RANK OF A MATRIX; ONE MORE EQUIVALENT STATEMENT

THEOREM

The row space and the column space of a matrix have the same dimension.

$$\boldsymbol{A} = \left.\overbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}^{n}\right\} m$$

Note that the row space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^n$ while the column space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^m$.

These two subspaces may have nothing to do with each other (if $m \neq n$) but the theorem states that they have the same dimension.

PROOF

Strategy: We will use what we learnt in the previous section to find bases for both subspaces.

Let $\boldsymbol{R}$ be a row echelon form of $\boldsymbol{A}$.

$$\underset{\boldsymbol{A}}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \underset{\boldsymbol{R}}{\begin{pmatrix} \otimes & * & & & & * \\ & & \otimes & * & & * \\ & 0 & & & \ddots & * \\ & & & & \otimes \quad * & * \\ 0 & \ldots & \ldots & \ldots & 0 & 0 \end{pmatrix}}$$

PROOF

Question: What is a basis for the row space of $\boldsymbol{A}$?

Answer: The non-zero rows of $\boldsymbol{R}$.

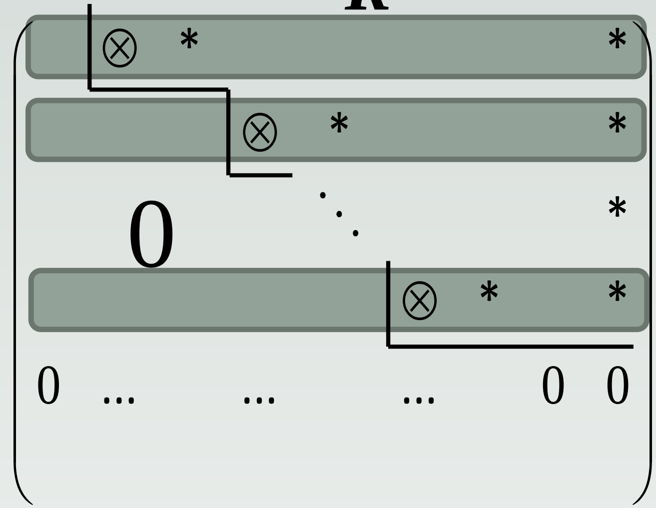
Question: What is a dimension of the row space of $\boldsymbol{A}$?

Answer: The number of non-zero rows of $\boldsymbol{R}$.

$=$ number of leading entries in $\boldsymbol{R}$.

$$\begin{pmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}}$$

$\boldsymbol{R}$

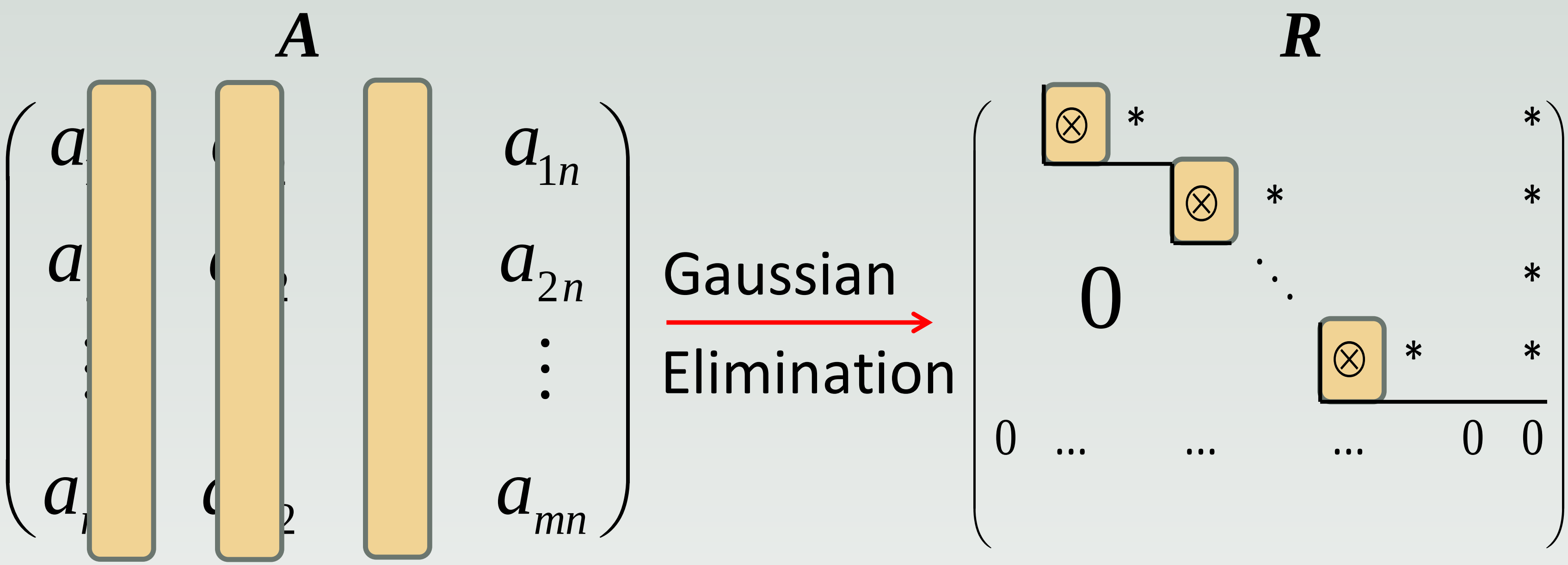


PROOF

Question: What is a basis for the column space of $\boldsymbol{A}$?

