MA1512 LECTURE NOTES

CHAPTER 1

DIFFERENTIAL EQUATIONS

1.1 Introduction

A *differential* equation is an equation that contains one or more derivatives of a differentiable function. [In this chapter we deal only with ordinary DEs, NOT partial DEs.]

The *order* of a d.e. is the order of the equation's highest order derivative; and a d.e. is *linear* if it can be put in the form

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y^{(1)}(x) + a_0 y(x) = F,$$

where a_i, $0 \leq i \leq n$, and F are all functions of x.

For example, $y' = 5y$ and $xy' - \sin x = 0$ are first order linear d.e.; $(y''')^2 + (y'')^5 - y' = e^x$ is third order, nonlinear.

We observe that in general, a d.e. has many solutions, e.g. $y = \sin x + c$, c an arbitrary constant, is a solution of $y' = \cos x$.

Such solutions containing arbitrary constants are called *general solution* of a given d.e.. Any solution obtained from the general solution by giving specific values to the arbitrary constants is called a *particular solution* of that d.e. e.g.

$y = \sin x + 1$ is a particular solution of $y' = \cos x$.

Basically, differential equations are solved using integration, and it is clear that there will be as many integrations as the order of the DE. Therefore, THE GENERAL SOLUTION OF AN nth-ORDER DE WILL HAVE n ARBITRARY CONSTANTS.

1.2 Separable equations

A first order d.e. is *separable* if it can be written in the form $M(x) - N(y)y' = 0$ or equivalently, $M(x)dx = N(y)dy$. When we write the d.e. in this form, we say that we have *separated the variables*, because everything involving x is on one side, and everything involving y is on the other.

We can solve such a d.e. by integrating w.r.t. x:

$$\int M(x)dx = \int N(y)dy + c.$$

Example 1. Solve $y' = (1 + y^2)e^x$.

Solution. We separate the variables to obtain

$$e^x dx = \frac{1}{1 + y^2} dy.$$

Integrating w.r.t. x gives

$$e^x = \tan^{-1} y + c,$$

or

$$\tan^{-1} y = e^x - c,$$

or

$$y = \tan(e^x - c).$$

Example 2. Experiments show that a radioactive substance decomposes at a rate proportional to the amount present. Starting with 2 mg at certain time, say $t = 0$, what can be said about the amount available at a later time?

Example 3. A copper ball is heated to

100°C. At $t = 0$ it is placed in water which is maintained at 30°C. At the end of 3 mins the temperature of the ball is reduced to 70°C. Find the time at which the temperature of the ball is 31°C.

Physical information: Experiments show that the rate of change dT/dt of the temperature T of the ball w.r.t. time is proportional to the difference between T and the temp T_0 of the surrounding medium. Also, heat flows so rapidly in copper that at any time the temperature is practically the same at all points of the ball.

Example 4. Suppose that a sky diver falls from rest toward the earth and the parachute opens at an instant $t = 0$, when sky diver's speed is $v(0) = v_0 = 10\,\mathrm{m/s}$. Find the speed of the sky diver at any later time t.

Physical assumptions and laws:

weight of the man + equipment = 712N,

air resistance = bv^2, where $b = 30\,\mathrm{kg/m}$.

Using Newton's second law we obtain

$$m\frac{dv}{dt} = mg - bv^2.$$

Thus $\dfrac{dv}{dt} = -\dfrac{b}{m}(v^2 - k^2),\ k^2 = \dfrac{mg}{b}$

$$\frac{1}{v^2 - k^2}\,dv = -\frac{b}{m}\,dt$$

$$\frac{1}{2k}\left(\frac{1}{v-k}-\frac{1}{v+k}\right)dv=-\frac{b}{m}\,dt.$$

Integrating gives

$$\ell n\left|\frac{v-k}{v+k}\right|=-\frac{2kb}{m}\,t+c_1,$$

or

$$\frac{v-k}{v+k}=ce^{-pt},\ p=\frac{2kb}{m}.$$

($c=\pm e^{c_1}$ according as the ratio on the left is positive or negative).

