SOLVING SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS

EXAMPLE 1

Suppose a particle is moving in a planar force field and its position vector $\boldsymbol{X}$ satisfies $\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X}$ and $\boldsymbol{X}(0) = \boldsymbol{X}_0$, where

$$\boldsymbol{A} = \begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix}, \qquad \boldsymbol{X}_0 = \begin{pmatrix} 2.9 \\ 2.6 \end{pmatrix}$$

Solve this initial value problem.

EXAMPLE 1

1) Find all the eigenvalues of $\boldsymbol{A}$:

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \det\begin{pmatrix} \lambda - 4 & 5 \\ 2 & \lambda - 1 \end{pmatrix}$$

$$= (\lambda - 4)(\lambda - 1) - 10 = \lambda^2 - 5\lambda - 6$$

$$= (\lambda - 6)(\lambda + 1)$$

So the eigenvalues of $\boldsymbol{A}$ are -1 and 6.

Recall: If $\boldsymbol{x}_1$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue λ_1, then $\boldsymbol{X}_1 = e^{\lambda_1 t}\boldsymbol{x}_1$ is a solution to $\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X}$.

EXAMPLE 1

2) Find all the linearly independent eigenvectors associated with each eigenvalue λ.

$\lambda = 6:$

$$(6\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0} \Leftrightarrow \begin{pmatrix} 2 & 5 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x & = & -\dfrac{5s}{2} \\ y & = & s \end{cases}$$

So $E_6 = \text{span}\left\{\begin{pmatrix} -5 \\ 2 \end{pmatrix}\right\}$

Recall: If $\boldsymbol{x}_1$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue λ_1, then $\boldsymbol{X}_1 = e^{\lambda_1 t}\boldsymbol{x}_1$ is a solution to $\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X}$.

EXAMPLE 1

2) Find all the linearly independent eigenvectors associated with each eigenvalue λ.

$\lambda = -1$:

$$(-\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0} \Leftrightarrow \begin{pmatrix} -5 & 5 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x & = & s \\ y & = & s \end{cases}$$

So $E_{-1} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$

Recall: If $\boldsymbol{x}_1$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue λ_1, then $\boldsymbol{X}_1 = e^{\lambda_1 t}\boldsymbol{x}_1$ is a solution to $\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X}$.

EXAMPLE 1

3) Construct linear combinations of the solutions $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$.

$$E_6 = \text{span}\left\{\begin{pmatrix} -5 \\ 2 \end{pmatrix}\right\} \quad \boldsymbol{X}_1 = \begin{pmatrix} -5 \\ 2 \end{pmatrix} e^{6t}$$

$$E_{-1} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \quad \boldsymbol{X}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

For any $k_1, k_2 \in \mathbb{R}$,

$$\boldsymbol{X} = k_1 \begin{pmatrix} -5 \\ 2 \end{pmatrix} e^{6t} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

is a solution to $\boldsymbol{X}' = \boldsymbol{AX}$.

Recall: If $\boldsymbol{x}_1$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue λ_1, then $\boldsymbol{X}_1 = e^{\lambda_1 t}\boldsymbol{x}_1$ is a solution to $\boldsymbol{X}' = \boldsymbol{AX}$.

EXAMPLE 1

4) Use the given initial conditions to solve for k_1, k_2.

$$\boldsymbol{X} = k_1 \begin{pmatrix} -5 \\ 2 \end{pmatrix} e^{6t} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} \qquad \boldsymbol{X}_0 = \begin{pmatrix} 2.9 \\ 2.6 \end{pmatrix}$$

$$\boldsymbol{X}(0) = k_1 \begin{pmatrix} -5 \\ 2 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.9 \\ 2.6 \end{pmatrix} \Rightarrow \begin{pmatrix} -5 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 2.9 \\ 2.6 \end{pmatrix}$$

Solving, we have $k_1 = -\frac{3}{70}, k_2 = \frac{188}{70}$.

