Unit 064 More on System of Linear Differential Equations

Slide 01: In this unit, we will discuss two special situations that may arise when we are solving a system of linear differential equations using eigenvalues an eigenvectors.

(#)

Slide 02: First, what happens when we encounter $\lambda = 0$ as one of the eigenvalues of $\boldsymbol{A}$ while solving the system $\boldsymbol{Y'} = \boldsymbol{AY}$?

(#)

Recall from a previous unit that if $\boldsymbol{A}$ has an eigenvalue of λ and $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ associated with λ, then $e^{\lambda t}\boldsymbol{x}$ is a solution to the system of linear differential equations.

(#)

In the event that λ is zero, this solution simply reduces to the eigenvector $\boldsymbol{x}$, since e^0 is 1.

Slide 03: Let us consider a simple example. We would like to solve $\boldsymbol{Y'} = \boldsymbol{AY}$ where $\boldsymbol{A}$ is the 2×2 matrix as shown.

(#)

We first find the eigenvalues of $\boldsymbol{A}$ as per normal by computing the characteristic polynomial of $\boldsymbol{A}$

(#)

and set it to zero. We find that the roots of the characteristic equation are 0 and 5.

(#)

For the eigenvalue 0, we consider the eigenspace E_0

(#)

and solve the associated homogeneous linear system, whose augemented matrix is shown here.

Slide 04: Solving the homogeneous linear system by Gauss-Jordan Elimination, we arrive at the following reduced row-echelon form.

(#)

It is easily seen that the vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ forms a basis for the eigenspace E_0.

(#)

For the eigenspace E_5, we do the same and solve the associated homogeneous linear system

(#)

resulting in the following reduced row-echelon form

(#)

and obtain the vector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ forming a basis for this eigenspace.

(#)

We are now able to write down a general solution for the system of linear differential equations as $\boldsymbol{Y}$ equals to some scalar k_1 times the eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ plus another scalar k_2 times the eigenvector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ with the factor of e^{5t}.

Slide 05: The second situation that we may encounter when computing the eigenvalues of $\boldsymbol{A}$ is that we may find that $\boldsymbol{A}$ has complex-valued eigenvalues. Suppose, for example, $\lambda = a + ib$ is an eigenvalue of $\boldsymbol{A}$

(#)

To manage this situation, we first introduce the following theorem on complex eigenvalues of a matrix $\boldsymbol{A}$ and also what happens in relation to the system of linear differential equations $\boldsymbol{Y'} = \boldsymbol{AY}$. If λ is an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ associated with λ, then

(#)

the complex conjugate of λ, that is $\bar{\lambda}$ will also be an eigenvalue of $\boldsymbol{A}$. Furthermore, the complex conjugate of $\boldsymbol{x}$, that is $\bar{\boldsymbol{x}}$, will be an eigenvector of $\boldsymbol{A}$ associated with $\bar{\lambda}$.

Slide 06: In addition, we know that both $e^{\lambda t}\boldsymbol{x}$ and its complex conjugate $e^{\bar{\lambda} t}\bar{\boldsymbol{x}}$ are both solutions of $\boldsymbol{Y'} = \boldsymbol{AY}$ and any linear combinations of these two solutions will also be a solution to the system.

Slide 07: Now consider the following linear combination of $e^{\lambda t}\boldsymbol{x}$ and $e^{\bar{\lambda} t}\bar{\boldsymbol{x}}$. Here we see that $\boldsymbol{Y_1}$ is $\frac{1}{2}$ times the sum of the conjuate pair

(#)

while $\boldsymbol{Y_2}$ is $\frac{1}{2i}$ times the difference of the conjugate pair.

(#)

Recall what this will give us. For example, if we take $\frac{1}{2}$ of the sum of a conjugate pair of complex numbers $a + ib$ and $a - ib$,

(#)

we obtain a

(#)

which is the real part of $a + ib$.

(#)

On the other hand, if we take $\frac{1}{2i}$ of the difference of the pair,

(#)

we obtain b

(#)

which is the imaginary part of $a + ib$.

Slide 08: So we now see that the two linear combinations $\boldsymbol{Y_1}$ and $\boldsymbol{Y_2}$ are basically the real and imaginary parts of the particular solution $e^{\lambda t}\boldsymbol{x}$.

(#)

These two solutions $\boldsymbol{Y_1}$ and $\boldsymbol{Y_2}$ are therefore real-valued functions of $\boldsymbol{Y'} = \boldsymbol{AY}$. More precisely, we can compute

(#)

$e^{\lambda t}\boldsymbol{x}$, representing the complex eigenvalue λ as $a + ib$,

(#)

which upon simplification,

(#)

using Euler's formula gives us the following expression.

(#)

The complex vector $\boldsymbol{x}$ can also be written in terms of its real part and imaginary part,

(#)

giving the following expression, which is obtained by grouping all the real terms together and the imaginary terms separately.

Slide 09: Since we know that $\boldsymbol{Y_1}$ is the real part of $e^{\lambda t}\boldsymbol{x}$, we have the following expression for $\boldsymbol{Y_1}$, while

(#)

$\boldsymbol{Y_2}$ is the imaginary part of $e^{\lambda t}\boldsymbol{x}$, we have the expression highlighted in yellow for $\boldsymbol{Y_2}$. This provides us with a quick and efficient method of computing two linearly independent real-valued solutions to the system of linear differential equations $\boldsymbol{Y'} = \boldsymbol{AY}$.

Slide 10: Let us consider an example. Here we wish to find a general solution to the system of linear differential equations where the matrix $\boldsymbol{A}$ is given as shown.

(#)

As before, we solve for the eigenvalues of $\boldsymbol{A}$ by computing the characteristic polynomial

(#)

and setting it to 0. The pair of complex conjugate eigenvalues of $\boldsymbol{A}$ are $2+i$ and $2-i$.

(#)

We may choose λ to be $2+i$ and consider the eigenspace E_λ.

(#)

Solving the homogeneous linear system,

Slide 11: We have the following row-echelon form of the augmented matrix,

(#)

and the general solution of the system

(#)

gives us a basis for the eigenspace. Let $\boldsymbol{x}$ be the basis vector as shown.

(#)

Now $e^{\lambda t}\boldsymbol{x}$, which is $e^{(2+i)t}\boldsymbol{x}$

(#)

can be written as follows using Euler's formula.

(#)

This allows us to write the solution $e^{\lambda t}\boldsymbol{x}$ as a two-dimensional vector in the complex space $\mathbb{C}^2$ as shown here.

Slide 12: Upon simplification,

(#)

we obtain the real part

(#)

and the imaginary part of the solution, and by our earlier discussion, these two are

(#)

precisely $\boldsymbol{Y_1}$

(#)

and $\boldsymbol{Y_2}$, which are two linearly independent solutions to the system of linear differential equations.

(#)

Any linear combinations of these two real-valued solutions will also be a solution to the system $\boldsymbol{Y'} = \boldsymbol{AY}$.

Slide 13: Let us summarise the main points in this unit.

(#)

We first discuss the situation when one of the eigenvalues of $\boldsymbol{A}$ is 0 and how this translates into the general solution of the system $\boldsymbol{Y'} = \boldsymbol{AY}$.

(#)

Next, we discussed and found a general solution of $\boldsymbol{Y'} = \boldsymbol{AY}$ when $\boldsymbol{A}$ has a pair of complex conjugate eigenvalues.