Unit 058 Eigenspaces

Slide 01: In this unit, we introduce the concept of eigenspaces of a matrix. The eigenspace of a matrix is closely related to eigenvalues and eigenvectors, both of which were discussed in earlier units.

Slide 02: Let $\boldsymbol{A}$ be a square matrix of order n and λ be an eigenvalue of $\boldsymbol{A}$.
(#)
Recall that the matrix $(\lambda\boldsymbol{I} - \boldsymbol{A})$, whose determinant is the characteristic polynomial of $\boldsymbol{A}$. Now, use this matrix as the coefficient matrix to form a homogeneous linear system as shown.
(#)
Before we proceed, note that since λ is an eigenvalue of $\boldsymbol{A}$, it would make the matrix $(\lambda\boldsymbol{I} - \boldsymbol{A})$ singular. This implies that the homogeneous linear system would have infinitely many solutions.
(#)
Consider the solution space of the homogeneous linear system $(\lambda\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$. This solution space is called the eigenspace of $\boldsymbol{A}$ associated with the eigenvalue λ. We will denote this solution space as E_λ.

Slide 03: Let us look at the definition of the eigenspace of $\boldsymbol{A}$ again. Remember that it is the solution space of a homoegenous linear system whose coefficient matrix is $(\lambda\boldsymbol{I} - \boldsymbol{A})$.
(#)
Thus, a non zero vector $\boldsymbol{v}$ belongs to E_λ if and only if $\boldsymbol{v}$ is a solution of the homogeneous linear system. In other words, $(\lambda\boldsymbol{I} - \boldsymbol{A})\boldsymbol{v} = \boldsymbol{0}$.
(#)
This is equivalent to $\lambda\boldsymbol{I}\boldsymbol{v} - \boldsymbol{A}\boldsymbol{v} = \boldsymbol{0}$
(#)
which can be rearranged to give $\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$.
(#)
Thus, the eigenspace E_λ contains all the eigenvectors of $\boldsymbol{A}$ that are associated with the eigenvalue λ.
(#)
Recall the definition of the nullspace of a matrix. It should be clear now that E_λ is also the nullspace of the matrix $(\lambda\boldsymbol{I} - \boldsymbol{A})$.

Slide 04: Let us revisit some of the matrices we have seen in previous units. First the 2×2 matrix $\boldsymbol{A}$ from the population movement example. We have found that $\boldsymbol{A}$ has two eigenvalues 1 and 0.95. For $\lambda = 1$, let us investigate the eigenspace E_1.
(#)
We need to evaluate the coefficient matrix $1\boldsymbol{I} - \boldsymbol{A}$,
(#)
as shown
(#)
and then solve the homogeneous linear system by Gaussian elimination. The reduced row-echelon form of the augmented matrix is shown.

(#)

The first row of the matrix tells us that the two variables are related by the equation $x_1 - \frac{1}{4}x_2 = 0$.

(#)

Thus a vector $\boldsymbol{x}$ belonging to the eigenspace E_1 is of the form $(\frac{t}{4}, t)$, where t is an arbitrary real number.

(#)

In conclusion, we know that E_1 is the linear span of the vector $(\frac{1}{4}, 1)$.

(#)

Note that this is a one dimensional subspace of $\mathbb{R}^2$.

Slide 05: We move on to the second eigenvalue 0.95 and investigate the eigenspace $E_{0.95}$.

(#)

First computing the coefficient matrix $(0.95\boldsymbol{I} - \boldsymbol{A})$

(#)

we have the following

(#)

and then we solve the homogeneous linear system as we did previously, we obtain the reduced row-echelon form as shown.

(#)

The first row of the matrix tells us that the two variables are related by the equation $x_1 + x_2 = 0$.

(#)

Thus a vector $\boldsymbol{x}$ that belongs to the eigenspace $E_{0.95}$ is of the form $(-t, t)$, where t is an arbitrary real number.

(#)

In conclusion, we know that $E_{0.95}$ is the linear span of the vector $(-1, 1)$.

(#)

Note that this eigenspace is also a one dimensional subspace of $\mathbb{R}^2$.

Slide 06: Next we consider the 3×3 matrix $\boldsymbol{B}$ which we have also found to have 2 eigenvalues, namely 3 and 0. We start by investigating the eigenspace E_3.

(#)

The coefficient matrix is $3\boldsymbol{I} - \boldsymbol{B}$

(#)

and when we solve the homogeneous linear system, we arrive at the following reduced row-echelon form. Notice that it has two non zero rows and one non pivot column on the left hand side.

