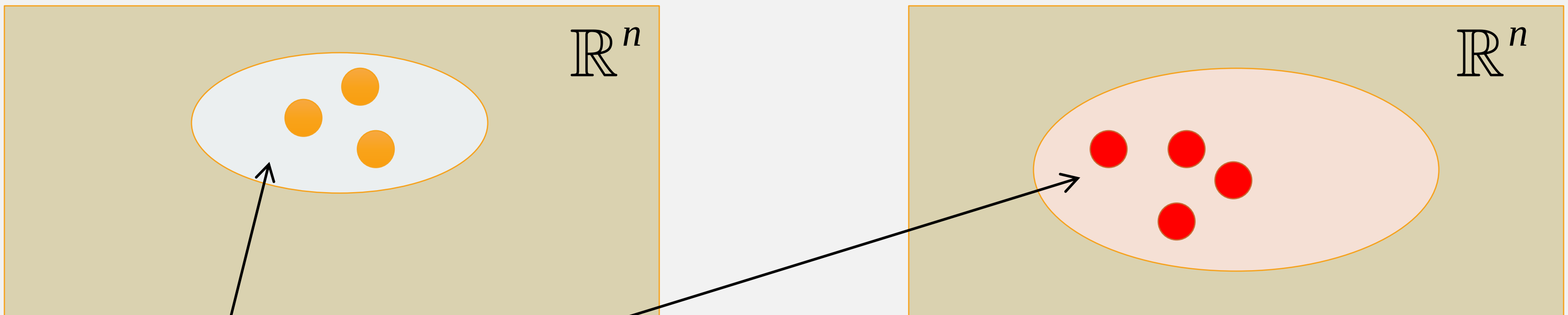
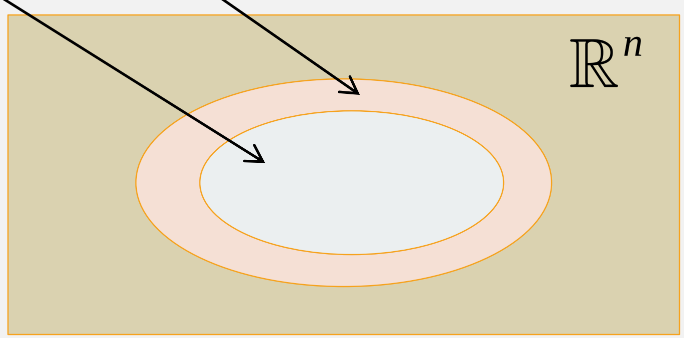
LINEAR SPAN III

THEOREM

Let $S_1 = \{u_1, u_2, \ldots, u_k\}$ and $S_2 = \{v_1, v_2, \ldots, v_m\}$ be subsets of $\mathbb{R}^n$.



Then $\mathrm{span}(S_1) \subseteq \mathrm{span}(S_2) \Leftrightarrow$ each u_i is a linear combination of $v_1, v_2, \ldots, v_m$.



EXAMPLE

Let $\boldsymbol{u}_1 = (1,0,1), \boldsymbol{u}_2 = (1,1,2), \boldsymbol{u}_3 = (-1,2,1), \boldsymbol{v}_1(1,2,3), \boldsymbol{v}_2 = (2,-1,1)$.

Show that $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

Idea: The previous theorem gives us a necessary and sufficient condition for

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}.$$

Each of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2$.

EXAMPLE

Let $\boldsymbol{u}_1 = (1,0,1), \boldsymbol{u}_2 = (1,1,2), \boldsymbol{u}_3 = (-1,2,1), \boldsymbol{v}_1(1,2,3), \boldsymbol{v}_2 = (2,-1,1)$.

Show that $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

Each of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2$.

