ANSWERS TO MA1512 TUTORIAL 5

Question 1.

The original ODE describing this situation in the absence of friction was

$$M\ddot{x} \;=\; Mg \;-\; \rho\, A\,(d+x)g,$$

where the downward direction is positive. We need an extra term to account for the sudden force exerted by the rogue wave. Since the force is exerted suddenly, this suggests that we need a Dirac delta function, so the force will be proportional to $\delta(t-T)$. Now F = ma, Newton's law, can be written as F = [time derivative of mv], where v is the velocity, so the change in the momentum is equal to the time integral of the force. Recall that $\int_0^\infty \delta(t-T)dt = 1$, so clearly in our case $F = -P\delta(t-T)$, since P is the given change in the momentum. [Integrate both sides to verify this, and remember that x(t) here is the DOWNWARD displacement so the upward force of the wave [as stated in the problem] is negative, like the buoyancy force. Note that the units here are correct since the delta function has units of 1/time; this is because the time integral of the delta function is a pure number.] So we have

$$M\ddot{x} \;=\; Mg \;-\; \rho\, A\,(d+x)g - P\delta(t-T),$$

which, as in Tutorial 3, simplifies to

$$\ddot{x} \;=\; -\frac{\rho\, A\, g}{M}\, x - \frac{P}{M}\delta(t-T).$$

Taking the Laplace transform of both sides, remembering that the ship is initially at rest, we have

$$s^2 X = -\frac{\rho A g}{M} X - \frac{P}{M} e^{-Ts},$$

or

$$X(s) = -\frac{P}{M}\frac{e^{-Ts}}{s^2+\omega^2} = -\frac{P}{\omega M}\frac{\omega e^{-Ts}}{s^2+\omega^2},$$

where ω is the natural frequency of oscillation of the ship, $\sqrt{\rho\, A\, g/M}$. Using the t-shifting theorem, we can find the inverse Laplace transform:

$$x(t) = -\frac{P}{\omega M} sin[\omega(t-T)]u(t-T).$$

The amplitude is P/ωM, so this is the maximum distance the ship goes down if it doesn't sink.

QUESTION 2.
We have to deal with the equation

$$V(t) = RI + L\dot{I} + \frac{1}{C}\int_0^t I\ dt.$$

The problem here is that we don't actually know V(t); all we know is that it is some multiple of the Dirac delta function $\delta(t-2)$ [since it was applied to the system, and turned off, almost instantaneously, at $t = 2$]. So we set $V(t) = A\delta(t-2)$ where A is some unknown constant. Thus we have

$$A\delta(t-2) = RI + L\dot{I} + \frac{1}{C}\int_0^t I\ dt.$$

Take the Laplace transform of both sides, and let Θ(s) denote the transform of I(t); we get

$$Ae^{-2s} = R\Theta(s) + Ls\Theta(s) + \frac{1}{sC}\Theta(s),$$

where we have used the fact that $I(0) = 0$ [see the given formula for $I(t)$]. Solving for Θ(s) we get

$$\Theta(s) = \frac{Ase^{-2s}}{Ls^2 + Rs + (1/C)}.$$

But the Laplace transform of the given current is [using both shifting theorems]

$$\Theta(s) = \frac{(s+1)e^{-2s}}{(s+1)^2+1} - \frac{e^{-2s}}{(s+1)^2+1} = \frac{se^{-2s}}{s^2+2s+2} = \frac{Ase^{-2s}}{As^2+2As+2A},$$

from which we see immediately that R = 2A; since TAL knows that R = 2, we see that A = 1. Clearly C must have been 1/2 and L must have been 1 in the appropriate units.

QUESTION 3.
(a) Setting $u(x,y) = X(x)Y(y)$, the p.d.e. $yu_x - xu_y = 0$ becomes $yX'Y - xXY' = 0$. Dividing by XY gives

$$\begin{aligned} y\frac{X'}{X} - x\frac{Y'}{Y} &= 0 \\ \text{i.e.} \quad \frac{1}{x}\cdot\frac{X'}{X} &= \frac{1}{y}\cdot\frac{Y'}{Y} = k_1 \quad \text{(constant)} \end{aligned}$$

