MORE ON SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS

0 AS AN EIGENVALUE

Suppose when we are solving a system of linear differential equations $\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}$, we encounter $\boxed{\lambda = 0 \text{ as an eigenvalue of } \boldsymbol{A}}$. How does this change the solution of the system?

Recall that if $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue λ, then $\boldsymbol{Y}_1 = e^{\lambda t}\boldsymbol{x}$ is a solution to $\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}$.

In this case where λ=0, the solution becomes $\underline{\boldsymbol{Y}_1 = \boldsymbol{x}}$

EXAMPLE

Solve the system of linear differential equations $\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}$ where

$$\boldsymbol{A} = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}.$$

We first find the eigenvalues of $\boldsymbol{A}$:

$$\begin{vmatrix} \lambda - 1 & 2 \\ 2 & \lambda - 4 \end{vmatrix} = \lambda^2 - 5\lambda \ = 0 \Leftrightarrow \lambda = 0 \text{ or } 5$$

$\lambda = 0:$ Consider the eigenspace E_λ : $\left(\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right)$

EXAMPLE

$$\left(\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad E_0 = \operatorname{span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$$

$\lambda = 5:$ Consider the eigenspace E_λ : $\left(\begin{array}{cc|c} 4 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right)$

$$\downarrow$$

$$\left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \qquad E_5 = \operatorname{span}\left\{\begin{pmatrix} -1 \\ 2 \end{pmatrix}\right\}$$

For any $k_1, k_2 \in \mathbb{R}$,

$$\boldsymbol{Y} = k_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + k_2\begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{5t}$$

is a solution to $\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}$.

THEOREM (COMPLEX EIGENVALUES)

Suppose when we are solving a system of linear differential equations $\boldsymbol{Y}' = \boldsymbol{AY}$, we encounter $\lambda = a + ib \in \mathbb{C}$ as a complex eigenvalue of $\boldsymbol{A}$. What can we do?

Theorem:

If λ is an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{x}$ is an eigenvector associated with λ, then $\bar{\lambda}$ is also an eigenvalue of $\boldsymbol{A}$ and $\bar{\boldsymbol{x}}$ is an eigenvector of $\boldsymbol{A}$ associated with $\bar{\lambda}$.

THEOREM (COMPLEX EIGENVALUES)

Theorem:

If λ is an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{x}$ is an eigenvector associated with λ, then $\overline{\lambda}$ is also an eigenvalue of $\boldsymbol{A}$ and $\overline{\boldsymbol{x}}$ is an eigenvector of $\boldsymbol{A}$ associated with $\overline{\lambda}$.

Furthermore, we know that $e^{\lambda t}\boldsymbol{x}$ and $e^{\overline{\lambda} t}\overline{\boldsymbol{x}}$ are both (conjugate) solutions of $\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}$ and any linear combinations of these two solutions will also be a solution.

COMPLEX EIGENVALUES

Consider the following linear combinations of $e^{\lambda t}\boldsymbol{x}$ and $e^{\bar{\lambda} t}\overline{\boldsymbol{x}}$:

$$\boldsymbol{Y}_1 = \frac{1}{2}(e^{\lambda t}\boldsymbol{x} + e^{\bar{\lambda} t}\overline{\boldsymbol{x}}) \qquad \boldsymbol{Y}_2 = \frac{1}{2i}(e^{\lambda t}\boldsymbol{x} - e^{\bar{\lambda} t}\overline{\boldsymbol{x}})$$

$$\frac{1}{2}\left[(a+ib)+(a-ib)\right] = \frac{1}{2}\left[2a\right] = a = \mathrm{Re}(a+ib)$$

$$\frac{1}{2i}\left[(a+ib)-(a-ib)\right] = \frac{1}{2i}\left[2ib\right] = b = \mathrm{Im}(a+ib)$$

COMPLEX EIGENVALUES

Consider the following linear combinations of $e^{\lambda t}\boldsymbol{x}$ and $e^{\bar{\lambda} t}\overline{\boldsymbol{x}}$:

$$\boldsymbol{Y}_1 = \frac{1}{2}(e^{\lambda t}\boldsymbol{x} + e^{\bar{\lambda} t}\overline{\boldsymbol{x}}) = \mathrm{Re}(e^{\lambda t}\boldsymbol{x}) \qquad \boldsymbol{Y}_2 = \frac{1}{2i}(e^{\lambda t}\boldsymbol{x} - e^{\bar{\lambda} t}\overline{\boldsymbol{x}}) = \mathrm{Im}(e^{\lambda t}\boldsymbol{x})$$

Then $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ will be real-valued solutions of $\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}$.

