W07-02

Slide 01: In this unit, we will discuss a procedure that will allow us to construct an orthogonal basis for a vector space.

Slide 02: Let us start off with the simplest type of subspaces, namely those that are one-dimensional. Suppose V is spanned by a single non zero vector $\boldsymbol{u_1}$.

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We know that V is a line and $\boldsymbol{u_1}$ lies on the line.

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The single vector $\boldsymbol{u_1}$ forms an orthogonal basis for V.

Slide 03: Suppose V is two-dimensional, that is V is spanned by two linearly independent vectors $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$. Clearly, $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ forms a basis for V.

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We can visualise the subspace V as shown, with the two basis vectors $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$.

Slide 04: Consider the subspace span$\{\boldsymbol{u_1}\}$ spanned by the vector $\boldsymbol{u_1}$ alone.

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Then this is a one-dimensional subspace of V and $\{\boldsymbol{u_1}\}$ is an orthogonal basis for this subspace.

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Since we have an orthogonal basis for this one-dimensional subspace, we can apply the orthogonal projection theorem to project $\boldsymbol{u_2}$ onto this subspace. By the orthogonal projection theorem, the projection of $\boldsymbol{u_2}$ onto span$\{\boldsymbol{u_1}\}$ is the vector $\frac{\boldsymbol{u_2}\cdot\boldsymbol{u_1}}{||\boldsymbol{u_1}||^2}\boldsymbol{u_1}$.

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This projection is represented by the green vector in the diagram.

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As this is the orthogonal projection of $\boldsymbol{u_2}$ onto span$\{\boldsymbol{u_1}\}$, if we take the difference between $\boldsymbol{u_2}$ and its projection, we would obtain the yellow vector as shown

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which will be orthogonal to span$\{\boldsymbol{u_1}\}$, and in particular, orthogonal to $\boldsymbol{u_1}$.

Slide 05: We have started off with a basis for the two-dimensional space V, comprising of vectors $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$. After applying orthogonal projection, we now have an orthogonal basis for the same space V. The vectors in this orthogonal basis are $\boldsymbol{u_1}$ and the difference between $\boldsymbol{u_2}$ and its projection onto span$\{\boldsymbol{u_1}\}$.

Slide 06: Let us take this one step further and consider V as a three-dimensional subspace of $\mathbb{R}^n$. Suppose V is spanned by three linearly independent vectors $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ and $\boldsymbol{u_3}$.

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We first consider V' to be the subspace of V spanned by just $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$. Note that V' is two dimensional and $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ forms a basis for V'.

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We will ignore $\boldsymbol{u_3}$ for the time being, but it is useful to note that $\boldsymbol{u_3}$, as shown by the green vector, does not lie on V', since $\boldsymbol{u_3}$ is not a linear combination of $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$.

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Now
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let $\boldsymbol{v_1}$ be equal to $\boldsymbol{u_1}$, represented by the red vector in the figure.

Slide 07: We have already seen how we can convert a basis for a 2-dimensional subspace into an orthogonal basis, thus
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applying the same principle, we can now have $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ to form an orthogonal basis for the space V'. Note that $\boldsymbol{v_1}$ is just $\boldsymbol{u_1}$ while $\boldsymbol{v_2}$ is the difference between $\boldsymbol{u_2}$ and its projection onto span$\{\boldsymbol{v_1}\}$.
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It is now time to bring back $\boldsymbol{u_3}$ into the picture. As mentioned earlier, $\boldsymbol{u_3}$ does not belong to V'.
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Since we now have an orthogonal basis for V', in the form of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$, we can apply the orthogonal projection theorem to compute the projection of $\boldsymbol{u_3}$ onto V'. The projection is given by the linear combination of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$
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as shown by the pink vector.

Slide 08: Just as before,
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we will consider the difference between $\boldsymbol{u_3}$ and its projection onto span$\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$.
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Let this vector be $\boldsymbol{v_3}$, as represented by the orange vector in the figure. Again, by our understanding of orthogonal projection, the vector $\boldsymbol{v_3}$ will be orthogonal to the space V', which means that $\boldsymbol{v_3}$, together with $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ would be an orthogonal basis for the three-dimensional subspace V.

