NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

Module: MA1508E Linear Algebra for Engineering
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Tutorial: 11

1. Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ a & 1 \end{pmatrix}$, where $a \in \mathbb{R}$. Find all values of a such that

   (a) $\boldsymbol{A}$ has only one eigenvalue.

   (b) $\boldsymbol{A}$ has two eigenvalues -1 and 2. In this case, compute $\boldsymbol{A}^{-10}$ using diagonalisation.

   (c) $\boldsymbol{A}$ has a pair of complex eigenvalues.

   The chracteristic equation of $\boldsymbol{A}$ is $(\lambda \boldsymbol{I} - \boldsymbol{A})$ is $\lambda^2 - \lambda - a = 0$.

   (a) For $\boldsymbol{A}$ to have only one eigenvalue, the 'discriminant' of $\boldsymbol{A}$ is 0. That is, $(-1)^2 - 4(1)(-a) = 0 \Leftrightarrow a = -\frac{1}{4}$.

   (b) For $\boldsymbol{A}$ to have two eigenvalues -1 and 2, the characteristic polynomial must be $(\lambda+1)(\lambda-2) = \lambda^2 - \lambda - 2$. Thus $a = 2$. In this case, we find that $\boldsymbol{A}$ is diagonalizable and by letting $\boldsymbol{P} = \begin{pmatrix} -1 & -1 \\ 1 & -2 \end{pmatrix}$, $\boldsymbol{P}^{-1} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} \end{pmatrix}$, we have

   $$\boldsymbol{A} = \boldsymbol{P} \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \boldsymbol{P}^{-1} \Leftrightarrow \boldsymbol{A}^{-1} = \boldsymbol{P}\boldsymbol{D}^{-1}\boldsymbol{P}^{-1} = \boldsymbol{P} \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \boldsymbol{P}^{-1}.$$

   So

   $$\boldsymbol{A}^{-10} = \boldsymbol{P} \begin{pmatrix} (-1)^{10} & 0 \\ 0 & (\frac{1}{2})^{10} \end{pmatrix} \boldsymbol{P}^{-1} = \begin{pmatrix} \frac{2}{3} + \frac{1}{3}(\frac{1}{2})^{10} & -\frac{1}{3} + \frac{1}{3}(\frac{1}{2})^{10} \\ -\frac{2}{3} + \frac{1}{3}(\frac{1}{2})^{9} & \frac{1}{3} + \frac{1}{3}(\frac{1}{2})^{9} \end{pmatrix}.$$

   (c) For $\boldsymbol{A}$ to have a pair of complex eigenvalues, the 'discriminant' of $\boldsymbol{A}$ is negative. That is, $a < -\frac{1}{4}$.

2. Each matrix $\boldsymbol{A}$ below has complex eigenvalues. Find a matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{A}$ and determine $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$.

   (a) $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; (b) $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$; (c) $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{pmatrix}$.

   (a) Eigenvalues are $-i$ and i.

   Let $\boldsymbol{P} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$.

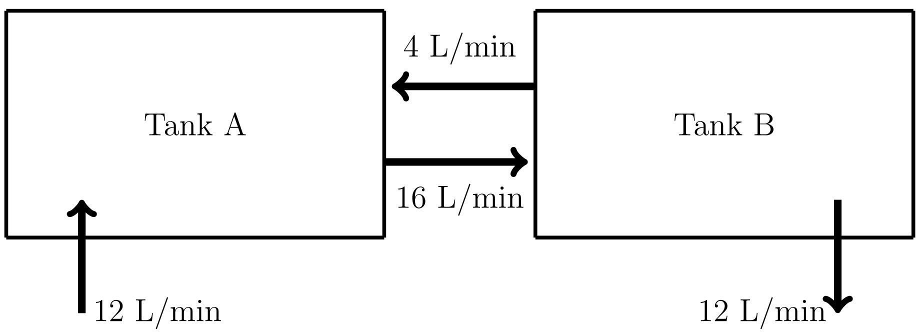
(b) Eigenvalues are $2-i$ and $2+i$.

Let $\boldsymbol{P} = \begin{pmatrix} 1+i & 1-i \\ 2 & 2 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} 2-i & 0 \\ 0 & 2+i \end{pmatrix}$.

(c) Eigenvalues are 0, $2-i$ and $2+i$.

Let $\boldsymbol{P} = \begin{pmatrix} 1 & 1+3i & 1-3i \\ 0 & 5i & -5i \\ 0 & 5 & 5 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2-i & 0 \\ 0 & 0 & 2+i \end{pmatrix}$.

