NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

SEMESTER I, 2018/2019
MA1508E MID-TERM TEST

Full Name : ______________________________

Matric/Student Number : ______________

Tutorial Group : __________

INSTRUCTIONS

PLEASE READ CAREFULLY

- Write your **full name, matric number and tutorial group** clearly above on this cover page.
- There are **4** questions printed on **2** pages. Answer **all** questions.
- You must show all your working clearly, failure to do so will result in marks deducted.
- Use pen for this test.
- All answers and working have to be written on the answer book provided.
- Start on a new page for each question.
- Tie this cover page (and question paper) together with your answer book before submission.

Question 1 (12 marks)

(a) Solve the following linear system by Gaussian Elimination.

$$\left\{\begin{array}{rcrcrcrcr} x_1 & - & x_2 & + & 3x_3 & + & x_4 & = & 0 \\ x_1 & - & x_2 & + & x_3 & - & 2x_4 & = & 0 \\ -5x_1 & + & x_2 & - & 3x_3 & - & x_4 & = & 0 \end{array}\right.$$

(b) Consider the linear system below, where a is a real number.

$$\left\{\begin{array}{lcccccc} ax & - & y & + & z & = & 3 \\ 2x & & & + & (a+2)z & = & -1 \\ & & (a-1)y & + & 3z & = & 2 \end{array}\right.$$

(i) Find all values of a such that Cramer's Rule **cannot** be used to solve the system.

(ii) Solve the linear system when $a = 1$.

(a)

$$\left(\begin{array}{cccc|c} 1 & -1 & 3 & 1 & 0 \\ 1 & -1 & 1 & -2 & 0 \\ -5 & 1 & -3 & -1 & 0 \end{array}\right) \xrightarrow[R_3+5R_1]{R_2-R_1} \xrightarrow{R_2 \leftrightarrow R_3} \xrightarrow[-\frac{1}{2}R_3]{-\frac{1}{4}R_2} \xrightarrow[R_2+3R_3]{R_1-3R_3} \xrightarrow{R_1+R_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{7}{2} & 0 \\ 0 & 0 & 1 & \frac{3}{2} & 0 \end{array}\right)$$

So a general solution to the linear system is

$$\left\{\begin{array}{lcl} x_1 & = & 0 \\ x_2 & = & -\frac{7t}{2} \\ x_3 & = & -\frac{3t}{2} \\ x_4 & = & t, \quad t \in \mathbb{R} \end{array}\right.$$

(b) (i) Let $\boldsymbol{A} = \begin{pmatrix} a & -1 & 1 \\ 2 & 0 & a+2 \\ 0 & a-1 & 3 \end{pmatrix}$. Then Cramer's Rule cannot be used if and only if $\det(\boldsymbol{A}) = 0$.

$$\begin{aligned} \det(\boldsymbol{A}) &= a\begin{vmatrix} 0 & a+2 \\ a-1 & 3 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ a-1 & 3 \end{vmatrix} \\ &= (a+2)(-a+2)(a+1) \end{aligned}$$

Thus Cramer's Rule cannot be used if and only if $a = 2, -2, -1$.

(ii) When $a = 1$, $\boldsymbol{A} = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 0 & 3 \\ 0 & 0 & 3 \end{vmatrix}$ and $\det(\boldsymbol{A}) = 6$.

$$\left|\boldsymbol{A_1}\right| = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 0 & 3 \\ 2 & 0 & 3 \end{vmatrix} = -9, \quad \left|\boldsymbol{A_2}\right| = \begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & 3 \\ 0 & 2 & 3 \end{vmatrix} = -23, \quad \left|\boldsymbol{A_3}\right| = \begin{vmatrix} 1 & -1 & 3 \\ 2 & 0 & -1 \\ 0 & 0 & 2 \end{vmatrix} = 4.$$

So by Cramer's rule,

$$x = -\frac{3}{2} \quad y = -\frac{23}{6}, \quad z = \frac{2}{3}.$$

Question 2 (9 marks)

