Week 10

MA1508E LINEAR ALGEBRA FOR ENGINEERING

Week 09

IVLE Quiz Discussion

Review of Week 09 (Units 045-049) content

- How to find a basis for the column space of ***A***? The transpose method.
- Column space s are not preserved under row operations.
- Correspondence between linear independent columns in two row equivalent matrices.
- Correspondence between basis column in two row equivalent matrices.
- Using the result above, we can find a basis for the column space of ***A***.
- Something in common between the row space and column space of ***A.***
- Rank of a matrix. What it means for a matrix to be "full rank".
- One more equivalent statement to "***A*** is an invertible square matrix of order *n*"
- Relationship between the column space of ***A*** and the consistency of the linear system ***Ax*** = ***b***. Consistency in terms of ranks.

Review of Week 09 (Units 045-049) content

- Nullspace of $\boldsymbol{A}$. This is essentially the solution space of $\boldsymbol{Ax} = \boldsymbol{0}$
- Nullity of a matrix and the Dimension Theorem for matrices.
- Extending a set to a basis (using row space method)
- Relationship between general solution of $\boldsymbol{Ax} - \boldsymbol{b}$ and the nullspace of $\boldsymbol{A}$.
- Recall what is an orthogonal/orthonormal set. Orthogonal sets are linearly independent.
- Since bases are just sets, we can have orthogonal/orthonormal bases.
- How to check if a set is an orthogonal basis (if we know the dimension of the space)?
- Revisit: writing a vector as linear combination of a set of basis vectors.
- What if the basis is an orthogonal/orthonormal bases? Writing a vector as a linear combination of a set of orthogonal basis vectors.

Week 10 (units 050-054) overview

050 Orthogonal projection

- What it means for a vector to be orthogonal to a vector space
- Definition of orthogonal projection onto a vector space

051 Computing orthogonal projection

- How to compute orthogonal projection onto a vector space V provided we have an orthogonal/orthonormal basis for V.

052 Gram-Schmidt Process

- Using orthogonal projection, Gram-Schmidt Process is a procedure to convert a basis into an orthogonal basis.

Week 10 (units 050-054) overview

053 Best approximation

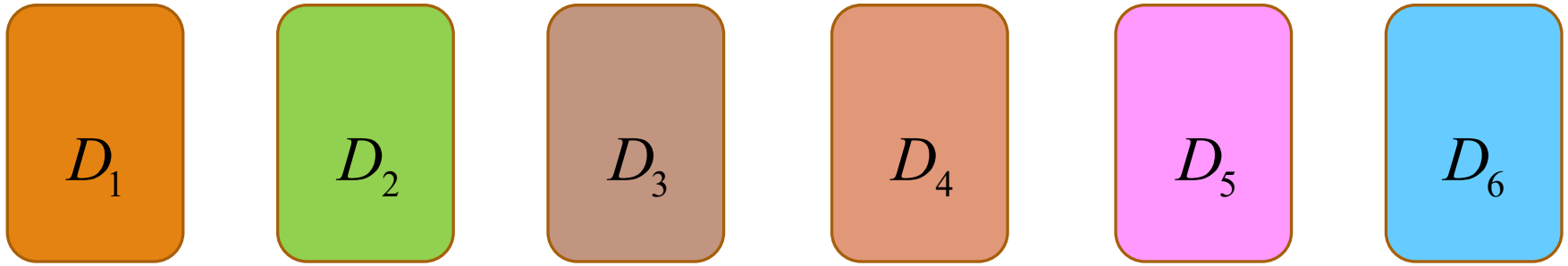
- If V is a subspace of $\mathbf{R}^n$ and $\boldsymbol{u}$ is a vector in $\mathbf{R}^n$, then the projection of $\boldsymbol{u}$ onto V is the best approximation of $\boldsymbol{u}$ in V.
- Definition of a least squares solution to a linear system $\boldsymbol{Ax} = \boldsymbol{b}$

054 Least squares solution to a linear system

- A least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$ is given by the vector $\boldsymbol{u}$ that satisfies $\boldsymbol{Au} = \boldsymbol{p}$ where $\boldsymbol{p}$ is the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$.
- Finding a least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$ by solving the normal equation $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$

How “close” are two vectors? (Document search)

Problem: Suppose there are six documents that we would like to search through and determine which one of the documents is 'closest' to my search criteria, which is defined in terms of a few keywords.



Five keywords we may want to look for in our documents:

$$k_1, k_2, k_3, k_4, k_5.$$

How "close" are two vectors? (Document search)

Approach: We represent each document D_j by a vector $\boldsymbol{d}_j \in \sim^5$ (because there are 5 keywords we are interested in).

