NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

Module: MA1508E Linear Algebra for Engineering
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Tutorial: 7

1. Let

$$\boldsymbol{u_1} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \quad \boldsymbol{u_2} = \begin{pmatrix} -1 \\ 1 \\ 3 \\ 1 \end{pmatrix}, \quad \boldsymbol{u_3} = \begin{pmatrix} 4 \\ 5 \\ -6 \\ -1 \end{pmatrix}.$$

(a) Show that $\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$ is a linearly independent set. Is $\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$ a basis for $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$? What is the dimension of $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$? Write down a basis for $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$.

(b) Show that $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is a linearly dependent set. What is the dimension of $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$? Write down a basis for $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$.

(c) Find a vector $\boldsymbol{u_4}$ such that the dimension of $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_4}\}$ is 3.

(d) Find a basis for $\mathbb{R}^4$ that contains $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$.

(a) It is clear that $\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$ is a linearly independent set since $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ are not multiples of each other. Yes $\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$ is a basis for $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$. The dimension of $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$ is 2. A basis for $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$ is $\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$.

(b) It can be shown easily that $\boldsymbol{u_3} = 3\boldsymbol{u_1} - \boldsymbol{u_2}$. So $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is a linearly dependent set. The dimension of $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is the same as that of $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$, which is 2 and $\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$ is a basis.

(c) We find $\boldsymbol{u_4}$ such that $\boldsymbol{u_4}$ is not a linear combination of $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$.

$$\left(\begin{array}{cc|c} 1 & -1 & x \\ 2 & 1 & y \\ -1 & 3 & z \\ 0 & 1 & w \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -1 & x \\ 0 & 3 & y-2x \\ 0 & 0 & z - \frac{2y}{3} + \frac{7x}{3} \\ 0 & 1 & w \end{array}\right)$$

So we can choose $\boldsymbol{u_4} = (0, 0, 1, 0)$.

(d) We can use the row space method as follows. Create a 3×4 matrix $\boldsymbol{A}$ such that the rows of $\boldsymbol{A}$ are the vectors $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_4}$. Find a row-echelon form $\boldsymbol{R}$ of $\boldsymbol{A}$ and identify which column does not have a leading entry:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & -1 & 0 \\ -1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -\frac{2}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{pmatrix} = \boldsymbol{R}$$

Since the fourth column of $\boldsymbol{R}$ does not have a leading entry, we see that $(0, 0, 0, 1)$ does not belong to $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_4}\}$. So a basis for $\mathbb{R}^4$ is $\{\boldsymbol{u_1}, \boldsymbol{u_2}, (0, 0, 1, 0), (0, 0, 0, 1)\}$.

2. Let V and W be subspaces of $\mathbb{R}^n$. Suppose S_1 and S_2 are two sets such that $\text{span}(S_1) = V$ and $\text{span}(S_2) = W$. Define the set $V + W$ as

$$V + W = \{\boldsymbol{v} + \boldsymbol{w} \mid \boldsymbol{v} \in V, \boldsymbol{w} \in W\}.$$

(a) Show that $S_1 \cup S_2$ spans $V + W$, that is, $V + W = \text{span}(S_1 \cup S_2)$. This would establish the result that $V + W$ is always a subspace.

(b) For each of the following,

(i) Find S_1 and S_2 that spans V and W respectively. Check if S_1 and S_2 are bases for V and W respectively. What is the dimension of V and W?

(ii) Write $V + W$ as a linear span. Find a basis for $V + W$ and state its dimension.

(iii) Is $V \cap W$ a subspace of $\mathbb{R}^n$? Explain your answer. If $V \cap W$ is a subspace, find a basis for $V \cap W$ and state its dimension.

(1) $V = \{(s, 0) \mid s \in \mathbb{R}\}$, $W = \{(0, t) \mid t \in \mathbb{R}\}$.

(2) $V = \{(x, y, z) \mid 2x - y + 3z = 0\}$, $W = \{(a, a, a) \mid a \in \mathbb{R}\}$.

(3) $V = \{(a, b, c, d) \mid a - 2b + c - d = 0 \;\text{ and }\; 2a + c + 2d = 0\}$,
$W = \{(r, 2r, r, -r) \mid r \in \mathbb{R}\}$.

