MA1512 TUTORIAL 1

KEY CONCEPTS – CHAPTER 1 DIFFERENTIAL EQUATIONS

Theory

- **General solutions** to DE contain arbitrary constants.
- Substituting specific values into constants of general solutions produces a **particular solution**.
- The general solution of an n^{th} order DE will contain n arbitrary constants.

Solving 1st order DEs

Technique 1:	Technique 2:
Separable Equations $y' = M(x)N(y)$ $\Rightarrow \int \frac{1}{N(y)} dy = \int M(x) dx$	Linear Change of Variable Form: $y' = f(ax + by + c)$ If $b \neq 0$, the equation will be reduced to a separable form. *Strategy*: Substitution Let $u = ax + by + c$, simplifying the equation into a separable form: $\frac{du}{dx} = \frac{d}{dx}(ax + by + c)a + b\frac{dy}{dx}a + bf(u)$

Typical Applications

Context	Differential Equation	Solution
Hot/Cold object left in environment: T is the temperature of object.	$\frac{dT}{dt} = -k(\mathrm{T} - T_{env})$	$T = T_{env} + Ae^{kt}$
Radioactive Decay: x is the amount of substance.	$\frac{dx}{dt} = -kx$	$x = Ae^{-kt}, k = \frac{ln2}{t_{1/2}}$

TUTORIAL PROBLEMS

Question 1

Solve the following differential equations:

(a) $x(x+1)y' = 1$

(b) $(secx)y' = cos5x$

(c) $y' = e^{x-3y}$

(d) $(1+y)y' + (1-2x)y^2 = 0$

Solutions

(a) This is a separable equation.

$$x(x+1)y' = 1$$

$$y' = \frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

$$\int dy = \int \frac{1}{x} - \frac{1}{x+1} dx$$

$$y = ln|x| - ln|x+1| + C$$

$$y = ln\left|\frac{x}{x+1}\right| + C$$

- One of the important standard integrals to remember is $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$.

(b) This is a separable equation.

$$(secx)y' = cos5x$$

$$y' = cosxcos5x = \frac{1}{2}(cos6x + cos4x)$$

$$\int dy = \frac{1}{2}\int (cos6x + cos4x)dx$$

$$y = \frac{1}{2}\left(\frac{1}{6}sin6x + \frac{1}{4}sin4x\right) + C$$

(c) This is a separable equation.

$$y' = e^{x-3y} = e^x \cdot e^{-3y}$$

$$e^{3y}y' = e^x$$

$$\int e^{3y} dy = \int e^x \, dx$$

$$\frac{1}{3}e^{3y} = e^x + C$$

(d) This is a separable equation.

$$(1+y)y' + (1-2x)y^2 = 0$$

$$\left(\frac{1+y}{y^2}\right)y' = 2x-1, \;\; y \neq 0$$

$$\int \left(\frac{1}{y^2} + \frac{1}{y}\right) dy = \int (2x-1)dx$$

$$\frac{-1}{y} + ln|y| = x^2 - x + C, \;\; y \neq 0$$

Why is $y \neq 0$? Is $y \neq 0$ a solution to the DE?

- Through the above method, we can notice that $y = 0$ cannot be a solution and must be excluded in the final solution. This is because in the second line of the working above, $y = 0$ will render the RHS expression undefined. **Question** So is $y = 0$ a solution to the above DE?
- We will directly substitute $y = 0$ into the above DE. We will realise that $y = 0$ will also satisfy the DE. As such, $y = 0$ is also a solution to the DE. The main takeaway for this part is that the solution $y = 0$ cannot be obtained through solving the DE. It can be obtained by deduction and observation.
- Note that $y \equiv 0$ is known as the **trivial solution** of the above DE, and will also be included into the solution.

