ANSWERS TO MA1506 TUTORIAL 5

Question 1

Following the standard equations for the Malthus Model [Chapter 3]:

$$\begin{aligned}
&N = \hat{N}e^{kt}; N(0) = 10000 = \hat{N} \\
&N(2.5) = 10000e^{2.5k} = 11000 \\
&\Rightarrow e^{2.5k} = 1.1 \Rightarrow k = \frac{1}{2.5}\ell n(1.1) \\
&\qquad\qquad\qquad\quad = 0.0381 \\
&N(10) = 10000e^{10k} = 10000e^{10(0.0381)} \approx 14600 \\
&20000 = 10000e^{kt} \to t = \frac{1}{k}\ell n(2) \\
&\qquad\qquad\qquad\qquad\quad = 18.18 \text{ hours}
\end{aligned}$$

Question 2

The logistic equation has 3 kinds of solution, one increasing, one constant, and one decreasing. Since the number of bugs in this problem clearly increases, the relevant solution of the logistic equation is

$$N = \frac{N_\infty}{1 + \left(\frac{N_\infty}{\hat{N}} - 1\right)e^{-Bt}}$$

Here $\hat{N} = 200$, $B = 1.5$, so at $t = 2$ we have

$$\begin{aligned}
360 &= \frac{N_\infty}{1 + \left(\frac{N_\infty}{200} - 1\right)e^{-1.5\times 2}} \\
\Rightarrow 360 &+ \frac{360}{200}e^{-3}N_\infty - 360e^{-3} = N_\infty \\
N_\infty &= \frac{360(1 - e^{-3})}{1 - \frac{360}{200}e^{-3}} \approx 376 \\
N(3) &= \frac{N_\infty}{1 + \left(\frac{N_\infty}{200} - 1\right)e^{-4.5}} \approx 372
\end{aligned}$$

Question 3 First compare 80 with $\dfrac{B^2}{4s}$.

From Question 2 we know $B = 1.5$ and and $N_\infty = 376$, so $N_\infty = B/s \Rightarrow s = \frac{1.5}{376} \Rightarrow \frac{B^2}{4s} = 141$. This is the maximum number we can kill without causing extinction.

Setting $E = 80$,

$$\begin{matrix}\beta_1 \\ \beta_2\end{matrix} = \frac{B \mp \sqrt{B^2 - 4Es}}{2s} = \begin{matrix}64 \\ 312\end{matrix}.$$

Since the initial number of bugs was 200, which is between these two values, we see that the limiting number is $\beta_2 = 312$, since this is the stable equilibrium.

Question 4

We have $B_\infty = \frac{B}{s} = 194600$ so since $B = 0.09866, s = \frac{B}{N_\infty} = \frac{0.09866}{194600}$.
Maximum hunting rate is

$$\frac{B^2}{4s} = \frac{(0.09866)^2}{4 \times \frac{0.09866}{194600}} = 4800$$

Since $10000 > 4800$, birds are doomed.

Question 5

For the fish to survive a 10% downward fluctuation, we must have (in the extreme case)

$\beta_1 = 90\% \quad \beta_2$ i.e.

$$\begin{aligned}
\frac{B - \sqrt{B^2 - 4Es}}{2s} &= 0.9 \left[\frac{B + \sqrt{B^2 - 4Es}}{2s} \right] \\
B - \sqrt{} &= 0.9B + 0.9\sqrt{}
\end{aligned}$$

$$\begin{aligned}
0.1B &= 1.9\sqrt{} \\
0.01B^2 &= 3.61(B^2 - 4Es) = 3.61B^2 - 14.44Es \\
14.44E &= 3.6B^2/s \\
E &= 0.2493074\frac{B^2}{s} \\
&= 0.997 \times \left(\frac{B^2}{4s} \right)
\end{aligned}$$

So a less than 1% drop in the catch below E* will give a 10% margin of safety.

Question 6.

(a) We shall use the following s-Shifting property:

$L(f(t)) = F(s) \Rightarrow L(e^{ct}f(t)) = F(s-c)$

$\therefore \quad L(t^2) = \dfrac{2}{s^3} \Rightarrow$ use $L(t^n) = \dfrac{n!}{s^{n+1}}$

$\therefore \quad L(t^2e^{-3t}) = L(e^{-3t}t^2) = \dfrac{2}{(s+3)^3}$

(b) Here u denotes the Unit Step Function given by

$u(t-a) \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$

We shall use the following t-Shifting property:

$L(f(t)) = F(s) \Rightarrow L\{f(t-a)u(t-a)\} = e^{-as}F(s)$

$\text{Let} f(t-2) = t$

$\therefore \quad f(t) = t+2$

$\therefore \quad L(f(t)) = L(t+2) = L(t) + 2L(1)$

$= \dfrac{1}{s^2} + \dfrac{2}{s}$

$\therefore \quad L(tu(t-2)) = L\{f(t-2)u(t-2)\}$

$= e^{-2s}(\dfrac{1}{s^2} + \dfrac{2}{s})$

Question 7. (a)