3. Consider two large tanks that are connected as shown in the figure below.



Tank A is initially filled with 100 L (litres) of water and 40 g (grams) of salt was dissolved in it. Tank B is initially filled with 100 L of water and 20 g of salt was dissolved in it. The well-mixed solution from Tank A is constantly pumped into Tank B at the rate of 16 L per minute while the solution in Tank B is pumped back into Tank A at the rate of 4 L per minute. Pure water is constantly pumped into Tank A at the rate of 12 L per minute while water exits the system from Tank B at the rate of 12 L per minute.

At t minutes after the start of the mixing, let $a(t)$ and $b(t)$ be the amount of salt in Tanks A and B respectively. Construct a system of linear first order differential equations to evaluate $a(t)$ and $b(t)$ for each t.

Hence deduce that the amount of salt in Tank B will always be less than twice the amount of salt in Tank A.

Consider tank A:

$$\begin{aligned} \text{rate of salt flowing in} &= \frac{4b(t)}{100} = \frac{2b(t)}{50} \\ \text{rate of salt flowing out} &= \frac{16a(t)}{100} = \frac{8a(t)}{50} \end{aligned}$$

Consider tank B:

$$\begin{aligned} \text{rate of salt flowing in} &= \frac{16a(t)}{100} = \frac{8a(t)}{50} \\ \text{rate of salt flowing out} &= \frac{16b(t)}{100} = \frac{8b(t)}{50} \end{aligned}$$

So

$$\begin{cases} a'(t) &=& -\frac{8a(t)}{50} &+& \frac{2b(t)}{50} \\ b'(t) &=& \frac{8a(t)}{50} &-& \frac{8b(t)}{50} \end{cases} = \begin{pmatrix} -\frac{8}{50} & \frac{2}{50} \\ \frac{8}{50} & -\frac{8}{50} \end{pmatrix} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}.$$

So we have $\boldsymbol{Y'} = \boldsymbol{AY}$ where

$$\boldsymbol{A} = \begin{pmatrix} -\frac{8}{50} & \frac{2}{50} \\ \frac{8}{50} & -\frac{8}{50} \end{pmatrix} \quad \text{and} \quad \boldsymbol{Y}(0) = \begin{pmatrix} a(0) \\ b(0) \end{pmatrix} = \begin{pmatrix} 40 \\ 20 \end{pmatrix}.$$

$$\begin{aligned} \det(\lambda \boldsymbol{I} - \boldsymbol{A}) &= \begin{vmatrix} \lambda + \frac{8}{50} & -\frac{2}{50} \\ -\frac{8}{50} & \lambda + \frac{8}{50} \end{vmatrix} \\ &= \lambda^2 + \frac{16\lambda}{50} + \frac{48}{2500} \\ &= (\lambda + \frac{12}{50})(\lambda + \frac{4}{50}) \end{aligned}$$

So the eigenvalues of $\boldsymbol{A}$ are $\lambda_1 = -\frac{12}{50}$ and $\lambda_2 = -\frac{4}{50}$.

Consider the eigenspace E_{λ_1}:

$$\left(\begin{array}{cc|c} -\frac{4}{50} & -\frac{2}{50} & 0 \\ -\frac{8}{50} & -\frac{4}{50} & 0 \end{array} \right) \longrightarrow \left(\begin{array}{cc|c} 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array} \right)$$

So $E_{\lambda_1} = \text{span}\{(-1,2)^T\}$.

Consider the eigenspace E_{λ_2}:

$$\left(\begin{array}{cc|c} \frac{4}{50} & -\frac{2}{50} & 0 \\ -\frac{8}{50} & \frac{4}{50} & 0 \end{array} \right) \longrightarrow \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array} \right)$$

So $E_{\lambda_2} = \text{span}\{(1,2)^T\}$.

Thus

$$\begin{aligned} \boldsymbol{Y} &= k_1 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-\frac{12t}{50}} + k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-\frac{4t}{50}} \\ \boldsymbol{Y}(0) = \begin{pmatrix} 40 \\ 20 \end{pmatrix} &= \begin{pmatrix} -k_1 + k_2 \\ 2k_1 + 2k_2 \end{pmatrix} \end{aligned}$$

So $100 = 4k_2 \Rightarrow k_2 = 25$ and $k_1 = -15$. Thus

$$\boldsymbol{Y} = -15 \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-\frac{12t}{50}} + 25 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-\frac{4t}{50}}$$

$$\begin{aligned} a(t) &= 15e^{-\frac{12t}{50}} + 25e^{-\frac{4t}{50}} \\ b(t) &= -30e^{-\frac{12t}{50}} + 50e^{-\frac{4t}{50}} \end{aligned}$$

Since $2a(t) - b(t) = 60e^{-\frac{12t}{50}} > 0$, we conclude that $2a(t) > b(t)$ for all t which implies that the amount of salt in tank B will always be less than twice the amount of salt in tank A.

