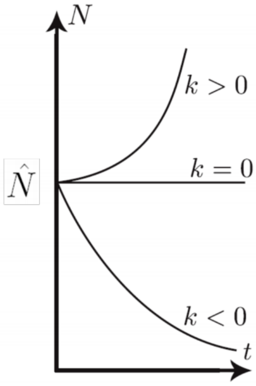
MA1512 TUTORIAL 4

KEY CONCEPTS – CHAPTER 3 MATHEMATICAL MODELLING

Malthus Model of Population:



$$\frac{dN}{dt} = (B - D)N = kN$$

- Particular solution: $N(t) = N(0)e^{kt}$

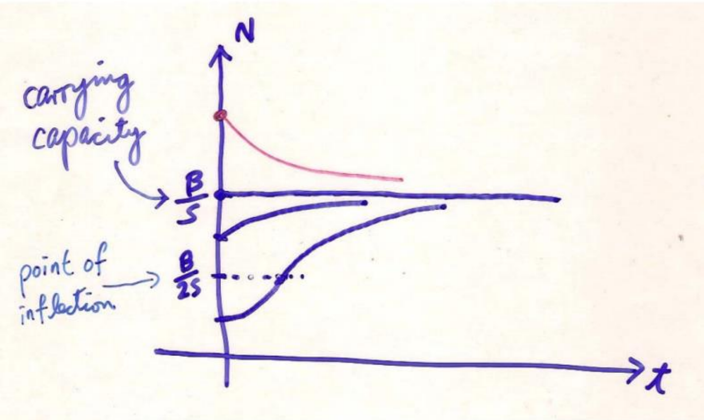
Logistic Growth Model:

$$\frac{dN}{dt} = (B - D)N = (B - sN)N = BN - sN^2$$

- A logistic growth model has (1) increasing initial growth; (2) slower growth as saturation steps in; (3) growth stops upon maturation.
- Particular solution (assuming $N(0) \neq 0$):

$$N = \frac{N_\infty}{1 + \left(\frac{N_\infty}{N_0} - 1\right)e^{-Bt}}$$

  where $N_\infty = B/s$ and $N(0) = N_0$.
- The graph of N against t is shown below, with different initial starting values of N_0.

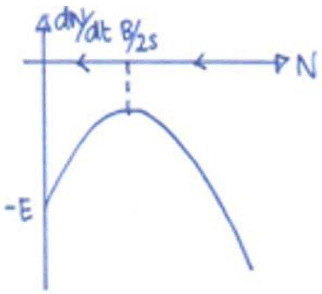


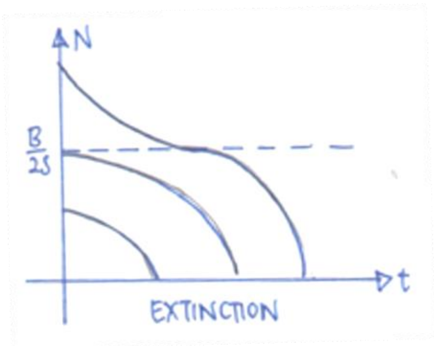
Harvesting Model:

$$\frac{dN}{dt} = BN - sN^2 - E$$

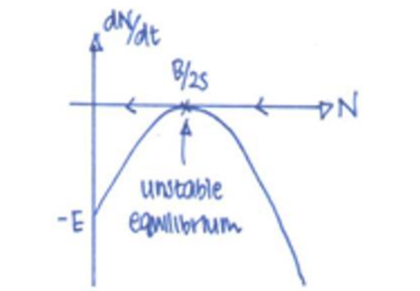
To find equilibrium solutions, we consider $BN - sN^2 - E = 0$. The discriminant $B^2 - 4sE$ is of concern, as it will tell us the number of equilibrium solutions we will get, as well as the behaviour of N with time t.

