CHAPTER 3

MATHEMATICAL MODELLING

3.1. MALTHUS MODEL OF POPULATION

The total population of a country is clearly a function of time, N(t) [NOTE: N may be measured in millions, so values of N less than 1 are meaningful!]. Given the population now, can we predict what it will be in the future?

Suppose that B is a function giving the PER CAPITA BIRTH-RATE in a given society, *ie* B is the number of babies born per second, divided by the total population of the country at that moment. Note that B could be small in a big country and large in a small country - it depends on whether there is a strong social pressure on people to get married and have kids. Now B could depend on time and it could depend on N. But SUPPOSE YOU DON'T BELIEVE THESE THINGS: suppose you think that people will always have as many kids as they can, no matter what. Then B is constant. Now just as

$$\textsc{Distance} = \textsc{Speed} \times \textsc{Time}$$

when SPEED IS CONSTANT, so also we have

$$\#\text{babies born in time } \delta t = BN\delta t$$

Similarly let D be the death rate per capita; again, it could be a function of t (better medicine, fewer smokers) or N (overcrowding leads to famine/disease) but if we assume that it is constant, then

$$\#\text{deaths in time } \delta t = DN\delta t$$

So the change in N, δN, during δt is

$$\delta N = \#birth - \#deaths$$

PROVIDED there is no emigration or immigration. Thus,

$$\delta N = (B - D)N\delta t$$

and so $\frac{\delta N}{\delta t} = (B - D)N$ or in the limit as $\delta t \to 0$,

$$\frac{dN}{dt} = (B - D)N = kN \qquad (1)$$

if $k = B - D$.

This model of society was put forward by THOMAS MALTHUS in 1798. Clearly Malthus was assuming a socially STATIC society in which human reproductive behaviour never changes with time or overcrowding, poverty etc... What does Malthus' model predict? Suppose that the population NOW is $\hat{N}$, and let $t = 0$ NOW. From $\frac{dN}{dt} = kN$ we have $\int \frac{dN}{N} = \int k dt = k \int dt = kt + c$

so $ln(N) = kt + c$ and thus $N(t) = Ae^{kt}$. Since $\hat{N} = N(0) = A$, we get:

$$N(t) = \hat{N}e^{kt} \tag{2}$$

with graphs as shown on figure 1.

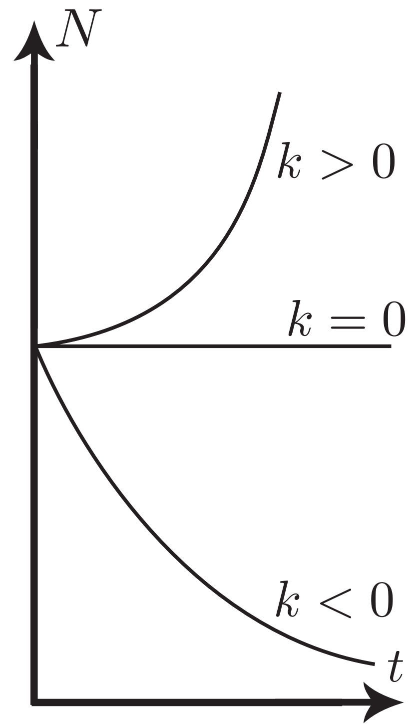


Figure 1: Graphs of $N(t)$, for different values of k

The population collapses if $k < 0$ (more deaths than births per capita), remains stable if (and only if) $k = 0$, and it EXPLODES if $k > 0$ (more births than deaths). Malthus observed that the population of Europe was increasing, so he predicted a catastrophic POPULATION EX-

PLOSION; since the food supply could not be expanded so fast, this would be disastrous.

In fact, this didn't happen (in Europe). So Malthus' model is wrong: many millions went to the US, many millions died in wars.

Second, the "static society" assumption has turned out to be wrong in many societies, with B and D both declining as time passed after WW2.

SUMMARY: The Malthus model of population is based on the idea that per capita birth and death rated are independent of time and N. It leads to EXPONENTIAL growth or decay of N.

