EQUIVALENT STATEMENTS (PART II)

A set of equivalent statements

Recall the following:

If $\boldsymbol{A}$ is a square matrix, then the following statements are equivalent.

1) $\boldsymbol{A}$ is invertible.

2) $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.

3) The reduced row-echelon form of $\boldsymbol{A}$ is $\boldsymbol{I}$.

4) $\boldsymbol{A}$ can be expressed as a product of elementary matrices.

5) $\det(\boldsymbol{A}) \neq 0$

Theorem

A square matrix $\boldsymbol{A}$ is invertible if and only if $\det(\boldsymbol{A}) \neq 0$.

Recall that we have already proven this:

$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if

$$ad - bc = \det(\boldsymbol{A}) \neq 0.$$

Theorem

A square matrix $\boldsymbol{A}$ is invertible if and only if $\det(\boldsymbol{A}) \neq 0$.

Proof: First assume that $\boldsymbol{A}$ is invertible

$$\Rightarrow \boldsymbol{A} = \boldsymbol{E}_1 \boldsymbol{E}_2 \ldots \boldsymbol{E}_k$$

$$\Rightarrow \det(\boldsymbol{A}) = \det(\boldsymbol{E}_1 \boxed{\boldsymbol{E}_2 \ldots \boldsymbol{E}_k})$$

$$= \det(\boldsymbol{E}_1)\underline{\det(\boxed{\boldsymbol{E}_2 \ldots \boldsymbol{E}_k})}$$

$$= \det(\boldsymbol{E}_1)\det(\boldsymbol{E}_2)\det(\boldsymbol{E}_3 \ldots \boldsymbol{E}_k)$$

$$= \det(\boldsymbol{E}_1)\det(\boldsymbol{E}_2)\det(\boldsymbol{E}_3)\ldots\det(\boldsymbol{E}_k)$$

1) $\boldsymbol{A}$ is invertible.

$\equiv$

4) $\boldsymbol{A}$ can be expressed as a product of elementary matrices.

$$\det(\boldsymbol{E}\boldsymbol{A}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$$

$$\det(\boldsymbol{A}) = \det(\boldsymbol{E}_1)\det(\boldsymbol{E}_2)\det(\boldsymbol{E}_3)\ldots\det(\boldsymbol{E}_k)$$

Theorem

A square matrix $\boldsymbol{A}$ is invertible if and only if $\det(\boldsymbol{A}) \neq 0$.

Proof: First assume that $\boldsymbol{A}$ is invertible

$$\det(\boldsymbol{A}) = \underbrace{\det(\boldsymbol{E}_1)}_{\neq 0}\underbrace{\det(\boldsymbol{E}_2)}_{\neq 0}\underbrace{\det(\boldsymbol{E}_3)}_{\neq 0}\ldots\underbrace{\det(\boldsymbol{E}_k)}_{\neq 0}$$

$$\Rightarrow \det(\boldsymbol{A}) \neq 0$$

So we have shown that
if $\boldsymbol{A}$ is invertible, then $\det(\boldsymbol{A}) \neq 0$.

$$\det(\boldsymbol{E}) = \begin{cases} c & (c \neq 0,\ \text{first type}) \\ -1 & (\text{second type}) \\ 1 & (\text{third type}) \end{cases}$$

Theorem

A square matrix $\boldsymbol{A}$ is invertible if and only if $\det(\boldsymbol{A}) \neq 0$.

Proof: Now assume that $\boldsymbol{A}$ is singular.

We will show that $\det(\boldsymbol{A}) = 0$.

Let $\boldsymbol{R} = \boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A}$

$\Rightarrow \det(\boldsymbol{R}) = \det(\boldsymbol{E}_k \boxed{\boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A}})$

$= \det(\boldsymbol{E}_k)\det(\boxed{\boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1 \boldsymbol{A}})$

$= \det(\boldsymbol{E}_k)\det(\boldsymbol{E}_{k-1})\ldots\det(\boldsymbol{E}_1)\det(\boldsymbol{A})$

$\boldsymbol{R} =$ reduced row-echelon form of $\boldsymbol{A}$

$\boldsymbol{E}_i =$ elementary matrices

$\det(\boldsymbol{E}\boldsymbol{A}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$

$\det(\boldsymbol{R}) = \det(\boldsymbol{E}_k)\det(\boldsymbol{E}_{k-1})\ldots\det(\boldsymbol{E}_1)\det(\boldsymbol{A})$

Theorem

A square matrix $\boldsymbol{A}$ is invertible if and only if $\det(\boldsymbol{A}) \neq 0$.

Proof: Now assume that $\boldsymbol{A}$ is singular.

$$0 = \det(\boldsymbol{R}) = \underbrace{\det(\boldsymbol{E}_k)}_{\neq 0}\underbrace{\det(\boldsymbol{E}_{k-1})}_{\neq 0}\ldots\underbrace{\det(\boldsymbol{E}_1)}_{\neq 0}\underbrace{\det(\boldsymbol{A})}_{=0}$$

$\boldsymbol{A}$ is singular $\Rightarrow$ $\boldsymbol{R}$ has at least one row of zeros

$$\Rightarrow \det(\boldsymbol{R}) = 0 \Rightarrow \det(\boldsymbol{A}) = 0$$

So we have shown that if $\boldsymbol{A}$ is singular, then $\det(\boldsymbol{A}) = 0$.

$$\det(\boldsymbol{E}) = \begin{cases} c & (c \neq 0,\ \text{first type}) \\ -1 & (\text{second type}) \\ 1 & (\text{third type}) \end{cases}$$

One more equivalent statement

If $\boldsymbol{A}$ is a square matrix, then the following statements are equivalent.

1) $\boldsymbol{A}$ is invertible.

2) $\boldsymbol{Ax} = \mathbf{0}$ has only the trivial solution.

3) The reduced row-echelon form of $\boldsymbol{A}$ is $\boldsymbol{I}$.

4) $\boldsymbol{A}$ can be expressed as a product of elementary matrices.

5) $\det(\boldsymbol{A}) \neq 0$

Summary

1) Proved the equivalence between "$\boldsymbol{A}$ is invertible" and "$\det(\boldsymbol{A}) \neq 0$". We now have a collection of five equivalent statements.