Unit 014 Inverse of a matrix

Slide 01: In this unit, we will introduce a very important concept for matrices, namely the inverse of a matrix.

Slide 02: For real numbers, we are all familiar with solving an equation like $2x = 5$. While you may know that $x = 2.5$ is the answer to the equation, how this answer came about was basically because we could multiply $\frac{1}{2}$ to both sides of the equation. This $\frac{1}{2}$ was chosen because it corresponds to the 2 in $2x$.

(#)

Since $\frac{1}{2}$ times 2 is 1, we now have $x = \frac{5}{2}$.

Slide 03: For a matrix equation like the one shown here, solving for $\boldsymbol{X}$ is not as straightforward. How do we go about solving for the matrix $\boldsymbol{X}$?

(#)

Consider the following approach. If there is another matrix, denoted by the question mark here, which we can pre-multiply to both sides of the matrix equation.

Slide 04: Suppose this question mark matrix is chosen such that when premultiplied with the 3×3 matrix $2, -1, 3,\ 0, 1, 3,\ 3, 1, 2$ results in a new matrix which is the identity matrix.

(#)

This would allow us to simplify the left hand side into identity matrix times $\boldsymbol{X}$ which is just $\boldsymbol{X}$. Computing the product of the question mark matrix with the right hand side matrix of the equation would therefore allow us to solve for $\boldsymbol{X}$.

(#)

The key point of this step is obviously the choice of the question mark matrix which will allow us to simplify the left hand side of the equation to become $\boldsymbol{X}$. Note that this idea is similar to the real number example, where the choice of $\frac{1}{2}$ to multiply to 2 is so that $\frac{1}{2}$ times 2 is equal to 1.

Slide 05: Let us define formally the inverse of a square matrix. Suppose $\boldsymbol{A}$ is a square matrix of order n. Then $\boldsymbol{A}$ is said to be invertible if there exists another square matrix $\boldsymbol{B}$ of the same order such that both the products $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are equal to the identity matrix of order n.

(#)

If such a $\boldsymbol{B}$ exists, it is called an inverse of $\boldsymbol{A}$.

(#)

If $\boldsymbol{A}$ has no inverse, we say that it is singular.

Slide 06: On the point of whether $\boldsymbol{A}$ has an inverse or not, note that the definition we have just seen is an existential definition.

(#)

This means that trying very hard to find such a matrix $\boldsymbol{B}$ but unsuccessfully

(#)

is not enough to conclude that such a $\boldsymbol{B}$ does not exist.

(#)

Even if such a $\boldsymbol{B}$ has been found, the interesting question would also be whether there can be more than one such $\boldsymbol{B}$ that satisfies the definition of being an inverse of $\boldsymbol{A}$? We will answer this question later on in this unit.

Slide 07: Consider the following example. We are simply checking if the blue matrix is an inverse of the green matrix.

(#)

Clearly, in this case, we simply need to check if the blue matrix, when pre- or post-multiplied to the green matrix will give the desired outcome of an identity matrix. We first check pre-multiplication. It is easily verified that when the blue matrix is pre-multiplied to the green matrix, we have the desired outcome.

(#)

Next, we post-multiply the blue matrix to the green matrix. Again we have the identity matrix as the outcome. Thus the conclusion is that the blue matrix is indeed an inverse of the green matrix.

Slide 08: Consider the following matrix equation, which we are trying to solve for the matrix $\boldsymbol{X}$. Note that the green matrix here is the same as the previous example.

(#)

Since we have already established that the blue matrix is an inverse of the green matrix, we can now make use of this information by pre-multiplying the blue matrix on both sides of the equation. You must remember that if you pre-multiply on the left hand side of the equation, you must also pre-multiply on the right hand side.

(#)

Since the pre-multiplication of the blue matrix to the green matrix results in the identity matrix, we now have $\boldsymbol{IX}$ on the left hand side and upon evaluation the product of the two matrices on the right hand side, we have

(#)

the solution for $\boldsymbol{X}$ as such.

Slide 09: For this next example, we would like to show that this simple 2×2 matrix is singular. Remember that the existential definition of an invertible matrix makes it somewhat challenging for us, at least for the time being, to show that a matrix is actually singular.

