W04-02

Slide 01: In this unit, we continue our discussion on Euclidean vectors.

Slide 02: We start off with a vector $\boldsymbol{u}$ in $\mathbb{R}^2$ with components u_1 and u_2. The length of the vector is the square root of $u_1^2 + u_2^2$.

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It is easy to see that geometrically, by putting the vector with its initial point at the origin, the length of the vector, by pythagora's theorem is simply the square root of the sum of squares of u_1 and u_2.

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The notation for the length of the vector $\boldsymbol{u}$ is to have two vertical lines on either side of $\boldsymbol{u}$.

Slide 03: For a $\boldsymbol{u}$ in $\mathbb{R}^3$, the length of $\boldsymbol{u}$ is the square root of the sum of the squares of the components u_1, u_2 and u_3.

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Now that you have seen how length is defined for vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$, how do you think the length of a vector can be defined if the vector belongs to $\mathbb{R}^n$?

Slide 04: Not surprisingly, the way we define the lengt of a vector $\boldsymbol{u}$ in $\mathbb{R}^n$ is as follows. Basically, we take the square root of the sum of squares of all the n components of the vector.

Slide 05: How do we measure the distance between two vectors? If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, the distance between them is defined to be the length of the vector $\boldsymbol{u} - \boldsymbol{v}$. It is intuitively clear that the further $\boldsymbol{u}$ and $\boldsymbol{v}$ are apart, the longer will be the vector $\boldsymbol{u} - \boldsymbol{v}$.

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We will use $d(\boldsymbol{u}, \boldsymbol{v})$ to represent the distance between $\boldsymbol{u}$ and $\boldsymbol{v}$. Therefore $d(\boldsymbol{u}, \boldsymbol{v})$ is equal to the length of the vector $\boldsymbol{u} - \boldsymbol{v}$.

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When vectors are in $\mathbb{R}^n$, the definition of distance between vectors is similar.

Slide 06: If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{R}^2$, with components u_1, u_2 and v_1, v_2 respectively,

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then the distance between $\boldsymbol{u}$ and $\boldsymbol{v}$ as defined to be the length of the vector $\boldsymbol{u} - \boldsymbol{v}$ can be computed as follows.

Slide 07: Similarly, if $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{R}^3$, with components u_1, u_2, u_3 and v_1, v_2, v_3,

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the distance between $\boldsymbol{u}$ and $\boldsymbol{v}$ can be computed in a similar manner.

Slide 08: We now move on to discuss the angle between two vectors. Once again, starting with vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, let the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ be θ and the side facing the angle θ is the vector $\boldsymbol{u} - \boldsymbol{v}$.

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By applying cosine rule, we have the following equation that relates the lengths of the 3 sides of the triangle as well as the cosine of the angle θ.

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This allows us to write cosine θ as follows,

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and θ as the arccosine of the expression shown here.

Slide 09: Let us keep in mind the expression for θ obtained from the previous slide. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{R}^2$,

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the expression in the bracket, involving the length of $\boldsymbol{u}$, $\boldsymbol{v}$ and $(\boldsymbol{u}-\boldsymbol{v})$ can be written as follows

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and further simplified to $u_1v_1 + u_2v_2$ divided by the length of $\boldsymbol{u}$ times the length of $\boldsymbol{v}$.

Slide 10: If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{R}^3$,

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we can go through the same process of writting out the expression in the bracket and upon simplification,

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we have $u_1v_1 + u_2v_2 + u_3v_3$ divided by the length of $\boldsymbol{u}$ times the length of $\boldsymbol{v}$.

Slide 11: Let us put the expressions we obtained for $\mathbb{R}^2$ and $\mathbb{R}^3$ side by side.

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You would notice that the numerators for both expressions follow a similar format, one that involves the multiplication of the corresponding components of the two vectors before summing up.

Slide 12: We will now formally define the dot product between two vectors in the more general $\mathbb{R}^n$. Suppose $\boldsymbol{u}$ and $\boldsymbol{v}$ are two vectors in $\mathbb{R}^n$ with their respective components as shown.

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The dot product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is the value $u_1v_1 + u_2v_2$ and so on till u_nv_n. Note that we have seen such an expression in the previous slide. The dot product of two vectors gives a scalar and thus sometimes it is also called the scalar product, or the inner product.

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The length of a vector, as we have discussed earlier in this unit is defined as follows. The length of a vector is sometimes also known as the norm,

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A vector with length equal to 1 is called a unit vector.

Slide 13: The distance between two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, as we have defined earlier, is the length of the vector $(\boldsymbol{u} - \boldsymbol{v})$.

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The angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is the arccosine of the expression $\boldsymbol{u} \cdot \boldsymbol{v}$ divided by the length of $\boldsymbol{u}$ times the length of $\boldsymbol{v}$.

Slide 14: You should probably be familiar with computing matrix product by now. Suppose we have two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ with components written in a row as shown here. In this case, we say the vectors are written as row vectors.

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The dot product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is the following expression,

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which can also be obtained if we consider the premultiplication of a row matrix to a column matrix as shown.

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In other words, the same numerical value is obtained for the dot product as well as the matrix product. It should be noted that since the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ were written as rows, the row matrix $\boldsymbol{u}$ is premultiplied to the column matrix $\boldsymbol{v}^T$. Thus the dot product $\boldsymbol{u} \cdot \boldsymbol{v}$ is equal to the matrix product $\boldsymbol{u}\boldsymbol{v}^T$ in this case.

Slide 15: On the other hand, if the two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are such that their components are written in a column as shown here, then we say that the vectors are written as column vectors.

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The dot product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is still the same expression,

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and similarly obtained by considering the matrix product.

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However, we now have $\boldsymbol{u}^T$ as a row matrix premultiplied to $\boldsymbol{v}$ as a column matrix in order to obtain the numerical value for $\boldsymbol{u} \cdot \boldsymbol{v}$. So in this case, the dot product $\boldsymbol{u} \cdot \boldsymbol{v}$ is equal to the matrix product $\boldsymbol{u}^T\boldsymbol{v}$.

Slide 16: Let us summarise the main points in this unit.

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We defined the following terminologies.

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The norm or the length of an euclidean vector.

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The distance between two vectors.

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The angle between two vectors,

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and the dot product, or inner product between two vectors.

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We also saw how we can interpret the dot product of two vectors as a matrix product.