MORE ON COFACTOR EXPANSION AND DETERMINANTS

An example

$$\text{Evaluate} \begin{vmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{vmatrix}.$$

By cofactor expansion,

$$\begin{vmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{vmatrix} = (-1)\cdot(-1)^{1+1}\begin{vmatrix} 4 & 1 \\ 2 & -9 \end{vmatrix} + (3)\cdot(-1)^{1+2}\begin{vmatrix} 2 & 1 \\ -4 & -9 \end{vmatrix}$$

$$+(-4)\cdot(-1)^{1+3}\begin{vmatrix} 2 & 4 \\ -4 & 2 \end{vmatrix}$$

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$$+(-4)\cdot(-1)^{1+3}\begin{vmatrix} 2 & 4 \\ -4 & 2 \end{vmatrix}$$

$$= -(4\times -9 - 1\times 2) - 3(2\times -9 - 1\times -4)$$

$$-4(2\times 2 - 4\times -4)$$

$$= 0.$$

A way to remember (3x3 matrices)

What is the determinant of the following matrix?

$$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Answer:

$$\begin{array}{ccccc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}$$

$$\det(\boldsymbol{A}) = aei + bfg + cdh - ceg - afh - bdi$$

Verify this expression using cofactor expansion!

What about 4x4 matrices?

What is the determinant of the following matrix?

$$\boldsymbol{A} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}$$

Answer:

No 'special formula'! Use cofactor expansion!

Cofactor expansion along first row

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

$$\det(\boldsymbol{A}) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \dots + a_{1n}A_{1n} & \text{if } n \geq 2 \end{cases}$$

This is actually performing cofactor expansion <u>along the first row of $\boldsymbol{A}$.</u>

Theorem

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{pmatrix}$$

It turns out that we can compute det($\boldsymbol{A}$) by performing cofactor expansion along any row or any column of $\boldsymbol{A}$.

cofactor expansion

$$\det(\boldsymbol{A}) = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in}$$ along ith row

$$= a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj}$$ along jth column

Back to an earlier example

Check that $\begin{vmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{vmatrix} = 0$ by cofactor expansion

along second row.

$$\begin{vmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{vmatrix} = 2\cdot(-1)^{2+1}\begin{vmatrix} 3 & -4 \\ 2 & -9 \end{vmatrix} + 4\cdot(-1)^{2+2}\begin{vmatrix} -1 & -4 \\ -4 & -9 \end{vmatrix}$$

$$+1\cdot(-1)^{2+3}\begin{vmatrix} -1 & 3 \\ -4 & 2 \end{vmatrix}$$

Back to an earlier example

$$\begin{vmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{vmatrix} = 2 \cdot (-1)^{2+1} \begin{vmatrix} 3 & -4 \\ 2 & -9 \end{vmatrix} + 4 \cdot (-1)^{2+2} \begin{vmatrix} -1 & -4 \\ -4 & -9 \end{vmatrix}$$

$$+1 \cdot (-1)^{2+3} \begin{vmatrix} -1 & 3 \\ -4 & 2 \end{vmatrix}$$

$$= -2(-27+8) + 4(9-16) - 1(-2+12)$$

$$= 38 - 28 - 10$$

$$= 0$$

Try cofactor expansion along another row or column!

Determinant of special matrices

If $\boldsymbol{A}$ is a triangular matrix, then $\det(\boldsymbol{A})$ is the product of its diagonal entries.

$$\boldsymbol{A} = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 3 & & \\ & \text{\Huge 0} & & 0 & \\ & & & & 3 \end{pmatrix} \qquad \boldsymbol{A} = \begin{pmatrix} 2 & & & & \\ & 3 & & \text{\Huge 0} & \\ & & \frac{1}{2} & & \\ & & & -1 & \\ & & & & 4 \end{pmatrix}$$

$$\det(\boldsymbol{A}) = 0 \qquad \det(\boldsymbol{A}) = 2 \cdot 3 \cdot \tfrac{1}{2} \cdot -1 \cdot 4 = -12$$

Determinant of special matrices

If $\boldsymbol{A}$ is a square matrix, then

$$\boxed{\det(\boldsymbol{A}) = \det(\boldsymbol{A}^T).}$$

$$\begin{vmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{vmatrix} = 0 \qquad \begin{vmatrix} -1 & 2 & -4 \\ 3 & 4 & 2 \\ -4 & 1 & -9 \end{vmatrix} = 0$$

Determinant of special matrices

The determinant of a square matrix with two identical rows is 0.

The determinant of a square matrix with two identical columns is 0.

$$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 0 & 1 & -1 \\ 2 & -3 & 2 & 3 \\ 4 & 1 & 4 & 3 \\ -1 & 2 & -1 & 0 \end{vmatrix} = 0$$

Summary

1) A way to remember the formula for the determinant of a 3×3 matrix

2) Cofactor expansion along any row or column

3) Determinant of

   (a) triangular matrices;

   (b) a matrix and its transpose

   (c) matrices with identical rows or columns.