$\dfrac{s}{s^2+10s+26} = \dfrac{s}{(s+5)^2+1} = \dfrac{(s+5)-5}{(s+5)^2+1}$

$\text{Let} F(s) = \dfrac{s-5}{s^2+1}$

$\therefore \quad L^{-1}(\dfrac{s}{s^2+10s+26}) = L^{-1}(F(s+5))$

$= L^{-1}(F(s-(-5))$

$= e^{-5t}L^{-1}(F(s)) \rightarrow$ use s-shifting

$= e^{-5t}L^{-1}(\dfrac{s}{s^2+1} - \dfrac{5}{s^2+1})$

$= e^{-5t}\{L^{-1}(\dfrac{s}{s^2+1}) - 5L^{-1}(\dfrac{1}{s^2+1})\}$

$= e^{-5t}(\text{cost} - 5\sin t)$

(b) Let $F(s) = \frac{1+2s}{s^3}$

$= \frac{1}{s^3} + \frac{2}{s^2}$

$\therefore \quad L^{-1}(F(s)) = \frac{t^2}{2} + 2t \quad \rightarrow \quad (\text{use } L(t^n) = \frac{n!}{s^{n+1}})$

Let $f(t) = \frac{t^2}{2} + 2t$

Using t-shifting,

$$L^{-1}(e^{-2s}\frac{1+2s}{s^3}) = L^{-1}(e^{-2s}F(s))$$
$$= f(t-2)u(t-2)$$
$$= \{\frac{(t-2)^2}{2} + 2(t-2)\}u(t-2)$$
$$= \frac{1}{2}(t^2-4)u(t-2)$$
$$= (\frac{1}{2}t^2-2)u(t-2)$$

Question 8. (a)

Let $L(y(t)) = Y(s)$

We shall use $L(y'(t)) = sY(s) - y(0)$.

We have

$$L(y') = L(tu(t-2))$$
$$\Rightarrow sY(s) - 4 = e^{-2s}(\frac{1}{s^2} + \frac{2}{s})$$
$$\Rightarrow Y(s) = e^{-2s}(\frac{1+2s}{s^3}) + \frac{4}{s}$$
$$\therefore \quad y(t) = L^{-1}(Y(s))$$
$$= L^{-1}\{e^{-2s}(\frac{1+2s}{s^3})\} + 4L^{-1}(\frac{1}{s})$$
$$= (\frac{1}{2}t^2-2)u(t-2) + 4$$

(by a previous question.)

(b) We shall use

$$L(y'') = s^2Y - sy(0) - y'(0)$$

We have

$$L(y'' - 2y') = L(4)$$

$$\Rightarrow s^2Y - sy(0) - y'(0) - 2\{sY - y(0)\} = \frac{4}{s}$$

$$\Rightarrow s^2Y - s - 2sY + 2 = \frac{4}{s}$$

$$\Rightarrow (s^2 - 2s)Y = \frac{4}{s} + s - 2 = \frac{4 + s^2 - 2s}{s}$$

$$\Rightarrow Y = \frac{s^2 - 2(s-2)}{s^2(s-2)}$$

$$= \frac{1}{s-2} - \frac{2}{s^2}$$

$$\therefore \quad y = L^{-1}(\frac{1}{s-2} - \frac{2}{s^2})$$

$$= e^{2t} - 2t$$

Question 9 Solution

Dividing the equation by $m_0 - \alpha t$ yields

$$\frac{dv}{dt} = -g + \frac{\alpha\beta}{m_0 - \alpha t}$$

$$\Rightarrow v(t) = \int_0^t \left(-g + \frac{\alpha\beta}{m_0 - \alpha s} \right) ds + v(0)$$

Thus,

$$v(t) = \left[-gs - \beta \ln(m_0 - \alpha s) \right] \Big|_{s=0}^{s=t} = -gt + \beta \ln \frac{m_0}{m_0 - \alpha t}$$

where we used the condition $0 \leq t < m_0/\alpha$ so that $m_0 - \alpha t > 0$.
Since the height h(t) of the rocket satisfies h(0) = 0, we find

$$\begin{aligned} h(t) &= \int_0^t v(s) ds = \int_0^t \left(-gs + \beta \ln \frac{m_0}{m_0 - \alpha s} \right) ds \\ &= \left[-\frac{gs^2}{2} + \beta s \ln m_0 + \frac{\beta}{\alpha}(m_0 - \alpha s) \ln \frac{m_0 - \alpha s}{e} \right]_{s=0}^{s=t} \\ &= \beta t - \frac{gt^2}{2} - \frac{\beta}{\alpha}(m_0 - \alpha t) \ln \frac{m_0}{m_0 - \alpha t} \end{aligned}$$

Question 10.

The cubic in N on the right side clearly passes through the origin, and its slope is negative immediately to the right of the origin, so it [and therefore dN/dt] is negative for small values of N; therefore the population always decreases when N is sufficiently small, which describes depensation. So we have the right kind of equation.

Roots of this cubic:

$$N = 0, N = [b \pm \sqrt{b^2 - 4ac}]/2a.$$

Since we are assuming $b^2 > 4ac$, both non-zero roots are real and, since $\sqrt{b^2 - 4ac} < b$, both are positive; the graph of the cubic goes down to a minimum [after passing through the origin], then up through the point $N = [b - \sqrt{b^2 - 4ac}]/2a$ on the N axis, reaching a maximum and then going down again to cut the N axis once more at $N = [b + \sqrt{b^2 - 4ac}]/2a$. So there are two non-zero equilibrium populations, one stable, one unstable, EVEN THOUGH THERE IS NO HARVESTING. The unstable equilibrium is at $N = [b - \sqrt{b^2 - 4ac}]/2a$; it is unstable because a population slightly above that will grow away from it, while a population slightly below it will drop towards zero. We see that the tigers will become extinct if the population ever falls below that value, EVEN IF WE DON'T HUNT THEM! So we have a good model of depensation.