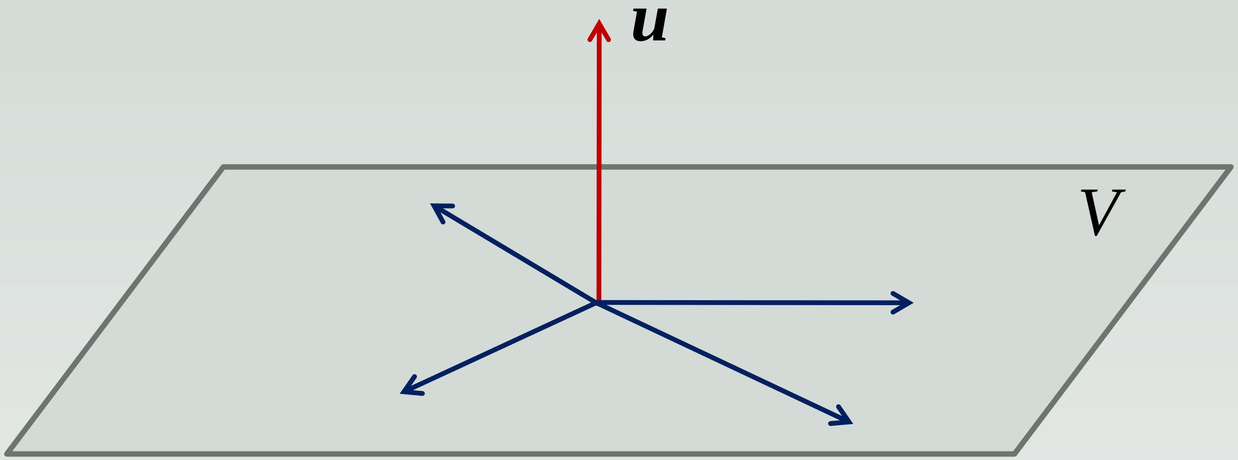
ORTHOGONAL PROJECTION

DEFINITION (VECTOR ORTHOGONAL TO A SPACE)

Let V be a subspace of $\mathbb{R}^n$. A vector $\boldsymbol{u}$ is orthogonal (or perpendicular) to V if $\boldsymbol{u}$ is orthogonal to all vectors in V.



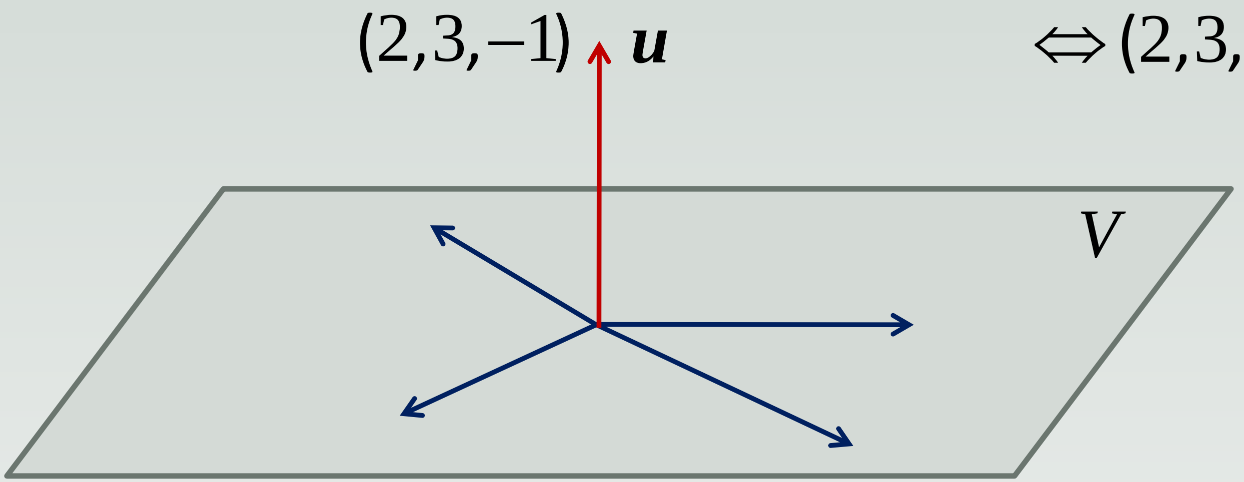
EXAMPLE (VECTOR ORTHOGONAL TO A SPACE)

