NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

SEMESTER II, 2017/2018
MA1508E MID-TERM TEST

Full Name : ______________________________

Matric/Student Number : ______________

Tutorial Group : __________

INSTRUCTIONS

PLEASE READ CAREFULLY

- Write your **full name, matric number and tutorial group** clearly above on this cover page.
- There are **4** questions printed on **3** pages. Answer **all** questions.
- You must show all your working clearly, failure to do so will result in marks deducted.
- Use pen for this test.
- All answers and working have to be written on the answer book provided.
- Start on a new page for each question.
- Tie this cover page (and question paper) together with your answer book before submission.

Question 1

(i) Let

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -2 & -1 \\ -2 & 1 & 6 & 7 \\ 3 & -2 & -5 & -3 \end{pmatrix}.$$

Find three elementary matrices $\boldsymbol{E_1}, \boldsymbol{E_2}, \boldsymbol{E_3}$ such that $\boldsymbol{E_3 E_2 E_1 A}$ is a matrix in row-echelon form.

(ii) Hence, solve the following linear system

$$\begin{cases} x & & - & 2z & = & -1 \\ -2x & + & y & + & 6z & = & 7 \\ 3x & - & 2y & - & 5z & = & -3 \end{cases}$$

(iii) **Use your answer in (ii)** to solve the following linear system

$$\begin{cases} 2a & & + & 4c & = & -1 \\ -4a & + & 3b & - & 12c & = & 7 \\ 6a & - & 6b & + & 10c & = & -3 \end{cases}$$

Warning: You should not solve the linear system directly (that is, perform any further elementary row operations).

(iv) Use your answer in (i) to write down the $\boldsymbol{LU}$ factorisation of $\begin{pmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{pmatrix}$. Hence solve the following linear system:

$$\begin{pmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix}.$$

Solutions:

(i)

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -2 & -1 \\ -2 & 1 & 6 & 7 \\ 3 & -2 & -5 & -3 \end{pmatrix} \underset{R_3 - 3R_1}{\xrightarrow{R_2 + 2R_1}} \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & -2 & 1 & 0 \end{pmatrix} \xrightarrow{R_3 + 2R_2} \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 5 & 10 \end{pmatrix} = \boldsymbol{R}.$$

So we have

$$\boldsymbol{E_1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{E_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{E_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}.$$

So $\boldsymbol{E_3 E_2 E_1 A} = \boldsymbol{R}$ which is in row-echelon form.

(ii) The augmented matrix of the linear system is $\boldsymbol{A}$. Thus from the row-echelon form $\boldsymbol{R}$, we have

$$5z = 10 \Leftrightarrow z = 2, \quad y + 2z = 5 \Leftrightarrow y = 5 - 2z = 1, \quad x - 2z = -1 \Leftrightarrow x = -1 + 2z = 3.$$

(iii) If we let $x = 2a, y = 3b, z = -2c$, the linear system in (iii) becomes the linear system in (ii). So the solution to the linear system in (iii) is

$$\begin{cases} 2a &= 3 \\ 3b &= 1 \\ -2c &= 2 \end{cases} \Leftrightarrow \begin{cases} a &= \frac{3}{2} \\ b &= \frac{1}{3} \\ c &= -1 \end{cases}$$

(iv)

$$\begin{aligned}
\begin{pmatrix} 1 & 0 & -2 \\ -2 & 1 & 6 \\ 3 & -2 & -5 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{pmatrix} \\
&= \boldsymbol{LU}
\end{aligned}$$

Let $\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$. Solving $\boldsymbol{Ly} = \boldsymbol{b}$ and $\boldsymbol{Ux} = \boldsymbol{y}$:

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -5 \end{pmatrix}$$

$$\Rightarrow y_1 = 2, -2y_1 + y_2 = 4 \Rightarrow y_2 = 8, 3y_1 - 2y_2 + y_3 = -5 \Rightarrow y_3 = 5.$$

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 5 \end{pmatrix}$$

$$\Rightarrow 5x_3 = 5 \Rightarrow x_3 = 1, x_2 + 2x_3 = 8 \Rightarrow x_2 = 6, x_1 - 2x_3 = 2 \Rightarrow x_1 = 4.$$

So the solution to the linear system is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 1 \end{pmatrix}$.