3.2. IMPROVING ON MALTHUS

Malthus' model is interesting because it shows that static behaviour patterns can lead to disaster. But precisely because e^{kt} grows so quickly, Malthus' assumptions must eventually go wrong - obviously there is a limit to the possible population. Eventually, if we don't control B, then D will have to increase. So we have to assume that D is a function of N.

Clearly, D must be an increasing function of N... but WHICH function? Well, surely the SIMPLEST POSSIBLE CHOICE (Remember: always go for the SIMPLE model before trying a complicated one!) is

$$\boxed{D = sN, \quad \begin{array}{c}\text{(LOGISTIC)} \\ \text{ASSUMPTION} \\ s = \text{constant}\end{array}} \tag{3}$$

This represents the idea that, in a world with FINITE RESOURCES, large N will eventually cause starvation and disease and so increase D.

Remark: In modelling, it is often useful to take note of $\boxed{\text{units}}$. Units of D are (#dead people) / second / (total # people) = $(\text{sec})^{-1}$. Units of N are # (ie no units). So if $D = sN$, units of s must be $(\text{sec})^{-1}$.

As before, let $\hat{N}$ be the value of N at $t = 0$.

We have to solve

$$\frac{dN}{dt} = BN - DN = BN - sN^2$$

with the condition $N(0) = \hat{N}$

We can and will solve this, but let's try to GUESS what the solution will look like (a useful skill - in many other cases you won't be able to solve exactly!). Suppose that $\hat{N}$ is very small. Then (by continuity) $N(t)$ will be very small for t near to zero.

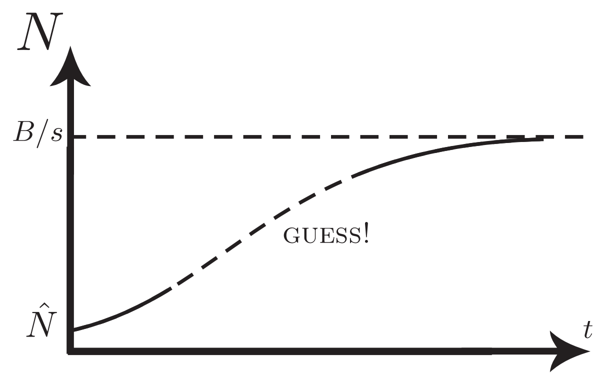
Of course if N is small, N^2 is much smaller and can be neglected. [Remember that N may be measured in millions or billions, so N can be small.] So at early times, our ODE is ALMOST linear and so

$$\frac{dN}{dt} \approx BN \to N(t) \approx \hat{N}e^{Bt}$$

So AT FIRST the population explodes, as Malthus predicted. On the other hand, if N continues to grow, since N^2 grows faster than N, we will reach a point where $sN^2 \approx BN$ ie $N \approx B/S$. At that point , since $\frac{dN}{dt} = BN - SN^2$, the population will stop growing. So B/S should measure the MAXIMUM population possible. So we GUESS that the solution should look like this:

ie it starts out exponentially and ends up approaching B/S asymptotically. The dotted part is a reasonable GUESS!

OK, now that we know what to expect, let's



actually solve it!

$$\frac{dN}{dt} = BN - sN^2 \to t = \int \frac{dN}{N(B - sN)} + c$$

Write $\frac{1}{N(B-sN)} = \frac{\alpha}{N} + \frac{\beta}{B-sN}$

$$\begin{aligned} 1 &= \alpha(B - sN) + \beta N \\ &= \alpha B + (\beta - \alpha s)N \to 1 = \alpha B \ , \ \beta = \alpha s \\ \alpha &= 1/B \ , \ \beta = s/B, \text{ so} \end{aligned}$$

$$
\begin{aligned}
\int \frac{dN}{N(B-sN)} &= \frac{1}{B}\int \frac{dN}{N} + \frac{s}{B}\int \frac{dN}{B-sN} \\
&= \frac{1}{B}\ln N - \frac{1}{B}\ln|B-sN|
\end{aligned}
$$

