# ELEMENTARY MATRICES (PART II)

# Are elementary matrices invertible?

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \boldsymbol{I}_4$$

$$\begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{I}_3$$

# Are elementary matrices invertible?

$$\boldsymbol{E} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & \cdots & \cdots & 0 \\ \vdots & & c & & & \vdots \\ \vdots & & & 1 & & \vdots \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix}$$

$\boldsymbol{E}$ is invertible and $\boldsymbol{E}^{-1}$ is shown below (check it!)

$$\boldsymbol{E}^{-1} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & & \cdots & \cdots & 0 \\ \vdots & & \frac{1}{c} & & & \vdots \\ \vdots & & & 1 & & \vdots \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix}$$

# Are elementary matrices invertible?

$$\begin{pmatrix} \boxed{0 & 0 & 1 & 0} \\ 0 & 1 & 0 & 0 \\ \boxed{1 & 0 & 0 & 0} \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \boxed{0 & 0 & 1 & 0} \\ 0 & 1 & 0 & 0 \\ \boxed{1 & 0 & 0 & 0} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \boldsymbol{I}_4$$

$$\begin{pmatrix} 1 & 0 & 0 \\ \boxed{0 & 0 & 1} \\ \boxed{0 & 1 & 0} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \boxed{0 & 0 & 1} \\ \boxed{0 & 1 & 0} \end{pmatrix} = \boldsymbol{I}_3$$

# Are elementary matrices invertible?

$$\boldsymbol{E} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

$\boldsymbol{E}$ is invertible and $\boldsymbol{E}^{-1}$ is shown below (check it!)

$$\boldsymbol{E}^{-1} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & 1 & & \\ & & & \ddots & & & \\ & & 1 & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

# Are elementary matrices invertible?

$$\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \boldsymbol{I}_4$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \boldsymbol{I}_3$$

# Are elementary matrices invertible?

$$\boldsymbol{E} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & k & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

$\boldsymbol{E}$ is invertible and $\boldsymbol{E}^{-1}$ is shown below (check it!)

$$\boldsymbol{E}^{-1} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & -k & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

# Are elementary matrices invertible?

$$\boldsymbol{E} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & k & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

$\boldsymbol{E}$ is invertible and $\boldsymbol{E}^{-1}$ is shown below (check it!)

$$\boldsymbol{E}^{-1} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & -k & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

# Theorem

All elementary matrices are invertible and their inverses are themselves elementary matrices.

$$\boldsymbol{I}_m \xrightarrow{cR_i} \boldsymbol{E}_1 \qquad \boldsymbol{I}_m \xrightarrow{\frac{1}{c}R_i} \boldsymbol{E}_1^{-1}$$

$$\boldsymbol{I}_m \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E}_2 \qquad \boldsymbol{I}_m \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{E}_2^{-1}$$

$$\boldsymbol{I}_m \xrightarrow{R_j + kR_i} \boldsymbol{E}_3 \qquad \boldsymbol{I}_m \xrightarrow{R_j - kR_i} \boldsymbol{E}_3^{-1}$$

Note that if an elementary matrix $\boldsymbol{E}$ represents a single elementary row operation $X$, $\boldsymbol{E}^{-1}$ represents the elementary row operation that 'un-do' $X$.

# Example

Let $\boldsymbol{A} = \begin{pmatrix} 0 & 4 & 8 & -4 \\ 1 & 0 & -1 & 4 \\ -3 & 0 & 3 & -6 \end{pmatrix}$. Find a sequence of elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_k$ such that $\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A}$ is the reduced row-echelon form of $\boldsymbol{A}$.

Write down the inverses of each of the elementary matrices and describe which elementary row operation these inverses represent.