4. Two species of fish, species A and species B, live in the same ecosystem (e.g. a pond) and compete with each other for food, water and space. Let the population of species A and B at time t years be given by $a(t)$ and $b(t)$ respectively.

   In the absence of species B, species A's growth rate is $4a(t)$ but when species B are present, the competition slows the growth of species A to $a'(t) = 4a(t) - 2b(t)$. In a similar manner, when species A is absent, species B's growth rate is $3b(t)$ but in the presence of species A, the growth rate reduces to $b'(t) = 3b(t) - a(t)$.

   (i) Write down a system of linear differential equations involving $a(t), b(t), a'(t)$ and $b'(t)$.

   (ii) Represent the system in (i) as $\boldsymbol{x'}(t) = \boldsymbol{A}\boldsymbol{x}(t)$ where

   $$\boldsymbol{A} \text{ is a } 2 \times 2 \text{ matrix and } \boldsymbol{x}(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}, \quad \boldsymbol{x'}(t) = \begin{pmatrix} a'(t) \\ b'(t) \end{pmatrix}.$$

   (iii) Solve the system using the initial condition $a(0) = 60$, $b(0) = 120$.

(i)

$$\begin{cases} a'(t) &= & 4a(t) & - & 2b(t) \\ b'(t) &= & -a(t) & + & 3b(t) \end{cases}$$

(ii) Let

$$\boldsymbol{x}'(t) = \begin{pmatrix} a'(t) \\ b'(t) \end{pmatrix} = \boldsymbol{A}\boldsymbol{x}(t) = \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}.$$

(iii) We first find the eigenvalues of $\boldsymbol{A}$:

$$\begin{aligned} \begin{vmatrix} \lambda - 4 & 2 \\ 1 & \lambda - 3 \end{vmatrix} &= (\lambda - 4)(\lambda - 3) - 2 \\ &= \lambda^2 - 7\lambda + 12 - 2 \\ &= \lambda^2 - 7\lambda + 10 = (\lambda - 2)(\lambda - 5) \end{aligned}$$

So $\boldsymbol{A}$ has two distinct eigenvalues $\lambda = 2$ and $\lambda = 5$.
Solving $(2\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = x_2 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad s \in \mathbb{R}.$$

So $\{(1, 1)^T\}$ is a basis for E_2.
Solving $(5\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_1 = -2x_2 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad s \in \mathbb{R}.$$

So $\{(-2, 1)^T\}$ is a basis for E_5.

A general solution to the given system is

$$\boldsymbol{x}(t) = A\begin{pmatrix}1\\1\end{pmatrix}e^{2t} + B\begin{pmatrix}-2\\1\end{pmatrix}e^{5t}$$

$$\text{i.e.} \qquad \begin{pmatrix}a(t)\\b(t)\end{pmatrix} = \begin{pmatrix}Ae^{2t} - 2Be^{5t}\\Ae^{2t} + Be^{5t}\end{pmatrix}$$

Using the given initial conditions:

$$\begin{cases} a(0) = 60 = A - 2B \\ b(0) = 120 = A + B \end{cases}$$

We find that $B = 20$, $A = 100$. So

$$\boldsymbol{x}(t) = 100\begin{pmatrix}e^{2t}\\e^{2t}\end{pmatrix} + 20\begin{pmatrix}-2e^{5t}\\e^{5t}\end{pmatrix} = \begin{pmatrix}100e^{2t} - 40e^{5t}\\100e^{2t} + 20e^{5t}\end{pmatrix}.$$

5. **(Repeated eigenvalues)** This question illustrates what we should do if a system of linear differential equations $\boldsymbol{Y'} = \boldsymbol{AY}$ (where $\boldsymbol{A}$ is a 2×2 matrix) is such that $\boldsymbol{A}$ has only 1 eigenvalue λ and $\dim(E_\lambda) = 1$.

   Suppose $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue λ. Let $\boldsymbol{u}$ be a non zero vector in $\mathbb{R}^2$ such that

   $$(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{u} = \boldsymbol{v}.$$

   Prove that

   $$\boldsymbol{Y}(t) = c_1e^{\lambda t}\boldsymbol{v} + c_2e^{\lambda t}(t\boldsymbol{v} + \boldsymbol{u}), \qquad c_1, c_2 \in \mathbb{R}$$

   satisfies $\boldsymbol{Y'} = \boldsymbol{AY}$ and is thus a solution to the system of linear differential equations. We call this solution a **generalised** eigenvector of $\boldsymbol{A}$ associated with λ.