Answer: The columns of $\boldsymbol{A}$ corresponding to the pivot columns of $\boldsymbol{R}$.

Question: What is a dimension of the column space of $\boldsymbol{A}$?



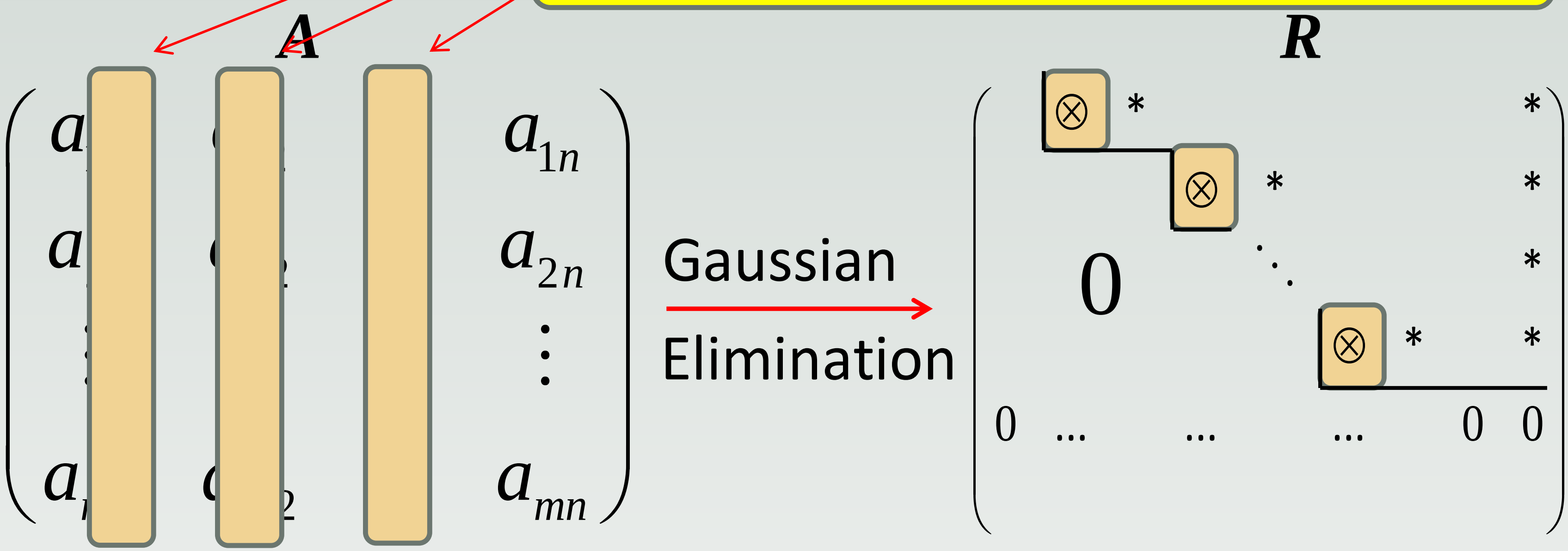
PROOF

Question: What is a dimension of the column space of $\boldsymbol{A}$?

Answer: The number of columns here.

$=$ The number of pivot columns in $\boldsymbol{R}$.

$=$ The number of leading entries in $\boldsymbol{R}$.



PROOF

Question: What is a dimension of the row space of $\boldsymbol{A}$?

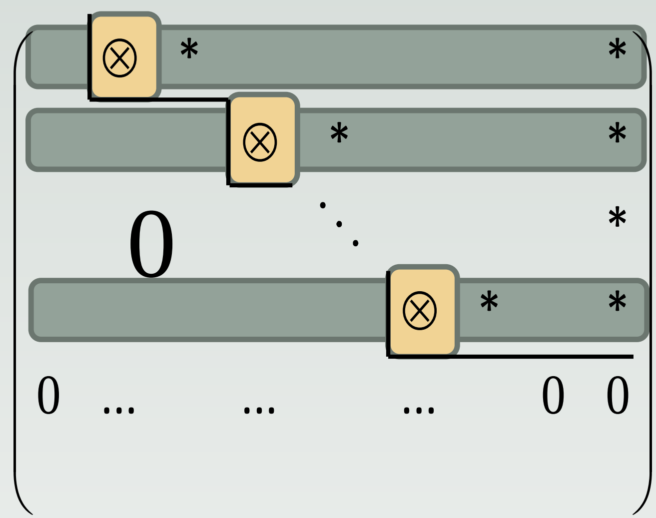
Answer: The number of leading entries in $\boldsymbol{R}$.