Solving for v : $v=k\,\dfrac{1+ce^{-pt}}{1-ce^{-pt}}$. (Note that $v\to k$ as $t\to\infty$). From $v(0)=v_0$, $c=\dfrac{v_0-k}{v_0+k}$. Now $k^2=\dfrac{mg}{b}=\dfrac{712}{30}$, so $k=4.87\,\text{m/s}$, $v_0=10\,\text{m/s}$, $c=0.345$, $p=4.02$.

$$\therefore\ v(t)=4.87\,\frac{1+0.345e^{-4.02t}}{1-0.345e^{-4.02t}}.$$

Example 5. The **orbit** of a planet is the shape it traces out as it moves around the Sun. The best way to describe an orbit is by using plane **polar coordinates** [please revise if necessary!]. These give the position of a point in the plane by specifying its distance r from the origin together with the angle θ made by its position vector with the x axis. A shape or graph in the plane is given by a function of the form $r(\theta)$ [just as, in Cartesian coordinates, a graph is given by a function $y(x)$]. Using his own laws of motion, Isaac Newton discovered that every planet has an orbit which satisfies an equation of the following form:

$$(\frac{du}{d\theta})^2 \ + \ (u \ - \ A)^2 \ = \ B^2,$$

where $u(\theta)$ is the reciprocal of $r(\theta)$ and where A, B are positive constants [with the same units as u, namely 1/[length]] with $B/A < 1$. Solve this equation.

<u>Solution.</u> This is a separable equation:

$$d\theta \ = \ \frac{du/B}{\sqrt{1 \ - \ (\frac{u-A}{B})^2}},$$

and so we have

$$\theta \ + \ C \ = \ arcsin(\frac{u - A}{B}),$$

where C is the constant of integration. Recalling the definition of u we get finally

$$r = \frac{A^{-1}}{1 + \frac{B}{A} sin(\theta + C)}.$$

If you sketch this you will find that it is an egg-shaped curve called an **ellipse**. [Note that it is important here that $B/A < 1$; otherwise you will get other shapes, such as a hyperbola]. Thus all of the planets have ellipse-shaped orbits. This is called Kepler's First Law.

Reduction to separable form

Certain first order d.e. are not separable but can be made separable by a simple change of variable. This holds for equations of the form

$$y' = g(\frac{y}{x}) \qquad (1)$$

where g is any function of $\frac{y}{x}$. We set $\frac{y}{x} = v$, then $y = vx$ and $y' = v + xv'$. Thus (1) becomes $v + xv' = g(v)$, which is separable. Namely, $\frac{dv}{g(v) - v} = \frac{dx}{x}$. We can now solve for v, hence obtain y.

Example 6.

Solve $2xyy' - y^2 + x^2 = 0$. $\quad [x^2 + y^2 = cx]$

Linear Change of Variable

A d.e. of the form $y' = f(ax + by + c)$, where f is continuous and $b \neq 0$ (if $b = 0$, the equation is separable) can be solved by setting $u = ax + by + c$.

Example 7. $(2x - 4y + 5)y' + x - 2y + 3 = 0$.

Set $x - 2y = u$, we have

$$(2u+5)\frac{1}{2}(1-u') + u + 3 = 0,$$
$$(2u+5)u' = 4u + 11.$$

Separating variables and integrating :

$$\left(1 - \frac{1}{4u+11}\right) du = 2dx.$$

Thus $\quad u - \frac{1}{4}\ell n|4u+11| = 2x + c_1,$

or $\quad 4x + 8y + \ell n|4x - 8y + 11| = c.$

1.3 Linear First Order ODEs

A d.e. which can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{1}$$

where P and Q are functions of x, is called a linear first order d.e. Relation (1) above is the standard form of such a d.e.

To solve (1), define a new function $R(x)$ by $R(x) = e^{\int^x P(s)ds}$ and note that $R' = RP$ by the chain rule. So $(Ry)' = RPy + Ry'$. Hence if we multiply both sides of (1) by R we get

$$Ry' + RPy = RQ$$

or

$$(Ry)' = RQ.$$

Now you can integrate both sides and then divide by R to obtain y. The function R is called the INTEGRATING FACTOR for this equation.

Example 8. Solve

(i) $xy' - 3y = x^2,\ x > 0.$

(ii) $y' - y = e^{2x}.$

Example 9. Consider an object of mass m

dropped from rest in a medium that offers a resistance proportional to the magnitude of the instantaneous velocity of the object. The goal is to find the position $x(t)$ and velocity $v(t)$ at any time t.

Solution. Newton's second law of motion gives $m\frac{dv}{dt} = mg - kv$. The initial conditions are $v(0) = 0$ and $x(0) = 0$. The equation is separable. Solving it directly or by multiplying it by an integrating factor $e^{\frac{k}{m}t}$, we obtain $v = \frac{mg}{k}(1 - e^{-\frac{kt}{m}})$, where $v(0) = 0$ has been used.