# Example

$$A = \begin{pmatrix} 0 & 4 & 8 & -4 \\ 1 & 0 & -1 & 4 \\ -3 & 0 & 3 & -6 \end{pmatrix} \xrightarrow[E_1]{R_1 \leftrightarrow R_2} \overset{E_1 A}{\begin{pmatrix} 1 & 0 & -1 & 4 \\ 0 & 4 & 8 & -4 \\ -3 & 0 & 3 & -6 \end{pmatrix}}$$

$$\xrightarrow[E_2]{R_3 + 3R_1}$$

$$\underset{E_4E_3E_2E_1A}{\begin{pmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}} \xleftarrow[E_4]{\frac{1}{6}R_3} \underset{E_3E_2E_1A}{\begin{pmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 6 \end{pmatrix}} \xleftarrow[E_3]{\frac{1}{4}R_2} \underset{E_2E_1A}{\begin{pmatrix} 1 & 0 & -1 & 4 \\ 0 & 4 & 8 & -4 \\ 0 & 0 & 0 & 6 \end{pmatrix}}$$

# Example

$$\begin{pmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[\boldsymbol{E}_5]{R_2 + R_3} \begin{pmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{E}_4\boldsymbol{E}_3\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A} \qquad \boldsymbol{E}_5\boldsymbol{E}_4\boldsymbol{E}_3\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A}$$

$$\xrightarrow[\boldsymbol{E}_6]{R_1 - 4R_3}$$

(Reduced row-echelon form)

$$\boldsymbol{R} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{R} = \boldsymbol{E}_6\boldsymbol{E}_5\boldsymbol{E}_4\boldsymbol{E}_3\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A} \qquad \boldsymbol{E}_6\boldsymbol{E}_5\boldsymbol{E}_4\boldsymbol{E}_3\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{A}$$

# Example

$$\boldsymbol{A} = \begin{pmatrix} 0 & 4 & 8 & -4 \\ 1 & 0 & -1 & 4 \\ -3 & 0 & 3 & -6 \end{pmatrix}$$

$$\boldsymbol{I}_3 \xrightarrow{R_1 \leftrightarrow R_2} \boldsymbol{E}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{I}_3 \xrightarrow{\frac{1}{6} R_3} \boldsymbol{E}_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{6} \end{pmatrix}$$

$$\boldsymbol{I}_3 \xrightarrow{R_3 + 3R_1} \boldsymbol{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{I}_3 \xrightarrow{R_2 + R_3} \boldsymbol{E}_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{I}_3 \xrightarrow{\frac{1}{4} R_2} \boldsymbol{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{I}_3 \xrightarrow{R_1 - 4R_3} \boldsymbol{E}_6 = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Example

Let $\boldsymbol{A} = \begin{pmatrix} 0 & 4 & 8 & -4 \\ 1 & 0 & -1 & 4 \\ -3 & 0 & 3 & -6 \end{pmatrix}$. Find a sequence of elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_k$ such that $\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A}$ is the reduced row-echelon form of $\boldsymbol{A}$. Done!

Write down the inverses of each of the elementary matrices and describe which elementary row operation these inverses represent.

$$\boldsymbol{E}_1^{-1} = ? \qquad \boldsymbol{I}_3 \xrightarrow{R_1 \leftrightarrow R_2} \boldsymbol{E}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Example

$$I_3 \xrightarrow{\boxed{R_1 \leftrightarrow R_2}} E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (R_1 \leftrightarrow R_2) \; E_1^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I_3 \xrightarrow{\boxed{R_3 + 3R_1}} E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \qquad (R_3 - 3R_1) \; E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

$$I_3 \xrightarrow{\boxed{\frac{1}{4}R_2}} E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (4R_2) \; E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Example

$$I_3 \xrightarrow{\boxed{\frac{1}{6}R_3}} E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{6} \end{pmatrix} \qquad \boxed{(6R_3)}\ E_4^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$I_3 \xrightarrow{\boxed{R_2 + R_3}} E_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \boxed{(R_2 - R_3)}\ E_5^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I_3 \xrightarrow{\boxed{R_1 - 4R_3}} E_6 = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \boxed{(R_1 + 4R_3)}\ E_6^{-1} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# Summary

1) All elementary matrices $\boldsymbol{E}$ are invertible...

2) ...and their inverse $\boldsymbol{E}^{-1}$ is also an elementary matrix.

3) If an elementary matrix $\boldsymbol{E}$ represents a single elementary row operation $X$, $\boldsymbol{E}^{-1}$ represents the elementary row operation that does the 'opposite' of $X$.