Question 2

Experiments show that the rate of change of the temperature of a small iron ball is proportional to the difference between its temperature $T(t)$ and that of its environment, T_{env} (which is constant). Write down a differential equation describing this situation. Show that $T = T_{env}$ is a solution. Does this make sense? The ball is heated to $300°F$ and then left to cool in a room at $75°F$. Its temperature falls to $200°F$ in half an hour. Show that its temperature will be $81.6°F$ after 3 hours of cooling.

Solutions

Part 1 Write down a differential equation to describe the situation.

$$\frac{dT}{dt} = -k(T - T_{env}),$$

Why is there a '–' sign?

where $k > 0$.

Question Does it make sense that the 'rate of change of the temperature of a small iron ball is proportional to the difference between its temperature and that of its environment'? What will happen if we put a hot object in a cold environment? What will happen if we put a cold object in a hot environment? Check the units!

Part 2 Show that $T = T_{env}$ is a solution. Does this make sense?

Substitute $T = T_{env}$ into the differential equation.

LHS of DE: $\frac{dT}{dt} = 0$ since T_{env} is a constant.

RHS of DE: $-k(T - T_{env}) = -k(T_{env} - T_{env}) = -k \cdot 0 = 0$

Since LHS RHS, we can conclude that $T = T_{env}$ is a solution to the DE.

It certainly does make sense. If the temperature of the small iron ball has the same temperature as the environment, there should not be any temperature change within the small iron ball. This is because the small iron ball does not spontaneously heat up or cool down when there is no temperature difference.

Part 3 Solve the differential equation using the initial conditions given.

Given Information: $T(0) = 300 \quad T_{env} = 75 \qquad T\left(\frac{1}{2}\right) = 200$

$$\frac{dT}{dt} = -k(T - 75)$$

Note that the differential equation is separable. Assuming $T \neq T_{env}$, (Why is this necessary?)

$$\frac{1}{T-75}\frac{dT}{dt} = -k$$

$$\int \frac{1}{T-75} dT = -\int k dt$$

$$ln|T - 75| = -kt + C$$

$$T - 75 = e^{-kt+C} = e^{-kt}e^{C}$$

$$T = 75 + Ae^{-kt}$$

The two unknown constants, A and k can be solved using the two conditions $T(0) = 300$ and $T\left(\frac{1}{2}\right) = 200$. Solving the two equations simultaneously, we get $A = 225$ and $k = 1.1756$.

$$T = 75 + 225e^{-1.1756t} \;--(1)$$

Part 4 Finally show that the temperature of the ball will be $81.6°F$ after 3 hours of cooling.

Substituting $t = 3$ into (1), we get $T = 81.6$. ■

- **Question** What if we decide to the change the units from Fahrenheit to Celsius? What will change? Will the sign of k change?
- As engineers, it is important to check the units all the time. Wrong units can often lead to fatal end-results.

Question 3
In very dry regions, the phenomenon called Virga is very important because it can endanger airplanes. Virga is rain in air that is so dry that the raindrops evaporate before they can reach the ground. Suppose that the volume of a raindrop is proportional to the 3/2 power of its surface area. (Why is this reasonable? Note: raindrops are not spherical, but let us assume that they always have the same shape, no matter what their size may be.) Suppose that the rate of reduction of the volume of a raindrop is proportional to its surface area. (Why is this reasonable?) Find a formula for the amount of time it takes for a Virga raindrop to evaporate completely, expressed in terms of the constants you introduced and the initial surface area of a raindrop. Check that the units of your formula are correct. Suppose somebody suggests that the rate of reduction of the volume of a raindrop is proportional to the square of the surface area. Argue that this cannot be correct.

Solutions

Long questions can be intimating at first sight. Try to read the whole question in one shot first to get the gist of the question, then slowly dissect out the different portions of the question. It is good to also write out the information provided to you in terms of mathematics.

