LINEAR SPAN I

SET OF *ALL* LINEAR COMBINATIONS

Consider $\boldsymbol{u}=(1,2,-1)$, $\boldsymbol{v}=(0,2,5)$, $\boldsymbol{w}=(1,0,-2)$.

How many different linear combinations of $\boldsymbol{u},\boldsymbol{v}$ and $\boldsymbol{w}$ are there?

Quite a bit...

What if I put ALL different linear combinations of $\boldsymbol{u},\boldsymbol{v}$ and $\boldsymbol{w}$ into a set?

Wow...

LINEAR SPAN

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a set of vectors in $\mathbb{R}^n$.

The set of all linear combinations of $\boldsymbol{u}_1, \boldsymbol{u}_2 \ldots, \boldsymbol{u}_k$,

$$\{c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k \mid c_1, c_2, \ldots, c_k \in \mathbb{R}\}$$

is called the linear span of S (or linear span of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$).

This set is denoted by span(S) or span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$.

Intuitively, span(S) or span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ is the set of ALL vectors that can be "generated" by linearly combining the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$.

EXAMPLE

Consider $\boldsymbol{u} = (2,1,3),\ \boldsymbol{v} = (1,-1,2),\ \boldsymbol{w} = (3,0,5).$

Question: Is $(3,3,4)$ a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$? Yes!

So $(3,3,4) \in \text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$

Question: Is $(1,2,4)$ a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$? No!

So $(1,2,4) \notin \text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$

EXAMPLE

$S = \{(1,1,0),(2,-1,1)\}$.

$\text{span}(S) =$ set of all linear combinations of $(1,1,0)$ and $(2,-1,1)$

Every vector in $\text{span}(S)$ is of the form

$a(1,1,0)+b(2,-1,1)$ where a,b are any real numbers.

So $\text{span}(S) = \{a(1,1,0)+b(2,-1,1) \mid a,b \in \mathbb{R}\}$

EXAMPLE

$$V = \{(2a+b, a, 3b-a) \mid a, b \in \mathbb{R}\}$$

V is a subset of $\mathbb{R}^3$. Can V be written as a linear span?

$$(2a+b, a, 3b-a)$$

$$= a(2,1,-1) + b(1,0,3)$$

So $V = \{a(2,1,-1) + b(1,0,3) \mid a, b \in \mathbb{R}\}$

$$= \text{span}\{(2,1,-1), (1,0,3)\}$$

EXAMPLE

Consider $\boldsymbol{u}=(1,2,-1),\ \boldsymbol{v}=(0,2,5),\ \boldsymbol{w}=(1,0,-2).$

Yes!

Question: Is every vector in $\mathbb{R}^3$ a linear combination of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$?

$$a(1,2,-1)+b(0,2,5)+c(1,0,-2)=(x,y,z)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & x \\ 2 & 2 & 0 & y \\ -1 & 5 & -2 & z \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & x \\ 0 & 2 & -2 & y-2x \\ 0 & 0 & -4 & z-\frac{5y}{2}+6x \end{array}\right)$$

always consistent, regardless of the values of x,y,z.

$$\boxed{\text{span}\{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\}=\mathbb{R}^3}$$

EXAMPLE

Consider $\boldsymbol{u}=(3,6,2),\ \boldsymbol{v}=(-1,0,1),\ \boldsymbol{w}=(3,12,7).$

No!

Question: Is every vector in $\mathbb{R}^3$ a linear combination of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$?

$$a(3,6,2)+b(-1,0,1)+c(3,12,7)=(x,y,z)$$

$$\left(\begin{array}{ccc|c} 3 & -1 & 3 & x \\ 6 & 0 & 12 & y \\ 2 & 1 & 7 & z \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 3 & -1 & 3 & x \\ 0 & 2 & 6 & y-2x \\ 0 & 0 & 0 & \boxed{z-\frac{5y}{6}+x} \end{array}\right)$$

will be inconsistent, for some values of x,y,z.

$\boxed{\text{span}\{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\}\neq\mathbb{R}^3}$

EXAMPLE

Show that $\operatorname{span}\{(1,0,1),(1,1,0),(0,1,1)\}=\mathbb{R}^3$.

We need to show that every vector in $\mathbb{R}^3$ can be written as a linear combination of $(1,0,1),(1,1,0),(0,1,1)$.

$$a(1,0,1)+b(1,1,0)+c(0,1,1)=(x,y,z)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 1 & 1 & y \\ 1 & 0 & 1 & z \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 1 & 1 & y \\ 0 & 0 & 2 & z-x+y \end{array}\right)$$

Linear system is consistent regardless of the values of x,y,z. So $\operatorname{span}\{(1,0,1),(1,1,0),(0,1,1)\}=\mathbb{R}^3$.

EXAMPLE

Show that $\text{span}\{(1,1,1),(1,2,0),(2,1,3),(2,3,1)\} \neq \mathbb{R}^3$.

We need to show that there is some vector in $\mathbb{R}^3$ that cannot be written as a linear combination of $(1,1,1),(1,2,0),(2,1,3),(2,3,1)$.

$$a(1,1,1)+b(1,2,0)+c(2,1,3)+d(2,3,1)=(x,y,z)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & x \\ 1 & 2 & 1 & 3 & y \\ 1 & 0 & 3 & 1 & z \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & x \\ 0 & 1 & -1 & 1 & y-x \\ 0 & 0 & 0 & 0 & y+z-2x \end{array}\right)$$

$$(1,0,0) \notin \text{span}\{(1,1,1),(1,2,0),(2,1,3),(2,3,1)\}.$$

MORE MAY NOT BE BETTER!

Show that $\operatorname{span}\{(1,0,1),(1,1,0),(0,1,1)\} = \mathbb{R}^3$.

Show that $\operatorname{span}\{(1,1,1),(1,2,0),(2,1,3),(2,3,1)\} \neq \mathbb{R}^3$.

SUMMARY

1) The linear span of a set of vectors.

2) How to write a set as a linear span (when possible).

3) How to check whether a vector belongs to span(S).

4) How to show span$(S)=\mathbb{R}^n$ (or not).