MA1512 TUTORIAL 5

Key Concepts – Chapter 5 Partial Differential Equations

Separation of Variables

Step 1 Assume that the solution is in the form $u(x,y) = X(x) \cdot Y(y)$.

Step 2 Substitute the form above into the PDE.

Step 3 Perform separation of variables across equal signs, and equate it to a constant. (**Question** Why must we equate to a constant?)

Step 4 Separate the variables to obtain ODEs with their boundary conditions.

Step 5 Obtain solutions to both ODEs and combine them to give the solution u.

D'Alembert's solution for wave equation	Using separation of variables for heat equation:
$$u_{tt} = c^2 u_{xx}, \qquad 0 \le x \le \pi,\ t > 0$$ $$u(0,t) = 0,\ u(\pi,t) = 0,$$ $$u(x,0) = f(x),\ u_t(x,0) = 0.$$ Then, $\boxed{u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]}$ is a solution, where f is an odd extension with period 2π.	$$u_t = c^2 u_{xx},$$ $$u(0,t) = 0,\ u(L,t) = 0,$$ $$u(x,0) = f(x).$$ Then $\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) e^{-c^2\left(\frac{n\pi}{L}\right)^2 t}}$ is a solution.

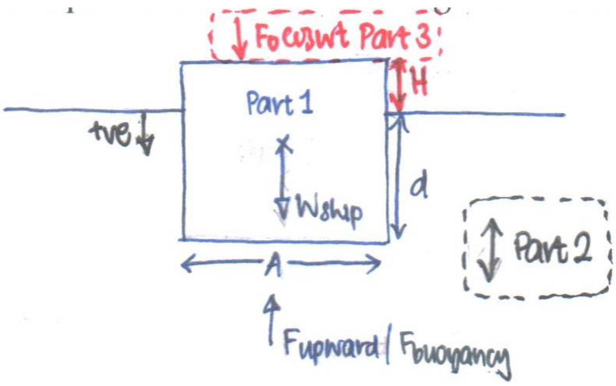
Tutorial Problems

Question 1

The oil tanker in Tutorial 3 is at rest in an almost calm sea. Suddenly, at time $t = T > 0$, it is hit by a single rogue wave which imparts to it a vertical [upward] momentum P, doing so almost instantaneously. Neglecting friction, solve for $x(t)$, the downward displacement of the ship, and graph it. How far down does the ship go [if it doesn't sink!]? [Hint: according to Newton's second law, momentum is the time integral of force. To get the force as a function of time in this problem, you have to find a function which is zero except at $t = T$, and which has an integral equal to P. Note that the delta function has units of 1/time.]

<u>Solutions</u>

Try to re-read Tutorial 3 Question 5 to refresh your memory about this question, especially on how the differential equation describing the system is derived. The following diagram is reproduced from Tutorial 3 Question 5.



Recall that the equation that governs the behaviour of this system is given by

$$M\ddot{x} = Mg - \rho A\big(d + x(t)\big)g.$$

Imparting a single rouge wave at time $t = T > 0$ will mean that there is a sudden spike in force introduced into the system, prompting the use of the Dirac delta function $\delta(t-T)$. Suppose the force is proportional to $\delta(t-T) \Rightarrow F = \alpha\delta(t-T)$. Note that

$$F = m\frac{dv}{dt}$$

$$\Rightarrow \int_0^\infty F\,dt = mv = P$$

$$\alpha\int_0^\infty \delta(t-T)\,dt = P$$

Thus, we have $\alpha = P$ (**Question** What property are we using?). The impulsive **force** at $t = T$ with momentum P is given by $P\delta(t-T)$, then our differential equation becomes

$$M\ddot{x} = Mg - \rho A\big(d + x(t)\big)g - P\delta(t-T)$$

$$M\ddot{x} = Mg - \rho A\left(\frac{M}{\rho A} + x(t)\right)g - P\delta(t-T)$$

$$\ddot{x} = -\frac{\rho A g}{M}x - \frac{P}{M}\,\delta(t-T)$$

Why is the sign before $P\delta(t-T)$ negative?