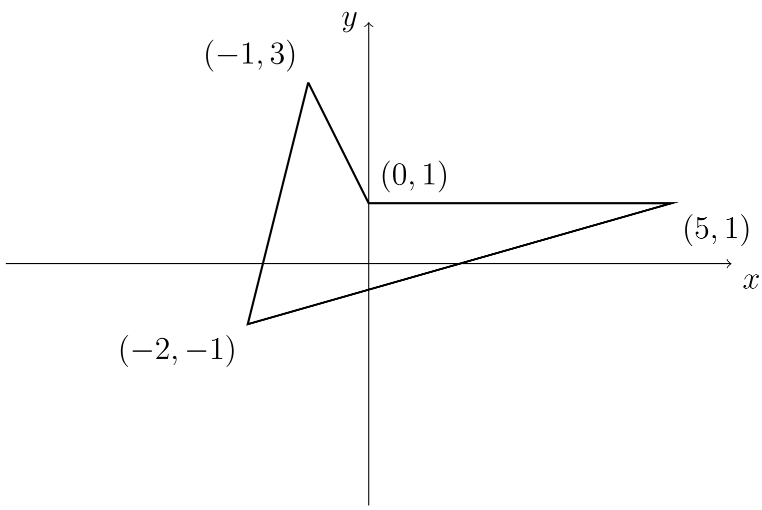
(a) For each of the statements below, determine if the statement is true or false. If it is true, provide justification. If it is false, provide a counterexample.

(i) If $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices of the same size such that $\boldsymbol{u}$ is a non trivial solution to both $\boldsymbol{Ax} = \boldsymbol{0}$ and $\boldsymbol{Bx} = \boldsymbol{0}$, then $\boldsymbol{u}$ is a non trivial solution to $(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{x} = \boldsymbol{0}$.

(ii) If the reduced row-echelon form of a matrix $\boldsymbol{C}$ has a zero row, then $\boldsymbol{Cx} = \boldsymbol{0}$ has infinitely many solutions.

(iii) If $\boldsymbol{A}$ is row equivalent to $\boldsymbol{B}$, then $\boldsymbol{A}^T$ will be row equivalent to $\boldsymbol{B}^T$.

(b) Find the area of the following quadrilateral.



(a) (i) True. Let $\boldsymbol{u}$ be a non trivial solution to $\boldsymbol{Ax} = \boldsymbol{0}$ and $\boldsymbol{Bx} = \boldsymbol{0}$. Then $\boldsymbol{Au} = \boldsymbol{0}$ and $\boldsymbol{Bu} = \boldsymbol{0}$. Since $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{u} = \boldsymbol{Au}+\boldsymbol{Bu} = \boldsymbol{0}$, $\boldsymbol{u}$ is also a non trivial solution to $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{x} = \boldsymbol{0}$.

(ii) False. Consider $\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\boldsymbol{C}$ is in reduced row-echelon form and has a zero row, but $\boldsymbol{Cx} = \boldsymbol{0}$ has only the trivial solution.

(iii) False. Let

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix} \xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \boldsymbol{B}.$$

So $\boldsymbol{A}$ is row equivalent to $\boldsymbol{B}$. On the other hand

$$\boldsymbol{A}^T = \begin{pmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{pmatrix} \underset{R_3 - R_1}{\xrightarrow{R_2 - R_1}} \begin{pmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \boldsymbol{B}^T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix} \underset{R_3 - R_1}{\xrightarrow{R_2 - R_1}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Since the reduced row-echelon form of $\boldsymbol{A}^T$ is not equal to the reduced row-echelon form of $\boldsymbol{B}^T$, $\boldsymbol{A}^T$ is not row equivalent to $\boldsymbol{B}^T$.

(b) We draw a line from the point $(0, 1)$ to $(-2, -1)$. This will divide the figure into two triangles. The area of the two triangles are, respectively,

$$\left| \frac{1}{2} \begin{vmatrix} -1 & 3 & 1 \\ 0 & 1 & 1 \\ -2 & -1 & 1 \end{vmatrix} \right| \quad \text{and} \quad \left| \frac{1}{2} \begin{vmatrix} 5 & 1 & 1 \\ 0 & 1 & 1 \\ -2 & -1 & 1 \end{vmatrix} \right|$$

The required area is thus 3+5=8 square units.

Question 3 (12 marks)

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -3 \end{pmatrix}$.

