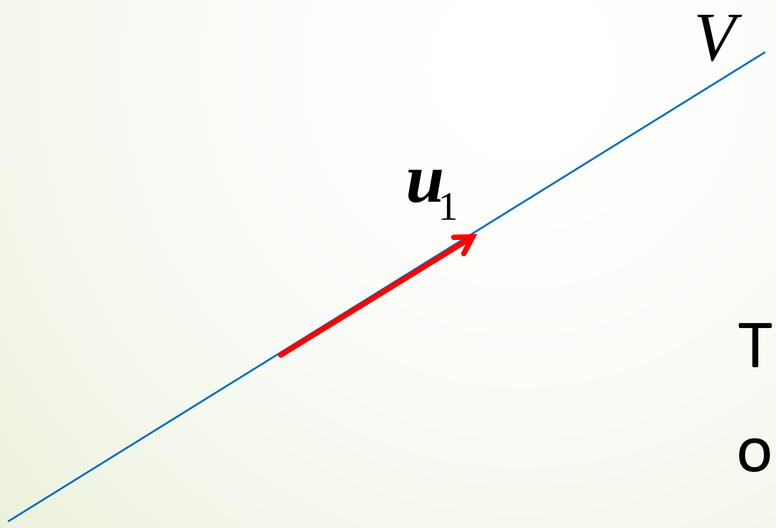
Gram-Schmidt Process

Finding orthogonal bases

Let V be a subspace of $\mathbb{R}^n$.

1) V is one-dimensional, that is, $V = \text{span}\{\boldsymbol{u}_1\}$, $\boldsymbol{u}_1 \neq \mathbf{0}$.

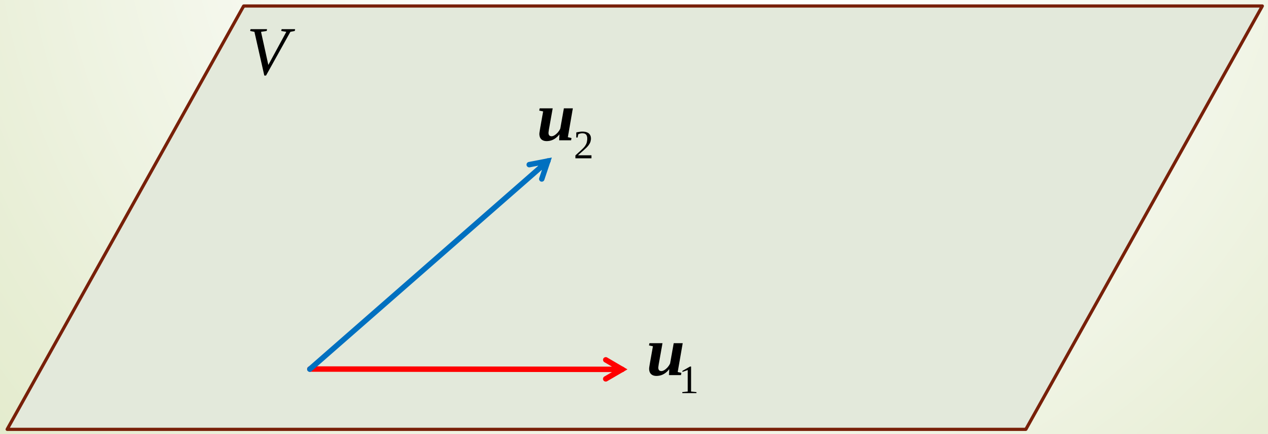


Trivially, $\{\boldsymbol{u}_1\}$ is an orthogonal basis for V.

Finding orthogonal bases

Let V be a subspace of $\mathbb{R}^n$.

2) V is two-dimensional, that is, $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are linearly independent vectors.

