W03-09

Slide 01: In this unit, we revisit the list of equivalent statements to $\boldsymbol{A}$ is invertible and add one more to the list stated in terms of determinants.

Slide 02: Let us recall the following result established in an earlier unit. If $\boldsymbol{A}$ is a square matrix, then the following 4 statements are equivalent, meaning that if one of them is true, then so are the others. Likewise if one of them is false, the rest will also be false.

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We will now add a fifth statement to the collection, namely that the determinant of $\boldsymbol{A}$ will be non zero.

Slide 03: To state it as a theorem, we will prove that a square matrix $\boldsymbol{A}$ is invertible if and only if the determinant of $\boldsymbol{A}$ is non zero.

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Recall that we actually have already establish the result for 2×2 matrices $\boldsymbol{A}$. More precisely, we have proven that $\boldsymbol{A}$ is invertible if and only if $ad - bc$, which is the determinant of $\boldsymbol{A}$, is non zero.

Slide 04: Suppose $\boldsymbol{A}$ is invertible.

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We will use the equivalence between the invertibilty of $\boldsymbol{A}$ and the fact that $\boldsymbol{A}$ can be expressed as a product of elementary matrices.

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Write $\boldsymbol{A}$ as a product of elementary matrices $\boldsymbol{E_1}$, $\boldsymbol{E_2}$ and so on until $\boldsymbol{E_k}$.

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So the determinant of $\boldsymbol{A}$ will be the determinant of the product of the elementary matrices.

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We now use a result established in an earlier unit, which is when an elementary matrix $\boldsymbol{E}$ is premultiplied to $\boldsymbol{A}$, the determinant of this product is equal to the determinant of $\boldsymbol{E}$ multiplied by the determinant of $\boldsymbol{A}$.

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We consider the product of the elementary matrices as the pre-multiplication of $\boldsymbol{E_1}$ to the rest of the elementary matrices.

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Applying the result, we now have the determinant of $\boldsymbol{A}$ equal to the determinant of $\boldsymbol{E_1}$ multiplied by the determinant of the product of the remaining elementary matrices.

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We now apply the same result on the term underlined in red.

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This would enable us to rewrite the right hand side as the determinant of $\boldsymbol{E_1}$ times the determinant of $\boldsymbol{E_2}$ times the determinant of the product of the remaining elementary matrices.

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Continue to apply the result until we arrive at the following.

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We now have the determinant of $\boldsymbol{A}$ to be equal to the product of the respective determinants of the k elementary matrices.

Slide 05: Now recall that an elementary matrix, depending on which type of elementary row operation that it represents, can have different values as its determinant. If it represents the elementary row operation cR_i, then its determinant will be c, where c is non zero. If it represents a row swap, then its determinant is -1 while if it represents the addition of a multiple of one row to another row, then its determinant will be 1.

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Regardless, the right hand side of the equation represents the product of k terms, all of which are non zero.

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This implies that the determinant of $\boldsymbol{A}$ is non zero and

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we have proven one implication, namely that if $\boldsymbol{A}$ is invertible, then the determinant of $\boldsymbol{A}$ will be non zero.

Slide 06: To prove the converse, we assume that $\boldsymbol{A}$ is singular and attempt to show that in this case, the determinant of $\boldsymbol{A}$ must be zero. Note that if we can establish this, we would have completed the proof that $\boldsymbol{A}$ is invertible if and only if determinant of $\boldsymbol{A}$ is non zero.

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Suppose $\boldsymbol{R}$ is the reduced row-echelon form of $\boldsymbol{A}$ which is obtained by performing a series of elementary row operations on $\boldsymbol{A}$. More precisely, let $\boldsymbol{E_1}$, $\boldsymbol{E_2}$ and so on until $\boldsymbol{E_k}$ represent the k elementary row operations performed on $\boldsymbol{A}$ in order to arrive at $\boldsymbol{R}$.

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Then the determinant of $\boldsymbol{R}$ will be equal to the determinant of the product of the matrices on the right.

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Once again, we apply the result like we did before on the product of matrices on the right, by considering the product of the matrices as $\boldsymbol{E_k}$ premultiplied to the remaining matrices.

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This allows us to rewrite the right hand side as determinant of $\boldsymbol{E_k}$ multiplied to the determinant of the product of the remaining matrices.

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By applying the same result repeatedly, the right hand side is now the product of the determinants of the k elementary matrices together with the determinant of $\boldsymbol{A}$,

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which is equal to the determinant of $\boldsymbol{R}$ on the left side.

Slide 07: Continuing on from the previous slide

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since we assume that the square matrix $\boldsymbol{A}$ is singular, $\boldsymbol{R}$, which is the reduced row-echelon form of $\boldsymbol{A}$ must have at least one row of zeros.

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This implies that the determinant of $\boldsymbol{R}$ must be zero, since we can simply perform cofactor expansion along the row of zeros.

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Once again, we observe that the determinant of all elementary matrices are non zero,

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while we have zero on the left hand side of the equation, this implies that the last term on the right hand side, namely determinant of $\boldsymbol{A}$ must be zero.

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We have now shown that if $\boldsymbol{A}$ is singular, then the determinant of $\boldsymbol{A}$ must be zero.

Slide 08: With this result, our collection of equivalent statements now has a total of 5 statements. Thus, the determinant of a square matrix tells us whether the matrix is invertible or singular.

Slide 09: To summarise,

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in this unit, we established the equivalence between $\boldsymbol{A}$ is invertible and the determinant of $\boldsymbol{A}$ being non zero.