Question 2

(i) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be row equivalent, square matrices of order 3 such that

$$\boldsymbol{A} \xrightarrow{R_3 - 3R_1} \xrightarrow{R_2 \leftrightarrow R_3} \xrightarrow{\frac{1}{4}R_2} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \end{pmatrix} = \boldsymbol{B}.$$

Find $\det(\boldsymbol{A})$ and $\boldsymbol{A}$.

(ii) Find $\boldsymbol{B}^{-1}$ and write down $\operatorname{adj}(\boldsymbol{B})$.

(iii) If $\boldsymbol{C}$ is row equivalent to $\boldsymbol{A}$, then $\boldsymbol{C}$ must be invertible. Is this statement true or false? Justify your answer.

(iv) If $\boldsymbol{D}$ is row equivalent to $\boldsymbol{B}$ and $\det(\boldsymbol{D}) = 1$, then $\boldsymbol{D} = \boldsymbol{I_3}$. Is this statement true or false? Justify your answer.

Solutions:

(i) $\det(\boldsymbol{B}) = -5$.

$$1 \cdot \frac{1}{4} \cdot (-1) \cdot 1 \cdot \det(\boldsymbol{A}) = \det(\boldsymbol{B}).$$

So $\det(\boldsymbol{A}) = -5 \cdot -4 = 20$. To find $\boldsymbol{A}$:

$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \end{pmatrix} \xrightarrow{R_3 + R_2} \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & -4 \end{pmatrix} \xrightarrow{4R_2} \begin{pmatrix} 1 & -3 & 2 \\ 0 & 4 & 4 \\ 0 & 1 & -4 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 0 & 4 & 4 \end{pmatrix} \xrightarrow{R_3 + 3R_1} \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 3 & -5 & 10 \end{pmatrix} = \boldsymbol{A}.$$

(ii)

$$\left(\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -5 & 0 & 0 & 1 \end{array}\right) \xrightarrow{-\frac{1}{5}R_3} \left(\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} \end{array}\right)$$

$$\xrightarrow[R_2 - R_3]{R_1 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & -3 & 0 & 1 & 0 & \frac{2}{5} \\ 0 & 1 & 0 & 0 & 1 & \frac{1}{5} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} \end{array}\right) \xrightarrow{R_1 + 3R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 1 \\ 0 & 1 & 0 & 0 & 1 & \frac{1}{5} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{5} \end{array}\right)$$

So $\boldsymbol{B}^{-1} = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & -\frac{1}{5} \end{pmatrix}$. Since $\boldsymbol{B}^{-1} = \frac{1}{\det(\boldsymbol{B})}\operatorname{adj}(\boldsymbol{B})$, we have

$$\operatorname{adj}(\boldsymbol{B}) = -5 \begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & -\frac{1}{5} \end{pmatrix} = \begin{pmatrix} -5 & -15 & -5 \\ 0 & -5 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(iii) True, since $\det(\boldsymbol{A}) \neq 0$ implies that $\boldsymbol{A}$ is invertible. Since $\boldsymbol{C}$ is row equivalent to $\boldsymbol{A}$, then the reduced row-echelon form of $\boldsymbol{C}$ is also $\boldsymbol{I}$. Thus $\boldsymbol{C}$ is also invertible.