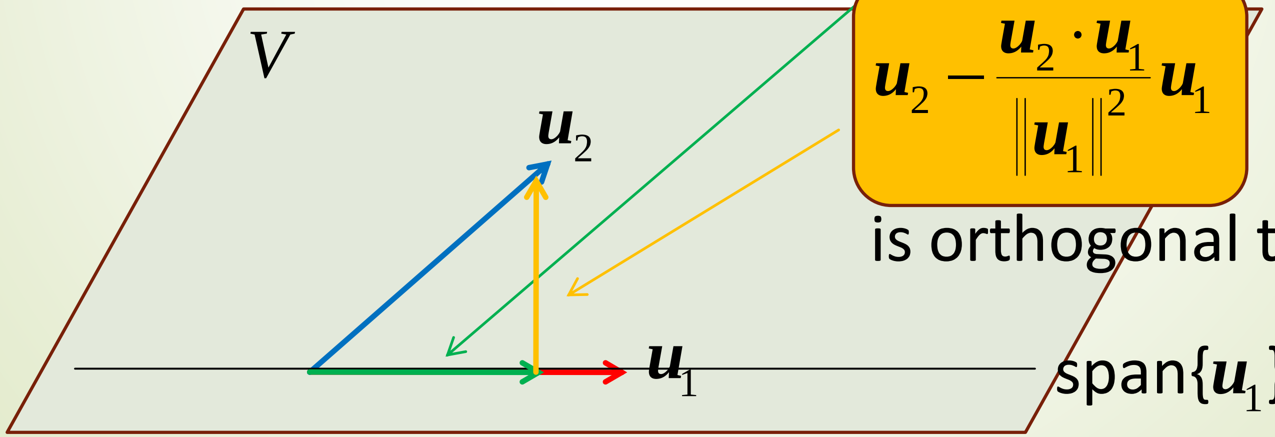


Finding orthogonal bases

span$\{\boldsymbol{u}_1\}$ is a subspace of V and $\{\boldsymbol{u}_1\}$ is an orthogonal basis for span$\{\boldsymbol{u}_1\}$.

By orthogonal projection theorem, $\dfrac{\boldsymbol{u}_2 \cdot \boldsymbol{u}_1}{\|\boldsymbol{u}_1\|^2}\boldsymbol{u}_1$ is the projection of $\boldsymbol{u}_2$ onto span$\{\boldsymbol{u}_1\}$

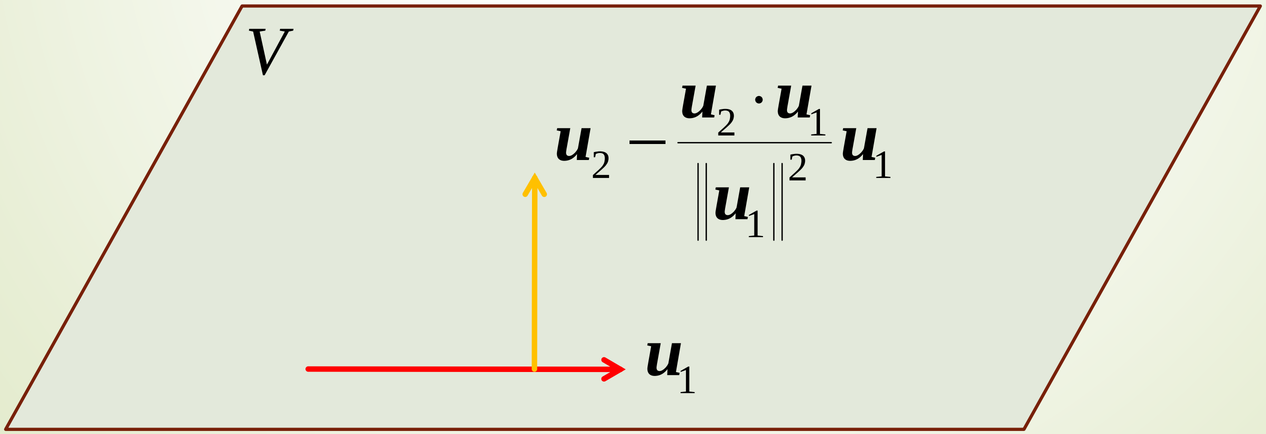
$\boldsymbol{u}_2 - \dfrac{\boldsymbol{u}_2 \cdot \boldsymbol{u}_1}{\|\boldsymbol{u}_1\|^2}\boldsymbol{u}_1$ is orthogonal to $\boldsymbol{u}_1$



