W05-04

Slide 01: In this unit, we will look at linear independence for sets with a small number of vectors.

Slide 02: We have already seen from a previous unit that for vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, if a set S contains exactly two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, then S is a linearly dependent set if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are multiples of each other.

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For example, they could be pointing in the same direction,

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or in the opposite direction. For such cases, the two vectors lie on the same line.

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If $\boldsymbol{u}$ and $\boldsymbol{v}$ do not lie on the same line, then they are linearly independent.

Slide 03: Let us consider a set of three vetors in $\mathbb{R}^3$. You should already know that if the three vectors are from $\mathbb{R}^2$, then they are already linearly dependent. What if they are in $\mathbb{R}^3$? We will see that the three vectors are linearly dependent if and only if they lie on the same line or the same plane.

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Let us start with one vector in the set and gradually add vectors into S. With only $\boldsymbol{u}$ in the set, the set is linearly independent as long as $\boldsymbol{u}$ is not the zero vector.

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Let us add in the second vector $\boldsymbol{v}$. If $\boldsymbol{v}$ lies on the same line as $\boldsymbol{u}$, the set $\{\boldsymbol{u}, \boldsymbol{v}\}$ will become a linearly dependent set.

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Regardless of $\boldsymbol{w}$, the set with all three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ would still be linearly dependent. It could be that $\boldsymbol{w}$ also lies on the same line as $\boldsymbol{u}$ and $\boldsymbol{v}$, as shown here

Slide 04: Or it could be that $\boldsymbol{w}$ does not lie on the same line as $\boldsymbol{u}$ and $\boldsymbol{v}$, in which case $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ would lie on the same plane as shown. Nevertheless the three vectors still form a linearly dependent set.

Slide 05: What if our second vector $\boldsymbol{v}$ did not lie on the same line as $\boldsymbol{u}$? Then the two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ would form a linearly independent set as shown here. The linear span of $\boldsymbol{u}$ and $\boldsymbol{v}$ would be a plane.

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If we add in a third vector $\boldsymbol{w}$, which lies on the same plane spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$, then $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ would become a linearly dependent set. This could happen regardless of whether $\boldsymbol{w}$ is on the same line as either $\boldsymbol{u}$ or $\boldsymbol{v}$, as shown here,

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or it could be that $\boldsymbol{w}$ is not on the same line as both $\boldsymbol{u}$ and $\boldsymbol{v}$, as shown here.

Slide 06: So when will it be that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ forms a linearly independent set? How must $\boldsymbol{w}$ be in relation to the linear span of $\boldsymbol{u}$ and $\boldsymbol{v}$?

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As seen from the previous case, we require $\boldsymbol{w}$ to **NOT** belong to the linear span of $\boldsymbol{u}$ and $\boldsymbol{v}$ for this to happen. This means that while $\boldsymbol{u}$ and $\boldsymbol{v}$ span a plane, the third vector $\boldsymbol{w}$ must be a vector that does not lie on the plane, or if it is easier to visualise, we can say that $\boldsymbol{w}$ sticks out from the plane spanned by $\boldsymbol{u}$ and $\boldsymbol{v}$.

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Only in this case will $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ form a linearly independent set. As you can see from the discussion above, intuitively, we now see that when a set contains more vectors, it is more likely that the set will be linearly dependent. So a natural question is if we already have a linearly independent set, how, if possible, can we extend the set by adding in a new vector but at the same time preserve the linear independence property?

Slide 07: This theorem will answer the question we have just posed. Suppose $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$ are k linearly independent vectors in $\mathbb{R}^n$.

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Suppose a new vector $\boldsymbol{u_{k+1}}$ is not a linear combination of $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$, then the extended collection of $k+1$ vectors, namely, $\boldsymbol{u_1}$ to $\boldsymbol{u_{k+1}}$ will still be a linearly independent collection.

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To proof this result, we will do so using the method of contradition. We start off by supposing that $\boldsymbol{u_1}$ to $\boldsymbol{u_{k+1}}$ are actually linearly dependent.

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By the definition, this would mean that the following vector equation will have non trivial solutions for the coefficients c_1 to c_{k+1}.

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Let d_1 to d_{k+1} be one such non trivial solution to the vector equation. By non trivial solution, this means that d_1 to d_{k+1} are not all zero.

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Note that since d_1 to d_{k+1} is a solution, they satisfy the vector equation, so we will have the following.

Slide 08: We consider two cases, depending on the value of d_{k+1}. If $d_{k+1} = 0$, this would mean that d_1 to d_k will not all be zero.

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The original equation will now have one term less, since $d_{k+1} = 0$ and now we have a vector equation involving only $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$ where the solutions d_1 to d_k are not all zero.

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This is a contradiction to what we are told about $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$. They are supposed to be linearly independent and thus not possible for them to be linearly combined in a non trivial way to obtain the zero vector.

Slide 09: For case 2, we suppose $d_{k+1} \neq 0$.

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In this case, the original equation can now to rewritten with $\boldsymbol{u_{k+1}}$ on one side and all the other terms on the other side. Note that all the coefficients on the right hand side are valid since d_{k+1} is not zero.

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We now see that $\boldsymbol{u_{k+1}}$ is a linear combination of $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$, which is a contradiction to the condition stated in the theorem, where the new vector $\boldsymbol{u_{k+1}}$ is known not to be a linear combination of the first k vectors. In conclusion, we have established the result which tells us how we can extend a set of linearly independent vectors.

Slide 10: Let us discuss this result briefly. While it tells us that as long as we can find $\boldsymbol{u_{k+1}}$ that is not a linear combination of $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$, the linear independence property of the collection $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$ can be preserved even after adding in the new vector.

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We also have the intuitive understanding that having linear independence property for a collection of k vectors becomes increasingly difficult when k becomes larger. So what do you think is the largest value of k that will still make it possible to have k linearly independent vectors?

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In other words, we are considering when will we no longer be able to find a new vector $\boldsymbol{u_{k+1}}$ that is not a linear combination of the first k vectors? These questions will be answered in the following units.

Slide 11: To summarise this unit.

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We discussed linear independence for sets with 2 or 3 vectors. We discussed this by progressively adding vectors to extend a set.

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The discussion led to the general result which tells us when we can add vectros into a linearly independent set and preserve that independence property of the set.