NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

Module: MA1508E Linear Algebra for Engineering
Year/Semester: 2018-2019 (Semester 2)
Tutorial: 3

1. For each of the following matrices $\boldsymbol{A}$, use elementary row operations to determine if $\boldsymbol{A}$ is invertible, and if so, find $\boldsymbol{A}^{-1}$. For the matrices that are invertible, express them as a product of elementary matrices.

(a) $\begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & -4 & 0 & 0 \\ 1 & 2 & 12 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & -4 & -5 \end{pmatrix}$

(a) Matrix is singular (b) $\begin{pmatrix} 1 & 3 & 1 \\ 0 & 1 & -1 \\ -2 & 2 & 0 \end{pmatrix}$.

(c) $\begin{pmatrix} \frac{7}{2} & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} \frac{1}{4} & \frac{1}{2} & -3 & 0 \\ -\frac{1}{8} & \frac{1}{4} & -\frac{3}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{40} & -\frac{1}{20} & -\frac{1}{10} & -\frac{1}{5} \end{pmatrix}$.

2. For each of the following matrices $\boldsymbol{B}$, find all values of k such that $\boldsymbol{B}$ is invertible and find the matrix $\boldsymbol{B}^{-1}$ (in terms of k).

(a) $\begin{pmatrix} k & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{pmatrix}$ (c) $\begin{pmatrix} k & k & k \\ 1 & k & k \\ 1 & k & k \end{pmatrix}$

(a) $\boldsymbol{B}$ is invertible if and only if $k \neq 0$. When $k \neq 0$, $\boldsymbol{B}^{-1} = \begin{pmatrix} \frac{1}{k} & -\frac{1}{k} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{k} & -\frac{1}{k} \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(b) $\boldsymbol{B}$ is invertible if and only if $k \neq 0$. When $k \neq 0$, $\boldsymbol{B}^{-1} = \frac{1}{k^4}\begin{pmatrix} k^3 & 0 & 0 & 0 \\ -k^2 & k^3 & 0 & 0 \\ k & -k^2 & k^3 & 0 \\ -1 & k & -k^2 & k^3 \end{pmatrix}$.

(c) The matrix is singular for all values of k.

3. For each of the following matrices $\boldsymbol{C}$, find $\det(\boldsymbol{C})$ by cofactor expansion.

(a) $\begin{pmatrix} 1 & 3 & 2 \\ 4 & 1 & -2 \\ 2 & 1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -1 & 2 \\ 1 & 3 & 2 \\ 5 & 1 & 6 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ -1 & 2 & -2 & 1 \\ -3 & 2 & 3 & 1 \end{pmatrix}$.

(a) -39 (b) 0 (c) 8 (d) 20

4. Let $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ be 2×2 matrices and let

$$\boldsymbol{C} = \begin{pmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad \boldsymbol{D} = \begin{pmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \boldsymbol{E} = \begin{pmatrix} 0 & \gamma_1 \\ \gamma_2 & 0 \end{pmatrix},$$

where $\gamma_1, \gamma_2 \in \mathbb{R}$.

(a) Show that $\det(\boldsymbol{A}+\boldsymbol{B}) = \det(\boldsymbol{A}) + \det(\boldsymbol{B}) + \det(\boldsymbol{C}) + \det(\boldsymbol{D})$.

(b) Show that if $\boldsymbol{B} = \boldsymbol{E}\boldsymbol{A}$, then $\det(\boldsymbol{A}+\boldsymbol{B}) = \det(\boldsymbol{A}) + \det(\boldsymbol{B})$.

(a) $\boldsymbol{A}+\boldsymbol{B} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$. So

$$\begin{aligned}
\det(\boldsymbol{A}+\boldsymbol{B}) &= (a_{11}+b_{11})(a_{22}+b_{22}) - (a_{21}+b_{21})(a_{12}+b_{12}) \\
&= a_{11}a_{22} + b_{11}b_{22} + a_{11}b_{22} + b_{11}a_{22} - (a_{21}a_{12} + a_{21}b_{12} + b_{21}a_{12} + b_{21}b_{12}) \\
\det(\boldsymbol{A}) &= a_{11}a_{22} - a_{12}a_{21} \\
\det(\boldsymbol{B}) &= b_{11}b_{22} - b_{12}b_{21} \\
\det(\boldsymbol{C}) &= a_{11}b_{22} - a_{12}b_{21} \\
\det(\boldsymbol{D}) &= b_{11}a_{22} - a_{21}b_{12} \\
\Rightarrow \det(\boldsymbol{A}+\boldsymbol{B}) &= \det(\boldsymbol{A}) + \det(\boldsymbol{B}) + \det(\boldsymbol{C}) + \det(\boldsymbol{D})
\end{aligned}$$