Finding orthogonal bases

2) V is two-dimensional, that is, $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are linearly independent vectors.

$$\left\{\boxed{\boldsymbol{u}_1}\ ,\ \boxed{\boldsymbol{u}_2 - \frac{\boldsymbol{u}_2 \cdot \boldsymbol{u}_1}{\|\boldsymbol{u}_1\|^2}\boldsymbol{u}_1}\right\}$$ is an orthogonal basis for V.

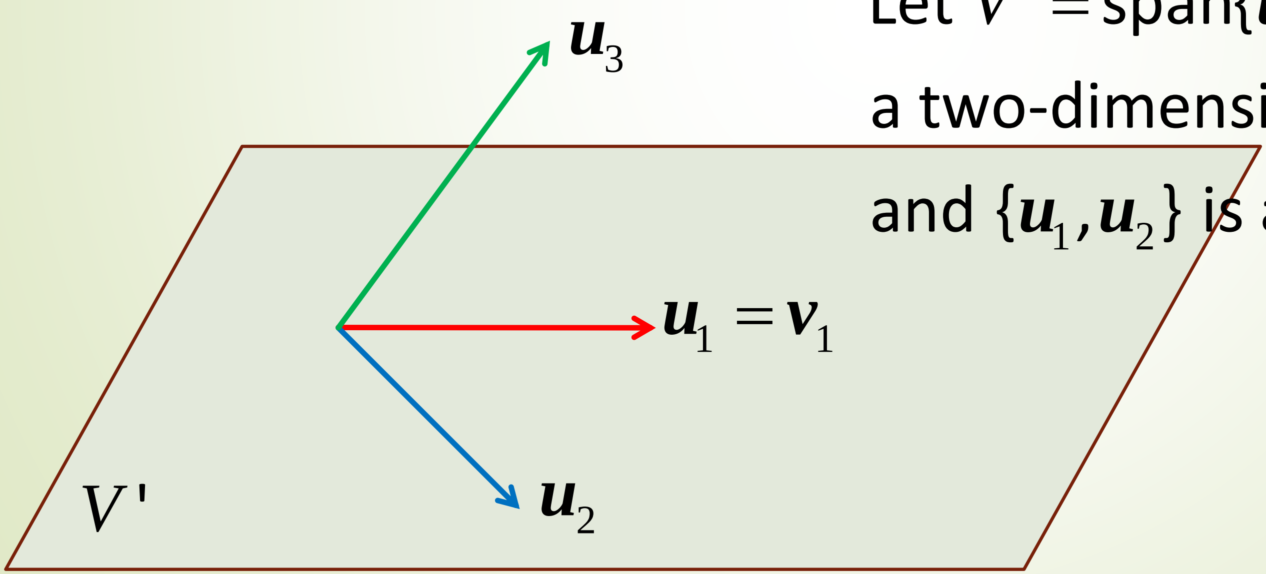


Finding orthogonal bases

Let V be a subspace of $\mathbb{R}^n$.

3) V is three-dimensional, that is, $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$, $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ are linearly independent vectors.

Let $V' = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$, then V' is a two-dimensional subspace of V and $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ is a basis for V'.

