ONE MORE EQUIVALENT STATEMENT; MORE ON EIGENVALUES

THEOREM

Let $\boldsymbol{A}$ be an $n \times n$ matrix. The following statements are equivalent.

1) $\boldsymbol{A}$ is invertible

2) $\boldsymbol{Ax} = \boldsymbol{0}$ has only trivial solution

3) RREF of $\boldsymbol{A}$ is $\boldsymbol{I}$

4) $\boldsymbol{A}$ can be written as produce of elementary matrices

5) $\det(\boldsymbol{A}) \neq 0$

6) Rows of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$

7) Columns of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$

8) $\text{rank}(\boldsymbol{A}) = n$

9) 0 is not an eigenvalue of $\boldsymbol{A}$

THEOREM

Proof: We will show "0 is not an eigenvalue of $\boldsymbol{A}$" is equivalent to "$\det(\boldsymbol{A}) \neq 0$".

$$
\begin{aligned}
0 \text{ is not an eigenvalue of } \boldsymbol{A} &\Leftrightarrow \det(0\boldsymbol{I} - \boldsymbol{A}) \neq 0 \\
&\Leftrightarrow \det(-\boldsymbol{A}) \neq 0 \\
&\Leftrightarrow (-1)^n \det(\boldsymbol{A}) \neq 0 \\
&\Leftrightarrow \det(\boldsymbol{A}) \neq 0
\end{aligned}
$$

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix}$$

The eigenvalues of $\boldsymbol{A}$ are 1 and 0.95.

$\boldsymbol{A}$ is invertible

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

The eigenvalues of $\boldsymbol{B}$ are 0 and 3.

$\boldsymbol{B}$ is singular

$$\boldsymbol{C} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

The eigenvalues of $\boldsymbol{C}$ are 1, $\sqrt{2}$ and $-\sqrt{2}$.

$\boldsymbol{C}$ is invertible

THEOREM (EIGENVALUES OF TRIANGULAR MATRICES)

If $\boldsymbol{A}$ is a triangular matrix, the eigenvalues of $\boldsymbol{A}$ are the diagonal entries of $\boldsymbol{A}$.

Proof: Suppose $\boldsymbol{A}$ is a triangular matrix.

Then $(\lambda \boldsymbol{I} - \boldsymbol{A})$ is also a triangular matrix.

THEOREM (EIGENVALUES OF TRIANGULAR MATRICES)

If $\boldsymbol{A}$ is a triangular matrix, the eigenvalues of $\boldsymbol{A}$ are the diagonal entries of $\boldsymbol{A}$.

So the eigenvalues of $\boldsymbol{A}$ are:

$$a_{11}, a_{22}, \ldots, a_{nn}$$

Proof: Suppose $\boldsymbol{A}$ is a triangular matrix.

Then $(\lambda \boldsymbol{I} - \boldsymbol{A})$ is also a triangular matrix.

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ 0 & & & a_{nn} \end{pmatrix} \qquad \lambda \boldsymbol{I} - \boldsymbol{A} = \begin{pmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ & \lambda - a_{22} & \cdots & -a_{2n} \\ & & \ddots & \vdots \\ 0 & & & \lambda - a_{nn} \end{pmatrix}$$

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = (\lambda - a_{11})(\lambda - a_{22})\ldots(\lambda - a_{nn})$$

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

The eigenvalues of $\boldsymbol{A}$ are: $1, -2, 3$

$$\boldsymbol{B} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 \\ 3 & 1 & 2 & 0 \\ 0 & -3 & 0 & 6 \end{pmatrix}$$

The eigenvalues of $\boldsymbol{B}$ are: $0, 2, 6$

WHAT MANY OF YOU WILL ASK

Since eigenvalues of $\boldsymbol{A}$ can be obtained easily if $\boldsymbol{A}$ is triangular...

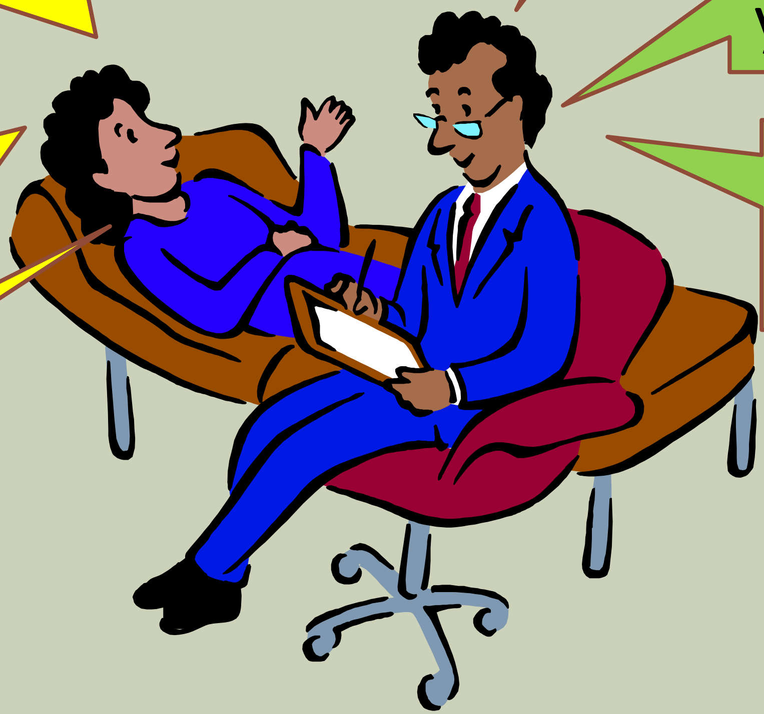
Can I perform E.R.O. on $\boldsymbol{A}$...

Can?

So?

I know what you are thinking...

Don't even THINK about it!



ELEMENTARY ROW OPERATIONS AND EIGENVALUES

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

The eigenvalues of $\boldsymbol{A}$ are: $1, 4$

$$\Big\downarrow \; R_1 \leftrightarrow R_2$$

$$\boldsymbol{B} = \begin{pmatrix} 0 & 4 \\ 1 & 0 \end{pmatrix}$$

$$\det(\lambda \boldsymbol{I} - \boldsymbol{B}) = \begin{vmatrix} \lambda & -4 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 4$$

$$= (\lambda - 2)(\lambda + 2)$$

The eigenvalues of $\boldsymbol{B}$ are: $2, -2$

ELEMENTARY ROW OPERATIONS AND EIGENVALUES

$$\boldsymbol{A} = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix} \qquad \det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda - 2 & 2 \\ 1 & \lambda - 3 \end{vmatrix} = \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4)$$

$$\Big\downarrow R_1 \leftrightarrow R_2$$

The eigenvalues of $\boldsymbol{A}$ are: $1, 4$

$$\boldsymbol{B} = \begin{pmatrix} -1 & 3 \\ 2 & -2 \end{pmatrix} \qquad \det(\lambda \boldsymbol{I} - \boldsymbol{B}) = \begin{vmatrix} \lambda + 1 & -3 \\ -2 & \lambda + 2 \end{vmatrix} = \lambda^2 + 3\lambda - 4 = (\lambda - 1)(\lambda + 4)$$

The eigenvalues of $\boldsymbol{B}$ are: $1, -4$

SUMMARY

1) One more equivalent statement to "$\boldsymbol{A}$ is invertible".

2) Eigenvalues of triangular matrices.