Part 1 "volume of a raindrop is proportional to the 3/2 power of its surface area"

What should the sign of k_1 be?

$$V = k_1 A^{3/2} - -(2)$$

Dimensional analysis tells us that this is reasonable. Volume of the raindrop, V has the units of $[L^3]$, while surface area of a raindrop, A has the units of $[L^2]$. The power 3/2 will balance out the length dimensions on both sides of equation (2), with k_1 being dimensionless.

- This model may be reasonable, but it does not make it true for all cases. That is why it is called 'a model'.
- We are not assuming that the raindrop is spherical, so $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$ cannot be used.

Part 2 "rate of reduction of the volume of a raindrop is proportional to its surface area"

$$\frac{dV}{dt} = -k_2 A - -(3)$$

This DE is reasonable. The phenomenon we are concerned with, evaporation, takes place on the surface of the raindrop. The rate of change of volume should be related to the surface area of the raindrop somehow.

Part 3 Solve the differential equation (3). We will now try to relate (2) and (3) together.

Why is there a '–' sign in (3)? What are the units of k_2?

Differentiating (2) with respect to time and using *chain rule*, we get

$$\frac{dV}{dt} = k_1 \left(\frac{3}{2}\right) A^{1/2} \frac{dA}{dt} - -(4)$$

Substituting (4) into (3), we get

$$\frac{3}{2} k_1 A^{1/2} \frac{dA}{dt} = -k_2 A$$

$$A^{-1/2} \frac{dA}{dt} = \frac{-2k_2}{3k_1} =: k$$

What should the units and sign of k be?

$$\int A^{-1/2}\, dA = \int k dt$$

$$2A^{1/2} = kt + C$$

Often, the initial conditions are not given in modelling problems. Hence, we need to set these conditions ourselves. Suppose $A(0) = A_0$, we get $C = 2A_0^{1/2}$. Thus, the DE is expressed as

$$2A^{1/2} = kt + 2A_0^{1/2}$$

Part 4 To find time when there is complete evaporation, we let $A = 0$.

$$0 = kt + 2A_0^{1/2}$$

$$t = \frac{2A_0^{1/2}}{-k} = \frac{3k_1}{k_2} A_0^{1/2} - -(5)$$

- Try to do dimensional analysis by yourself to convince yourself that both sides of equation (5) have the same units.

- In the solution presented above, we change all the variables so we will eventually get a DE in terms of surface area, A. An alternative method will be to express all variables in terms of volume, V. It can be slightly more complicated, but it will be a good practice.

Part 5 New suggestion: "rate of reduction of the volume of a raindrop is proportional to the square of the surface area"

What is the new units of k_2 now? Did the sign of k_2 change?

$$\frac{dV}{dt} = -k_2 A^2 \quad --(6)$$

The following steps are pretty much the same. Substituting (2) into (6), we get

$$\frac{3}{2} k_1 A^{1/2} \frac{dA}{dt} = -k_2 A^2$$

$$A^{-3/2} \frac{dA}{dt} = \frac{-2k_2}{3k_1} =: k$$

$$\int A^{-3/2}\, dA = \int k dt$$

$$-2A^{-1/2} = kt + C$$

Suppose $A(0) = A_0$, we get $C = -2A_0^{-1/2}$. Thus, the DE is expressed as

$$-2A^{-1/2} = kt - 2A_0^{-1/2}$$

For complete evaporation, we let $A = 0$.

$$0 = kt - 2A_0^{-1/2}$$

$$t = \frac{2A_0^{-1/2}}{k} = \frac{-3k_1}{k_2} A_0^{-1/2}$$

- The value of *t* when complete evaporation occurs is a negative value. This does not make sense, hence the model that produces differential equation (6) is invalid.
- Can you find other methods to discredit the model that produces equation (6)? The solution presented above is one of the many ways to discredit the model. It is okay to have different methods, as long as all your steps are logical and justified.

Question 4

One theory about the behavior of moths states that they navigate at night by keeping a fixed angle between their velocity vector and the direction of the Moon. A certain moth flies near to a candle and mistakes it for the Moon. What will happen to the moth?