Set $v = \frac{dx}{dt}$ in the above and integrate, and using $x(0) = 0$, we get $x(t) = \frac{mg}{k}t - \frac{m^2 g}{k^2}(1 - e^{-\frac{kt}{m}})$.

Example 10. At time $t = 0$ a tank contains 20 lbs of salt dissolved in 100 gal of water. Assume that water containing 0.25 lb of salt per gallon is entering the tank at a rate of 3 gal/min and the well stirred solution is leaving the tank at the same rate. Find the amount of salt at any time t.

Solution. Let the amount of salt at time t be $Q(t)$. The time rate of change $\frac{dQ}{dt}$ equals the inflow minus the outflow.

We obtain $\frac{dQ}{dt} = 3 \times 0.25 - \frac{3Q}{100}$, with $Q(0) = 20$. Solving, $Q(t) = 25 - 5e^{-\frac{3t}{100}}$. Note that $\lim_{t\to\infty} Q(t) = 25$. Thus after sufficiently long time, the salt concentration remains constant at 25 lbs/100 gal.

Example 11. In Example 2 in Section 1.2, we saw that radioactive substances typically decay at a rate proportional to the amount present. Sometimes the product of a radioactive decay is itself a radioactive substance which in turn decays (at a different rate). An interesting example of this is provided by *Uranium-Thorium dating*, which is a method used by palaeontologists to determine how old certain fossils [especially ancient corals] are. Corals filter the sea-water in which they live. Sea-water contains a tiny amount of a certain kind of Uranium [Uranium 234] and the corals absorb this into their bodies. Uranium 234 decays, with a half-life of 245000 years, into Thorium 230, which itself decays with a half-life of 75000 years. Thorium is not found in sea-water; so when the coral dies, it has a certain amount of Uranium in it but no Thorium [because the lifetime of a coral polyp is negligible compared with 245000 years]. It is possible to measure the ratio of the amounts of Uranium and Thorium in any given sample. From this ratio we want to work out the age of the sample [the time when it died]. This is important if we want to know whether global warming is causing corals to die now. [Maybe they die off regularly over long periods of time and the current deaths have nothing to do with global warming.]

Let U(t) be the number of atoms of Uranium in a particular sample of ancient coral and let T(t) be the number of atoms of Thorium. Because each decay of one Uranium atom produces one Thorium atom, Thorium atoms are being born at exactly the same rate at which Uranium atoms die: so we have

$$\frac{dU}{dt} = -k_U U, \tag{1}$$

$$\frac{dT}{dt} = +k_U U - k_T T, \tag{2}$$

where k_U, k_T are constants [related to the half-lives] with $k_U \neq k_T$, and $U(0) = U_0$, $T(0) = 0$. We want to find t given that we know the ratio of T(t) to U(t) at the present time.

Solving (4) with the given data gives $U =$

$U_0\, e^{-k_U t}$. From this we see that $U_0/2 = U_0\, e^{-k_U \times 245000}$ so $k_U = ln(2)/245000$ and similarly $k_T = ln(2)/75000$. So we know these numbers. Notice that k_U is smaller than k_T.

Now (5) becomes

$$\frac{dT}{dt} + k_T T = k_U U_0 e^{-k_U t}.$$

An integrating factor is $e^{k_T t}$. Solving, with $T(0) = 0$, gives

$$T(t) = \frac{k_U}{k_T - k_U} U_0 (e^{-k_U t} - e^{-k_T t}).$$

Unfortunately we don't know U_0 but luckily that goes away when we take the ratio:

$$T/U = \frac{k_U}{k_T - k_U}[1 - e^{(k_U - k_T)t}].$$

[Check that the expression on the right side is positive after t = 0; note also that while both U and T tend to zero, their ratio does not.] So now if we measure the ratio T/U at the present time, we can solve this for t and we have our answer. This method is good for coral fossils up to about half a million years old.

Reduction to linear form

Certain nonlinear d.e.s can be reduced to a linear form. The most important class of such equations are the Bernoulli equations of the form $y' + p(x)y = q(x)y^n$ where n is any real number.

If $n = 0$ or 1, the equation is linear, otherwise it is nonlinear. To solve, we rewrite this as

$y^{-n}y' + y^{1-n}p(x) = q(x)$, and set $y^{1-n} = z$. Then $(1-n)y^{-n}y' = z'$, and the given d.e. becomes $z' + (1-n)p(x)z = (1-n)q(x)$, which is a first order linear d.e.

