Technique 1:
Separable Equations $y'=M(x)N(y)$
$\Rightarrow \int 1N(y)\,dy = \int M(x)\,dx$
Technique 2: Linear Change of Variable
Form: $y'=f(ax+by+c)$
If $b\neq 0$, the equation will be reduced to a separable form.
Strategy: Substitution
Let $u=ax+by+c$, simplifying the equation into a separable form:
$$\frac{du}{dx}=\frac{d}{dx}(ax+by+c)=a+b\frac{dy}{dx}$$
$$=a+bf(u)$$

Technique 3: Integrating Factor $\frac{dy}{dx}+P(x)y=Q(x)$

Step 1 Define **Integrating Factor**:
$R(x)=e^{\int^x P(s)ds}$, where $R'=RP$ (chain rule)
Step 2 We get $y=\frac{1}{R}(\int RQ\,dx+C)$.
Technique 4: Reduction to Linear Form
Bernoulli's Equation: $y'+p(x)y=q(x)yn$
Step 1 Let $z=z=\ y^{1-n}$ to get $\frac{dz}{dx}=\frac{(1-n)}{y^n}\frac{dy}{dx}$. Then, the DE is transformed into:
$$z'+(1-n)p(x)z=(1-n)q(x).$$
Step 2 Solve the first order linear differential equation using the integrating factor method.

Context	Differential Equation	Solution
Hot/Cold object left in environment	$\frac{dT}{dt}=k\lvert T-Tenv\rvert$	$T=Tenv+Ae^{kt}$
Radioactive Decay *x is the amount of substance.*	$\frac{dx}{dt}=-kx$	$x=Ae-kt,\ k=\frac{\ln 2}{t_{1/2}}$

Newton 2[nd] law: $F(v,t)=m\frac{dv}{dt},\quad F\left(x,t,\frac{dv}{dt}\right)=m\frac{d^2x}{dt^2}$

Decay application: $\frac{dU}{dt}=-k_uU,\ \frac{dT}{dt}=-k_TT+k_uU$

$$\frac{T}{U}=\frac{k_u}{k_t-k_u}(1-e^{(k_u+k_t)t})$$

- If $r(x)\equiv 0$, this is known as a **homogeneous DE**. Then, the general solution is given by: $y=C1y1+C2y2$
where $y1$ and $y2$ are linearly independent solutions to the DE.
- • If $r(x)\not\equiv 0$, the DE becomes **non-homogeneous**. Thus, the general solution is given by:
$$y=yh+yp=(C1y1+C2y2)+yp$$
where yh is the general solution to the homogeneous DE and yp is the particular solution satisfying the non-homogeneous DE.

- **Homogenous 2nd Order Linear ODEs with Constant Real Coefficients**:
$$y''+ay'+by=0,\qquad a,b\in\mathbb{R}$$
Step 1 Find out the characteristic equation $\lambda^2+a\lambda+b=0$. Then solve for λ.
Step 2 Choose case based on λ.

A: $\lambda 1,\lambda 2$ real and distinct	Case B: λ real, repeated	Case C: $\lambda=\alpha\pm\beta i$
G.S.: $y=C_1e^{\lambda_1x}+C_2e^{\lambda_2x}$	G.S.: $y=C_1e^{\lambda_1x}+C_2e^{\lambda_2x}$	G.S.: $y=C_1e^{\lambda_1x}\cos\beta x+C_2e^{\lambda_2x}\sin\beta x$

Method of Variation of Parameters: $y''+p(x)y'+q(x)y=r(x),\ r(x)\not\equiv 0$
Step 1 Find the general solution $yh=C1y1+C2y2$ to homogeneous DE $y''+ay'+by=0$.
Step 2 Find Wronskian $W(y1,y2)=y1y2'-y1'y2$.
Step 3 Use the formula to find u and v.
$y_p(x)=u(x)y_1(x)+v(x)y_2(x)$
$$u=\int\frac{y_2r}{y_1y_2'-y_1'y_2}dx,\qquad v=\int\frac{y_1r}{y_1y_2'-y_1'y_2}dx$$

Step 4 The general solution is given by $y=(C_1y_1+C_2y_2)+(uy_1+vy_2)=\ y_h+y_p$

