NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

Module: MA1508E Linear Algebra for Engineering
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Tutorial: 6

1. For each of the following sets $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$, determine the values of the constants a and b such that the set S is a linearly dependent set.

   (a) $\boldsymbol{u_1} = (1, 0, 1)$, $\boldsymbol{u_2} = (a, 1, 1)$, $\boldsymbol{u_3} = (1, 1, 3a)$.

   (b) $\boldsymbol{u_1} = (1, 0, 0)$, $\boldsymbol{u_2} = (a, 1, -a)$, $\boldsymbol{u_3} = (1, 2a, 3a + 1)$.

   (a)
$$\left(\begin{array}{ccc|c} 1 & a & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 3a & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & a & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 4a-2 & 0 \end{array}\right)$$

   So the set S is linearly dependent if and only if $a = \frac{1}{2}$.

   (b)
$$\left(\begin{array}{ccc|c} 1 & a & 1 & 0 \\ 0 & 1 & 2a & 0 \\ 0 & -a & 3a+1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & a & 1 & 0 \\ 0 & 1 & 2a & 0 \\ 0 & 0 & 2a^2+3a+1 & 0 \end{array}\right)$$

   Since $2a^2 + 3a + 1 = (2a + 1)(a + 1)$, the set S is linearly dependent if and only if $a = -\frac{1}{2}$ or -1.

2. Let $\boldsymbol{u_1} = (1, -2, 1, 1, 2)$, $\boldsymbol{u_2} = (-1, 3, 0, 2, -2)$, $\boldsymbol{u_3} = (0, 1, 1, 3, 4)$.

   (a) Show that $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is a linearly independent set.

   (b) Find a vector $\boldsymbol{u_4}$ such that $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$ is a linearly independent set.

   (c) Find a vector $\boldsymbol{u_5}$ such that $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}, \boldsymbol{u_5}\}$ is a basis for $\mathbb{R}^5$.

   (a) Solving $c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + c_3\boldsymbol{u_3} = \boldsymbol{0}$,

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -2 & 3 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 2 & -2 & 4 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

   So $c_1 = c_2 = c_3 = 0$ is the only solution and thus $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is a linearly independent set.

(b) We find $\boldsymbol{x} = (x_1, x_2, x_3, x_4, x_5)$ such $c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + c_3\boldsymbol{u_3} = \boldsymbol{x}$ is inconsistent.

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & x_1 \\ -2 & 3 & 1 & x_2 \\ 1 & 0 & 1 & x_3 \\ 1 & 2 & 3 & x_4 \\ 2 & -2 & 4 & x_5 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & x_1 \\ 0 & 1 & 1 & x_2 + 2x_1 \\ 0 & 0 & 4 & x_5 - 2x_1 \\ 0 & 0 & 0 & x_3 - x_2 - 3x_1 \\ 0 & 0 & 0 & x_4 - 3x_2 - 7x_1 \end{array}\right).$$

So we may choose $\boldsymbol{u_4} = (1, 0, 0, 0, 0)$.

(c) We find a vector $\boldsymbol{y} = (y_1, y_2, y_3, y_4, y_5)$ such that $c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + c_3\boldsymbol{u_3} + c_4\boldsymbol{u_4} = \boldsymbol{y}$ is inconsistent.

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & y_1 \\ -2 & 3 & 1 & 0 & y_2 \\ 1 & 0 & 1 & 0 & y_3 \\ 1 & 2 & 3 & 0 & y_4 \\ 2 & -2 & 4 & 0 & y_5 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & y_1 \\ 0 & 1 & 1 & 2 & y_2 + 2y_1 \\ 0 & 0 & 4 & -2 & -2y_1 + y_5 \\ 0 & 0 & 0 & -7 & -7y_1 - 3y_2 + y_4 \\ 0 & 0 & 0 & 0 & \frac{2}{7}y_2 + y_3 - \frac{3}{7}y_4 \end{array}\right)$$

So we may choose $\boldsymbol{u_5} = (0, 1, 0, 0, 0)$.

3. Let $\boldsymbol{v_1} = (1, 2, 3)$, $\boldsymbol{v_2} = (2, 4, 6)$, $\boldsymbol{v_3} = (2, 5, 7)$, $\boldsymbol{v_4} = (3, 5, 9)$, $\boldsymbol{v_5} = (1, 4, 5)$.

(a) Show that $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}, \boldsymbol{v_5}\}$ is a linearly dependent set.

(b) Remove one **redundant** vector from S to obtain S' such that $\text{span}(S) = \text{span}(S')$.

(c) Explain why S' is still a linearly dependent set.

(d) Remove one more redundant vector from S' to obtain S'' such that $\text{span}(S') = \text{span}(S'')$.

(e) Determine if S'' is a basis for $\mathbb{R}^3$.

(a) Since S contains 5 vectors from $\mathbb{R}^3$, it is immediate that S is a linearly dependent set.

(b) By observation, $\boldsymbol{v_2} = 2\boldsymbol{v_1}$, so we may remove $\boldsymbol{v_2}$, that is, let $S' = \{\boldsymbol{v_1}, \boldsymbol{v_3}, \boldsymbol{v_4}, \boldsymbol{v_5}\}$ and we have $\text{span}(S') = \text{span}(S)$.

(c) S' is still a linearly dependent set since it contains 4 vectors from $\mathbb{R}^3$.

