NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

Module: MA1508E Linear Algebra for Engineering
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Tutorial: 10

1. For each of the following matrices $\boldsymbol{A}$, determine if $\boldsymbol{A}$ is diagonalizable. If $\boldsymbol{A}$ is diagonalizable, find an invertible matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{A}$.

   (a) $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$ (b) $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{pmatrix}$

   (c) $\boldsymbol{A} = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix}$ (d) $\boldsymbol{A} = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$.

   (a) $\boldsymbol{A}$ is diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

   (b) $\boldsymbol{A}$ is not diagonalizable. The characteristic polynomial is $(\lambda - 1)(\lambda - 2)^2$ but the dimension of E_2 is only 1.

   (c) $\boldsymbol{A}$ is diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

   (d) $\boldsymbol{A}$ is diagonalizable. Let $\boldsymbol{P} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$.

2. Let $\boldsymbol{A} = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$.

   (a) Show that 2 is an eigenvalue of $\boldsymbol{A}$.

   (b) Find a basis for the eigenspace associated with 2.

   (c) If $\boldsymbol{B}$ is another 3×3 matrix with an eigenvalue λ such that the dimension of the eigenspace associated with λ is 2, show that $2 + \lambda$ is an eigenvalue of the matrix $\boldsymbol{A} + \boldsymbol{B}$.

   (a) Since $\det(2\boldsymbol{I} - \boldsymbol{A}) = 0$, 2 is an eigenvalue of $\boldsymbol{A}$.

(b) $\{(1,2,0)^T, (-3,0,1)^T\}$ is a basis for the eigenspace associated with 2.

(c) Let E_2 be the eigenspace of $\boldsymbol{A}$ associated with 2 and let E'_λ be the eigenspace of $\boldsymbol{B}$ associated with λ.

Since E_2 and E'_λ are subspaces of $\mathbb{R}^3$ and have dimension 2, they are two planes in $\mathbb{R}^3$ that contain the origin. So $E_2 \cap E'_\lambda$ is either a line through the origin or a plane containing the origin. In both cases, we can find a nonzero vector $\boldsymbol{u} \in E_2 \cap E'_\lambda$, i.e. $\boldsymbol{Au} = 2\boldsymbol{u}$ and $\boldsymbol{Bu} = \lambda \boldsymbol{u}$, such that

$$(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{u} = \boldsymbol{Au} + \boldsymbol{Bu} = 2\boldsymbol{u} + \lambda\boldsymbol{u} = (2+\lambda)\boldsymbol{u}.$$

So $2+\lambda$ is an eigenvalue of $\boldsymbol{A}+\boldsymbol{B}$.

3. Two square matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be similar if ther exists an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{AP} = \boldsymbol{B}$. (**Note:** This means that any matrix $\boldsymbol{A}$ that is diagonalizable will be similar to its diagonal form $\boldsymbol{D}$.)

(a) Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are similar matrices.

(i) Show that $\boldsymbol{A}^n$ is similar to $\boldsymbol{B}^n$ for all positive integers n.

(ii) If $\boldsymbol{A}$ is invertible, show that $\boldsymbol{B}$ is also invertible. Furthermore, $\boldsymbol{A}^{-1}$ will be similar to $\boldsymbol{B}^{-1}$.

(iii) If $\boldsymbol{A}$ is diagonalizable, show that $\boldsymbol{B}$ is also diagonalizable.

(b) Show that the following two matrices are similar.

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(a) (i) $\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{AP} \;\Rightarrow\; \boldsymbol{B}^n = \underbrace{(\boldsymbol{P}^{-1}\boldsymbol{AP})(\boldsymbol{P}^{-1}\boldsymbol{AP})\cdots(\boldsymbol{P}^{-1}\boldsymbol{AP})}_{n \text{ times}} = \boldsymbol{P}^{-1}\boldsymbol{A}^n\boldsymbol{P}$

So $\boldsymbol{A}^n$ is similar to $\boldsymbol{B}^n$.

(ii) $\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{AP} \;\Rightarrow\; \boldsymbol{B}^{-1} = (\boldsymbol{P}^{-1}\boldsymbol{AP})^{-1} = \boldsymbol{P}^{-1}\boldsymbol{A}^{-1}\boldsymbol{P}$

So $\boldsymbol{A}^{-1}$ is similar to $\boldsymbol{B}^{-1}$.