Now here we begin to feel uneasy - what if $N = B/s$ at some time? (ln(0) is not defined). In fact we should have worried about this when we first wrote $\frac{1}{B-sN}$ - how do we know that we are not dividing by zero?? Let's not worry about that just now: let's ASSUME (temporarily) THAT $B - sN$ IS NEVER ZERO. That is, we assume either that N is always either LESS THAN B/s or MORE THAN B/s. OK, let's take LESS THAN first. So $|B - sN| = B - sN$, and we get

$$\begin{aligned} t &= \frac{1}{B}\ln N - \frac{1}{B}\ln(B - sN) + c \\ &= \frac{1}{B}\ln\frac{N}{B - sN} + c \end{aligned}$$

So

$\frac{N}{B-sN} = Ke^{Bt}$. Since $\hat{N} = N(0)$, $\frac{\hat{N}}{B-s\hat{N}} = K$

so

$$\frac{N}{B - sN} = \frac{\hat{N}}{B - s\hat{N}}e^{Bt}$$

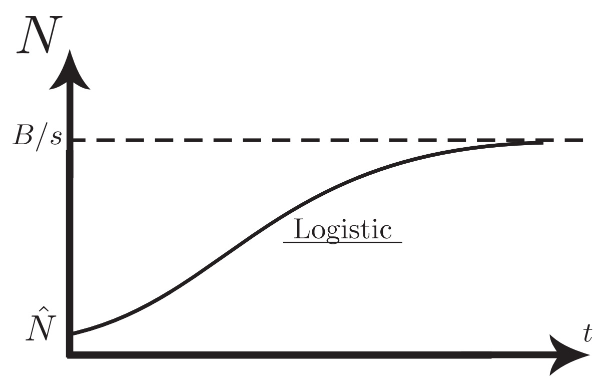
Solve for N,

$$N(t) = \frac{B}{s + \left(\frac{B}{\hat{N}} - s\right)e^{-Bt}} \qquad (4)$$

REMARK: It is a very good habit to check that your solution agrees with your assumptions - to guard against mistakes!

Check: $N(0) = \frac{B}{s+\left(\frac{B}{\hat{N}}-s\right)} = \hat{N}$ correct! Check: If $B-sN > 0$ is true at $t=0$ then $B-s\hat{N} > 0$ so $\left(\frac{B}{\hat{N}} - s\right) > 0$ so $\frac{B}{s+\left(\frac{B}{\hat{N}}-s\right)e^{-Bt}} < \frac{B}{s}$ for all t *ie* $N(t) < B/s$ which is consistent.

The graph of (4) is easy to sketch:



This is the famous LOGISTIC CURVE; $N(t)$ given by (4) is called the LOGISTIC FUNCTION; and $\frac{dN}{dt} = BN - sN^2$ is the LOGISTIC EQUATION.

It's easy to see what is happening here. Initially the population is small, plenty of food and space, so we get a Malthusian population explosion. But eventually the death rate rises until it is almost equal to the birth rate (*ie* $sN \approx B$ or $N \approx B/s$) and then the population approaches a fixed limit.

This situation is what people usually mean when they use the word "LOGISTIC". But we are not done yet: on page 12 we ASSUMED that $N(t) < B/s$. What if $N(t) > B/s$?

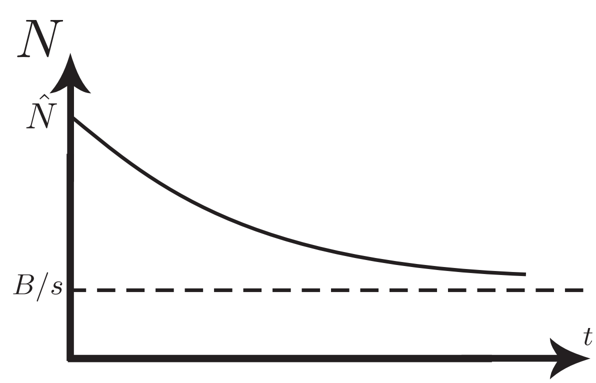
Then $|B - sN| = -(B - sN)$ so:

$$\begin{aligned} t &= \frac{1}{B}\ln N - \frac{1}{B}\ln(sN - B) + c \\ &= \frac{1}{B}\ln\frac{N}{sN - B} + c \Rightarrow \end{aligned}$$

$$N(t) = \frac{B}{s - \left(s - \frac{B}{\hat{N}}\right)e^{-Bt}} \tag{5}$$

Check $N(0) = \hat{N}$ and $N(t) > B/s$

And now the graph is



Again, the meaning is clear: the initial population was so big that the death rate exceeded the birth rate, so of course the population declines until it gets near to the long-term SUSTAINABLE value.