More precisely,

Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{\lambda t}\boldsymbol{x} = e^{(a+ib)t}\boldsymbol{x} = e^{at}e^{ibt}\boldsymbol{x}$$

$$= e^{at}((\cos bt) + (i\sin bt))\boldsymbol{x}$$

$$= e^{at}((\cos bt) + (i\sin bt))(\mathrm{Re}(\boldsymbol{x}) + i\,\mathrm{Im}(\boldsymbol{x}))$$

$$= e^{at}\left[(\cos bt)\mathrm{Re}(\boldsymbol{x}) - (\sin bt)\mathrm{Im}(\boldsymbol{x}) + i((\cos bt)\mathrm{Im}(\boldsymbol{x}) + (\sin bt)\mathrm{Re}(\boldsymbol{x}))\right]$$

COMPLEX EIGENVALUES

Consider the following linear combinations of $e^{\lambda t}\boldsymbol{x}$ and $e^{\bar{\lambda} t}\overline{\boldsymbol{x}}$:

$$\boxed{\boldsymbol{Y}_1 = \frac{1}{2}\left(e^{\lambda t}\boldsymbol{x} + e^{\bar{\lambda} t}\overline{\boldsymbol{x}}\right) = \operatorname{Re}\left(e^{\lambda t}\boldsymbol{x}\right)}$$

$$\boxed{\boldsymbol{Y}_2 = \frac{1}{2i}\left(e^{\lambda t}\boldsymbol{x} - e^{\bar{\lambda} t}\overline{\boldsymbol{x}}\right) = \operatorname{Im}\left(e^{\lambda t}\boldsymbol{x}\right)}$$

$$e^{\lambda t}\boldsymbol{x}$$

$$= e^{at}\left[\boxed{(\cos bt)\operatorname{Re}(\boldsymbol{x}) - (\sin bt)\operatorname{Im}(\boldsymbol{x})} + i\left(\boxed{(\cos bt)\operatorname{Im}(\boldsymbol{x}) + (\sin bt)\operatorname{Re}(\boldsymbol{x})}\right)\right]$$

$$\boxed{\operatorname{Re}\left(e^{\lambda t}\boldsymbol{x}\right) = e^{at}\left[(\cos bt)\operatorname{Re}(\boldsymbol{x}) - (\sin bt)\operatorname{Im}(\boldsymbol{x})\right] = \boldsymbol{Y}_1}$$

$$\boxed{\operatorname{Im}\left(e^{\lambda t}\boldsymbol{x}\right) = e^{at}\left[(\cos bt)\operatorname{Im}(\boldsymbol{x}) + (\sin bt)\operatorname{Re}(\boldsymbol{x})\right] = \boldsymbol{Y}_2}$$

EXAMPLE

Solve the system of linear differential equations $\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}$ where

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}.$$

We first find the eigenvalues of $\boldsymbol{A}$:

$$\begin{vmatrix} \lambda - 1 & -1 \\ 2 & \lambda - 3 \end{vmatrix} = \lambda^2 - 4\lambda + 5 \quad = 0 \Leftrightarrow \lambda = 2 + i \text{ or } 2 - i$$

$\lambda = 2 + i$: Consider the eigenspace E_λ:
$$\left(\begin{array}{cc|c} 1+i & -1 & 0 \\ 2 & -1+i & 0 \end{array}\right)$$

EXAMPLE

$$\left(\begin{array}{cc|c} 1+i & -1 & 0 \\ 2 & -1+i & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1+i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$(1+i)x - y = 0 \Leftrightarrow \begin{cases} x & = & s \\ y & = & (1+i)s \end{cases}$$

$$E_\lambda = \text{span}\left\{\begin{pmatrix} 1 \\ 1+i \end{pmatrix}\right\}$$

$$\text{Let } \boldsymbol{x} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

$$e^{\lambda t}\boldsymbol{x}$$

$$= e^{(2+i)t}\boldsymbol{x}$$

$$= e^{2t}(\cos t + i\sin t)\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

$$= \begin{pmatrix} e^{2t}(\cos t + i\sin t) \\ e^{2t}(\cos t + i\sin t + i\cos t - \sin t) \end{pmatrix}$$

EXAMPLE

$$e^{\lambda t}\boldsymbol{x} = \begin{pmatrix} e^{2t}(\cos t + i\sin t) \\ e^{2t}(\cos t + i\sin t + i\cos t - \sin t) \end{pmatrix}$$

$$= \begin{pmatrix} e^{2t}\cos t + ie^{2t}\sin t \\ e^{2t}(\cos t - \sin t + ie^{2t}(\cos t + \sin t) \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} e^{2t}\cos t \\ e^{2t}(\cos t - \sin t) \end{pmatrix}} + i \boxed{\begin{pmatrix} e^{2t}\sin t \\ e^{2t}(\cos t + \sin t) \end{pmatrix}}$$

$$\boldsymbol{Y}_1 = \mathrm{Re}(e^{\lambda t}\boldsymbol{x}) \qquad \boldsymbol{Y}_2 = \mathrm{Im}(e^{\lambda t}\boldsymbol{x})$$

Any linear combination $\boldsymbol{Y} = c_1\boldsymbol{Y}_1 + c_2\boldsymbol{Y}_2$ will be a solution to the system.

SUMMARY

1) How to deal with a system of Linear Differential Equations $\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}$ with 0 as an eigenvalue of $\boldsymbol{A}$.

2) How to deal with a system of Linear Differential Equations $\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}$ with $\lambda = a + ib \in \mathbb{C}$ as an eigenvalue of $\boldsymbol{A}$.