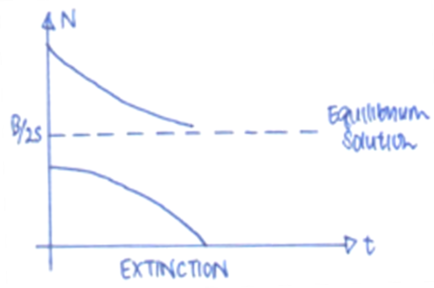
Case 1: $B^2 - 4sE < 0 \Leftrightarrow E > \dfrac{B^2}{4s}$



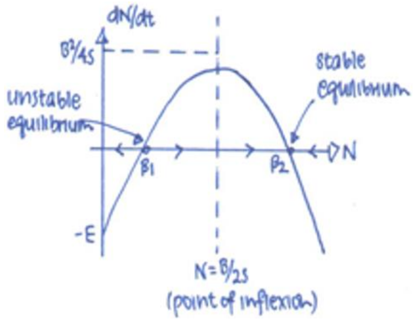


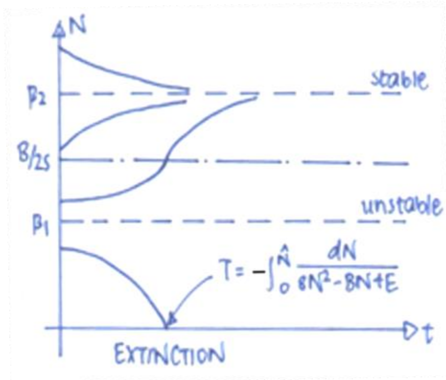
Case 2: $B^2 - 4sE = 0 \Leftrightarrow E = \dfrac{B^2}{4s}$





Case 3: $B^2 - 4sE > 0 \Leftrightarrow E < \dfrac{B^2}{4s}$





Key Concepts – Chapter 4 Laplace Transform

- Laplace transform $L(f)$ is a mapping L which maps a function $f(t)$ to a function $F(s)$, where $F(s)$ is given by

$$L(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

- The inverse Laplace transform is given by $L^{-1}(F(s)) = f(t)$.
- The Laplace transform and inverse Laplace's transform has the **linearity** property (α and β are constants):

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g) \qquad L^{-1}(\alpha f + \beta g) = \alpha L^{-1}(f) + \beta L^{-1}(g)$$

- Solving **Initial** Value Problems

  Step 1 Perform Laplace transform on the DE (in terms of t) and transform it into an equation in terms of s.

  Step 2 Substitute in the appropriate initial values from the problem and make $L(f)$ the subject of the formula (in terms of s).

  Step 3 Perform inverse Laplace transform on the resultant equation to obtain the solution (in terms of t) for the differential equation.

- The standard Laplace transforms and its inverses are presented in the table below.

$L(k) = \frac{k}{s}, k \in \mathbb{R}$	$L^{-1}\left(\frac{k}{s}\right) = k, k \in \mathbb{R}$
$L(t^n) = \frac{n!}{s^{n+1}}$	$L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$
$L(e^{at}) = \frac{1}{s-a}, s > a$	$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
$L(\cos at) = \frac{s}{s^2+a^2}, s > 0$	$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$
$L(\sin at) = \frac{a}{s^2+a^2}, s > 0$	$L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$
$L(\cosh at) = \frac{s}{s^2-a^2}, s > \lvert a \rvert$	$L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$
$L(\sinh at) = \frac{a}{s^2-a^2}, s > \lvert a \rvert$	$L^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh at$
$L\big(f(t-a)\cdot u(t-a)\big) = e^{-as}\cdot F(s)$	$L^{-1}(e^{-as}F(s)) = f(t-a)\cdot u(t-a)$
$L\big(\delta(t-a)\big) = e^{-as}$	$L^{-1}(e^{-as}) = \delta(t-a)$
$L\big(f^{(n)}\big) = s^n L(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$	
Fundamental Theorem of Calculus: $L\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}L(f), s > 0$	What is the difference between the unit step function and the Dirac delta function?
Frequency-Shifting (s-shifting): $L\big(e^{ct}f(t)\big) = F(s-c)$	Time-shifting (t-shifting): $L^{-1}\big(e^{-as}F(s)\big) = f(t-a)\cdot u(t-a)$
$L(e^{ct}t^n) = \frac{n!}{(s-c)^{n+1}}$	$L^{-1}\left(\frac{1}{(s-c)^n}\right) = \frac{e^{ct}t^{n-1}}{(n-1)!}$
$L(e^{ct}\cos\omega t) = \frac{s-c}{(s-c)^2+\omega^2}$	$L^{-1}\left(\frac{s-c}{(s-c)^2+\omega^2}\right) = e^{ct}\cos\omega t$
$L(e^{ct}\sin\omega t) = \frac{\omega}{(s-c)^2+\omega^2}$	$L^{-1}\left(\frac{\omega}{(s-c)^2+\omega^2}\right) = e^{ct}\sin\omega t$
$L\big(t^n f(t)\big) = (-1)^n F^{(n)}(s)$	