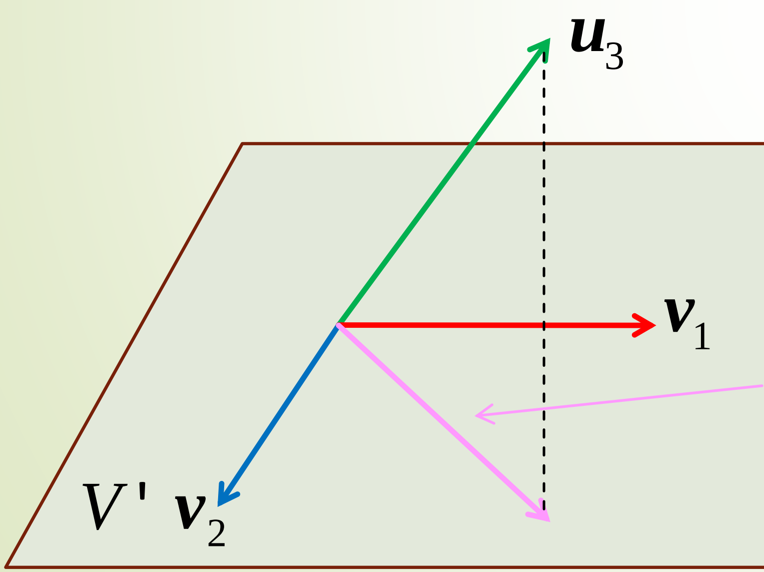


Let $\boldsymbol{v}_1 = \boldsymbol{u}_1$.

Finding orthogonal bases

Let V be a subspace of $\mathbb{R}^n$.

By previous discussion, $\{\boldsymbol{v}_1, \underbrace{\boxed{\boldsymbol{u}_2 - \frac{\boldsymbol{u}_2 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1}}_{\boldsymbol{v}_2}\}$ is an orthogonal basis for V'.



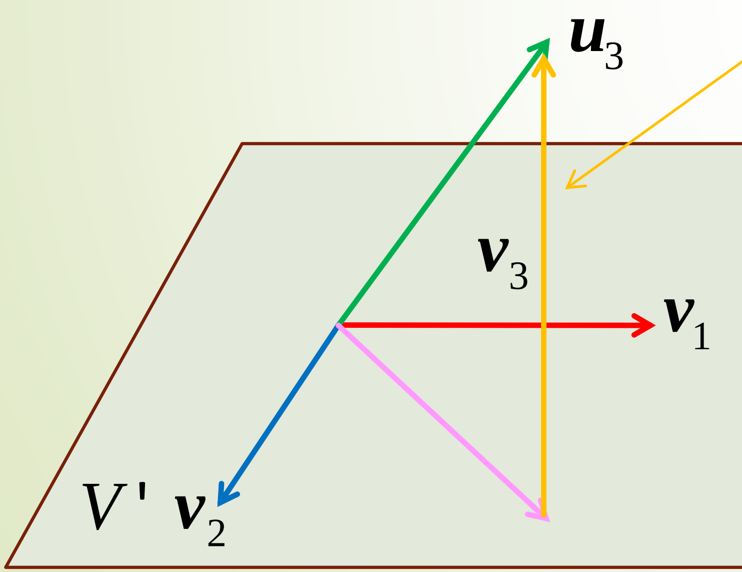
By orthogonal projection theorem, $\boxed{\frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 + \frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_2}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2}$ is the projection of $\boldsymbol{u}_3$ onto V'.

Finding orthogonal bases

$\boldsymbol{u}_3 - (\frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 + \frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_2}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2)$ is orthogonal to $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$

Let $\boxed{\boldsymbol{v}_3} = \boldsymbol{u}_3 - (\frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 + \frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_2}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2)$.

Then $\boldsymbol{v}_3$ is orthogonal to V'.



By orthogonal projection theorem,

$\frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 + \frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_2}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2$ is the projection

of $\boldsymbol{u}_3$ onto V'.

Finding orthogonal bases

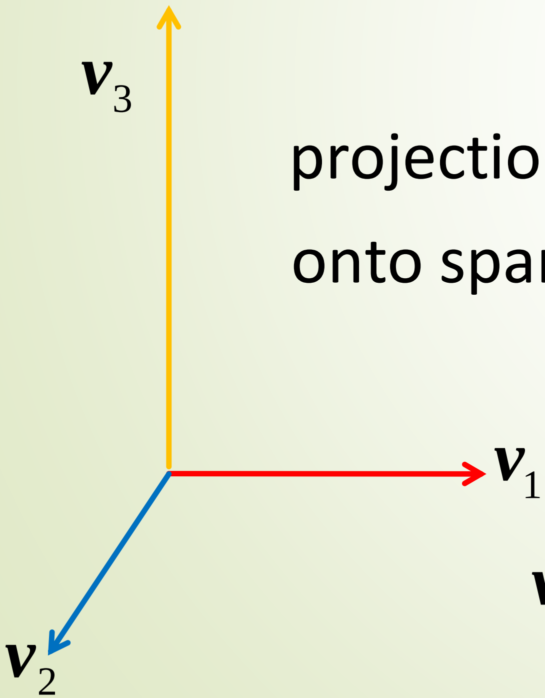
$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is an orthogonal basis for V.

