Week 08

MA1508E LINEAR ALGEBRA FOR ENGINEERING

Week 7

IVLE Quiz Discussion

Review of Week 07 (Units 035-039) content

- Recall the notion of redundancy → linear independence
- Notion of linear independence → solutions to a vector equation
- Set with only one vector → when is it a linearly independent set?
- Set with only two vectors → when is it a linearly independent set?
- Linear independence defined in terms of redundancy
- Guaranteed dependence (more than n vectors in $\boldsymbol{R}^n$)
- Linear (in)dependence in $\boldsymbol{R}^2$ (two or more vectors)
- Linear (in)dependence in $\boldsymbol{R}^3$ (two or more vectors)
- Intuitively, more vectors means more likely to be linearly dependent. When can we add vectors to a linearly independent set and preserve independent property? When will we no longer be able to do this?

Review of Week 07 (Units 035-039) content

- What is a vector space?
- Finding a ‘small’ set that is able to span a vector space.
- Definition of a basis
- Uniqueness of expression when writing a vector in a vector space in terms of a set of basis vectors.
- Definition of coordinate vectors (with respect to a basis)
- Different ‘representations’ of the same vector.
- What is a standard basis for $\boldsymbol{R}^n$?

Week 08 (units 040-044) overview

040 Dimensions Part I

- All bases for the same vector space have the same number of vectors
- Definition of dimension (of a vector space)
- Finding a basis for the solution space of a homogeneous linear system.

041 Dimensions Part II

- Knowing the dimension of a vector space V helps in determining whether a set S can be a basis for V
- The dimension of all subspaces of a vector space V does not exceed the dimension of V
- The only subspace of a vector space V that has the same dimension as V is V itself.

042 Equivalent statements Part III

- $\boldsymbol{A}$ is an invertible matrix of order n if and only if the rows (columns) of $\boldsymbol{A}$ forms a basis for $\boldsymbol{R}^n$

Week 08 (units 040-044) overview

043 Row space and column space

- Definition of the row space and column space of a matrix
- If the rows of $\boldsymbol{A}$ are linearly independent, then they form a basis for the row space of $\boldsymbol{A}$
- If $\boldsymbol{R}$ is a matrix in row-echelon form, then the non-zero rows of $\boldsymbol{R}$ form a basis for the row space of $\boldsymbol{R}$

044 Finding a basis for the row space

- Row equivalent matrices have the same row space
- A method to find a basis for the row space of a matrix.

Discussion

Question: To find a basis for and determine the dimension of the solution space of a homogeneous linear system.

$$\left(\begin{array}{ccccc|c} \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & & \vdots & & & \vdots \\ \cdots & \cdots & \cdots & \cdots & .. & 0 \end{array}\right) \xrightarrow{\text{GJE}} \left(\begin{array}{cccccc|c} * & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & * & \cdots & 0 & \cdots & 0 \\ & & 0 & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & * & \cdots & 0 \end{array}\right)$$

pivot columns

How do we write out a general solution?

Discussion

Assign variables corresponding to non pivot columns arbitrary parameters.

Let's say there are a total of 3 arbitrary parameters s,t,u (it does not matter how many) assigned to variables x_i,x_j,x_k.

$$
\begin{array}{ccccccccc}
 & x_i & & x_j & & x_k & & \\
\end{array}
$$

$$
\left(\begin{array}{cccccc|c}
* & \boxed{\cdots} & 0 & \boxed{\cdots} & 0 & \boxed{\cdots} & 0 \\
0 & \cdots & * & \cdots & 0 & \cdots & 0 \\
 & & 0 & & \vdots & & \vdots \\
0 & \cdots & 0 & \cdots & * & \cdots & 0
\end{array}\right)
$$

How do we write out a general solution?

Discussion

3 arbitrary parameters s,t,u

(it does not matter how many)

assigned to variables x_i, x_j, x_k.

$$
\begin{array}{cccccccc}
 & x_i & & x_j & & x_k & & \\
\end{array}
$$

$$
\left(\begin{array}{ccccccc|c}
* & \boxed{\cdots} & 0 & \boxed{\cdots} & 0 & \boxed{\cdots} & & 0 \\
0 & \boxed{\cdots} & * & \boxed{\cdots} & 0 & \boxed{\cdots} & & 0 \\
 & & 0 & & \vdots & & & \vdots \\
0 & \boxed{\cdots} & 0 & \boxed{\cdots} & * & \boxed{\cdots} & & 0
\end{array}\right)
$$

$$
\begin{cases}
\vdots & & \vdots \\
x_i & = & s \\
\vdots & & \vdots \\
x_j & = & t \\
\vdots & & \vdots \\
x_k & = & u \\
\vdots & & \vdots
\end{cases}
\qquad
\begin{pmatrix} \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_k \\ \vdots \end{pmatrix}
= s \begin{pmatrix} \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix}
+ t \begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \end{pmatrix}
+ u \begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \end{pmatrix}
$$

