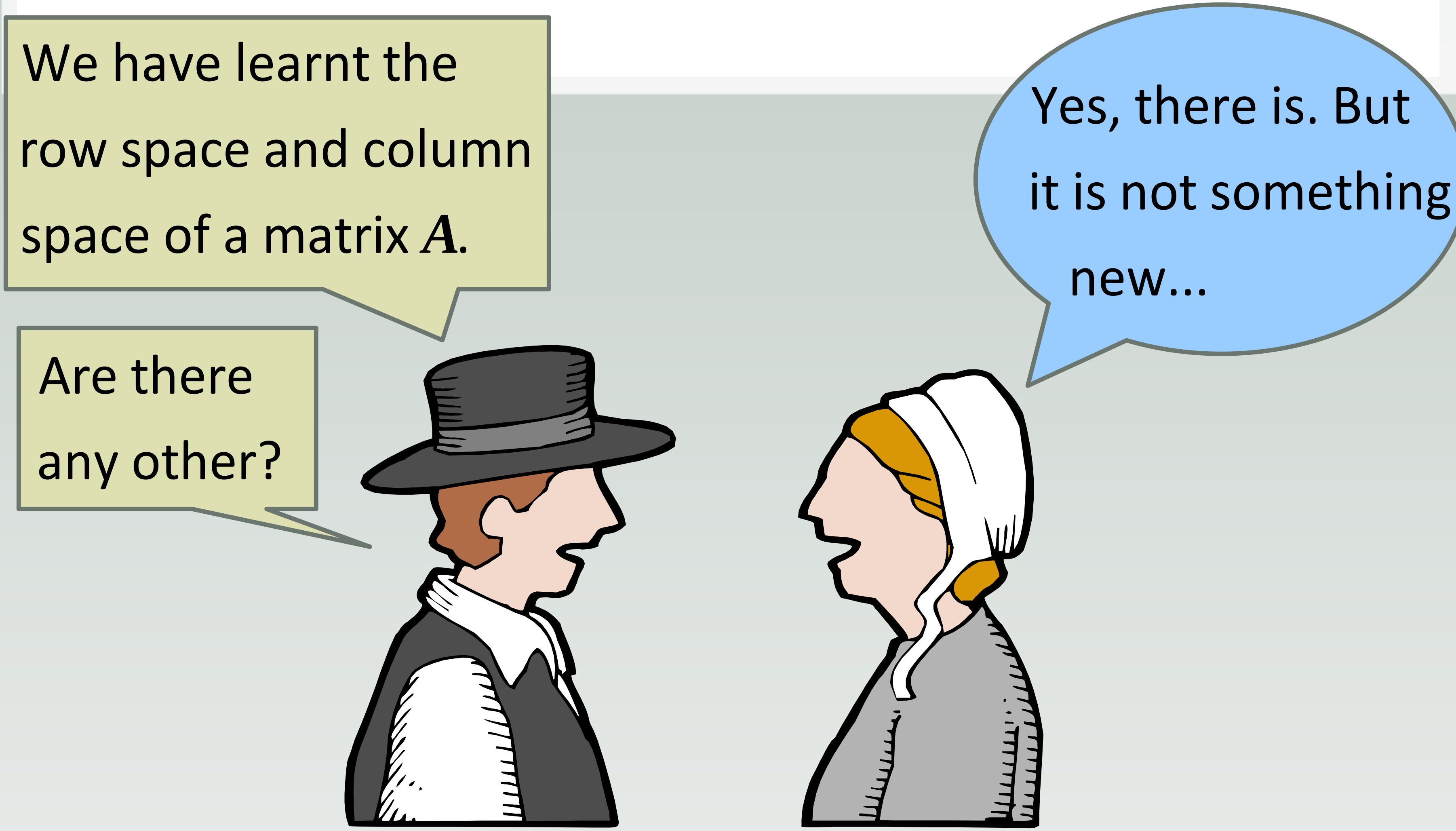
NULLSPACE OF A MATRIX

ANOTHER SUBSPACE



DEFINITION

Let $\boldsymbol{A}$ be a $m \times \boxed{n}$ matrix.

$\boldsymbol{Ax} = \mathbf{0}$ is a homogeneous linear system with $\boxed{n}$ variables.

