NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

Module: MA1508E Linear Algebra for Engineering
Year/Semester: 2018-2019 (Semester 2)
Tutorial: 8

1. For each of the following matrices $\boldsymbol{A}$, determine a basis for each of the following subspaces (i) row space of $\boldsymbol{A}$; (ii) row space of $\boldsymbol{A}^T$; (iii) nullspace of $\boldsymbol{A}$; (iv) nullspace of $\boldsymbol{A}^T$. State also the dimension of each of these subspaces.

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & -2 \\ 1 & 3 \\ 2 & 1 \\ 3 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 2 & 1 & 4 \end{pmatrix}$

2. For each of the following $\boldsymbol{A}$ and $\boldsymbol{b}$,

(i) Find a basis for the row space of $\boldsymbol{A}$.

(ii) Find a basis for the column space of $\boldsymbol{A}$.

(iii) Determine nullity($\boldsymbol{A}$). If nullity($\boldsymbol{A}$) > 0, find a basis for the nullspace of $\boldsymbol{A}$.

(iv) Solve the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ and express $\boldsymbol{b}$ as a linear combination of the columns of $\boldsymbol{A}$.

(v) If nullity($\boldsymbol{A}$) > 0, use the basis you obtained in part (iii) to write down the solution set of $\boldsymbol{Ax} = \boldsymbol{b}$.

(a) $\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 5 \\ 0 & 1 & 3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$.

(b) $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -3 & 3 \\ 3 & 1 & 0 & 4 \\ 1 & 1 & 6 & -2 \\ 3 & 0 & -1 & 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ 0 \\ -4 \\ -2 \end{pmatrix}$.

(c) $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 3 \\ 4 & 4 & -1 & 9 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

3. Let $\boldsymbol{A}$ be a 3×4 matrix. Suppose $x_1 = 1$, $x_2 = 0$, $x_3 = -1$, $x_4 = 0$ is a solution to a non-homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{b}$ and that the homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$ has a general solution $x_1 = t - 2s$, $x_2 = s + t$, $x_3 = s$, $x_4 = t$, where s, t are arbitrary parameters.

   (a) Find a basis for the nullspace of $\boldsymbol{A}$ and determine the nullity of $\boldsymbol{A}$.

   (b) Find a general solution for the system $\boldsymbol{Ax} = \boldsymbol{b}$.

   (c) Write down the reduced row-echelon form of $\boldsymbol{A}$.

   (d) Find a basis for the row space of $\boldsymbol{A}$ and determine the rank of $\boldsymbol{A}$.

   (e) Do we have enough information for us to find the column space of $\boldsymbol{A}^T$?

4. Let $\boldsymbol{A}$ be a $m \times n$ matrix. Show that

   (a) If $\boldsymbol{x}$ belongs to the nullspace of $\boldsymbol{A}^T\boldsymbol{A}$, then $\boldsymbol{Ax}$ belongs to both the column space of $\boldsymbol{A}$ and the nullspace of $\boldsymbol{A}^T$.

   (b) Nullspace of $\boldsymbol{A}^T\boldsymbol{A}$ is equal to the nullspace of $\boldsymbol{A}$.

   (c) Rank of $\boldsymbol{A}$ is equal to the rank of $\boldsymbol{A}^T\boldsymbol{A}$.

   (d) If $\boldsymbol{A}$ has linearly independent columns, then $\boldsymbol{A}^T\boldsymbol{A}$ is invertible.

5. Let $\boldsymbol{w} = (0, 1, 2, 3)$.

   (a) Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$ where $\boldsymbol{u_1} = (2, 1, 0, 0)$, $\boldsymbol{u_2} = (-1, 0, 0, 1)$, $\boldsymbol{u_3} = (2, 0, -1, 1)$, $\boldsymbol{u_4} = (0, 0, 1, 1)$. Show that S is a basis for $\mathbb{R}^4$. Is S an orthogonal basis for $\mathbb{R}^4$? Compute $(\boldsymbol{w})_S$.

   (b) Let $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$ where $\boldsymbol{v_1} = (1, 2, 2, -1)$, $\boldsymbol{v_2} = (1, 1, -1, 1)$, $\boldsymbol{v_3} = (-1, 1, -1, -1)$, $\boldsymbol{v_4} = (-2, 1, 1, 2)$. Show that T is a basis for $\mathbb{R}^4$. Is T an orthogonal basis for $\mathbb{R}^4$? Compute $(\boldsymbol{w})_T$.