(d) We put the vectors $\boldsymbol{v_1}, \boldsymbol{v_3}, \boldsymbol{v_4}, \boldsymbol{v_5}$ as columns of a matrix:

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 5 & 5 & 4 \\ 3 & 7 & 9 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We see that $\boldsymbol{v_5} = -3\boldsymbol{v_1} + 2\boldsymbol{v_3} + 0\boldsymbol{v_4}$, so $\boldsymbol{v_5}$ can be removed to give $S'' = \{\boldsymbol{v_1}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$ such that $\text{span}(S') = \text{span}(S'')$.

(e) Consider $c_1\boldsymbol{v_1} + c_2\boldsymbol{v_3} + c_3\boldsymbol{v_4} = (x, y, z)$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & x \\ 2 & 5 & 5 & y \\ 3 & 7 & 9 & z \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & x \\ 0 & 1 & -1 & y-2x \\ 0 & 0 & 1 & z-y-x \end{array}\right).$$

So S'' spans $\mathbb{R}^3$. It is also easy to see from the above working (set $x = y = z = 0$) that S'' is a linearly independent set. Thus S'' is a basis for $\mathbb{R}^3$.

4. Let $V = \{(w+x, w+y, y+z, x+z) \mid w, x, y, z \in \mathbb{R}\}$ and $S = \{(1, 1, 0, 0), (1, 0, -1, 0), (0, -1, 0, 1)\}$.

   (a) Show that V is a subspace of $\mathbb{R}^4$ by writing it as a linear span.

   (b) Show that S is a basis for V.

   (c) Find the coordinate vector of $\boldsymbol{u} = (1, 2, 3, 2)$ relative to S.

   (d) Find a vector $\boldsymbol{v}$ such that $(\boldsymbol{v})_S = (1, 3, -1)$.

   (a) $V = \text{span}\{(1, 1, 0, 0), (1, 0, 0, 1), (0, 1, 1, 0), (0, 0, 1, 1)\}$ and hence is a subspace of $\mathbb{R}^4$.

   (b)

$$\left(\begin{array}{cccc|c|c|c} 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c|c|c} 1 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{ccc|c|c|c|c} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c|c|c|c} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

   Thus we have shown that $\text{span}(S) = V$. It is also easy to check that S is linearly independent. So S is a basis for V.

   (c) $(4, -3, 2)$.

   (d) $(4, 2, -3, -1)$.

5. (All vectors in this question are written as column vectors.) Let $\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}$ be vectors in $\mathbb{R}^n$ and $\boldsymbol{P}$ is a square matrix of order n. Note that $\boldsymbol{Pu_1}, \boldsymbol{Pu_2}, \cdots, \boldsymbol{Pu_k}$ are also (column) vectors in $\mathbb{R}^n$.

   (a) Show that if $\boldsymbol{Pu_1}, \boldsymbol{Pu_2}, \cdots, \boldsymbol{Pu_k}$ are linearly independent, then $\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}$ are linearly independent.

   (b) Let us investigate the converse of (a). Suppose $\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}$ are linearly independent.

      (i) Show that if $\boldsymbol{P}$ is invertible, then $\boldsymbol{Pu_1}, \boldsymbol{Pu_2}, \cdots, \boldsymbol{Pu_k}$ are linearly independent.

(ii) If $\boldsymbol{P}$ is singular, are $\boldsymbol{Pu_1}, \boldsymbol{Pu_2}, \cdots, \boldsymbol{Pu_k}$ linearly independent?

(a) Note that

$$\begin{aligned} & c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \boldsymbol{0} \\ \Rightarrow \quad & \boldsymbol{P}(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}) = \boldsymbol{P0} \\ \Rightarrow \quad & c_1\boldsymbol{Pu_1} + c_2\boldsymbol{Pu_2} + \cdots + c_k\boldsymbol{Pu_k} = \boldsymbol{0}. \end{aligned}$$

Since $\boldsymbol{Pu_1}, \boldsymbol{Pu_2}, \ldots, \boldsymbol{Pu_k}$ are linearly independent, we conclude that $c_1 = 0$, $c_2 = 0$, $\ldots$, $c_k = 0$. Thus $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ are linearly independent.

(b) (i) Note that

$$\begin{aligned} & c_1\boldsymbol{Pu_1} + c_2\boldsymbol{Pu_2} + \cdots + c_k\boldsymbol{Pu_k} = \boldsymbol{0} \\ \Rightarrow \quad & \boldsymbol{P}(c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}) = \boldsymbol{0}. \\ \Rightarrow \quad & c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k} = \boldsymbol{0} \qquad \text{(because } \boldsymbol{P} \text{ is invertible).} \end{aligned}$$

Since $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ are linearly independent, we conclude that $c_1 = 0$, $c_2 = 0$, $\ldots$, $c_k = 0$. Thus $\boldsymbol{Pu_1}, \boldsymbol{Pu_2}, \ldots, \boldsymbol{Pu_k}$ are linearly independent.

(ii) No conclusion.

For example, let $\boldsymbol{u_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\boldsymbol{u_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. It is obvious that $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ are linearly independent.

If $\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then $\boldsymbol{Pu_1}$ and $\boldsymbol{Pu_2}$ are linearly independent.

If $\boldsymbol{P} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then $\boldsymbol{Pu_1}$ and $\boldsymbol{Pu_2}$ are linearly dependent.