Prove that in polar coordinates (r, θ) with the candle at the origin, the formula for the angle ψ between the radius vector and the velocity vector is given by $tan\psi = r\frac{d\theta}{dr}$.

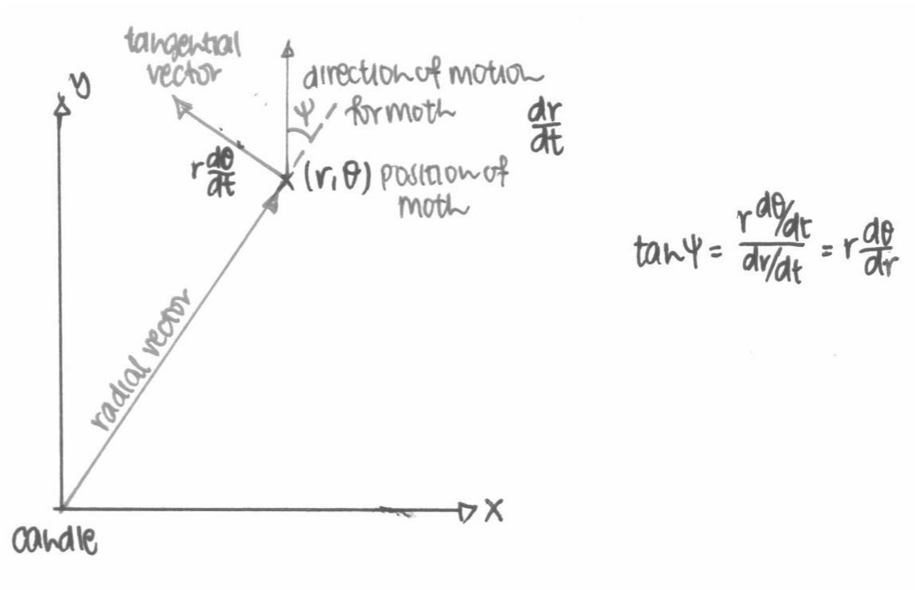
[If you want to derive this formula, remember that the tangential component of a small displacement in polar coordinates $(r, \theta) \to (r + dr, \theta + d\theta)$ is $rd\theta$, and the radial component is just dr. Now use simple trigonometry.]

Use the formula to solve for r as a function of θ, and discuss what will happen to the moth.

Solutions

Motivation We know for sure that ψ is a constant value, and based on this information, we will like to relate the various variables together to establish a DE. As ψ is an angle, it is natural for us to consider the polar coordinates. If we were to use Cartesian coordinates, we will expect the equations to generate complicated functions such as square roots etc.

- Please see the official solutions for a different approach in proving $tan\psi = r\frac{d\theta}{dr}$.



- Remember that ψ is the angle between the radius vector of the moth (pointing outwards) and her velocity.

Step 1 Set up the differential equation and solve.

$$tan\psi = r\frac{d\theta}{dr}$$

Note that in the DE above, $\psi \neq 90°$. Otherwise, $tan\psi \to \infty$. Also note that ψ is a constant.

$$\int \frac{1}{r}dr = \frac{1}{tan\psi}\int d\theta$$

$$ln|r| = \frac{\theta}{tan\psi} + C$$

$$r = Ae^{\frac{\theta}{tan\psi}}$$

Step 2 Once again, we are not provided with the initial conditions. As such, we will set them as $r(0) = R$. Thus, we get $A = R$. The DE becomes

$$r = Re^{\frac{\theta}{tan\psi}}$$

Step 3 Now we perform a simple analysis.

Case 1 $\psi < 90° \Longrightarrow tan\psi > 0$

This will mean that an increase in θ causes an increase in r.

This makes sense because for $\psi < 90°$, the moth is moving in the direction away from the candle. The moth will spiral away from the candle.