Examples. To solve

(i) $y' - Ay = -By^2$, A, B constants.

<u>Solution.</u> (i) Observe that $n = 2$. Set $z = y^{1-2} = y^{-1}$. Then $-y^{-2}y' = z'$. (i) becomes $z' + Az = B$, which is separable. Solving, $z = \frac{B}{A} + ce^{-Ax}$.

$$\therefore \quad y = \frac{1}{z} = \frac{1}{\frac{B}{A} + ce^{-Ax}}.$$

(ii) $y' + y = x^2y^2$. $[y(Ae^x + x^2 + 2x + 2) = 1]$

1.4 Second order linear differential equations

The general form of a second order linear d.e. is

$$y'' + p(x)y' + q(x)y = F(x). \qquad (1)$$

If $F(x) \equiv 0$, the linear d.e. is called *homogeneous* otherwise it is called *nonhomogeneous*. The d.e.

$$y'' + 4y = e^{-x}\sin x$$

is a nonhomogeneous linear second order d.e.; $(1-x^2)y'' - 2xy' + 6y = 0$ or in the above standard form $y'' - \dfrac{2x}{1-x^2}y' + \dfrac{6}{1-x^2}y = 0$ is homogeneous; whereas $x(y''y+(y')^2)+2y'y = 0$

and $y'' = \sqrt{1 + (y')^2}$ are nonlinear. Note that (1) is *linear* in the sense that it is linear in y and its derivatives.

A *solution* of a second order d.e. on some interval I is a function $y = h(x)$ with derivatives $y' = h'(x)$ and $y'' = h''(x)$ satisfying the d.e. for all x in I.

Homogeneous d.e.s

The general solutions of homogeneous equations can be found with the help of the **Superposition** or **linearity** principle, which is contained in the following theorem.

Theorem. For a homogeneous linear d.e.

$$y'' + p(x)y' + q(x)y = 0, \tag{2}$$

any linear combination of two solutions on an open interval I is also a solution on I. In particular for such an equation, sums and constant multiples of solutions are again solutions.

Proof. Let y_1 and y_2 be solution of (2) on I. Then $y_1''+py_1'+qy_1 = 0$ and $y_2''+py_2'+qy_2 = 0$. Substituting $y = c_1y_1 + c_2y_2$ in the left of (2), we get

$$(c_1y_1 + c_2y_2)'' + p(c_1y_1 + c_2y_2)' + q(c_1y_1 + c_2y_2)$$
$$= c_1y_1'' + c_2y_2'' + pc_1y_1' + pc_2y_2' + qc_1y_1 + qc_2y_2$$
$$= c_1(y_1'' + py_1' + qy_1) + c_2(y_2'' + py_2' + qy_2)$$

$$= c_1 \cdot 0 + c_2 \cdot 0 = 0.$$

Thus $c_1 y_1 + c_2 y_2$ is also a solution of (2).

Caution

The above result does not hold for nonhomogeneous or nonlinear d.e.s. For example, $y = 1 + \cos x$ and $y = 1 + \sin x$ are solutions of the nonhomogeneous linear d.e. $y'' + y = 1$, but $2(1 + \cos x)$ and $2 + \cos x + \sin x$ are not its solutions. Similarly, $y = 1$ and $y = x^2$ are solutions of the nonlinear d.e. $yy'' - xy' = 0$. But $-x^2$ and $x^2 + 1$ are not its solutions.

Example 12. Solve the initial value problem $y'' - y = 0$, $y(0) = 5$, $y'(0) = 3$.

Solution. It is easy to see that e^x and e^{-x} are solutions of $y'' - y = 0$. Thus $y = c_1 e^x + c_2 e^{-x}$ is also a solution. From $y(0) = 5$ we get $c_1 + c_2 = 5$, and $y'(0) = 3$ gives $c_1 - c_2 = 3$. Solving, $c_1 = 4$, $c_2 = 1$. The required solution is $y = 4e^x + e^{-x}$.

General solution of homogeneous linear second order d.e.

Let $y_1(x)$ and $y_2(x)$ be defined on some interval I. Then y_1 and y_2 are said to be *linearly dependent* on I if one of them is a CONSTANT MULTIPLE OF THE OTHER ONE. Otherwise they are LINEARLY INDEPENDENT.

A *general solution* of $y'' + py' + qy = 0$ on an open interval I is $y = c_1 y_1 + c_2 y_2$, where y_1 and y_2 are linearly independent solutions of the d.e. and c_1, c_2 are arbitrary constants.