The number B/s is called the CARRYING CAPACITY or the SUSTAINABLE POPULATION - in all cases, it is the value approached by $N(t)$ as $t \to \infty$. If we set

$$N_\infty = B/s \tag{6}$$

then our solutions are:

$$\begin{aligned} N(t) &= \frac{N_\infty}{1 + \left(\frac{N_\infty}{\hat{N}} - 1\right) e^{-Bt}} (\hat{N} < N_\infty) \quad (7) \\ N(t) &= \frac{N_\infty}{1 - \left(1 - \frac{N_\infty}{\hat{N}}\right) e^{-Bt}} (\hat{N} > N_\infty) \quad (8) \end{aligned}$$

(obtained by dividing numerator and denominator by s in 4 and 5)

BUT we aren't finished yet! We had to ASSUME (page 12) that N is never equal to B/s, *ie* to N_∞. So we have to think about this.

First, let's ask what happens if $\hat{N} = N_\infty$ *ie* $N(t)$ is initially N_∞.

Intuitively, since N_∞ is the sustainable population, you would expect that $N(t) = \text{constant} = \hat{N} = N_\infty$ should be possible ! Indeed, substitute $N = N_\infty$ into $\frac{dN}{dt} = BN - sN^2$ and the left side is zero while the right side is $BN_\infty - sN_\infty^2 = N_\infty[B - sN_\infty] = N_\infty[B - B] = 0$. So we have

$$N(t) = N_\infty \;\; (\hat{N} = N_\infty) \qquad (9)$$

Clearly (7) (8) (9) cover all possible values of $\hat{N}$.

SUMMARY: A simple way to improve on Malthus is to replace his assumption $D =$ constant by the LOGISTIC ASSUMPTION $D = sN$, $s =$ constant. If $\hat{N} < B/s = N_\infty$, then the graph of $N(t)$ is the "S-shape" on page 14.

3.3. HARVESTING

A major application of modelling is in dealing with populations of animals e.g. fish. We want to know how many we can eat without wiping them out. Let's build on our logistic model, *ie* $\boxed{\text{assume}}$ that the fish population WOULD follow that model if we didn't catch any. Next, $\boxed{\text{assume}}$ that we catch E (constant) fish per year. Then we have:

$$\frac{dN}{dt} = (B - sN)N - E \quad \boxed{\begin{array}{c}\text{Basic}\\ \text{Harvesting}\\ \text{Model}\end{array}}$$

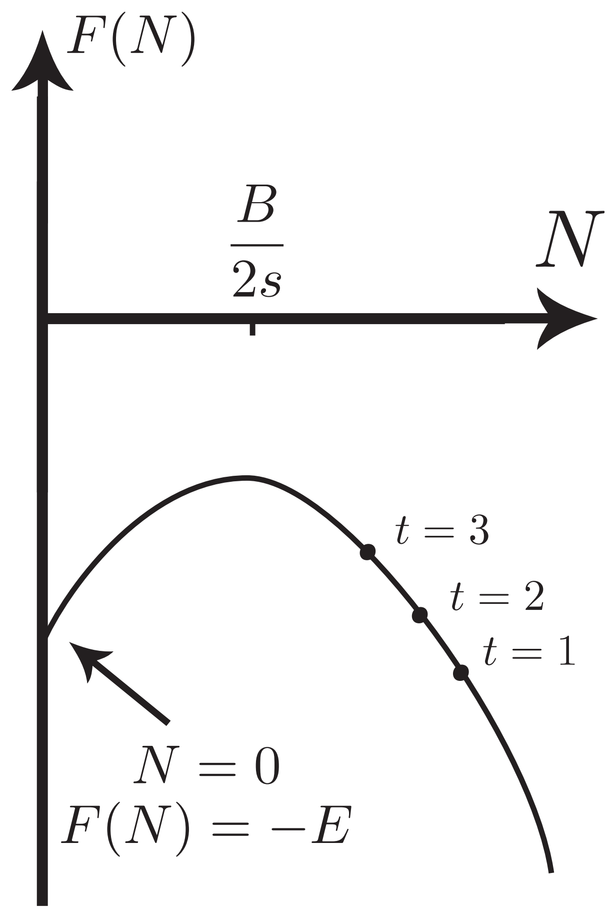
We'll now try to guess what the solutions should look like. We are particularly interested in the long term - will the harvesting eventually exterminate the fish? Consider the function:

$$F(N) = (B - sN)N - E$$

. The graph of $F(N)$ can take one of 3 forms.

In the first case, the quadratic

$$-sN^2 + BN - E$$

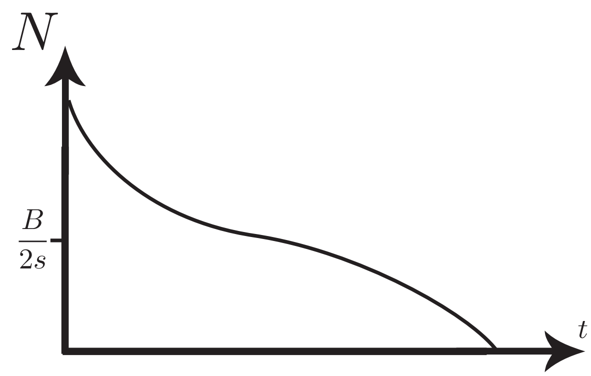


has no solutions, *ie*

$$B^2 - 4sE < 0 \text{ or } E > \frac{B^2}{4s}$$

.

Since $\frac{dN}{dt} = F(N)$, we see that in this case the population always declines. Note that there is no t-axis in this picture, but you should imagine time passing by thinking of a moving spot on the graph. Notice that as we move through $t = 1, 2, 3$ we have to move to the left since $\frac{dN}{dt} < 0$ always. But $|\frac{dN}{dt}|$ is DECREASING, so the rate of decline slows down as time goes on, until we pass $N = \frac{B}{2s}$. After that, $|\frac{dN}{dt}|$ increases and the value of N decreases rapidly to zero. Congratulations - you have wiped out your fish! [In drawing this picture, I assumed that $\hat{N}$, the initial value of N, was greater than $\frac{B}{2s}$, the maximum point on the graph of $F(N)$. Note that



$\frac{dN}{dt} = F(N)$ implies

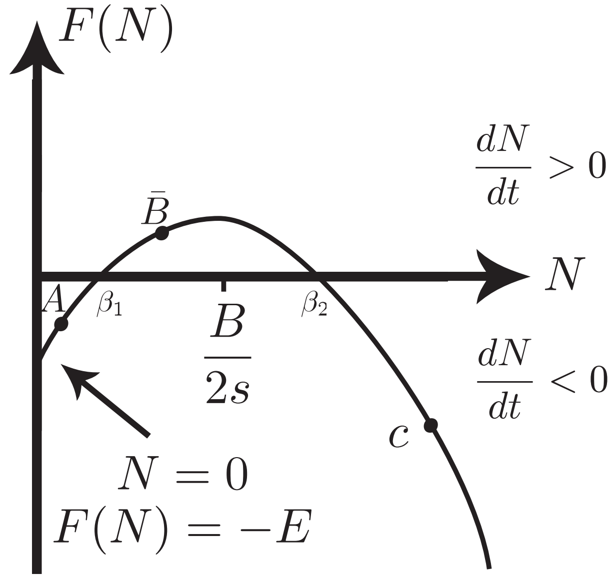
$$\frac{d^2N}{dt^2} = F'(N)\frac{dN}{dt} = F'(N)F(N)$$

so watch for "points of inflection" on the graph of $N(t)$ at values of N where $F'(N)$ or $F(N)$ vanish.]

