MA1512 TUTORIAL 4

Question 1

The bacteria in a certain culture number 10000 initially. Two and a half hours later there are 11000 of them. Assuming a Malthus model, how many bacteria will there be 10 hours after the start of the experiment? How long will it take for the number to reach 20000?

Question 2

You have 200 bugs in a bottle. Every day you supply them with food and count them. After two days you have 360 bugs. It is known that the birth rate for this kind of bug is 150% per day. [Is this a sensible way of stating a birth rate per capita? Why?] Assuming that the population is given by a logistic model, find the number of bugs after 3 days. Predict how many bugs you will have eventually.

Question 3

In question 2, let us assume that you are keeping the bugs not as a hobby, but because you are developing a new insecticide. Suppose that you remove 80 bugs per day from the bottle, and that all of these bugs die as a result of being sprayed with this insecticide. What is the limiting population in this case? What is the maximum number of bugs you can put to death per day without causing the population to die out?

Question 4

The sandhill crane is a beautiful Canadian bird with an unfortunate liking for farm crops. For many years the cranes were protected by law, and eventually they settled down to a logistic equilibrium population of 194,600 with birth rate per capita 9.866% per year. Eventually the patience of the farmers was exhausted and they managed to have the hunting ban lifted. The farmers happily shot 10000 cranes per year, which they argued was reasonable enough since it only represents about 5% of the original population. Show that the sandhill crane is doomed.

Question 5

Suppose that Peruvian fishermen take a fixed number of anchovies per year from an anchovy stock which would otherwise behave logistically, apart from occasional natural disasters. According to our lecture notes, any fishing rate $\geq \mathrm{B}^2/4\mathrm{s}$ will be disastrous. Let's call this number E^*. The fishermen want to take as many anchovies as they **safely** can, meaning that they want the fish to be able to bounce back from a natural disaster that pushes their population down by 10%. Advise them. That is, tell them the maximum number of fish they can take, expressed as a percentage of E^*.

6. Find the Laplace transforms of the following functions [where u denotes the unit step function and the answers are given in brackets]:

   (a) $t^2 e^{-3t}$. $\qquad [\frac{2}{(s+3)^3}]$

   (b) $tu(t-2)$. $\qquad [e^{-2s}\{\frac{1}{s^2} + \frac{2}{s}\}]$

7. Find the inverse Laplace transforms of the following functions:

   (a) $\frac{s}{s^2 + 10s + 26}$. $\qquad [e^{-5t}(\cos t - 5\sin t)]$

   (b) $e^{-2s}\frac{1+2s}{s^3}$. $\qquad [(\frac{1}{2}t^2 - 2)u(t-2)]$

8. Solve the following initial value problems using Laplace transforms:

   (a) $y' = tu(t-2), \; y(0) = 4$. $\qquad [(\frac{1}{2}t^2 - 2)u(t-2) + 4]$

   (b) $y'' - 2y' = 4, \; y(0) = 1, \; y'(0) = 0$. $\qquad [e^{2t} - 2t]$

Question 9 (Suggested by FoE)

Rocket Flight A model rocket having initial mass m_0 kg is launched vertically from the ground. The rocket expels gas at a constant rate of α kg/sec and at a constant velocity of β m/sec relative to the rocket. Assume that the magnitude of the gravitational force is proportional to the mass with proportionality constant g. Because the mass is not constant, Newton's second law leads to the equation

$$(m_0 - \alpha t)\frac{dv}{dt} - \alpha\beta = -g\,(m_0 - \alpha t)$$

where $v = dx/dt$ is the velocity of the rocket, x is its height above the ground, and $m_0 - \alpha t$ is the mass of the rocket at t sec after launch. If the initial velocity is zero, solve the above equation to determine the velocity of the rocket and its height above ground for $0 \leq t < m_0/\alpha$

Question 10

In the harvesting model we considered in the lectures, the population will rebound if all harvesting is stopped. Unhappily, this is not always true: for some animals, if you drive their population down too low, they will have trouble finding mates, or they will be forced to breed with relatively close kin, which reduces genetic variability and hence their ability to resist disease. For such animals [for example, certain rare species of tigers] extinction will result if the population falls too low, even if all harvesting is forbidden. Biologists call this **depensation**. Show that this situation can be modelled by the ODE

$$\frac{dN}{dt} = -aN^3 + bN^2 - cN,$$

where N is the population and a, b, and c are positive constants such that $b^2 > 4ac$. Find the population below which extinction will occur.