(a) Let $\boldsymbol{u} \in V + W$, then $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$ for some $\boldsymbol{v} \in V$ and $\boldsymbol{w} \in W$. Let $S_1 = \{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_k}\}$ and $S_2 = \{\boldsymbol{b_1}, \boldsymbol{b_2}, \cdots, \boldsymbol{b_r}\}$. Since S_1 spans V and S_2 spans W,

$$\boldsymbol{v} = c_1\boldsymbol{a_1} + c_2\boldsymbol{a_2} + \cdots + c_k\boldsymbol{a_k}; \quad \text{and}$$

$$\boldsymbol{w} = d_1\boldsymbol{b_1} + d_2\boldsymbol{b_2} + \cdots + d_r\boldsymbol{b_r},$$

for some real numbers $c_1, c_2, \cdots, c_k, d_1, d_2, \cdots, d_r$. Thus

$$\boldsymbol{u} = c_1\boldsymbol{a_1} + \cdots + c_k\boldsymbol{a_k} + d_1\boldsymbol{b_1} + \cdots + d_r\boldsymbol{b_r}.$$

Thus every vector in $V + W$ is a linear combination of vectors in $S_1 \cup S_2$, that is, $S_1 \cup S_2$ spans $V + W$.

(b) (1) (i) $S_1 = \{(1, 0)\}$ spans V. $S_2 = \{(0, 1)\}$ spans W. S_1 and S_2 are bases for V and W respectively. The dimension of V and W are both equals to 1.

(ii) As discussed in part (a), $V + W = \text{span}\{(1, 0), (0, 1)\}$. Since $\{(1, 0), (0, 1)\}$ is a linearly independent set, it forms a basis for $V + W$. The dimension of $V + W$ is 2. In fact, $V + W = \mathbb{R}^2$.

(iii) $V \cap W = \{\mathbf{0}\}$, which is the zero subspace of $\mathbb{R}^2$. The empty set is a basis for $V \cap W$, whose dimension is 0.

(2) (i) $S_1 = \{(\frac{1}{2}, 1, 0), (-\frac{3}{2}, 0, 1)\}$ spans V, and since it is a linearly independent set, it forms a basis for V. $S_2 = \{(1, 1, 1)\}$ spans W and is also a basis for W. The dimension of V is 2 while the dimension of W is 1.

(ii) As discussed in part (a), $\{(\frac{1}{2},1,0),(-\frac{3}{2},0,1),(1,1,1)\}$ spans $V+W$. To check if the three vectors are linearly independent, we can form the following square matrix of order 3 and compute its determinant:

$$\begin{vmatrix} \frac{1}{2} & -\frac{3}{2} & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} \neq 0.$$

So the 3 vectors (columns of the matrix) are linearly independent and thus forms a basis for $V+W$. The dimension of $V+W$ is 2. In fact, $V+W=\mathbb{R}^3$.

(iii) Since $\{(\frac{1}{2},1,0),(-\frac{3}{2},0,1),(1,1,1)\}$ is a linearly independent set, the two subspaces V and W only have the zero vector in common. Thus $V\cap W=\{\mathbf{0}\}$, whose dimension is 0.

(3) (i) $S_1=\{(-\frac{1}{2},\frac{1}{4},1,0),(-1,-1,0,1)\}$ spans and is a basis for V. $S_2=\{(1,2,1,-1)\}$ spans and is a basis for W. The dimension of V is 2 while the dimension of W is 1.

(ii) $\{(-\frac{1}{2},\frac{1}{4},1,0),(-1,-1,0,1),(1,2,1,-1)\}$ spans $V+W$. It can be checked easily that this is a linearly independent spanning set of $V+W$, thus forming a basis for $V+W$. The dimension of $V+W$ is 3.

(iii) Since $\{(-\frac{1}{2},\frac{1}{4},1,0),(-1,-1,0,1),(1,2,1,-1)\}$ is a linearly independent set, V and W only have the zero vector in common. Thus $V\cap W=\{\mathbf{0}\}$, whose dimension is 0.

Remark: For (3) ask students to try what happens when $W=\{(r,-2r,-4r,r)\mid r\in\mathbb{R}\}$ instead.