Clearly it is NOT a good idea to harvest at a rate $E > \frac{B^2}{4s}$ (Check units). So let's assume that our fishermen ease off and harvest at a rate $E < \frac{B^2}{4s}$. (The special case E EXACTLY EQUALS $\frac{B^2}{4s}$

is clearly impossible in reality, but we will come back to it later anyway!)

Now the graph of $F(N)$ is as shown in the next diagram.



Again, remembering that $\frac{dN}{dt} = F(N)$, we see that

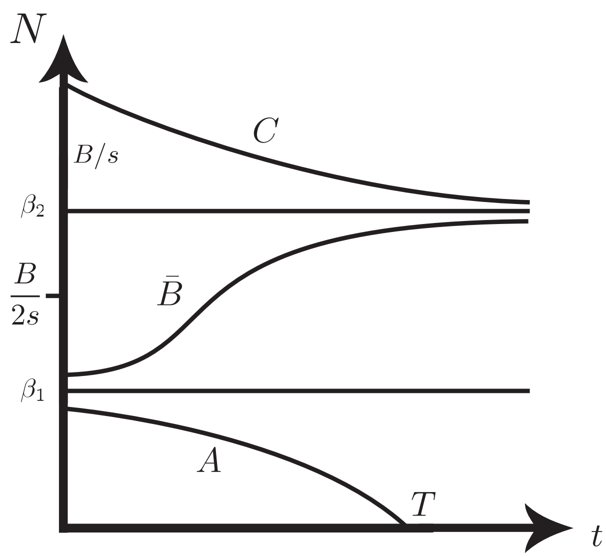
$$
\begin{aligned}
\frac{dN}{dt} &< 0 \text{ if } 0 < N < \beta_1 \\
\frac{dN}{dt} &> 0 \text{ if } \beta_1 < N < \beta_1 \\
\frac{dN}{dt} &< 0 \text{ if } N > \beta_2
\end{aligned}
$$

and of course $\frac{dN}{dt} = 0$ at $N = \beta_1$ and β_2, where

$$
\begin{matrix}\beta_1 \\ \beta_2\end{matrix} = \frac{-B \pm \sqrt{B^2 - 4Es}}{-2s} = \frac{B \mp \sqrt{B^2 - 4Es}}{2s}
$$

Now suppose $\hat{N} = N(0)$ is large, so we start at point C on the diagram. Then $\frac{dN}{dt} < 0$ so the fish stocks decline toward β_2. Suppose on the other hand that $\hat{N}$ is small, but still more than β_1. Then we might start at $\bar{B}$ and the number of fish will INCREASE until we reach β_2. If $\hat{N}$

is very small, however, then we are at a point like A, and the fish population will collapse to zero. So we get a picture like this:

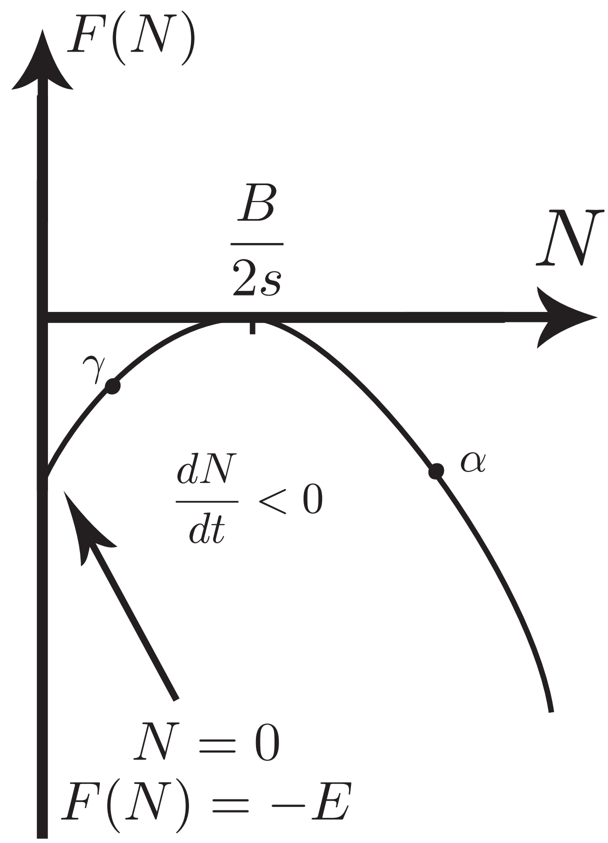


Of course, if $\hat{N} = \beta_1$ or β_2, then since $F(\beta_1) = 0$ and $F(\beta_2) = 0$, $\frac{dN}{dt} = F(N)$ has solutions $N(t) = \beta_1$ and $N(t) = \beta_2$, the constant solutions. We call β_1 and β_2 the EQUILIBRIUM POPULATIONS: given a fixed harvesting rate, if the initial population is either β_1 or β_2, then IN THEORY the population remains steady - which is good! BUT there is a vast DIFFERENCE between β_1 and β_2!! Look at the diagram and suppose you have exactly β_2 fish. Now suppose a SMALL number of new fish arrive from somewhere else. Then the diagram shows that the population will decline back to β_2. If some fish go away, the population will INCREASE back to β_2. Of course, such things happen all the time, so it's VERY GOOD to have this kind of behaviour! We say that β_2 is a STABLE EQUILIBRIUM POPULATION. BUT NOW LOOK AT β_1! If a few more fish arrive, also fine - in fact the population INCREASES (to β_2) BUT suppose a few fish decide to move on. THEN YOUR FISH STOCKS BECOME EXTINCT!! Note that this can happen though E is relatively small (we are assuming $E < \frac{B^2}{4s}$. We say that β_1 is an UNSTABLE EQUILIBRIUM POPULATION. The time required to reach $N = 0$ is called the EXTINCTION TIME. It can be computed: since $\frac{dN}{dt} = N(B - sN) - E$, we have:

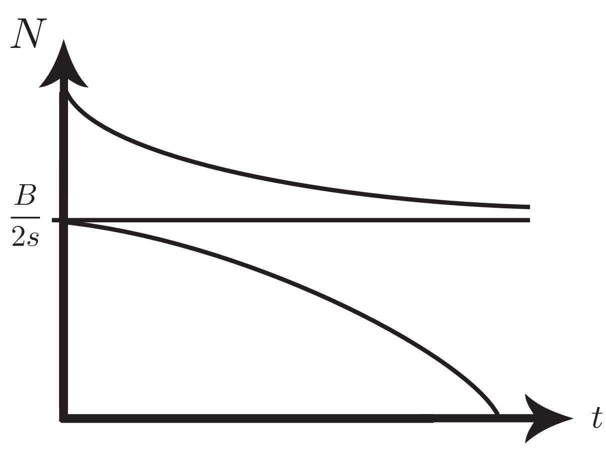
$$\int_0^T dt = T = \int_{\hat{N}}^0 \frac{dN}{N(B - sN) - E}$$

Of course it is very important to know T - it is the amount of time you have to save the situation!

Finally, let's consider the third possible graph for $F(N)$:



Clearly $\hat{N} = \frac{B}{2s}$ is a solution; it is the only equilibrium population. If $\hat{N} > \frac{B}{2s}$ (for example, at the point α) then the population declines to $\frac{B}{2s}$, asymptotically. But if we start at the point γ, then (since $\frac{dN}{dt} < 0$ everywhere below the axis - always remember that $\frac{dN}{dt} = F(N)$) the population will collapse to zero. So we have an UNSTABLE equilibrium at $\hat{N} = \frac{B}{2s}$. [We only call it stable if it is stable to perturbations in BOTH directions!]. The graph of N is as shown.



Clearly, the first and third cases are bad! The first case was $E > \frac{B^2}{4s}$. So we want the second case, with $E < \frac{B^2}{4s}$, and we want STABLE equilibrium, that is, a population which fluctuates around $\beta_2 = \frac{B+\sqrt{B^2-4Es}}{2s}$.