Unit 013 Block multiplication

Slide 01: In this unit, we will discuss block multiplication of matrices.

Slide 02: In a previous unit, we saw that if $\boldsymbol{A}$ is a $m \times p$ matrix and $\boldsymbol{B}$ is a $p \times n$ matrix, then $\boldsymbol{AB}$ will be a $m \times n$ matrix.

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Suppose we consider the matrix $\boldsymbol{A}$ in terms of its rows. Let $\boldsymbol{a_1}$, $\boldsymbol{a_2}$ and so on until $\boldsymbol{a_m}$ be the rows of $\boldsymbol{A}$. In particular, $\boldsymbol{a_i}$ is the ith row of $\boldsymbol{A}$ with entries a_{i1}, a_{i2} and so on till a_{ip}.

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Similarly, let us consider the matrix $\boldsymbol{B}$ in terms of its columns. Let $\boldsymbol{b_1}$, $\boldsymbol{b_2}$ and so on until $\boldsymbol{b_n}$ be the columns of $\boldsymbol{B}$. In particular, $\boldsymbol{b_j}$ is the jth column of $\boldsymbol{B}$ with entries b_{1j}, b_{2j} and so on till b_{pj}.

Slide 03: The first way of computing the product $\boldsymbol{AB}$ is to do it entry by entry. This is precisely how we have defined matrix multiplication.

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To recap, the (i, j)-entry of $\boldsymbol{AB}$ is obtained by identifying the i-th row of $\boldsymbol{A}$, which is $\boldsymbol{a_i}$ and the j-th column of $\boldsymbol{B}$, which is $\boldsymbol{b_j}$ before multiplying the corresponding entries and summing up the terms.

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Thus $\boldsymbol{a_i b_j}$ is the summation of $a_{ik}b_{kj}$ from $k = 1$ to p.

Slide 04: The second way of computing the product $\boldsymbol{AB}$ is to do it row by row.

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In order to do this, we wiil write $\boldsymbol{A}$ in terms of its rows. So by prem-ultiplying the rows of $\boldsymbol{A}$ to $\boldsymbol{B}$, we will have the rows of $\boldsymbol{AB}$. In particular, the i-th row of $\boldsymbol{AB}$, denoted by $\boldsymbol{a_i B}$ is obtained by pre-mutiplying the i-th row of $\boldsymbol{A}$, $\boldsymbol{a_i}$ to $\boldsymbol{B}$. Note that $\boldsymbol{a_i B}$ will be a row matrix with n entries.

Slide 05: The third way of computing the product $\boldsymbol{AB}$ is to do it column by column.

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In order to do this, we will write $\boldsymbol{B}$ in terms of its columns. So by post-multiplying the columns of $\boldsymbol{B}$ to $\boldsymbol{A}$, we will have the columns of $\boldsymbol{AB}$. In particular, the j-th column of $\boldsymbol{AB}$, denoted by $\boldsymbol{Ab_j}$ is obtained by post-multiplying the j-th column of $\boldsymbol{B}$, $\boldsymbol{b_j}$ to $\boldsymbol{A}$. Note that $\boldsymbol{Ab_j}$ is a column matrix with m entries.

Slide 06: Consider the following example where $\boldsymbol{A}$ is a 3×3 matrix and $\boldsymbol{B}$ is a 3×2 matrix. We first compute $\boldsymbol{AB}$ column by column. Consider the two columns of $\boldsymbol{B}$, $\boldsymbol{b_1}$ and $\boldsymbol{b_2}$. The first column of $\boldsymbol{AB}$ will be $\boldsymbol{Ab_1}$ while the second column of $\boldsymbol{AB}$ is $\boldsymbol{Ab_2}$.

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Post-multiplying $\boldsymbol{b_1}$ to $\boldsymbol{A}$

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and similarly, post-multiplying $\boldsymbol{b_2}$ to $\boldsymbol{A}$,

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we have the two columns of $\boldsymbol{AB}$.

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Thus $\boldsymbol{AB}$ is the 3×2 matrix as shown here.

Slide 07: We now compute the same matrix $\boldsymbol{AB}$ row by row. Consider the three rows of $\boldsymbol{A}$, $\boldsymbol{a_1}$, $\boldsymbol{a_2}$ and $\boldsymbol{a_3}$. The first row of $\boldsymbol{AB}$ will be $\boldsymbol{a_1 B}$, the second row of $\boldsymbol{AB}$ will be $\boldsymbol{a_2 B}$ while the third row of $\boldsymbol{AB}$ will be $\boldsymbol{a_3 B}$.