Question: What is a dimension of the column space of $\boldsymbol{A}$?

Answer: The number of leading entries in $\boldsymbol{R}$.

$$\underset{\boldsymbol{A}}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}} \xrightarrow[\text{Elimination}]{\text{Gaussian}}$$

$\boldsymbol{R}$



EXAMPLE

$$C = \begin{pmatrix} 2 & 0 & 3 & -1 & 8 \\ 2 & 1 & 1 & -2 & 5 \\ -4 & -3 & 0 & 5 & -7 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 2 & 0 & 3 & -1 & 8 \\ 0 & 1 & -2 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A basis for the row space of C is:

$$\{(2,0,3,-1,8),(0,1,-2,-1,-3)\}$$

A basis for the column space of C is:

$$\left\{ \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}$$

The dimension of both the row space and column space of C is 2.

DEFINITION

The rank of a matrix is the dimension of its row space (or column space).

The rank of $\boldsymbol{A}$ is denoted by rank($\boldsymbol{A}$).

By our earlier discussion,

rank($\boldsymbol{A}$) $=$ number of non zero rows in a row echelon form of $\boldsymbol{A}$

$=$ number of pivot columns in a row echelon form of $\boldsymbol{A}$

EXAMPLES (RANK)

1) $\operatorname{rank}(\mathbf{0}) = 0$

2) $\operatorname{rank}(\boldsymbol{I}_n) = n$

3) $\operatorname{rank}(\boldsymbol{C}) = 2$

$$\boldsymbol{C} = \begin{pmatrix} 2 & 0 & 3 & -1 & 8 \\ 2 & 1 & 1 & -2 & 5 \\ -4 & -3 & 0 & 5 & -7 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 2 & 0 & 3 & -1 & 8 \\ 0 & 1 & -2 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

EXAMPLES (RANK)

$$\boldsymbol{A} = \left.\overbrace{\begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdot\cdot & \cdots & \cdots \end{pmatrix}}^{m}\right\} n$$

$$m \geq n$$

What is the largest possible value for rank($\boldsymbol{A}$)?

$$\boxed{n}$$

$$\boldsymbol{A} = \left.\overbrace{\begin{pmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots \end{pmatrix}}^{m}\right\} n$$

$$n \geq m$$

What is the largest possible value for rank($\boldsymbol{A}$)?

$$\boxed{m}$$

REMARKS

1) For a $m \times n$ matrix $\boldsymbol{A}$, $\text{rank}(\boldsymbol{A}) \leq \min\{m, n\}$.

2) For a $m \times n$ matrix $\boldsymbol{A}$, if $\text{rank}(\boldsymbol{A}) = \min\{m, n\}$,

we say that $\boldsymbol{A}$ is of full rank.

3) $\text{rank}(\boldsymbol{A}) =$ dimension of row space of $\boldsymbol{A}$

$\parallel$

$=$ dimension of column space of $\boldsymbol{A}^T$

$= \text{rank}(\boldsymbol{A}^T)$

REMARKS

4) A square matrix $\boldsymbol{A}$ (of order n) is of full rank

if and only if $\det(\boldsymbol{A}) \neq 0$.

Proof: $\text{rank}(\boldsymbol{A}) = n$

$\Leftrightarrow$ dimension of row space of $\boldsymbol{A}$ is n

$\Leftrightarrow$ $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ is a basis for the row space of $\boldsymbol{A}$

$\Leftrightarrow$ $\boldsymbol{A}$ is row equivalent to $\boldsymbol{I}_n$

$\Leftrightarrow$ RREF of $\boldsymbol{A}$ is $\boldsymbol{I}_n$ $\Leftrightarrow$ $\det(\boldsymbol{A}) \neq 0$

ONE MORE STATEMENT

Let $\boldsymbol{A}$ be an $n \times n$ matrix. The following statements are equivalent.

1) $\boldsymbol{A}$ is invertible
2) $\boldsymbol{Ax} = \boldsymbol{0}$ has only trivial solution
3) RREF of $\boldsymbol{A}$ is $\boldsymbol{I}$
4) $\boldsymbol{A}$ can be written as produce of elementary matrices
5) $\det(\boldsymbol{A}) \neq 0$
6) Rows of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$
7) Columns of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$
8) $\boxed{\text{rank}(A) = n}$

SUMMARY

1) The row space and column space of a matrix have the same dimension.

2) Definition of the rank of a matrix.

3) One more equivalent statement added in terms of the rank of $\boldsymbol{A}$.