$V = \{(x, y, z) \mid 2x + 3y - z = 0\}$ is a subspace of $\mathbb{R}^3$

(it is a plane in $\mathbb{R}^3$ containing the origin).

$$(x, y, z) \in V \Leftrightarrow 2x + 3y - z = 0$$

$$\Leftrightarrow (2,3,-1) \cdot (x, y, z) = 0$$



$(2,3,-1)$ is orthogonal to all vectors in V.

So $(2,3,-1)$ is orthogonal to V.

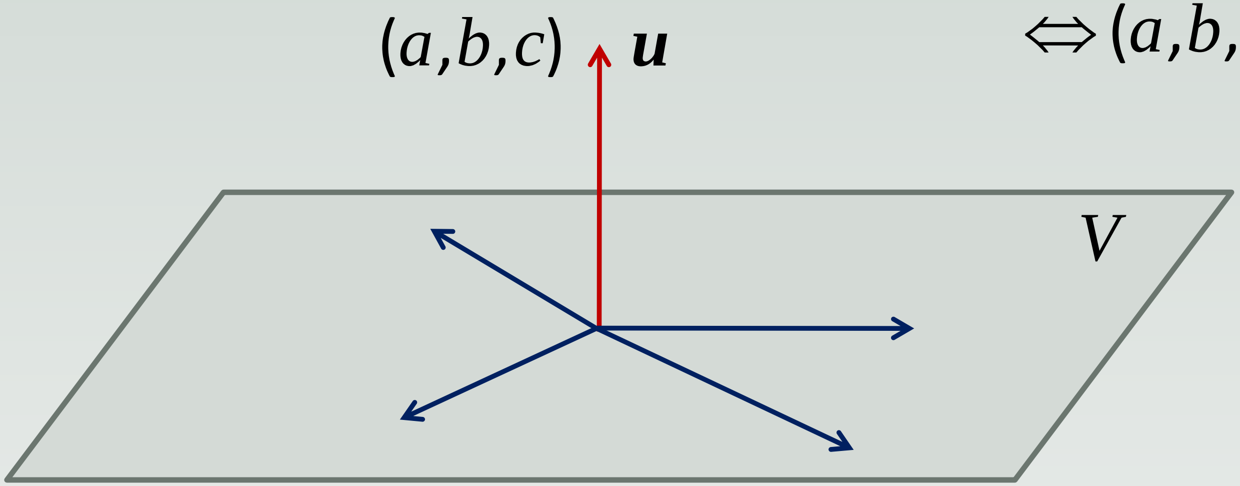
EXAMPLE (VECTOR ORTHOGONAL TO A SPACE)

$V = \{(x, y, z) \mid ax + by + cz = 0\}$ is a subspace of $\mathbb{R}^3$

(it is a plane in $\mathbb{R}^3$ containing the origin).

$$(x, y, z) \in V \Leftrightarrow ax + by + cz = 0$$

$$\Leftrightarrow (a, b, c) \cdot (x, y, z) = 0$$



(a, b, c) is orthogonal to all vectors in V.

So (a, b, c) is orthogonal to V.

EXAMPLE (VECTOR ORTHOGONAL TO A SPACE)

$V = \{(x,y,z) \mid ax+by+cz=0\}$ is a subspace of $\mathbb{R}^3$

Let $\boldsymbol{n} = (a,b,c)$.

$$V = \{(x,y,z) \mid ax+by+cz=0\}$$

$$= \{\boldsymbol{u} \in \mathbb{R}^3 \mid \boldsymbol{n}\cdot\boldsymbol{u} = 0\}$$

$\boldsymbol{n}$ is orthogonal to V. We say that $\boldsymbol{n}$ is a normal vector of V.