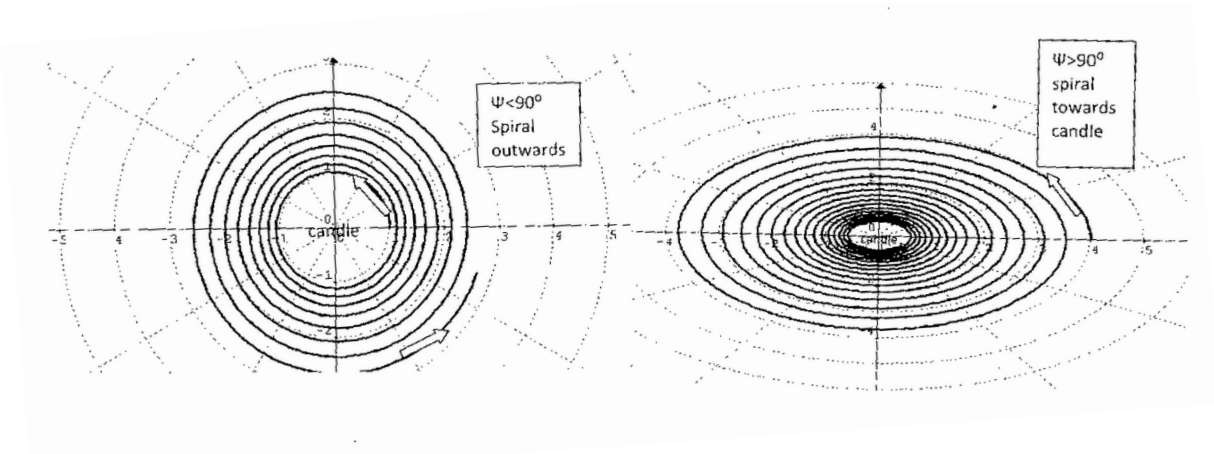
Case 2 $\psi > 90° \Longrightarrow tan\psi < 0$

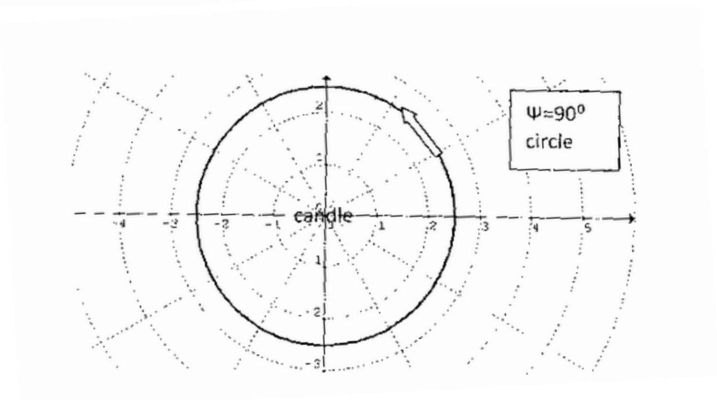
This will mean that an increase in θ causes a decrease in r.

This makes sense because for $\psi > 90°$, the moth is moving in the direction towards from the candle. The moth will spiral towards the candle.

Case 3 $\psi = 90°$

From geometry, we can infer that $r = R$, which is essentially a circle of radius R units with centre at $(0,0)$.





Question 5

Solve the following differential equations:

(a) $y' = \frac{1-2y-4x}{1+y+2x}$ *(b)* $y' = \left(\frac{x+y+1}{x+y+3}\right)^2$

Solutions

All the parts in this question requires a linear change of variable.

(a) $y' = \frac{1-2y-4x}{1+y+2x}$

Use the substitution $v = y + 2x \Longrightarrow v' = y' + 2$.

$$y' = \frac{1-2y-4x}{1+y+2x}$$

$$v' - 2 = \frac{1-2v}{1+v}$$

$$v' = \frac{3}{1+v}$$

$$\int (1+v)\,dv = \int 3dx$$

$$v + \frac{1}{2}v^2 = 3x + C$$

$$(y+2x) + \frac{1}{2}(y+2x)^2 = 3x + C$$

$$y + \frac{1}{2}(y+2x)^2 = x + C$$

Will the substitution $v = 1 - 2y - 4x$ work? Try it.

