GAUSSIAN AND GAUSS-JORDAN ELIMINATION

RECAP

1) Row-echelon forms allows us to determine how many (if any) solutions a linear system has.

2) Row-echelon forms helps us to write down a general solution for a linear system that is consistent.

3) We now need to develop a systematic way of finding an augmented matrix in row-echelon form that is row equivalent to the augmented matrix of the original linear system.

GAUSSIAN ELIMINATION

Gaussian Elimination is an algorithm (systematic way of doing things) to reduce an augmented matrix to a row-echelon form using elementary row operations.

Again, this is not restricted to augmented matrices.

STEP 1

Locate the leftmost column that is not entirely zero.

$$\begin{pmatrix} \boxed{\begin{matrix} 0 \\ 1 \\ 2 \end{matrix}} & \begin{matrix} 2 \\ -1 \\ 3 \end{matrix} & \begin{matrix} \dots \\ \dots \\ \dots \end{matrix} & \begin{matrix} \dots \\ \dots \\ \dots \end{matrix} \end{pmatrix} \qquad \begin{pmatrix} \boxed{\begin{matrix} 2 \\ -1 \\ 3 \end{matrix}} & \begin{matrix} 1 \\ 1 \\ 2 \end{matrix} & \begin{matrix} \dots \\ \dots \\ \dots \end{matrix} & \begin{matrix} \dots \\ \dots \\ \dots \end{matrix} \end{pmatrix}$$

$$\begin{pmatrix} \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & \boxed{\begin{matrix} 0 \\ 1 \\ 2 \end{matrix}} & \begin{matrix} \dots \\ \dots \\ \dots \end{matrix} & \begin{matrix} \dots \\ \dots \\ \dots \end{matrix} \end{pmatrix} \qquad \begin{pmatrix} \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & \boxed{\begin{matrix} 1 \\ 2 \\ -1 \end{matrix}} & \begin{matrix} \dots \\ \dots \\ \dots \end{matrix} \end{pmatrix}$$

STEP 2

Look at the topmost entry in the column identified in Step 1.

(i) if the entry is non zero, do nothing. This entry is your leading entry.

$$\begin{pmatrix} 0 & 2 & \dots & \dots \\ 1 & -1 & \dots & \dots \\ 2 & 3 & \dots & \dots \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 & \dots & \dots \\ -1 & 1 & \dots & \dots \\ 3 & 2 & \dots & \dots \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \dots & \dots \\ 0 & 1 & \dots & \dots \\ 0 & 2 & \dots & \dots \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & \dots \\ 0 & 0 & 2 & \dots \\ 0 & 0 & -1 & \dots \end{pmatrix}$$

STEP 2

Look at the topmost entry in the column identified in Step 1.

(ii) if the entry is zero, interchange the top row with another row to bring a non zero entry to the top.

$$\begin{pmatrix} 0 & 2 & \dots & \dots \\ 1 & -1 & \dots & \dots \\ 2 & 3 & \dots & \dots \end{pmatrix} \xrightarrow[\text{rows 1 and 2}]{\text{interchange}} \begin{pmatrix} 1 & -1 & \dots & \dots \\ 0 & 2 & \dots & \dots \\ 2 & 3 & \dots & \dots \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \dots & \dots \\ 0 & 1 & \dots & \dots \\ 0 & 2 & \dots & \dots \end{pmatrix} \xrightarrow[\text{rows 1 and 2}]{\text{interchange}} \begin{pmatrix} 0 & 1 & \dots & \dots \\ 0 & 0 & \dots & \dots \\ 0 & 2 & \dots & \dots \end{pmatrix}$$

STEP 2

At the end of Step 2, you would have identified a pivot column and a leading entry in that column.

$$\begin{pmatrix} 2 & 1 & \cdots & \cdots \\ -1 & 1 & \cdots & \cdots \\ 3 & 2 & \cdots & \cdots \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 & \cdots & \cdots \\ 0 & 2 & \cdots & \cdots \\ 2 & 3 & \cdots & \cdots \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & \cdots \\ 0 & 0 & 2 & \cdots \\ 0 & 0 & -1 & \cdots \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots \\ 0 & 2 & \cdots & \cdots \end{pmatrix}$$