Question: If $\boldsymbol{n}$ is orthogonal to V, is $c\boldsymbol{n}$ orthogonal to V for every $c \neq 0$?

YES!

$$\boldsymbol{n}\cdot\boldsymbol{u} = 0 \Leftrightarrow (c\boldsymbol{u})\cdot\boldsymbol{u} = 0 \quad (c \neq 0)$$

EXAMPLE (VECTOR ORTHOGONAL TO A SPACE)

$V = \text{span}\{(1,1,-1,0),(0,1,1,-1)\}$ is a subspace of $\mathbb{R}^4$.

Question: Find all vectors orthogonal to V.

Remember:

$\boldsymbol{v} = (w,x,y,z)$ is orthogonal to V

$\Leftrightarrow$

$\boldsymbol{v} = (w,x,y,z)$ is orthogonal to every vector in V.

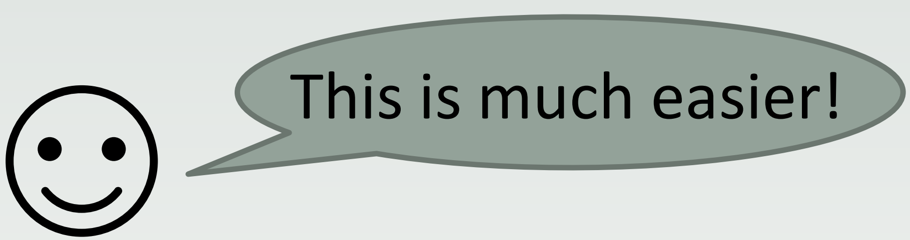
Wow! Isn't this very difficult to check??

EXAMPLE (VECTOR ORTHOGONAL TO A SPACE)

$V = \text{span}\{(1,1,-1,0),(0,1,1,-1)\}$ is a subspace of $\mathbb{R}^4$.

Question: Find all vectors orthogonal to V.

In general, if $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$, then
$\boldsymbol{v}$ is orthogonal to V if and only if $\boldsymbol{v}$ is orthogonal to
$\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$, that is, $\boldsymbol{v} \cdot \boldsymbol{u}_i = 0$ for all $i = 1, \ldots, k$.



EXAMPLE (VECTOR ORTHOGONAL TO A SPACE)

$V = \text{span}\{(1,1,-1,0),(0,1,1,-1)\}$ is a subspace of $\mathbb{R}^4$.

Question: Find all vectors orthogonal to V.

$\boldsymbol{u}$ is orthogonal to $V \Leftrightarrow \boldsymbol{u}$ is orthogonal to the vectors that spans V

Let $\boldsymbol{u} = (w, x, y, z)$.

$(w,x,y,z)\cdot(1,1,-1,0)=0$ and $(w,x,y,z)\cdot(0,1,1,-1)=0$

$$\begin{cases} w + x - y \phantom{{}-z} = 0 \\ \phantom{w+{}} x + y - z = 0 \end{cases}$$

EXAMPLE (VECTOR ORTHOGONAL TO A SPACE)

$V = \text{span}\{(1,1,-1,0),(0,1,1,-1)\}$ is a subspace of $\mathbb{R}^4$.

Question: Find all vectors orthogonal to V.

$$\begin{cases} w + x - y \phantom{{}-z} = 0 \\ \phantom{w+{}} x + y - z = 0 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right)$$

$$\begin{cases} w = 2s - t \\ x = -s + t \\ y = s \\ z = t, \quad s,t \in \mathbb{R} \end{cases}$$

EXAMPLE (VECTOR ORTHOGONAL TO A SPACE)

$V = \text{span}\{(1,1,-1,0),(0,1,1,-1)\}$ is a subspace of $\mathbb{R}^4$.

Question: Find all vectors orthogonal to V.

