W06-07

Slide 01: In this unit, we will discuss a special category of bases.

Slide 02: Recall that in an earlier unit, we have seen what is meant when we say two vectors are orthogonal. A set can also be orthogonal if the vectors in the set are pairwise orthogonal. Finally, an orthogonal set is said to be orthonormal if every vector in the set is a unit vector.

Slide 03: The standard basis for $\mathbb{R}^n$ is clearly an orthonormal set since
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every pair of vectors in the basis are orthogonal because $\boldsymbol{e_i} \cdot \boldsymbol{e_j}$ is zero whenever $i \neq j$.
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In addition, each vector $\boldsymbol{e_i}$ is clearly a unit vector, that is, $\boldsymbol{e_i}$ has length 1.

Slide 04: This theorem establishes the result that an orthogonal set of vectors, as long as none of the vectors is the zero vector, will always be a linearly independent set.
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To prove this, let S be an orthogonal set of non zero vectors, consisting of vectors $\boldsymbol{u_1}, \boldsymbol{u_2}$ to $\boldsymbol{u_k}$.
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To show that S is a linearly independent set, consider the vector equation $(*)$. We will show that $(*)$ has only the trivial solution for the coefficients c_1 to c_k.
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From equation $(*)$, we will take the dot product with vector $\boldsymbol{u_i}$ on both sides. Note that i can be any integer from 1 to k.
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After applying distributive law, the left hand side of the expression simplifies to only one term because for all the other terms when $i \neq j$, the dot product between $\boldsymbol{u_i}$ and $\boldsymbol{u_j}$ is zero. So on the left hand side we have the term c_i times the dot product of $\boldsymbol{u_i}$ with itself. On the right hand side, we have the scalar zero since any vector dot product with the zero vector gives the scalar zero.
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Now this equation implies that the coefficient c_i must be zero since $\boldsymbol{u_i}$ is not the zero vector means that the dot product of $\boldsymbol{u_i}$ with itself cannot be zero, so this would mean that c_i must be zero. As the conclusion that $c_i = 0$ holds for any i from 1 to k, we have shown that vector equation $(*)$ has only the trivial solution. This establishes that S is indeed a linearly independent set.

Slide 05: We have already seen what is a basis for a vector space. Since a basis is just a set of vectors, it makes sense for us to define what is an orthogonal basis, which is simply a basis that is an orthogonal set.
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Likewise, an orthonormal basis is a basis that is an orthonormal set.

Slide 06: How can we determine if a given set of vectors is an orthogonal basis for a vector space V? Once again, if we know the dimension of V, our work would be made much easier. Suppose the dimension of V is known to be k.

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Now if S is an orthogonal set of non zero vectors belonging to V,

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note that such a set is a linearly independent set by the theorem we have proven earlier in this unit.

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In addition, S has the correct number of vectors, namely, k;

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Then we can now conclude that S is an orthogonal basis for V.

Slide 07: Consider the following example where S contains three vectors $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ and $\boldsymbol{u_3}$. It is easy to check that S is an orthogonal set of non zero vectors. Clearly, $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ and $\boldsymbol{u_3}$ are three vectors in $\mathbb{R}^3$.

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Since the dimension of $\mathbb{R}^3$ is 3, which matches with how many vectors we have in S,

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we can conclude that S is an orthogonal basis for $\mathbb{R}^3$.

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Furthermore, if we normalise the vectors in S by dividing each vector in S by their respective length, we will have three vectors $\boldsymbol{v_1}$, $\boldsymbol{v_2}$ and $\boldsymbol{v_3}$

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that will form an orthonormal basis for $\mathbb{R}^3$.

Slide 08: Now consider this set S with three vectors in $\mathbb{R}^4$. Suppose S is a basis for some subspace V of $\mathbb{R}^4$.

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Then for any vector $\boldsymbol{w}$ in V, we can certainly write $\boldsymbol{w}$ as a linear combination of $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ and $\boldsymbol{u_3}$. How do we go about doing it?

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Well, of course we need to write down the following vector equation and try to solve for the coefficients a, b and c.

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The vector equation would look like this,

Slide 09: and when we write down the equivalent linear system, we have one with 3 unknowns and 4 equations.

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Solving the linear system can now proceed via Gaussian elimination.

Slide 10: However, if we take another look at the set S, it turns out that it is actually an orthogonal set. This means that S is in fact an orthogonal basis for the subspace V. Is there anything special about S that will make the writing of $\boldsymbol{w}$ in terms of $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ and $\boldsymbol{u_3}$ easier?

Slide 11: There certainly is. We will present it in the form of the following theorem. Let S be an orthogonal basis for a vector space V. Then for any vector $\boldsymbol{w}$ in V,

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$\boldsymbol{w}$ can be written as a linear combination of $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$ in the following way.

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Notice that the coefficients of $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$ in the expression are all expressible as a ratio of some dot product divided by the square of length of a vector. This means that these coefficients can be obtained without having to solve any linear systems.

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This is indeed convenient and is one of the reason why orthogonal bases are easy to use.

Slide 12: Recall that the coordinate vector with respect to a basis is formed when we arrange the coefficients in an ordered manner. Therefore, the coordinate vector of $\boldsymbol{w}$ relative to the basis S is as shown.

Slide 13: What if T is an orthonormal basis for V? Note that an orthonormal basis is an orthogonal basis with the additional requirement that each vector in the basis is of length 1. Thus it should be no surprise that the expression of $\boldsymbol{w}$ in terms of the orthonormal basis vectors is derived from the previous one but with the denominators in each of the coefficients equal to 1. The coordinate vector of $\boldsymbol{w}$ relative to the orthonormal basis T is as shown, where each component is just the dot product between $\boldsymbol{w}$ and the respective vectors in T.

Slide 14: As an example, consider the set S which we have already shown to be an orthogonal basis for $\mathbb{R}^3$. We wish to express the vector $\boldsymbol{w} = (1, 2, 3)$ as a linear combination of the orthogonal basis vectors $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ and $\boldsymbol{u_3}$.

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The coefficient of $\boldsymbol{u_1}$ in the expression would be $\boldsymbol{w} \cdot \boldsymbol{u_1}$ divided by the square of the length of $\boldsymbol{u_1}$. This gives us $\frac{1}{2}$.

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The coefficent of $\boldsymbol{u_2}$ in the expression would be $\boldsymbol{w} \cdot \boldsymbol{u_2}$ divided by the square of the length of $\boldsymbol{u_2}$, which gives $\frac{5}{2}$.

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Likewise the coefficient of $\boldsymbol{u_3}$ can be found to be $-\frac{1}{2}$.

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Thus $\boldsymbol{w}$ is $\frac{1}{2}\boldsymbol{u_1} + \frac{5}{2}\boldsymbol{u_2} - \frac{1}{2}\boldsymbol{u_3}$

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and the coordinate vector $(\boldsymbol{w})_S$ follows immediately.

Slide 15: To summarise the main points in this unit,

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We first proved that an orthogonal set of non zero vectors is always a linearly independent set.

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We defined what is an orthogonal and orthonormal basis for a vector space.

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Lastly, we found out that writing linear combinations in terms of orthogonal or orthonormal basis vectors can be done conveniently without solving linear system.