(#)

You should be able to write down a general solution very quickly. A vector $\boldsymbol{x}$ belongs to this eigenspace if $\boldsymbol{x}$ is of the form (t, t, t) where t is any real number.

(#)

Thus E_3 is the linear span of the vector $(1, 1, 1)$

(#)

and it is a one dimensional subspace of $\mathbb{R}^3$.

Slide 07: Moving on to E_0.
(#)
The coefficient matrix is $0\boldsymbol{I} - \boldsymbol{B}$
(#)
and when we solve the homogeneous linear system, we arrive at the following reduced row-echelon form. Notice that it has one non zero row and two non pivot columns on the left hand side. So how many arbitrary parameters are there in a general solution for the linear system?

Slide 08: From the reduced row-echelon form, we have the equation $x_1 + x_2 + x_3 = 0$.
(#)
A general solution to the linear system is as shown, where $x_1 = -s-t, x_2 = s, x_3 = t$, s and t are arbitrary parameters.
(#)
Thus a vector $\boldsymbol{x}$ belongs to E_0 if and only if $\boldsymbol{x}$ is a linear combination of $(-1, 1, 0)$ and $(-1, 0, 1)$.
(#)
The eigenspace E_0 is the linear span of $(-1, 1, 0)$ and $(-1, 0, 1)$
(#)
and the dimension of E_0 is 2.

Slide 09: Without showing the details, we see here the other 3×3 matrix $\boldsymbol{C}$, which has 3 distinct eigenvalues 1, $\sqrt{2}$ and $-\sqrt{2}$. The eigenspace E_1, $E_{\sqrt{2}}$ and $E_{-\sqrt{2}}$ are all one dimensional subspaces of $\mathbb{R}^3$.

Slide 10: Consider this 2×2 matrix $\boldsymbol{M}$. Since it is a lower triangular matrix, we can immediately conclude that $\boldsymbol{M}$ has only one eigenvalue 2.
(#)
To investigate the eigenspace E_2, we solve the homogeneous linear system
(#)
with coefficient matrix $(2\boldsymbol{I} - \boldsymbol{M})$.
(#)
We see that a vector $\boldsymbol{x}$ belongs to the eigenspace E_2 if and only if $\boldsymbol{x}$ is of the form $(0, s)$ where s is any real number.
(#)
So E_2 is the linear span of $(0, 1)$ and it is a one dimensional subspace of $\mathbb{R}^2$.

Slide 11: Before we end this unit, recall that leading up to this unit, we were concerned with the question of whether for a given square matrix $\boldsymbol{A}$, we could find an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{A}$ can be written as $\boldsymbol{PDP}^{-1}$ where $\boldsymbol{D}$ is a diagonal matrix. Recall that if this can be done, it would help us to compute powers of $\boldsymbol{A}$ efficiently.
(#)
$\boldsymbol{A} = \boldsymbol{PDP}^{-1}$ can equivalently be written as $\boldsymbol{P}^{-1}\boldsymbol{AP} = \boldsymbol{D}$.
(#)
While we will not answer this question in this unit, the examples we have seen so far are in fact illustrative enough to give you a hint as to when it is or not possible for $\boldsymbol{A}$ to be written as $\boldsymbol{PDP}^{-1}$.

Slide 12: The matrix $\boldsymbol{A}$ is a 2×2 matrix. It has two eigenvalues and each of the two eigenspaces are one dimensional.

(#)

The matrix $\boldsymbol{B}$ is a 3×3 matrix. It has two eigenvalues 3 and 0.

(#)

The dimension of E_3 is 1 while the dimension of E_0 is 2.

(#)

The matrix $\boldsymbol{C}$ is a 3×3 matrix. It has three eigenvalues, 1, $\sqrt{2}$ and $-\sqrt{2}$.

(#)

The dimension of each of the three eigenspaces is 1. We will discover, in a later unit that all the three matrices here can be written as $\boldsymbol{PDP}^{-1}$. Perhaps you would like to pause for a while and see what do the three matrices have in common. As a hint, you may wish to look at the highlighted dimensions of the various eigenspaces.

Slide 13: Contrastingly, this matrix $\boldsymbol{M}$ which we have seen in this unit is one that cannot be written as $\boldsymbol{PDP}^{-1}$. Do you know why? Once again, the hint lies in the dimension of the eigenspace highlighted. We will answer these questions in the subsequent units.

Slide 14: In this unit,

(#)

We defined what is an eigenspace of a matrix associated with a particular eigenvalue λ.