**W06-04**

**Slide 01:** In this unit, we will define what is the rank of a matrix.

**Slide 02:** Recall that we have already discussed methods to find bases for the row space and the column space of a matrix. While you should be aware by now that if the matrix $\boldsymbol{A}$ is not a square matrix, then the row and column spaces of $\boldsymbol{A}$ are subspaces of different Euclidean spaces and thus have no vectors in common. While that is true, this theorem states that the dimension of these two subspaces are actually the same.

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Remember that if $\boldsymbol{A}$ is a $m \times n$ matrix, then the row space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^n$

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while the column space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^m$.

**Slide 03:** To prove the theorem, let $\boldsymbol{R}$ be a row-echelon form of $\boldsymbol{A}$. Pay attention to the leading entries in $\boldsymbol{R}$.

**Slide 04:** Using the information in $\boldsymbol{R}$, what is a basis for the row space of $\boldsymbol{A}$?

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We have already seen that a basis for the row space of $\boldsymbol{A}$ can be formed by taking the non zero rows of $\boldsymbol{R}$. These are, of course, the rows containing the leading entries in $\boldsymbol{R}$.

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Once we have a basis for the row space, we will be able to determine the dimension of the row space, since the dimension is just the number of vectors in the basis.

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Hence, the dimension of the row space of $\boldsymbol{A}$ is the number of non zero rows in $\boldsymbol{R}$, which is the number of leading entries we see in $\boldsymbol{R}$.

**Slide 05:** Let's move on to the column space of $\boldsymbol{A}$. What is a basis for the column space of $\boldsymbol{A}$? This question can again be answered by using the information from $\boldsymbol{R}$.

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We have seen that the columns of $\boldsymbol{A}$ that corresponds to the pivot columns in $\boldsymbol{R}$ will form a basis for the column space of $\boldsymbol{A}$.

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So what is the dimension of the column space of $\boldsymbol{A}$? Again we will know the answer by looking at how many vectors are there in a basis for the column space.

**Slide 06:** The number of vectors in the basis is the number of columns in $\boldsymbol{A}$

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that corresponds to the number of pivot columns in $\boldsymbol{R}$.

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How many pivot columns are there in $\boldsymbol{R}$? This is simply the number of leading entries there are in $\boldsymbol{R}$.

**Slide 07:** Thus, we have found that the dimension of both the row space and column space of $\boldsymbol{A}$ is equal to the number of leading entries we see in $\boldsymbol{R}$. This implies that the row space and column space of $\boldsymbol{A}$ have the same dimension.

**Slide 08:** Consider the matrix $\boldsymbol{C}$ with its row-echelon form shown.
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A basis for the row space of $\boldsymbol{C}$ can be obtained by taking he non zero rows of $\boldsymbol{C}$, so we have the two vectors $(2, 0, 3, -1, 8)$ and $(0, 1, -2, -1, -3)$.
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While a basis for the column space of $\boldsymbol{C}$ can be obtained by taking the first two columns of $\boldsymbol{C}$, since the pivot columns in the row-echelon form are columns 1 and 2. Thus the vectors $(2, 2, -4)$ and $(0, 1, -3)$ form a basis for the column space of $\boldsymbol{C}$.
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As per the theorem we have just proven, the dimension of both the row and column space of $\boldsymbol{C}$ is 2.

**Slide 09:** This gives rise to the definition of the rank of a matrix, which is the dimension of the matrix's row space or column space.
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By our discussion in this unit, we know that the rank of $\boldsymbol{A}$ is the number of non zero rows in a row-echelon form of $\boldsymbol{A}$,
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or we can also say it is the number of pivot columns in a row-echelon form of $\boldsymbol{A}$.

**Slide 10:** Clearly, the rank of the zero matrix is 0 while the identity matrix of order $n$ has precisely $n$ leading entries and thus the rank of $\boldsymbol{I_n}$ is $n$.
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The rank of the matrix $\boldsymbol{C}$ which we have seen earlier is 2.

**Slide 11:** For a matrix with $n$ rows and $m$ columns, where $m \geq n$, what would be the largest value that rank of $\boldsymbol{A}$ can take? To answer this question, we need to understand that the number of leading entries in any row-echelon form of $\boldsymbol{A}$ is limited by the number of rows $\boldsymbol{A}$ has, since $\boldsymbol{A}$ has at least as many columns as rows.
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Thus the largest possible value of rank $\boldsymbol{A}$ is $n$.
(#)
On the other hand, if $\boldsymbol{A}$ has at least as many rows as columns, then the number of leading entries in any row-echelon of $\boldsymbol{A}$ will be limited by the number of columns in $\boldsymbol{A}$.
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Thus the largest possible value of rank $\boldsymbol{A}$ in this case would be $m$.

**Slide 12:** With this, we know that for a $m \times n$ matrix $\boldsymbol{A}$, the rank of $\boldsymbol{A}$ is no larger than the smaller of the two numbers $m$ and $n$.
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For a matrix that attains the maximum rank it can attain, we say that it is of full rank.
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Next, we already know that the rank of $\boldsymbol{A}$ is the dimension of the row space of $\boldsymbol{A}$.
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However, we also know that the row space of $\boldsymbol{A}$ is the column space of $\boldsymbol{A}^T$,
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which in turn is the rank of $\boldsymbol{A}^T$. Thus we have shown that a matrix $\boldsymbol{A}$ and its transpose always have the same rank.

**Slide 13:** If $\boldsymbol{A}$ is a square matrix of order $n$, then $\boldsymbol{A}$ is of full rank if and only if the determinant of $\boldsymbol{A}$ is non zero.

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To see why this is so, notice that since $\boldsymbol{A}$ is a square matrix of order $n$, $\boldsymbol{A}$ having full rank means rank of $\boldsymbol{A}$ is $n$.

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This is equivalent to the fact that the dimension of the row space of $\boldsymbol{A}$ is $n$.

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This would mean that the row space of $\boldsymbol{A}$ must be the entire $\mathbb{R}^n$ and thus the standard basis for $\mathbb{R}^n$ is a basis for the row space of $\boldsymbol{A}$.

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The standard basis for $\mathbb{R}^n$ are precisely the rows of the identity matrix, thus $\boldsymbol{A}$ is row equivalent to $\boldsymbol{I_n}$.

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This is equivalent to the fact that the unique reduced row-echelon form of $\boldsymbol{A}$ must be $\boldsymbol{I_n}$

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which is one of the equivalent statements we have seen in earlier units, another one being the determinant of $\boldsymbol{A}$ being non zero.

**Slide 14:** We have now in fact just added one more equivalent statement to our collection. This latest statement to be added is in terms of the rank of the square matrix $\boldsymbol{A}$.

**Slide 15:** Summarising this unit,

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We first proved that the row space and column space of a matrix always have the same dimension.

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This led us to define what is the rank of a matrix.

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Lastly, we added one more equivalent statement to our collection, the latest one in terms of the rank of the matrix.