Unit 057 One more equivalent statement; more on eigenvalues

Slide 01: In this unit, we continue to discuss eigenvalues and add one more equivalent statement to our collection of statements equivalent to "$\boldsymbol{A}$ is invertible".

Slide 02: Recall that up to this point, we have a collection of 8 equivalent statements as shown. So it does seem that knowing that $\boldsymbol{A}$ is invertible is knowing a lot of other information about the matrix $\boldsymbol{A}$. We are now ready to add one more to this collection.

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The ninth statement in this collection is the statement that 0 is **not** an eigenvalue of $\boldsymbol{A}$.

Slide 03: To prove this, we will show that 0 is not an eigenvalue of $\boldsymbol{A}$ is equivalent to the statement $\det(\boldsymbol{A}) \neq 0$, which is one of the statement in the existing collection.

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By our understanding on characteristic equations, 0 is not an eigenvalue of $\boldsymbol{A}$ would be equivalent to the fact that $\det(0\boldsymbol{I} - \boldsymbol{A})$ is not zero.

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$0\boldsymbol{I}$ is just the zero matrix so this is equivalent to $\det(-\boldsymbol{A}) \neq 0$. Now how is the determinant of $\boldsymbol{A}$ related to the determinant of $-\boldsymbol{A}$?

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They are related by a factor of $(-1)^n$ as $\boldsymbol{A}$ is a square matrix of order n and thus we have the equivalent statement as shown.

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The factor $(-1)^n$ is either 1 or -1 which means it is not zero. So $(-1)^n\det(\boldsymbol{A}) \neq 0$ is logically equivalent to $\det(\boldsymbol{A}) \neq 0$. We have thus completed the proof. The significance of having this statement added to the collection is that we can now determine if a matrix is invertible by finding all its eigenvalues and checking if 0 is one of them.

Slide 04: This is the 2×2 matrix $\boldsymbol{A}$ from the population example. We have found, in an earlier unit, that 1 and 0.95 are the only two eigenvalues of $\boldsymbol{A}$.

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Since 0 is not an eigenvalue of $\boldsymbol{A}$, we can conclude from here that $\boldsymbol{A}$ is invertible.

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This 3×3 matrix $\boldsymbol{B}$ has also been discussed previously. We have found that 0 and 3 are the only two eigenvalues of $\boldsymbol{B}$.

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Since 0 is one of the eigenvalues of $\boldsymbol{B}$, we can conclude that it is singular. Notice that we could have also concluded that $\boldsymbol{B}$ is singular immediately, since $\boldsymbol{B}$ has identical rows.

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We have also found all the 3 eigenvalues of this matrix $\boldsymbol{C}$.

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Since 0 is not an eigenvalue of $\boldsymbol{C}$, we can conclude that $\boldsymbol{C}$ is invertible.

Slide 05: We next present a theorem on the eigenvalues of triangular matrices. It turns out that the eigenvalues of a triangular matrix is simply its diagonal entries.

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The proof is quite simple. First, observe that since $\boldsymbol{A}$ is a triangular matrix, $(\lambda\boldsymbol{I}-\boldsymbol{A})$ will also be a triangular matrix.

Slide 06: The $\boldsymbol{A}$ shown here is upper triangular. You can see that $\lambda\boldsymbol{I} - \boldsymbol{A}$ is also upper triangular. For lower triangular matrices, the observation is similar.

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We already know from our discussion on determinants, that the determinant of a triangular matrix is just the product of its diagonal entries. Thus, the determinant of $(\lambda\boldsymbol{I} - \boldsymbol{A})$ can be written dowm immediately. You see that in this case, we have already factorised the characteristic polynomial of $\boldsymbol{A}$ into its factors, which makes it easy for us to find the eigenvalues of $\boldsymbol{A}$.

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Setting the characteristic polynomial to zero, we have the characteristic equation and the roots of the equation are just a_{11}, a_{22} and so on till a_{nn}. These are the eigenvalues of $\boldsymbol{A}$ and also the diagonal entries of $\boldsymbol{A}$. The proof is thus complete.

Slide 07: The matrix $\boldsymbol{A}$ is upper triangular. The eigenvalues of $\boldsymbol{A}$ are its diagonal entries, namely, $1, -2$ and 3.

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The matrix $\boldsymbol{B}$ is lower triangular. The eigenvalues of $\boldsymbol{B}$ are its diagonal entries, namely, $0, 2$ and 6.

Slide 08: Some of you would be wondering now.

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The eigenvalues of a triangular matrix can be obtained easily, using the previous theorem.

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Just like when we wanted to compute the determinant of a matrix,

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we can perform elementary row operations on the matrix $\boldsymbol{A}$ and find a row-echelon form of $\boldsymbol{A}$, which is triangular.

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Would it be possible to know how elementary row operations would affect the eigenvalues of a matrix, like we did before for determinants

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and then backtrack to find the eigenvalues of the matrix $\boldsymbol{A}$?

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Unfortunately, there is no way to determine how a particular elementary row operation would affect the eigenvalues of a matrix.

Slide 09: Consider the following matrix $\boldsymbol{A}$. It is easy to see, since $\boldsymbol{A}$ is triangular, that the eigenvalues of $\boldsymbol{A}$ are 1 and 4.

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Suppose we perform a row swap on $\boldsymbol{A}$ to obtain the matrix $\boldsymbol{B}$. Let us see what the eigenvalues of $\boldsymbol{B}$ will be.

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We find the characteristic polynomial of $\boldsymbol{B}$ as follows,

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and obtain the polynomial $\lambda^2 - 4$.

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This can be factorised into $(\lambda - 2)(\lambda + 2)$.

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Thus the eigenvalues of $\boldsymbol{B}$ are 2 and -2. So this example shows that elementary row operation changes the eigenvalues of a matrix. Will the same elementary row operation, perform on another matrix change the eigenvalues in the same way?

Slide 10: Consider this other 2×2 matrix $\boldsymbol{A}$. To find the eigenvalues of $\boldsymbol{A}$, we again find the characteristic polynomial

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and obtain $\lambda^2 - 5\lambda + 4$

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which can be factorised into $(\lambda - 1)(\lambda - 4)$. Thus the eigenvalues of $\boldsymbol{A}$ are 1 and 4. Note that this matrix has the same two eigenvalues as the previous matrix $\boldsymbol{A}$ in the earlier slide. What happens when we perform the same row swap on $\boldsymbol{A}$?

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We obtain the matrix $\boldsymbol{B}$ as shown. Let us find the eigenvalues of $\boldsymbol{B}$.

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The characteristic polynomial can be found in the usual manner

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and then subsequently factorised to give $(\lambda - 1)(\lambda + 4)$. Thus the eigenvalues of $\boldsymbol{B}$ are 1 and -4. Notice that the eigenvalues of $\boldsymbol{A}$ has also been changed after the elementary row operation was performed but the changes are clearly different from what we saw in the previous slide, despite that same elementary row operation was performed.

Slide 11: In summary,

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we started this unit by adding one more equivalent statement to $\boldsymbol{A}$ is invertible. This new statement is in terms of the eigenvalues of $\boldsymbol{A}$.

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We also discovered the eigenvalues of a triangular matrix can be obtained easily.