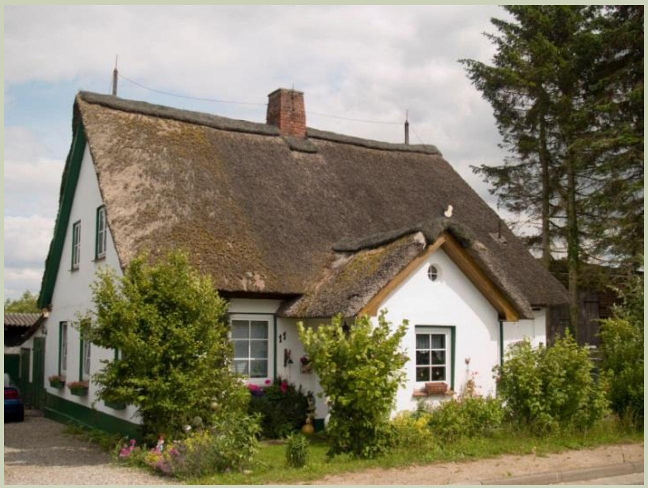
EIGENVALUES AND EIGENVECTORS

A REAL LIFE EXAMPLE

Movement of people between rural and urban district:



1%

4%

rural

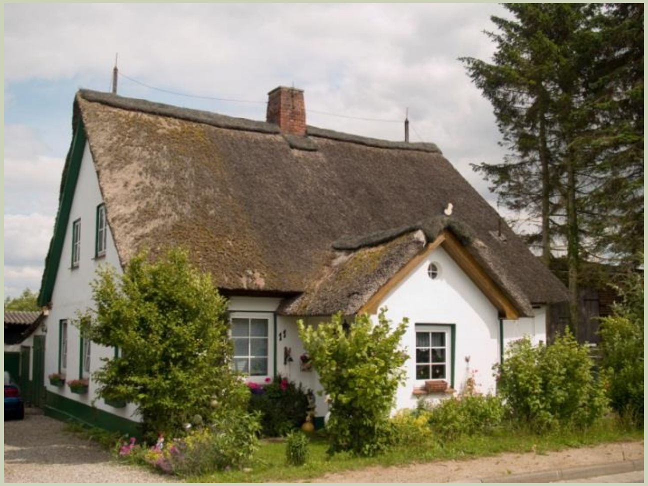
urban

Assume: Total population is a constant.

Question: What is going to happen in the long run?

A REAL LIFE EXAMPLE

Movement of people between rural and urban district:



1%

4%

rural

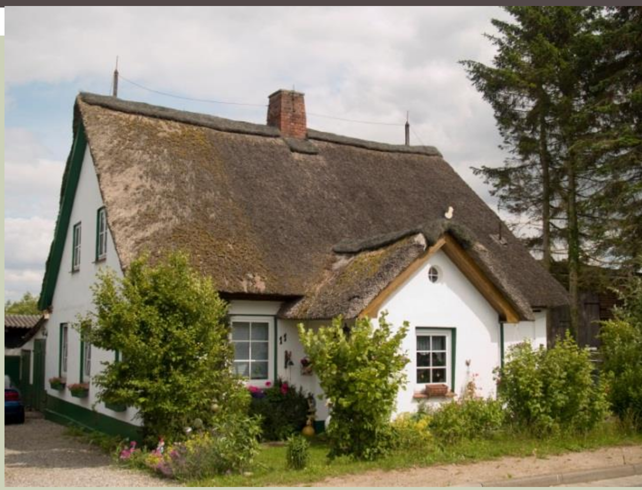
urban

Meaning of a_0 and b_0

Let b_n be the rural population after n years.

Let a_n be the urban population after n years.