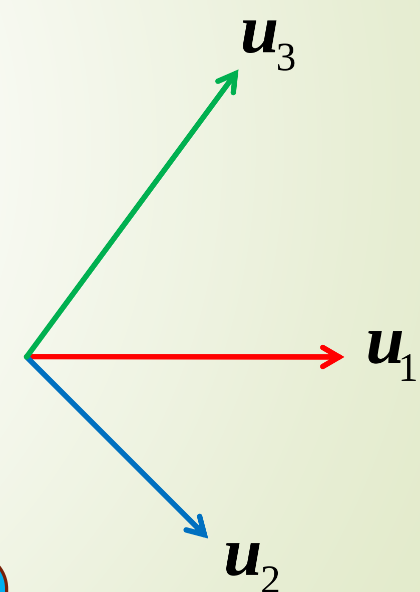
$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is a basis for V.

$$\boldsymbol{v}_1 = \boldsymbol{u}_1 \qquad \boldsymbol{v}_2 = \boldsymbol{u}_2 - \boxed{\frac{\boldsymbol{u}_2 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1}$$

projection of $\boldsymbol{u}_2$ onto span$\{\boldsymbol{v}_1\}$

projection of $\boldsymbol{u}_3$ onto span$\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$





$$\boldsymbol{v}_3 = \boldsymbol{u}_3 - \boxed{\left(\frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 + \frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_2}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2\right)}$$

Theorem (Gram-Schmidt Process)

Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a basis for a vector space V.

Let $\boldsymbol{v}_1 = \boldsymbol{u}_1$;

$$\boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\boldsymbol{u}_2 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1; \qquad \boldsymbol{v}_3 = \boldsymbol{u}_3 - \left(\frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 + \frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_2}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2\right);$$

$$\vdots$$

$$\boldsymbol{v}_k = \boldsymbol{u}_k - \left(\frac{\boldsymbol{u}_k \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 + \frac{\boldsymbol{u}_k \cdot \boldsymbol{v}_2}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2 + \ldots + \frac{\boldsymbol{u}_k \cdot \boldsymbol{v}_{k-1}}{\|\boldsymbol{v}_{k-1}\|^2}\boldsymbol{v}_{k-1}\right);$$

Then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is an orthogonal basis for V.

Theorem (Gram-Schmidt Process)

Then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is an orthogonal basis for V.

Let $\boldsymbol{v}_1 = \boldsymbol{u}_1$;

$$\boldsymbol{v}_2 = \boldsymbol{u}_2 - \frac{\boldsymbol{u}_2 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1; \qquad \boldsymbol{v}_3 = \boldsymbol{u}_3 - \left(\frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 + \frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_2}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2\right);$$

$$\vdots$$

$$\boldsymbol{v}_k = \boldsymbol{u}_k - \left(\frac{\boldsymbol{u}_k \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 + \frac{\boldsymbol{u}_k \cdot \boldsymbol{v}_2}{\|\boldsymbol{v}_2\|^2}\boldsymbol{v}_2 + \ldots + \frac{\boldsymbol{u}_k \cdot \boldsymbol{v}_{k-1}}{\|\boldsymbol{v}_{k-1}\|^2}\boldsymbol{v}_{k-1}\right);$$

$\left\{\frac{1}{\|\boldsymbol{v}_1\|}\boldsymbol{v}_1, \frac{1}{\|\boldsymbol{v}_2\|}\boldsymbol{v}_2, \ldots, \frac{1}{\|\boldsymbol{v}_k\|}\boldsymbol{v}_k\right\}$ is an orthonormal basis for V.

