**W03-07**

**Slide 01:** In this unit, we will have more discussion on cofactor expansion and determinants.

**Slide 02:** Let's begin with an example to compute the determinant of this $3 \times 3$ matrix.

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By cofactor expansion, the first term in the expansion will be the $(1,1)$-entry, which is $-1$, multiplied by the $(1,1)$-cofactor. The $(1,1)$-cofactor is $(-1)^2$ times the determinant of the $2 \times 2$ matrix $\boldsymbol{M_{11}}$.

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The second term in the expansion is the $(1,2)$-entry multiplied by the $(1,2)$-cofactor. The $(1,2)$-cofactor is $(-1)^3$ times the determinant of the $2 \times 2$ matrix $\boldsymbol{M_{12}}$.

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The third and final term in the expansion can be found in a similar manner.

**Slide 03:** Notice that the determinants of all the $2 \times 2$ matrices can be computed using the expression $ad - bc$, as discussed in a previous unit.

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Thus, the three terms in the expansion can be computed

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and it simplifies to give 0. Thus the determinant of this $3 \times 3$ matrix is 0.

**Slide 04:** We will now demonstrate a simple way to compute the determinant of a $3 \times 3$ matrix. Consider the general $3 \times 3$ matrix $\boldsymbol{A}$ as shown. To obtain an expression for the determinant of $\boldsymbol{A}$, first copy down the 9 entries in $\boldsymbol{A}$, followed by repeating columns 1 and 2 of $\boldsymbol{A}$ on the right side as shown by the two highlighted columns.

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The determinant of $\boldsymbol{A}$ can now be written as an expression involving the 9 entries in the following way. First we have the sum of three terms $aei + bfg + cdh$. This can be remembered easily by drawing three diagonal lines as shown.

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Next we have the negative of the sum of three terms, $-ceg - afh - bdi$. This can be remembered easily by drawing another three diagonal lines as shown.

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These six terms will give the determinant of the matrix $\boldsymbol{A}$. You may wish to verify this expression with that obtained using cofactor expansion.

**Slide 05:** Well you may be wondering if this method will work for $4 \times 4$ or even larger matrices.

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Unfortunately, it does not. Thus for larger matrices, cofactor expansion is still required for the computation of determinants.

**Slide 06:** Let us relook at the definition for the determinant of a square matrix $\boldsymbol{A}$. Notice that in this definition, we are using the entries in the first row of $\boldsymbol{A}$ and the corresponding first row cofactors of $\boldsymbol{A}$, that is, the $A_{11}$, $A_{12}$ and so on.

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We say we are performing cofactor expansion along the first row of $\boldsymbol{A}$.

**Slide 07:** It turns out that the determinant of $\boldsymbol{A}$ can be computed by performing cofactor expansion along any row or any column of $\boldsymbol{A}$, not necessarily the first row.

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For example, this expression uses cofactor expansion along the $i$th row of $\boldsymbol{A}$. Here you see each of the $i$th row entries of $\boldsymbol{A}$ multiplied to their corresponding $i$th row cofactors.

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The next expression here computes the determinant of $\boldsymbol{A}$ using cofactor expansion along the $j$th column of $\boldsymbol{A}$. Once again, the $j$th column entries of $\boldsymbol{A}$ are multiplied to their corresponding $j$th column cofactors.

**Slide 08:** Revisiting the same matrix in at the beginning of this unit, let us perform cofactor expansion along the second row to evaluate the determinant.

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Once again, we will have three terms. You see that the three entries in the second row of the matrix are used in the expansion. They are multiplied to their corresponding $(2,1)$, $(2,2)$ and $(2,3)$ cofactors. Remember that each cofactor is $(-1)^{i+j}$ times the determinant of a $2 \times 2$ matrix $\boldsymbol{M_{ij}}$.

**Slide 09:** Using $ad - bc$ to compute the determinant of each $2 \times 2$ matrix, we can simplify our computation as follows.

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Finally it simplifies to 0 which is consistent with what we have computed earlier using cofactor expansion along the first row.

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As a form of practice, you may wish to try cofactor expansion along another row or column.

**Slide 10:** We now present some results on the determinat of special matrices, starting with triangular matrices. If $\boldsymbol{A}$ is a triangular matrix, then the determinant of $\boldsymbol{A}$ is simply the product of its diagonal entries.

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The two examples shown here illustrates this fact. Note that the matrix can be either upper or lower triangular.

**Slide 11:** This next result states that a square matrix $\boldsymbol{A}$ and its transpose $\boldsymbol{A}^T$ has the same determinant.

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Thus, the matrix we used as an example in this unit, as well as its transpose shown here, both have 0 as their determinant.

**Slide 12:** The next result states that the determinant of a square matrix with at least two identical rows or identical columns will be 0. With this result it is easy to see why the two matrices shown here both have determinant equal to 0. The proofs of these three results just presented is non trivial and will not be discussed in this unit.

**Slide 13:** Let us summarise the main points in this unit.

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We introduced a simple way to remember the formula for the determinant of a $3 \times 3$ matrix.

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We also learnt that to evaluate the determinant of a matrix, we can perform cofactor expansion along any row or any column of the matrix.

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Lastly, we presented three results on the determinant of some special matrices.