NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

Module: MA1508E Linear Algebra for Engineering
Year/Semester: 2018-2019 (Semester 2)
Tutorial: 4

1. (**$\boldsymbol{LU}$ factorisation**) $\boldsymbol{LU}$ factorisation is a way to solve a given linear system $\boldsymbol{Ax} = \boldsymbol{b}$ efficiently. The discussion below only deals with the special case where $\boldsymbol{A}$ is a square matrix but can be extended to other sizes of $\boldsymbol{A}$ as well.

   (a) Let $\boldsymbol{A} = \begin{pmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{pmatrix}$. Peform exactly **three** elementary row operations on $\boldsymbol{A}$ to reduce $\boldsymbol{A}$ into row-echelon form.

   (b) Let the row-echelon form of $\boldsymbol{A}$ obtained in (a) be $\boldsymbol{U}$. Write down three elementary matrices $\boldsymbol{E_1}$, $\boldsymbol{E_2}$ and $\boldsymbol{E_3}$ such that
   $$\boldsymbol{E_3 E_2 E_1 A} = \boldsymbol{U}. \tag{*}$$

   (c) Find the inverses of $\boldsymbol{E_1}$, $\boldsymbol{E_2}$, $\boldsymbol{E_3}$ such that
   $$\boldsymbol{A} = \boldsymbol{E_1}^{-1}\boldsymbol{E_2}^{-1}\boldsymbol{E_3}^{-1}\boldsymbol{U}.$$

   (d) Compute the product $\boldsymbol{E_1}^{-1}\boldsymbol{E_2}^{-1}\boldsymbol{E_3}^{-1}$ and check that it is lower triangular. Since it is lower triangular, we have successfully factorised $\boldsymbol{A}$ as $\boldsymbol{LU}$ where $\boldsymbol{U}$ is upper triangular and $\boldsymbol{L}$ is lower triangular. In fact, all the diagonal entries of $\boldsymbol{L}$ are equal to 1. We call such a matrix, a **unit lower triangular** matrix.

2. (**Use of $\boldsymbol{LU}$ factorisation**) To see why $\boldsymbol{LU}$ factorisation is useful, consider a linear system $\boldsymbol{Ax} = \boldsymbol{b}$, where the coefficient matrix $\boldsymbol{A}$ has an $\boldsymbol{LU}$ factorisation. We can rewrite the system $\boldsymbol{Ax} = \boldsymbol{b}$ as $\boldsymbol{L}(\boldsymbol{Ux}) = \boldsymbol{b}$. If we now define $\boldsymbol{y} = \boldsymbol{Ux}$, then we can solve for $\boldsymbol{x}$ in two stages:

   (1) Solve $\boldsymbol{Ly} = \boldsymbol{b}$ for $\boldsymbol{y}$ using *forward substitution.*

   (2) Solve $\boldsymbol{Ux} = \boldsymbol{y}$ for $\boldsymbol{x}$ using *back substitution.*

   Use the $\boldsymbol{LU}$ factorisation to solve the following system:
   $$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 2 & 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 4 \end{pmatrix}.$$

   Remark: You will obtain an unique solution for this linear system. Do you think **LU** factorisation can be used if the linear system is inconsistent? Or has infinitely many solutions?

3. Find the determinant for each of the following square matrices by first reducing the matrix into row-echelon form.

(a) $\begin{pmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{pmatrix}$.

4. Suppose we know that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6.$$

Evaluate the determinant of the following matrices.

(a) $\begin{pmatrix} d & e & f \\ g & h & i \\ a & b & c \end{pmatrix}$ (b) $\begin{pmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{pmatrix}$ (c) $\begin{pmatrix} a & b & c \\ d & e & f \\ 2a & 2b & 2c \end{pmatrix}$

(d) $\begin{pmatrix} a+d & b+e & c+f \\ -d & -e & -f \\ g & h & i \end{pmatrix}$ (e) $\begin{pmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{pmatrix}$

(f) $\begin{pmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{pmatrix}$

5. Determine whether the following subsets of $\mathbb{R}^4$ are equal to each other.

$$S = \{(p, q, p, q) \mid p, q \in \mathbb{R}\},$$

$$T = \{(x, y, z, w) \mid x + y - z - w = 0\},$$

$$V = \left\{ (a, b, c, d) \;\middle|\; \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ a & b & c & d \end{vmatrix} = 0 \right\}.$$

Briefly explain why one subset is equal (or not equal) to another subset.

6. Consider a triangle in $\mathbb{R}^4$ with vertices $A = (1, 1, 0, 0)$, $B = (1, -1, 0, 0)$ and $C = (2, 0, 0, 1)$.

   (a) Find the lengths of the sides of the triangle.
   (b) Find the angle between AB and AC.
   (c) Verify the cosie rule: $2|AB||AC|\cos\theta = |AB|^2 + |AC|^2 - |BC|^2$, where θ is the angle between AB and AC.

7. Let $\boldsymbol{u_1} = (1, 3, -2, 0, 2, 0)$, $\boldsymbol{u_2} = (2, 6, -5, -2, 4, -3)$, $\boldsymbol{u_3} = (0, 0, 5, 10, 0, 15)$, $\boldsymbol{u_4} = (2, 6, 0, 8, 4, 18)$ and $\boldsymbol{v} = (-3, -1, -2, 1, 1, 0)$.

(a) Verify that $\boldsymbol{v}$ is orthogonal to $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ and $\boldsymbol{u_4}$.

(b) Construct a 4×6 matrix $\boldsymbol{A}$ with the vectors $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}$ as the rows of $\boldsymbol{A}$. Furthermore, write the vector $\boldsymbol{v}$ as a column matrix $\boldsymbol{v}$.

(c) What do you think is the matrix product $\boldsymbol{Av}$?

(d) Generalise this observation in terms of any homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$ and its solutions. (**Note:** This idea will be discussed in greater detail later in the course.)