$\boldsymbol{u}$ is orthogonal to $V \Leftrightarrow \boldsymbol{u}$ is orthogonal to the vectors that spans V

So $\boldsymbol{u}$ is orthogonal to V if and only if $\boldsymbol{u} = s\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

EXAMPLE (VECTOR ORTHOGONAL TO A SPACE)

$V = \text{span}\{(1,1,-1,0),(0,1,1,-1)\}$ is a subspace of $\mathbb{R}^4$.

Question: Find all vectors orthogonal to V.

$\boldsymbol{u}$ is orthogonal to $V \Leftrightarrow \boldsymbol{u}$ is orthogonal to the vectors that spans V

So $\boldsymbol{u}$ is orthogonal to V if and only if $\boldsymbol{u} \in \text{span}\left\{\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right\}$.

DEFINITION (ORTHOGONAL PROJECTION)

Let V be a subspace of $\mathbb{R}^n$.

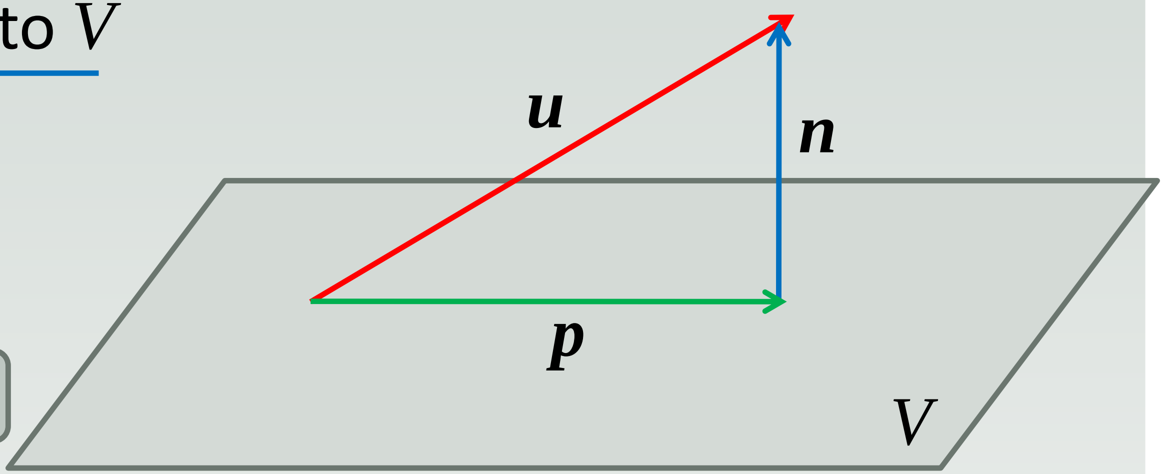
Every vector $\boldsymbol{u} \in \mathbb{R}^n$ can be uniquely written as

$$\boldsymbol{u} = \boldsymbol{n} + \boldsymbol{p}$$

where $\boldsymbol{n}$ is orthogonal to V

and $\boldsymbol{p}$ belongs to V.

$\boldsymbol{p}$ is called the (orthogonal) projection of $\boldsymbol{u}$ onto V.



WAIT A MINUTE…

Let V be a subspace of $\mathbb{R}^n$.