Method of Undetermined Coefficients: $y''+ay'+by=r(x)$
Step 1 Find the general solution yh to homogeneous DE $y''+ay'+by=0$.
Step 2 Choose case based on λ.
Case A: $r(x)$ is a polynomial
Guess y_p to be a polynomial with unknown constant coefficients with the same highest power as the highest order in the DE.
Case B: $r(x)=P(x)e^{kx}$
Guess $y_p=ue^{kx}$, where u is a polynomial.
Case C: $(x)\equiv P(x)e^{\alpha x}\sin\beta x$ or $r(x)\equiv P(x)\,e^{\alpha x}\cos\beta x$
Guess $y_p=ue^{(\alpha+i\beta)x}$.
If $r(x)$ has $\sin\beta x$, then $yp=\mathrm{Im}\,ue^{(\alpha+i\beta)x}$..
If $r(x)$ has $\cos\beta x$, then $yp=\mathrm{Re}\,ue^{(\alpha+i\beta)x}$..

2: Harmonic Oscillator

$\ddot{x}+\omega^2x=0,\qquad \omega^2=\frac{k}{m}$
$x(t)=A\cos\omega t+B\sin\omega t$
$x(t)=\sqrt{A^2+B^2}\cos(\omega t+\psi),$
$x(t)=\sqrt{A^2+B^2}\sin(\omega t+\delta),$
$Amplitude=\sqrt{A^2+B^2},\qquad \delta=\psi\pm\frac{\pi}{2},\qquad phase$

$$\boxed{\begin{gathered}\omega\tau=2\pi\\ \tau=2\pi\sqrt{\frac{m}{k}}\end{gathered}}$$

Pendulum: $\theta''=-\omega^2\sin\theta$
Stable: $\theta=0, small\ \measuredangle:\sin\theta=\theta,\theta''=-\omega^2\theta$
Unstable: $\theta=\pi,\sin\theta=\sin(\pi-\theta),\theta''=-\omega^2(\theta-\pi)$
$$\theta=Ae^{\omega t}+Be^{-\omega t}+\pi$$

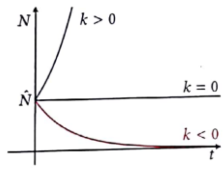
2.1: Damped Oscillation
$\ddot{x}+2\gamma\dot{x}+\omega^2x=0,\qquad \gamma=\frac{C}{2m},\qquad \omega^2=\frac{k}{m}$
$\lambda=\frac{-2\gamma\pm\sqrt{4\gamma^2-4\omega^2}}{2}$
$\lambda=-\gamma\pm\sqrt{\gamma^2-\omega^2}$
$if\ \omega^2<\gamma^2,\qquad C_1e^{\lambda_1x}+C_2e^{\lambda_2x}, over-damped$
$if\ \omega^2=\gamma^2,\qquad C_1e^{\lambda_1x}+C_2xe^{\lambda_2x}, critical\ damped$
$if\ \omega^2>\gamma^2,\qquad e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x), under-damped$

2.2: Forced Oscillation (undamped)
$\ddot{x}+\omega^2x=\frac{F_0}{m}\cos(\omega_0t),\qquad \omega^2=\frac{k}{m}$
$x=\frac{F_0}{m(\omega^2-\omega_0^2)}(\cos\omega_0t-\cos\omega t)$
Resonance: $\omega_0=\omega, x(t)=\frac{F_0}{2m\omega}t\sin(\omega t)$

Natural Frequency

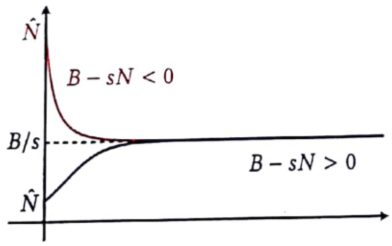
Driving Frequency

3: Population
3.1: Malthus model: $\frac{dN}{dt}=(B-D)N$
$N(t)=\hat{N}e^{(B-D)t},\ \hat{N}\ is\ pop\ at\ t=0$



$B>D$	*Population explodes*
$B=D$	*Population constant*
$B<D$	*Population collapse*

3.2: Logistic Model: $\frac{dN}{dt}=(B-sN)N$
$$N(t)=\frac{B}{s+\left(\frac{B}{\hat{N}}-s\right)e^{-Bt}}$$
$$N(t)=\frac{N_\infty}{1+\left(\frac{N_\infty}{\hat{N}}-1\right)e^{-Bt}},\quad N_\infty=\frac{B}{s}\ (carrying\ capacity)$$