$(\boldsymbol{u}_1)$ $(1,0,1) = a(1,2,3) + b(2,-1,1)$

$(\boldsymbol{u}_2)$ $(1,1,2) = a(1,2,3) + b(2,-1,1)$

$(\boldsymbol{u}_3)$ $(-1,2,1) = a(1,2,3) + b(2,-1,1)$

$$\begin{cases} a + 2b = 1 \\ 2a - b = 0 \\ 3a + b = 1 \end{cases}$$

$$\begin{cases} a + 2b = 1 \\ 2a - b = 1 \\ 3a + b = 2 \end{cases} \qquad \begin{cases} a + 2b = -1 \\ 2a - b = 2 \\ 3a + b = 1 \end{cases}$$

EXAMPLE

$$\begin{array}{ccccc} & & (\boldsymbol{u}_1) & (\boldsymbol{u}_2) & (\boldsymbol{u}_3) \\ \end{array}$$

$$\left(\begin{array}{cc|c|c|c} 1 & 2 & 1 & 1 & -1 \\ 2 & -1 & 0 & 1 & 2 \\ 3 & 1 & 1 & 2 & 1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{cc|c|c|c} 1 & 0 & \boxed{\frac{1}{5}} & \boxed{\frac{3}{5}} & \boxed{\frac{3}{5}} \\ 0 & 1 & \boxed{\frac{2}{5}} & \boxed{\frac{1}{5}} & \boxed{\frac{-4}{5}} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$(1,0,1) = \boxed{\tfrac{1}{5}}(1,2,3) + \boxed{\tfrac{2}{5}}(2,-1,1)$$

$$(1,1,2) = \boxed{\tfrac{3}{5}}(1,2,3) + \boxed{\tfrac{1}{5}}(2,-1,1)$$

$$(-1,2,1) = \boxed{\tfrac{3}{5}}(1,2,3) \boxed{-\tfrac{4}{5}}(2,-1,1)$$

Since each of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$

is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2$,

span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq$ span$\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

EXAMPLE

Let $\boldsymbol{u}_1 = (1,0,1), \boldsymbol{u}_2 = (1,1,2), \boldsymbol{u}_3 = (-1,2,1), \boldsymbol{v}_1(1,2,3), \boldsymbol{v}_2 = (2,-1,1)$.

Show that $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

Can we show

$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \supseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$?

Shown:

$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

Each of $\boldsymbol{v}_1, \boldsymbol{v}_2$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$.

$(\boldsymbol{v}_1)$ $\quad (1,2,3) = a(1,0,1) + b(1,1,2) + c(-1,2,1)$

$(\boldsymbol{v}_2)$ $\quad (2,-1,1) = a(1,0,1) + b(1,1,2) + c(-1,2,1)$

EXAMPLE

Let $\boldsymbol{u}_1=(1,0,1), \boldsymbol{u}_2=(1,1,2), \boldsymbol{u}_3=(-1,2,1), \boldsymbol{v}_1(1,2,3), \boldsymbol{v}_2=(2,-1,1)$.

$(\boldsymbol{v}_1)\quad (1,2,3)=a(1,0,1)+b(1,1,2)+c(-1,2,1)$

$(\boldsymbol{v}_2)\quad (2,-1,1)=a(1,0,1)+b(1,1,2)+c(-1,2,1)$

$$\left\{\begin{array}{ccccccc} a & + & b & - & c & = & 1 \\ & & b & + & 2c & = & 2 \\ a & + & 2b & + & c & = & 3 \end{array}\right. \qquad \left\{\begin{array}{ccccccc} a & + & b & - & c & = & 2 \\ & & b & + & 2c & = & -1 \\ a & + & 2b & + & c & = & 1 \end{array}\right.$$

$$\begin{array}{c} \qquad\qquad\quad (\boldsymbol{v}_1)\,(\boldsymbol{v}_2) \\ \left(\begin{array}{ccc|c|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 & -1 \\ 1 & 2 & 1 & 3 & 1 \end{array}\right) \end{array}$$