(b) $\boldsymbol{B} = \begin{pmatrix} \gamma_1 a_{21} & \gamma_1 a_{22} \\ \gamma_2 a_{11} & \gamma_2 a_{12} \end{pmatrix}$. So

$$\begin{aligned}
\boldsymbol{A}+\boldsymbol{B} &= \begin{pmatrix} a_{11}+\gamma_1 a_{21} & a_{12}+\gamma_1 a_{22} \\ a_{21}+\gamma_2 a_{11} & a_{22}+\gamma_2 a_{12} \end{pmatrix} \\
\det(\boldsymbol{A}+\boldsymbol{B}) &= (a_{11}+\gamma_1 a_{21})(a_{22}+\gamma_2 a_{12}) - (a_{12}+\gamma_1 a_{22})(a_{21}+\gamma_2 a_{11}) \\
&= a_{11}a_{22} - (\gamma_1\gamma_2)a_{11}a_{22} - a_{12}a_{21} + (\gamma_1\gamma_2)a_{21}a_{12} \\
&= (1-\gamma_1\gamma_2)a_{11}a_{22} - (1-\gamma_1\gamma_2)a_{12}a_{21} \\
&= (1-\gamma_1\gamma_2)\det(\boldsymbol{A})
\end{aligned}$$

Since $\det(\boldsymbol{B}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$, we have

$$\det(\boldsymbol{B}) = -\gamma_1\gamma_2\det(\boldsymbol{A}) \Rightarrow \det(\boldsymbol{A}+\boldsymbol{B}) = \det(\boldsymbol{A}) - \gamma_1\gamma_2\det(\boldsymbol{A}) = \det(\boldsymbol{A}) + \det(\boldsymbol{B}).$$

5. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square matrices of order n and $\boldsymbol{M}$ be the square matrix of order $2n$ defined as

$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0}_n \\ \boldsymbol{0}_n & \boldsymbol{B} \end{pmatrix}.$$

Use the result in Unit 18 (Equivalent Statements Part I), show that if either $\boldsymbol{A}$ or $\boldsymbol{B}$ is singular, then $\boldsymbol{M}$ must be singular.

Suppose $\boldsymbol{A}$ is singular, then there exists a non trivial solution to $\boldsymbol{Ax} = \boldsymbol{0}$. Let $\boldsymbol{x'} \neq \boldsymbol{0}$ be one such non trivial solution, thus $\boldsymbol{Ax'} = \boldsymbol{0}$. Let

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{x'} \\ \boldsymbol{0}_{n\times 1} \end{pmatrix}.$$

Then

$$\boldsymbol{MX} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0}_n \\ \boldsymbol{0}_n & \boldsymbol{B} \end{pmatrix} \begin{pmatrix} \boldsymbol{x'} \\ \boldsymbol{0}_{n\times 1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{Ax'} + \boldsymbol{0}_{n\times 1} \\ \boldsymbol{0}_{n\times 1} \end{pmatrix} = \boldsymbol{0}_{2n\times 1} \quad (\text{since } \boldsymbol{Ax'} = \boldsymbol{0}.)$$

So $\boldsymbol{X}$ is a non trivial solution to $\boldsymbol{Mx} = \boldsymbol{0}$ which implies that $\boldsymbol{M}$ is singular. The case where $\boldsymbol{B}$ is singular is similarly done by considering

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{0}_{n\times 1} \\ \boldsymbol{x'} \end{pmatrix} \qquad \text{where } \boldsymbol{x'} \text{ is a non trivial solution to } \boldsymbol{Bx} = \boldsymbol{0}.$$

6. Let $\boldsymbol{A}, \boldsymbol{C}, \boldsymbol{D}$ be square matrices of order n, and let $\boldsymbol{I}$ and $\boldsymbol{0}$ denote the identity and zero matrices of order n. Let $|\boldsymbol{X}|$ denote the determinant of $\boldsymbol{X}$. Show that

   (a) $\begin{vmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{vmatrix} = |\boldsymbol{A}|.$

   (Hint: Start by performing cofactor expansion along last row.)