A *particular solution* of the d.e. on I is obtained if specific values are assigned to c_1 and c_2.

For example, $y_1 = \cos x$ and $y_2 = \sin x$ are linearly independent solutions of $y'' + y = 0$. A general solution is $y = c_1 \cos x + c_2 \sin x$.

A particular solution is, for example, $y = 2\cos x + \sin x$, (which satisfies $y(0) = 2$ and $y'(0) = 1$).

Homogeneous d.e. with constant coefficients

Consider

$$y'' + ay' + by = 0, \ \ a, \ b \ \text{ constants.} \qquad (1)$$

Recall that a first order linear d.e. $y' + ky = 0$, k constant, has $y = e^{-kx}$ as a solution. We now try the function $y = e^{\lambda x}$ as a solution of (1). Substituting $y = e^{\lambda x}$ in (1) we obtain $(\lambda^2 + a\lambda + b)e^{\lambda x} = 0$, which implies that $e^{\lambda x}$ is a solution if λ is a solution of

$$\lambda^2 + a\lambda + b = 0. \qquad (2)$$

This equation is called the *characteristic* equation (or *auxiliary* equation) of (1).

The roots of (2) are

$$\lambda_1 = \tfrac{1}{2}(-a + \sqrt{a^2 - 4b}\,),$$
$$\lambda_2 = \tfrac{1}{2}(-a - \sqrt{a^2 - 4b}\,).$$

We obtain $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ as solutions of (1).

Depending on the sign of a^2-4b, equation (2) will have

Case 1: two real roots if $a^2 - 4b > 0$,

Case 2: a real double root (i.e. $\lambda_1 = \lambda_2$) if $a^2-4b=0$,

Case 3: complex conjugate roots if $a^2-4b<0$.

<u>Case 1.</u> (2) has two distinct real roots λ_1 and

λ_2.

In this case $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are linearly independent solutions of (1) on any interval. The corresponding general solution of (1) is $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.

Example 13. Solve $y'' + y' - 2y = 0$, with $y(0) = 4$, $y'(0) = -5$.

<u>Solution.</u> The characteristic equation is $\lambda^2 + \lambda - 2 = 0$ $(a^2 - 4b = 9 > 0)$. The roots are $\lambda_1 = 1$ and $\lambda_2 = -2$. The general solution is $y = c_1 e^x + c_2 e^{-2x}$. Initial conditions $y(0) = 4$ and $y'(0) = -5$ give $c_1 + c_2 = 4$ and $c_1 - 2c_2 = -5$. Solving: $c_1 = 1$, $c_2 = 3$. Thus $y = e^x + 3e^{-2x}$

is the solution.

<u>Case 2.</u> (2) has a real double root $\lambda_1\,(=\lambda_2)$.

This occurs when $a^2-4b=0$, and $\lambda_1=\lambda_2=-\frac{a}{2}$, from which we get one solution $y_1=e^{-\frac{a}{2}x}$. To find a second solution y_2, we try $y_2=xe^{-\frac{ax}{2}}$. This does work: $y_2'=e^{-\frac{ax}{2}}-\frac{a}{2}xe^{-\frac{ax}{2}}$, and

$$\begin{aligned} y_2'' &= -\tfrac{a}{2}e^{-\frac{ax}{2}}-\tfrac{a}{2}e^{-\frac{ax}{2}}+\tfrac{a^2}{4}xe^{-\frac{ax}{2}} \\ &= \left(-a+\tfrac{a^2}{4}x\right)e^{-\frac{ax}{2}} \end{aligned}$$

So

$$\begin{aligned} y_2''+ay_2'+by_2 &= \left[-a+\tfrac{a^2}{4}x+a-\tfrac{a^2}{2}x+bx\right]e^{-\frac{ax}{2}} \\ &= \left[b-\tfrac{a^2}{4}\right]xe^{-\frac{ax}{2}}=0 \end{aligned}$$

because $b=\frac{a^2}{4}$.

Thus in this case when $a^2 - 4b = 0$, a linearly independent pair of solutions of $y'' + ay' + by = 0$ on any interval is $e^{-\frac{ax}{2}}$, $xe^{-\frac{ax}{2}}$. The corresponding general solution is $y = (c_1 + c_2 x)e^{-\frac{ax}{2}}$.

Example 14.

(i) Solve $y'' + 8y' + 16y = 0$.

