W04-06

Slide 01: In this unit, we will continue to discuss the concept of linear span.

Slide 02: Recall that in a previous unit, we had several examples to determine if a set of vectors would span the entire Euclidean n-space. Let us discuss this problem in a more general setting. Suppose S is a set of vectors from $\mathbb{R}^n$. S contains vectors $\boldsymbol{u_1}, \boldsymbol{u_2}$ and so on till $\boldsymbol{u_k}$ and the components of these vectors are as shown. We wish to determine if span(S) is equal to $\mathbb{R}^n$.

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As per what was done in the numerical examples, we consider an arbitrary vector in $\mathbb{R}^n$, in this case, $\boldsymbol{v}$, with components v_1, v_2 and so on till v_n. We write down the vector equation as shown. Recall that we wish to check if coefficients c_1, c_2 to c_k can be found to satisfy this vector equation regardless of the values of v_1, v_2 to v_n.

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Let's write down the vectors explicitly with their respective components.

Slide 03: As before, we will compare components on both sides of this vector equation in order to write down a linear system.

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The linear system, as can be seen here will be one that consists of n equations and k unknowns. Note that we have n equations because there are n components to compare and we have k unknowns c_1, c_2 to c_k arising from the k vectors $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ to $\boldsymbol{u_k}$.

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You should also observe that the n components from the first vector $\boldsymbol{u_1}$ appears in the first column of the linear system,

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similarly, the components from the second vector $\boldsymbol{u_2}$ appears in the second column and likewise

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the components from the last vector $\boldsymbol{u_k}$ appears in the last column on the left side of the linear system.

Slide 04: When the linear system is represented as an augmented matrix, the matrix will have a total of n rows and k columns on the left side of the vertical line.

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Let us represent the left side of the augmented matrix by $\boldsymbol{A}$, so that $\boldsymbol{A}$ is a $n \times k$ matrix.

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From the examples discussed in an earlier unit, we know that if a row-echelon form of $\boldsymbol{A}$ does not have a zero row,

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it would indicate that the linear system and consequently, the vector equation will always be consistent regardless of the components in the vector $\boldsymbol{v}$.

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This would allow us to conclude that span(S) is the entire $\mathbb{R}^n$.

Slide 05: On the other hand, if a row-echelon form of $\boldsymbol{A}$ has at least one zero row,
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this would indicate that the vector equation will not always be consistent. There will be some choice of vector $\boldsymbol{v}$ that will make the vector equation inconsistent.
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Thus in this case, $\text{span}(S)$ will not be equal to $\mathbb{R}^n$.

Slide 06: Let us revisit from an earlier example in the previous unit. In that example, we wanted to show that the linear span of $(1, 0, 1)$, $(1, 1, 0)$ and $(0, 1, 1)$ is equal to $\mathbb{R}^3$. As per our discussion, we have an augmented matrix representing the linear system with 3 equations and 3 unknowns. Notice that the three vectors appears as the three columns on the left side of the augmented matrix.
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Upon performing Gaussian elmination, we arrive at a row-echelon form of the augmented matrix which did not have a zero row and this led to the conclusion that the linear span of the three vectors is indeed $\mathbb{R}^3$.

Slide 07: Also from the previous unit, we showed that the linear span of these 4 vectors is not equal to $\mathbb{R}^3$. Following the same procedure, we have the augmented matrix with the vectors as columns on the left hand side.
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A row-echelon form of the augmented matrix shows the existence of a zero row on the left and this means that the linear span is not equal to $\mathbb{R}^3$.

Slide 08: The discussion thus far in this unit gives rise to the following theorem. Suppose S is a set of k vectors taken from $\mathbb{R}^n$. Comparing the two integers k and n, if $k < n$, then we can immediately conclude that S cannot span $\mathbb{R}^n$.
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The proof of this theorem stems entirely from our earlier discussion. We note that as we set up the vector equation and subsequently the linear system and the augmented matrix representing the system, the key consideration is the matrix $\boldsymbol{A}$ found on the left side of the augmented matrix. If a row-echelon form of $\boldsymbol{A}$ has at least one zero row, then the vector equation will not always be consistent, indicating that the set S will not span $\mathbb{R}^n$.
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Now if $k < n$, then the matrix $\boldsymbol{A}$ will have strictly more rows than columns. In other words, any row-echelon form of $\boldsymbol{A}$ will have at most k leading entries and since we have strictly more rows than k, we know that there will be at least one row without a leading entry, in other words, a zero row.
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Thus the conclusion that S cannot span $\mathbb{R}^n$ follows.