STEP 3

For each row below the row with the leading entry, add a suitable multiple of the row with the leading entry so that all the entries below the leading entry (in the same column) becomes zero.

$$-2 = -\left(\frac{2}{1}\right) \qquad \begin{pmatrix} 0 & 1 & \dots & \dots \\ 0 & 0 & \dots & \dots \\ 0 & 2 & \dots & \dots \end{pmatrix}$$

Add -2 times of row 1 to row 3 so that 2 becomes 0.

STEP 3

For each row below the row with the leading entry, add a suitable multiple of the row with the leading entry so that all the entries below the leading entry (in the same column) becomes zero.

$$\frac{1}{2} = -\left(\frac{-1}{2}\right)$$

$$\begin{pmatrix} 2 & 1 & \cdots & \cdots \\ -1 & 1 & \cdots & \cdots \\ 3 & 2 & \cdots & \cdots \end{pmatrix}$$

$$-\frac{3}{2} = -\left(\frac{3}{2}\right)$$

Add $\frac{1}{2}$ times of row 1 to row 2 so that (-1) becomes 0.

Add $-\frac{3}{2}$ times of row 1 to row 3 so that 3 becomes 0.

STEP 4

Cover the row containing the leading entry and apply Step 1 again to the remaining submatrix.

Add $\frac{1}{2}$ times of row 1 to row 2 so that (-1) becomes 0.

$$\begin{pmatrix} 2 & 1 & \cdots & \cdots \\ -1 & 1 & \cdots & \cdots \\ 3 & 2 & \cdots & \cdots \end{pmatrix} \longrightarrow \begin{pmatrix} & & & \\ 0 & \frac{3}{2} & \cdots & \cdots \\ 0 & \frac{1}{2} & \cdots & \cdots \end{pmatrix}$$

remaining submatrix

Add $-\frac{3}{2}$ times of row 1 to row 3 so that 3 becomes 0.

Continue this way until the entire matrix is in row-echelon form.

GAUSS-JORDAN ELIMINATION

Gauss-Jordan Elimination is an algorithm to reduce an augmented matrix to the reduced row-echelon form using elementary row operations.

Again, this is not restricted to augmented matrices.

STEP 5

First, apply Gaussian Elimination (Steps 1 to 4) to obtain a row-echelon form for the augmented matrix.

row-echelon forms
$$\begin{pmatrix} 2 & 1 & \ldots & \ldots \\ 0 & \frac{3}{2} & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 1 & \ldots \\ 0 & 0 & -\frac{11}{3} & \ldots \\ 0 & 0 & 0 & \ldots \end{pmatrix}$$

Multiply each (non zero) row by a suitable constant so that each leading entry becomes 1.

$$\begin{pmatrix} 2 & 1 & \ldots & \ldots \\ 0 & \frac{3}{2} & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots \end{pmatrix} \xrightarrow[\text{Multiply row 2 by } \frac{2}{3}]{\text{Multiply row 1 by } \frac{1}{2}} \begin{pmatrix} 1 & \frac{1}{2} & \ldots & \ldots \\ 0 & 1 & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots \end{pmatrix}$$

STEP 5

First, apply Gaussian Elimination (Steps 1 to 4) to obtain a row-echelon form for the augmented matrix.

row-echelon forms
$$\begin{pmatrix} 2 & 1 & \ldots & \ldots \\ 0 & \frac{3}{2} & \ldots & \ldots \\ 0 & 0 & \ldots & \ldots \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 1 & \ldots \\ 0 & 0 & -\frac{11}{3} & \ldots \\ 0 & 0 & 0 & \ldots \end{pmatrix}$$

Multiply each (non zero) row by a suitable constant so that each leading entry becomes 1.