This gives two *separable* o.d.e., the first of which is $\frac{1}{x}\cdot\frac{X'}{X} = k_1$. Integrating this yields

$$\begin{aligned} \ln|X| &= \frac{1}{2}k_1x^2 + c_1 \\ \text{i.e.} \quad X &= \pm e^{c_1}e^{cx^2}, \text{where } c = \frac{1}{2}k_1 \end{aligned}$$

Similarly, integrating the second *separable* o.d.e. $\frac{1}{y}\cdot\frac{Y'}{Y} = k_1$ gives $Y = \pm e^{c_2}e^{cy^2}$. Thus,

$$\begin{aligned} u(x,y) &= XY = \pm e^{c_1}e^{cx^2}e^{c_2}e^{cy^2} \\ &= ke^{c(x^2+y^2)}. \end{aligned}$$

(b) Setting $u(x,y) = X(x)Y(y)$, the p.d.e. $u_x = yu_y$ becomes $X'Y = yXY'$. Dividing by XY gives

$$\frac{X'}{X} = y\frac{Y'}{Y} = c \quad \text{(constant)}$$

This gives two *separable* o.d.e., the first of which is $\frac{X'}{X} = c$. Integrating this yields

$$\begin{aligned} \ln|X| &= cx + k_1 \\ \text{i.e.} \qquad X &= \pm e^{k_1} e^{cx}. \end{aligned}$$

Similarly, integrating the second *separable* o.d.e. $\frac{Y'}{Y} = \frac{c}{y}$ gives

$$\begin{aligned} \ln|Y| &= c\ln|y| + k_2 \\ \text{i.e.} \qquad Y &= \pm e^{k_2} y^c. \end{aligned}$$

Thus,

$$\begin{aligned} u(x,y) &= XY = \pm e^{k_1} e^{cx} e^{k_2} y^c \\ &= k y^c e^{cx}. \end{aligned}$$

(c) Setting $u(x,y) = X(x)Y(y)$, the p.d.e. $u_{xy} = u$ becomes $X'Y' = XY$. Dividing by XY gives

$$\frac{X'}{X} \cdot \frac{Y'}{Y} = 1.$$

This implies that both $\frac{X'}{X}$ and $\frac{Y'}{Y}$ are nonzero constants and we set $\frac{X'}{X} = c$ and $\frac{Y'}{Y} = \frac{1}{c}$. This gives two *separable* o.d.e., the first of which is $\frac{X'}{X} = c$. Integrating this yields

$$\begin{aligned} \ln|X| &= cx + k_1 \\ \text{i.e.} \qquad X &= \pm e^{k_1} e^{cx}. \end{aligned}$$

Similarly, integrating the second *separable* o.d.e. $\frac{Y'}{Y} = \frac{1}{c}$ gives

$$\begin{aligned} \ln|Y| &= \frac{y}{c} + k_2 \\ \text{i.e.} \qquad Y &= \pm e^{k_2} e^{y/c}. \end{aligned}$$

Thus,

$$\begin{aligned} u(x,y) &= XY = \pm e^{k_1} e^{cx} e^{k_2} e^{y/c} \\ &= k e^{cx + y/c}. \end{aligned}$$

(d) Setting $u(x,y) = X(x)Y(y)$, the p.d.e. $xu_{xy} + 2yu = 0$ becomes $xX'Y' + 2yXY = 0$. Dividing by $-2yXY$ gives

$$\left(x\frac{X'}{X}\right)\left(-\frac{1}{2y} \cdot \frac{Y'}{Y}\right) = 1.$$

This implies that both $x\dfrac{X'}{X}$ and $\dfrac{-1}{2y}\cdot\dfrac{Y'}{Y}$ are nonzero constants. We set $x\dfrac{X'}{X} = c$ and $\dfrac{-1}{2y}\cdot\dfrac{Y'}{Y} = \dfrac{1}{c}$, which respectively give two *separable* o.d.e. $\dfrac{X'}{X} = \dfrac{c}{x}$ and $\dfrac{Y'}{Y} = -\dfrac{2y}{c}$.

Integrating the first o.d.e. $\dfrac{X'}{X} = \dfrac{c}{x}$ yields

$$\begin{aligned} \ln|X| &= c\ln x + k_1 \\ \text{i.e.} \qquad X &= \pm e^{k_1}x^c. \end{aligned}$$

Similarly, integrating the second *separable* o.d.e. $\dfrac{Y'}{Y} = -\dfrac{2y}{c}$ gives

$$\begin{aligned} \ln|Y| &= -\frac{y^2}{c} + k_2 \\ \text{i.e.} \qquad Y &= \pm e^{k_2}e^{-y^2/c}. \end{aligned}$$

Thus,

$$\begin{aligned} u(x,y) &= XY = \pm e^{k_1}x^c e^{k_2}e^{-y^2/c} \\ &= kx^c e^{-y^2/c}. \end{aligned}$$

QUESTION 4.