Slide 09: The previous theorem immediately tells us that one vector can never span $\mathbb{R}^2$,
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and similarly, one or two vectors can never span $\mathbb{R}^3$.

Slide 10: To end this unit, we will present another theorem to extend our understanding of linear span. Let S be a set of k vectors taken from $\mathbb{R}^n$.

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The first result states that the zero vector is always an element found in span(S).

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To see why this is so, remember that span(S) contains all possible linear combinations of $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$.

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In particular, it contains the linear combination

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$0\boldsymbol{u_1} + 0\boldsymbol{u_2}$ and so on until $0\boldsymbol{u_k}$, which is obviously the zero vector. Thus, any linear span always contains the zero vector.

Slide 11: The second result is as follows. If $\boldsymbol{v_1}$, $\boldsymbol{v_2}$ and so on till $\boldsymbol{v_r}$ are vectors taken from span(S), then for any choice of real numbers c_1 to c_r, the linear combination $c_1\boldsymbol{v_1}+c_2\boldsymbol{v_2}$ and so on till $c_r\boldsymbol{v_r}$ will still remain as a vector inside span(S). Intuitively, what this result means is that span(S) behaves like a close-ecosystem of vectors where linearly combining vectors in span(S) will never result in a vector that lies outside span(S).

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To prove this result, we go back to the definition of what it means to be a vector in span(S). Since span(S) is the set of all linear combinations of $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$, the vector $\boldsymbol{v_1}$ belonging to span(S) means that $\boldsymbol{v_1}$ can be written as a linear combination of $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$. So let $\boldsymbol{v_1}$ be written as shown here where d_{11} to d_{1k} are some scalars.

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Likewise, we can do the same for $\boldsymbol{v_2}$,

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and all the way until $\boldsymbol{v_r}$.

Slide 12: Recall that we would like to linearly combine the vectors $\boldsymbol{v_1}$ to $\boldsymbol{v_r}$. Now that we know how each $\boldsymbol{v_i}$ can be expressed in terms of the vectors $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$,

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we can perform the substitution as follows, as shown here, replacing $\boldsymbol{v_1}$ in terms of $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$

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and similarly, replacing $\boldsymbol{v_2}$ in terms of $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$

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and all the way until $\boldsymbol{v_r}$.

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Let us group all the expressions involving $\boldsymbol{u_1}$ together. It is easily verified that the coefficient for $\boldsymbol{u_1}$ is as shown.

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Similarly, we group all the expressions involving $\boldsymbol{u_2}$ together

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and all the way, doing the same until we reach $\boldsymbol{u_k}$.

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What have we shown here? Well, on the left side of the equation is a linear combination of $\boldsymbol{v_1}$ to $\boldsymbol{v_r}$. While on the right, we now have a linear combination of $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$, which by definition of linear span, will be a vector inside span(S). We have thus shown that any linear combination of $\boldsymbol{v_1}$ to $\boldsymbol{v_r}$ will remain inside span(S).

Slide 13: Let us briefly discuss the significance of this theorem. The first statement tells us that the zero vector will always be an element in any linear span. In other words, any subset of $\mathbb{R}^n$ that does not contain the zero vector will never be a linear span.

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The second statement is really about the closure property of a linear span.

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As we have seen, span(S) is closed under linear combinations, so linearly combining vectors in span(S) will always result in some vector inside span(S).

Slide 14: Let us summarise the main points.

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We started with a detailed discussion in general on the problem of checking whether span(S) is equal to $\mathbb{R}^n$.

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The discussion eventually led to the theorem that we can never span $\mathbb{R}^n$ with less than n vectors.

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We then established two properties of linear span, one involving the zero vector and the other on the closure property of linear spans.