- Unit function: $u(t-a) = \begin{cases} 0, t < a \\ 1, t \geq a \end{cases}$. Thus, $g(t)[u(t-a) - u(t-b)] = \begin{cases} 0, t < a \\ g(t), a < t < b \\ 0, t > b \end{cases}$.
- The Dirac Delta function has properties $\delta(t-a) = 0$ for $t \neq a$ and an infinitely large magnitude (height) at the time $t = a$. Also, it has the property $\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1$ and $\int_{-\infty}^{\infty} \delta(t-a)g(t)\,dt = g(a)$.

TUTORIAL PROBLEMS

Question 1

The bacteria in a certain culture number 10000 initially. Two and a half hours later there are 11000 of them. Assuming a **Malthus model**, how many bacteria will there be 10 hours after the start of the experiment? How long will it take for the number to reach 20000?

Solutions

Since we are assuming the **Malthus model**, the following DE models the above problem:

$$\frac{dN}{dt} = kN$$

The solution (verify it) is given by (with 2 unknowns N_0 and k):

$$N = N_0 e^{kt}$$

We are given two conditions: $N(0) = 10000$ and $N\left(\frac{5}{2}\right) = 11000$. Substituting into the solution above and solving simultaneously, we get $N_0 = 10000$ and $k = \frac{2}{5}\ln 1.1 \approx 0.0381$.

To find the number of bacteria after 10 hours, we let $t = 10$.

$$N(10) = 10000e^{(0.0381)(10)} \approx \mathbf{14641}$$

To find the time when the number of bacteria reaches 20000, let $N(t) = 20000$.

$$10000e^{0.0381t} = 20000 \Longrightarrow t \approx \mathbf{18.18\ hrs}$$

Question 2

You have 200 bugs in a bottle. Every day you supply them with food and count them. After two days you have 360 bugs. It is known that the birth rate for this kind of bug is 150% per day. [Is this a sensible way of stating a birth rate per capita? Why?] Assuming that the population is given by a **logistic model**, find the number of bugs after 3 days. Predict how many bugs you will have eventually.

Solutions

Since the number of bugs is increasing, we will use the following formula:

$$N(t) = \frac{N_\infty}{1 + \left(\frac{N_\infty}{N_0} - 1\right)e^{-Bt}}$$

The way of stating birth rate is sensible. It means that there is a population increase due to birth by 1.5 times of the current population. Hence, substituting with $N_0 = 200, B = 1.5$, and taking $N(2) = 360$ we will obtain $N_\infty = B/s \approx \mathbf{376}$. Also we get $s \approx 0.00399$.

To find out number of bugs after 3 days, let $t = 3$. We will get $N(3) \approx \mathbf{372}$.

Question 3

In Question 2, let us assume that you are keeping the bugs not as a hobby, but because you are developing a new insecticide. Suppose that you remove 80 bugs per day from the bottle, and that all of these bugs die as a result of being sprayed with this insecticide. What is the limiting population in this case? What is the maximum number of bugs you can put to death per day without causing the population to die out?

Solutions

We will use the logistic model with harvesting.

Part 1 Find out which case of the harvesting model this question is about.

