Fourth Week

CHAPTER 4

THE LAPLACE TRANSFORM

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\text{ODE} \arrow[r, "\text{Laplace Transform}"] \arrow[d, dashed, "\text{Solve directly}"'] & \text{Algebraic Equation} \arrow[d, "\text{Solve by simple algebra}"] \\
\text{Answer to ODE} & \text{Answer to Algebraic Equation} \arrow[l, "\text{Inverse Laplace Transform}"]
\end{tikzcd}
$$

Definition. Let f be a function defined for all $t \geq 0$. The Laplace transform of f is the function $F(s)$ defined by

$$F(s) = L(f) = \int_0^\infty e^{-st} f(t)dt, \qquad (1)$$

provided the improper integral on the right exists.

Recall: $\int_0^\infty e^{-st} f(t)\,dt$ is convergent

means that $\lim_{b\to\infty} \int_0^b e^{-st} f(t)\,dt$ is

finite.

The original function $f(t)$ in (1) is called the inverse transform or inverse of $F(s)$ and is denoted by $L^{-1}(F)$; i.e.,

$$f(t) = L^{-1}(F(s)).$$

Notation: Original functions are denoted by lower case letters and their Laplace transforms by the same letters in capitals. Thus $F(s) = L(f(t))$, $Y(s) = L(y(t))$ etc.

Example 1. Let $f(t) = e^{at}$, when $t \geq 0$.

Find $F(s)$.

Solution

Let $L(e^{at}) = F$

Then $F(s) = \int_0^\infty e^{-st} e^{at} \, dt$

$$= \int_0^\infty e^{(a-s)t} \, dt$$

Case 1: $s < a$

$$F(s) = \frac{1}{a-s} e^{(a-s)t} \Big|_0^\infty = \infty$$

Case 2: $s = a$

$$F(s) = \int_0^\infty dt = t \Big|_0^\infty = \infty$$

Case 3: $s > a$

$$F(s) = \frac{1}{a-s} e^{(a-s)t} \Big|_0^\infty$$

$$= \frac{1}{a-s} \left\{ e^{-\infty} - e^0 \right\} = \frac{1}{s-a}$$

$$\boxed{L(e^{at}) = \frac{1}{s-a} \quad \text{for } s > a}$$

$$L(e^{at}) = \frac{1}{s-a} \qquad L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

$$L(\cos at) = \frac{s}{s^2+a^2} \qquad L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

$$L(\sin at) = \frac{a}{s^2+a^2} \qquad L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$$

$$L(\cosh at) = \frac{s}{s^2-a^2} \qquad L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$$

$$L(\sinh at) = \frac{a}{s^2-a^2} \qquad L^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh at$$

$$L(t^n) = \frac{n!}{s^{n+1}} \qquad L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$$

$$\boxed{L(f') = sL(f) - f(0)}$$

$$\boxed{L(f'') = s^2 L(f) - sf(0) - f'(0)}$$

$$L\left(\int_0^t f(u)\,du\right) = \frac{1}{s} L(f(t))$$

Theorem. (*s*-**Shifting)**

If $f(t)$ has the transform $F(s)$, $s > a$, then

$$L(e^{ct}f(t)) = F(s-c), \quad s-c > a.$$

$$\boxed{L(e^{ct}t^n) = \frac{n!}{(s-c)^{n+1}}}$$

$$\boxed{L^{-1}\left\{\frac{1}{(s-c)^n}\right\} = \frac{e^{ct}t^{n-1}}{(n-1)!}}$$

$$\boxed{L\left(e^{ct}\cos\omega t\right) = \frac{s-c}{(s-c)^2+\omega^2}}$$

$$\boxed{L^{-1}\left(\frac{s-c}{(s-c)^2+\omega^2}\right) = e^{ct}\cos\omega t}$$

$$\boxed{L(e^{ct}\sin\omega t) = \frac{\omega}{(s-c)^2+\omega^2}}$$

$$\boxed{L^{-1}\left(\frac{\omega}{(s-c)^2+\omega^2}\right) = e^{ct}\sin\omega t}$$

Example 13. Solve $y'' - 2y' + y = e^t + t$, $y(0) = 1$, $y'(0) = 0$.

Solution

Set $L(y) = Y$.

We have

$$L(y'' - 2y' + y) = L(e^t + t)$$

$$\therefore s^2 Y - s y(0) - y'(0)$$

$$- 2\{sY - y(0)\} + Y = \frac{1}{s-1} + \frac{1}{s^2}$$

$$\therefore (s^2 - 2s + 1)Y - s + 2 = \frac{1}{s-1} + \frac{1}{s^2}$$

$$\therefore \quad Y = \frac{s-2}{(s-1)^2} + \frac{1}{(s-1)^3} + \frac{1}{s^2(s-1)^2}$$

$$= \frac{(s-1)-1}{(s-1)^2} + \frac{1}{(s-1)^3} + \frac{1}{s^2(s-1)^2}$$

$$= \frac{1}{s-1} - \frac{1}{(s-1)^2} + \frac{1}{(s-1)^3}$$

$$+ \frac{1}{s^2(s-1)^2}$$

Let $\dfrac{1}{s^2(s-1)^2} \equiv \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s-1} + \dfrac{D}{(s-1)^2}$

