ORTHOGONAL AND ORTHONORMAL BASES

RECALL

1) Two vectors $\boldsymbol{u}, \boldsymbol{v}$ are said to be orthogonal if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

2) A set S of vectors in $\mathbb{R}^n$ is said to be orthogonal if every pair of distinct vectors in S are orthogonal.

$$S = \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, \boldsymbol{x}\} \qquad \boldsymbol{u} \cdot \boldsymbol{v} = 0,\ \boldsymbol{u} \cdot \boldsymbol{w} = 0,\ \boldsymbol{u} \cdot \boldsymbol{x} = 0$$

$$\boldsymbol{v} \cdot \boldsymbol{w} = 0,\ \boldsymbol{v} \cdot \boldsymbol{x} = 0,\ \boldsymbol{w} \cdot \boldsymbol{x} = 0$$

3) A set S of vectors in $\mathbb{R}^n$ is said to be orthonormal if S is orthogonal and every vector in S is a unit vector.

EXAMPLE (AN ORTHONORMAL SET)

The standard basis for $\mathbb{R}^n$, $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n\}$ is an orthonormal set.

$$\boldsymbol{e}_1 = (1,0,\ldots,0),\ \boldsymbol{e}_2 = (0,1,\ldots,0), \ldots,\ \boldsymbol{e}_n = (0,0,\ldots,1)$$

1) orthogonal: $\boldsymbol{e}_i \cdot \boldsymbol{e}_j = 0$ if $i \neq j$;

2) unit vectors: $\|\boldsymbol{e}_i\| = \sqrt{0^2 + \ldots + 1^2 + \ldots + 0^2} = 1.$

THEOREM

Let S be an orthogonal set of non zero vectors in a vector space. Then S is a linearly independent set.

Proof: Let $S=\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be an orthogonal set of non zero vectors.

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k = \mathbf{0} \qquad (*)$$

$$\Rightarrow \boldsymbol{u}_i \cdot (c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k) = \boldsymbol{u}_i \cdot \mathbf{0} \quad \text{for all } i = 1, \ldots, k$$

$$\Rightarrow c_i(\boldsymbol{u}_i \cdot \boldsymbol{u}_i) = 0 \quad (\boldsymbol{u}_i \cdot \boldsymbol{u}_j = 0 \text{ if } i \neq j)$$

$$\Rightarrow c_i = 0 \quad (\boldsymbol{u}_i \cdot \boldsymbol{u}_i \neq 0 \text{ since } \boldsymbol{u}_i \neq \mathbf{0})$$

DEFINITION (ORTHOGONAL AND ORTHONORMAL BASES)

1) A basis S for a vector space is an orthogonal basis if S is an orthogonal set.

2) A basis S for a vector space is an orthonormal basis if S is an orthonormal set.

AN EASY WAY TO CHECK ORTHOGONAL BASIS

If we know that a vector space V has dimension k and

1) S is an orthogonal set of non zero vectors in V (all vectors in S belong to V);

> Let S be an orthogonal set of non zero vectors in a vector space. Then S is a linearly independent set.

2) $|S| = k$;

Then we can conclude that S is an orthogonal basis for V.

EXAMPLE (ORTHOGONAL AND ORTHONORMAL BASES)

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ where

$$\boldsymbol{u}_1 = (2,0,0); \qquad \boldsymbol{u}_2 = (0,1,1); \qquad \boldsymbol{u}_3 = (0,1,-1).$$

S is an orthogonal set of 3 non zero vectors in $\mathbb{R}^3$.

S is an orthogonal basis for $\mathbb{R}^3$.

$\dim(\mathbb{R}^3) = 3$

Let $S' = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ where

$$\boldsymbol{v}_1 = \frac{\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|}; \qquad \boldsymbol{v}_2 = \frac{\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|}; \qquad \boldsymbol{v}_3 = \frac{\boldsymbol{u}_3}{\|\boldsymbol{u}_3\|}.$$

S' is an orthonormal basis for $\mathbb{R}^3$.

HOW WOULD YOU ANSWER THIS QUESTION?

Assume that you already know that $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\}$ where

$$\boldsymbol{u}_1=(1,2,2,-1),\ \boldsymbol{u}_2=(1,1,-1,1),\ \boldsymbol{u}_3=(-1,1,-1,-1)$$

is a basis for a subspace V of $\mathbb{R}^4$.