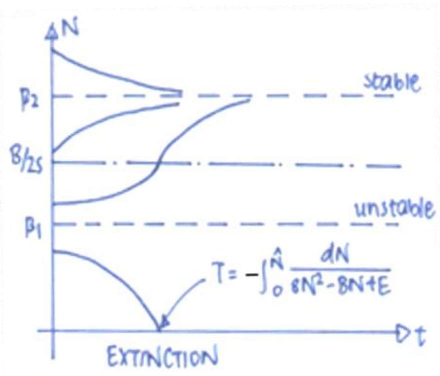
We will now assume the logistic model with harvesting, $E = 80$. We will calculate $B^2 - 4sE = 0.9732 > 0$ (using B and s from Question 2) which is **Case 3**.

Part 2 Classify the initial data in order to find the limiting population.

First, we need to find the values of β_1 and β_2, by solving $1.5N - 0.00399N^2 - 80 = 0$. Thus, we will obtain $\beta_1 = 64.3472$ and $\beta_2 = 311.5926$. Since the initial population is $N(0) = 200$ and that $\beta_1 < N(0) < \beta_2$, the limiting population in this case will be $\beta_2 = 311.5926 \approx \mathbf{312}$.

Part 3 Find the maximum number of bugs you can kill without causing population to die out.

Compute $\frac{B^2}{4s}$ and make sure that $E < \frac{B^2}{4s} \approx 140.98$. The answer is **140**.



Question 4

The sandhill crane is a beautiful Canadian bird with an unfortunate liking for farm crops. For many years the cranes were protected by law, and eventually they settled down to a logistic equilibrium population of 194,600 with birth rate per capita 9.866% per year. Eventually the patience of the farmers was exhausted and they managed to have the hunting ban lifted. The farmers happily shot 10000 cranes per year, which they argued was reasonable enough since it only represents about 5% of the original population. Show that the sandhill crane is doomed. How long will it take, from the legalisation of hunting, to exterminate them?

Solutions

Since the logistic equilibrium $\frac{B}{s} = 194600$ and that $B = 0.09866$, we get $s = 5.07 \times 10^{-7}$.

Consider the logistic model with harvesting $E = 10000$.

Step 1 Calculate $B^2 - 4sE$ to determine the case of the harvesting model.

Since $B^2 - 4sE = -0.0105 < 0$, this is **Case 1**, where the population of birds will decrease to zero eventually, regardless of the initial population.

Step 2 Find the time required to exterminate the species.

$$T = \int_{N_0}^{0} \frac{dN}{BN - sN^2 - E} = \int_{194600}^{0} \frac{dN}{0.09866N - (5.07 \times 10^{-7})N^2 - 10000} \approx \mathbf{29.81}$$

Question 5

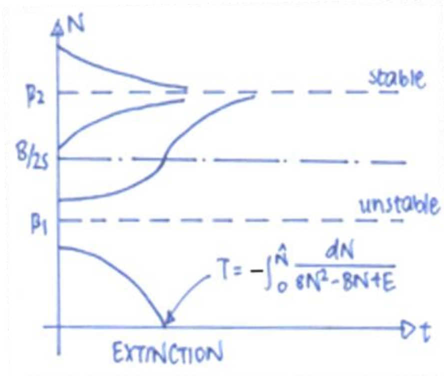
Suppose that Peruvian fishermen take a fixed number of anchovies per year from an anchovy stock which would otherwise behave logistically, apart from occasional natural disasters. According to our lecture notes, any fishing rate $\geq B^2/4s$ will be disastrous. Let us call this number E^*. The fishermen want to take as many anchovies as they **safely** can, meaning that they want the fish to be able to bounce back from a natural disaster that pushes their population down by 10%. Advise them. That is, tell them the maximum number of fish they can take, expressed as a percentage of E^*.

[Hint: assume that you start with the stable equilibrium population β_2, and compute the value of E, the harvesting rate, such that β_1, the **unstable** equilibrium population, becomes 90% of β_2.]