Example (Gram-Schmidt Process)

Apply Gram-Schmidt Process to transform

$$\{(1,0,1),(0,1,2),(2,1,0)\}$$

into an orthogonal basis for $\mathbb{R}^3$.

Remark: You may choose

$$\boldsymbol{u}_1=(1,0,1),\ \boldsymbol{u}_2=(0,1,2),\ \boldsymbol{u}_3=(2,1,0).$$

Example (Gram-Schmidt Process)

$\boldsymbol{u}_1 = (1,0,1),\ \boldsymbol{u}_2 = (0,1,2),\ \boldsymbol{u}_3 = (2,1,0).$

$\boldsymbol{v}_1 = \boldsymbol{u}_1 = (1,0,1)$

$\boldsymbol{v}_2 = \boldsymbol{u}_2 - (\text{projection of } \boldsymbol{u}_2 \text{ onto span}\{\boldsymbol{v}_1\})$

Check: $\boldsymbol{v}_2 \cdot \boldsymbol{v}_1 = 0$

$$= \boldsymbol{u}_2 - \left(\frac{\boldsymbol{u}_2 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2} \right) \boldsymbol{v}_1 \quad = (0,1,2) - \left(\frac{2}{2} \right)(1,0,1) \ = (-1,1,1)$$

$\boldsymbol{v}_3 = \boldsymbol{u}_3 - (\text{projection of } \boldsymbol{u}_3 \text{ onto span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\})$

$$= \boldsymbol{u}_3 - \left[\left(\frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2} \right) \boldsymbol{v}_1 - \left(\frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_2}{\|\boldsymbol{v}_2\|^2} \right) \boldsymbol{v}_2 \right]$$

Example (Gram-Schmidt Process)

$$\boldsymbol{v}_1 = \boldsymbol{u}_1 = (1,0,1)$$

$\boldsymbol{v}_2 = \boldsymbol{u}_2 -$ (projection of $\boldsymbol{u}_2$ onto span$\{\boldsymbol{v}_1\}$)

Check: $\boldsymbol{v}_2 \cdot \boldsymbol{v}_1 = 0$

$$= \boldsymbol{u}_2 - \left(\frac{\boldsymbol{u}_2 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\right)\boldsymbol{v}_1 \quad = (0,1,2) - \left(\frac{2}{2}\right)(1,0,1) \; = (-1,1,1)$$

$\boldsymbol{v}_3 = \boldsymbol{u}_3 -$ (projection of $\boldsymbol{u}_3$ onto span$\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$)

$$= \boldsymbol{u}_3 - \left[\left(\frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\right)\boldsymbol{v}_1 + \left(\frac{\boldsymbol{u}_3 \cdot \boldsymbol{v}_2}{\|\boldsymbol{v}_2\|^2}\right)\boldsymbol{v}_2\right]$$

Check: $\boldsymbol{v}_3 \cdot \boldsymbol{v}_1 = 0$

and $\boldsymbol{v}_3 \cdot \boldsymbol{v}_2 = 0$

$$= (2,1,0) - \left[\left(\frac{2}{2}\right)(1,0,1) + \left(\frac{-1}{3}\right)(-1,1,1)\right] = (\frac{2}{3}, \frac{4}{3}, -\frac{2}{3})$$

Example (Gram-Schmidt Process)

Apply Gram-Schmidt Process to transform

$$\{(1,0,1),(0,1,2),(2,1,0)\}$$

into an orthogonal basis for $\mathbb{R}^3$.

$\boldsymbol{v}_1 = (1,0,1)$ $\quad$ $\boldsymbol{v}_2 = (-1,1,1)$ $\quad$ $\boldsymbol{v}_3 = (\frac{2}{3},\frac{4}{3},-\frac{2}{3})$

$\{\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

Summary

1) Using orthogonal projection, Gram-Schmidt Process is a procedure to convert a basis into an orthogonal basis.