We want to solve

$$c^2 y_{xx} = y_{tt},$$

subject to the four conditions

$$y(t,0) = y(t,\pi) = 0, \quad y(0,x) = f(x), \quad y_t(0,x) = 0,$$

where f(x) is a given function which is zero at 0 and π, and which we extend when necessary to be an odd periodic function of period 2π. D'Alembert claimed that the following object satisfies all of the above:

$$y(t,x) = \frac{1}{2}\Big[f(x+ct) + f(x-ct)\Big].$$

First, note that if y(t,x) is given by this formula, then

$$y_{tt} = \frac{c^2}{2}\Big[f''(x+ct) + f''(x-ct)\Big],$$

while

$$y_{xx} = \frac{1}{2}\Big[f''(x+ct) + f''(x-ct)\Big].$$

So it does satisfy the equation. We also see that

$$y(t,0) = \frac{1}{2}\Big[f(ct) + f(-ct)\Big],$$

which is zero because f(x) is odd. Then

$$y(t,\pi) = \frac{1}{2}\Big[f(\pi+ct) + f(\pi-ct)\Big] = \frac{1}{2}\Big[f(\pi+ct) + f(-\pi-ct)\Big],$$

because f(x) is periodic with period 2π, and this too vanishes because f(x) is odd. Next, clearly

$$y(0,x) = \frac{1}{2}\Big[f(x) + f(x)\Big] = f(x),$$

and finally

$$y_t(0,x) = \frac{1}{2}\Big[cf'(x) - cf'(x)\Big] = 0.$$

So D'Alembert was right.

QUESTION 5.

As in the last example in the notes, we know that

$$u_n(x,t) \;=\; b_n e^{-2n^2\pi^2 t/9} \sin\frac{n\pi x}{3} \qquad (HS_n)$$

is a solution for each $n = 1, 2, 3, \ldots$. that satisfies all the conditions except perhaps the initial condition $u(x,0) = \sin^5 \pi x$. This time, to get a solution that satisfies the initial condition, it is insufficient to use only a single $u_n(x,t)$ from (HS_n). Instead, we need to take three appropriate $u_n(x,t)$ and take their linear combination. By the superposition principle, this combination will still satisfy the p.d.e and the boundary conditions.

If we choose $n = 3,\ \ n = 9,\ \ n = 15$ separately, then we get the three terms in the initial condition. Therefore, the particular solution is of the form

$$\begin{aligned} u(x,t) &= u_3(x,t) + u_9(x,t) + u_{15}(x,t) \\ &= b_3 e^{-2\pi^2 t}\sin\pi x + b_9 e^{-18\pi^2 t}\sin 3\pi x + b_{15} e^{-50\pi^2 t}\sin 5\pi x \end{aligned}$$

By comparing coefficients, we need to choose $b_3 = \frac{5}{8},\ b_9 = -\frac{5}{16}, b_{15} = \frac{1}{16}$ in order to satisfy the initial condition. That is, the particular solution in this case is

$$u(x,t) = \frac{5}{8}e^{-2\pi^2 t}\sin\pi x - \frac{5}{16}e^{-18\pi^2 t}\sin 3\pi x + \frac{1}{16}e^{-50\pi^2 t}\sin 5\pi x.$$

Question 6 Solution

$m\dfrac{d^2x}{dt^2} = -A\dfrac{dx}{dt} + B\delta(t)$ with $x(0) = 0$ and $x'(0) = 0$

Laplace transform of the above eqn

$$ms^2 X = -AsX + B$$

$$X = \frac{B}{s(mS + A)} = \frac{c_1}{s} + \frac{c_2}{ms + A}$$

$$c_1 = \frac{B}{A} \qquad c_2 = -\frac{mB}{A}$$

$$X = \frac{B}{A}\left(\frac{1}{s} - \frac{1}{s + A/m}\right)$$

Inverse Laplace transform of the above $x = \dfrac{B}{A}(1 - e^{-At/m})$