W04-04

Slide 01: In this unit, we will introduce an importance concept known as linear combinations.

Slide 02: Consider the following two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^3$. It is easy to compute vectors $2\boldsymbol{u}+3\boldsymbol{v}$ and $\boldsymbol{u}-2\boldsymbol{v}$ as shown.

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The vectors $(2,10,13)$ and $(1,-2,-11)$, which are $2\boldsymbol{u}+3\boldsymbol{v}$ and $\boldsymbol{u}-2\boldsymbol{v}$, respectively were obtained by linearly combining vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Thus, they are called linear combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$.

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Let us define what is a linear combination in general. Suppose $\boldsymbol{u_1}, \boldsymbol{u_2}$ and so on till $\boldsymbol{u_k}$ are vectors in $\mathbb{R}^n$. For any choice of real numbers c_1, c_2 and so on till c_k, the vector $c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2}$ and so on till $c_k\boldsymbol{u_k}$

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is called a linear combination of $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$.

Slide 03: Consider the following example where $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors as shown. It is easy to compute the linear combination $2\boldsymbol{u}+3\boldsymbol{v}-\boldsymbol{w}$.

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We simply evaluate and simplify

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and obtain the vector $(1,10,15)$.

Slide 04: However, if the question asked was instead to check if $(0,4,8)$ is a linear combination of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$, how should we go about checking this?

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From our understanding of linear combinations, we need to check whether we can find real numbers a, b and c such that combining the three vectors in the way $a\boldsymbol{u}+b\boldsymbol{v}+c\boldsymbol{w}$ will result in the vector $(0,4,8)$.

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Writing out the vectors $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$, we have the following equation. This is known as a vector equation since it equates a vector on the right with a vector on the left. How do we check if such a, b and c can be found?

Slide 05: Recall that two vectors are said to be equal if and only if all their corresponding components are equal. Therefore for this vector equation to hold, we need the constants a, b and c

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to satisfy the following linear system. How did we arrive at this linear system?

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The first equation is obtained by comparing the first component on both sides of the equation. Thus we have $a+c=0$.

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The second equation is obtained by comparing the second component, giving us $2a+2b=4$.

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And lastly the third equation comes from comparing the third component, giving us $-a + 5b - 2c = 8$.

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We will apply our the Gauss-Jordan elimination method to check if this linear system can be solved. This is the starting augmented matrix representing the linear system.

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While we skip the intermediate steps, which you should try to fill in,

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the reduced row-echelon form of the augemented matrix is shown here.

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From the reduced row-echelon form, we can see that the linear system has a unique solution $a = \frac{1}{2}$, $b = \frac{3}{2}$ and $c = -\frac{1}{2}$.

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This means that we have found the constants a, b and c that satisfies the vector equation. More precisely, we now know that $\frac{1}{2}\boldsymbol{u} + \frac{3}{2}\boldsymbol{v} - \frac{1}{2}\boldsymbol{w}$ is equal to $(0, 4, 8)$

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and indeed $(0, 4, 8)$ is a linear combination of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

Slide 06: Let us consider another similar example. Once again, we are given vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ and we would like to check if the vector $(3, 3, 4)$ is a linear combination of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

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Like before, we set up a vector equation $a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w} = (3, 3, 4)$ and try to solve for a, b and c.

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By comparing the components on both sides of the equation, we derive the following linear system,

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and the corresponding augmented matrix.

Slide 07: Once again, we omit the intermediate steps involved performing Gaussian elimination and show a row-echelon form of the augmented matrix here.

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The outcome here is a little different from the previous example. The first important question is whether the linear system is consistent, since that was the question we started off with, as we wanted to determine if the vector $(3, 3, 4)$ is a linear combination of $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$. The row-echelon form tells us that indeed the linear system is consistent but unlike the previous example, we have infnitely many solutions in this case.

Slide 08: In fact, if we proceed to write down a general solution of the linear system, we can obtain one as shown here. Note that t is an arbitrary parameter that can take on any real number.

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For example, when we choose $t = 0$, then $a = 2, b = -1, c = 0$ is a solution to the system, meaning that we can express $(3, 3, 4)$ as $2\boldsymbol{u} - \boldsymbol{v} + 0\boldsymbol{w}$. This is just one way of writing $(3, 3, 4)$ as a linear combination of $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$.

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Another solution could be obtained by taking $t = 1$. So with $a = 1, b = -2, c = 1$, we have another way of writing $(3, 3, 4)$ as a linear combination of $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$. It is now clear that not only is $(3, 3, 4)$ a linear combination of $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$, there are actually infinitely many ways of writing $(3, 3, 4)$ in terms of the three vectors.