To solve the DE, we will take Laplace transform on both sides:

$$s^2X - sx(0) - x'(0) = -\frac{\rho A g}{M}X - \frac{P}{M}e^{-Ts}$$

Noting that both $x(0) = x'(0) = 0$ because the ship is at rest when $t = 0$, we have

$$X(s) = -\frac{P}{M}\frac{e^{-Ts}}{s^2+\omega^2} = -\frac{P}{\omega M}\frac{\omega}{s^2+\omega^2}e^{-Ts}$$

where ω is the natural frequency of the oscillation of the ship, $\omega = \sqrt{\frac{\rho A g}{M}}$. Performing inverse Laplace transform using the t-shifting theorem $L^{-1}\big(e^{-as}F(s)\big) = f(t-a)\cdot u(t-a)$:

$$x(t) = -\frac{P}{\omega M}\sin[\omega(t-T)]\,u(t-T).$$

Observing the equation, there is no movement of the ship for time $t < T$. However, for $t \geq T$, the ship will oscillate (bob up and down) which is a simple harmonic motion (SHM) according to the frequency ω with the amplitude, which is $\frac{P}{\omega M}$.

Question 2

Billionaire engineer Tan Ah Lian attributes her enormous success to the fact that she never talked in class when she was an Engineering student at NUS. One day in the lecture the professor announces that a certain gadget contains an electrical circuit with a resistance, capacitance, and inductance, with values of R, C, L which were all stated. [For those who do not know the physics: the current in an electrical circuit is a function of time, $I(t)$, which satisfies the equation

$$V(t) = RI + L\dot{I} + \frac{1}{C}\int_0^t I\,dt.$$

Here $V(t)$ is a given function of time called the voltage, and R, L, and C are certain constants called the resistance, inductance, and capacitance; as usual $\dot{I}$ is the time derivative of I. You do not need to know the physics to solve this problem.] Sadly, Ah Lian could not hear all of the numbers mentioned due to the incessant babbling of a talkative minority; all she could hear was that the resistance R is 2 Ω. Undeterred, she steals back into the room after class and quickly switches the gadget on and off at $t = 2$, thus firing a short burst of voltage into it, and observes that the resulting current at $t > 0$ is

$$I(t) = u(t-2)\left[e^{-(t-2)}\cos(t-2) - e^{-(t-2)}\sin(t-2)\right],$$

where $u(t)$ is the step function. She then deduces what the professor must have said about the inductance and the capacitance, L and C. What are her answers? [Hint: Recall the formula for the Laplace transform of an integral.]

<u>Solutions</u>

First, we know that Ah Lian switches the gadget on and off quickly at $t = 2$ to introduce voltage, which implies that the Dirac delta function is involved in $V(t)$. We do not know the magnitude of this small burst of voltage, hence we will denote it by A. Then, $V(t) = A\delta(t-2)$, giving the DE

$$A\delta(t-2) = 2I + L\dot{I} + \frac{1}{C}\int_0^t I\,dt.$$

Question What kind of equation is this? What are the dependent and independent variables? What are we solving for?

Taking Laplace transform on both sides, where $\Theta(s)$ is the transform of $I(t)$, we get

$$Ae^{-2s} = 2\Theta(s) + L[s\Theta(s) - I(0)] + \frac{1}{sC}\Theta(s)$$

Noting that $I(0) = 0$,

$$\Theta(s) = \frac{Ase^{-2s}}{Ls^2 + 2s + 1/C}$$

Since the question provides $I(t) = u(t-2)\left[e^{-(t-2)}\cos(t-2) - e^{-(t-2)}\sin(t-2)\right]$, it is easier to perform Laplace transform on $I(t)$ instead of trying to perform inverse Laplace transform to $\Theta(s)$ above. With the help of t-shifting $L\big(f(t-a)\cdot u(t-a)\big) = e^{-as}F(s)$ and s-shifting $L\big(e^{ct}f(t)\big) = F(s-c)$,

$$\Theta(s) = \frac{(s+1)e^{-2s}}{(s+1)^2+1} - \frac{e^{-2s}}{(s+1)^2+1} = \frac{se^{-2s}}{s^2+2s+2} \equiv \frac{Ase^{-2s}}{Ls^2+2s+1/C}.$$

By comparison, $\boxed{A = 1, L = 1}$ and $\boxed{C = \frac{1}{2}}$.

Question 3

Using the method of separation of variables, solve the following partial differential equations:

Solutions

(a) $yu_x - xu_y = 0$

Let $u(x,y) = X(x) \cdot Y(y)$. Then,

$$yX'(x)Y(y) - xX(x)Y'(y) = 0$$
$$\frac{X'}{xX} = \frac{Y'}{yY} \coloneqq \lambda$$

Thus, the first ODE we obtain is $\frac{X'}{X} = \lambda x$. Integrating both sides with respect to x:

$$\ln|X| = \frac{\lambda}{2}x^2 + C_1$$
$$X = Ae^{\frac{\lambda}{2}x^2}.$$

The second ODE obtained is $\frac{Y'}{yY} \coloneqq \lambda$. Similarly, we should get $Y = Be^{\frac{\lambda}{2}y^2}$.

