MA1512 Tutorial 3 Solutions

(1a)

$y''+6y'+9y=0 \qquad\qquad \text{Set } y=e^{\lambda t}$

$\lambda^2+6\lambda+9=0 \to \lambda=-3$

$\to y=(A+Bx)e^{-3x} \to y'=Be^{-3x}-3(A+Bx)e^{-3x}$

$y(0)=1 \Rightarrow A=1$

$y'(0)=-1 \Rightarrow B-3A=-1 \Rightarrow B=2 \to y=(1+2x)e^{-3x}$

(1b)

$\lambda^2-2\lambda+(1+4\pi^2)=0 \to \lambda=1\pm 2\pi i$

$\to y=e^x[A\cos 2\pi x+B\sin 2\pi x]$

$y'=y+e^x[-2\pi A\sin 2\pi x+2\pi B\cos 2\pi x]$

$y(0)=-2 \Rightarrow A=-2$

$y'(0)=2(3\pi-1) \Rightarrow 2(3\pi-1)=-2+2\pi B$

$\Rightarrow B=3 \Rightarrow y=e^x[-2\cos 2\pi x+3\sin 2\pi x]$

(2a)

Try $y=Ax^2+Bx+C$

$y''+2y'+10y$

$=2A+2(2Ax+B)+10(Ax^2+Bx+C)$

$=25x^2+3$

$\to 10A=25, \quad 4A+10B=0, \quad 2A+2B+10C=3$

$\to A=5/2, \quad B=-1, \quad C=0$

$\to y=\dfrac{5}{2}x^2-x$

(2b)

Try $y=(Ax^2+Bx+C)e^{3x}$

$y'=(2Ax+B)e^{3x}+3(Ax^2+Bx+C)e^{3x}$

$y''=2Ae^{3x}+3(2Ax+B)e^{3x}+3(2Ax+B)e^{3x}+9(Ax^2+Bx+C)e^{3x}$

$9A-18A+8A=1 \to A=-1$

$6A+6A+9B-12A-18B+8B=0 \to B=0$

$2A+3B+3B+9C-6B-18C+8C=0 \to C=-2$

$y=(-x^2-2)e^{3x}$

(2c)

$y''-y=2x\,\mathrm{Im}\,e^{ix}\qquad (\mathrm{Im}=\text{imaginary part})$

if we can solve the complex equation $z''-z=2xe^{ix}$ then Imz satisfies the above.

Try $z=(Ax+B)e^{ix}$

$z'=Ae^{ix}+i(Ax+B)e^{ix}$

$z''=Aie^{ix}+iAe^{ix}-(Ax+B)e^{ix}$

$z''-z=(2Ai-Ax-B-Ax-B)e^{ix}=2xe^{ix}$

$\rightarrow A=-1$

$-2i-2B=0\rightarrow B=-i$

$\rightarrow z=(-x-i)e^{ix}=-x\cos x+\sin x+i[-\cos x-x\sin x]$

$\mathrm{Im}\,z=-\cos x-x\sin x\rightarrow y=-\cos x-x\sin x$

(2d)

$y''+4y=\frac{1}{2}(1-\cos 2x)=\frac{1}{2}-\frac{1}{2}\mathrm{Re}(e^{2ix})$

Solve $z''+4z=\frac{1}{2}-\frac{1}{2}e^{2ix}\Leftrightarrow$ Try $z=A+Bxe^{2ix}$

$z''=-4Bxe^{2ix}+4iBe^{2ix}\rightarrow z''+4z=-4Bxe^{2ix}+4A+4Bxe^{2ix}+4iBe^{2ix}$

$\rightarrow 4A=\frac{1}{2}\rightarrow A=\frac{1}{8}$

$\rightarrow -\frac{1}{2}=4iB\rightarrow B=\frac{1}{8}i$

$z=\frac{1}{8}+\frac{1}{8}ixe^{2ix}=\frac{1}{8}(1+x(i\cos 2x-\sin 2x))$

$y=\mathrm{Re}\,z=\frac{1}{8}-\frac{1}{8}x\sin 2x$

(3a)

Variation of parameters: first solve $y''+4y=0\rightarrow y=A\cos 2x+B\sin 2x$

Promote A and B to functions A(x), B(x).

Then $A(x)\cos(2x)+B(x)\sin 2x$ is *a* solution of $y''+4y=\frac{1}{2}(1-\cos 2x)$

if A(x) and B(x) are chosen to satisfy

$$A'=\frac{-[\frac{1}{2}\overbrace{(1-\cos 2x)}^{(RHS)}]\sin 2x}{W},\qquad B'=\frac{+[\frac{1}{2}\overbrace{(1-\cos 2x)}^{(RHS)}]\cos 2x}{W}$$

where $W = (\cos 2x)x(\sin 2x)' - (\cos 2x)' \sin 2x = 2$
so

$$A' = -\frac{1}{4}\sin 2x + \frac{1}{4}\cos 2x \sin 2x = -\frac{1}{4}\sin 2x + \frac{1}{8}\sin 4x$$

$$B' = \frac{1}{4}\cos 2x - \frac{1}{4}\cos^2(2x) = \frac{1}{4}\cos 2x - \frac{1}{8}(\cos 4x + 1)$$

$$\to A = \frac{1}{8}\cos 2x - \frac{1}{32}\cos 4x$$

$\to B = \frac{1}{8}\sin 2x - \frac{1}{32}\sin 4x - \frac{x}{8}$ so the solution is

$$A\cos 2x + B\sin 2x = \frac{1}{8}[(\cos 2x - \frac{1}{4}\cos 4x)\cos 2x + (\sin 2x - \frac{1}{4}\sin 4x - x)\sin 2x]$$

which is the same as in (2d) since

$$\frac{1}{8}(\cos^2 2x - \frac{1}{4}\cos 4x \cos 2x + \sin^2 2x - \frac{1}{4}\sin 4x \sin 2x) - \frac{x}{8}\sin 2x$$