EXAMPLE

Let $\boldsymbol{u}_1 = (1,0,1), \boldsymbol{u}_2 = (1,1,2), \boldsymbol{u}_3 = (-1,2,1), \boldsymbol{v}_1(1,2,3), \boldsymbol{v}_2 = (2,-1,1).$

$$\begin{array}{c} \begin{array}{ccccc} & & & (\boldsymbol{v}_1) & (\boldsymbol{v}_2) \end{array} \\ \left(\begin{array}{ccc|c|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 & -1 \\ 1 & 2 & 1 & 3 & 1 \end{array}\right) \end{array} \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c|c} 1 & 0 & -3 & -1 & 3 \\ 0 & 1 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\boldsymbol{v}_1 = a(1,0,1) + b(1,1,2) + c(-1,2,1)$$

$$\begin{cases} a = -1 + 3s \\ b = 2 - 2s \\ c = s, \quad s \in \mathbb{R} \end{cases} \qquad \begin{cases} a = -1 \\ b = 2 \\ c = 0 \end{cases}$$

$$\boldsymbol{v}_1 = -(1,0,1) + 2(1,1,2) + 0(-1,2,1)$$

EXAMPLE

Let $\boldsymbol{u}_1=(1,0,1), \boldsymbol{u}_2=(1,1,2), \boldsymbol{u}_3=(-1,2,1), \boldsymbol{v}_1(1,2,3), \boldsymbol{v}_2=(2,-1,1)$.

$$\begin{array}{cccccc} & & & & (\boldsymbol{v}_1) & (\boldsymbol{v}_2) \end{array}$$

$$\left(\begin{array}{ccc|c|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 & -1 \\ 1 & 2 & 1 & 3 & 1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c|c} 1 & 0 & -3 & -1 & 3 \\ 0 & 1 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\boldsymbol{v}_1=a(1,0,1)+b(1,1,2)+c(-1,2,1) \qquad \boldsymbol{v}_2=a(1,0,1)+b(1,1,2)+c(-1,2,1)$$

$$\begin{cases} a & = & -1+3s \\ b & = & 2-2s \\ c & = & s, \quad s\in\mathbb{R} \end{cases} \quad \begin{cases} a & = & -1 \\ b & = & 2 \\ c & = & 0 \end{cases} \quad \begin{cases} a & = & 3+3s \\ b & = & -1-2s \\ c & = & s, \quad s\in\mathbb{R} \end{cases} \quad \begin{cases} a & = & 3 \\ b & = & -1 \\ c & = & 0 \end{cases}$$

$$\boldsymbol{v}_1=-(1,0,1)+2(1,1,2)+0(-1,2,1) \qquad \boldsymbol{v}_2=3(1,0,1)-1(1,1,2)+0(-1,2,1)$$

EXAMPLE

Let $\boldsymbol{u}_1 = (1,0,1), \boldsymbol{u}_2 = (1,1,2), \boldsymbol{u}_3 = (-1,2,1), \boldsymbol{v}_1(1,2,3), \boldsymbol{v}_2 = (2,-1,1)$.

$$\begin{array}{ccccc} & & & (\boldsymbol{v}_1) & (\boldsymbol{v}_2) \end{array}$$

$$\left(\begin{array}{ccc|c|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 2 & -1 \\ 1 & 2 & 1 & 3 & 1 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c|c} 1 & 0 & -3 & -1 & 3 \\ 0 & 1 & 2 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Since each of $\boldsymbol{v}_1, \boldsymbol{v}_2$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$,

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \supseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}.$$

Together with $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} \subseteq \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, we have shown

$$\boxed{\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \text{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}.}$$

EXAMPLE

Let $\boldsymbol{u}_1=(1,0,0,1),\boldsymbol{u}_2=(0,1,-1,2),\boldsymbol{u}_3=(2,1,-1,4),$
$\boldsymbol{v}_1=(1,1,1,1),\boldsymbol{v}_2=(-1,1,-1,1),\boldsymbol{v}_3=(-1,1,1,-1).$

Show that span$\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\}\not\subset$ span$\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\}$.