$B-sN>0\Rightarrow B/s>\hat{N}$	$Pop\ is\ small\Rightarrow explosion\Rightarrow approach\ B/s$
$B-sN<0\Rightarrow B/s<\hat{N}$	$Over-pop\Rightarrow decline\Rightarrow approach\ B/s$
$B-sN=0\Rightarrow B/s=N$	$Pop\ remain\ constant$

3: Harvesting: $\frac{dN}{dt}=(B-D)N-E=-sN^2+BN-E$
$$N=\frac{B\mp\sqrt{B^2-4sE}}{2s}$$

$B^2-4sE<0$	No real sol[tn] $\Rightarrow$ Pop goes to extinction
$B^2-4sE=0$	1 real sol[tn] : Unstable equi $N=\beta=\frac{B}{2s}\Rightarrow$ if $\hat{N}<\beta$, pop goes to extinction
$B^2-4sE>0$	2 real sol[tn]: stable equi $\hat{N}=\beta_2$, Unstable equi $\hat{N}=\beta_1$, if $\hat{N}<\beta_1$, pop goes to extinction

4: Laplace: $\mathcal{L}[f(t)]=F(s)=\int_0^\infty e^{-st}f(t)dt$

Table of Laplace Transforms

$f(t)=\mathcal{L}^{-1}[F(s)]$	$F(s)=\mathcal{L}[f(s)]$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$
e^{at}	$\frac{1}{s-a}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
$t\cos\omega t$	$\frac{s^2-a^2}{s^2+\omega^2}$
$t\sin\omega t$	$\frac{2\omega s}{s^2+\omega^2}$
$f^{(n)}(t)$	$S^nf(s)-S^{n-1}f(0)-\cdots-f^{n-1}(0)$
y'	$sL(y)-y(0)$
y''	$s^2L(y)-sy(0)-y'(0)$
$\int_0^t f(\tau)\,d\tau$	$\frac{1}{s}\mathcal{L}[f(t)]$
$u(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t-a)$	e^{-as}
Frequency-Shifting (s-shifting): $L(e^{ct}f(t))=F(s-c)$	Time-shifting (t-shifting): $L^{-1}(e^{-as}F(s))=f(t-a)\cdot u(t-a)$
$L(e^{ct}t^n)=\frac{n!}{(s-c)^{n+1}}$	$L^{-1}\left(\frac{1}{(s-c)^n}\right)=\frac{e^{ct}t^{n-1}}{(n-1)!}$
$L(e^{ct}\cos\omega t)=\frac{s-c}{(s-c)^2+\omega^2}$	$L^{-1}\left(\frac{s-c}{(s-c)^2+\omega^2}\right)=e^{ct}\cos\omega t$
$L(e^{ct}\sin\omega t)=\frac{\omega}{(s-c)^2+\omega^2}$	$L^{-1}\left(\frac{\omega}{(s-c)^2+\omega^2}\right)=e^{ct}\sin\omega t$
$L(t^nf(t))=(-1)^nF^{(n)}(s)$	

1st Shifting Theorem: $\mathcal{L}[e^{at}f(t)] = F(s-a)$

2nd Shifting Theorem: : $\mathcal{L}[f(t-a)\,u(t-a)] = e^{-as}F(s)$

4.5: Unit Step (Heaviside) function

$$u(t-c) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

4.6: Dirac delta function

$$f_\mathcal{E}(t) = \begin{cases} 1/\mathcal{E}, & 0 \leq t \leq \mathcal{E} \\ 0, & t > \mathcal{E} \end{cases}$$

$$\int_{-\infty}^{\infty} f(t)\cdot\delta(t-c)dt = f(c)$$

5: PDE

Superposition principle:

$if\ u_1\ \&\ u_2\ are\ solutions\ of\ lin.hom.DE$

$then\ u = c_1u_1 + c_2u_2\ is\ also\ a\ sol^{tn}.$

5.2: Separation of Variables

1. $Suppose\ u(x,y) = X(x)Y(y)$
2. $Replace\ "u"\ with\ X, Y$
3. $Manipulate\ to\ separable\ form, equate\ to\ k$
4. $Solve\ ODE\ separately\ with\ k$
5. $Solution\ is\ u = XY$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