$$\begin{pmatrix} 0 & 1 & 1 & \ldots \\ 0 & 0 & -\frac{11}{3} & \ldots \\ 0 & 0 & 0 & \ldots \end{pmatrix} \xrightarrow{\text{Multiply row 2 by } -\frac{3}{11}} \begin{pmatrix} 0 & 1 & 1 & \ldots \\ 0 & 0 & 1 & \ldots \\ 0 & 0 & 0 & \ldots \end{pmatrix}$$

STEP 6

Starting with the leading entry in the 'lowest' row, add suitable multiples of this row to rows above so that entries (in the same column) above the leading entry becomes 0.

$$\begin{pmatrix} 1 & -2 & \cdots & \cdots & -4 & \cdots \\ & 1 & \cdots & \cdots & 3 & \cdots \\ & & \ddots & & \vdots & \vdots \\ & & & 1 & 2 & \cdots \\ & & & & 1 & \cdots \end{pmatrix}$$

$$-2 = -\left(\frac{2}{1}\right)$$

Add (–2) times row $\boxed{}$ to this row so that 2 becomes 0.

STEP 6

Starting with the leading entry in the 'lowest' row, add suitable multiples of this row to rows above so that entries (in the same column) above the leading entry becomes 0.

$$\begin{pmatrix} 1 & -2 & \dots & \dots & -4 & \dots \\ & 1 & \dots & \dots & 3 & \dots \\ & & \ddots & & \vdots & \vdots \\ & & & 1 & 2 & \dots \\ & & & & 1 & \dots \end{pmatrix}$$

$$-3 = -\left(\frac{3}{1}\right)$$

Add (-3) times row $\boxed{}$ to this row so that 3 becomes 0.

STEP 6

Starting with the leading entry in the 'lowest' row, add suitable multiples of this row to rows above so that entries (in the same column) above the leading entry becomes 0.

$$\begin{pmatrix} 1 & -2 & \dots & \dots & -4 & \dots \\ & 1 & \dots & \dots & 3 & \dots \\ & & \ddots & & \vdots & \vdots \\ & & & 1 & 2 & \dots \\ & & & & 1 & \dots \end{pmatrix}$$

Add (4) times row $\boxed{}$ to this row so that -4 becomes 0.

$$4 = -\left(\frac{-4}{1}\right)$$

STEP 6

Repeat the same procedure with the 'next lowest' leading entry.

$$\begin{pmatrix} 1 & -2 & \dots & \dots & 0 & \dots \\ & 1 & \dots & \dots & 0 & \dots \\ & & \ddots & & \vdots & \vdots \\ & & & 1 & 0 & \dots \\ & & & & 1 & \dots \end{pmatrix}$$

Continue this way until reduced row-echelon form is attained.

NOTATIONS FOR ELEMENTARY ROW OPERATIONS

Please use the following notations when performing elementary row operations:

1) When you want to multiply row i by a non zero constant c, write cR_i.

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 0 \end{array}\right) \xrightarrow[2R_3]{-\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \end{array}\right)$$

NOTATIONS FOR ELEMENTARY ROW OPERATIONS

Please use the following notations when performing elementary row operations:

2) When you want interchange rows i and j,

write $R_i \leftrightarrow R_j$.

$$\left(\begin{array}{ccc|c} 0 & 0 & -1 & 2 \\ 1 & -2 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & \frac{1}{2} & 0 \end{array}\right)$$

NOTATIONS FOR ELEMENTARY ROW OPERATIONS

Please use the following notations when performing elementary row operations:

3) When you want add k times of row i to row j,

write $R_j + kR_i$.

Remember that in this case,

row j changes but row i does not.

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 1 & -2 & 0 & 1 \\ -2 & 0 & \frac{1}{2} & 0 \end{array}\right) \xrightarrow[R_3+2R_1]{R_2-R_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -3 & 1 & -1 \\ 0 & 2 & -\frac{3}{2} & 4 \end{array}\right)$$

SUMMARY

1) Gaussian elimination

2) Gauss-Jordan elimination

3) Standard notations to use while performing elementary row operations.