DIMENSIONS PART II

Theorem

Let V be a vector space of dimension k and S a subset of V.

The following statements are equivalent:

1) S is a basis for V.

2) S is linearly independent and $|S| = k$.

3) S spans V and $|S| = k$.

How can we use this theorem?

How to use the theorem

Let V be a vector space of dimension k and S a subset of V.

Once we know the $\boxed{\text{dimension of } V \text{ is } k}$:

Any subset S of V with $\boxed{\text{exactly } k}$ linearly independent vectors will be a basis for V.

Any subset S of V with $\boxed{\text{exactly } k}$ vectors that spans V will be a basis for V.

Example

Show that $\boldsymbol{u}_1 = (2,0,-1), \boldsymbol{u}_2 = (4,0,7), \boldsymbol{u}_3 = (-1,1,4)$ form a basis for $\mathbb{R}^3$.

If we go by definition, we need to show that the $\boxed{3}$ vectors are linearly independent and spans $\mathbb{R}^3$.

But now we know $\dim(\mathbb{R}^3) = \boxed{3}$

So showing either linearly independence or span will do.

We will show that the three vectors are linearly independent.

Example

Show that $\boldsymbol{u}_1 = (2,0,-1), \boldsymbol{u}_2 = (4,0,7), \boldsymbol{u}_3 = (-1,1,4)$ are linearly independent.

$$a(2,0,-1) + b(4,0,7) + c(-1,1,4) = (0,0,0)$$

$$\begin{cases} 2a + 4b - c = 0 \\ c = 0 \\ -a + 7b + 4c = 0 \end{cases}$$

$\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ are linearly independent.

Since $\dim(\mathbb{R}^3) = 3$,

3 linearly independent vectors in $\mathbb{R}^3$ always form a basis for $\mathbb{R}^3$.

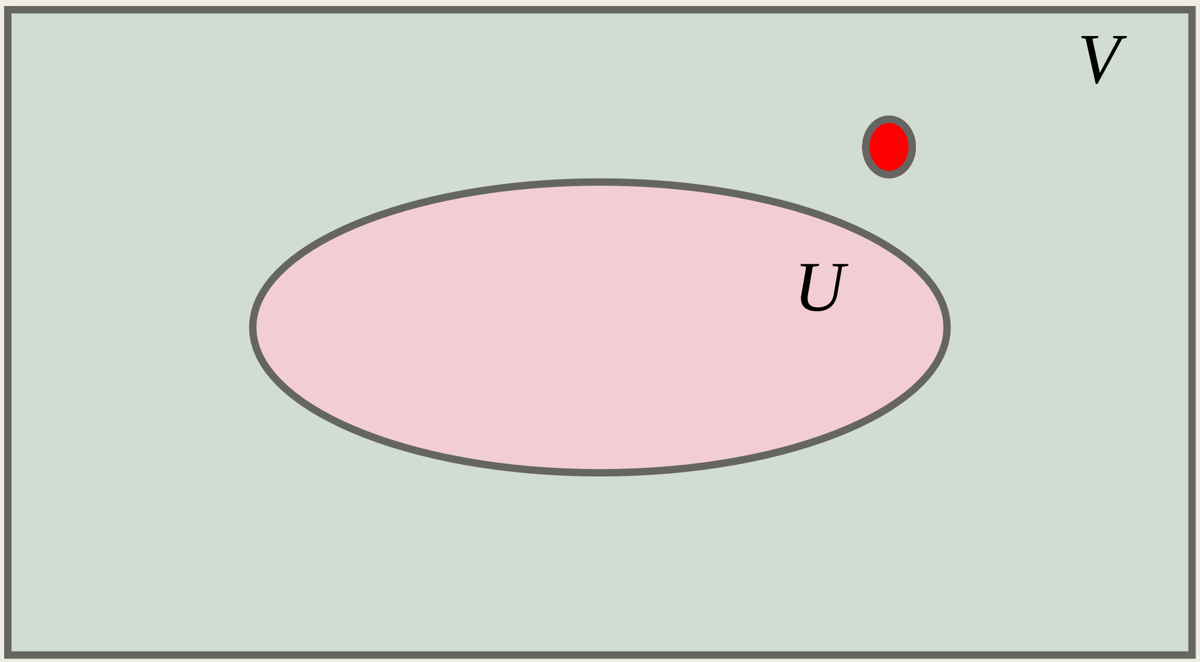
$$\left(\begin{array}{ccc|c} 2 & 4 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 7 & 4 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Theorem

Let U be a subspace of a vector space V.

Then $\boxed{\dim(U) \le \dim(V).}$

Furthermore, if $\underline{U \neq V}$, then $\boxed{\dim(U) < \dim(V).}$



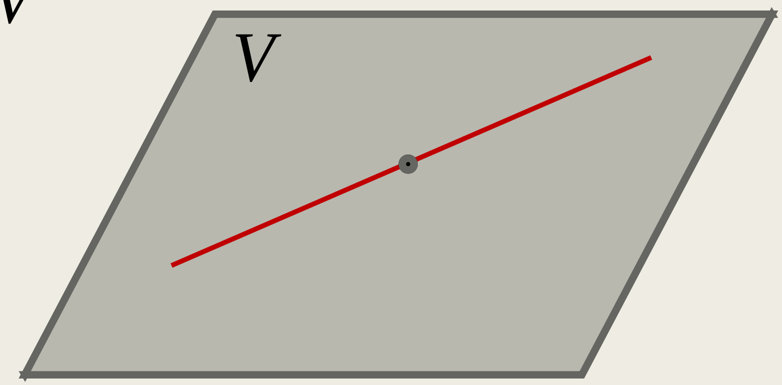
Example

Let V be a plane in $\mathbb{R}^3$ containing the origin.

Suppose U is a subspace of V

and $U \neq V$.

Then $\dim(U) < 2$.



$\dim(V) = 2$

If $\dim(U) = 0$

$\Rightarrow U$ is the zero subspace (that is, just the origin).

If $\dim(U) = 1$

$\Rightarrow U$ is a straight line passing through the origin.

Summary

1) Knowing the dimension of a vector space V helps in determining whether a set S is a basis for V.

2) The dimension of all subspaces of a vector space V does not exceed the dimension of V.

3) The only subspace of a vector space V that has the same dimension as V is V itself.