5.3 Wave Equations: $y_{tt} = C^2y_{xx}$ (D' Alembert's solution)

$$y(t,x) = \sum_{n=1}^{\infty} b_n \sin(nx)\cos(nct)\,dx$$

Boundary conditions(x): $u(t,0) = u(t,\pi) = 0$

Initial conditions(t): $u(0,x) = f(x)$

$u_t(0,x) = 0$

General solution: $y(t,x) = \frac{1}{2}[f(x+ct) + f(x-ct)]$

5.4 Heat Equation: $U_t = C^2U_{xx}$

$$u(x,t) = e^{c^2kt}\left(\alpha\cos x\sqrt{-k} + \beta\sin x\sqrt{-k}\right)$$

Boundary conditions(x): $u(t,0) = u(t,\ell) = 0$

Initial conditions(t): $u(0,x) = f(x)$

General Solution: $u(x,t) = e^{\frac{-C^2n^2\pi^2t}{\ell^2}}\beta_n\sin(\frac{n\pi}{\ell}x)$

$\int(ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)a} + C \quad (n \neq -1)$	$\int\csc^2(ax+b)dx = -\frac{1}{a}\cot(ax+b) + C$
$\int\frac{1}{ax+b}dx = \frac{1}{a}\ln\lvert ax+b\rvert + C$	$\int\csc(ax+b)\cdot\cot(ax+b)dx = -\frac{1}{a}\csc(ax+b) + C$
$\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$	$\int\frac{1}{a^2+(x+b)^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x+b}{a}\right) + C$
$\int\sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C$	$\int\frac{1}{\sqrt{a^2-(x+b)^2}}dx = \sin^{-1}\left(\frac{x+b}{a}\right) + C$
$\int\cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$	$\int\frac{-1}{\sqrt{a^2-(x+b)^2}}dx = \cos^{-1}\left(\frac{x+b}{a}\right) + C$
$\int\sec(ax+b)dx = \frac{1}{a}\ln\lvert\sec(ax+b) + \tan(ax+b)\rvert$	$\int\frac{1}{a^2-(x+b)^2}dx = \frac{1}{2a}\ln\left\lvert\frac{x+b+a}{x+b-a}\right\rvert + C$
$\int\csc(ax+b)dx = -\frac{1}{a}\ln\lvert\csc(ax+b) + \cot(ax+b)$	$\int\frac{1}{(x+b)^2-a^2}dx = \frac{1}{2a}\ln\left\lvert\frac{x+b-a}{x+b+a}\right\rvert + C$
$\int\cot(ax+b)dx = -\frac{1}{a}\ln\lvert\csc(ax+b)\rvert + C$	$\int\frac{1}{\sqrt{(x+b)^2+a^2}}dx = \ln\left\lvert(x+b) + \sqrt{(x+b)^2+a^2}\right\rvert + C$
$\int\tan(ax+b)dx = \frac{1}{a}\ln\lvert\sec(ax+b)\rvert + C$	$\int\frac{1}{\sqrt{(x+b)^2-a^2}}dx = \ln\left\lvert(x+b) + \sqrt{(x+b)^2-a^2}\right\rvert + C$
$\int\sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b) + C$	
$\int\sec(ax+b)\cdot\tan(ax+b)dx = \frac{1}{a}\sec(ax+b) + C$	
$\sinh x = \frac{e^x - e^{-x}}{2}$	$\frac{d}{dx}\ or \int\sinh x\,dx = \cosh x + C$
$\cosh x = \frac{e^x + e^{-x}}{2}$	$\frac{d}{dx}\ or \int\cosh x = \sinh x + C$
$\tanh x = \frac{\sinh x}{\cosh x}$	$\frac{d}{dx}\ or \int\tanh x = \ln(\cosh x) + C$
$\int\frac{1}{\sqrt{x^2+1}}dx = \sinh^{-1}x + c$	$\int\frac{1}{\sqrt{x^2-1}}dx = \cosh^{-1}x + c$
$\int\frac{1}{1-x^2}dx = \tanh^{-1}x + c$	$\int\ln x\,dx = x\ln x - x + C$

Integration by parts: $\int u\,dv = uv - \int v\,du$

$\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$

$\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$

$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

$\sin 2A \equiv 2\sin A\cos A$

$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - \sin^2 A$

$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$

$$\int xe^x dx = (x-1)e^x \quad (52)$$

$$\int xe^{ax}dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax} \quad (53)$$

$$\int x^2e^x dx = \left(x^2 - 2x + 2\right)e^x \quad (54)$$

$$\int x^2e^{ax}dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)e^{ax} \quad (55)$$

$$\int x^3e^x dx = \left(x^3 - 3x^2 + 6x - 6\right)e^x \quad (56)$$

$$\int x^ne^{ax}\,\mathrm{d}x = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,\mathrm{d}x \quad (57)$$

