see that the inverses of elementary matrices are themselves elementary matrices. Thus, suppose $\boldsymbol{E_1}$, $\boldsymbol{E_2}$, $\boldsymbol{E_3}$ are elementary matrices representing the three types of elementary row operations respectively, as shown here.

(#)

Then, ${\boldsymbol{E_1}}^{-1}$ is also an elementary matrix. Furthermore, if $\boldsymbol{E_1}$ represents the elementary row operation cR_i, then ${\boldsymbol{E_1}}^{-1}$ represents the elementary row operation $\frac{1}{c}R_i$.

(#)

The elementary matrix $\boldsymbol{E_2}$, representing a row swap between rows i and j is also invertible and its inverse is precisely itself.

(#)

The elementary matrix $\boldsymbol{E_3}$, which represents the operation $R_j + kR_i$, is invertible and ${\boldsymbol{E_3}}^{-1}$ is the elementary matrix that represents the operation $R_j - kR_i$.

(#)

It is now clear that if $\boldsymbol{E}$ represents the elementary row operation X, then $\boldsymbol{E}^{-1}$ would be the elementary matrix that represents the elementary row operation that does the opposite of what X does.

Slide 10: Consider the following example. $\boldsymbol{A}$ is a 3×4 matrix. We are required to find a sequence of elementary matrices such that premultiplying the sequence of the elementary matrices to $\boldsymbol{A}$ results in the reduced row-echelon form of $\boldsymbol{A}$. Essentially, we need to perform Gauss Jordan elimination on $\boldsymbol{A}$, keeping track of each elementary row operation and the corresponding elementary matrix.

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The problem also require us to write down the inverse of each elementary matrix and describe which elementary row operation the inverses represent.

Slide 11: Let us begin with performing Gauss Jordan elimination on $\boldsymbol{A}$.

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The first elementary row operation performed is to swap rows 1 and 2.

(#)

This row operation corresponds to an elemntary matrix $\boldsymbol{E_1}$ and pre-multiplying $\boldsymbol{E_1}$ to $\boldsymbol{A}$ gives the current matrix with the two rows swapped.

(#)

We now add 3 times R_1 to R_3.

(#)

This second row operation corresponds to $\boldsymbol{E_2}$ and the latest matrix shown here is the result of pre-multiplying $\boldsymbol{E_2}$ to $\boldsymbol{E_1 A}$. You should note that $\boldsymbol{E_2 E_1 A}$ is actually in row-echelon form already. But since the question requires reduced row-echelon form, we will continue.

(#)

We next multiply R_2 by $\frac{1}{4}$.

(#)

This third elementary row operation corresponds to $\boldsymbol{E_3}$ and the latest matrix shown is $\boldsymbol{E_3 E_2 E_1 A}$.

(#)

The next elementary row operation performed is $\frac{1}{6}R_3$.

(#)

Representing $\frac{1}{6}R_3$ is the elementary matrix $\boldsymbol{E_4}$.

Slide 12: Now all the leading entries have been changed to 1, we will proceed with the elimination of entries above each leading entry.

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The next elementary row operation performed will be to add R_3 to R_2.

(#)

This elementary row operation is represented by $\boldsymbol{E_5}$ and the latest matrix shown here is $\boldsymbol{E_5 E_4 E_3 E_2 E_1 A}$.

(#)

The final elementary row operation that is required is to add -4 times of R_3 to R_1.

(#)

Corresponding to this elementary row operation, we have $\boldsymbol{E_6}$.

(#)

We can check that the current matrix $\boldsymbol{E_6 E_5 E_4 E_3 E_2 E_1 A}$ is indeed the reduced row-echelon form of $\boldsymbol{A}$, denoted by $\boldsymbol{R}$.

(#)

Thus we have found a sequence of elementary matrices $\boldsymbol{E_1}$ to $\boldsymbol{E_6}$ such that $\boldsymbol{E_6 E_5 E_4 E_3 E_2 E_1 A}$ is $\boldsymbol{R}$.

