MATRIX MULTIPLICATION

CAN WE MULTIPLY MATRICES?

We have already seen that we can add and subtract matrices pretty much like what we do for real numbers. What about multiplication?

For real numbers x, y, $\boxed{xy \text{ is always defined.}}$

What about for matrices $\boldsymbol{A}, \boldsymbol{B}$?

DEFINITION

Let $\boldsymbol{A}=(a_{ij})_{m\times p}$ and $\boldsymbol{B}=(b_{ij})_{p\times n}$ be two matrices.

$$\boldsymbol{A}=\begin{pmatrix} & & & \\ \boxed{a_{i1} & a_{i2} & \dots & a_{ip}} \\ & & & \end{pmatrix} \qquad \boldsymbol{B}=\begin{pmatrix} & \boxed{\begin{matrix} b_{1j} \\ b_{2j} \\ \vdots \\ \vdots \\ b_{pj} \end{matrix}} & \end{pmatrix}$$

$i-$th

row of $\boldsymbol{A}$

$j-$th column of $\boldsymbol{B}$

Then $\boldsymbol{AB}$ is a $m\times n$ matrix

whose (i, j)-entry is given by:

$$\boxed{a_{i1}b_{1j}+a_{i2}b_{2j}+\dots+a_{ip}b_{pj}} \quad \boxed{=\sum_{k=1}^{p}a_{ik}b_{kj}}$$

EXAMPLE

Compute $\boldsymbol{AB}$ where

$$\boldsymbol{A} = \begin{pmatrix} 2 & 3 & -1 \\ 0 & 1 & -1 \\ 2 & 2 & -1 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \\ 2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\boldsymbol{AB} = \begin{pmatrix} 1 & -3 \\ -1 & -3 \\ 0 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

REMARK

From the way $\boldsymbol{AB}$ is defined, it is clear that the number of columns in $\boldsymbol{A}$ must be EQUAL to the number of rows in $\boldsymbol{B}$.

number of columns in $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{pmatrix} \\ \overbrace{\boxed{a_{i1} \quad a_{i2} \quad \cdots \quad a_{ip}}} \\ \\ \end{pmatrix}$$

$$\boldsymbol{B} = \left.\begin{pmatrix} & \boxed{\begin{matrix} b_{1j} \\ b_{2j} \\ \vdots \\ \vdots \\ b_{pj} \end{matrix}} & \end{pmatrix}\right\} \text{number of rows in } \boldsymbol{B}$$

REMARK

We know that for real numbers x, y we have $xy = yx$.

However, for matrices $\boldsymbol{A}, \boldsymbol{B}$, even when both $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are defined, $\boldsymbol{AB}$ does not necessarily equal $\boldsymbol{BA}$.

$$\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} -2 & 2 \\ -1 & 3 \end{pmatrix}$$

$$\boldsymbol{AB} = \begin{pmatrix} -5 & 7 \\ -6 & 6 \end{pmatrix} \qquad \boldsymbol{BA} = \begin{pmatrix} 2 & -2 \\ 7 & -1 \end{pmatrix}$$

REMARK

We know that for real numbers x, y if $xy = 0$, then either $x = 0$ or $y = 0$ (or both $= 0$).

However, if $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$,

then $\boldsymbol{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}$.

So $\boldsymbol{AB} = \mathbf{0}$ but neither $\boldsymbol{A}$ nor $\boldsymbol{B}$ is the zero matrix.

PRE AND POST MULTIPLICATION

$\boxed{\text{Pre-multiply}}$ of $\boldsymbol{A}$ to $\boldsymbol{B} \rightarrow \boldsymbol{AB}$

$\boxed{\text{Post-multiply}}$ of $\boldsymbol{A}$ to $\boldsymbol{B} \rightarrow \boldsymbol{BA}$

MATRIX MULTIPLICATION LAWS

(Associative Law for Matrix Multiplication)

$$\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$$

(Distributive Law for Matrix Addition and Multiplication)

$$\boldsymbol{A}(\boldsymbol{B}_1 + \boldsymbol{B}_2) = \boldsymbol{AB}_1 + \boldsymbol{AB}_2$$

$$(\boldsymbol{C}_1 + \boldsymbol{C}_2)\boldsymbol{A} = \boldsymbol{C}_1\boldsymbol{A} + \boldsymbol{C}_2\boldsymbol{A}$$

MATRIX MULTIPLICATION LAWS

If a is a scalar, then

$$a(\boldsymbol{AB}) = (a\boldsymbol{A})\boldsymbol{B} = \boldsymbol{A}(a\boldsymbol{B})$$

Let $\boldsymbol{A}$ be a $m \times n$ matrix.

$$\boldsymbol{A0} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{0A} = \boldsymbol{0}$$

$$\boldsymbol{AI} = \boldsymbol{A} \quad \text{and} \quad \boldsymbol{IA} = \boldsymbol{A}$$

DEFINITION

Let $\boldsymbol{A}$ be a square matrix and n a nonnegative integer.

We define

$$\boldsymbol{A}^n = \begin{cases} \boldsymbol{I} & \text{if } n = 0 \\ \underbrace{\boldsymbol{A}\boldsymbol{A}\ldots\boldsymbol{A}}_{n \text{ times}} & \text{if } n \geq 1 \end{cases}$$

1) $\boldsymbol{A}^m \boldsymbol{A}^n = \boldsymbol{A}^{(m+n)}$

2) In general, $(\boldsymbol{A}\boldsymbol{B})^m \neq \boldsymbol{A}^m \boldsymbol{B}^m$

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \boldsymbol{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \boldsymbol{AB} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$$

$$(\boldsymbol{AB})^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\boldsymbol{A}^2\boldsymbol{B}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

SUMMARY

1) How matrices can be multiplied.

2) Matrix multiplication laws.

3) Instances where real number multiplication and matrix multiplication differ.