$$\int \ln ax\,dx = x\ln ax - x \quad (42)$$

$$\int\frac{\ln ax}{x}dx = \frac{1}{2}(\ln ax)^2 \quad (43)$$

$$\int\ln(ax+b)dx = \left(x + \frac{b}{a}\right)\ln(ax+b) - x, a \neq 0 \quad (44)$$

$$\int\ln(x^2+a^2)\,\mathrm{dx} = x\ln(x^2+a^2) + 2a\tan^{-1}\frac{x}{a} - 2x \quad (45)$$

$$\int\ln(x^2-a^2)\,\mathrm{dx} = x\ln(x^2-a^2) + a\ln\frac{x+a}{x-a} - 2x \quad (46)$$

$$\int\sin ax\,dx = -\frac{1}{a}\cos ax \quad (63)$$

$$\int\sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a} \quad (64)$$

$$\int\cos ax\,dx = \frac{1}{a}\sin ax \quad (67)$$

$$\int\cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a} \quad (68)$$

$$\int\cos^3 ax\,dx = \frac{3\sin ax}{4a} + \frac{\sin 3ax}{12a} \quad (70)$$

$$\int\sin^2 x\cos x\,dx = \frac{1}{3}\sin^3 x \quad (73)$$

$$\int\tan ax\,dx = -\frac{1}{a}\ln\cos ax \quad (78)$$

$$\int\cos^2 ax\sin ax\,dx = -\frac{1}{3a}\cos^3 ax \quad (75)$$

$$\int\tan^2 ax\,dx = -x + \frac{1}{a}\tan ax \quad (79)$$

$$\int\sec^3 x\,\mathrm{dx} = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln\lvert\sec x + \tan x\rvert \quad (84)$$

$$\int\sec x\tan x\,dx = \sec x \quad (85)$$

$$\int\sec^2 x\tan x\,dx = \frac{1}{2}\sec^2 x \quad (86)$$

$$\int\sec^n x\tan x\,dx = \frac{1}{n}\sec^n x, n \neq 0 \quad (87)$$

$$\int\csc^2 ax\,dx = -\frac{1}{a}\cot ax \quad (89)$$

$$\int\csc^3 x\,dx = -\frac{1}{2}\cot x\csc x + \frac{1}{2}\ln\lvert\csc x - \cot x\rvert \quad (90)$$

$$\int\sec x\,dx = \ln\lvert\sec x + \tan x\rvert = 2\tanh^{-1}\left(\tan\frac{x}{2}\right) \quad (82)$$

$$\int x\cos x\,dx = \cos x + x\sin x \quad (93)$$

$$\int x\cos ax\,dx = \frac{1}{a^2}\cos ax + \frac{x}{a}\sin ax \quad (94)$$

$$\int x^2\cos x\,dx = 2x\cos x + \left(x^2 - 2\right)\sin x \quad (95)$$

$$\int x^2\cos ax\,dx = \frac{2x\cos ax}{a^2} + \frac{a^2x^2 - 2}{a^3}\sin ax \quad (96)$$

$$\int x\sin x\,dx = -x\cos x + \sin x \quad (99)$$

$$\int x\sin ax\,dx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2} \quad (100)$$

$$\int x^2\sin x\,dx = \left(2 - x^2\right)\cos x + 2x\sin x \quad (101)$$

$$\int x^2\sin ax\,dx = \frac{2 - a^2x^2}{a^3}\cos ax + \frac{2x\sin ax}{a^2} \quad (102)$$

$$\int e^x\cos x\,dx = \frac{1}{2}e^x(\sin x + \cos x) \quad (106)$$

$$\int e^{bx}\cos ax\,dx = \frac{1}{a^2+b^2}e^{bx}(a\sin ax + b\cos ax) \quad (107)$$

$$\int xe^x\sin x\,dx = \frac{1}{2}e^x(\cos x - x\cos x + x\sin x) \quad (108)$$

$$\int xe^x\cos x\,dx = \frac{1}{2}e^x(x\cos x - \sin x + x\sin x) \quad (109)$$

Trigonometric Identities for Integration

$\tan^2 A = \sec^2 A - 1$

$\cot^2 A = \csc^2 A - 1$

$\sin A\cos A = \frac{1}{2}\sin 2A$

$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

$\sin A\cdot\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$\cos A\cdot\sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$

$\cos A\cdot\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$\sin A\cdot\sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$