WRITING SOLUTIONS FROM ROW-ECHELON FORMS

WHAT CAN ROW-ECHELON FORM TELL US?

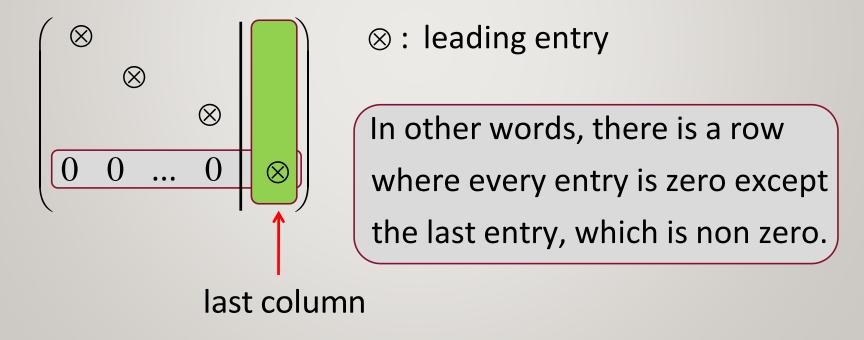
Remember that any linear system has either

- (i) no solution (that is, inconsistent); or
- (ii) exactly one solution (that is, an unique solution); or
- (iii) infinitely many solutions.

By looking at a row-echelon form of the augmented matrix of the linear system, we can determine which of the above holds for the linear system.

INCONSISTENT LINEAR SYSTEMS

If the augmented matrix of a linear system has a row-echelon form whose last column is a pivot column, then the linear system is inconsistent.



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If the augmented matrix of a linear system has a row-echelon form whose last column is a pivot column, then the linear system is inconsistent.

$$\begin{cases} x & - & y & = 2 \\ & y & = 0 \\ 0x & + & 0y & + & 0z & = 2 \end{cases} \qquad \begin{pmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

CONSISTENT LINEAR SYSTEMS

If the augmented matrix of a linear system has a row-echelon form whose last column is NOT a pivot column, then the linear system is consistent.

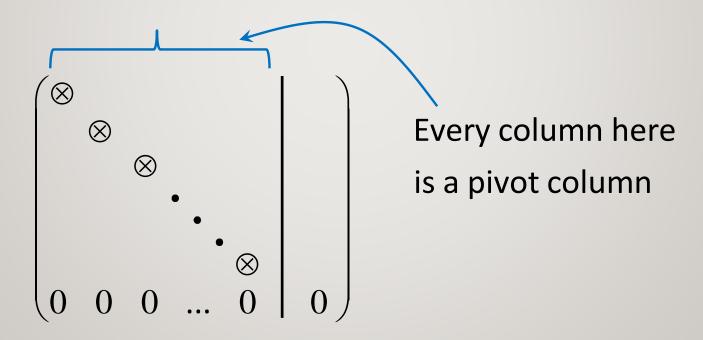
Consistent = Only one (unique) solution

OR

Infinitely many solutions?

CONSISTENT LINEAR SYSTEMS (UNIQUE SOLUTION)

If the augmented matrix of a consistent linear system has a row-echelon form where every column (except the last) is a pivot column, then the linear system has a unique (that is, exactly one) solution.



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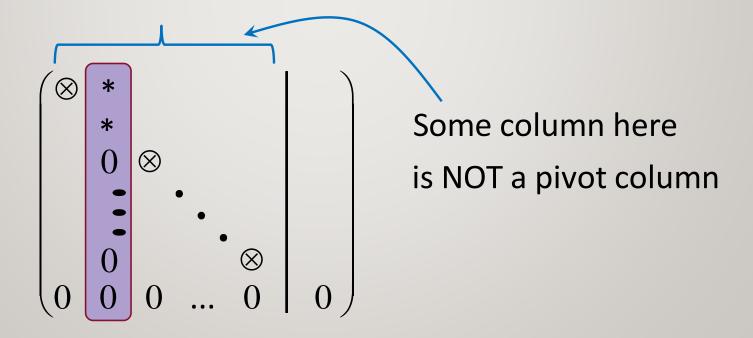
$$\begin{bmatrix}
-1 & 0 & 1 & 4 & | & 0 \\
0 & 1 & -1 & 3 & | & 1 \\
0 & 0 & 3 & 2 & | & -1 \\
0 & 0 & 0 & 1 & | & 0 \\
0 & 0 & 0 & 0 & | & 0
\end{bmatrix}$$

$$\begin{bmatrix}
-w & + & y & + & 4z & = & 0 \\
x & - & y & + & 3z & = & 1 \\
3y & + & 2z & = & -1 \\
z & = & 0$$
Back substitution

 $z = 0 \rightarrow$ solve for $y \rightarrow$ solve for $x \rightarrow$ solve for w

CONSISTENT LINEAR SYSTEMS (INFINITELY MANY SOLUTIONS)

If the augmented matrix of a consistent linear system has a row-echelon form where some column (other than the last) is NOT a pivot column, then the linear system has infinitely many solutions.