(iii) Suppose there exists an invertible matrix $\boldsymbol{Q}$ such that $\boldsymbol{Q}^{-1}\boldsymbol{AQ}$ is a diagonal matrix. Let $\boldsymbol{R} = \boldsymbol{P}^{-1}\boldsymbol{Q}$. Then $\boldsymbol{R}$ is invertible and $\boldsymbol{R}^{-1}\boldsymbol{BR} = \boldsymbol{Q}^{-1}\boldsymbol{PBP}^{-1}\boldsymbol{Q} = \boldsymbol{Q}^{-1}\boldsymbol{AQ}$ is a diagonal matrix.

(B) Since $\boldsymbol{A}$ is a triangular matrix, its eigenvalues are 0, 1 and -1. Also it is easy to find from the characteristic equation of $\boldsymbol{B}$ that the eigenvalues of $\boldsymbol{B}$ are 0, 1 and -1. Since both $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices of order 3 and has 3 distinct eigenvalues, both $\boldsymbol{A}$ and $\boldsymbol{B}$ are diagonalizable. So there exist invertible matrices $\boldsymbol{R}$ and $\boldsymbol{Q}$ such that

$$\boldsymbol{R}^{-1}\boldsymbol{AR} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \boldsymbol{Q}^{-1}\boldsymbol{BQ}.$$

Let $\boldsymbol{P} = \boldsymbol{R}\boldsymbol{Q}^{-1}$. Then $\boldsymbol{P}$ is invertible matrix and $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{Q}\boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{R}\boldsymbol{Q}^{-1} = \boldsymbol{B}$.

4. A DVD rental shop rents out four types of DVD movies: romance, action, horror and animation. Each rental is for the duration of one week. At the end of each week, each customer will return and rent another DVD for another week.

   This rental process can be viewed as one with four possible outcomes. The probability of each outcome can be estimated by reviewing the previous rental records of each customer. The records indicate that 80 percent of the customers currently renting romance movies will continue doing so in the next week. Furthermore, 10 percent of the customers currently renting an action movie will switch to a romance movie in the next week. In addition, 5 percent of the customers currently renting a horror (and similarly those currently renting an animation) movie will also switch to a romance movie next week. These results are summarised in the first row of the table below. The seocnd row indicates the percentages of customers that will rent an action movie next week, and the final two rows give the percentages that will rent horror and animation movies, respectively.

Current week's rental				
Romance	Action	Horror	Animation	Next week's rental
0.80	0.10	0.05	0.05	Romance
0.10	0.80	0.05	0.05	Action
0.05	0.05	0.80	0.10	Horror
0.05	0.05	0.10	0.80	Animation

   Suppose that initially (say, in week 0) there are 200 Romance movies rented out and 100 each of the other three types of movies rented out (so there are 500 customers). Assume that there is no increase or decrease in the number of customers each week.

   (a) Find the estimated number of movies of each type that will be rented out in Week 2.

   (b) (**Computation challenge**) Find the estimated number of movies of each type that will be rented out in the long run. (You may need to use some computational tools (e.g. MATLAB or Python) to help you with this part.)

   (a) We let $\boldsymbol{A} = \begin{pmatrix} 0.8 & 0.1 & 0.05 & 0.05 \\ 0.1 & 0.8 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.8 & 0.1 \\ 0.05 & 0.05 & 0.1 & 0.8 \end{pmatrix}$ and $\boldsymbol{x_0} = \begin{pmatrix} 200 \\ 100 \\ 100 \\ 100 \end{pmatrix}$. Then the process can be modelled by $\boldsymbol{x_i} = \boldsymbol{A}\boldsymbol{x_{i-1}}$ where $i = 1, 2, \ldots$ and $\boldsymbol{x_i}$ gives the number of customers renting movies of each type in week i. To find out the numbers in week 2, we have

$$\boldsymbol{x_1} = \boldsymbol{A}\boldsymbol{x_0} = \begin{pmatrix} 180 \\ 110 \\ 105 \\ 105 \end{pmatrix} \Rightarrow \boldsymbol{x_2} = \boldsymbol{A}\boldsymbol{x_1} = \begin{pmatrix} 166 \\ 116 \\ 109 \\ 109 \end{pmatrix}.$$