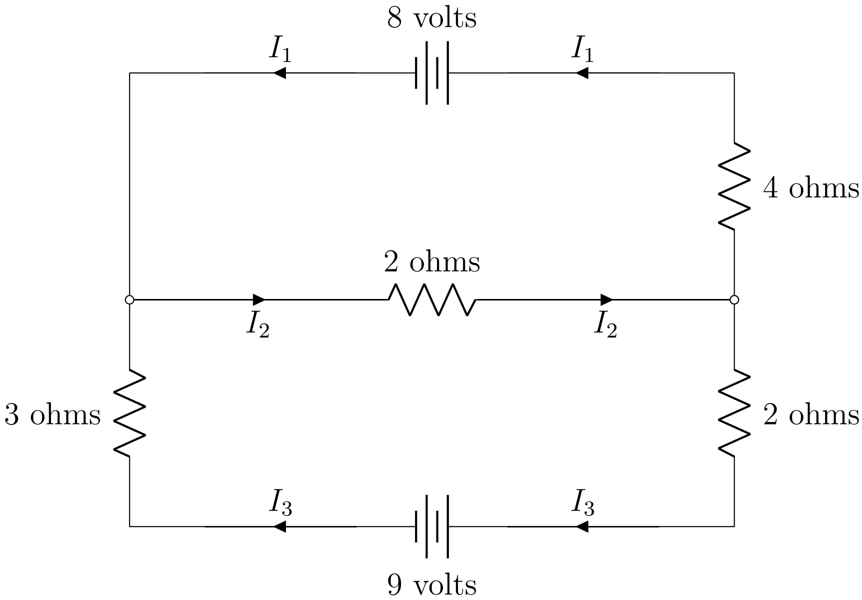
(iv) False. Let

$$\boldsymbol{D} = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $\boldsymbol{D}$ is row equivalent to $\boldsymbol{B}$ and $\det(\boldsymbol{D}) = 1$ but $\boldsymbol{D} \neq \boldsymbol{I}$.

Question 3

Consider the following electrical network.



(i) Write down a linear system with 3 equations involving unknowns I_1, I_2 and I_3 using KCL and KVL.

(ii) Solve the linear system using Gauss-Jordan Elimination.

Solutions:

(i) Note that direction of the currents I_1 and I_3 are in the opposite direction of what they are supposed to be since current flows out from the positive side of the battery. Thus, in order to use KVL and KCL, we need to make the following substitution $i_1 = -I_1$, $i_2 = -I_2$, $i_3 = -I_3$. Then the linear system (involving I_1, I_2, I_3) would be

$$\begin{cases} -I_1 + I_2 - I_3 = 0 \\ -4I_1 - 2I_2 = 8 \\ -2I_2 - 5I_3 = 9 \end{cases}$$

(ii)

$$\left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ -4 & -2 & 0 & 8 \\ 0 & -2 & -5 & 9 \end{array}\right) \xrightarrow{R_2 - 4R_1} \left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & -6 & 4 & 8 \\ 0 & -2 & -5 & 9 \end{array}\right) \xrightarrow{R_3 - \frac{1}{3}R_2} \left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & -6 & 4 & 8 \\ 0 & 0 & -\frac{19}{3} & \frac{19}{3} \end{array}\right)$$

So $I_3 = -1$, $-6I_2 + 4I_3 = 8 \Rightarrow I_2 = -2$, $-I_1 + I_2 - I_3 = 0 \Rightarrow I_1 = -1$.

Question 4

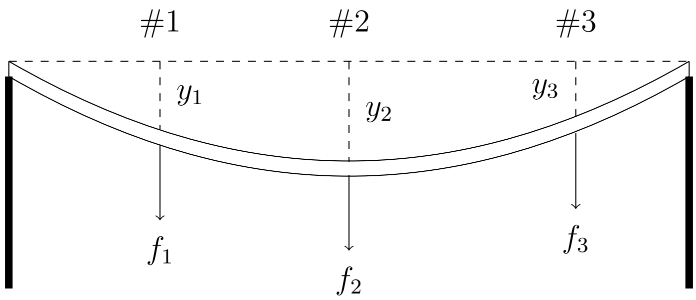
Let

$$\boldsymbol{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{v} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{w} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

(i) Find $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ such that

$$a_1\boldsymbol{u} + a_2\boldsymbol{v} + a_3\boldsymbol{w} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad b_1\boldsymbol{u} + b_2\boldsymbol{v} + b_3\boldsymbol{w} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad c_1\boldsymbol{u} + c_2\boldsymbol{v} + c_3\boldsymbol{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(ii) Recall the definition of a 3×3 *flexibility matrix* $\boldsymbol{D}$ used to study the flexibility of a horizontal elastic beam shown in the figure below.



We know that, by Hooke's law,

$$\boldsymbol{y} = \boldsymbol{D}\boldsymbol{f}$$

where $\boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$ represents the forces applied at the 3 points #1, #2 and #3 and $\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ represents the amount of deflection of the beam at the 3 points when it is subjected to $\boldsymbol{f}$.

Suppose we wish to determine the flexibility matrix $\boldsymbol{D}$ of an elastic beam. Three sets of experiments were conducted, where different units of force were applied at the 3 points and each time, the deflections were measured. The results of the experiment are given in the table below.