The solution set of $\boldsymbol{Ax} = \mathbf{0}$ is a subspace of $\mathbb{R}^n$

and is also called the solution space of $\boldsymbol{Ax} = \mathbf{0}$.

This is also called the nullspace of the matrix $\boldsymbol{A}$.

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

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Since the nullspace of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^n$,

its dimension is $\leq n$.

The dimension of the nullspace of $\boldsymbol{A}$ is called the nullity of $\boldsymbol{A}$

and denoted by nullity($\boldsymbol{A}$).

EXAMPLE

Find a basis for the nullspace and determine the nullity of the following matrix:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 5 & 0 \\ -1 & 3 & 2 & -4 \\ 2 & 1 & 0 & 8 \\ 3 & 1 & -1 & 12 \end{pmatrix}$$

EXAMPLE

The reduced row echelon form of the augmented matrix

$(\boldsymbol{A}\,|\,\mathbf{0})$ is
$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4s \\ 0 \\ 0 \\ s \end{pmatrix} = s\begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad s \in \mathbb{R}.$$

A general solution for $\boldsymbol{Ax} = \mathbf{0}$ is
$$\begin{cases} x_1 &= -4s \\ x_2 &= 0 \\ x_3 &= 0 \\ x_4 &= s, \quad s \in \mathbb{R} \end{cases}$$

EXAMPLE

The reduced row echelon form of the augmented matrix

$(\boldsymbol{A} \mid \mathbf{0})$ is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -4s \\ 0 \\ 0 \\ s \end{pmatrix} = s\begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad s \in \mathbb{R}.$$

So a basis for the nullspace of $\boldsymbol{A}$ is $\left\{\begin{pmatrix} -4 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

$\text{nullity}(\boldsymbol{A}) = 1$

1 non pivot column

What is the rank of $\boldsymbol{A}$?

$\text{rank}(\boldsymbol{A}) = 3$

3 pivot columns

EXAMPLE

Find a basis for the nullspace and determine the nullity of the following matrix:

$$\boldsymbol{B} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix}$$

EXAMPLE

The reduced row echelon form of the augmented matrix $(\boldsymbol{B}\,|\,\boldsymbol{0})$ is

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

A general solution for $\boldsymbol{Bx}=\boldsymbol{0}$ is

$$\begin{cases} x_1 &= -s-t \\ x_2 &= s \\ x_3 &= -t \\ x_4 &= 0 \\ x_5 &= t, \qquad s,t\in\mathbb{R} \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{pmatrix} = s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

EXAMPLE

The reduced row echelon form of the augmented matrix

$(\boldsymbol{B}\,|\,\mathbf{0})$ is $\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

So a basis for the nullspace of $\boldsymbol{B}$ is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\text{nullity}(\boldsymbol{B}) = 2$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{pmatrix} = s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

What is the rank of $\boldsymbol{B}$?

$\text{rank}(\boldsymbol{B}) = 3$

2 non pivot columns

3 pivot columns

THEOREM

(Dimension Theorem for matrices)

Let $\boldsymbol{A}$ be a matrix with n columns. Then

$$\boxed{\text{rank}(\boldsymbol{A}) + \text{nullity}(\boldsymbol{A}) = n}$$

Proof: Let $\boldsymbol{R}$ be the reduced row-echelon form of $\boldsymbol{A}$.

The n columns in $\boldsymbol{R}$ can be classified into two groups.

Pivot columns Non pivot columns

\# of pivot columns $+$ # of non pivot columns $= n$

THEOREM

Proof:

The n columns in $\boldsymbol{R}$ can be classified into two groups.

Pivot columns — Non pivot columns

\# of pivot columns $+$ # of non pivot columns $= n$

$\parallel$ — $\parallel$

\# of leading entries in $\boldsymbol{R}$ $+$ # of arbitrary parameters in a general solution to $\boldsymbol{Ax} = \mathbf{0}$

$\parallel$ — $\parallel$

Rank($\boldsymbol{A}$) $+$ # of vectors in a basis for the solution space of $\boldsymbol{Ax} = \mathbf{0}$

$\parallel$

$\boxed{\text{Rank}(\boldsymbol{A})} + \boxed{\text{Nullity}(\boldsymbol{A})} \boxed{= n}$

SUMMARY

1) The nullspace and nullity of a matrix.

2) Dimension Theorem for matrices.