(b) We find that $\boldsymbol{A}$ can be diagonalized as follows:

$$\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1} = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{4}{5} & 0 & 0 \\ 0 & 0 & \frac{7}{10} & 0 \\ 0 & 0 & 0 & \frac{7}{10} \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{pmatrix}.$$

So $\boldsymbol{x_n} = \boldsymbol{A}^n \boldsymbol{x_0}$ which, when n is large, can be approximated by

$$\boldsymbol{x_n} \approx \boldsymbol{P} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \boldsymbol{P}^{-1} \begin{pmatrix} 200 \\ 100 \\ 100 \\ 100 \end{pmatrix} = \begin{pmatrix} 125 \\ 125 \\ 125 \\ 125 \end{pmatrix}.$$

So in the long run, one quarter of the customers will be renting movies of each type.

5. I have a supply of three kind of tiles:

(1) 1×1 red-colored (square) tiles;

(2) 1×2 blue-colored (rectangular) tiles; and

(3) 1×2 green-colored (rectangular) tiles.

We represent each red tile by $(1R)$ (R for red, 1 since it is 1×1), each blue tile by $(2B)$ (B for blue, 2 since it is 2×2) and each green tile by $(2G)$ (G for green, 2 since it is 2×2).

I intend to tile a pavement that is $1 \times n$ units long and would like to know how many ways are there to tile the entire pavement with the colored tiles. Let b_n represent the number of differnent ways to tile a $1 \times n$ pavement. For example, $b_1 = 1$ since I can only tile it using $(1R)$; $b_2 = 3$ since there are three ways to tile a 1×2 pavement, namely $(1R)(1R)$, $(2B)$ and $(2G)$.

(a) Determine the value of b_3.

(b) Recall in Week 11 F2F lecture, we introduced the Fibonacci sequence (a_n), where the terms in the sequence are related via a **linear recurrence relation**:

$$a_n = a_{n-1} + a_{n-2}.$$

Write down a linear recurrence relation involving 3 consecutive terms (b_n, b_{n-1} and b_{n-2}) in the sequence (b_n).

(c) Solve the linear recurrence relation you obtained in part (b) and use it to find the number of ways to tile a 1×100 pavement.

(a) $b_3 = 5$.

(b) $b_n = b_{n-1} + 2b_{n-2}$.

(c) Let $\boldsymbol{X_1} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and for $n = 1, 2, \cdots,$

$$\boldsymbol{X_n} = \begin{pmatrix} b_n \\ b_{n+1} \end{pmatrix}, \quad \boldsymbol{X_{n+1}} = \begin{pmatrix} b_{n+1} \\ b_{n+2} \end{pmatrix} = \begin{pmatrix} b_{n+1} \\ b_{n+1} + 2b_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} b_n \\ b_{n+1} \end{pmatrix} = \boldsymbol{A X_n}.$$

Solving $\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$, we find that $\boldsymbol{A}$ has two eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$. Following this, we find that

$$E_{\lambda_1} = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}, \quad E_{\lambda_2} = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}.$$

So $\boldsymbol{P} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$ diagonalizes $\boldsymbol{A}$. Since $\boldsymbol{X_2} = \boldsymbol{A X_1}$, $\boldsymbol{X_3} = \boldsymbol{A}^2 \boldsymbol{X_1}$ and so on, we have

$$\boldsymbol{X_n} = \boldsymbol{A}^{n-1} \boldsymbol{X_1} = \boldsymbol{P D}^{n-1} \boldsymbol{P}^{-1} \boldsymbol{X_1}$$

$$\Rightarrow \boldsymbol{X_n} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2^{n-1} & 0 \\ 0 & (-1)^{n-1} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{2}{3}(2^n) + \frac{1}{3}(-1)^n \\ \frac{2}{3}(2^{n+1}) + \frac{1}{3}(-1)^{n+1} \end{pmatrix}.$$

So there are

$$b_{100} = \frac{2}{3}(2^{100}) + \frac{1}{3} = \frac{1}{3}(2^{101} + 1)$$

ways to pave a 1×100 pavement.