$\therefore \; 1 \equiv As(s-1)^2 + B(s-1)^2$

$\qquad + Cs^2(s-1) + Ds^2$

$s = 0 \Rightarrow B = 1$

$s = 1 \Rightarrow D = 1$

Compare coefficients:

s^3 : $0 = A + C$

s : $0 = A - 2B \Rightarrow A = 2$

$\therefore C = -2$

$$\therefore Y = \frac{1}{s-1} - \frac{1}{(s-1)^2} + \frac{1}{(s-1)^3}$$

$$+ \frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{1}{(s-1)^2}$$

$$= \frac{2}{s} + \frac{1}{s^2} - \frac{1}{s-1} + \frac{1}{(s-1)^3}$$

$$\therefore \underline{\underline{y(t) = 2 + t - e^t + \frac{1}{2}t^2 e^t}}$$

Theorem (t-Shifting)

If $L(f(t)) = F(s)$, then

$$\boxed{L\{f(t-a)u(t-a)\} = e^{-as}F(s)}.$$

i.e. $$\boxed{L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)}.$$

$$\boxed{L(u(t-a)) = \frac{e^{-as}}{s}}$$

$$\boxed{L^{-1}\left(\frac{e^{-as}}{s}\right) = u(t-a)}$$

$\therefore \quad \boxed{L(\delta(t-a)) = e^{-as}}$

$$\boxed{L^{-1}(e^{-as}) = \delta(t-a)}$$

EXAMPLE: INJECTIONS!

Suppose that a doctor injects, almost instantly, 100 mg of morphine into a patient. He does it again 24 hours later. Suppose that the HALF-LIFE of morphine in the patient's body is 18 hours. Find the amount of morphine in the patient at any time.

Suppose c mg of medicine is injected into a person at time $t=a$.

We try to model this mathematically.

We observe that although the injection is done at $t=a$, the whole process still needs a bit of time, say h seconds, to complete. So actually the medicine enters the blood during the time interval a to $a+h$.

Since c mg enters the blood during h sec.,
the average rate is $c/h = c f_h(t-a)$

Now $\lim_{h \to 0} f_h(t-a) = \delta(t-a)$

$\therefore \quad c/h \simeq c\,\delta(t-a)$ when h is small

Therefore, we can use $c\,\delta(t-a)$ to model
the rate of this injection.

Solution: Half-life refers to the exponential function e^{-kt}. "Half-life 18 hours" $= 0.75$ days means $\frac{1}{2} = e^{-k\times 0.75}$, that is, k $= \dfrac{\ell n(2)}{0.75} = 0.924$. So without the injections,

$$\frac{dy}{dt} = -ky, \quad k = 0.924. \quad y(0) = 0.$$

Next with two injections at $t=0$ and $t=1$ at 100 mg each time, we have

$$\frac{dy}{dt} = -(0.924)y + 100\,\delta(t) + 100\,\delta(t-1).$$

Let $L(y) = Y$.

$$\therefore\ sY - y(0) = -0.924Y + 100 + 100e^{-s}$$

$$\therefore\ (s+0.924)Y = 100 + 100e^{-s}$$

$$\therefore\ Y = \frac{100}{s+0.924} + 100\,\frac{e^{-s}}{s+0.924}$$

$$\therefore y = 100\,L^{-1}\left(\frac{1}{s+0.924}\right)$$

$$+100\,L^{-1}\left(\frac{1}{s+0.924}e^{-s}\right)$$

$$\because L^{-1}\left(\frac{1}{s+0.924}\right) = e^{-0.924t}$$

$$\because L^{-1}\left(\frac{1}{s+0.924}e^{-s}\right) = e^{-0.924(t-1)}\,u(t-1)$$

(by the t-shifting theorem)

$$\therefore\ y = 100e^{-0.924t} + 100e^{-0.924(t-1)}u(t-1)$$

$$= 100e^{-0.924t} + 100e^{0.924}e^{-0.924t}u(t-1)$$

$$= \begin{cases} 100e^{-0.924t}, & 0 < t < 1 \\ 100e^{-0.924t}(1+e^{0.924}), & t > 1 \end{cases}$$

Tutorial 4

Question 10

In the harvesting model we considered in the lectures, the population will rebound if all harvesting is stopped. Unhappily, this is not always true: for some animals, if you drive their population down too low, they will have trouble finding mates, or they will be forced to breed with relatively close kin, which reduces genetic variability and hence their ability to resist disease. For such animals [for example, certain rare species of tigers] extinction will result if the population falls too low, even if all harvesting is forbidden. Biologists call this **depensation**. Show that this situation can be modelled by the ODE

$$\frac{dN}{dt} = -aN^3 + bN^2 - cN,$$

where N is the population and a, b, and c are positive constants such that $b^2 > 4ac$. Find the population below which extinction will occur.

$$\frac{dN}{dt} = -aN^3 + bN^2 - cN$$

$$= -aN\left\{N^2 - \frac{b}{a}N + \frac{c}{a}\right\}$$

$$= -aN\left\{N^2 - \frac{b}{a}N + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right\}$$

$$= -aN\left\{\left(N - \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right\}$$

$$= -aN\left\{N - \frac{b-\sqrt{b^2-4ac}}{2a}\right\}\left\{N - \frac{b+\sqrt{b^2-4ac}}{2a}\right\}$$