Give a vector $\boldsymbol{w}=(w_1,w_2,w_3,w_4)\in V$, how do we write $\boldsymbol{w}$ as a linear combination of $\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3$?

$$a\boldsymbol{u}_1+b\boldsymbol{u}_2+c\boldsymbol{u}_3=\boldsymbol{w}$$

$$\Rightarrow a(1,2,2,-1)+b(1,1,-1,1)+c(-1,1,-1,-1)=(w_1,w_2,w_3,w_4)$$

HOW WOULD YOU ANSWER THIS QUESTION?

$$\Rightarrow a(1,2,2,-1)+b(1,1,-1,1)+c(-1,1,-1,-1)=(w_1,w_2,w_3,w_4)$$

$$\Rightarrow \begin{cases} a & + & b & - & c & = & w_1 \\ 2a & + & b & + & c & = & w_2 \\ 2a & - & b & - & c & = & w_3 \\ -a & + & b & - & c & = & w_4 \end{cases}$$

We now solve this linear system for a, b, c.

HOW WOULD YOU ANSWER THIS QUESTION?

Assume that you already know that $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ where

$$\boldsymbol{u}_1 = (1,2,2,-1),\ \boldsymbol{u}_2 = (1,1,-1,1),\ \boldsymbol{u}_3 = (-1,1,-1,-1)$$

is a basis for a subspace V of $\mathbb{R}^4$.

Give a vector $\boldsymbol{w} = (w_1, w_2, w_3, w_4) \in V$, how do we write $\boldsymbol{w}$ as a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$?

What if we know further that S is an orthogonal basis?
(is it?)

THEOREM

1) If $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ is an orthogonal basis for a vector space V, then for any vector $\boldsymbol{w} \in V$,

$$\boldsymbol{w} = \left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|^2}\right)\boldsymbol{u}_1 + \left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|^2}\right)\boldsymbol{u}_2 + \ldots + \left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_k}{\|\boldsymbol{u}_k\|^2}\right)\boldsymbol{u}_k$$

WOW!

No need to solve a linear system for these coefficients!

THEOREM

1) If $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}$ is an orthogonal basis for a vector space V, then for any vector $\boldsymbol{w}\in V$,

$$\boldsymbol{w}=\left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|^2}\right)\boldsymbol{u}_1+\left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|^2}\right)\boldsymbol{u}_2+\ldots+\left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_k}{\|\boldsymbol{u}_k\|^2}\right)\boldsymbol{u}_k$$

So $(\boldsymbol{w})_S=\left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|^2},\frac{\boldsymbol{w}\cdot\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|^2},\ldots,\frac{\boldsymbol{w}\cdot\boldsymbol{u}_k}{\|\boldsymbol{u}_k\|^2}\right)$ (coordinate vector of $\boldsymbol{w}$ with respect to basis S).

THEOREM

2) If $T = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ is an orthonormal basis for a vector space V, then for any vector $\boldsymbol{w} \in V$,

$$\boldsymbol{w} = (\boldsymbol{w} \cdot \boldsymbol{v}_1)\boldsymbol{v}_1 + (\boldsymbol{w} \cdot \boldsymbol{v}_2)\boldsymbol{v}_2 + \ldots + (\boldsymbol{w} \cdot \boldsymbol{v}_k)\boldsymbol{v}_k$$

So $(\boldsymbol{w})_T = (\boldsymbol{w} \cdot \boldsymbol{v}_1, \boldsymbol{w} \cdot \boldsymbol{v}_2, \ldots \boldsymbol{w} \cdot \boldsymbol{v}_k)$

EXAMPLE (USING ORTHOGONAL BASIS)

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ where

$$\boldsymbol{u}_1 = (2,0,0); \qquad \boldsymbol{u}_2 = (0,1,1); \qquad \boldsymbol{u}_3 = (0,1,-1).$$

S is an orthogonal basis for $\mathbb{R}^3$.

Express $\boldsymbol{w} = (1,2,3)$ as a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$.

$$\frac{\boldsymbol{w} \cdot \boldsymbol{u}_1}{\|\boldsymbol{u}_1\|^2} = \frac{2}{4} = \boxed{\frac{1}{2}} \qquad \frac{\boldsymbol{w} \cdot \boldsymbol{u}_2}{\|\boldsymbol{u}_2\|^2} = \boxed{\frac{5}{2}} \qquad \frac{\boldsymbol{w} \cdot \boldsymbol{u}_3}{\|\boldsymbol{u}_3\|^2} = \boxed{\frac{-1}{2}}$$

Thus $\boldsymbol{w} = \boxed{\frac{1}{2}}\boldsymbol{u}_1 + \boxed{\frac{5}{2}}\boldsymbol{u}_2 \boxed{-\frac{1}{2}}\boldsymbol{u}_3.$ $\qquad (\boldsymbol{w})_S = (\tfrac{1}{2}, \tfrac{5}{2}, -\tfrac{1}{2})$

SUMMARY

1) An orthogonal set (of non zero) vectors is a linearly independent set.

2) Orthogonal basis and orthonormal basis.

3) Writing linear combinations in terms of orthogonal basis vectors.