(#)

For this example, we will adopt what is known as a proof by contradiction technique. In this technique, we first assume what we want to establish, which is the matrix being singular, is false. Thus we assume that the matrix is actually invertible. If this assumption eventually leads to something that is non-sensical or clearly wrong, then we know that what we assumed must have been wrong. Now since the matrix is assumed to be invertible, there must be, by definition, another 2×2 matrix, say a, b, c, d such that the pre-multiplication of this matrix with the matrix $1, 0, 1, 0$ results in the identity matrix.

(#)

Note that such a matrix a, b, c, d is guranteed to exist due to the definition of an invertible matrix.

Slide 10: By simplifying the product of the two matrices on the left hand side, we have the following matrix equation.

(#)

However, we now realise that we have arrived at a point where there is something that is clearly wrong. The two matrices here cannot be equal because there is at least one entry, namely the $(2,2)$-entry that are different in both matrices. This is the contradication that we are looking for.

(#)

The assumption that led to this contradiction is therefore a wrong assumption and thus what must have been correct is that this simple 2×2 matrix does not have an inverse, in other words, it is singular.

Slide 11: For the two examples we have just seen, the matrix $\boldsymbol{A}$ was found to be invertible because we were given specifically another matrix $\boldsymbol{B}$ to test whether $\boldsymbol{BA}$ and $\boldsymbol{AB}$ both resulted in $\boldsymbol{I}$. Fortunately both products did result in $\boldsymbol{I}$ and we were able to conclude that $\boldsymbol{A}$ is invertible. You may ask, what if we were not so lucky?

(#)

The next example saw a very simple 2×2 matrix which was eventually shown to be singular but the way it was done was ad-hoc and certainly not feasible for matrices that are bigger and more complicated. So do we have a more efficient way to determine whether a matrix is invertible or singular?

Slide 12: Before that, let us now answer a question that was posed earlier in this unit. If a matrix $\boldsymbol{A}$ is invertible, is it possible for $\boldsymbol{A}$ to have more than 1 inverse? The answer is no, as this result affirms. The statement here states that if $\boldsymbol{B}$ and $\boldsymbol{C}$ are matrices that both matrices satisfy the condition of being an inverse of $\boldsymbol{A}$, then it is necessary that $\boldsymbol{B}$ is equal to $\boldsymbol{C}$.

(#)

This establishes the uniqueness of the inverse of an invertible matrix.

(#)

Henceforth, for an invertible matrix $\boldsymbol{A}$ we will denote its unique inverse as $\boldsymbol{A}^{-1}$. Let us see how we can establish this result.

Slide 13: Suppose $\boldsymbol{B}$ and $\boldsymbol{C}$ are both inverses of $\boldsymbol{A}$. By the definition of an inverse, we can say

(#)

that $\boldsymbol{BA}$ is equal to $\boldsymbol{I}$

(#)

and likewise $\boldsymbol{AC}$ is equal to $\boldsymbol{I}$.

(#)

From $\boldsymbol{BA} = \boldsymbol{I}$, we post-multiply $\boldsymbol{C}$ to both sides of the equation, giving us the following matrix equation.

(#)

By the associative law for matrix multiplication, the product on the left can be written as the matrix $\boldsymbol{AC}$ post-multiplied to $\boldsymbol{B}$ while the right hand side is simply $\boldsymbol{C}$.

(#)

Using the fact that $\boldsymbol{AC}$ is equal to $\boldsymbol{I}$, we now have $\boldsymbol{BI}$ equal to $\boldsymbol{C}$
(#)
and thus we now have the desired result that the matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ are actually the same matrix. This completes the proof.

Slide 14: It turns out that to verify whether a given square matrix $\boldsymbol{B}$, having the same size as $\boldsymbol{A}$, is indeed the unique inverse of $\boldsymbol{A}$, we would only need to check either $\boldsymbol{AB} = \boldsymbol{I}$ or $\boldsymbol{BA} = \boldsymbol{I}$ and not both.
(#)
The reason for this will be explained in a later unit.

Slide 15: To summarise this unit,
(#)
We defined what is meant by an invertible matrix $\boldsymbol{A}$. Recall that the definition is an existential one because it relies on whether another matrix $\boldsymbol{B}$ can be found such that $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are both equal to $\boldsymbol{I}$.
(#)
At this moment, you should appreciate that it is not so easy to determine whether a matrix is invertible or singular.
(#)
We did, however, establish that if a matrix is invertible, then it has a unique inverse.