CHECKING INVERSE

Recall from a previous unit

It turns out that to check whether a given square matrix $\boldsymbol{B}$ is the inverse of $\boldsymbol{A}$, we only need to check either

$$\boldsymbol{AB} = \boldsymbol{I} \qquad \boxed{\text{OR}} \qquad \boldsymbol{BA} = \boldsymbol{I}$$

The reason will be explained in a ~~later~~ unit.

now!

Theorem

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square matrices of the same size.

If $\boldsymbol{AB} = \boldsymbol{I}$, then

$$\boldsymbol{BA} = \boldsymbol{I} \qquad \boldsymbol{A} = \boldsymbol{B}^{-1} \qquad \boldsymbol{B} = \boldsymbol{A}^{-1}$$

Proof:

Consider the homogeneous linear system $\boldsymbol{Bx} = \mathbf{0}$.

Strategy: If we can show $\boldsymbol{Bx} = \mathbf{0}$ has only the trivial solution, then $\boldsymbol{B}$ is invertible.

Theorem

If $\boldsymbol{AB} = \boldsymbol{I}$, then

$$\boldsymbol{BA} = \boldsymbol{I} \qquad \boldsymbol{A} = \boldsymbol{B}^{-1} \qquad \boldsymbol{B} = \boldsymbol{A}^{-1}$$

Proof:

Consider the homogeneous linear system $\boldsymbol{Bx} = \boldsymbol{0}$.

Let $\boldsymbol{u}$ be a solution to $\boldsymbol{Bx} = \boldsymbol{0}$.

$$\boldsymbol{Bu} = \boldsymbol{0}$$

$$\Rightarrow \boldsymbol{ABu} = \boldsymbol{A0}$$

$$\Rightarrow \boldsymbol{Iu} = \boldsymbol{0} \Rightarrow \boldsymbol{u} = \boldsymbol{0}$$

Strategy: If we can show $\boldsymbol{Bx} = \boldsymbol{0}$ has only the trivial solution, then $\boldsymbol{B}$ is invertible.

So $\boldsymbol{Bx} = \boldsymbol{0}$ has only the trivial solution $\boldsymbol{u} = \boldsymbol{0}$ and thus $\boldsymbol{B}$ is invertible (that is, $\boldsymbol{B}^{-1}$ exists).

Theorem

If $\boldsymbol{AB} = \boldsymbol{I}$, then

$\boxed{\boldsymbol{BA} = \boldsymbol{I}}$ $\boxed{\boldsymbol{A} = \boldsymbol{B}^{-1}}$ $\boxed{\boldsymbol{B} = \boldsymbol{A}^{-1}}$

Proof:

So $\boldsymbol{Bx} = \boldsymbol{0}$ has only the trivial solution $\boldsymbol{u} = \boldsymbol{0}$ and

thus $\boldsymbol{B}$ is invertible (that is, $\boldsymbol{B}^{-1}$ exists).

$$\boldsymbol{AB} = \boldsymbol{I} \Rightarrow \boldsymbol{ABB}^{-1} = \boldsymbol{IB}^{-1} \Rightarrow \boldsymbol{AI} = \boldsymbol{B}^{-1} \Rightarrow \boxed{\boldsymbol{A} = \boldsymbol{B}^{-1}}$$

Since $\boldsymbol{A} = \boldsymbol{B}^{-1}$, $\boldsymbol{A}$ is invertible and

$$\boxed{\boldsymbol{A}^{-1} = (\boldsymbol{B}^{-1})^{-1} = \boldsymbol{B}}$$

Finally,

$$\boxed{\boldsymbol{BA} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}}$$

Example

If $\boldsymbol{A}$ is a square matrix such that

$$\boldsymbol{A}^2 - 6\boldsymbol{A} + 8\boldsymbol{I} = \boldsymbol{0},$$

prove that $\boldsymbol{A}$ is invertible.

$$\boldsymbol{A}^2 - 6\boldsymbol{A} + 8\boldsymbol{I} = \boldsymbol{0} \Rightarrow \boldsymbol{A}^2 - 6\boldsymbol{A} = -8\boldsymbol{I}$$

$$\Rightarrow \boldsymbol{A}(\boldsymbol{A} - 6) = -8\boldsymbol{I}$$

something wrong??

$$\Rightarrow \boldsymbol{A}(\boldsymbol{A} - 6\boldsymbol{I}) = -8\boldsymbol{I}$$

$$\boldsymbol{A}^{-1} = \boxed{[-\tfrac{1}{8}(\boldsymbol{A} - 6\boldsymbol{I})]} \qquad \Rightarrow \boldsymbol{A}\boxed{[-\tfrac{1}{8}(\boldsymbol{A} - 6\boldsymbol{I})]} = \boldsymbol{I}$$

Theorem

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two square matrices of the same order. If $\boldsymbol{A}$ is singular, then

$\boldsymbol{AB}$ and $\boldsymbol{BA}$ are both singular.

Proof: We will first show that $\boldsymbol{AB}$ is singular.

Suppose $\boldsymbol{AB}$ is invertible.

Then there is a square matrix $\boldsymbol{C}$ of the same size as $\boldsymbol{A}$ and $\boldsymbol{B}$ such that

$$(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{I} \Rightarrow \boldsymbol{A}(\boldsymbol{BC}) = \boldsymbol{I}$$

Theorem

Proof: We will first show that $\boldsymbol{AB}$ is singular.

Suppose $\boldsymbol{AB}$ is invertible.

Can you show $\boldsymbol{BA}$ is singular in the same way?

Then there is a square matrix $\boldsymbol{C}$ of the same size as $\boldsymbol{A}$ and $\boldsymbol{B}$ such that

$$(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{I} \Rightarrow \boldsymbol{A}(\boldsymbol{BC}) = \boldsymbol{I}$$

But $(\boldsymbol{BC})$ is the same size as $\boldsymbol{A}$ so $\boldsymbol{A}(\boldsymbol{BC}) = \boldsymbol{I}$ implies that

$\boldsymbol{A}$ is invertible

which is a contradiction since we known that $\boldsymbol{A}$ is singular.

Summary

1) To check whether $\boldsymbol{B}$ is the inverse of $\boldsymbol{A}$, it suffices to check either $\boldsymbol{AB} = \boldsymbol{I}$ or $\boldsymbol{BA} = \boldsymbol{I}$.

2) If $\boldsymbol{A}$ is singular and $\boldsymbol{B}$ is the same size as $\boldsymbol{A}$, then both $\boldsymbol{AB}$ and $\boldsymbol{BA}$ will be singular.