Slide 09: Now, with the same $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ from the previous example, we will again investigate if the vector $(1, 2, 4)$ is a linear combination of $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$. We will follow the same procedure of setting up the vector equation, the corresponding linear system and also write down the augement matrix as shown.

Slide 10: For this example, upon performing Gaussian elimination on the augemented matrix, we arrive at the row-echelon form as shown. Is the linear system consistent in this case?

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It is easy to see that as the last column on the right hand side of the vertical line is a pivot column, the linear system is unfortunately, inconsistent. This means that the linear system and thus, the vector equation cannot be solved and thus we conclude that the vector $(1, 2, 4)$ is not a linear combination of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

Slide 11: Consider the following 4 vectors in $\mathbb{R}^4$. These are interesting vectors as $\boldsymbol{e_1}$ has zero in the second, third and fourth components and a 1 in the first component. The other vectors $\boldsymbol{e_2}$, $\boldsymbol{e_3}$ and $\boldsymbol{e_4}$ follow a similar format, having only a 1 in a particular component and zero everywhere else. To write the vector $(1, 2, 3, 4)$ as a linear combination of $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$ and $\boldsymbol{e_4}$ is easily done due to the format of the four vectors.

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Similarly, to write $(-3, \frac{1}{3}, 0, 2)$ as a linear combination of the four vectors is trivial.

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In fact, for any vector (w, x, y, z) in $\mathbb{R}^4$,

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we can write it as $w\boldsymbol{e_1} + x\boldsymbol{e_2} + y\boldsymbol{e_3} + z\boldsymbol{e_4}$.

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Thus, each and evey single vector in $\mathbb{R}^4$ is a linear combination of $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$ and $\boldsymbol{e_4}$.

Slide 12: Let us consider another example, with vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in $\mathbb{R}^3$. We would like to see if evey vector in $\mathbb{R}^3$ can be written as a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

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We need to start by considering an arbitrary vector in $\mathbb{R}^3$ and set up the vector equation as we did before. Note that the vector (x, y, z) represents any arbitrary vector chosen from $\mathbb{R}^3$.

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The next step is similar to before, where we compare components on both sides of the vector equation and write down the linear system. Remember the objective is to solve for the unknowns a, b and c.

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The augemented matrix representing the linear system is shown here.

Slide 13: We will proceed with Gaussian elimination as usual. Note that the x, y, z on the right hand side should just be treated as any other number when the elimination steps are performed.

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This is how a row-echelon form of the augmented matrix will look like.

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The key observation to be made is that at row-echelon form, each of the 3 columns on the left side is a pivot column.

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This implies that regardless of the values of x, y and z, the vector equation can always be solved to obtain values of a, b and c. Obviously, different values of x, y and z, will correspond to different solutions to the vector equation.

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Nevertheless, we are now certain that every vector in $\mathbb{R}^3$ can be written as a linear combination of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

Slide 14: We will re-work the same problem with a different set of vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$. Recall that we wish to determine if every vector in $\mathbb{R}^3$ is a linear combination of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

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We write down the vector equation, again using an arbitrary vector (x, y, z) on the right hand side.

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The corresponding linear system,

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and augmented matrix follows.

Slide 15: Performing Gaussian elimination on the augmented matrix, we derive the following row-echelon form. It is immediately noted that there is a non-pivot column on the left side of the vertical line. This means that there is one row in the row-echelon form, where every entry on the left is zero and on the right, we have an expression involving x, y and z.

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In other words, since x, y and z are any arbitrary real numbers, representing any arbitrary vector in $\mathbb{R}^3$, we can conclude that the vector equation can be inconsistent for some choices of values taken by x, y and z. The choice just needs to be made such that the expression highlighted in blue, namely $z - \frac{5y}{6} + x$ is non zero.

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Thus, the conclusion in this case is that not every vector in $\mathbb{R}^3$ is a linear combination of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$. One such vector, is, for example, $(1, 0, 0)$.

Slide 16: Let us summarise the main points.

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We defined what is a linear combination of vectors.

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We then had a few examples on how to check whether a given vector is a linear combination of some other vectors.

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Recall that in that process, we started off with a vector equation and then wrote down an equivalent linear system which we then proceed to solve by Gaussian elimination.

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We then had several examples to illustrate how to check whether every vector in $\mathbb{R}^n$ is a linear combination of some vectors.