EIGENSPACES

DEFINITION

Let $\boldsymbol{A}$ be a square matrix of order n and λ an eigenvalue of $\boldsymbol{A}$.

$(\lambda \boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$: homogeneous linear system with coefficient matrix $(\lambda \boldsymbol{I} - \boldsymbol{A})$

Note that $(\lambda \boldsymbol{I} - \boldsymbol{A})$ is singular and thus this homogeneous linear system has infinitely many solutions.

The solution space of $(\lambda \boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$ is called the eigenspace of $\boldsymbol{A}$ associated with λ and is denoted by E_λ.

REMARK

The solution space of $(\lambda \boldsymbol{I} - \boldsymbol{A})x = \boldsymbol{0}$ is called the eigenspace of $\boldsymbol{A}$ associated with λ and is denoted by E_λ.

Note that a non zero vector $\boldsymbol{v}$ belongs to E_λ

$$\Leftrightarrow (\lambda \boldsymbol{I} - \boldsymbol{A})\boldsymbol{v} = \boldsymbol{0} \quad \Leftrightarrow \lambda \boldsymbol{I}\boldsymbol{v} - \boldsymbol{A}\boldsymbol{v} = \boldsymbol{0}$$

$$\Leftrightarrow \boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}$$

So E_λ contains ALL the eigenvectors of $\boldsymbol{A}$ associated with λ.

E_λ is also the nullspace of $(\lambda \boldsymbol{I} - \boldsymbol{A})$

EXAMPLE

The eigenvalues of $\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix}$ are $\lambda = 1$ and $\lambda = 0.95$.

$E_1 = \text{span}\left\{ \begin{pmatrix} \frac{1}{4} \\ 1 \end{pmatrix} \right\}$

For $\lambda = 1$, we investigate E_1:

$$1\boldsymbol{I} - \boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix} = \begin{pmatrix} 0.04 & -0.01 \\ -0.04 & 0.01 \end{pmatrix}$$

$\dim(E_1) = 1$

Solving $(1\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$,

$$\boldsymbol{x} = \begin{pmatrix} \frac{t}{4} \\ t \end{pmatrix} = t \begin{pmatrix} \frac{1}{4} \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

$$\left(\begin{array}{cc|c} 0.04 & -0.01 & 0 \\ -0.04 & 0.01 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & \frac{-1}{4} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$x_1 - \tfrac{1}{4} x_2 = 0$$

EXAMPLE

$\dim(E_{0.95}) = 1$

The eigenvalues of $\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix}$ are $\lambda = 1$ and $\lambda = 0.95$.

$E_{0.95} = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$

For $\lambda = 0.95,$ we investigate $E_{0.95}$:

$$0.95\boldsymbol{I} - \boldsymbol{A} = \begin{pmatrix} 0.95 & 0 \\ 0 & 0.95 \end{pmatrix} - \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix} = \begin{pmatrix} -0.01 & -0.01 \\ -0.04 & -0.04 \end{pmatrix}$$

Solving $(0.95\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$,

$$\boldsymbol{x} = \begin{pmatrix} -t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

$$\left(\begin{array}{cc|c} -0.01 & -0.01 & 0 \\ -0.04 & -0.04 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$x_1 + x_2 = 0$$

EXAMPLE

The eigenvalues of $\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ are $\lambda = 3$ and $\lambda = 0$.

For $\lambda = 3$, we investigate E_3 :

$$3\boldsymbol{I} - \boldsymbol{B} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Solving $(3\boldsymbol{I} - \boldsymbol{B})\boldsymbol{x} = \boldsymbol{0}$

$$\left(\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\boldsymbol{x} = \begin{pmatrix} t \\ t \\ t \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

$$E_3 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$$

$$\boxed{\dim(E_3) = 1}$$

EXAMPLE

The eigenvalues of $\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ are $\lambda = 3$ and $\lambda = 0$.

