

MORE ON COLUMN SPACE AND RANK

COLUMN SPACE AND LINEAR SYSTEMS

$$\begin{cases} 2x - y & = -1 \\ x - y + 3z & = 4 \\ -5x + y & = -2 \\ x & + z = 3 \end{cases}$$

$\Leftrightarrow Ax = b$ where

$$\Leftrightarrow x \begin{pmatrix} 2 \\ 1 \\ -5 \\ 1 \end{pmatrix} + y \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -2 \\ 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$b = \begin{pmatrix} -1 \\ 4 \\ -2 \\ 3 \end{pmatrix}$$

COLUMN SPACE AND LINEAR SYSTEMS

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$\mathbf{Ax} = \mathbf{b}$ is consistent means
 x, y, z can be found
to satisfy (*)

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$\Rightarrow \mathbf{b}$ is a linear combination
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That is, \mathbf{b} belongs to
the column space of \mathbf{A} .

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THEOREM

Let \mathbf{A} be a $m \times n$ matrix. Then the column space of \mathbf{A} is

$$\left\{ \mathbf{A} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \mid u_1, u_2, \dots, u_n \in \mathbb{R} \right\} = \{ \mathbf{A}\mathbf{u} \mid \mathbf{u} \in \mathbb{R}^n \}$$

A system of linear equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ is consistent if and only if \mathbf{b} lies in the column space of \mathbf{A} .

EXAMPLE

Consider the following linear system

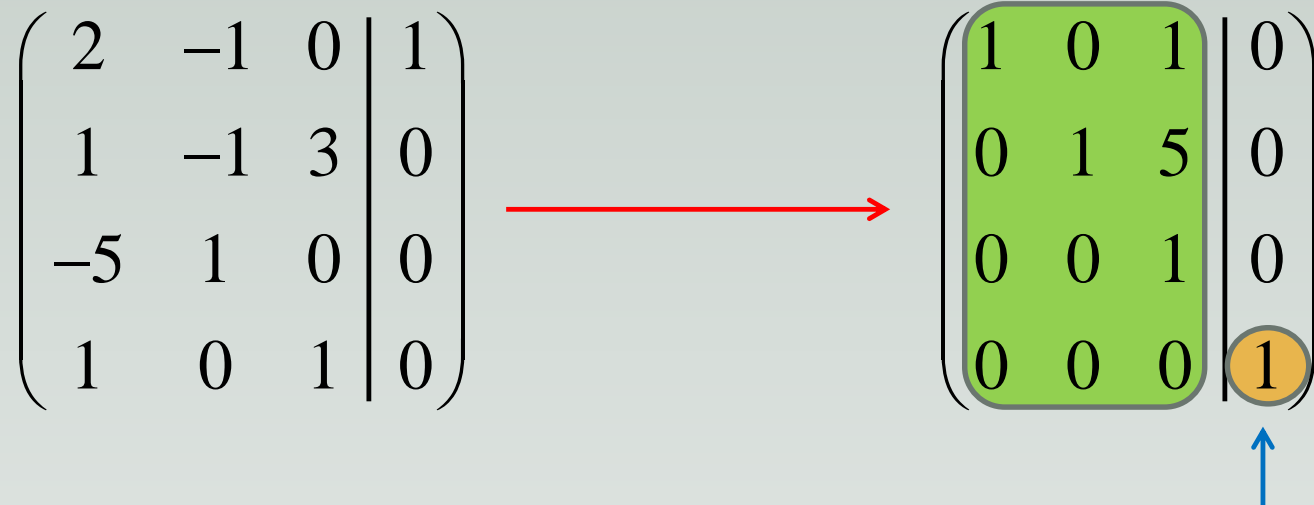
$$\begin{cases} 2x & - & y & & = & 1 \\ x & - & y & + & 3z & = & 0 \\ -5x & + & y & & = & 0 \\ x & & & + & z & = & 0 \end{cases}$$

Rewriting as $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

EXAMPLE

Solving $A\mathbf{x} = \mathbf{b}$ using Gaussian Elimination,

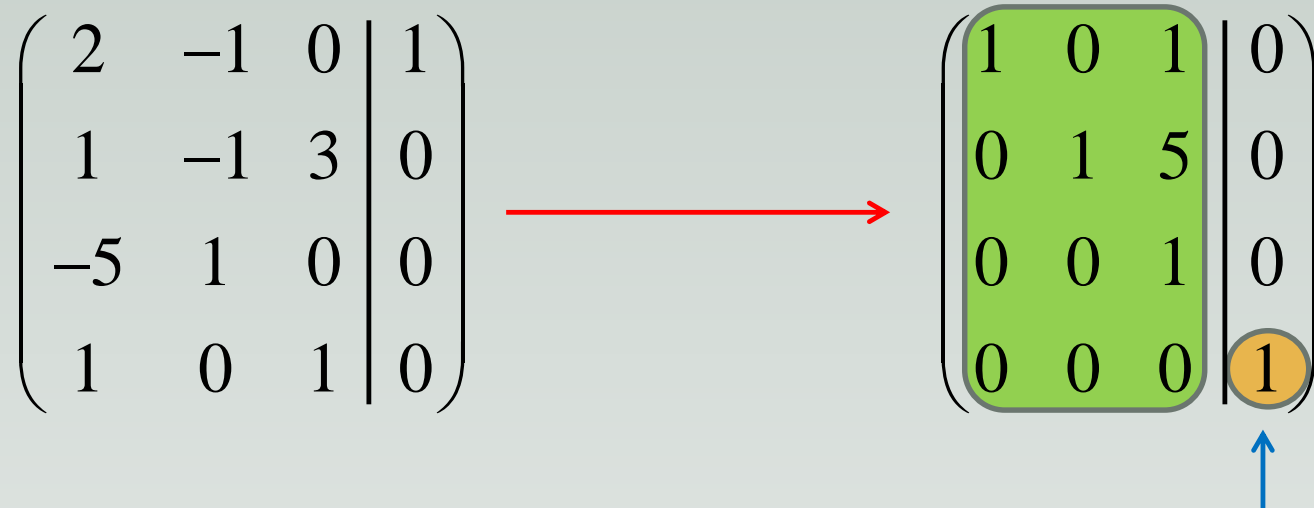
$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 1 & -1 & 3 & 0 \\ -5 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array} \right) \xrightarrow{\hspace{1cm}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$


The linear system is inconsistent since the last column of a row echelon form of $(A | \mathbf{b})$ is a **pivot column**.

 is a row echelon form of A , so $\text{rank}(A) = 3$.

EXAMPLE

Solving $A\mathbf{x} = \mathbf{b}$ using Gaussian Elimination,

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 1 & -1 & 3 & 0 \\ -5 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array} \right) \xrightarrow{\hspace{1cm}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$


The linear system is inconsistent since the last column of a row echelon form of $(A \mid \mathbf{b})$ is a **pivot column**.

Since the last column is a pivot column, $\text{rank}(A \mid \mathbf{b}) = \text{rank}(A) + 1 = 4$

REMARK

A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if A and the augmented matrix $(A \mid \mathbf{b})$ have the same rank.

SUMMARY

- 1) $A\mathbf{x} = \mathbf{b}$ is consistent if and only if \mathbf{b} belongs to the column space of A .
- 2) $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\text{rank}(A) = \text{rank}(A \mid \mathbf{b})$.