	Force applied			Deflection observed		
	f_1	f_2	f_3	y_1	y_2	y_3
Experiment 1	1	0	1	0.5	0.3	0.5
Experiment 2	0	1	2	0.1	0.3	0.7
Experiment 3	2	1	0	0.7	0.3	0.1

Use your answer in part (i) to find the flexibility matrix $\boldsymbol{D}$.

Solutions:

(i) We solve three linear systems:

$$\begin{cases} a_1 \quad\quad + 2a_3 = 1 \\ \quad\quad a_2 + a_3 = 0 \\ a_1 + 2a_2 \quad\quad = 0 \end{cases} \qquad \begin{cases} b_1 \quad\quad + 2b_3 = 0 \\ \quad\quad b_2 + b_3 = 1 \\ b_1 + 2b_2 \quad\quad = 0 \end{cases}$$

$$\begin{cases} c_1 \quad\quad + 2c_3 = 0 \\ \quad\quad c_2 + c_3 = 0 \\ c_1 + 2c_2 \quad\quad = 1 \end{cases}$$

$$\left(\begin{array}{ccc|c|c|c} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 - 2R_2} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -4 & -1 & -2 & 1 \end{array}\right) \xrightarrow{-\frac{1}{4}R_3} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{array}\right)$$

$$\underset{R_2 - R_3}{\xrightarrow{R_1 - 2R_3}} \left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{array}\right)$$

So we have

$$\begin{aligned} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= \frac{1}{2}\boldsymbol{u} - \frac{1}{4}\boldsymbol{v} + \frac{1}{4}\boldsymbol{w} \\ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} &= -\boldsymbol{u} + \frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w} \\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} &= \frac{1}{2}\boldsymbol{u} + \frac{1}{4}\boldsymbol{v} - \frac{1}{4}\boldsymbol{w} \end{aligned}$$

(ii) From the given information, we have

$$\boldsymbol{Du} = \boldsymbol{D}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.3 \\ 0.5 \end{pmatrix}, \boldsymbol{Dv} = \boldsymbol{D}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.3 \\ 0.7 \end{pmatrix}, \boldsymbol{Dw} = \boldsymbol{D}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.7 \\ 0.3 \\ 0.1 \end{pmatrix}.$$

So

$$
\begin{aligned}
\boldsymbol{D}\begin{pmatrix}1\\0\\0\end{pmatrix} &= \text{first column of } \boldsymbol{D}\\
&= \boldsymbol{D}(\frac{1}{2}\boldsymbol{u} - \frac{1}{4}\boldsymbol{v} + \frac{1}{4}\boldsymbol{w})\\
&= \frac{1}{2}\begin{pmatrix}0.5\\0.3\\0.5\end{pmatrix} - \frac{1}{4}\begin{pmatrix}0.1\\0.3\\0.7\end{pmatrix} + \frac{1}{4}\begin{pmatrix}0.7\\0.3\\0.1\end{pmatrix}\\
&= \begin{pmatrix}0.4\\0.15\\0.1\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{D}\begin{pmatrix}0\\1\\0\end{pmatrix} &= \text{second column of } \boldsymbol{D}\\
&= \boldsymbol{D}(-\boldsymbol{u} + \frac{1}{2}\boldsymbol{v} + \frac{1}{2}\boldsymbol{w})\\
&= -\begin{pmatrix}0.5\\0.3\\0.5\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0.1\\0.3\\0.7\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0.7\\0.3\\0.1\end{pmatrix}\\
&= \begin{pmatrix}-0.1\\0\\-0.1\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{D}\begin{pmatrix}0\\0\\1\end{pmatrix} &= \text{third column of } \boldsymbol{D}\\
&= \boldsymbol{D}(\frac{1}{2}\boldsymbol{u} + \frac{1}{4}\boldsymbol{v} - \frac{1}{4}\boldsymbol{w})\\
&= \frac{1}{2}\begin{pmatrix}0.5\\0.3\\0.5\end{pmatrix} + \frac{1}{4}\begin{pmatrix}0.1\\0.3\\0.7\end{pmatrix} - \frac{1}{4}\begin{pmatrix}0.7\\0.3\\0.1\end{pmatrix}\\
&= \begin{pmatrix}0.1\\0.15\\0.4\end{pmatrix}
\end{aligned}
$$

Thus $\boldsymbol{D} = \begin{pmatrix}0.4 & -0.1 & 0.1\\ 0.15 & 0 & 0.15\\ 0.1 & -0.1 & 0.4\end{pmatrix}$.