Solutions

First, we will find β_1 and β_2.

$$\beta_1 = \frac{B - \sqrt{B^2 - 4sE}}{2s} \qquad \beta_2 = \frac{B + \sqrt{B^2 - 4sE}}{2s}$$



Next, we need to find the value of E such that β_1 becomes 90% of β_2.

$$\begin{aligned}
\beta_1 &= 0.9\beta_2 \\
\frac{B - \sqrt{B^2 - 4sE}}{2s} &= 0.9\frac{B + \sqrt{B^2 - 4sE}}{2s} \\
B - \sqrt{B^2 - 4sE} &= 0.9B + 0.9\sqrt{B^2 - 4sE} \\
1.9\sqrt{B^2 - 4sE} &= 0.1B
\end{aligned}$$

$$E = \frac{1}{4s}\left(B^2 - \frac{1}{361}B^2\right) = \frac{360}{361}\left(\frac{B^2}{4s}\right) = \frac{360}{361}E^*$$

Thus, the maximum number of fish that the fishermen can harvest is about **99.7% of E^***.

Question 6

Find the Laplace transforms of the following functions [where u denotes the unit step function and the answers are given in brackets]:

Solutions

(a) $t^2 e^{-3t}$

Using the formula $L(t^n e^{ct}) = \frac{n!}{(s-c)^{n+1}}$, we get $L(t^2 e^{-3t}) = \boxed{\frac{2}{(s+3)^3}}$.

(b) $t \cdot u(t-2)$

Using the formula $L\big(f(t-a)\cdot u(t-a)\big) = e^{-as}\cdot F(s)$, we get

$L\big(t\cdot u(t-2)\big) = L\big((t-2+2)\cdot u(t-2)\big)$,

And so we know that $f(t-2) = (t-2)+2$, so $f(t) = t+2$ and $F(s) = L\big(f(t)\big) = \frac{1}{s^2} + \frac{2}{s}$. And we get

$$L\big(t\cdot u(t-2)\big) = L\big((t-2+2)\cdot u(t-2)\big) = \boxed{e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right)}$$

Question 7

Find the inverse Laplace transforms of the following functions:

Solutions

(a) $\frac{s}{s^2+10s+26}$

Since $\frac{s}{s^2+10s+26} \equiv \frac{s}{(s+5)^2+1} \equiv \frac{(s+5)-5}{(s+5)^2+1}$,we expect to use the formula $L^{-1}\left(\frac{s-c}{(s-c)^2+\omega^2}\right) = e^{ct}\cos\omega t$ and $L^{-1}\left(\frac{\omega}{(s-c)^2+\omega^2}\right) = e^{ct}\sin\omega t$. Thus we get

$$\begin{aligned} L^{-1}\left(\frac{s}{s^2+10s+26}\right) &= L^{-1}\left(\frac{(s+5)-5}{(s+5)^2+1}\right) \\ &= L^{-1}\left(\frac{(s+5)}{(s+5)^2+1}\right) - 5L^{-1}\left(\frac{1}{(s+5)^2+1}\right) \\ &= e^{-5t}\cos t - 5e^{-5t}\sin t \\ &= \boxed{e^{-5t}(\cos t - 5\sin t)} \end{aligned}$$

(b) $e^{-2s}\frac{1+2s}{s^3}$

We will use the formula $L\big(f(t-a)\cdot u(t-a)\big) = e^{-as}\cdot F(s)$.

Since $e^{-2s}\frac{1+2s}{s^3} = e^{-2s}\left(\frac{1}{s^3}+\frac{2}{s^2}\right)$, define $F(s) = \frac{1}{s^3}+\frac{2}{s^2}$.