EXAMPLE 1

So the solution to $\boldsymbol{X}' = \boldsymbol{AX}$ satisfying the initial condition is:

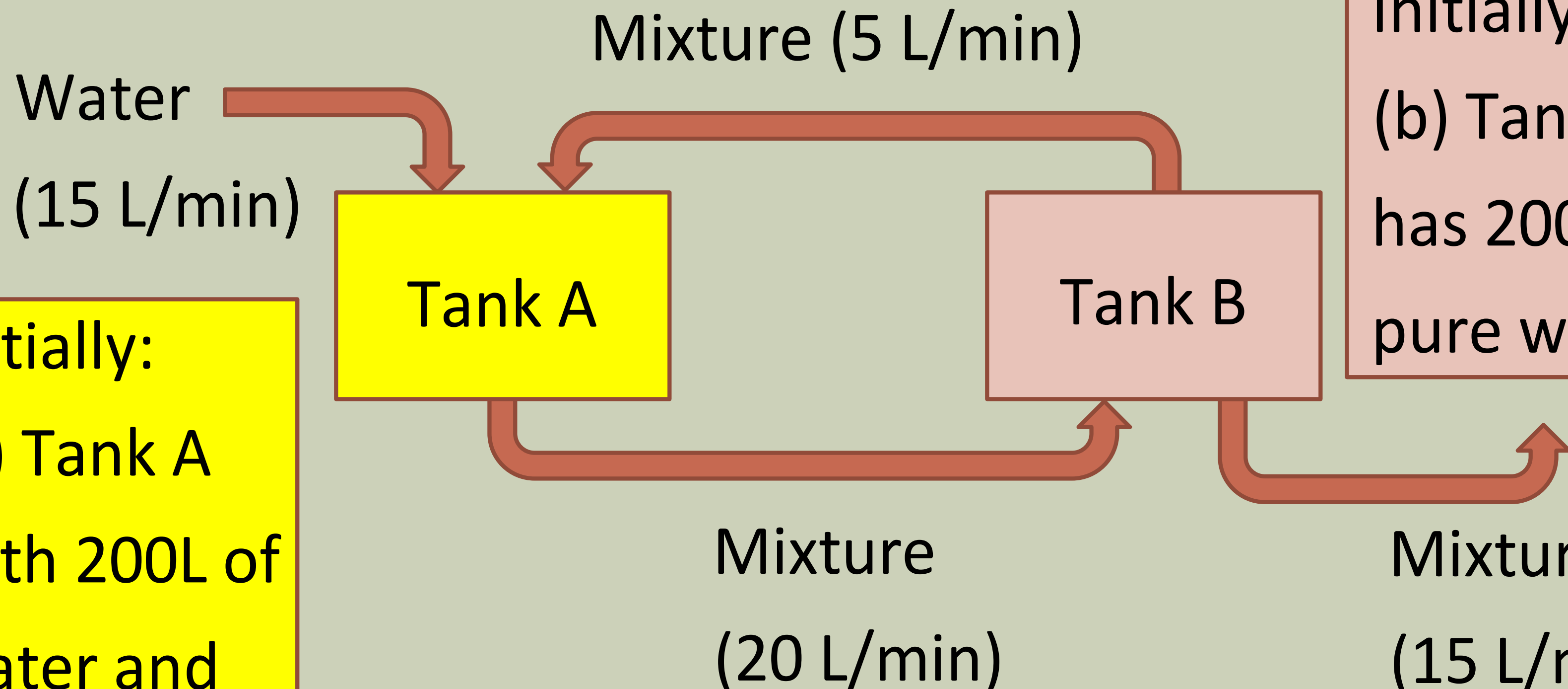
$$\boxed{\boldsymbol{X} = -\frac{3}{70}\begin{pmatrix} -5 \\ 2 \end{pmatrix} e^{6t} + \frac{188}{70}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}}$$

$$\boldsymbol{X} = k_1 \begin{pmatrix} -5 \\ 2 \end{pmatrix} e^{6t} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

Solving, we have $k_1 = -\frac{3}{70}, k_2 = \frac{188}{70}$

EXAMPLE 2

Two tanks of water are connected as shown below.

