NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

Module: MA1508E Linear Algebra for Engineering
Year/Semester: 2018-2019 (Semester 2)
Practice Problem Set: 2

Name:

Matriculation Number:

Tutorial Group:

Write down the solutions to the problems below, showing all working involved (imagine it is an examination question). Hand in your answers **together with this question paper** before you leave the classroom. You may refer to any materials while answering the questions. You may also discuss with your friends but do not **copy blindly**.

1. A horizontal elastic beam is supported at each end and is subjected to forces at points #1, #2 and #3 as shown in the figure below. Let

$$\boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

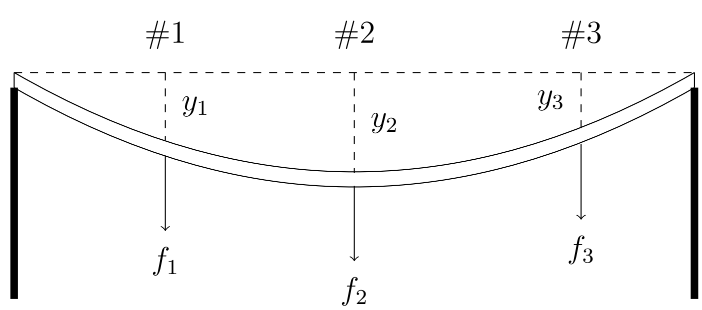
represent the forces at the 3 points and

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

represent the amounts of deflection (that is, movement) of the beam at the 3 points. By Hooke's law (from physics), it can be shown that

$$\boldsymbol{y} = \boldsymbol{D}\boldsymbol{f}$$

where $\boldsymbol{D}$ is a 3×3 *flexibility matrix*. The inverse of $\boldsymbol{D}$ is called the *stiffness matrix*.



Suppose we write

$$\boldsymbol{D} = \boldsymbol{D}\boldsymbol{I_3} = \begin{pmatrix} \boldsymbol{D}\boldsymbol{e_1} & \boldsymbol{D}\boldsymbol{e_2} & \boldsymbol{D}\boldsymbol{e_3} \end{pmatrix} \quad \text{where } \boldsymbol{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Interpret the column matrix $\boldsymbol{D}\boldsymbol{e_1}$ (the first column of $\boldsymbol{D}$) in terms of force and deflection experienced by the elastic beam. Similarly, interpret $\boldsymbol{D}\boldsymbol{e_2}$ and $\boldsymbol{D}\boldsymbol{e_3}$.

(b) Likewise, we now write

$$\boldsymbol{D}^{-1} = \boldsymbol{D}^{-1}\boldsymbol{I_3} = \begin{pmatrix} \boldsymbol{D}^{-1}\boldsymbol{e_1} & \boldsymbol{D}^{-1}\boldsymbol{e_2} & \boldsymbol{D}^{-1}\boldsymbol{e_3} \end{pmatrix}.$$

Interpret the columns of the matrix $\boldsymbol{D}^{-1}$ in terms of force and deflection experienced by the elastic beam.

(c) Suppose the flexibility of the beam is measured in millimeters (**mm**) of deflection per kilogram (**kg**) of load and the flexibility matrix of the beam is given below

$$\boldsymbol{D} = \begin{pmatrix} 0.005 & 0.002 & 0.001 \\ 0.002 & 0.004 & 0.002 \\ 0.001 & 0.002 & 0.005 \end{pmatrix}.$$

If loads of 30kg, 50kg and 20kg are applied to points #1, #2 and #3 respectively, calculate the corresponding deflections at the 3 points.

Solution:

(a) $\boldsymbol{e_1}$ is when unit force is applied downward at position #1 with zero force applied at the other two positions. Thus $\boldsymbol{D}\boldsymbol{e_1}$ is the matrix that lists the beam deflections (at the three points) due to this unit force at position #1 and zero force at the other two positions. $\boldsymbol{D}\boldsymbol{e_2}$ and $\boldsymbol{D}\boldsymbol{e_3}$ is interpreted similarly.

