NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

Module: MA1508E Linear Algebra for Engineering
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Tutorial: 5

1. Let $\boldsymbol{u_1} = (1, 2, -1)$, $\boldsymbol{u_2} = (6, 4, 2)$, $\boldsymbol{u_3} = (9, 2, 7)$, $\boldsymbol{u_4} = (4, -1, 8)$, $\boldsymbol{u_5} = (1, 2, 3)$.

   (a) Is $\boldsymbol{u_3}$ a linear combination of $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$? Is $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\} = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$? Is either $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$ or $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ equals to $\mathbb{R}^3$?

   (b) Is $\boldsymbol{u_4}$ a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}$ and $\boldsymbol{u_3}$? Is $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\} = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$? Is $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\} = \mathbb{R}^3$?

   (c) Is $\boldsymbol{u_5}$ a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ and $\boldsymbol{u_4}$? Is $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\} = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}, \boldsymbol{u_5}\}$? Is $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}, \boldsymbol{u_5}\} = \mathbb{R}^3$?

   (a) $\left(\begin{array}{cc|c} 1 & 6 & 9 \\ 2 & 4 & 2 \\ -1 & 2 & 7 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$ So $\boldsymbol{u_3}$ is a linear combination of $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$. Yes, $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\} = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\} \neq \mathbb{R}^3$ since two vectors are not enough to span $\mathbb{R}^3$.

   (b) $\left(\begin{array}{ccc|c} 1 & 6 & 9 & 4 \\ 2 & 4 & 2 & -1 \\ -1 & 2 & 7 & 8 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ So $\boldsymbol{u_4}$ is not a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}$ and $\boldsymbol{u_3}$. No, $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ is not equal to $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$, which is equal to $\mathbb{R}^3$.

   (c) Since $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}$ spans $\mathbb{R}^3$, we know for sure that $\boldsymbol{u_5}$ is a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}$ and the span of $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}, \boldsymbol{u_5}\}$ and $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$ are both equal to $\mathbb{R}^3$.

2. For each of the following matrices $\boldsymbol{A}$, express the solution space of $\boldsymbol{Ax} = \boldsymbol{0}$ as a linear span. Give a geometrical interpretation of the solution space (in other words, describe the geometrical object represented by the linear span).

   (a) $\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{pmatrix}$ (b) $\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & 6 \end{pmatrix}$

   (c) $\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{pmatrix}$ (d) $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(a) The solution space is $S = \{(x, y, z) \mid x - 2y + 3z = 0\}$. Solving $x - 2y + 3z = 0$, we have $S = \text{span}\{(2, 1, 0), (-3, 0, 1)\}$, which is a plane (with equation $x - 2y + 3z = 0$) in $\mathbb{R}^3$.

(b) The solution space is $S = \{(0, 0, 0)\}$, which is $\text{span}\{\mathbf{0}\}$. This is the zero space, which is the origin in $\mathbb{R}^3$.

(c) A general solution for the linear system $\boldsymbol{Ax} = \mathbf{0}$ is $x_1 = -\frac{7t}{9}$, $x_2 = \frac{10t}{9}$, $x_3 = t$, where $t \in \mathbb{R}$. Thus $S = \text{span}\{(-7, 10, 9)\}$, which is a straight line in $\mathbb{R}^3$ passing through the origin and the point $(-7, 10, 9)$.

(d) The solution space is $\{(x, y, z) \mid x, y, z \in \mathbb{R}\}$, which is the entire Euclidean space $\mathbb{R}^3$.

3. Let $V = \{(x, y, z) \mid 2x - y + 3z = 0\}$ be a subset of $\mathbb{R}^3$.

(a) Is V a subspace of $\mathbb{R}^3$? If so, describe the subspace geometrically.

(b) Let $S = \{(1, -1, -1), (1, 2, 0)\}$. Show that $\text{span}(S) = V$.

(c) Let $\boldsymbol{u} = (0, 3, a)$, where a is a real number. Suppose $T = S \cup \{\boldsymbol{u}\}$. Find all values of a such that

(i) $\text{span}(T) = \mathbb{R}^3$.

(ii) $\text{span}(T) = V$.

(a) Yes, V is a subspace of $\mathbb{R}^3$ since it is the solution space of a homogeneous linear system. It is a plane in $\mathbb{R}^3$ that contains the origin.