We try to write each $\boldsymbol{v}_i$ as a linear combination of $\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3$.

$$a(1,0,0,1)+b(0,1,-1,2)+c(2,1,-1,4)=(1,1,1,1) \qquad (\boldsymbol{v}_1)$$

$$\begin{array}{c} (\boldsymbol{v}_1) \end{array}$$

$$\begin{cases} a \qquad\qquad\quad + \ 2c = 1 \\ \qquad\quad b + c = 1 \\ \quad -b - c = 1 \\ a + 2b + 4c = 1 \end{cases}$$

EXAMPLE

Let $\boldsymbol{u}_1 = (1,0,0,1), \boldsymbol{u}_2 = (0,1,-1,2), \boldsymbol{u}_3 = (2,1,-1,4),$

$\boldsymbol{v}_1 = (1,1,1,1), \boldsymbol{v}_2 = (-1,1,-1,1), \boldsymbol{v}_3 = (-1,1,1,-1).$

Show that span$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\} \not\subset$ span$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$.

We try to write each $\boldsymbol{v}_i$ as a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$.

$$a(1,0,0,1) + b(0,1,-1,2) + c(2,1,-1,4) = (-1,1,-1,1) \qquad (\boldsymbol{v}_2)$$

$$\begin{array}{c} (\boldsymbol{v}_2) \\ \left\{\begin{array}{rrrrrrrr} a & & & + & 2c & = & -1 \\ & & b & + & c & = & 1 \\ & - & b & - & c & = & -1 \\ a & + & 2b & + & 4c & = & 1 \end{array}\right. \end{array}$$

EXAMPLE

Let $\boldsymbol{u}_1=(1,0,0,1),\boldsymbol{u}_2=(0,1,-1,2),\boldsymbol{u}_3=(2,1,-1,4),$
$\boldsymbol{v}_1=(1,1,1,1),\boldsymbol{v}_2=(-1,1,-1,1),\boldsymbol{v}_3=(-1,1,1,-1).$

Show that $\operatorname{span}\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\}\not\subset\operatorname{span}\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\}$.

We try to write each $\boldsymbol{v}_i$ as a linear combination of $\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3$.

$$a(1,0,0,1)+b(0,1,-1,2)+c(2,1,-1,4)=(-1,1,1,-1) \qquad (\boldsymbol{v}_3)$$

$$\begin{array}{r} (\boldsymbol{v}_3) \end{array}$$

$$\begin{cases} a & & & + & 2c & = & -1 \\ & & b & + & c & = & 1 \\ & - & b & - & c & = & 1 \\ a & + & 2b & + & 4c & = & -1 \end{cases}$$

EXAMPLE

Let $\boldsymbol{u}_1=(1,0,0,1),\boldsymbol{u}_2=(0,1,-1,2),\boldsymbol{u}_3=(2,1,-1,4),$

$\boldsymbol{v}_1=(1,1,1,1),\boldsymbol{v}_2=(-1,1,-1,1),\boldsymbol{v}_3=(-1,1,1,-1).$

Show that span$\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\}\not\subset$ span$\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\}$.

We try to write each $\boldsymbol{v}_i$ as a linear combination of $\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3$.

$$\left(\begin{array}{ccc|c|c|c} 1 & 0 & 2 & 1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 4 & 1 & 1 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c|c|c} 1 & 0 & 2 & 1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Which $\boldsymbol{v}_i$ is NOT a linear combination of $\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3$?

$\boldsymbol{v}_1$ is NOT a linear combination of $\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3$

THEOREM

Suppose $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are vectors taken from $\mathbb{R}^n$.

If $\boldsymbol{u}_k$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}$, then

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}\} = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k\}$$

set of ALL linear combinations of = set of ALL linear combinations of

THEOREM

Proof:

If $\boldsymbol{u}_k$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}$, then

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}\} = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k\}$$

($\subseteq$) To show $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}\} \subseteq \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k\}$

Done!

