W06-05

Slide 01: In this unit, we continue our discussion on the column space of a matrix and also its rank.

Slide 02: Consider the following linear system.

(#)

We have seen in earlier units that one way of representing a linear system is to use a matrix equation $\boldsymbol{Ax} = \boldsymbol{b}$ where $\boldsymbol{A}$ is the coefficient matrix, $\boldsymbol{x}$ is the variable matrix while $\boldsymbol{b}$ represents the right hand side constant matrix.

(#)

The linear system can also be written in the form of a vector equation as shown here. On the left side of the equation, we have

(#)

the first variable x multiplied to the first column of $\boldsymbol{A}$ plus

(#)

the second variable y multiplied to the second column of $\boldsymbol{A}$ plus

(#)

the third variable z multiplied to the third column of $\boldsymbol{A}$.

(#)

The right hand side of the vector equation is the constant matrix $\boldsymbol{b}$. The expression on the left side of the equation is essentially a linear combination of the columns of $\boldsymbol{A}$.

Slide 03: Now that we see that the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ can be expressed as a vector equation $(*)$,

(#)

$\boldsymbol{Ax} = \boldsymbol{b}$ is consistent would mean that vector equation $(*)$ can be satisfied by some x, y and z.

(#)

The values of x, y an z that satisfies equation $(*)$ would be how the constant matrix $\boldsymbol{b}$ can be written as a linear combination of the columns of $\boldsymbol{A}$.

(#)

By definition, since the column space of $\boldsymbol{A}$ contains all the possible linear combinations of the columns of $\boldsymbol{A}$, since $\boldsymbol{b}$ is a linear combination of the columns of $\boldsymbol{A}$, this would mean that $\boldsymbol{b}$ belongs to the column space of $\boldsymbol{A}$.

Slide 04: Conversely, if we start off with the knowledge that $\boldsymbol{b}$ belongs to the column space of $\boldsymbol{A}$,

(#)

this would imply that $\boldsymbol{b}$ is a linear combination of the columns of $\boldsymbol{A}$.

(#)

By writing $\boldsymbol{b}$ as a linear combinaiton of the columns of $\boldsymbol{A}$, the coefficients x, y and z would be the solutions to vector equation $(*)$,

(#)

which means that, equivalently, $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent. Although this argument arises from a numerical example, the same idea can be applied to prove the following theorem.

Slide 05: Let $\boldsymbol{A}$ be a $m \times n$ matrix. From the previous discussion, we observe that any vector in the column space of $\boldsymbol{A}$ can be represented by $\boldsymbol{A}\boldsymbol{u}$ for some vector $\boldsymbol{u}$ in $\mathbb{R}^n$ with n components. Thus the column space of $\boldsymbol{A}$ is the set of all $\boldsymbol{A}\boldsymbol{u}$ where $\boldsymbol{u}$ takes on all possible vectors in $\mathbb{R}^n$. We also saw that a linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ will be consistent if and only if $\boldsymbol{b}$ is a vector in the column space of $\boldsymbol{A}$.

Slide 06: In this example, the linear system with 4 equations and 3 unknowns is first rewritten as a matrix equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

Slide 07: We then solve the linear system by Gaussian elimination.

(#)

The conclusion we obtain is that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is inconsistent since the last column of a row-echelon form of the augemented matrix is a pivot column.

(#)

If we only look at the left hand side of the vertical line, the portion of the matrix in green is in fact a row-echelon form of $\boldsymbol{A}$. Since it has three leading entries, we see that the rank of $\boldsymbol{A}$ is 3.

Slide 08: If we look at the entire augmented matrix as a whole, the rank of the augmented matrix is 4, which is 1 more than the rank of $\boldsymbol{A}$.

Slide 09: Thus, we now have another way of chracterising consistent linear systems. Basically, a linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if the coefficient matrix $\boldsymbol{A}$ and the augmented matrix $(\boldsymbol{A} \mid \boldsymbol{b})$ have the same rank.

Slide 10: To summarise this unit,

(#)

we presented a theorem that gave an equivalent statement to the statement that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is consistent. The equivalent statement was stated in terms of the right hand side $\boldsymbol{b}$, thought of as a vector, belonging to the column space of the coefficient matrix $\boldsymbol{A}$.

(#)

Another way of characterising a consistent linear system was in terms of the ranks of $\boldsymbol{A}$ and the augmented matrix $(\boldsymbol{A} \mid \boldsymbol{b})$.