Discussion

These 3 vectors will span the solution space.

$$\begin{pmatrix} \vdots \\ x_i \\ \vdots \\ x_j \\ \vdots \\ x_k \\ \vdots \end{pmatrix} = s\begin{pmatrix} \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} + t\begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} + u\begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \end{pmatrix}$$

$$\begin{array}{cccccccc} & x_i & & x_j & & x_k & & \\ \end{array}$$

$$\left(\begin{array}{cccccc|c} * & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & * & \cdots & 0 & \cdots & 0 \\ & & 0 & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & * & \cdots & 0 \end{array}\right)$$

Are they always linearly independent?

Discussion

$$\begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} = c_1 \begin{pmatrix} \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} + c_2 \begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} + c_3 \begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \end{pmatrix}$$

$$\Rightarrow c_1 = 0 \quad , c_2 = 0 \quad , c_3 = 0$$

$$\begin{array}{c} \begin{array}{ccccccc} & x_i & & x_j & & x_k & \end{array} \\ \left(\begin{array}{ccccccc|c} * & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & * & \cdots & 0 & \cdots & 0 \\ & & 0 & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & * & \cdots & 0 \end{array}\right) \end{array}$$

Are they always linearly independent?

YES!

So, this method always produces a basis for the solution space

Discussion

$$\begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} = c_1 \begin{pmatrix} \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} + c_2 \begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} + c_3 \begin{pmatrix} \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 1 \\ \vdots \end{pmatrix}$$

$$\begin{array}{ccccccc} & x_i & & x_j & & x_k & \\ \end{array}$$

$$\left(\begin{array}{ccccccc|c} * & \cdots & 0 & \cdots & 0 & \cdots & & 0 \\ 0 & \cdots & * & \cdots & 0 & \cdots & & 0 \\ & & 0 & & \vdots & & & \vdots \\ 0 & \cdots & 0 & \cdots & * & \cdots & & 0 \end{array}\right)$$

Are they always linearly independent?

YES!

$\Rightarrow c_1 = 0 \;, c_2 = 0 \;, c_3 = 0$ AND

the dimension is always the number of non pivot columns in

Example 7.1

Find a basis and determine the dimension of the solution space of the following homogeneous linear system:

$$\begin{cases} x_1 + 3x_2 - x_3 + 2x_4 = 0 \\ \quad\;\; - 3x_2 + x_3 = 0 \\ x_1 \qquad\qquad\quad - x_4 = 0 \end{cases}$$

Example 7.2

Let $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ be a basis for a vector space V. Determine whether $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is a basis for V if

(a) $\boldsymbol{v}_1 = \boldsymbol{u}_1,\ \boldsymbol{v}_2 = \boldsymbol{u}_1 + \boldsymbol{u}_2,\ \boldsymbol{v}_3 = \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3$

(b) $\boldsymbol{v}_1 = \boldsymbol{u}_1 - \boldsymbol{u}_2,\ \boldsymbol{v}_2 = \boldsymbol{u}_2 - \boldsymbol{u}_3,\ \boldsymbol{v}_3 = \boldsymbol{u}_3 - \boldsymbol{u}_1$

Example 7.3

Let $\boldsymbol{u}_1 = (1,0,1,1)$, $\boldsymbol{u}_2 = (-3,3,7,1)$, $\boldsymbol{u}_3 = (-1,3,9,3)$, $\boldsymbol{u}_4 = (-5,3,5,-1)$ and let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$ and $V = \text{span}(S)$.

(a) Find a non-trivial solution to the equation

$$a\boldsymbol{u}_1 + b\boldsymbol{u}_2 + c\boldsymbol{u}_3 + d\boldsymbol{u}_4 = \boldsymbol{0}$$

(b) Express $\boldsymbol{u}_3$ and $\boldsymbol{u}_4$ (separately) as linear combinations of $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$.

(c) Find a basis for and determine the dimension of V.

(d) Find a subspace W of $\mathbb{R}^4$ such that $\dim(W) = 3$ and $\dim(W \cap V) = 2$. Justify your answer.

Example 7.4

Let $A = \begin{pmatrix} 1 & 0 & -1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 3 \\ -1 & 0 & 2 & -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$.

(a) Show that A and B has the same row space.

(b) Without performing any further computation, write down a basis for the row space of A and state its dimension.

Finally…

THE END