Week 11 IVLE Quiz

1. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are both eigenvectors of $\boldsymbol{A}$ associated with the eigenvalue λ, which of the following statements is/are definitely true?

   (I) $\boldsymbol{x} + \boldsymbol{y}$ is an eigenvector of $\boldsymbol{A}$ associated with eigenvalue $\lambda + \lambda = 2\lambda$.

   (II) If the eigenspace associated with λ is two dimensional, then $\boldsymbol{x}$ is not a scalar multiple of $\boldsymbol{y}$.

   (III) If the eigenspace associated with λ is one dimensional, then $\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{y}$.

   (A) (II) and (III) only.

   (B) (I) and (II) only.

   (C) (III) only.

   (D) None of the given combinations is correct.

   Answer: (C). (I) is incorrect. $\boldsymbol{x}+\boldsymbol{y}$ is still associated with λ. (II) is incorrect since E_λ is two dimensional does not imply that $\boldsymbol{x}$ and $\boldsymbol{y}$ must be linearly independent. (III) is correct since if E_λ is one dimensional, then $\boldsymbol{x}$ and $\boldsymbol{y}$ must be multiples of each other.

2. If λ is an eigenvalue of a matrix $\boldsymbol{A}$, which of the statements below is/are definitely correct?

   (I) $(\lambda \boldsymbol{I} - \boldsymbol{A})$ is a singular matrix.

   (II) $(\boldsymbol{A} - \lambda \boldsymbol{I})$ is a singular matrix.

   (III) λ is an eigenvalue of $\boldsymbol{A}^T$.

   (A) (I) only.

   (B) (I) and (II) only.

   (C) (I) and (III) only.

   (D) All three statements are correct.

   Answer: (D). (I) is true, since by defintion, any eigenvalue λ satisfies the equation $\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$. (II) is also true, since $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = (-1)^n \det(\boldsymbol{A} - \lambda \boldsymbol{I})$. (III) is true as $\boldsymbol{A}$ and $\boldsymbol{A}^T$ has the same set of eigenvalues.

3. Suppose $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are all 3×3 matrices such that 0 is an eigenvalue of $\boldsymbol{A}$, 1 is an eigenvalue of $\boldsymbol{B}$ and 2 is an eigenvalue of $\boldsymbol{C}$. Which of the following statements is/are definitely correct?

(I) 0×1 is an eigenvalue of $\boldsymbol{AB}$.

(II) 1×2 is an eigenvalue of $\boldsymbol{BC}$.

(III) $0 + 1$ is an eigenvalue of $\boldsymbol{A} + \boldsymbol{B}$.

(A) All three statements are correct.

(B) (I) and (II) only.

(C) (I) and (III) only.

(D) None of the given combinations is correct.

Answer: (D). (I) is correct since $\boldsymbol{AB}$ is singular (because $\boldsymbol{A}$ is singular) which means that 0 must be an eigenvalue of $\boldsymbol{AB}$. (II) is not necessarily true since there is no assurance that there will be a common eigenvector between E_1 (of $\boldsymbol{B}$) and E_2 (of $\boldsymbol{C}$). Similarly, for (III), there is no assurance that there will be a common eigenvector between E_0 (of $\boldsymbol{A}$) and E_1 (of $\boldsymbol{B}$).

4. How many statements below is/are correct?

(I) If there exists an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{AP} = \boldsymbol{PD}$ where $\boldsymbol{D}$ is a diagonal matrix, then $\boldsymbol{A}$ is diagonalizable.

(II) Every diagonal matrix is diagonalizable.

(III) If there exists an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{AP} = \boldsymbol{0}$, then the only eigenvalue that $\boldsymbol{A}$ has is 0.

(A) All three.

(B) Exactly two.

(C) Exactly one.

(D) None.

Answer: (A). (I) is correct since $\boldsymbol{AP} = \boldsymbol{PD}$ is equivalent of $\boldsymbol{A} = \boldsymbol{PDP}^{-1}$. (II) is correct since we can write $\boldsymbol{D} = \boldsymbol{IDI}^{-1}$, so any diagonal matrix is trivially diagonalizable. (III) is correct since $\boldsymbol{0}$ is a diagonal matrix and the diagonal of $\boldsymbol{0}$ (which are all zeros) are the eigenvalues of $\boldsymbol{A}$.

5. Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be **all** the eigenvalues of a $n \times n$ matrix $\boldsymbol{A}$. Suppose we know that $\boldsymbol{A}$ is diagonalizable, which of the statements below is/are always true?

(I) $\boldsymbol{A}$ has exactly n distinct eigenvalues (that is, $k = n$).

(II) The linear system $(\boldsymbol{A} - \lambda_1 \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$ has non trivial solutions.

(A) (I) only.

(B) (II) only.

(C) Both (I) and (II)

(D) None of them.

Answer: (B). (I) is incorrect, see for example, the matrix $\boldsymbol{B}$ in slide number 4 of unit 059. (II) is correct since $(\boldsymbol{A} - \lambda_1 \boldsymbol{I})$ is singular.