(b) As long as we have two vectors that belongs to V and are not multiples of each other, these two vectors will span V. Indeed, we check that both $(1, -1, -1)$ and $(1, 2, 0)$ both satisfy the equation $2x - y + 3z = 0$, so $\text{span}(S) = V$.

(c) (i) We require $\boldsymbol{u}$ **NOT** to be a linear combination of $(1, -1, -1)$ and $(1, 2, 0)$.

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & 2 & 3 \\ -1 & 0 & a \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & a-1 \end{array}\right)$$

So $\text{span}(T) = \mathbb{R}^3$ if and only if $a \neq 1$.

(ii) It follows from the previous part that $\text{span}(T) = V$ if and only if $a = 1$.

4. Let $\boldsymbol{u_1} = (2, 0, 2, -4)$, $\boldsymbol{u_2} = (1, 0, 2, 5)$, $\boldsymbol{u_3} = (0, 3, 6, 9)$, $\boldsymbol{u_4} = (1, 1, 2, -1)$, $\boldsymbol{v_1} = (-1, 2, 1, 0)$, $\boldsymbol{v_2} = (3, 1, 4, 0)$, $\boldsymbol{v_3} = (0, 1, 1, 3)$, $\boldsymbol{v_4} = (-4, 3, -1, 6)$. Determine if the following are true.

(a) $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\} \subseteq \text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$.

(b) $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\} \subseteq \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$.

(c) $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\} = \mathbb{R}^4$.

(d) $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\} = \mathbb{R}^4$.

(a) $\left(\begin{array}{cccc|c} -1 & 3 & 0 & -4 & 1 \\ 2 & 1 & 1 & 3 & 0 \\ 1 & 4 & 1 & -1 & 2 \\ 0 & 0 & 3 & 6 & 5 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{cccc|c} -1 & 3 & 0 & -4 & 1 \\ 0 & 7 & 1 & -5 & 2 \\ 0 & 0 & 3 & 6 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$

Since $\boldsymbol{u_2} \notin \text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$, $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\} \not\subseteq \text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$.

(b) $\left(\begin{array}{cccc|c|c|c|c} 2 & 1 & 0 & 1 & -1 & 3 & 0 & -4 \\ 0 & 0 & 3 & 1 & 2 & 1 & 1 & 3 \\ 2 & 2 & 6 & 2 & 1 & 4 & 1 & -1 \\ -4 & 5 & 9 & -1 & 0 & 0 & 3 & 6 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{cccc|c|c|c|c} 2 & 1 & 0 & 1 & -1 & 3 & 0 & -4 \\ 0 & 1 & 6 & 1 & 2 & 1 & 1 & 3 \\ 0 & 0 & 3 & 1 & 2 & 1 & 1 & 3 \\ 0 & 0 & 0 & 5 & 6 & 10 & 7 & 10 \end{array}\right)$

The systems are consistent and thus $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\} \subseteq \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$.

(c) $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\} = \mathbb{R}^4$.

(d) $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\} \neq \mathbb{R}^4$.

5. For each of the following subsets S of $\mathbb{R}^n$ determine if S is a subspace of $\mathbb{R}^3$ (or $\mathbb{R}^4$) and for those which are, write S as a linear span.

(a) $S = \{(a, b, c) \mid abc = 0\}$.

(b) $S = \{(x, y, z) \mid 4y = z\}$.

(c) $S = \{(a, b, c) \mid a \leq b \leq c\}$

(d) $S = \{(w, x, y, z) \mid 2x + 3y - z = 0 \text{ and } x + 2y - z = 0\}$.

(e) $S = \{\boldsymbol{Au} \mid \boldsymbol{u} \in \mathbb{R}^3\}$ where $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}$ (here $\boldsymbol{u}$ is written as a column vector).

(f) $S = \{\boldsymbol{u} \in \mathbb{R}^4 \mid \boldsymbol{Au} = \boldsymbol{u}\}$ where $\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (here $\boldsymbol{u}$ is written as a column vector).

(a) S is not a subspace since $(1, 0, 0)$ and $(0, 1, 1)$ belongs to S but $(1, 0, 0) + (0, 1, 1) = (1, 1, 1)$ does not.

(b) S is a subspace of $\mathbb{R}^3$ since it is the solution space of the homogeneous linear system $4y - z = 0$. Solving $4y - z = 0$, we have $S = \text{span}\{(1, 0, 0), (0, 1, 4)\}$.

