W05-07

Slide 01: In this unit, we will formally define what is called the dimension of a vector space.

Slide 02: We start off with a very important theorem, which we will state without proof. Let V be a vector space and suppose we know that V has a basis that contains exactly k vectors.

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Then, any subset of vectors from V with more than k vectors will always be linearly dependent. This means that such a subset cannot be a basis for V.

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Also, any subset of vectors from V that contains less than k vectors will not be able to span V and thus such a subset cannot be a basis for V too.

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In other words, for any subset of V to be a basis for V, it must first have the correct number of vectors. While we do not present the proof of this result, the significance cannot be understated. With this theorem, we now know that even though a vector space V can have infinitely many different bases, the different bases for V must all have the same number of vectors.

Slide 03: The preceding theorem gives rise to the definition of the dimension of a vector space. We define the dimension of V to be the number of vectors in a basis for V.

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The dimension of the zero space is defined to be zero. Recall that the empty set is a basis for the zero space.

Slide 04: For the Euclidean n-space, we have already seen one basis for it, namely the standard basis consisting of vectors $\boldsymbol{e_1}, \boldsymbol{e_2}$ to $\boldsymbol{e_n}$. Thus the dimension of $\mathbb{R}^n$ is n.

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We have also discussed and characterised all subspaces of $\mathbb{R}^2$.

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There is the zero subspace of $\mathbb{R}^2$, which has dimension 0.

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There are the lines in $\mathbb{R}^2$ that passes through the origin. These subspaces have dimension 1.

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Then there is $\mathbb{R}^2$ itself, which has dimension 2.

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We have also characterised all subspaces of $\mathbb{R}^3$.

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There is the zero subspace, which has dimension 0.

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All the lines in $\mathbb{R}^3$ that passes through the origin, which has dimension 1.

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All the planes in $\mathbb{R}^3$ that contains the origin, which has dimension 2.

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And finally $\mathbb{R}^3$ itself, which has dimension 3.

Slide 05: Consider the following problem where we would like to find a basis for and determine the dimension of the subspace W.

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We first try to rewrite W as a linear span. Notice that the vectors in W are of the form $(x, 2z, z)$ where x and z are any real number.

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This allows us to see that an arbitrary vector in W is of the form x times $(1, 0, 0)$ plus z times $(0, 2, 1)$.

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This shows that W is precisely the linear span of $(1, 0, 0)$ and $(0, 2, 1)$.

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In other words, we have found two vectors that spans W. Are they linearly independent vectors?

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Yes indeed they are. Do you recall how we can quickly check whether a set containing only two vectors is linearly independent? Are these two vectors multiples of each other?

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Since we have found a set of linearly independent vectors that spans W, we have gotten a basis for W and since this basis has two vectors, the dimension of W is 2.

Slide 06: For this next example, we would like to determine the dimension of the solution space of the following homogeneous linear system. Recall that we have already shown that the solution set of homogeneous linear systems are always subspaces. The linear system in this example is one that involves 5 variables, so the solution space will be a subspace of $\mathbb{R}^5$.

Slide 07: We solve this linear system as per normal and upon obtaining the reduced row-echelon form of the augemented matrix, as shown here,

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we notice that there are two non pivot columns on the left side. This means that our general solution will involve two arbitrary parameters.

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You should be able to write down a general solution of the system quickly, like the one we have here.

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Now an arbitrary vector in the solution space would be of the form as shown.

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This expression can be further rewritten as a linear combination of two particular vectors, namely $(-1, 1, 0, 0, 0)$ and $(-1, 0, -1, 0, 1)$.

Slide 08: Thus a vector belongs to the solution space of the linear system if and only if it is a linear combination of the two vectors.

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This means that the solution space is spanned by $(-1, 1, 0, 0, 0)$ and $(-1, 0, -1, 0, 1)$. We have found two vectors that spans the solution space. Are these two vectors linearly independent?

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Indeed they are since they are not multiples of each other.

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Thus we have found a basis for the solution space of the homogeneous linear system

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and since this basis has two vectors, the dimension of the solution space is 2. This solution space is a 2-dimensional subspace of $\mathbb{R}^5$.

Slide 09: As a concluding remark to this unit, note that when we are investigating the solution spaces of homogeneous linear systems, if we were to follow the method described in the previous example, a set of vectors found to span the solution space will always be a linearly independent set. In other words, such a set found will always be a basis for the solution space.

Slide 10: Let us summarise the main points in this unit.

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We presented a very important theorem which states that all bases for the same vector space will have the same number of vectors in them.

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This led us to define formally what is the dimension of a vector space.

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We went through an example in finding a basis for the solution space of a homogeneous linear system.