$= \frac{1}{8}(1 - \frac{1}{4}\cos 2x) - \frac{x}{8}\sin 2x$ and the extra $-\frac{1}{32}\cos 2x$ can be absorbed into the general solution
(arbitrary constant) x $\cos 2x$ + (arbitrary constant) x $\sin 2x$

(3b)
$A(x)\cos x + B(x)\sin(x)$ where

$$A' = \frac{-[\sec(x)]\sin x}{w} \qquad B' = \frac{+[\sec(x)]\cos x}{w}$$

$$w = (\cos x)(\sin x)' - (\cos x)'(\sin x) = +1$$

$$A = -\int \frac{\sin x}{\cos x}dx = \ln|\cos x|, \quad B = x$$

$$\to y = \cos x \ln|\cos x| + x\sin x$$

Question 4

The equation of motion is given by

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0.$$

Therefore $\theta = A\cos\sqrt{9.8}t + B\sin\sqrt{9.8}t$.
Using $\theta(0) = 0$ and $\frac{d\theta}{dt}(0) = 1$, we have $A = 0$ and $B = \frac{1}{\sqrt{9.8}}$.
So the solution is $\theta = \frac{1}{\sqrt{9.8}}\sin\sqrt{9.8}t$.
We find $\theta(0.8) = \frac{1}{\sqrt{9.8}}\sin 0.8\sqrt{9.8} = 0.190\,048\,03$.

Question 5

When the ship is at rest, the part of it which is under sea level has a volume of Ad [that is, the area of the base times the height]. Therefore, this is the volume of seawater that has been pushed aside by the ship. If the density of seawater is ρ, then the mass of seawater pushed aside is ρAd, and its weight is ρAdg. This upward force exactly balances the weight of the ship, so we have

$$\rho\,\mathrm{A\,d\,g} = \mathrm{M\,g}.$$

Thus

$$\mathrm{d} = \mathrm{M}/\rho\,\mathrm{A}.$$

Now if the ship is moving and the distance from sea level to the bottom of the ship is d + x, where x is a function of time, we have to use Force = mass $\times$ acceleration. Taking the downwards direction to be positive, we find that the buoyancy force is now $-\rho$A(d + x)g, so we have

$$\mathrm{M\ddot{x}} = \mathrm{Mg} - \rho\,\mathrm{A\,(d + x)g},$$

which, using our formula for d, is just

$$\mathrm{\ddot{x}} = -\frac{\rho\,\mathrm{A\,g}}{\mathrm{M}}\,\mathrm{x}$$

This represents simple harmonic motion with angular frequency $\sqrt{\rho\,\mathrm{A\,g/M}}$, as claimed. The ship will bob up and down at this frequency. Note the inverse dependence on M, which is to be expected, but also that the frequency increases if A is large, which is not so obvious.

Taking into account the force exerted by the waves, Force = mass $\times$ acceleration gives

$$\mathrm{M\ddot{x}} = \mathrm{Mg} - \rho\,\mathrm{A\,(d + x)g} + \mathrm{F_0}\cos(\omega\,\mathrm{t})$$

or

$$\mathrm{M\ddot{x}} + \rho\,\mathrm{A\,g\,x} = \mathrm{F_0}\cos(\omega\,\mathrm{t}).$$

This is exactly the equation studied in the notes when we studied resonance, except that k is replaced by ρAg. In the problem we are told that the input frequency [the frequency of the waves] is $\omega = \sqrt{\rho\,\mathrm{A\,g/M}}$, the same as the natural frequency of the ship, so we do indeed have resonance here.

From the resonance notes we have [when x and its derivative are both zero at $t = 0$, which is also the case here]

$$x(t) = \frac{F_0 t}{2M\omega} \sin(\omega t).$$

Because of the factor of t, this will eventually be larger than any fixed number. As soon as it reaches the value H, this means that the ship has gone down by a greater distance than the height of the deck, which means that water washes over the deck and the ship sinks. So t_{sink} is the smallest positive solution of the equation

$$H = \frac{F_0 t_{\text{sink}}}{2M\omega} \sin(\omega t_{\text{sink}}).$$

Question 6 solution

We have $EI\frac{d^2v}{dx^2} = 20x - 2x^2$.
Integrate once, we get $EI\frac{dv}{dx} = 10x^2 - \frac{2}{3}x^3 + A$.
Integrate again, we get $EIv = \frac{10}{3}x^3 - \frac{1}{6}x^4 + Ax + B$.
$v(0) = 0 \Rightarrow B = 0$
$v(10) = 0 \Rightarrow \frac{10}{3}(10)^3 - \frac{1}{6}(10)^4 + A(10) = 0$
$\frac{10}{3}(10)^3 - \frac{1}{6}(10)^4 + A(10) = 0$, Solution is: $\left\{A = -\frac{500}{3}\right\}$

$EIv = \frac{10}{3}x^3 - \frac{1}{6}x^4 - \frac{500}{3}x$
By symmetry, maximum deflection occurs at the midpoint $x = 5$.

$\frac{\frac{10}{3}5^3 - \frac{1}{6}5^4 - \frac{500}{3}5}{EI} = -\frac{3125}{6EI} = -\frac{520.83333}{EI} \approx -\frac{521}{EI}$.