Combining the two together, the solution is given by:

$$u(x,y) = XY = Ae^{\frac{\lambda}{2}x^2} \cdot Be^{\frac{\lambda}{2}y^2}$$
$$= Ce^{\frac{\lambda}{2}(x^2+y^2)} = \boxed{Ce^{d(x^2+y^2)}}.$$

(b) $u_x = yu_y, \quad y > 0$

Let $u(x,y) = X(x) \cdot Y(y)$. Then,

$$X'Y = yXY'$$
$$\frac{X'}{X} = \frac{yY'}{Y} \coloneqq \lambda.$$

The first ODE obtained is $\frac{X'}{X} = \lambda$. Solving the ODE, we have

$$\ln|X| = \lambda x + C_1$$
$$X = Ae^{\lambda x}.$$

The second ODE obtained is $\frac{yY'}{Y} = \lambda$. Solving the ODE, we have

$$\ln|Y| = \lambda \ln|y| + C_2$$
$$Y = By^{\lambda}.$$

Combining the two together, we have

$$u(x,y) = XY = Ae^{\lambda x} \cdot By^{\lambda}$$
$$= \boxed{Ce^{\lambda x}y^{\lambda}}.$$

(c) $u_{xy} = u$

Let $u(x,y) = X(x) \cdot Y(y)$. Then,

$$X'Y' = XY$$
$$\frac{X'}{X} = \frac{Y}{Y'} \coloneqq \lambda$$

The first ODE obtained is $\frac{X'}{X} = \lambda$. Solving, we get $X = Ae^{\lambda x}$ (from part (b)).

The second ODE will give us $\frac{Y'}{Y} = \frac{1}{\lambda}$. Solving, we get $Y = Be^{y/\lambda}$. Combining the two together, we have

$$u(x,y) = XY = Ae^{\lambda x} \cdot Be^{y/\lambda}$$
$$= \boxed{Ce^{\lambda x + y/\lambda}}.$$

(d) $xu_{xy} + 2yu = 0, \quad x > 0$

Let $u(x,y) = X(x) \cdot Y(y)$. Then,

$$xX'Y' = -2yXY$$
$$\frac{xX'}{X} = -\frac{2yY}{Y'} \coloneqq \lambda$$

The first ODE obtained is $\frac{xX'}{X} = \lambda$. From part (b), we get $X = Ax^{\lambda}$.

The second ODE obtained is $\frac{Y'}{Y} = -\frac{2}{\lambda}y$. Solving, we get

$$\ln|Y| = -\frac{1}{\lambda}y^2 + C_1$$
$$Y = Be^{-\frac{1}{\lambda}y^2}.$$

Combining the two together, we have

$$u(x,y) = XY = Ax^{\lambda} \cdot Be^{-\frac{1}{\lambda}y^2}$$
$$= \boxed{Cx^{\lambda}e^{-\frac{1}{\lambda}y^2}}.$$

Question 4

Show carefully that the d'Alembert solution of the wave equation, given in lectures, does satisfy the equation and the boundary and initial conditions.

Solutions

The d'Alembert solution $u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$ solves the problem:

$$u_{tt} = c^2 u_{xx}, \qquad 0 \le x \le \pi,\ \ t > 0$$
$$u(0,t) = 0,\ \ u(\pi,t) = 0,$$
$$u(x,0) = f(x),\ \ u_t(x,0) = 0.$$

Note that we extended $f(x)$ to be an odd periodic function of period 2π where necessary. First, we verify that the solution satisfies the PDE. Since by chain rule,

$$u_{tt} = \frac{c^2}{2}[f''(x+ct) + f''(x-ct)],$$
$$u_{xx} = \frac{1}{2}[f''(x+ct) + f''(x-ct)],$$

we can indeed verify that

$$u_{tt} - c^2 u_{xx} = \frac{c^2}{2}[f''(x+ct) + f''(x-ct)] - c^2 \cdot \frac{1}{2}[f''(x+ct) + f''(x-ct)] = 0.$$

Next, the boundary conditions: (**Note:** Odd functions $f(-x) = -f(x)$.)