CONSISTENT LINEAR SYSTEMS (INFINITELY MANY SOLUTIONS)

If the augmented matrix of a consistent linear system has a row-echelon form where some column (other than the last) is NOT a pivot column, then the linear system has infinitely many solutions.

the general solution?

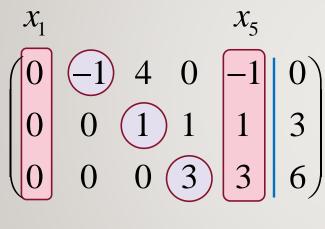
5 variables: x_1, x_2, x_3, x_4, x_5

Writing out the linear system:

$$\begin{cases} -x_2 + 4x_3 & -x_5 = 0 \\ x_3 + x_4 + x_5 = 3 \\ 3x_4 + 3x_5 = 6 \end{cases}$$

What happened to x_1 ?

For any 'variable' column (left side of vertical line) that is a non pivot column, we will assign an arbitrary parameter to that variable.



In this case, since x_1 does not appear in any of the equations, none of the variables will 'depend' on the value of x_1 .

$$\begin{cases} -x_2 + 4x_3 - x_5 = 0 \\ x_3 + x_4 + x_5 = 3 \\ 3x_4 + 3x_5 = 6 \end{cases}$$



Let $x_1 = s$, $x_5 = t$, where s, $t \in \mathbb{R}$.

$$\begin{cases}
-x_2 + 4x_3 - x_5 = 0 \\
x_3 + x_4 + x_5 = 3 \\
3x_4 + 3x_5 = 6
\end{cases}$$

Starting from the last equation:

$$3x_4 + 3t = 6 \Leftrightarrow 3x_4 = 6 - 3t \Leftrightarrow x_4 = 2 - t$$
.

Substitute into next 'higher' equation:

$$x_3 + (2-t) + t = 3 \Leftrightarrow x_3 = 1.$$

Substitute into next 'higher' equation:

$$-x_2 + 4(1) - t = 0 \Leftrightarrow x_2 = 4 - t$$
.

Let $x_1 = s$, $x_5 = t$, where s, $t \in \mathbb{R}$.

$$3x_4 + 3t = 6 \Leftrightarrow 3x_4 = 6 - 3t \Leftrightarrow x_4 = 2 - t$$
.

$$x_3 + (2-t) + t = 3 \iff x_3 = 1.$$

$$-x_2 + 4(1) - t = 0 \Leftrightarrow x_2 = 4 - t.$$

Back substitution

A general solution is:

Linear system has infinitely many solutions.

$$\begin{cases} x_1 &= s \\ x_2 &= 4-t \\ x_3 &= 1 \\ x_4 &= 2-t \\ x_5 &= t, \quad s, t \in \mathbb{R}. \end{cases}$$

5 variables: v, w, x, y, z

Writing out the linear system:

$$\begin{cases} v + w & = 0 \\ x + y & = -2 \\ z = 4 \end{cases}$$

Let w = s, y = t, $s, t \in \mathbb{R}$.

$$\begin{cases} v + w & = 0 \\ x + y & = -2 \\ z = 4 \end{cases}$$

Let w = s, y = t, s, $t \in \mathbb{R}$.

Starting from the last equation:

$$z = 4$$
.

$$x+t=-2 \Leftrightarrow x=-2-t$$
.

$$v + s = 0 \Leftrightarrow v = -s$$
.

substitution!

No back

A general
$$\begin{cases} x \\ x \end{cases}$$

$$z=4.$$
Next equation:
$$x+t=-2 \Leftrightarrow x=-2-t.$$
Next equation:
$$v+s=0 \Leftrightarrow v=-s.$$

$$x+t=-2 \Leftrightarrow x=-2-t.$$
A general solution is:
$$x+t=-2 \Leftrightarrow x=-2-t.$$

$$x=-2-t$$

$$y=t$$

$$z=4, \quad s,t\in\mathbb{R}.$$

SUMMARY

- 1) What row-echelon forms can tell us.
- 2) Last column is a pivot column inconsistent
- 3) Last column is non-pivot consistent.
- 4) Unique solution vs. Infinitely many solutions.
- 5) How to write down a general solution based on a (reduced) row-echelon form.