Slide 13: Now that we know what are the elementary row operations $\boldsymbol{E_1}$ to $\boldsymbol{E_6}$ represent, we can proceed to write down the 6 elementary matrices. Since $\boldsymbol{A}$ is a matrix with 3 rows, then it is clear that all the elementary matrices will be obtained by performing a single elementary row operation on $\boldsymbol{I_3}$.

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For $\boldsymbol{E_1}$, we swap rows 1 and 2 from $\boldsymbol{I_3}$.
(#)
This matrix will be $\boldsymbol{E_1}$.
(#)
Similarly, $\boldsymbol{E_2}$ is obtained by performing $R_3 + 3R_1$ on $\boldsymbol{I_3}$.
(#)
The matrix $\boldsymbol{E_2}$ is shown here.
(#)
To obtain $\boldsymbol{E_3}$, we perform $\frac{1}{4}R_2$ on $\boldsymbol{I_3}$.
(#)
This is the elementary matrix $\boldsymbol{E_3}$.
(#)
Continuing, we perform $\frac{1}{6}R_3$ to obtain $\boldsymbol{E_4}$,
(#)
shown here.
(#)
To obtain $\boldsymbol{E_5}$, we perform $R_2 + R_3$ on $\boldsymbol{I_3}$.
(#)
This is the elementary matrix $\boldsymbol{E_5}$.
(#)
And finallly to get $\boldsymbol{E_6}$, we perform $R_1 - 4R_3$ on $\boldsymbol{I_3}$.
(#)
The sixth and final elementary matrix $\boldsymbol{E_6}$ is shown here.

Slide 14: We are done with the first part of the question, where we have found 6 elementary matrices $\boldsymbol{E_1}$ to $\boldsymbol{E_6}$ such that pre-multiplying these matrices to $\boldsymbol{A}$, in the appropriate order, will give us the reduce row-echelon of $\boldsymbol{A}$.
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We now need to write down the inverses of each of the 6 elementary matrices and explain which elementary row operations these inverses represent. For example, recall that E_1 represents the row swap $R_1 \leftrightarrow R_2$.
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What will happen to $\boldsymbol{E_1}^{-1}$?

Slide 15: Consider the first three elementary matrices $\boldsymbol{E_1}$ to $\boldsymbol{E_3}$.
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Since E_1 represents $R_1 \leftrightarrow R_2$, $\boldsymbol{E_1}^{-1}$ represents the same elementary row operation.
(#)
$\boldsymbol{E_2}$ represents $R_3 + 3R_1$, so $\boldsymbol{E_2}^{-1}$ represents $R_3 - 3R_1$ and $\boldsymbol{E_2}^{-1}$ is shown here.
(#)
Moving on to $\boldsymbol{E_3}$, which is representing $\frac{1}{4}R_2$, $\boldsymbol{E_3}^{-1}$ represents $4R_2$, whose matrix is shown here.

Slide 16: Here are the elementary matrices $\boldsymbol{E_4}$ to $\boldsymbol{E_6}$ and how these matrices look like.
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$\boldsymbol{E_4}$ represent $\frac{1}{6}R_3$ so $\boldsymbol{E_4}^{-1}$ represents $6R_3$.
(#)
$\boldsymbol{E_5}$ represents $R_2 + R_3$, so $\boldsymbol{E_5}^{-1}$ represents $R_2 - R_3$.
(#)
Lastly, $\boldsymbol{E_6}$ is representing $R_1 - 4R_3$, so $\boldsymbol{E_6}^{-1}$ represents $R_1 + 4R_3$.

Slide 17: To sumarise this unit.
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We have shown that all elementary matrices are invertible.
(#)
Furthermore, their inverse $\boldsymbol{E}^{-1}$ is also an elementary matrix.
(#)
Lastly, if an elementary matrix $\boldsymbol{E}$ represents a single elementary row operation X, then $\boldsymbol{E}^{-1}$ will represent the elementary row operation that does the opposite of X.