$$u(0,t) = \frac{1}{2}[f(0+ct) + f(0-ct)] = \frac{1}{2}[f(ct) + f(-ct)] = \frac{1}{2}[f(ct) - f(ct)] = 0$$
$$u(\pi,t) = \frac{1}{2}[f(\pi+ct) + f(\pi-ct)] = \frac{1}{2}[f(\pi+ct) + f(-\pi-ct)] = \frac{1}{2}[f(\pi+ct) - f(\pi+ct)] = 0$$
$$u(x,0) = \frac{1}{2}[f(x+c\cdot 0) + f(x-c\cdot 0)] = \frac{1}{2}[f(x) + f(x)] = f(x)$$

Since $u_t(t,x) = \frac{c}{2}[f'(x+ct) - f'(x-ct)]$, we have

$$u_t(x,0) = \frac{c}{2}[f'(x+c\cdot 0) - f'(x-c\cdot 0)] = \frac{c}{2}[f'(x) - f'(x)] = 0.$$

Hence verified.

Question What are the main few properties we have used in this question?

Question 5

Solve $\begin{cases} u_t = 2u_{xx}, & 0 < x < 3,\ t > 0 \\ u(0,\ t) = 0,\ u(3,\ t) = 0 \\ u(x,\ 0) = \sin^5 \pi x \end{cases}$.

Hint: Use trigonometric identity
$$\sin^5 \pi x = \frac{5}{8}\sin \pi x - \frac{5}{16}\sin 3\pi x + \frac{1}{16}\sin 5\pi x.$$

Solutions

From the lecture notes, the solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) e^{-c^2\left(\frac{n\pi}{L}\right)^2 t} \ --(1)$$

satisfies $\begin{cases} u_t = 2u_{xx},\ 0 < x < 3,\ t > 0 \\ u(0,\ t) = 0,\ u(3,\ t) = 0 \end{cases}$ with $c^2 = 2$ and $L = 3$. We need to satisfy the last condition of

$$u(x,\ 0) = \sin^5 \pi x = \frac{5}{8}\sin \pi x - \frac{5}{16}\sin 3\pi x + \frac{1}{16}\sin 5\pi x. -- (2)$$

Substitute the solution (1) into the condition (2):

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{3}x\right) e^{-2\left(\frac{n\pi}{3}\right)^2 t}$$

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{3}x\right)$$

$$\frac{5}{8}\sin \pi x - \frac{5}{16}\sin 3\pi x + \frac{1}{16}\sin 5\pi x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{3}x\right)$$

By comparison of the sine functions, n can only take the values of 3, 9 and 15 to match the coefficients of x. Thus, their corresponding b values are

$$b_3 = \frac{5}{8}, \quad b_9 = -\frac{5}{16}, \quad b_{15} = \frac{1}{16}, \qquad \text{and the rest of the coefficients are zero.}$$

Thus, the particular solution to the problem is

$$u(x,t) = \frac{5}{8}\sin \pi x\, e^{-2\left(\frac{3\pi}{3}\right)^2 t} - \frac{5}{16}\sin 3\pi x\, e^{-2\left(\frac{9\pi}{3}\right)^2 t} + \frac{1}{16}\sin 5\pi x\, e^{-2\left(\frac{15\pi}{3}\right)^2 t}$$

$$\boxed{u(x,t) = \frac{5}{8}\sin \pi x\, e^{-2\pi^2 t} - \frac{5}{16}\sin 3\pi x\, e^{-18\pi^2 t} + \frac{1}{16}\sin 5\pi x\, e^{-50\pi^2 t}}.$$

Question 6

Consider a solid object with mass m immersed in water. A pulse horizontal force is applied initially. The dynamic equation is given by

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} -- = -A\frac{\mathrm{d}x}{\mathrm{d}t} + B\delta(t),$$

where the first term on the right is the hydrodynamic resistance force, and A and B are constants. The initial conditions are $x(0) = 0$ and $x'(0) = 0$. Apply Laplace transform to solve the above ODE to determine the location x as a function of time.

Solutions

Applying Laplace transform to the ODE, we get

$$m\left[s^2 X(s) - sx(0) - x'(0)\right] = -A\left[sX(s) - x(0)\right] + B$$

$$ms^2 X + AsX = B$$

$$X = \frac{B}{s(ms+A)} = \frac{B}{As} - \frac{mB}{A(ms+A)} = \frac{B}{A}\left(\frac{1}{s} - \frac{m}{ms+A}\right) = \frac{B}{A}\left(\frac{1}{s} - \frac{1}{s+A/m}\right)$$

Performing the inverse Laplace transform, we get

$$\underline{\underline{x(t) = \frac{B}{A}\left(1 - e^{-\frac{A}{m}t}\right)}}.$$

What formulas did we use?