Every vector $\boldsymbol{u} \in \mathbb{R}^n$ can be uniquely written as

$$\boxed{\boldsymbol{u}} = \boxed{\boldsymbol{n}} + \boxed{\boldsymbol{p}}$$

where $\boldsymbol{n}$ is orthogonal to V

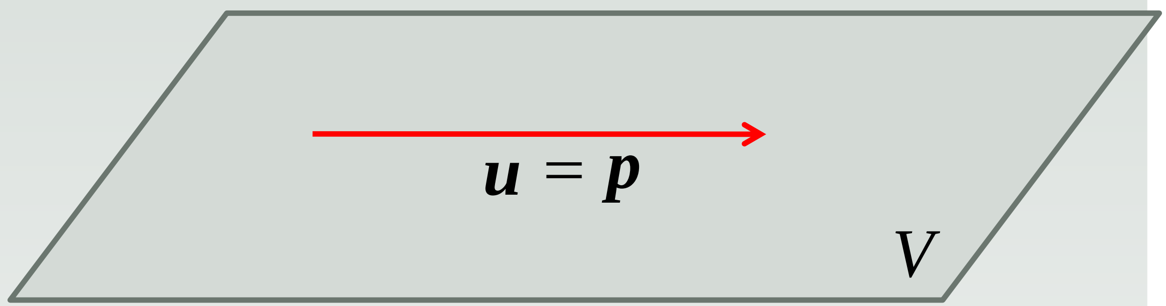
and $\boldsymbol{p}$ belongs to V.

$$\boldsymbol{n} = \mathbf{0}$$

$$\boxed{\boldsymbol{u}} = \boxed{\mathbf{0}} + \boxed{\boldsymbol{u}}$$

What happens if

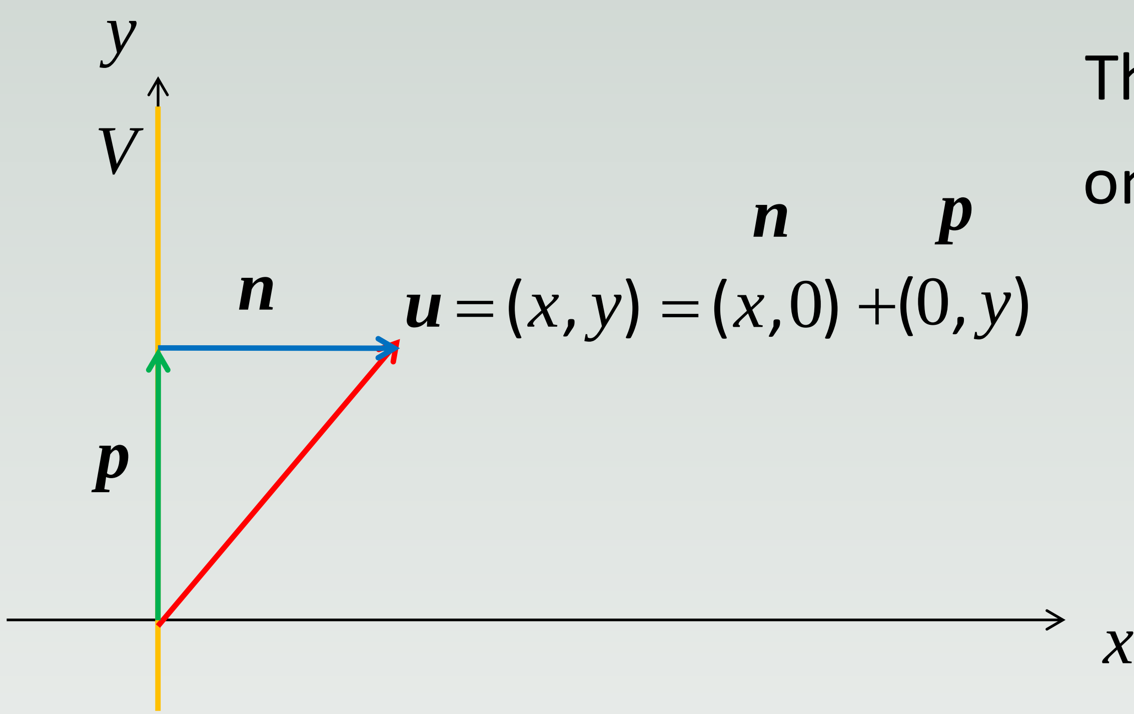
$\boldsymbol{u}$ belongs to V?