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Pre-multiplying $\boldsymbol{a_1}$ to $\boldsymbol{B}$,

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$\boldsymbol{a_2}$ to $\boldsymbol{B}$ and

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$\boldsymbol{a_3}$ to $\boldsymbol{B}$,

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we have the three rows of $\boldsymbol{AB}$.

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Thus $\boldsymbol{AB}$, computed row by row, is shown here.

Slide 08: While we have seen how we can partition a matrix into its rows or columns, this is not the only way a matrix can be partitioned for multiplication.

Slide 09: For example, this matrix $\boldsymbol{A}$ is partitioned into 6 submatrices, each of a different size.

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We have $\boldsymbol{A_{11}}$ being a 2×3 submatrix of $\boldsymbol{A}$, $\boldsymbol{A_{12}}$ is a 2×2 submatrix while $\boldsymbol{A_{13}}$ is a 2×1 submatrix of $\boldsymbol{A}$.

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Likewise, we have the following submatrices $\boldsymbol{A_{21}}$, $\boldsymbol{A_{22}}$ and $\boldsymbol{A_{23}}$ of $\boldsymbol{A}$.

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So we may write $\boldsymbol{A}$ in terms of its submatrices as follows.

Slide 10: These submatrices of $\boldsymbol{A}$ are also called blocks.

Slide 11: If two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same size are partitioned in the same way, then the blocks of $\boldsymbol{A}$ and $\boldsymbol{B}$ can be added or subtracted in the usual way.

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For example the matrix $\boldsymbol{A} + \boldsymbol{B}$ can be obtained by adding the corresponding blocks from $\boldsymbol{A}$ and $\boldsymbol{B}$. Note that the usual way of adding two matrices that we have defined previously was simply done by treating each entry as a block, so we are still adding the the two matrices block by block.

Slide 12: What about for matrix multiplication? Partitioned matrices can also be multiplied by the usual row-column matching rule as long as for a product $\boldsymbol{AB}$, the column partition of $\boldsymbol{A}$ matches the row partition of $\boldsymbol{B}$.

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The column partition of $\boldsymbol{A}$ means the number of partitions the columns of $\boldsymbol{A}$ are partitioned into while the row partition of $\boldsymbol{B}$ is the number of partitions the rows of $\boldsymbol{B}$ are partitioned into.

Slide 13: In this example, the columns of $\boldsymbol{A}$ are partitioned into two partitions. This matches with the row partition of $\boldsymbol{B}$, where the rows of $\boldsymbol{B}$ are partitioned into two partitions. Note that $\boldsymbol{A}$ is partitioned into 4 blocks while $\boldsymbol{B}$ is partitioned into two blocks.

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The product $\boldsymbol{AB}$ can be computed as follows. By writing $\boldsymbol{A}$ and $\boldsymbol{B}$ as blocks,

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the matrix $\boldsymbol{AB}$ can be computed block by block. In this case $\boldsymbol{AB}$ will have two blocks. The first block of $\boldsymbol{AB}$ will be $\boldsymbol{A_{11}B_{11}} + \boldsymbol{A_{12}B_{21}}$. Note that this block of $\boldsymbol{AB}$ will be a 2×2 submatrix. The second block of $\boldsymbol{AB}$ will be $\boldsymbol{A_{21}B_{11}} + \boldsymbol{A_{22}B_{21}}$. Note that this block of $\boldsymbol{AB}$ will be a 1×2 submatrix. Together, these two blocks will form the 3×2 matrix $\boldsymbol{AB}$.

Slide 14: Let us compute the product $\boldsymbol{AB}$ by block multiplication.

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$\boldsymbol{A_{11}B_{11}}$ gives the folowing 2×2 matrix.

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Similarly, we compute $\boldsymbol{A_{12}B_{21}}$ and obtain the following 2×2 matrix.

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Adding them up, we have the first block of $\boldsymbol{AB}$, the 2×2 submatrix.

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To find the second block of $\boldsymbol{AB}$, we will need to compute $\boldsymbol{A_{21}B_{11}} + \boldsymbol{A_{22}B_{21}}$. I will leave it to you to verify that this second block is

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the 1×2 submatrix $\begin{pmatrix} 2 & 1 \end{pmatrix}$. Thus we have completed the pre-multiplication of $\boldsymbol{A}$ to $\boldsymbol{B}$ using block multiplication.

Slide 15: To summarise the main points in this unit.

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We first saw how a matrix can be partitioned into blocks or submatrices.

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In the special case where we simply partition a matrix into either its rows or its columns, we saw how the matrix $\boldsymbol{AB}$ can be computed either row by row or column by column.

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After that, we generalised this to matrix multiplication by blocks, where each block does not have to be just a row or a column partition.