W03-06

Slide 01: In this unit we will introduce an important concept related to square matrices known as determinants.

Slide 02: Consider the following 2×2 matrix $\boldsymbol{A}$.

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If $ad - bc$ is not zero, define the following matrix $\boldsymbol{B}$. Note that we require $ad - bc$ to be non zero, for otherwise, the ratio $\frac{1}{ad-bc}$ will be undefined.

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Post-multiplying $\boldsymbol{B}$ to $\boldsymbol{A}$

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results in the following

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and after some simplification, we arrive at the identity matrix of order 2.

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As per discussed in a previous unit, it is not necessary to check pre-multiplication of $\boldsymbol{B}$ to $\boldsymbol{A}$. We are able to conclude that if $ad - bc$ is not zero, then the matrix $\boldsymbol{A}$ is invertible and $\boldsymbol{B}$ is the unique inverse of $\boldsymbol{A}$.

Slide 03: We will now show that the converse is also true, meaning that if $\boldsymbol{A}$ is invertible, then the expression $ad - bc$ must be non zero.

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If we manage to do so, we would have established the implication in both directions, namely that the 2×2 matrix $\boldsymbol{A}$ is invertible if and only if $ad - bc$ is not zero.

Slide 04: We will consider a few cases. For case 1, consider if both a and c are zero. In this case, the matrix simplifies to the following.

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Since the first column of the matrix is entirely zero, it is clear that this matrix will not have $\boldsymbol{I_2}$ as its reduced row-echelon form. Thus the matrix $\boldsymbol{A}$ in this case will not be invertible.

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As we are assuming the $\boldsymbol{A}$ is an invertible matrix, this case where a and c are both zero does not need to be considered any further.

Slide 05: Consider case 2, where at least one of a and c is not zero. First suppose a is not zero.

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We perform the following elementary row operation on $\boldsymbol{A}$. Note that this row operation is valid since we know that $\frac{c}{a}$ is well defined.

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The resulting matrix is shown here.

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By our assumption that a is not zero, the $(1,1)$-entry is definitely a leading entry. Since we assume that $\boldsymbol{A}$ is invertible, the $(2,2)$-entry highlighted must also be a leading entry and therefore cannot be zero. Thus we arrive at the conclusion that $ad - bc$ is non zero as desired.

Slide 06: Continuing with case 2, we now suppose $a = 0$ and $c \neq 0$. The matrix $\boldsymbol{A}$ simplifies to the following.

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Perform the row swap between rows 1 and 2,

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we have the following matrix.

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Once again, since we assume that $\boldsymbol{A}$ is invertible, we must have two leading entries in row-echelon form. Since $c \neq 0$ we know that the $(1, 1)$-entry is a leading entry. The $(2, 2)$-entry, namely b must also be non zero.

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For this case, since we are assuming that a is zero, the expression $ad - bc$ reduces to just $-bc$ and this will be non zero since b and c are both non zero.

Slide 07: We have thus established the result that the 2×2 matrix $\boldsymbol{A}$ is invertible if and only if $ad - bc$ is not zero.

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This quantity $ad - bc$ is known as the determinant of the 2×2 matrix.

Slide 08: Let us define the determinant of a square matrix formally. Let $\boldsymbol{A}$ be a square matrix of order n with entries denoted by a_{ij}. Define the matrix M_{ij} to be a square matrix obtained from $\boldsymbol{A}$ by removing the ith row and jth column from $\boldsymbol{A}$.

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For example, if $\boldsymbol{A}$ is the following 4×4 matrix, then

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$\boldsymbol{M_{11}}$ is the 3×3 matrix as shown, obtained when the first row and first column of $\boldsymbol{A}$ is removed.

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Similarly, we can have the matrix $\boldsymbol{M_{32}}$.

Slide 09: Let us return to the definition of the determinant of $\boldsymbol{A}$ as follows. If $\boldsymbol{A}$ is just a 1×1 matrix, then the determinant of $\boldsymbol{A}$ is just the one and only entry a_{11} in the matrix. If n is at least 2, the determinant of $\boldsymbol{A}$ is given by the expression as shown. Notice that in this expression,

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the small a_{ij}, namely a_{11}, a_{12} and so on are just the entries you find in the first row of the matrix $\boldsymbol{A}$.

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While the A_{ij}, namely A_{11}, A_{12} and so on is defined to be $(-1)^{i+j}$ multiplied by the determinant of the matrix M_{ij} where M_{ij} is a $(n-1) \times (n-1)$ matrix defined in the previous slide. Notice the $(-1)^{i+j}$ is either $+1$ or -1 depending on whether $i + j$ is even or odd.

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This A_{ij} is called the (i,j)-cofactor of the matrix $\boldsymbol{A}$. Thus the cofactors in the expression are precisely the $(1,j)$-cofactors of $\boldsymbol{A}$ where j ranges from 1 to n.

Slide 10: From this definition, you can see that to evaluate the determinant of a $n \times n$ matrix, we need to know how to compute the determinant of $(n-1)\times(n-1)$ matrices, since each $\boldsymbol{M_{ij}}$ is a $(n-1)\times(n-1)$ matrix.

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However, to evaluate the determinant of a $(n-1)\times(n-1)$ matrix, we need to know how to compute the determinant of $(n-2)\times(n-2)$ matrices.

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This means that the definition of the determinant of a matrix is recursive and at this point, you may find it to be something complicated to compute.

Slide 11: This definition of determinant, where the determinant of $\boldsymbol{A}$ is written in terms of its cofactors, is known as cofactor expansion.

Slide 12: The determinant of a matrix is usually denoted by using two vertical lines on either side of the matrix.

Slide 13: Let us use the same 2×2 matrix from the beginning of this unit and compute its determinant using cofactor expansion.

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In order to do so, we need to evaluate the $(1,1)$ and $(1,2)$ cofactors of $\boldsymbol{A}$. First the matrix $\boldsymbol{M_{11}}$ is simply the $(2,2)$-entry of $\boldsymbol{A}$, which is d. Thus the $(1,1)$-cofactor of $\boldsymbol{A}$, denoted by A_{11} is simply d.

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Similarly, the matrix $\boldsymbol{M_{12}}$ is the $(2,1)$-entry of $\boldsymbol{A}$, which is c. Thus the $(1,2)$-cofactor of $\boldsymbol{A}$, denoted by A_{12} is $-c$.

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By cofactor expansion, the determinant of $\boldsymbol{A}$ is the $(1,1)$-entry of $\boldsymbol{A}$ multiplied by the $(1,1)$-cofactor plus the $(1,2)$-entry of $\boldsymbol{A}$ multiplied by the $(1,2)$-cofactor. This gives $ad - bc$ which is consistent with what we have seen at the beginning of this unit.

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Thus, for a 2×2 matrix $\boldsymbol{A}$ we have shown that $\boldsymbol{A}$ is invertible if and only if the determinant of $\boldsymbol{A}$ is non zero. This result will be extended to all square matrices, not just those of order 2 in a subsequent unit.

Slide 13: To summarise the main points in this unit.

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We first established a necessary and sufficient condition for a 2×2 matrix to be invertible. This condition was given as $ad - bc$ not equal to zero.

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Next we defined the determinant of a square matrix in terms of cofactor expansion. It should be noted that this is a recursive definition.

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Finally, for the general 2×2 matrix $\boldsymbol{A}$, we now know that $\boldsymbol{A}$ is invertible if and only if the determinant of $\boldsymbol{A}$ is non zero.