Week 03

MA1508E LINEAR ALGEBRA FOR ENGINEERING

Week 03

IVLE Quiz Discussion

Review of last week's content

- Gaussian (resp. Gauss-Jordan) elimination are two algorithms to reduce an augmented matrix into row-echelon (resp. reduced row-echelon) form.
- A standard set of notations to represent the three types of elementary row operations.
- Definition of homogeneous linear systems and how they are special compared to a general linear system.
- Homogeneous linear systems with more variables than equations always has infinitely many solutions.
- What is a matrix? Terminologies related to a matrix e.g. entries, size, column, row etc.
- Representing the entries of a matrix $\boldsymbol{A}$ by a_{ij}

Review of last week’s content

- Special types of matrices: diagonal, scalar, identity, zero, symmetric, triangular
- Two matrices are equal if and only if they have the same size and identical corresponding entries.
- We (1) add matrices by adding corresponding entries; (2) subtract matrices by subtracting corresponding entries; (3) multiply a scalar to a matrix by multiplying the scalar to every entry in the matrix.
- Commutative laws for matrix addition; associative laws for matrix addition
- Matrix multiplication – when can this be performed; writing down the (i,j) entry of a matrix product.
- Non-commutative nature of matrix multiplication -> pre- and post-multiplication.

Review of last week’s content

- $\boldsymbol{AB} = \mathbf{0}$ does not imply $\boldsymbol{A} = \mathbf{0}$ or $\boldsymbol{B} = \mathbf{0}$
- Associative law for matrix multiplication; distributive law for matrix multiplication

Week 03 content (motivation)

- Looking at matrix multiplication in another way
- What is the inverse of a matrix? Are all matrices invertible?
- Laws involving matrix inverse.
- Transpose of a matrix.
- Representing an elementary row operation with a matrix -> elementary matrices.
- Are elementary matrices invertible? What properties do elementary matrices have?

Week 03 (units 013-017) overview

013 Block multiplication

- How to consider a matrix as blocks or submatrices
- Matrix multiplication via rows or columns
- More generally, matrix multiplication by blocks

014 Inverse of a matrix

- Definition for the inverse of a square matrix
- Not so easy to determine (at least for now) whether a matrix is invertible or singular
- Uniqueness of inverse

015 Matrix inverse laws

- Some laws involving the inverse of a matrix
- Transpose of a matrix and some laws
- Inverse of the powers of an invertible matrix

Week 03 (units 013-017) overview

016 Elementary matrices Part I

- For each elementary row operation X, there is a corresponding square matrix $\boldsymbol{E}$ such that performing X on $\boldsymbol{B}$ produces the same effect as pre-multiplying $\boldsymbol{E}$ to $\boldsymbol{B}$
- The matrix $\boldsymbol{E}$, defined as an elementary matrix is obtained by performing the corresponding elementary row operation X on $\boldsymbol{I}$

017 Elementary matrices Part II

- All elementary matrices are invertible…
- … and their inverses are also elementary matrices
- If an elementary matrix $\boldsymbol{E}$ represents a single elementary row operation X, then $\boldsymbol{E}^{-1}$ represents the elementary row operation that does the opposite of X

Linear system and Matrix equation

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{cases}$$

can be represented by a matrix equation $\boxed{\boldsymbol{Ax} = \boldsymbol{b}}$ where

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Linear system and Matrix equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

coefficient matrix

variable matrix

constant matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Linear system and Matrix equation

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$\boldsymbol{Ax} = \boldsymbol{b}$ can also be expressed as

$$x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Linear system and Matrix equation

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$x_1 \overset{\boldsymbol{c}_1}{\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}} + x_2 \overset{\boldsymbol{c}_2}{\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}} + \dots + x_n \overset{\boldsymbol{c}_n}{\begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \dots + x_n\boldsymbol{c}_n = \boldsymbol{b}$

where $\boldsymbol{c}_i = i$th column of $\boldsymbol{A}$

Example

$$\begin{cases} x + 2y - z = 1 \\ 2x - y = 2 \\ x + 2y - 3z = 2 \end{cases}$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 0 \\ 1 & 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$x\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + y\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Can you find x, y, z

that satisfies

Example

$$\begin{cases} x + 2y - z = 1 \\ 2x - y = 2 \\ x + 2y - 3z = 2 \end{cases}$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 0 \\ 1 & 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$x\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + y\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} + z\begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$x = \frac{9}{10}, y = -\frac{1}{5}, z = -\frac{1}{2}$$

is a solution.

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 0 \\ 1 & 2 & -3 \end{pmatrix}\begin{pmatrix} \frac{9}{10} \\ -\frac{1}{5} \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\frac{9}{10}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Example 3.1

Given that A is a 3×3 matrix such that

$$A\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix} \text{ and } A\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Find a matrix X such that

$$AX=\begin{pmatrix}1&0&3\\1&0&3\\1&0&7\end{pmatrix}.$$

Example 3.2

Let $\boldsymbol{A} = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$.

(a) Verify that $\boldsymbol{A}^2 - 6\boldsymbol{A} + 8\boldsymbol{I} = \mathbf{0}$.

(b) Show that $\boldsymbol{A}^{-1} = \frac{1}{8}(6\boldsymbol{I} - \boldsymbol{A})$ without computing the inverse of $\boldsymbol{A}$ explicitly.

Example 3.3

Let A be a square matrix.

(a) Show that if $A^2 = \mathbf{0}$, then $I - A$ is invertible and $(I - A)^{-1} = I + A$.

(b) Show that if $A^3 = \mathbf{0}$, then $I - A$ is invertible and $(I - A)^{-1} = I + A + A^2$. What can we say about $I - A$ if $A^n = \mathbf{0}$ for some $n \geq 4$?

(c) If there is a non zero scalar k such that $(A - kI)(A + kI) = \mathbf{0}$, is A invertible?

Example 3.4

Let $\boldsymbol{A}$ be a $m \times n$ matrix and $\boldsymbol{B}$ be a $n \times p$ matrix. Then $(\boldsymbol{AB})^T = \boldsymbol{B}^T \boldsymbol{A}^T$.

Example 3.5

Let $\boldsymbol{B} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ -1 & -1 & 4 \end{pmatrix}$. Find four elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3, \boldsymbol{E}_4$ such that $\boldsymbol{E}_4\boldsymbol{E}_3\boldsymbol{E}_2\boldsymbol{E}_1\boldsymbol{B} = \boldsymbol{R}$, where $\boldsymbol{R}$ is the reduced row-echelon form of $\boldsymbol{B}$.

Find four elementary matrices $\boldsymbol{F}_1, \boldsymbol{F}_2, \boldsymbol{F}_3, \boldsymbol{F}_4$ such that $\boldsymbol{B} = \boldsymbol{F}_4\boldsymbol{F}_3\boldsymbol{F}_2\boldsymbol{F}_1\boldsymbol{R}$.

Example 3.6

Suppose the augmented matrix of a linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is given by

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ -1 & 2 & -2 & 3 \end{array}\right)$$

(a) Find the unique solution to the linear system.

(b) If $\boldsymbol{x}$ is the solution found in (a), find elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_k$ such that $\boldsymbol{x} = \boldsymbol{E}_k \ldots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{b}$.

Finally…

THE END