LINEAR SPAN II

DISCUSSION

Suppose $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\} \subseteq \mathbb{R}^n$. We want to determine if $\text{span}(S) = \mathbb{R}^n$.

$$\boldsymbol{u}_1 = (a_{11}, a_{12}, \dots, a_{1n}) \quad \boldsymbol{u}_2 = (a_{21}, a_{22}, \dots, a_{2n}) \quad \dots \quad \boldsymbol{u}_k = (a_{k1}, a_{k2}, \dots, a_{kn})$$

For any $\boldsymbol{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$, consider the equation:

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \dots + c_k\boldsymbol{u}_k = \boldsymbol{v}$$

$$c_1(a_{11}, a_{12}, \dots, a_{1n}) + c_2(a_{21}, a_{22}, \dots, a_{2n}) + \dots + c_k(a_{k1}, a_{k2}, \dots, a_{kn})$$
$$= (v_1, v_2, \dots, v_n)$$

DISCUSSION

Suppose $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}\subseteq\mathbb{R}^n$. We want to determine if span$(S)=\mathbb{R}^n$.

$$c_1\boldsymbol{u}_1+c_2\boldsymbol{u}_2+\ldots+c_k\boldsymbol{u}_k=\boldsymbol{v}$$

$$c_1(a_{11},a_{12},\ldots,a_{1n})+c_2(a_{21},a_{22},\ldots,a_{2n})+\ldots+c_k(a_{k1},a_{k2},\ldots,a_{kn})$$
$$=(v_1,v_2,\ldots,v_n)$$

$$\begin{cases} a_{11}c_1 & + & a_{21}c_2 & + & \ldots & + & a_{k1}c_k & = & v_1 \\ a_{12}c_1 & + & a_{22}c_2 & + & \ldots & + & a_{k2}c_k & = & v_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{1n}c_1 & + & a_{2n}c_2 & + & \ldots & + & a_{kn}c_k & = & v_n \end{cases}$$

DISCUSSION

Suppose $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\} \subseteq \mathbb{R}^n$. We want to determine if $\text{span}(S) = \mathbb{R}^n$.

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \dots + c_k\boldsymbol{u}_k = \boldsymbol{v} \quad \boxed{(*)}$$

$$c_1(\boxed{a_{11}, a_{12}, \dots, a_{1n}}) + c_2(\boxed{a_{21}, a_{22}, \dots, a_{2n}}) + \dots + c_k(\boxed{a_{k1}, a_{k2}, \dots, a_{kn}})$$
$$= (v_1, v_2, \dots, v_n)$$

$$\begin{array}{cc} & k \text{ columns} \\ \begin{array}{c} n \\ \text{rows} \end{array} & \left(\begin{array}{c|c} \boldsymbol{A} & \begin{matrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{matrix} \end{array}\right) \end{array}$$

If a row-echelon form of $\boldsymbol{A}$ does not have a zero row, $\boxed{(*)}$ is always consistent regardless of $\boldsymbol{v}$

$$\boxed{\Rightarrow \text{span}(S) = \mathbb{R}^n}$$

DISCUSSION

Suppose $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\} \subseteq \mathbb{R}^n$. We want to determine if $\text{span}(S) = \mathbb{R}^n$.

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \dots + c_k\boldsymbol{u}_k = \boldsymbol{v} \quad (*)$$

$$c_1(a_{11}, a_{12}, \dots, a_{1n}) + c_2(a_{21}, a_{22}, \dots, a_{2n}) + \dots + c_k(a_{k1}, a_{k2}, \dots, a_{kn})$$
$$= (v_1, v_2, \dots, v_n)$$

$$\begin{array}{cc} & k \text{ columns} \\ n \text{ rows} & \left(\begin{array}{ccccc|c} & & & & & v_1 \\ & & \boldsymbol{A} & & & v_2 \\ & & & & & \vdots \\ 0 & 0 & \dots & 0 & 0 & v_n \end{array}\right) \end{array}$$

If a row-echelon form of $\boldsymbol{A}$ has at least one zero row,

(*) is not always consistent

$\Rightarrow \text{span}(S) \neq \mathbb{R}^n$

EXAMPLE

From earlier example:

Show that span$\{(1,0,1),(1,1,0),(0,1,1)\} \boxed{= \mathbb{R}^3.}$

We need to show that every vector in $\mathbb{R}^3$ can be written as a linear combination of $(1,0,1),(1,1,0),(0,1,1)$.

$$a(1,0,1)+b(1,1,0)+c(0,1,1)=(x,y,z)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 1 & 1 & y \\ 1 & 0 & 1 & z \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 0 & 1 & 1 & y \\ 0 & 0 & 2 & z-x+y \end{array}\right)$$

No zero row

EXAMPLE

From earlier example:

Show that $\text{span}\{(1,1,1),(1,2,0),(2,1,3),(2,3,1)\} \boxed{\neq \mathbb{R}^3.}$

We need to show that there is some vector in $\mathbb{R}^3$ that cannot be written as a linear combination of $(1,1,1),(1,2,0),(2,1,3),(2,3,1)$.

$$a(1,1,1)+b(1,2,0)+c(2,1,3)+d(2,3,1)=(x,y,z)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & x \\ 1 & 2 & 1 & 3 & y \\ 1 & 0 & 3 & 1 & z \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & 2 & 2 & x \\ 0 & 1 & -1 & 1 & y-x \\ 0 & 0 & 0 & 0 & y+z-2x \end{array}\right)$$

Has zero row

THEOREM

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a set of vectors in $\mathbb{R}^n$.