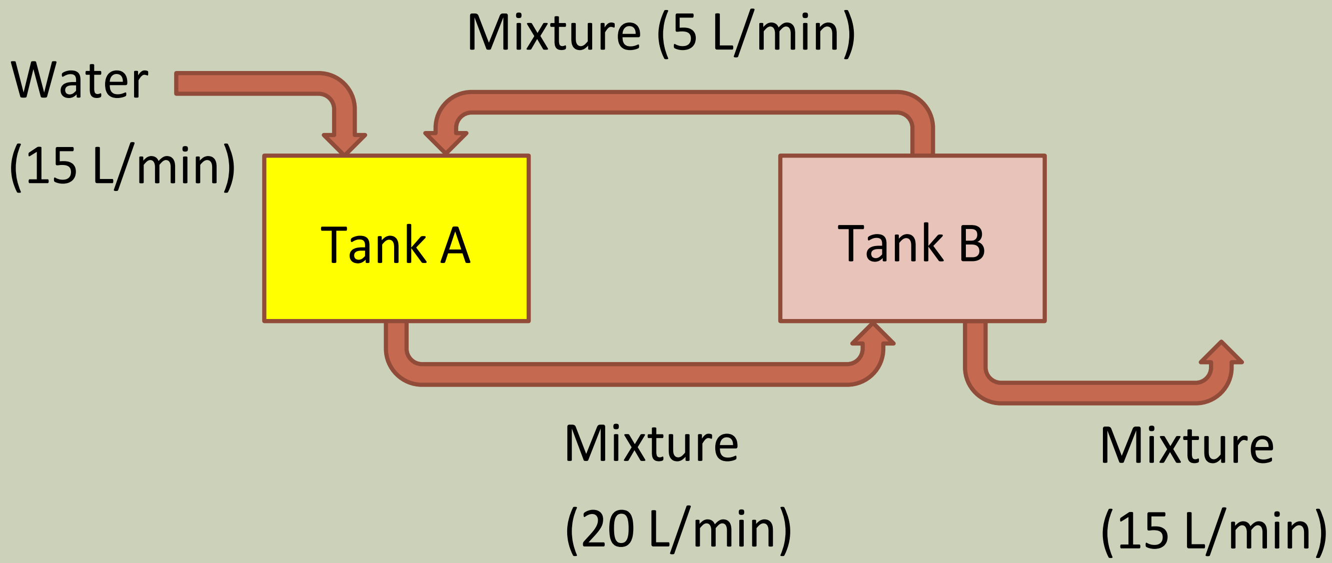


Initially:
(a) Tank A with 200L of water and 60g of salt dissolved

Initially:
(b) Tank B has 200L of pure water

Determine the amount of salt in each tank at time t.

EXAMPLE 2

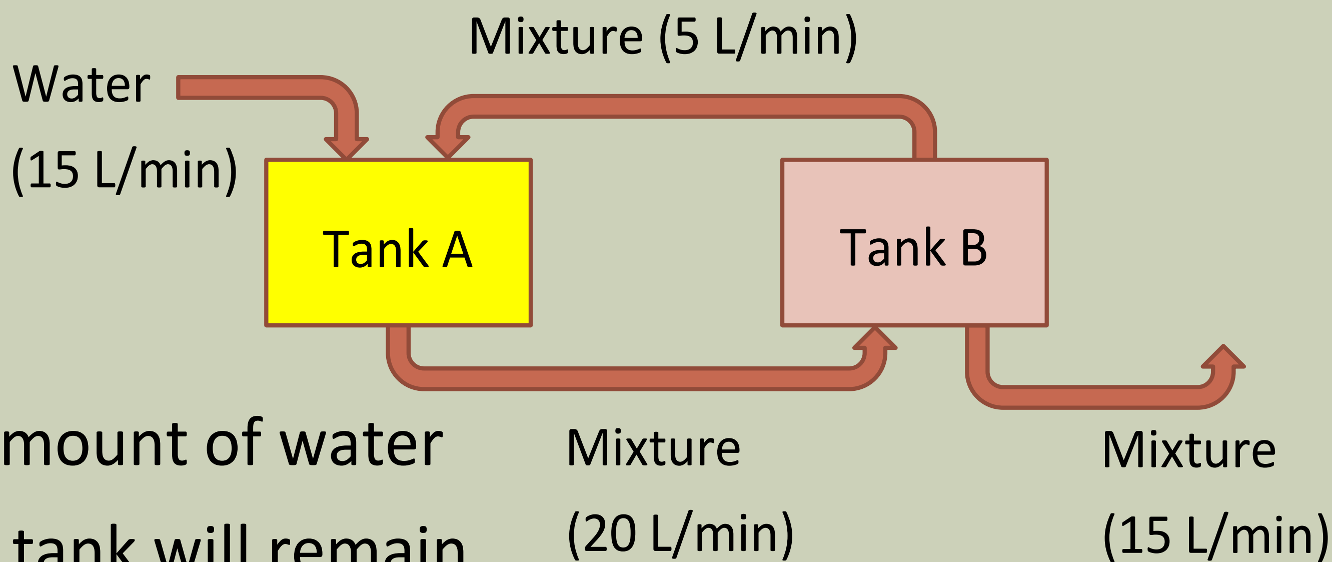


Let $y_1(t) =$ amount of salt (in g) in tank A at time t.