A REAL LIFE EXAMPLE



1%

4%

rural (b_n)

urban (a_n)

$$a_n = 0.96\, a_{n-1} + 0.01\, b_{n-1}$$

$$b_n = 0.04\, a_{n-1} + 0.99\, b_{n-1}$$

Let $\boldsymbol{x}_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$ Population distribution after n years

$\boldsymbol{x}_{n-1} = \begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix}$ $(n-1)$

A REAL LIFE EXAMPLE

Let $\boldsymbol{A} = \begin{pmatrix} \boxed{0.96} & \boxed{0.01} \\ \boxed{0.04} & \boxed{0.99} \end{pmatrix}$

$\boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$

Initial population distribution

Then $\boldsymbol{x}_n = \boldsymbol{A}\boldsymbol{x}_{n-1} = \boldsymbol{A}^2\boldsymbol{x}_{n-2} = \boldsymbol{A}^3\boldsymbol{x}_{n-3}$

$$\boldsymbol{x}_{n-1} = \boldsymbol{A}\boldsymbol{x}_{n-2}$$

$$\boldsymbol{x}_{n-2} = \boldsymbol{A}\boldsymbol{x}_{n-3}$$

$$a_n = 0.96\, a_{n-1} + 0.01\, b_{n-1}$$

$$b_n = 0.04\, a_{n-1} + 0.99\, b_{n-1}$$

Let $\boldsymbol{x}_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$

$\boldsymbol{x}_{n-1} = \begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix}$

Population distribution after n years

A REAL LIFE EXAMPLE

Let $\boldsymbol{A} = \begin{pmatrix} \boxed{0.96} & \boxed{0.01} \\ \boxed{0.04} & \boxed{0.99} \end{pmatrix}$

$$\boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$$

Initial population distribution

What happens after 1000 years?

First I have to compute $\boldsymbol{A}^{1000}$!

Let $\boldsymbol{x}_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$

$\boldsymbol{x}_{n-1} = \begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix}$

Population distribution after n years

A REAL LIFE EXAMPLE

Let $\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix}$

$$\boxed{\boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0}$$

Suppose somebody tells you that

$$\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.95 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}^{-1}$$

$$= \boldsymbol{PDP}^{-1}$$

($\boldsymbol{P}$ is invertible,

$\boldsymbol{D}$ is diagonal)

$$\begin{aligned} \boldsymbol{A}^n &= (\boldsymbol{PDP}^{-1})^n \\ &= (\boldsymbol{PDP}^{-1})(\boldsymbol{PDP}^{-1})\ldots(\boldsymbol{PDP}^{-1}) \\ &= (\boldsymbol{PD}\underbrace{\boldsymbol{P}^{-1}\boldsymbol{P}}\boldsymbol{D}\underbrace{\boldsymbol{P}^{-1}\ldots\boldsymbol{P}}\boldsymbol{DP}^{-1}) \\ &= (\boldsymbol{PDD}\ldots\boldsymbol{DP}^{-1}) \\ &= (\boldsymbol{PD}^n\boldsymbol{P}^{-1}) \end{aligned}$$

What is $\boldsymbol{D}^{1000}$?

A REAL LIFE EXAMPLE

$$\boldsymbol{D} = \begin{pmatrix} 1 & 0 \\ 0 & 0.95 \end{pmatrix} \qquad \boldsymbol{D}^2 = \begin{pmatrix} 1^2 & 0 \\ 0 & (0.95)^2 \end{pmatrix} \qquad \boldsymbol{D}^3 = \begin{pmatrix} 1^3 & 0 \\ 0 & (0.95)^3 \end{pmatrix}$$

$$\boldsymbol{D}^n = \begin{pmatrix} 1^n & 0 \\ 0 & (0.95)^n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & (0.95)^n \end{pmatrix}$$

$$\boldsymbol{A}^n = \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1} = \boldsymbol{P}\begin{pmatrix} 1 & 0 \\ 0 & (0.95)^n \end{pmatrix}\boldsymbol{P}^{-1}$$

So $\lim\limits_{n\to\infty} \boldsymbol{A}^n = \boldsymbol{P}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\boldsymbol{P}^{-1}$ since $\boxed{(0.95)^n \to 0 \text{ as } n \to \infty}$

A REAL LIFE EXAMPLE

Let $\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix}$

$\boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$

Which was the crucial step?