As such, we use $L(t^n) = \frac{n!}{s^{n+1}}$ to get:

$$L^{-1}\left(\frac{1}{s^3}+\frac{2}{s^2}\right) = \frac{t^2}{2} + 2t = f(t)$$

Then, using t-shifting: $L^{-1}\big(e^{-as}F(s)\big) = f(t-a)\cdot u(t-a)$ with $a = 2$, we have

$$\begin{aligned} L^{-1}\left(e^{-2s}\frac{1+2s}{s^3}\right) &= \left[\frac{(t-2)^2}{2} + 2(t-2)\right]u(t-2) \\ &= \boxed{\left(\frac{1}{2}t^2 - 2\right)u(t-2)} \end{aligned}$$

Question 8

Solve the following initial value problems using Laplace transforms:

Solutions

(a) $y' = t\cdot u(t-2) \qquad y(0) = 4$

Performing Laplace transform and assuming $L(y) = Y(s)$, we get

$$sY(s) - y(0) = L\big(t\cdot u(t-2)\big)$$

Notice that the RHS is exactly what we are asked to evaluate in Question 6(b). Thus,

$$\begin{aligned} sY(s) - 4 &= e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right) \\ Y(s) &= e^{-2s}\frac{1+2s}{s^3} + \frac{4}{s} \end{aligned}$$

Now, the first term on RHS is exactly what we are asked to evaluate in Question 7(b). Thus,

$$\boxed{y(t) = \left(\frac{1}{2}t^2 - 2\right)u(t-2) + 4}.$$

(b) $y'' - 2y' = 4 \qquad y(0) = 1 \qquad y'(0) = 0$

Performing Laplace transform, we get

$$s^2Y(s) - sy(0) - y'(0) - 2[sY(s) - y(0)] = \frac{4}{s}$$
$$s^2Y(s) - 2sY(s) = \frac{4}{s} + s - 2$$
$$Y(s) = \frac{4 - 2s + s^2}{s^2(s-2)} = \frac{1}{s-2} - \frac{2}{s^2}$$

Performing inverse Laplace transform (**Question** What formulas are you using now?), we get

$$\boxed{y(t) = e^{2t} - 2t}$$

- This method may be simpler (and shorter) compared to the method introduced in Chapter 1 (method of undetermined coefficients or variation of parameters). Try using methods in Chapter 1 for practice.

Question 9 [Rocket Flight]

A model rocket having initial mass m_0 kg is launched vertically from the ground. The rocket expels gas at a constant rate of α kg/s at a constant velocity of β m/s relative to the rocket. Assume that the magnitude of the gravitational force is proportional to the mass with proportionality constant g. Because the mass is not constant, Newton's second law leads to the equation

$$(m_0 - \alpha t)\frac{dv}{dt} - \alpha\beta = -g(m_0 - \alpha t),$$

where $v = dx/dt$ is the velocity of the rocket, x is its height above the ground, and $m_0 - \alpha t$ is the mass of the rocket t seconds after launch. If the initial velocity is zero, solve the above equation to determine the velocity of the rocket and its height above the ground for $0 \le t < m_0/\alpha$.

Solutions

The ODE is rewritten as

$$\frac{dv}{dt} = -g + \frac{\alpha\beta}{m_0 - \alpha t}$$
$$v = -gt - \frac{\alpha\beta}{\alpha}\ln|m_0 - \alpha t| + C$$

Since the initial velocity is zero, substitute initial condition $v(0) = 0$ into equation.

$$0 = -\beta\ln|m_0| + C \Rightarrow C = \beta\ln m_0.$$

Why is the condition $0 \le t < m_0/\alpha$ necessary?

Thus,

$$v(t) = -gt - \beta\ln(m_0 - \alpha t) + \beta\ln m_0$$
$$= -gt + \beta\ln\frac{m_0}{m_0 - \alpha t}$$

Since $v = dx/dt$ and we now solve for x:

$$\frac{dx}{dt} = -gt + \beta \ln m_0 - \beta \ln(m_0 - \alpha t)\,.$$

Using the fact that:

$$\int \ln(at+b)\,dt = \frac{(at+b)\ln(at+b) - at}{a} + C$$

$$x = -\frac{g}{2}t^2 + \beta t \ln m_0 + \frac{\beta}{\alpha}[(m_0 - \alpha t)\ln(m_0 - \alpha t) + \alpha t] + D$$

Since we define the starting position of model rocket to be our reference, we have initial condition $x(0) = 0$.

