PROPERTIES OF DETERMINANTS

Theorem

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two square matrices of order n and c is a scalar.

1) $\det(c\boldsymbol{A}) = c^n \det(\boldsymbol{A})$

Proof:

$$\boldsymbol{A} \xrightarrow{cR_1,\ cR_2, \ldots,\ cR_n} c\boldsymbol{A}$$

Each cR_i changes the determinant by a factor of c, so $\det(c\boldsymbol{A}) = c^n \det(\boldsymbol{A})$.

Theorem

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two square matrices of order n and c is a scalar.

2) $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$

Remark:

This results generalizes one that we had previously:

$$\boxed{\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})}$$

where $\boldsymbol{E}$ is an elementary matrix of the same order as $\boldsymbol{A}$.

Theorem

2) $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$

Proof:

If $\boldsymbol{A}$ is singular, we already know that $\boldsymbol{AB}$ is singular.

In this case, $\det(\boldsymbol{A}) = 0,$ $\det(\boldsymbol{AB}) = 0,$ so

$$0 = \det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B}) = 0$$

Next consider the case when $\boldsymbol{A}$ is invertible.

Theorem

2) $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$

Proof:

Next consider the case when $\boldsymbol{A}$ is invertible.

Since $\boldsymbol{A}$ is invertible, it can be expressed as a product of elementary matrices.

$$\boldsymbol{A} = \boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_2 \boldsymbol{E}_1$$

So

$$\boldsymbol{AB} = \boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{B}$$

$$\Rightarrow \det(\boldsymbol{AB}) = \det(\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{B})$$

Theorem

2) $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$

Proof:

We use the result $\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$

repeatedly on $\boxed{}$

$$\Rightarrow \det(\boldsymbol{AB}) = \boxed{\det(\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{B})}$$

$$= \det(\boldsymbol{E}_k)\det(\boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{B})$$

$$\vdots$$

$$= \det(\boldsymbol{E}_k)\det(\boldsymbol{E}_{k-1}) \ldots \det(\boldsymbol{E}_2)\det(\boldsymbol{E}_1)\det(\boldsymbol{B})$$

Theorem

2) $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$

Proof:

We again use the result $\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$

repeatedly on

$$\det(\boldsymbol{AB}) = \boxed{\det(\boldsymbol{E}_k)\det(\boldsymbol{E}_{k-1})\ldots\det(\boldsymbol{E}_2)\det(\boldsymbol{E}_1)}\det(\boldsymbol{B})$$

$$= \det(\boldsymbol{E}_k)\det(\boldsymbol{E}_{k-1})\ldots\det(\boldsymbol{E}_2\boldsymbol{E}_1)\det(\boldsymbol{B})$$

$$\vdots$$

$$= \det(\boldsymbol{E}_k\boldsymbol{E}_{k-1}\ldots\boldsymbol{E}_2\boldsymbol{E}_1)\det(\boldsymbol{B})$$

$$= \det(\boldsymbol{A})\det(\boldsymbol{B})$$

Theorem

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two square matrices of order n and c is a scalar.

3) If $\boldsymbol{A}$ is invertible, then $\det(\boldsymbol{A}^{-1}) = \dfrac{1}{\det(\boldsymbol{A})}$.

Proof:

$$\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I} \Rightarrow \det(\boldsymbol{A}^{-1}\boldsymbol{A}) = \det(\boldsymbol{I})$$

$$\Rightarrow \det(\boldsymbol{A}^{-1})\det(\boldsymbol{A}) = \det(\boldsymbol{I})$$

by $\det(\boldsymbol{AB})$
$= \det(\boldsymbol{A})\det(\boldsymbol{B})$

$$\Rightarrow \det(\boldsymbol{A}^{-1})\det(\boldsymbol{A}) = 1$$

$$\Rightarrow \det(\boldsymbol{A}^{-1}) = \frac{1}{\det(\boldsymbol{A})}$$

Example

Let $\boldsymbol{A} = \begin{pmatrix} -3 & -2 & 4 \\ 4 & 3 & 1 \\ 0 & 2 & 4 \end{pmatrix}$. We can check that $\det(\boldsymbol{A}) = 34$.

$\det(4\boldsymbol{A}) = 4^3 \det(\boldsymbol{A}) = 64 \cdot 34 = 2176$.

If $\boldsymbol{B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$, $\quad \det(\boldsymbol{B}) = -1$

$\det(\boldsymbol{A}\boldsymbol{B}) = -34 \qquad \det(\boldsymbol{A}^{-1}) = \dfrac{1}{34}$

Summary

1) Proved several results on determinants:

a) $\det(c\boldsymbol{A}) = c^n \det(\boldsymbol{A})$

b) $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$

c) If $\boldsymbol{A}$ is invertible, then $\det(\boldsymbol{A}^{-1}) = \dfrac{1}{\det(\boldsymbol{A})}$.