Is $\boldsymbol{u}_1$ a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k$?

$$\boldsymbol{u}_1 = 1\boldsymbol{u}_1 + 0\boldsymbol{u}_2 + \ldots + 0\boldsymbol{u}_{k-1} + 0\boldsymbol{u}_k$$

Is $\boldsymbol{u}_2$ a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k$?

$$\boldsymbol{u}_2 = 0\boldsymbol{u}_1 + 1\boldsymbol{u}_2 + \ldots + 0\boldsymbol{u}_{k-1} + 0\boldsymbol{u}_k$$

$$\vdots$$

THEOREM

Proof:

If $\boldsymbol{u}_k$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}$, then

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}\} = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k\}$$

($\supseteq$) To show $\text{span}\{\boxed{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}}\} \supseteq \text{span}\{\boxed{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k}\}$

Is $\boxed{\boldsymbol{u}_1}$ a linear combination of $\boxed{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}}$?

$$\boldsymbol{u}_1 = 1\boldsymbol{u}_1 + 0\boldsymbol{u}_2 + \ldots + 0\boldsymbol{u}_{k-1}$$

Is $\boxed{\boldsymbol{u}_2}$ a linear combination of $\boxed{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}}$?

$$\boldsymbol{u}_2 = 0\boldsymbol{u}_1 + 1\boldsymbol{u}_2 + \ldots + 0\boldsymbol{u}_{k-1}$$

$$\vdots$$

THEOREM

Proof:

If $\boldsymbol{u}_k$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}$, then

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}\} = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k\}$$

Does not add "value" to the linear span

($\supseteq$) To show $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}\} \supseteq \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k\}$

Done!

$\vdots$

Is $\boldsymbol{u}_k$ a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}$?

YES!

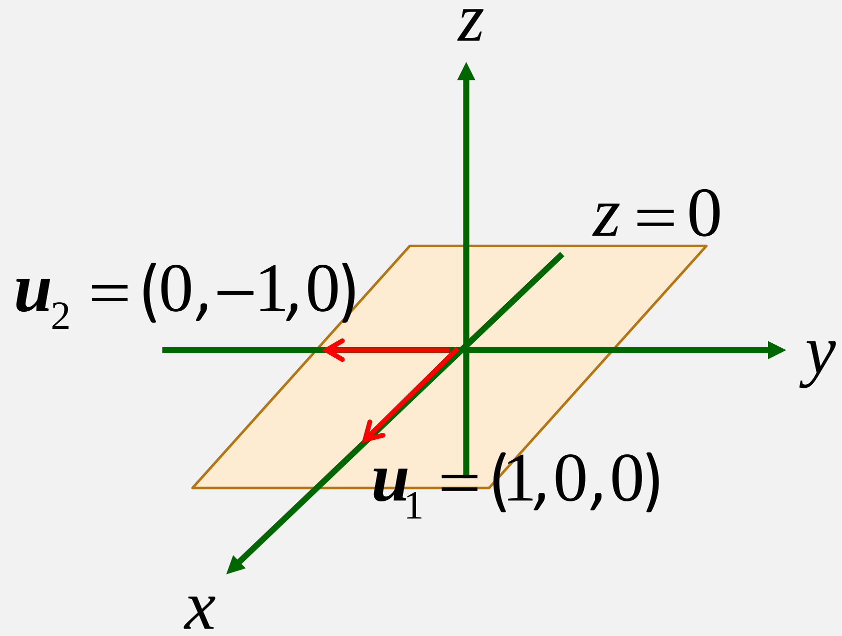
Shown: $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}\} = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k\}$

EXAMPLE

Let $\boldsymbol{u}_1 = (1,0,0), \boldsymbol{u}_2 = (0,-1,0), \boldsymbol{u}_3 = (2,3,0)$.

Clearly, $\boldsymbol{u}_3 = 2\boldsymbol{u}_1 - 3\boldsymbol{u}_2$. So $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$.

Can you describe $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ geometrically?



SUMMARY

1) Necessary and sufficient condition for span(S) $\subseteq$ span(T).

2) How to use the result in (1): to check span(S) $\subseteq$ span(T) or span(S) $\not\subset$ span(T).

3) When a vector does not add "value" to the linear span.