(i) Find exactly 3 elementary matrices $\boldsymbol{E_1}, \boldsymbol{E_2}, \boldsymbol{E_3}$ such that $\boldsymbol{E_3 E_2 E_1 A}$ is an upper triangular matrix.

(ii) Use your answer in (i) to find $\det(\boldsymbol{A})$. Explain why $\boldsymbol{A}$ is invertible.

(iii) Express $\boldsymbol{A}$ as $\boldsymbol{LU}$ where $\boldsymbol{L}$ and $\boldsymbol{U}$ are lower and upper triangular matrices respectively.

(iv) Use your answer in part (iii), solve the equation

$$\boldsymbol{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}.$$

(**Note:** You will not be given any marks if you solve the equation using other methods.)

(i)

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -3 \end{pmatrix} \underset{R_3 - R_1}{\xrightarrow{R_2 - 2R_1}} \xrightarrow{R_3 + \frac{1}{3}R_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & -5 \end{pmatrix}.$$

So

$$\boldsymbol{E_1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \boldsymbol{E_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \boldsymbol{E_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{pmatrix}$$

and $\boldsymbol{E_3 E_2 E_1 A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & -5 \end{pmatrix}$, a triangular matrix.

(ii)

$$\det(\boldsymbol{E_3})\det(\boldsymbol{E_2})\det(\boldsymbol{E_1})\det(\boldsymbol{A}) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & -3 \\ 0 & 0 & -5 \end{vmatrix} = 15.$$

Thus $\det(\boldsymbol{A}) = 15$. Since $\det(\boldsymbol{A}) \neq 0$, $\boldsymbol{A}$ is invertible.

(iii)

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{E_1}^{-1}\boldsymbol{E_2}^{-1}\boldsymbol{E_3}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & -5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & -5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & -5 \end{pmatrix} = \boldsymbol{LU} \end{aligned}$$

(iv)

$$\boldsymbol{A} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix} \Leftrightarrow \boldsymbol{LU} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}$$

We first solve

$$\boldsymbol{L} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -\frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}.$$

This implies $a = -1$, $2a + b = 4 \Leftrightarrow b = 6$ and $c = 10$. We now solve

$$\boldsymbol{U}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ 10 \end{pmatrix}.$$

This impies $z = -2$, $-3y - 3z = 6 \Leftrightarrow y = 0$ and $x = 1$. Thus $(x, y, z)^T = (1, 0, -2)^T$.

Question 4 (7 marks)

The lecturer of a module needs to assign grades to the students taking the module. There are only 3 possible grades (A, B and C) that can be assigned to each student. Show that it is impossible for the lecturer to assign grades to his students such that the following conditions are **all satisfied**.

(1) The number of students receiving A grade plus twice the number of students receiving B grade is 300.

(2) The total number of students receving B or C grade is 300.

(3) There are 300 more students receiving A grade than twice the number of students receiving C grade.

If you are allowed to change the '300' in all the 3 conditions above to another value x (for all the 3 conditions), is it possible to choose x such that the lecturer is now able to assign grades to his students in such a way that satisfies all the 3 conditions? Justify your answer.

Let the number of students given A,B,C grades be a, b, c respectively. Then the requirements are

$$\begin{cases} a + 2b = 300 \\ b + c = 300 \\ a - 2c = 300 \end{cases}$$

Solving the system

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 300 \\ 0 & 1 & 1 & 300 \\ 1 & 0 & -2 & 300 \end{array}\right) \xrightarrow{R_3 - R_1} \xrightarrow{R_3 + 2R_2} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 300 \\ 0 & 1 & 1 & 300 \\ 0 & 0 & 0 & 600 \end{array}\right).$$

Since the last column is a pivot column, the system is inconsistent.
Suppose 300 is changed to x:

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & x \\ 0 & 1 & 1 & x \\ 1 & 0 & -2 & x \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & x \\ 0 & 1 & 1 & x \\ 0 & 0 & 0 & 2x \end{array}\right).$$

If the system is consistent, then $x = 0$ but this is impossible as a and b are non negative integers so $a + 2b = 0$ would imply $a = b = 0$. Simlarly $b + c = 0$ implies $c = 0$. So $a + b + c = 0$ which is impossible.

END OF TEST