$y_2(t) =$ amount of salt (in g) in tank B at time t.

$$\boldsymbol{Y}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 60 \\ 0 \end{pmatrix}$$

EXAMPLE 2

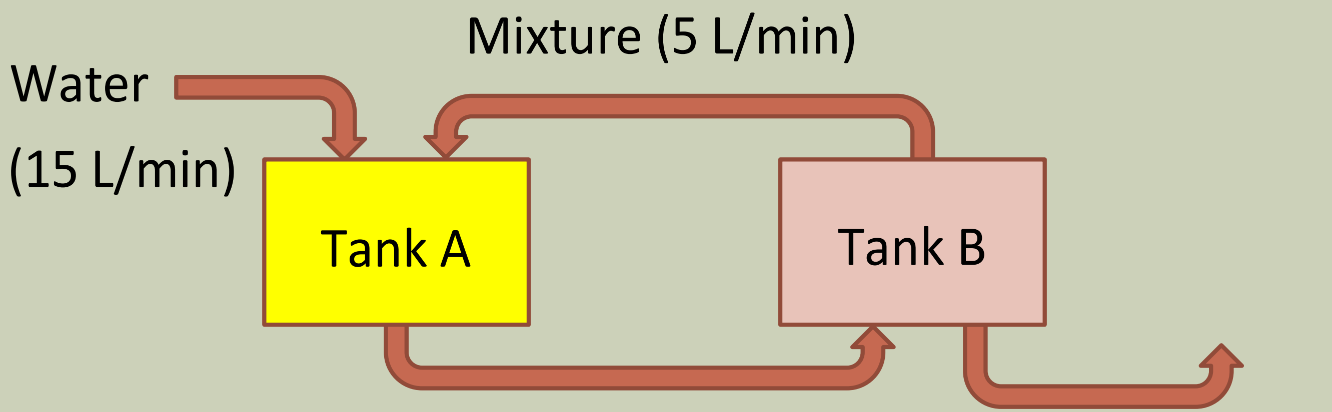


Note: amount of water in each tank will remain at 200L.

Rate of change in amount of salt in each tank

$=$ rate "in" minus rate "out"

EXAMPLE 2



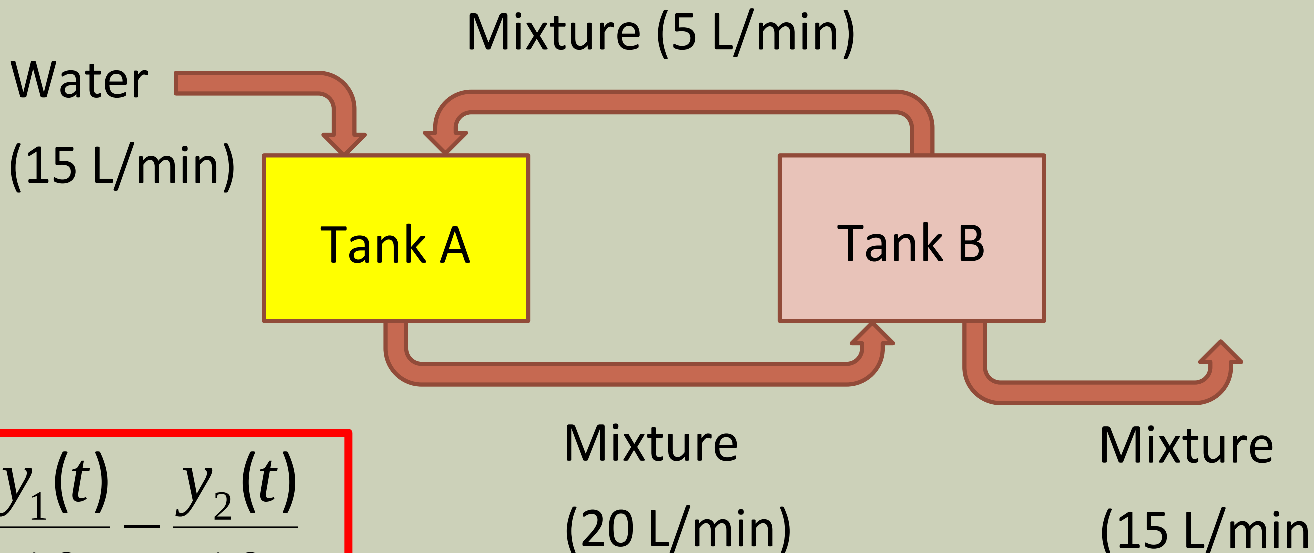
$$\boxed{y_1'(t) = \frac{y_2(t)}{40} - \frac{y_1(t)}{10}}$$