$$0 = \frac{\beta m_0}{\alpha}\ln m_0 + D \Longrightarrow D = -\frac{\beta m_0}{\alpha}\ln m_0$$

Then,

$$\begin{aligned}
x &= -\frac{g}{2}t^2 + \beta t \ln m_0 + \frac{\beta}{\alpha}[(m_0 - \alpha t)\ln(m_0 - \alpha t) + \alpha t] - \frac{\beta m_0}{\alpha}\ln m_0 \\
&= -\frac{g}{2}t^2 + \beta t \ln m_0 + \frac{\beta m_0}{\alpha}\ln(m_0 - \alpha t) - \beta t \ln(m_0 - \alpha t) + \beta t - \frac{\beta m_0}{\alpha}\ln m_0 \\
&= -\frac{g}{2}t^2 + \beta t + \beta t \ln\frac{m_0}{m_0 - \alpha t} + \frac{\beta m_0}{\alpha}\ln\frac{m_0 - \alpha t}{m_0} \\
&= \beta t - \frac{g}{2}t^2 + \beta t \ln\frac{m_0}{m_0 - \alpha t} - \frac{\beta m_0}{\alpha}\ln\frac{m_0}{m_0 - \alpha t}
\end{aligned}$$

$$\boxed{x = \beta t - \frac{g}{2}t^2 - \frac{\beta}{\alpha}(m_0 - \alpha t)\ln\frac{m_0}{m_0 - \alpha t}}\,.$$

Question 10

In the harvesting model we considered in the lectures, the population will rebound if all harvesting is stopped. Unhappily, this is not always true: for some animals, if you drive their population down too low, they will have trouble finding mates, or they will be forced to breed with relatively close kin, which reduces genetic variability and hence their ability to resist disease. For such animals [for example, certain rare species of tigers] extinction will result if the population falls too low, even if all harvesting is forbidden. Biologists call this **depensation**. Show that this situation can be modelled by the ODE

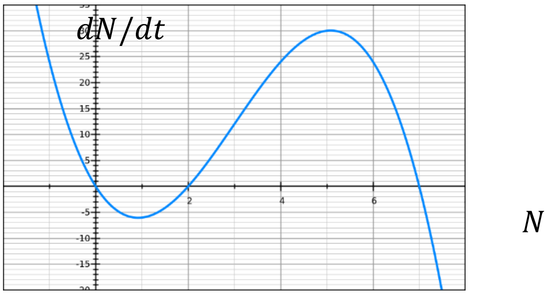
$$\frac{dN}{dt} = -aN^3 + bN^2 - cN$$

where N is the population and a, b, and c are positive constants such that $b^2 > 4ac$. Find the population below which extinction will occur.

Solutions

Part I Show that the situation can be modelled by $\frac{dN}{dt} = -aN^3 + bN^2 - cN$.

Depensation occurs if the population is low enough, the population will continue to fall till extinction. Translating into mathematics, it means that if N is small enough, $\frac{dN}{dt}$ will be negative. Clearly, for a, b, and c being positive constants, the DE $\frac{dN}{dt} = -aN^3 + bN^2 - cN$ gives negative $\frac{dN}{dt}$ for small values of N, as seen in the graph below.



As such, this is a suitable model for the phenomenon of depensation.

Part II Find out the population below which extinction will occur.

For extinction to occur, the rate of change of population $\frac{dN}{dt}$ should be decreasing till the population reaches zero. Observing the graph above, that population value we are looking for should be the smallest N-intercept other than $N = 0$. Hence, we need to solve

$$-aN^3 + bN^2 - cN = 0$$
$$-N(aN^2 - bB + c) = 0$$
$$N = 0 \text{ OR } N = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus, the required maximum population value is $\boxed{\frac{b-\sqrt{b^2-4ac}}{2a}}$.

Question Classify the stability of each of the equilibrium solutions.

- Notice that harvesting is not present in this whole situation. However, the behaviour of depensation is very similar to the logistic model with harvesting. Hence, this is a good model for depensation.