For $\lambda = 0$, we investigate E_0 :

$$0\boldsymbol{I} - \boldsymbol{B} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

Solving $(0\boldsymbol{I} - \boldsymbol{B})\boldsymbol{x} = \boldsymbol{0}$

$$\left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

EXAMPLE

For $\lambda = 0$, we investigate E_0 :

$$x_1 + x_2 + x_3 = 0$$

Solving $(0\boldsymbol{I} - \boldsymbol{B})\boldsymbol{x} = \boldsymbol{0}$

$$\left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 & = & -s-t \\ x_2 & = & s \\ x_3 & = & t, \ \ s,t \in \mathbb{R} \end{cases}$$

$$\boldsymbol{x} = \left\{ \begin{pmatrix} -s-t \\ s \\ t \end{pmatrix} \right\} = \left\{ s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \middle| \ s,t \in \mathbb{R} \right\}$$

$$E_0 = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\dim(E_0) = 2$$

EXAMPLE

The eigenvalues of $\boldsymbol{C} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ are $\lambda = 1, \sqrt{2}$ and $-\sqrt{2}$.

Following similar method as before, we have:

$$E_1 = \text{span}\left\{\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}\right\} \quad E_{\sqrt{2}} = \text{span}\left\{\begin{pmatrix} -1 \\ \sqrt{2} \\ 1 \end{pmatrix}\right\} \quad E_{-\sqrt{2}} = \text{span}\left\{\begin{pmatrix} -1 \\ -\sqrt{2} \\ 1 \end{pmatrix}\right\}$$

EXAMPLE

Let $\boldsymbol{M} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$. 2 is the only eigenvalue of $\boldsymbol{M}$.

Solving $(2\boldsymbol{I} - \boldsymbol{M})\boldsymbol{x} = \boldsymbol{0}$:

$$\begin{pmatrix} 2-2 & 0-0 \\ 0-1 & 2-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ s \end{pmatrix}$, $s \in \mathbb{R}$ So $E_2 = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ and $\boxed{\dim(E_2) = 1}$.

REMEMBER HOW IT ALL STARTED?

Given a square matrix $\boldsymbol{A}$, we wanted to know if it is possible to find an invertible matrix $\boldsymbol{P}$ such that

$$\boldsymbol{A} = \boldsymbol{PDP}^{-1} \text{ (}\boldsymbol{D}\text{ is a diagonal matrix)}$$

$$\text{or equivalently, } \boldsymbol{P}^{-1}\boldsymbol{AP} = \boldsymbol{D}$$

Let's look at a summary of the examples we have seen previously:

WHAT CAN YOU DEDUCE?

$$\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \boldsymbol{C} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$E_1 = \text{span}\left\{\begin{pmatrix} \frac{1}{4} \\ 1 \end{pmatrix}\right\}$$

$$\dim(E_1) = 1$$

$$E_{0.95} = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$$

$$\dim(E_{0.95}) = 1$$

$$E_3 = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$$

$$\dim(E_3) = 1$$

$$E_0 = \text{span}\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$$

$$\dim(E_0) = 2$$

$$E_1 = \text{span}\left\{\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}\right\}$$

$$\dim(E_1) = 1$$

$$E_{\sqrt{2}} = \text{span}\left\{\begin{pmatrix} -1 \\ \sqrt{2} \\ \end{pmatrix}\right\}$$

$$\dim(E_{\sqrt{2}}) = 1$$

$$E_{-\sqrt{2}} = \text{span}\left\{\begin{pmatrix} -1 \\ -\sqrt{2} \\ 1 \end{pmatrix}\right\}$$

$$\dim(E_{-\sqrt{2}}) = 1$$

WHAT CAN YOU DEDUCE?

$$\boldsymbol{M} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \qquad E_2 = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \qquad \boxed{\dim(E_2) = 1}$$

SUMMARY

1) Eigenspace of a matrix $\boldsymbol{A}$ associated with an eigenvalue λ.