   Use the technique above to solve the system of linear differential equations $\boldsymbol{Y'} = \boldsymbol{AY}$ where $\boldsymbol{A} = \begin{pmatrix}-1 & 1\\-4 & 3\end{pmatrix}$ and the system has the initial condition $y_1(0) = 1$ and $y_2(0) = 3$.

   Note that

   $$(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{u} = \boldsymbol{v} \Leftrightarrow \boldsymbol{Au} - \lambda\boldsymbol{u} = \boldsymbol{v}.$$

   We now check that $\boldsymbol{Y'} = \boldsymbol{AY}$. With the given $\boldsymbol{Y}$,

   $$\boldsymbol{Y'} = \lambda c_1e^{\lambda t}\boldsymbol{v} + c_2e^{\lambda t}\boldsymbol{v} + \lambda c_2e^{\lambda t}(t\boldsymbol{v} + \boldsymbol{u})$$

   While

   $$\begin{aligned}\boldsymbol{AY} &= c_1e^{\lambda t}(\boldsymbol{Av}) + c_2e^{\lambda t}(t\boldsymbol{Av} + \boldsymbol{Au})\\ &= c_1e^{\lambda t}(\lambda\boldsymbol{v}) + c_2e^{\lambda t}(t\lambda\boldsymbol{v} + \lambda\boldsymbol{u} + \boldsymbol{v})\\ &= \lambda c_1e^{\lambda t}\boldsymbol{v} + \lambda c_2e^{\lambda t}(t\boldsymbol{v} + \boldsymbol{u}) + c_2e^{\lambda t}\boldsymbol{v}\end{aligned}$$

   Thus indeed we have $\boldsymbol{Y'} = \boldsymbol{AY}$.

To solve $\boldsymbol{Y'} = \boldsymbol{AY}$, we first find the eigenvalues of $\boldsymbol{A}$:

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0 \Rightarrow (\lambda - 1)^2 = 0.$$

So $\lambda = 1$ is a repeated eigenvalue. Solve

$$(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0} \Rightarrow \boldsymbol{x} \in \operatorname{span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}.$$

So $E_1 = \operatorname{span}\{(1,2)^T\}$. We now find a non zero vector $\boldsymbol{u}$ such that $(\boldsymbol{A} - \boldsymbol{I})\boldsymbol{u} = \boldsymbol{v}$.

$$\left(\begin{array}{c|c} \boldsymbol{A} - \boldsymbol{I} & \boldsymbol{v} \end{array}\right) = \left(\begin{array}{cc|c} -2 & 1 & 1 \\ -4 & 2 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 2 & -1 & -1 \\ 0 & 0 & 0 \end{array}\right).$$

So $\boldsymbol{u} = (-\frac{1}{2} + \frac{s}{2}, s)^T$ where $s \in \mathbb{R}$. We may choose $\boldsymbol{u} = (0,1)^T$. Thus a solution to $\boldsymbol{Y'} = \boldsymbol{AY}$ can be

$$\boldsymbol{Y} = c_1 e^t \begin{pmatrix}1\\2\end{pmatrix} + c_2 e^t \left\{ t\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right\}.$$

Using the initial conditions, we find that $c_1 = 1$ and $c_2 = 1$. So the particular solution is

$$\boldsymbol{Y} = \begin{pmatrix} e^t + te^t \\ 3e^t + 2te^t \end{pmatrix}.$$

6. Solve the following systems of second order linear differential equations.

   (a) $y'' + 2y' + 5y = 0$;

   (b)

$$\begin{cases} y_1'' = & -2y_2 + y_1' + 2y_2' \\ y_2'' = 2y_1 & + 2y_1' - y_2' \end{cases}$$

   with initial conditions $y_1(0) = 1$, $y_2(0) = 0$, $y_1'(0) = -3$, $y_2'(0) = 2$.