   (b) $\begin{vmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{D} \end{vmatrix} = |\boldsymbol{D}|.$

   (Hint: Start by performing cofactor expansion along first row.)

   (c) $\begin{vmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{D} \end{vmatrix} = |\boldsymbol{A}|\,|\boldsymbol{D}|.$

   (Hint: Write the matrix as a product of two partitioned (block) matrices.)

   (d) $\begin{vmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{D} \end{vmatrix} = |\boldsymbol{A}|\,|\boldsymbol{D}|.$

   (Hint: Consider the transpose of the matrix in part (c).)

(**Remark:** Once we have established part (d), the result in Question 5 can be obtained immediately.)

(a) By cofactor expansion along the last row,

$$\begin{vmatrix} \boldsymbol{A} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{vmatrix} = \begin{vmatrix} \boldsymbol{A} & \mathbf{0}_{n\times n-1} \\ \mathbf{0}_{n-1\times n} & \boldsymbol{I}_{n-1} \end{vmatrix}.$$

Again, by cofactor expansion along the last row,

$$\begin{vmatrix} \boldsymbol{A} & \mathbf{0}_{n\times n-1} \\ \mathbf{0}_{n-1\times n} & \boldsymbol{I}_{n-1} \end{vmatrix} = \begin{vmatrix} \boldsymbol{A} & \mathbf{0}_{n\times n-2} \\ \mathbf{0}_{n-2\times n} & \boldsymbol{I}_{n-2} \end{vmatrix}.$$

Continuing this way (by performing cofactor expansion along the last row each time), we have the desired result.

(b) By cofactor expansion along the first row,

$$\begin{vmatrix} \boldsymbol{I} & \mathbf{0} \\ \boldsymbol{C} & \boldsymbol{D} \end{vmatrix} = \begin{vmatrix} \boldsymbol{I}_{n-1\times n-1} & \mathbf{0}_{n-1\times n} \\ \boldsymbol{C_1} & \boldsymbol{D} \end{vmatrix}.$$

Here, $\boldsymbol{C_1}$ is the $n \times n-1$ matrix obtained from $\boldsymbol{C}$ when the first column is removed. Again, by cofactor expansion along the first row,

$$\begin{vmatrix} \boldsymbol{I}_{n-1\times n-1} & \mathbf{0}_{n-1\times n} \\ \boldsymbol{C_1} & \boldsymbol{D} \end{vmatrix} = \begin{vmatrix} \boldsymbol{I}_{n-2\times n-2} & \mathbf{0}_{n-2\times n} \\ \boldsymbol{C_2} & \boldsymbol{D} \end{vmatrix}.$$

Here, $\boldsymbol{C_2}$ is the $n \times n-2$ matrix obtained from $\boldsymbol{C_1}$ when the first column is removed. Continuing this way (by performing cofactor expansion along the first row each time), we have the desired result.

(c)

$$\begin{pmatrix} \boldsymbol{A} & \mathbf{0} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & \mathbf{0} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix} \Rightarrow \begin{vmatrix} \boldsymbol{A} & \mathbf{0} \\ \boldsymbol{C} & \boldsymbol{D} \end{vmatrix} = \begin{vmatrix} \boldsymbol{A} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{vmatrix} \begin{vmatrix} \boldsymbol{I} & \mathbf{0} \\ \boldsymbol{C} & \boldsymbol{D} \end{vmatrix} = |\boldsymbol{A}|\,|\boldsymbol{D}|$$

(d) Note that

$$\begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \mathbf{0} & \boldsymbol{D} \end{pmatrix}^T = \begin{pmatrix} \boldsymbol{A}^T & \mathbf{0}^T \\ \boldsymbol{C}^T & \boldsymbol{D}^T \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}^T & \mathbf{0} \\ \boldsymbol{C}^T & \boldsymbol{D}^T \end{pmatrix}.$$

Since a matrix and its transpose has the same determinant,

$$\begin{vmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \mathbf{0} & \boldsymbol{D} \end{vmatrix} = \begin{vmatrix} \boldsymbol{A}^T & \mathbf{0} \\ \boldsymbol{C}^T & \boldsymbol{D}^T \end{vmatrix} = |\boldsymbol{A}^T|\,|\boldsymbol{D}^T| = |\boldsymbol{A}|\,|\boldsymbol{D}|.$$