(ii) Solve the initial problem $y'' - 4y' + 4y = 0$,

$y(0) = 3$, $y'(0) = 1$.

<u>Case 3.</u> (2) has two complex roots λ_1, λ_2.

This happens when $a^2 - 4b < 0$. We set $w = \sqrt{b - \frac{a^2}{4}}$. Then you can easily show that $\lambda_1, \lambda_2 = -\frac{a}{2} \pm iw$, where $i^2 = -1$. We try

$$y_1 = e^{-\frac{ax}{2}} \cos wx$$

$$y_2 = e^{-\frac{ax}{2}} \sin wx$$

We leave it to you to show by substitution that these are solutions of $y'' + ay' + by = 0$. The point to remember is that $-\frac{a}{2}$, the REAL part of λ_1 and λ_2, goes into the exponential part $\left(e^{-\frac{ax}{2}}\right)$ while the IMAGINARY part w goes into the cos and sin part, $\cos(wx)$ and $\sin(wx)$. For example if $\lambda_1, \lambda_2 = -1 \pm 2i$, then $y_1 = e^{-x}\cos(2x)$ and $y_2 = e^{-x}\sin(2x)$.

To conclude: set

$$\begin{aligned} y_1 &= e^{-\frac{a}{2}x} \cos wx, \\ y_2 &= e^{-\frac{a}{2}x} \sin wx. \end{aligned}$$

Then y_1 and y_2 are linearly independent. The corresponding general solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= e^{-\frac{a}{2}x}(c_1 \cos wx + c_2 \sin wx). \end{aligned}$$

Example 15.

(i) Solve $y'' + 2y' + 5y = 0$.

(ii) Solve $y''+2y'+5y = 0$, $y(0) = 1$, $y'(0) = 5$.

Nonhomogeneous equations

We consider

$$y'' + p(x)y' + q(x)y = r(x), \; r(x) \not\equiv 0. \quad (1)$$

The corresponding homogeneous equation is

$$y'' + p(x)y' + q(x)y = 0. \quad (2)$$

Let y_1 and y_2 be any two solutions of (1). Then

$$y_1'' + p(x)y_1' + q(x)y_1 = r(x), \; (3)$$

and

$$y_2'' + p(x)y_2' + q(x)y_2 = r(x). \; (4)$$

Subtracting (4) from (3):

$$y_1''-y_2''+p(x)(y_1'-y_2')+q(x)(y_1-y_2) = r(x)-r(x) = 0.$$

Thus $(y_1-y_2)''+p(x)(y_1-y_2)'+q(x)(y_1-y_2) = 0$, i.e. $y_1 - y_2$ is a solution of (2).

On the other hand, if y_0 is a solution of (2) and y_1 a solution of (1), then clearly $y_1 + y_0$ is again a solution of (1). This suggests the following definition:

Definition. A *general solution* of the non-homogeneous d.e. (1) is of the form

$$y(x) = y_h(x) + y_p(x), \tag{5}$$

where $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$ is a general solution of the homogeneous d.e. (2) and $y_p(x)$ is any solution of (1) containing no arbitrary constants.

Thus to solve (1), we have to solve the homogeneous equation (2) and find a (particular) solution of (1). The sum of these two is what we want.

Determination of $y_p(x)$

(I) **Method of undetermined coefficients**

This method applies to equations of the form $y'' + ay' + by = r(x)$, where a and b are constants, and $r(x)$ is a polynomial, exponential function, sine or cosine, or sums or products of such functions.

We denote $y'' + ay' + by$ by $L(y)(x)$, where a and b may be complex numbers. We'll make use of the Principle of super-position:

If $y_1(x)$ is a solution of $L(y)(x) = g_1(x)$ and

$y_2(x)$ is a solution of $L(y)(x) = g_2(x)$, then for any constants c_1 and c_2, $y = c_1y_1(x)+c_2y_2(x)$ is a solution of $L(y)(x) = c_1g_1(x)+c_2g_2(x)$. Using this principle, the problem is reduced to finding a particular solution of $L(y)(x) = p(x)e^{kx}$, where $p(x)$ is a polynomial in x and k is a real or complex constant. Method of undetermined coefficients is adequately described by considering the following three cases, which we illustrate with examples.

1. Polynomial case

In this case $k = 0$. The method begins with "try a polynomial with unknown coefficients".