(c) S is not a subspace of $\mathbb{R}^3$ since $(1, 2, 3)$ belongs to S but $-(1, 2, 3) = (-1, -2, -3)$ does not.

(d) S is a subspsace of $\mathbb{R}^4$ since it is the solution space of the homogeneous linear system

$$\begin{cases} 2x + 3y - z = 0 \\ x + 2y - z = 0 \end{cases}$$

Solving the system, we have the general solution $w = t, x = -s$, $y = s$, $z = s$ where $t, s \in \mathbb{R}$. Thus, $S = \text{span}\{(1, 0, 0, 0), (0, -1, 1, 1)\}$.

(e) Note that S is a subset of $\mathbb{R}^2$ in this case. For each $\boldsymbol{u} = (x, y, z)^T \in \mathbb{R}^3$, $\boldsymbol{A}\boldsymbol{u} = x\begin{pmatrix}1\\0\end{pmatrix} + y\begin{pmatrix}2\\1\end{pmatrix} + z\begin{pmatrix}3\\1\end{pmatrix}$. Thus $S = \text{span}\{(1,0)^T, (2,1)^T, (3,1)^T\}$, which is a subspace of $\mathbb{R}^2$. (In fact, $S = \mathbb{R}^2$.)

(f) Note that S is a subset of $\mathbb{R}^4$ and $\boldsymbol{u} \in S$ if and only if $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{u} \Leftrightarrow (\boldsymbol{A} - \boldsymbol{I})\boldsymbol{u} = \boldsymbol{0}$. So S is the solution space of the homogeneous linear system with coefficient matrix $(\boldsymbol{A} - \boldsymbol{I})$ and thus it is a subspace. Solving $(\boldsymbol{A} - \boldsymbol{I})\boldsymbol{u} = \boldsymbol{0}$, we have a general soliution $x_1 = s$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ where $s \in \mathbb{R}$. So $S = \text{span}\{(1,0,0,0)\}$.

6. Determine which of the following statements are true. Justify your answer.

(a) If $\boldsymbol{u}$ is a nonzero vector in $\mathbb{R}^1$, then $\text{span}\{\boldsymbol{u}\} = \mathbb{R}^1$.

(b) If $\boldsymbol{u}, \boldsymbol{v}$ are nonzero vectors in $\mathbb{R}^2$ such that $\boldsymbol{u} \neq \boldsymbol{v}$, then $\text{span}\{\boldsymbol{u}, \boldsymbol{v}\} = \mathbb{R}^2$.

(c) If S_1 and S_2 are finite subsets of $\mathbb{R}^n$, then $\text{span}(S_1 \cap S_2) = \text{span}\,(S_1) \cap \text{span}(S_2)$.

(d) If S_1 and S_2 are finite subsets of $\mathbb{R}^n$, then $\text{span}(S_1 \cup S_2) = \text{span}(S_1) \cup \text{span}(S_2)$.

(a) True, since $\text{span}\{\boldsymbol{u}\} = \{c\boldsymbol{u} \mid c \in \mathbb{R}\}$ which is the set of all real numbers.

(b) False, since $\boldsymbol{u}$ and $\boldsymbol{v}$ may be scalar multiples of each other. For example $\boldsymbol{u} = (1,0)$, $\boldsymbol{v} = (2,0)$, then $\text{span}\{\boldsymbol{u}, \boldsymbol{v}\} = \text{span}\{\boldsymbol{u}\} \neq \mathbb{R}^2$.

(c) False. For example, if $S_1 = \{(1,0)\}$, $S_2 = \{(2,0)\}$, then $S_1 \cap S_2 = \emptyset$ and $\text{span}(S_1 \cap S_2) = \{\boldsymbol{0}\}$. On the other hand $\text{span}(S_1) \cap \text{span}(S_2) = \text{span}\{(1,0)\}$.

(d) False. For example, if $S_1 = \{(1,0)\}$ and $S_2 = \{(0,1)\}$, then $\text{span}(S_1 \cup S_2) = \mathbb{R}^2$. On the other hand $\text{span}(S_1) \cup \text{span}(S_2)$ is the union of the x and y-axes in $\mathbb{R}^2$ which is not a subspace of $\mathbb{R}^2$.