Slide 09: Let us recap on what we have just discussed. We started off with a basis for the three-dimensional subspace V. This initial basis consists of vectors $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ and $\boldsymbol{u_3}$.
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We first let $\boldsymbol{v_1}$ equal to $\boldsymbol{u_1}$.
(#)
Then we project $\boldsymbol{u_2}$ onto the one-dimensional subspace spanned by $\boldsymbol{v_1}$, followed by taking the difference between $\boldsymbol{u_2}$ and its projection onto span$\{\boldsymbol{v_1}\}$. The resulting vector $\boldsymbol{v_2}$ will be orthogonal to $\boldsymbol{v_1}$. The two vectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ would form an orthogonal basis for a two-dimensional subspace.
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We then continue by computing the projection of $\boldsymbol{u_3}$ onto the subspace spanned by $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$. By taking the difference between $\boldsymbol{u_3}$ and its projection onto span$\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$, we have the resulting vector $\boldsymbol{v_3}$, which is orthogonal to both $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$.
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We have now successfully converted the initial basis $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ into an orthogonal basis $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$.

Slide 10: The procedure that we have just described does not have to stop at three dimensional subspaces. The same idea can be applied for higher dimensional subspaces. We will now present this procedure as a theorem. This procedure which converts a basis into an orthogonal basis, is known as the Gram-Schmidt Process. Suppose $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$ is a basis for V.

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We start by letting $\boldsymbol{v_1}$ be equal to $\boldsymbol{u_1}$.

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Then let $\boldsymbol{v_2}$ be equal to the difference between $\boldsymbol{u_2}$ and its projection onto span$\{\boldsymbol{v_1}\}$. At this point, if we have done all computations carefully, the new vector $\boldsymbol{v_2}$ should be orthogonal to $\boldsymbol{v_1}$.

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We then let $\boldsymbol{v_3}$ to be equal to the difference between $\boldsymbol{u_3}$ and its projection onto span$\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$. Note that we can compute this orthogonal projection because $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ forms an orthogonal basis for span$\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$. Let this newly constructed vector be $\boldsymbol{v_3}$.

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Continuing in this manner, constructing the new vectors $\boldsymbol{v_i}$ one at a time, we eventually arrive at the final vector $\boldsymbol{v_k}$, which is the difference between $\boldsymbol{u_k}$ and the projection of $\boldsymbol{u_k}$ onto the subspace spanned by $\boldsymbol{v_1}$ to $\boldsymbol{v_{k-1}}$. Note that by this time, we would have constructed an orthogonal basis $\boldsymbol{v_1}, \boldsymbol{v_2}$ to $\boldsymbol{v_{k-1}}$, which is an orthogonal basis for the $k-1$-dimensional subspace.

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The process has now ended and $\boldsymbol{v_1}$ to $\boldsymbol{v_k}$ will be an orthogonal basis for V.

Slide 11: Converting this orthogonal basis to an orthonormal basis is easy as we simply need to normalise all the vectors by dividing each of them by their length.

Slide 12: We will work through one example on the application of Gram-Schmidt Process. In this example, we wish to convert the three vectors $(1, 0, 1)$, $(0, 1, 2)$, $(2, 1, 0)$ into an orthogonal basis for $\mathbb{R}^3$.

Slide 13: Before we start, it should be noted that because of the sequential nature of the Gram-Schmidt Process, we need to decide which of the three vectors we would like to be $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ and $\boldsymbol{u_3}$. Needless to say, if we choose to label the three vectors differently, the resulting three vectors after conversion will be different but would still be correct. For this example, we will let $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ and $\boldsymbol{u_3}$ be the three vectors as shown. We start off by letting $\boldsymbol{v_1}$ be $\boldsymbol{u_1}$, which is $(1, 0, 1)$.

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Then $\boldsymbol{v_2}$ is the difference between $\boldsymbol{u_2}$ and the projection of $\boldsymbol{u_2}$ onto span$\{\boldsymbol{v_1}\}$.

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This is given by the expression as shown

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and when evaluated, we have

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the vector $(-1, 1, 1)$.

(#)

As mentioned previously it is good practice to check everytime we constructed a new vector that the new vector is orthogonal to all those constructed up to this point. So now we just need to check that $\boldsymbol{v_2}$ is orthogonal to $\boldsymbol{v_1}$, which is indeed the case, since the dot product between them is zero.

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Next, the vector $\boldsymbol{v_3}$ is the difference between $\boldsymbol{u_3}$ and its projection onto $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$.

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By the orthogonal projection theorem, the projection vector can be computed

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and when evaluated, we have

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the vector $(\frac{2}{3}, \frac{4}{3}, -\frac{2}{3})$.

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Again, we should now check that the new vector $\boldsymbol{v_3}$ is orthogonal to both $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ and indeed it is true.

Slide 15: We have come to the concluion of the Gram-Schmidt Process and the three vectors $\boldsymbol{v_1}, \boldsymbol{v_2}$ and $\boldsymbol{v_3}$ is now an orthogonal basis for $\mathbb{R}^3$.

Slide 16: In summary, for this unit,

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we apply the knowledge of orthogonal projection to come up with a procedure known as the Gram-Schmidt Process that will convert a basis for a vector space into an orthogonal basis.