LINEAR INDEPENDENCE I

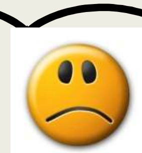
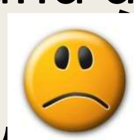
Recall from an earlier unit

Recall that if $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are vectors taken from $\mathbb{R}^n$.

If $\boldsymbol{u}_k$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}$, then

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}\} = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k\}$$

We say that $\boldsymbol{u}_k$ is redundant in the span of $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k\}$.

I am redundant 
Having me around does
not 'add value' 

$\boldsymbol{u}_k \Rightarrow$

The notion of redundancy is closely related to the concept that we will be introducing next.

Linear independence

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} \subseteq \mathbb{R}^n$. Consider the solutions to the following equation (values of $c_1, c_2, \ldots, c_k$)

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k = \mathbf{0} \qquad (*)$$

1) Clearly, $c_1 = 0, c_2 = 0, \ldots, c_k = 0$ is a solution. This is called the trivial solution to (*).

2) S is called a **linearly independent set** if (*) has only the trivial solution. In this case, we say that $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are **linearly independent vectors.**

Linear independence

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$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k = \boldsymbol{0} \qquad (*)$$

2) S is called a $\boxed{\text{linearly independent set}}$ if (*) has only the trivial solution. In this case, we say that $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are $\boxed{\text{linearly independent vectors.}}$

3) S is called a $\boxed{\text{linearly dependent set}}$ if (*) has non-trivial solutions. In this case, we say that $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are $\boxed{\text{linearly dependent vectors.}}$

Example

Determine whether $(1,-2,3),(5,6,-1),(3,2,1)$ are linearly independent vectors in $\mathbb{R}^3$.

Vector equation: $a(1,-2,3)+b(5,6,-1)+c(3,2,1)=(0,0,0)$

Linear system:

$$\begin{cases} a + 5b + 3c = 0 \\ -2a + 6b + 2c = 0 \\ 3a - b + c = 0 \end{cases}$$

Example

Determine whether $(1,-2,3),(5,6,-1),(3,2,1)$ are linearly independent vectors in $\mathbb{R}^3$.

Solving linear system:

$$\left(\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

How many solutions does the linear system have?

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Determine whether $(1,-2,3),(5,6,-1),(3,2,1)$ are linearly independent vectors in $\mathbb{R}^3$.

The vectors are linearly dependent.

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$$a(1,-2,3)+b(5,6,-1)+c(3,2,1)=(0,0,0)$$

Example

Determine whether $(1,0,0,1),(0,2,1,0),(1,-1,1,1)$ are linearly independent vectors in $\mathbb{R}^4$.

Vector equation: $a(1,0,0,1)+b(0,2,1,0)+c(1,-1,1,1)=(0,0,0,0)$

Linear system:
$$\begin{cases} a & & + & c & = & 0 \\ & 2b & - & c & = & 0 \\ & b & + & c & = & 0 \\ a & & + & c & = & 0 \end{cases}$$

Example

Determine whether $(1,0,0,1),(0,2,1,0),(1,-1,1,1)$ are linearly independent vectors in $\mathbb{R}^4$.

Solving linear system:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

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Example

Determine whether $(1,0,0,1),(0,2,1,0),(1,-1,1,1)$ are linearly independent vectors in $\mathbb{R}^4$.

The vectors are linearly independent.

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How many solutions does the vector equation have?

$$a(1,0,0,1)+b(0,2,1,0)+c(1,-1,1,1)=(0,0,0,0)$$

Set with only one vector

$S = \{\boldsymbol{u}\}$. When is S a linearly independent set?

When does the equation $c\boldsymbol{u} = \mathbf{0}$ have only the trivial solution $c = 0$?

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$\Leftrightarrow$

$\boldsymbol{u}$ is not the zero vector

$S = \{\boldsymbol{u}\}$ is a linearly independent set if and only if $\boldsymbol{u} \neq \mathbf{0}$.

Set with exactly two vectors

$S = \{\boldsymbol{u}, \boldsymbol{v}\}$. When is S a linearly independent set?

When does the equation $c_1\boldsymbol{u} + c_2\boldsymbol{v} = \boldsymbol{0}$ have non trivial solutions for c_1 and c_2?

Suppose $c_1 \neq 0$.

$$c_1\boldsymbol{u} + c_2\boldsymbol{v} = \boldsymbol{0} \iff \boldsymbol{u} = -\left(\frac{c_2}{c_1}\right)\boldsymbol{v} \iff \boldsymbol{u} \text{ is a scalar multiple of } \boldsymbol{v}$$

$S = \{\boldsymbol{u}, \boldsymbol{v}\}$ is a linearly dependent set if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are scalar multiples of each other.

Summary

1) Linear independence (definition)

2) To check whether a set of vectors are linearly independent:

Vector equation $\rightarrow$ linear system $\rightarrow$ Solve

Only trivial solution $\Rightarrow$ vectors are linearly independent

Non trivial solutions exist $\Rightarrow$ vectors are linearly dependent

3) Linear independence for sets with one or two vectors.