(b) $y' = \left(\frac{x+y+1}{x+y+3}\right)^2$

Use the substitution $v = x + y \Longrightarrow v' = 1 + y'$.

$$y' = \left(\frac{x+y+1}{x+y+3}\right)^2$$

$$v' - 1 = \left(\frac{v+1}{v+3}\right)^2$$

$$v' = \frac{2v^2+8v+10}{(v+3)^2}$$

$$\frac{(v+3)^2}{v^2+4v+5}v' = 2$$

$$\int \left(1 + \frac{2v+4}{v^2+4v+5}\right) dv = \int 2dx$$

$$v + \ln|v^2+4v+5| = 2x + C$$

$$y + ln|(x+y)^2 + 4x + 4y + 5| = x + C$$

- We will typically obtain a separable equation after the substitution.

Question 6

When a cake is removed from an oven, its temperature is measured at 130°C. Three minutes later, its temperature is 90°C. How long will it take for the cake to cool off to 26°C, with room temperature at 25°C?

Solutions

Similar to question 2, we recognise that the rate of change of temperature is proportional to the temperature difference compared to the room temperature. We thus set up the following DE:

$$\frac{dT}{dt} = -k(T - 25)$$

We will solve the separable DE as in question 2 to obtain:

$$T = 25 + Ae^{-kt}$$

Using the 2 pieces of information given in the question,

(1) When $t = 0$, $T = 130$. This implies that $A = 105$.

(2) When $t = 3$, $T = 90$. This implies that $k = \frac{1}{3}ln\frac{21}{13}$.

$$\therefore T = 25 + 105e^{\frac{t}{3}ln\frac{21}{13}}$$

Thus, when $T = 26$, we obtain $t = 29.1min$.

Question 7 (Persistence of a radioactive contaminant)

If we start with an initial concentration of C_i of radioactive radon-222, what would it concentration be after 5 days? The half-life for radon is 3.8 days.

Solutions

Note that the change of amount of radon-222 depends on amount of radon-222. Hence, the DE looks like

$$\frac{dR}{dt} = -kR$$

What is the sign of k?

Solving the separable equation, we have

$$\int \frac{1}{R} dR = -\int k\, dt$$

$$lnR = -kt + C$$

$$R = Ae^{-kt}$$

When $t = 0$, $R = C_i$. Thus, this implies that $A = C_i$.

$$R = C_i e^{-kt}$$

The half-life of 3.8 days implies that when $R = \frac{C_i}{2}$, $t = 3.8$.

$$\frac{C_i}{2} = C_i e^{-3.8k} \Longrightarrow k = \frac{ln2}{3.8}$$

$$\therefore R = C_i e^{\frac{-ln2}{3.8}t}$$

This is how the formula $k = \frac{ln2}{t_{1/2}}$ is derived.

When $t = 5$,

$$R = C_i e^{\frac{-ln2}{3.8}(5)} = 0.402 C_i$$

Question 8 (Historical world population growth rate)

It took the world about 300 years to increase its population from 0.5 billion to 4.0 billion. If we assume exponential growth at a constant rate over that period of time, what would that growth rate be?

Solutions

Question What are they asking for?

Note that the change in population depends on the population amount. Hence, the DE looks like

$$\frac{dP}{dt} = kP$$

Solving the separable equation, we have

$$\int \frac{1}{P} dP = \int k\, dt$$

$$lnP = kt + C$$

$$P = Ae^{kt}$$

Using the 2 pieces of information given in the question,

(1) When $t = 0$, $P = 0.5$. This implies that $A = 0.5$.

(2) When $t = 300$, $P = 4$. This implies that $k = \frac{ln2}{100} = 0.00693 = 0.693\%$.