(b) To interpret $\boldsymbol{D}^{-1}$, observe that

$$\boldsymbol{y} = \boldsymbol{D}\boldsymbol{f} \Leftrightarrow \boldsymbol{f} = \boldsymbol{D}^{-1}\boldsymbol{y}.$$

So the first column of $\boldsymbol{D}^{-1}$, which is $\boldsymbol{D}^{-1}\boldsymbol{e_1}$, is the force that must be applied at the three points that will result in one unit deflection at point #1 and zero deflection at the other two points. The second and third columns of $\boldsymbol{D}^{-1}$ can be interpreted similarly.

(c) The answer is

$$\begin{pmatrix} 0.005 & 0.002 & 0.001 \\ 0.002 & 0.004 & 0.002 \\ 0.001 & 0.002 & 0.005 \end{pmatrix} \begin{pmatrix} 30 \\ 50 \\ 20 \end{pmatrix} = \begin{pmatrix} 0.27 \\ 0.30 \\ 0.23 \end{pmatrix}.$$

Thus the deflections at points #1, #2 and #3 are 0.27mm, 0.30mm and 0.23mm respectively.

2. Another elastic beam is being examined for its suitability to be used in construction. A series of experiments were conducted to determine the forces required to be applied at 3 points #1, #2 and #3 in order to observe various desired deflections at the 3 points. The results of the experiments are given in the table below.

	Force applied			Deflection observed		
	f_1	f_2	f_3	y_1	y_2	y_3
Experiment 1	0.25	−0.125	0	1	0	0
Experiment 2	−0.125	0.375	−0.125	0	1	0
Experiment 3	0	−0.125	0.25	0	0	1

For example, in order to produce 1 unit of deflection at point #1 and no deflection and points #2 and #3, we need to apply a force of 0.25 units downwards at point #1, a force of 0.125 units **upwards** at point #2 and no force at point #3.

(a) Find the flexibility matrix $\boldsymbol{D}$ of the elastic beam.

(**Remark:** You may wish to use the fact that $\begin{pmatrix} 5 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 5 \end{pmatrix}$ is the inverse of $\frac{1}{8}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.)

(b) Determine f_1, f_2, f_3 if we wish to observe the deflection $y_1 = y_2 = y_3 = 1$.

(c) The beam will snap if $\max\{y_1, y_2, y_3\} \geq 3$ or $y_1 + y_2 + y_3 \geq 5$. Determine whether the beam snaps when the forces applied to the 3 points is given by

$$\boldsymbol{f} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} 0.5 \\ -0.2 \\ 0.3 \end{pmatrix}.$$

Solution:

(a) From the data given, we have

$$\boldsymbol{D}\begin{pmatrix} \frac{1}{4} \\ -\frac{1}{8} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{8} \\ 0 \end{pmatrix} = \boldsymbol{D}^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \text{first column of } \boldsymbol{D}^{-1}$$

SImilarly, we see that the second and third columns of $\boldsymbol{D}^{-1}$ are

$$\begin{pmatrix} -\frac{1}{8} \\ \frac{3}{8} \\ -\frac{1}{8} \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ -\frac{1}{8} \\ \frac{1}{4} \end{pmatrix} \text{ respectively.}$$

Hence

$$\boldsymbol{D}^{-1} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{8} & 0 \\ -\frac{1}{8} & \frac{3}{8} & -\frac{1}{8} \\ 0 & -\frac{1}{8} & \frac{1}{4} \end{pmatrix}.$$

By the fact given in the question, we have

$$\boldsymbol{D} = \begin{pmatrix} 5 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 5 \end{pmatrix}.$$

(b)

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \boldsymbol{D}\boldsymbol{f} \Rightarrow \boldsymbol{f} = \boldsymbol{D}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

So $f_1 = f_2 = f_3 = \frac{1}{8}$.

(c)

$$\boldsymbol{y} = \boldsymbol{D}\boldsymbol{f} = \begin{pmatrix} 5 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} 0.5 \\ -0.2 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 2.4 \\ 0.8 \\ 1.6 \end{pmatrix}.$$

Since $\max\{y_1, y_2, y_3\} = 2.4 < 3$ and $y_1 + y_2 + y_3 = 4.8 < 5$, we conclude that the beam will not snap.