Example 16.

$$y'' - 4y' + y = x^2 + x + 2.$$

Try $y = Ax^2 + Bx + C$ where A, B, C are constant. Then we have, after substitution,

$$2A - 4(2Ax + B) + Ax^2 + Bx + C = x^2 + x + 2$$

or

$$Ax^2 + (B - 8A)x + 2A - 4B + C = x^2 + x + 2.$$

Comparing coefficients, we have

$$\begin{aligned} A &= 1 \\ B - 8A &= 1 \Rightarrow B = 9 \\ 2A - 4B + C &= 2 \Rightarrow C = 36 \end{aligned}$$

So $Ax^2 + Bx + C = x^2 + 9x + 36$ is a particular solution.

Example 17. $y'' - 2y = 2x^3$. Try $Ax^3 + Bx^2 + Cx + D$, and we get

$$\begin{aligned} & 6Ax + 2B - 2Ax^3 - 2Bx^2 - 2Cx - 2D \\ = \ & 2x^3 + 0x^2 + 0x + 0. \end{aligned}$$

This means

$$\begin{aligned} -2A &= 2 \Rightarrow A = -1 \\ -2B &= 0 \Rightarrow B = 0 \\ 6A - 2C &= 0 \Rightarrow C = -3 \\ 2B - 2D &= 0 \Rightarrow D = 0 \end{aligned}$$

So $y = -x^3 - 3x$ is a particular solution.

2. **Exponential case**

Here k is real but not zero. The method begins with "put $y = ue^{kx}$, where $u = u(x)$." This substitution will remove e^{kx} from the equation and reduce the problem to the polynomial case 1 (above).

Example 18.

$$y'' - 4y' + 2y = 2x^3 e^{2x}. \tag{1}$$

Substituting $y = ue^{2x}$, we get

$$\begin{aligned} y' &= u'e^{2x} + 2ue^{2x} \\ y'' &= u''e^{2x} + 4u'e^{2x} + 4ue^{2x}. \end{aligned}$$

Thus (1) becomes:

$$u''e^{2x} - 2ue^{2x} = 2x^3 e^{2x},$$

or

$$u'' - 2u = 2x^3.$$

Using the result of example 17, $u = -x^3 - 3x$. Thus a particular solution is $y_p(x) = (-x^3 - 3x)e^{2x}$.

Example 19.

$$y'' - 4y' + 4y = 20x^3e^{2x}.$$

Set $y = ue^{2x}$, we get as above $u'' = 20x^3$. Integrating twice, $u = x^5 + Ax + B$. We may set

$A = 0 = B$ and take a particular solution

$y_p = x^5 e^{2x}$.

3. **Trigonometric case**

Here we use complex exponentials. Recall: $e^{s+it} = e^s(\cos t + i \sin t)$. We only need the differentiation property of e^{cx}, c complex.

Recall also: if $y = u(x) + iv(x)$ is a complex valued solution of $L(y)(x) = h_1(x) + ih_2(x)$ (where $u(x)$, $v(x)$, $h_1(x)$ and $h_2(x)$ are real valued functions), then

$$(u'' + iv'') + a(u' + iv') + b(u + iv)$$

$$= h_1 + ih_2,$$

$$\Rightarrow (u'' + au' + bu) + i(v'' + av' + bv)$$

$$= h_1 + ih_2.$$

Equating real and imaginary parts:

u is a solution of $L(y)(x) = h_1(x)$

v is a solution of $L(y)(x) = h_2(x)$

Example 20. Solve

$$y'' + 4y = 16x \sin 2x. \tag{1}$$

We solve instead

$$z'' + 4z = 16xe^{i2x}. \tag{2}$$

By the above discussion, the imaginary part of a solution of (2) will be a solution of (1). Set $z = ue^{i2x}$, $u = u(x)$. (recall case 2 above) computing z' and z'' and substituting in (2) give

$$u'' + 4iu' = 16x.$$

Solving this, we get $u = -2ix^2 + x$. Thus $z = (-2ix^2 + x)e^{i2x}$, and a particular solution of (1) is

$$y = Imz = x \sin 2x - 2x^2 \cos 2x.$$

Example 21.

$$y'' + 2y' + 5y = 16xe^{-x} \cos 2x. \qquad (1)$$

Solve instead

$$z'' + 2z' + 5z = 16xe^{(-1+2i)x}. \qquad (2)$$

$Re\,(z)$ will be a solution of (1). Put $z = ue^{(-1+2i)x}$, $u = u(x)$. [Solution is $y = Ae^{(-x)}cos(2x) +$

$Be^{(-x)}sin(2x) + xe^{(-x)}[2xsin(2x) + cos(2x).]$

(II) **Method of variation of parameters**

We consider

$$y'' + p(x)y' + q(x)y = r(x) \qquad (1)$$

where p, q, r are continuous functions of x on some open interval I.