PROJECTION IN R2 AND R3

Let V be a subspace of $\mathbb{R}^2$.

$$V = \{(0, y) \mid y \in \mathbb{R}\} \text{ (} V \text{ is the } y\text{-axis)}$$



$$\boldsymbol{u} = (x, y) = \overset{\boldsymbol{n}}{(x, 0)} + \overset{\boldsymbol{p}}{(0, y)}$$

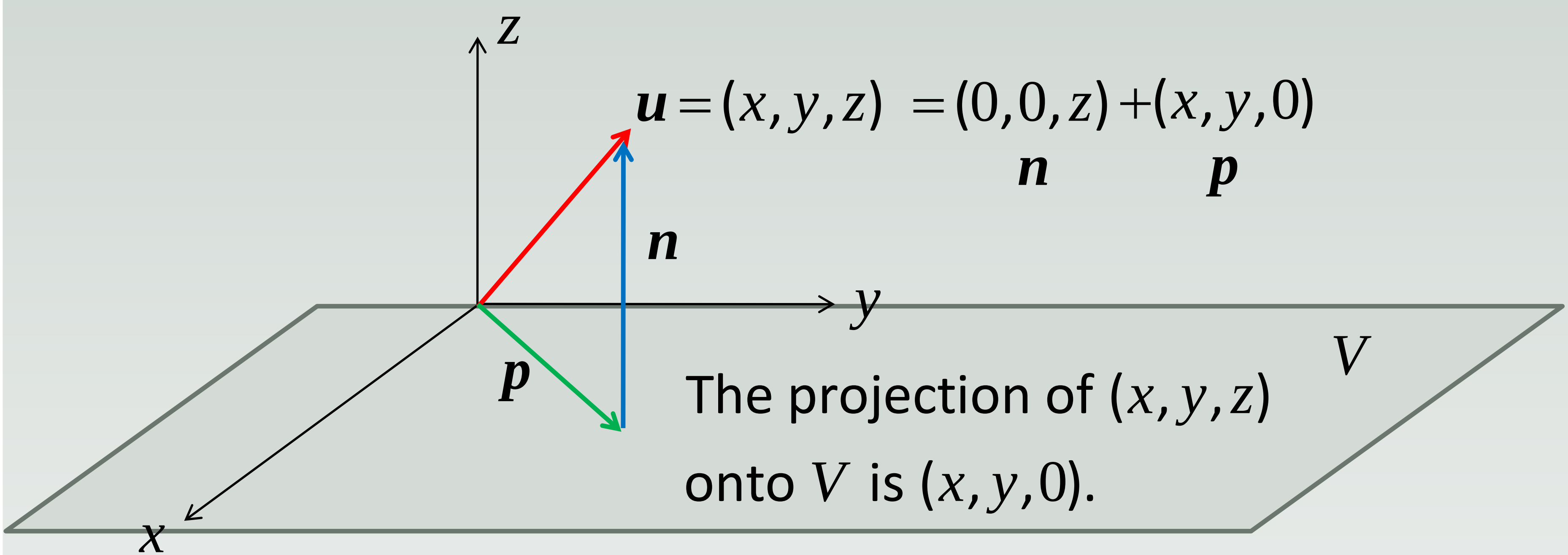
The projection of (x, y) onto V is $(0, y)$.

PROJECTION IN R^2 AND R^3

Let V be a subspace of $\mathbb{R}^3$.

$$V = \{(x, y, 0) \mid x, y \in \mathbb{R}\} \text{ (}V\text{ is the }xy\text{-plane)}$$

$$\boldsymbol{u} = (x, y, z) = \underbrace{(0,0,z)}_{\boldsymbol{n}} + \underbrace{(x,y,0)}_{\boldsymbol{p}}$$



The projection of (x, y, z) onto V is $(x, y, 0)$.

WAIT A(NOTHER) MINUTE…

Let V be a subspace of $\mathbb{R}^2$.

$$V = \{(0, y) \mid y \in \mathbb{R}\}$$ (V is the y-axis)

Let V be a subspace of $\mathbb{R}^3$.

$$V = \{(x, y, 0) \mid x, y \in \mathbb{R}\}$$ (V is the xy-plane)

Questions:

What if the subspaces are not so 'trivial'?

How can we compute orthgonal projections in general?

What is required?

SUMMARY

1) What it means for a vector to be orthogonal to
   a vector space.

2) Definition of orthogonal projection onto a vector space.