   (a) Let $z = y'$. Then $y'' = -5y - 2y' \Leftrightarrow z' = -5y - 2y'$. Together with $y' = z$, we have

$$\begin{cases} z' = -5y - 2z \\ y' = z \end{cases}$$

   Let $\boldsymbol{Y} = \begin{pmatrix}z\\y\end{pmatrix}$, then we have $\boldsymbol{Y'} = \boldsymbol{AY}$ where $\boldsymbol{A} = \begin{pmatrix}-2 & -5\\1 & 0\end{pmatrix}$. The eigenvalues of $\boldsymbol{A}$ are $\lambda = -1 + 2i$ and $\bar{\lambda} = -1 - 2i$. Solving

$$(\lambda \boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0} \Rightarrow \boldsymbol{x} \in \operatorname{span}\left\{\begin{pmatrix}-1+2i\\1\end{pmatrix}\right\}.$$

Thus $E_\lambda = \text{span}\{(-1+2i, 1)^T\}$. Let $\boldsymbol{y} = (-1+2i, 1)^T$. Two real solutions to the system of linear differential equations are $\text{Re}(e^{\lambda t}\boldsymbol{y})$ and $\text{Im}(e^{\lambda t}\boldsymbol{y})$, where

$$\begin{aligned}
e^{\lambda t}\boldsymbol{y} &= e^{(-1+2i)t}\begin{pmatrix}-1+2i\\ 1\end{pmatrix}\\
&= e^{-t(\cos 2t + i\sin 2t)}\begin{pmatrix}-1+2i\\ 1\end{pmatrix}\\
&= e^{-t}\begin{pmatrix}-\cos 2t - 2\sin 2t + i(2\cos 2t - \sin 2t)\\ \cos 2t + i\sin 2t\end{pmatrix}
\end{aligned}$$

So the two real solutions are

$$\boldsymbol{Y_1} = e^{-t}\begin{pmatrix}-\cos 2t - 2\sin 2t\\ \cos 2t\end{pmatrix} \text{ and } \boldsymbol{Y_2} = e^{-t}\begin{pmatrix}2\cos 2t - \sin 2t\\ \sin 2t\end{pmatrix}.$$

A general solution is

$$\begin{pmatrix}z\\ y\end{pmatrix} = \boldsymbol{Y} = c_1e^{-t}\begin{pmatrix}-\cos 2t - 2\sin 2t\\ \cos 2t\end{pmatrix} + c_2e^{-t}\begin{pmatrix}2\cos 2t - \sin 2t\\ \sin 2t\end{pmatrix}$$

and the solution to the original second-order differential equation is $y = c_1e^{-t}\cos 2t + c_2e^{-t}\sin 2t$ where $c_1, c_2 \in \mathbb{C}$.

(b) Let $y_3 = y_1'$ and $y_4 = y_2'$. Then we have

$$\begin{cases}
y_1' = y_3\\
y_2' = y_4\\
y_3' = -2y_2 + y_3 + 2y_4\\
y_4' = 2y_1 + 2y_3 - y_4
\end{cases}$$

Let $\boldsymbol{A} = \begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & -2 & 1 & 2\\ 2 & 0 & 2 & -1\end{pmatrix}$. Then the eigenvalues of $\boldsymbol{A}$ are $\lambda_1 = -2$, $\lambda_2 = 2$, $\lambda_3 = -1$, $\lambda_4 = 1$ and the corresponding eigenvectors are

$$\boldsymbol{x_1} = \begin{pmatrix}-1\\ 1\\ 2\\ -2\end{pmatrix}, \quad \boldsymbol{x_2} = \begin{pmatrix}-1\\ -1\\ -2\\ -2\end{pmatrix}, \quad \boldsymbol{x_3} = \begin{pmatrix}-2\\ 1\\ 2\\ -1\end{pmatrix}, \quad \boldsymbol{x_4} = \begin{pmatrix}-1\\ -2\\ -1\\ -2\end{pmatrix}.$$

Thus a general solution to the first-order system is of the form

$$c_1\boldsymbol{x_1}e^{-2t} + c_2\boldsymbol{x_2}e^{2t} + c_3\boldsymbol{x_3}e^{-t} + c_4\boldsymbol{x_4}e^{t}.$$

Now we use the initial condition provided to find c_1, c_2, c_3, c_4. When $t = 0$, we have

$$c_1\boldsymbol{x_1} + c_2\boldsymbol{x_2} + c_3\boldsymbol{x_3} + c_4\boldsymbol{x_4} = (1, 0, -3, 2)^T.$$

Solving, we have $c_1 = -1, c_2 = 1, c_3 = 0, c_4 = -1$. Thus

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} e^{-2t} - e^{2t} + e^t \\ -e^{-2t} - e^{2t} + 2e^t \\ -2e^{-2t} - 2e^{2t} + e^t \\ 2e^{-2t} - 2e^{2t} + 2e^t \end{pmatrix}.$$

In particular, the solution to the second-order system is

$$y_1(t) = e^{-2t} - e^{2t} + e^t; \quad y_2(t) = -e^{-2t} - e^{2t} + 2e^t.$$