The continuity of p and q implies that the corresponding homogeneous d.e. $y'' + p(x)y' + q(x)y = 0$ has a general solution $y_h(x) = c_1y_1(x) + c_2y_2(x)$ on I.

The method of variation of parameters involves replacing constants c_1 and c_2 by functions

$u(x)$ and $v(x)$ to be determined so that

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x)$$

is a particular solution of (1) on I.

Now $y_p' = u'y_1 + uy_1' + v'y_2 + vy_2'$. We impose

$$u'y_1 + v'y_2 = 0. \qquad (A)$$

Then $\qquad y_p' = uy_1' + vy_2',$

and $\qquad y_p'' = u'y_1' + uy_1'' + v'y_2' + vy_2''.$

Substituting these into (1) we obtain

$$(u'y_1' + v'y_2' + uy_1'' + vy_2'') + p(uy_1' + vy_2') + q(uy_1 + vy_2)$$

$$= u(y_1'' + py_1' + qy_1) + v(y_2'' + py_2' + qy_2) + (u'y_1' + v'y_2')$$

$$= r.$$

Thus

$$u'y_1' + v'y_2' = r. \qquad (B)$$

Solving (A) and (B) for u' and v':

$$u' = -\frac{y_2 r}{y_1 y_2' - y_1' y_2}, \; v' = \frac{y_1 r}{y_1 y_2' - y_1' y_2}. \qquad \text{(C)}$$

Since y_1 and y_2 are linearly independent, $y_1 y_2' - y_1' y_2 \neq 0$. [Convince yourself that this is true!]

Integrating (C) we obtain

$$\begin{aligned} u &= -\int \frac{y_2 r}{y_1 y_2' - y_1' y_2} \, dx, \\ v &= \int \frac{y_1 r}{y_1 y_2' - y_1' y_2} \, dx. \qquad (D) \end{aligned}$$

We obtain now y_p and hence a general solution of (1).

Note. The term $y_1 y_2' - y_1' y_2$ may be viewed as the determinant $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. It's called the Wronskian of y_1 and y_2.

Caution. When applying the above procedure to solve (1), make sure that the given d.e. is in standard form (1) where the coefficient of y'' is 1.

Example 22. Solve $y'' + y = \tan x$, $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Solution. The characteristic equation of $y'' + y = 0$ is $\lambda^2 + 1 = 0$. So $\lambda = \pm i$, and $y_h = c_1 \cos x + c_2 \sin x$. Here $y_1 = \cos x$, $y_2 = \sin x$.

$\therefore$ $y_1 y_2' - y_1' y_2 = \cos x \cos x - (-\sin x) \sin x = 1$.

Using (D) above,

$$u = -\int \sin x \tan x \, dx$$

$$
\begin{aligned}
&= -\int \frac{\sin^2 x}{\cos x}\, dx \\
&= \int \frac{\cos^2 x - 1}{\cos x}\, dx \\
&= \int (\cos x - \sec x)\, dx \\
&= \sin x - \ell n\,|\sec x + \tan x|, \\
v &= \int \cos x \tan x\, dx \\
&= -\cos x.
\end{aligned}
$$

A general solution is

$$
\begin{aligned}
y &= y_h + u \cos x + v \sin x \\
&= c_1 \cos x + c_2 \sin x - \cos x\, \ell n\,|\sec x + \tan x|.
\end{aligned}
$$

Example 23. Solve $y'' - y = e^{-x} \sin e^{-x} + \cos e^{-x}$.

Solution. We have $y_h = c_1 e^x + c_2 e^{-x}$. We take

$y_p = ue^x + ve^{-x}$, and determine u, v as in the above example:

We have $y_1y_2' - y_1'y_2 = -2$, and evaluating the integrals by means of a change of variable [let $z = e^{-x}$, so $dz = -zdx$] and using integration by parts, we get

$$\begin{aligned} u &= -\frac{1}{2}(2\sin e^{-x} - e^{-x}\cos e^{-x}) \\ v &= -\frac{1}{2}e^x \cos e^{-x} \\ y_p &= -e^x \sin e^{-x}. \end{aligned}$$

The general solution is $y = c_1e^x + c_2e^{-x} - e^x \sin e^{-x}$.