3. For each of the following cases, write down a matrix $\boldsymbol{A}$ with the required property or explain why no such matrix exists.

   (a) The column space of $\boldsymbol{A}$ contains vectors $(1,0,0)^T$, $(0,0,1)^T$ and the row space of $\boldsymbol{A}$ contains vectors $(1,1)$, $(1,2)$.

   (b) The column space $=\mathbb{R}^4$ and the row space $=\mathbb{R}^3$.

   (c) The column space of $2\boldsymbol{A}=$ the row space of $-\boldsymbol{A}=\operatorname{span}\{(1,2,3)\}$.

   (d) $\boldsymbol{A}$ is a square matrix of order 2 where the column space of $\boldsymbol{A}$ is the solution space of the homogeneous linear system $\boldsymbol{Ax}=\mathbf{0}$.

(a) $\boldsymbol{A}=\begin{pmatrix}1&0\\0&0\\0&1\end{pmatrix}$. Note that the row space of $\boldsymbol{A}$ is $\mathbb{R}^2$.

(b) Not possible, since for a matrix $\boldsymbol{A}$ to have the desired row space property, a row-echelon form of $\boldsymbol{A}$ would have 3 non zero rows. But this would result in 3 pivot columns, meaning that a basis for the column space of $\boldsymbol{A}$ would have only 3 vectors, so the column space of $\boldsymbol{A}$ will not be $\mathbb{R}^4$.

(c) Note that $\boldsymbol{A}$, $2\boldsymbol{A}$ and $-\boldsymbol{A}$ are all row equivalent matrices. So we require $\boldsymbol{A}$ such that the row space and column space of $\boldsymbol{A}$ are both equal to span$\{(1,2,3)\}$.

We can let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$.

(d) For example, $\boldsymbol{A} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then both the column space of $\boldsymbol{A}$ and the solution space of $\boldsymbol{Ax} = \boldsymbol{0}$ are equals to span$\{(0,1)\}$.

4. In $\mathbb{R}^4$, let X be the subspace of all vectors of the form $(x_1, x_2, 0, 0)$ and let Y be the subspace of all vectors of the form $(0, y_1, y_2, 0)$. What are the dimensions of X, Y, $X \cap Y$, $X + Y$? Find a basis for each of these four subspaces.

$X = \text{span}\{(1,0,0,0),(0,1,0,0)\}$. The dimension of X is 2 and a basis for X is $\{(1,0,0,0),(0,1,0,0)\}$.

$Y = \text{span}\{(0,1,0,0),(0,0,1,0)\}$. The dimension of Y is 2 and a basis for Y is $\{(0,1,0,0),(0,0,1,0)\}$.

For a vector $\boldsymbol{w} \in X \cap Y$, we must have

$$\boldsymbol{w} = (a, b, 0, 0) = (0, c, d, 0)$$

for some real numbers a, b, c, d. This implies $a = d = 0$ and $b = c$. Thus the $X \cap Y = \text{span}\{(0,1,0,0)\}$. The dimension of $X \cap Y$ is 1 and a basis for $X \cap Y$ is $\{(0,1,0,0)\}$.

$X + Y = \text{span}\{(1,0,0,0),(0,1,0,0),(0,0,1,0)\}$. The dimension of $X + Y$ is 3 and a basis for $X + Y$ is $\{(1,0,0,0),(0,1,0,0),(0,0,1,0)\}$.

5. Is it possible to find two subspaces V and W of $\mathbb{R}^3$, both having dimension 2, such that $V \cap W = \{\boldsymbol{0}\}$ (meaning that these two subspaces have only the zero vector in common)? Explain your answer.

No it is not possible. If V and W are both of dimension 2, then we have $S_1 = \{\boldsymbol{w}, \boldsymbol{x}\}$ forming a basis for V and $S_2 = \{\boldsymbol{y}, \boldsymbol{z}\}$ forming a basis for W. If $V \cap W = \{\boldsymbol{0}\}$, then the only solution to

$$a\boldsymbol{w} + b\boldsymbol{x} = c\boldsymbol{y} + d\boldsymbol{z}$$

is $a = b = c = d = 0$. This implies that $\{\boldsymbol{w}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$ is a linearly independent set, which is impossible in $\mathbb{R}^3$.