If $k < n$, then S cannot span $\mathbb{R}^n$.

> If $k < n$, then a row-echelon form of $\boldsymbol{A}$ has at least one zero row $\Rightarrow S$ cannot span $\mathbb{R}^n$.

Suppose $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} \subseteq \mathbb{R}^n$. We want to determine if span$(S) = \mathbb{R}^n$.

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k = \boldsymbol{v} \quad \boxed{(*)}$$

$$\begin{array}{cc} & k \text{ columns} \\ \begin{array}{c} n \\ \text{rows} \end{array} & \left(\begin{array}{ccccc|c} & & & & & v_1 \\ & & \boldsymbol{A} & & & v_2 \\ & & & & & \vdots \\ 0 & 0 & \ldots & 0 & 0 & v_n \end{array}\right) \end{array}$$

If a row-echelon form of $\boldsymbol{A}$ has at least one zero row, $\boxed{(*)}$ is not always consistent

EXAMPLE

1) One vector cannot span $\mathbb{R}^2$.

2) One or two vectors cannot span $\mathbb{R}^3$.

THEOREM

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} \subseteq \mathbb{R}^n$.

1) $\mathbf{0} \in$ span(S)

Proof:

Span(S) contains all linear combinations of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$.

In particular, it contains

$$0\boldsymbol{u}_1 + 0\boldsymbol{u}_2 + \ldots + 0\boldsymbol{u}_k \quad = \mathbf{0}$$

THEOREM

Let $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}\subseteq\mathbb{R}^n$.

2) For any $\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_r\in\text{span}(S)$ and $c_1,c_2,\ldots,c_r\in\mathbb{R}$,

$$c_1\boldsymbol{v}_1+c_2\boldsymbol{v}_2+\ldots+c_r\boldsymbol{v}_r\in\text{span}(S).$$

Proof:

$$\boldsymbol{v}_1\in\text{span}(S)\Rightarrow\boldsymbol{v}_1=d_{11}\boldsymbol{u}_1+d_{12}\boldsymbol{u}_2+\ldots+d_{1k}\boldsymbol{u}_k$$

$$\boldsymbol{v}_2\in\text{span}(S)\Rightarrow\boldsymbol{v}_2=d_{21}\boldsymbol{u}_1+d_{22}\boldsymbol{u}_2+\ldots+d_{2k}\boldsymbol{u}_k$$

$$\vdots$$

$$\boldsymbol{v}_r\in\text{span}(S)\Rightarrow\boldsymbol{v}_r=d_{r1}\boldsymbol{u}_1+d_{r2}\boldsymbol{u}_2+\ldots+d_{rk}\boldsymbol{u}_k$$

THEOREM

Proof:

$$\boldsymbol{v}_1 \in \text{span}(S) \Rightarrow \boldsymbol{v}_1 = d_{11}\boldsymbol{u}_1 + d_{12}\boldsymbol{u}_2 + \ldots + d_{1k}\boldsymbol{u}_k$$

$$\boldsymbol{v}_2 \in \text{span}(S) \Rightarrow \boldsymbol{v}_2 = d_{21}\boldsymbol{u}_1 + d_{22}\boldsymbol{u}_2 + \ldots + d_{2k}\boldsymbol{u}_k$$

$$\vdots$$

$$\boldsymbol{v}_r \in \text{span}(S) \Rightarrow \boldsymbol{v}_r = d_{r1}\boldsymbol{u}_1 + d_{r2}\boldsymbol{u}_2 + \ldots + d_{rk}\boldsymbol{u}_k$$

$$\begin{aligned} c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_r\boldsymbol{v}_r = {} & c_1(d_{11}\boldsymbol{u}_1 + d_{12}\boldsymbol{u}_2 + \ldots + d_{1k}\boldsymbol{u}_k) \\ & + c_2(d_{21}\boldsymbol{u}_1 + d_{22}\boldsymbol{u}_2 + \ldots + d_{2k}\boldsymbol{u}_k) \\ & + \ldots + c_r(d_{r1}\boldsymbol{u}_1 + d_{r2}\boldsymbol{u}_2 + \ldots + d_{rk}\boldsymbol{u}_k) \end{aligned}$$

$$= (c_1d_{11} + c_2d_{21} + \ldots + c_rd_{r1})\boldsymbol{u}_1 + (c_1d_{12} + c_2d_{22} + \ldots + c_rd_{r2})\boldsymbol{u}_2$$

$$+ \ldots + (c_1d_{1k} + c_2d_{2k} + \ldots + c_rd_{rk})\boldsymbol{u}_k \in \text{span}(S)$$

WHAT DOES IT MEAN?

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} \subseteq \mathbb{R}^n$.

1) $\mathbf{0} \in \text{span}(S)$

$\mathbf{0}$ is always an element in span(S). In other words, if a set does not contain $\mathbf{0}$, then it cannot be a linear span.

2) For any $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_r \in \text{span}(S)$ and $c_1, c_2, \ldots, c_r \in \mathbb{R}$,

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_r\boldsymbol{v}_r \in \text{span}(S).$$

span(S) (any linear span) is "closed" under linear combinations.

SUMMARY

1) Detailed discussion on checking if span$(S) = \mathbb{R}^n$.

2) We can never span $\mathbb{R}^n$ with less than n vectors.

3) Two properties of span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$.