LINEAR INDEPENDENCE II

Recall the notion of redundancy

$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are vectors taken from $\mathbb{R}^n$.

If $\boldsymbol{u}_k$ is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}$, then

$$\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}\} = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k\}$$

We say that $\boldsymbol{u}_k$ is redundant in the span of $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_k\}$.

Theorem

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a set of vectors in $\mathbb{R}^n$, where $k \geq 2$.

1) S is linearly dependent if and only if at least one $\boldsymbol{u}_i \in S$ can be written as a linear combination of the other vectors in S, that is,

$$\boldsymbol{u}_i = a_1\boldsymbol{u}_1 + a_2\boldsymbol{u}_2 + \ldots + a_{i-1}\boldsymbol{u}_{i-1} + a_{i+1}\boldsymbol{u}_{i+1} + \ldots + a_k\boldsymbol{u}_k$$

for some $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k \in \mathbb{R}$.

So,

$\text{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{i-1}, \boldsymbol{u}_i, \boldsymbol{u}_{i+1}, \ldots \boldsymbol{u}_k\} = \text{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{i-1}, \boldsymbol{u}_{i+1}, \ldots \boldsymbol{u}_k\}$

Theorem

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a set of vectors in $\mathbb{R}^n$, where $k \geq 2$.

2) S is linearly independent if and only if no vector in S can be written as a linear combination of the other vectors in S.

Remark

So a set of vectors is linearly dependent if and only if there exists at least one 'redundant' vector in the set.

A set of vectors is linearly independent if and only if there is no 'redundant' vector in the set.

Example

$S=\{(1,0),(0,4),(2,4)\}$. Is S a linearly independent set?

No, since $(2,4)=2(1,0)+1(0,4)$.

$S=\{(-1,0,0),(0,3,0),(0,0,7)\}$. Is S a linearly independent set?

Yes, since

$(\boxed{-1},0,0)$ is not a linear combination of $(\boxed{0},3,0)$ and $(\boxed{0},0,7)$

$(0,\boxed{3},0)$ is not a linear combination of $(-1,\boxed{0},0)$ and $(0,\boxed{0},7)$

$(0,0,\boxed{7})$ is not a linear combination of $(-1,0,\boxed{0})$ and $(0,3,\boxed{0})$

Theorem

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_{\boxed{k}}\}$ be a set of vectors in $\mathbb{R}^{\boxed{n}}$.

If $\boxed{k} > \boxed{n}$, then S is linearly dependent.

Proof:

Let $\boldsymbol{u}_1 = (u_{11,} u_{12}, \ldots, u_{1n})$ $\quad \boldsymbol{u}_2 = (u_{21,} u_{22}, \ldots, u_{2n})$

$\ldots\ \boldsymbol{u}_k = (u_{k1,} u_{k2}, \ldots, u_{kn})$

Vector equation: $c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k = \mathbf{0}$

$$c_1(u_{11}, u_{12}, \ldots, u_{1n}) + c_2(u_{21}, u_{22}, \ldots, u_{2n}) + \ldots + c_k(u_{k1}, u_{k2}, \ldots, u_{kn})$$
$$= (0, 0, \ldots, 0)$$

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$$= (0, 0, \ldots, 0)$$

Linear system:

One unknown for each vector $\boxed{k}$

One equation for each component $\boxed{n}$

$$\begin{cases} c_1 u_{11} & + & c_2 u_{21} & + & \ldots & + & c_k u_{k1} & = & 0 \\ c_1 u_{12} & + & c_2 u_{22} & + & \ldots & + & c_k u_{k2} & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ c_1 u_{1n} & + & c_2 u_{2n} & + & \ldots & + & c_k u_{kn} & = & 0 \end{cases}$$

Theorem

If $\boxed{k} > \boxed{n}$, then S is linearly dependent.

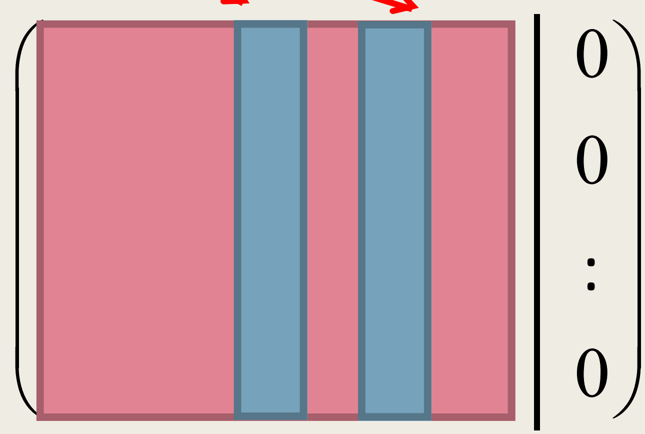
Proof:

$$c_1(u_{11}, u_{12}, \ldots, u_{1n}) + c_2(u_{21}, u_{22}, \ldots, u_{2n}) + \ldots + c_k(u_{k1}, u_{k2}, \ldots, u_{kn})$$
$$= (0, 0, \ldots, 0)$$

$\boxed{k} > \boxed{n} \Rightarrow$ more columns than rows

$\Rightarrow$ non pivot columns at row-echelon form

$$\begin{array}{cc} & \boxed{k} \\ \boxed{n} & \left(\begin{array}{cccc|c} u_{11} & u_{21} & \ldots & u_{k1} & 0 \\ u_{12} & u_{22} & \ldots & u_{k2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ u_{1n} & u_{2n} & \ldots & u_{kn} & 0 \end{array}\right) \end{array} \xrightarrow[\text{Elimination}]{\text{Gaussian}}$$



Theorem

If $\boxed{k} > \boxed{n}$, then S is linearly dependent.

Proof:

Linear system has non trivial solutions.

$$\begin{cases} c_1u_{11} & + & c_2u_{21} & + & \ldots & + & c_ku_{k1} & = & 0 \\ c_1u_{12} & + & c_2u_{22} & + & \ldots & + & c_ku_{k2} & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ c_1u_{1n} & + & c_2u_{2n} & + & \ldots & + & c_ku_{kn} & = & 0 \end{cases}$$

Vector equation has non trivial solutions.

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k = \boldsymbol{0}$$

Example

1) A set of three or more vectors in $\mathbb{R}^{2}$ is always linearly dependent.

2) A set of four or more vectors in $\mathbb{R}^{3}$ is always linearly dependent.

Summary

1) Linear independence and redundancy.

2) "Guaranteed" linear dependence.