So $\lim\limits_{n\to\infty} \boldsymbol{A}^n = \boldsymbol{P}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\boldsymbol{P}^{-1}$

$$= \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0.2 & 0.2 \\ 0.8 & 0.8 \end{pmatrix}$$

$$\lim_{n\to\infty} \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \lim_{n\to\infty} \boldsymbol{x}_n = \lim_{n\to\infty} \boldsymbol{A}^n \boldsymbol{x}_0$$

$$= \begin{pmatrix} 0.2 & 0.2 \\ 0.8 & 0.8 \end{pmatrix}\begin{pmatrix} a_0 \\ b_0 \end{pmatrix}$$

$a_n = 0.2(a_0 + b_0)$ 20% will stay in urban

$b_n = 0.8(a_0 + b_0)$ 80% will stay in rural

A REAL LIFE EXAMPLE

Let $\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix}$

$$\boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$$

Suppose somebody tells you that

$$\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.95 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}^{-1}$$

$$= \boldsymbol{PDP}^{-1}$$

($\boldsymbol{P}$ is invertible,

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$$\begin{aligned} \boldsymbol{A}^n &= (\boldsymbol{PDP}^{-1})^n \\ &= (\boldsymbol{PDP}^{-1})(\boldsymbol{PDP}^{-1})\ldots(\boldsymbol{PDP}^{-1}) \\ &= (\boldsymbol{PD}\underbrace{\boldsymbol{P}^{-1}\boldsymbol{P}}\boldsymbol{D}\underbrace{\boldsymbol{P}^{-1}\ldots}\boldsymbol{PDP}^{-1}) \\ &= (\boldsymbol{PDD}\ldots\boldsymbol{DP}^{-1}) \\ &= (\boldsymbol{PD}^n\boldsymbol{P}^{-1}) \end{aligned}$$

What is $\boldsymbol{D}^{1000}$?

DEFINITION

Let $\boldsymbol{A}$ be a square matrix of order n.

A nonzero column vector $\boldsymbol{u} \in \mathbb{R}^n$ is called an eigenvector of $\boldsymbol{A}$ if

$$\boldsymbol{Au} = \lambda \boldsymbol{u} \quad \text{for some scalar } \lambda.$$

'multiplying $\boldsymbol{A}$ to $\boldsymbol{u}$ results in some scalar multiple of $\boldsymbol{u}$.'

The scalar λ is called an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{u}$ is said to be an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue λ.

A QUESTION BEFORE WE PROCEED

If $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue λ,

$\boldsymbol{Au} = \lambda \boldsymbol{u}$

what about vectors like $2\boldsymbol{u},\ (-1.5)\boldsymbol{u},\ 300\boldsymbol{u}$?

$$\boldsymbol{A}(2\boldsymbol{u}) = 2(\boldsymbol{Au}) = 2(\lambda\boldsymbol{u}) = \lambda(2\boldsymbol{u})$$

$\boldsymbol{A}(2\boldsymbol{u}) = \lambda(2\boldsymbol{u})$

$2\boldsymbol{u},\ (-1.5)\boldsymbol{u},\ 300\boldsymbol{u}$ are also eigenvectors of $\boldsymbol{A}$ associated with the same eigenvalue λ.

All scalar multiples of $\boldsymbol{u}$ will also be an eigenvector of $\boldsymbol{A}$ associated with the same eigenvalue λ.

SUMMARY

1) A real life example on population movement and the need to compute the powers of a square matrix.

2) Definition of eigenvalue (of a matrix $\boldsymbol{A}$) and eigenvector (of a matrix $\boldsymbol{A}$) associated with the eigenvalue.