D_1 D_2 D_3 D_4 D_5 D_6

$$\boldsymbol{d}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Document D_1 contains k_1, k_2, k_4, k_5 but not k_3

Five keywords we may want to look for: k_1, k_2, k_3, k_4, k_5.

How "close" are two vectors? (Document search)

Approach: We represent each document D_j by a vector $\boldsymbol{d}_j \in \sim^5$ (because there are 5 keywords we are interested in).

D_1 D_2 D_3 D_4 D_5 D_6

$$\boldsymbol{d}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad \boldsymbol{d}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \boldsymbol{d}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \boldsymbol{d}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \boldsymbol{d}_5 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad \boldsymbol{d}_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

How “close” are two vectors? (Document search)

What if we have 207842147823 documents and 12462247621 keywords?

D_1 D_2 D_3 D_4 D_5 D_6

$$\boldsymbol{d}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad \boldsymbol{d}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \boldsymbol{d}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \boldsymbol{d}_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \boldsymbol{d}_5 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad \boldsymbol{d}_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

NOT EASY (by inspection)!

How "close" are two vectors? (Document search)

Let $\boldsymbol{Q}$ be a 5×6 matrix where $\boldsymbol{d}_j, j=1,2,\ldots,6$ is the jth column of the matrix $\boldsymbol{Q}$.

Since we are interested in keywords k_1, k_4 and k_5, let the search vector

$$\begin{array}{c} \begin{array}{cccccc} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \boldsymbol{d}_3 & \boldsymbol{d}_4 & \boldsymbol{d}_5 & \boldsymbol{d}_6 \end{array} \\ \boldsymbol{Q} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array} \qquad \boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

How “close” are two vectors? (Document search)

Note that $\boldsymbol{d}_j \cdot \boldsymbol{x}$ is the number of 'matches' there are in document D_j.

How do we put these information together systematically?

$$\boldsymbol{Q} = \begin{array}{c} \begin{array}{cccccc} \boldsymbol{d}_1 & \boldsymbol{d}_2 & \boldsymbol{d}_3 & \boldsymbol{d}_4 & \boldsymbol{d}_5 & \boldsymbol{d}_6 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \end{array} \qquad \boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\boldsymbol{d}_1 \cdot \boldsymbol{x} = 3$$

$$\boldsymbol{d}_6 \cdot \boldsymbol{x} = 2$$

How "close" are two vectors? (Document search)

Let $\boldsymbol{y} = \boldsymbol{Q}^T \boldsymbol{x}$.

$$\boldsymbol{Q}^T \boldsymbol{x} = \begin{array}{c} \boldsymbol{d}_1 \\ \\ \\ \\ \\ \\ \end{array} \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 1 \\ 3 \\ 2 \end{pmatrix} = \boldsymbol{y}$$

Conclusion:

D_1 and D_5 matches our search.

Question:

How can we distinguish between D_1 and D_5?

First, we need more information.

How “close” are two vectors? (Document search)

Suppose the entries in each $\boldsymbol{d}_j$ tells us how many times each keyword actually appears in document D_j :

D_1 D_2 D_3 D_4 D_5 D_6

$$\boldsymbol{d}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \quad \boldsymbol{d}_2 = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \boldsymbol{d}_3 = \begin{pmatrix} 0 \\ 1 \\ 5 \\ 0 \\ 0 \end{pmatrix} \quad \boldsymbol{d}_4 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \boldsymbol{d}_5 = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 5 \\ 1 \end{pmatrix} \quad \boldsymbol{d}_6 = \begin{pmatrix} 0 \\ 0 \\ 6 \\ 1 \\ 4 \end{pmatrix}$$

How “close” are two vectors? (Document search)

Note that if the angle between $\boldsymbol{d}_j$ and the search vector $\boldsymbol{x}$ is θ_j, then

$$\cos\theta_j = \frac{\boldsymbol{d}_j \cdot \boldsymbol{x}}{\|\boldsymbol{d}_j\|\|\boldsymbol{x}\|} = \frac{\boldsymbol{d}_j}{\|\boldsymbol{d}_j\|} \cdot \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$$

Idea: Let's divide each $\boldsymbol{d}_j$ and $\boldsymbol{x}$ by their respective lengths.