Tank A

$$\text{Rate "in"} = (5\ \text{L/min}) \cdot \left(\frac{y_2(t)}{200}\ \text{g/L} \right) = \frac{y_2(t)}{40}\ \text{g/min}$$

$$\text{Rate "out"} = (20\ \text{L/min}) \cdot \left(\frac{y_1(t)}{200}\ \text{g/L} \right) = \frac{y_1(t)}{10}\ \text{g/min}$$

EXAMPLE 2



$$\boxed{y_2'(t) = \frac{y_1(t)}{10} - \frac{y_2(t)}{10}}$$

Tank B

$$\text{Rate "in"} = (20\text{ L/min})\cdot\left(\frac{y_1(t)}{200}\text{ g/L}\right) = \frac{y_1(t)}{10}\text{ g/min}$$

$$\text{Rate "out"} = (5+15\text{ L/min})\cdot\left(\frac{y_2(t)}{200}\text{ g/L}\right) = \frac{y_2(t)}{10}\text{ g/min}$$

EXAMPLE 2

$$y_1'(t) = \frac{y_2(t)}{40} - \frac{y_1(t)}{10} \qquad y_2'(t) = \frac{y_1(t)}{10} - \frac{y_2(t)}{10}$$

$$\begin{cases} y_1'(t) & = & -\frac{1}{10} y_1(t) & + & \frac{1}{40} y_2(t) \\ y_2'(t) & = & \frac{1}{10} y_1(t) & - & \frac{1}{10} y_2(t) \end{cases} \Leftrightarrow \boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y} \text{ where}$$

$$\boldsymbol{A} = \begin{pmatrix} -\frac{1}{10} & \frac{1}{40} \\ \frac{1}{10} & -\frac{1}{10} \end{pmatrix} \qquad \boldsymbol{Y}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 60 \\ 0 \end{pmatrix}$$

EXAMPLE 2

1) Find all the eigenvalues of $\boldsymbol{A}$:

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0 \Leftrightarrow (\lambda + \frac{3}{20})(\lambda + \frac{1}{20})$$

So the eigenvalues of $\boldsymbol{A}$ are $-\frac{3}{20}$ and $-\frac{1}{20}$.

2) Find all the linearly independent eigenvectors associated with each eigenvalue λ.

$$E_{-\frac{3}{20}} = \text{span}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\} \qquad E_{-\frac{1}{20}} = \text{span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$$

EXAMPLE 2

3) Construct linear combinations of the solutions $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$.

A general solution to the system of linear differential equations is:

$$\boldsymbol{Y} = k_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t/20} + k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t/20}$$

4) Use the given initial conditions to solve for k_1, k_2.

$$\boldsymbol{Y}(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 60 \\ 0 \end{pmatrix} = \begin{pmatrix} k_1 + k_2 \\ -2k_1 + 2k_2 \end{pmatrix} \Rightarrow k_1 = k_2 = 30.$$

EXAMPLE 2

The solution to the initial value problem is:

$$\boldsymbol{Y} = 30\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-3t/20} + 30\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t/20}$$

$$= \begin{pmatrix} 30e^{-3t/20} + 30e^{-t/20} \\ -60e^{-3t/20} + 60e^{-t/20} \end{pmatrix}$$

amount of salt (in g) in tank A (resp. B) at time t

SUMMARY

1) Solving a system of linear differential equations $\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}$ with initial conditions.