$$\boldsymbol{Q}' = \begin{pmatrix} \frac{\boldsymbol{d}_1}{\|\boldsymbol{d}_1\|} & \frac{\boldsymbol{d}_2}{\|\boldsymbol{d}_2\|} & \frac{\boldsymbol{d}_3}{\|\boldsymbol{d}_3\|} & \frac{\boldsymbol{d}_4}{\|\boldsymbol{d}_4\|} & \frac{\boldsymbol{d}_5}{\|\boldsymbol{d}_5\|} & \frac{\boldsymbol{d}_6}{\|\boldsymbol{d}_6\|} \end{pmatrix}$$

$$\boldsymbol{x}' = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$$

How "close" are two vectors? (Document search)

Note that if the angle between $\boldsymbol{d}_j$ and the search vector $\boldsymbol{x}$ is θ_j, then

Let $\boldsymbol{y}' = \boldsymbol{Q}'^T \boldsymbol{x}'$

$$\cos\theta_j = \frac{\boldsymbol{d}_j \cdot \boldsymbol{x}}{\|\boldsymbol{d}_j\|\|\boldsymbol{x}\|} = \frac{\boldsymbol{d}_j}{\|\boldsymbol{d}_j\|} \cdot \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$$

cosine of angle between $\boldsymbol{d}_1$ and $\boldsymbol{x}$

$$\boldsymbol{Q}'^T \boldsymbol{x}' = \begin{pmatrix} \frac{\boldsymbol{d}_1}{\|\boldsymbol{d}_1\|} \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} \begin{pmatrix} \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \end{pmatrix} = \begin{pmatrix} \frac{\boldsymbol{d}_1}{\|\boldsymbol{d}_1\|} \cdot \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \cos\theta_1 \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}$$

How "close" are two vectors? (Document search)

$\cos\theta = 1 \Rightarrow$ angle between the two vectors is 0

$\Rightarrow$ we look for j such that $\cos\theta_j$ is as close to 1 as possible.

$$\cos\theta_j = \frac{\boldsymbol{d}_j \cdot \boldsymbol{x}}{\|\boldsymbol{d}_j\|\|\boldsymbol{x}\|} = \frac{\boldsymbol{d}_j}{\|\boldsymbol{d}_j\|} \cdot \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$$

cosine of angle between $\boldsymbol{d}_1$ and $\boldsymbol{x}$

$$\boldsymbol{Q}'^T \boldsymbol{x}' = \begin{pmatrix} \boxed{\dfrac{\boldsymbol{d}_1}{\|\boldsymbol{d}_1\|}} \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} \begin{pmatrix} \boxed{\dfrac{\boldsymbol{x}}{\|\boldsymbol{x}\|}} \end{pmatrix} = \begin{pmatrix} \dfrac{\boldsymbol{d}_1}{\|\boldsymbol{d}_1\|} \cdot \dfrac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \\ \vdots \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \cos\theta_1 \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}$$

How “close” are two vectors? (Document search)

For our example:

So the conclusion is $\boldsymbol{d}_1$ is 'closer' to $\boldsymbol{x}$ compared to $\boldsymbol{d}_5$.

$$\boldsymbol{y}' = \boldsymbol{Q}'^T \boldsymbol{x}' = \begin{pmatrix} \boxed{0.9130} \\ 0.1826 \\ 0 \\ 0.2582 \\ \boxed{0.8784} \\ 0.6868 \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \qquad \boldsymbol{d}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \qquad \boldsymbol{d}_5 = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 5 \\ 1 \end{pmatrix}$$

Do you know why?

Example 9.1

Let W be the subspace of $\mathbb{R}^4$ spanned by the following vectors

$\boldsymbol{u}_1 = (1,0,1,1) \qquad \boldsymbol{u}_2 = (1,-1,0,2) \qquad \boldsymbol{u}_3 = (1,2,3,-1).$

(a) Find the set U, containing all vectors that are orthogonal to W.

(b) Is U a subspace of $\mathbb{R}^4$? Is this true in general?

Example 9.2

Use Gram-Schmidt Process to transform $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ into an orthonormal basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ for $\mathbb{R}^3$ where

$$\boldsymbol{u}_1 = (1,1,1) \qquad \boldsymbol{u}_2 = (1,-1,1) \qquad \boldsymbol{u}_3 = (1,1,-1)$$

Example 9.3

Consider the following linear system

$$\begin{cases} x + y + z = 1 \\ \quad\;\; y + z = 1 \\ x - y - z = 1 \\ \qquad\quad\;\; z = 1 \end{cases}$$

(a) Show that the linear system is inconsistent.

(b) Find a least squares solution by using orthogonal projection.

(c) Find a least squares solution by solving the normal equation.

Finally…

THE END