Week 11

MA1508E LINEAR ALGEBRA FOR ENGINEERING

Week 10

IVLE Quiz Discussion

Review of Week 10 (Units 050-054) content

- (Definition) When is a vector orthogonal to a vector space? Normal vector to a vector space.
- A vector is orthogonal to a vector space if it is orthogonal to a set that spans the vector space.
- Finding all vectors that are orthogonal to a vector space → homogeneous linear system.
- (Definition) (Orthogonal) projection of a vector onto a vector space. The uniqueness of orthogonal projection.
- If the vector we want to project is already a vector in the vector space, what does orthogonal projection mean?
- What do we need to compute orthogonal projection onto a vector space?

Review of Week 10 (Units 050-054) content

- Orthogonal Projection Theorem and its relation to (writing a vector as a linear combination of a set of orthogonal basis vectors).
- What if we don’t have an orthogonal basis for the vector space that we wish to project on?
- Gram-Schmidt Process – using orthogonal projection to construct an orthogonal basis (sequentially).
- The concept of approximations.
- (Geometrical discussion) The shortest distance from a vector to a space (line or plane).
- (Theorem) The projection ***p*** of a vector ***u*** onto a space *V* is the best approximation of ***u*** among ll vectors in *V*. So *d*(***u*,*p***) is no larger than *d*(***u*,*v***) for all choices of ***v*** in *V*.

Review of Week 10 (Units 050-054) content

- An example on experiments. Find a line (or curve) of “best fit”.
- (Definition) The least squares solution to a linear system $\boldsymbol{Ax} = \boldsymbol{b}$
- We know that sometimes we need to find least squares solution to a linear system. How to find it?
- How “close” (to $\boldsymbol{b}$) is the projection $\boldsymbol{p}$ (of $\boldsymbol{b}$) onto a space V?

Week 11 (units 055-059) overview

055 Eigenvalues and eigenvectors

- A real-life example on population movement and the need to compute powers of a square matrix.
- Definition of eigenvalue (of a matrix ***A***) and eigenvector (of a matrix ***A***) associated with the eigenvalue.

056 Characteristic equation of a matrix

- Characteristic polynomial and characteristic equation of a square matrix
- How to find the eigenvalues of a square matrix using characteristic equation.

057 One more equivalent statement; more on eigenvalues

- ***A*** is invertible if and only if 0 is not an eigenvalue of ***A***
- Eigenvalues of triangular matrices.

Week 11 (units 055-059) overview

058 Eigenspaces

- Eigenspace of a matrix associated with an eigenvalue

059 Diagonalization Part I

- Definition of a diagonalizable matrix
- A necessary and sufficient condition for a square matrix of order n to be diagonalizable
- An algorithm to (a) determine if a matrix $\boldsymbol{A}$ is diagonalizable, and if it is; (b) find a matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{A}$

Example (Fibonacci numbers)

Fibonacci numbers: For $n \geq 2$, $a_n = a_{n-1} + a_{n-2}$

$a_0 = 0 \quad a_1 = 1 \quad a_2 = a_1 + a_0 = 1 \quad a_3 = a_2 + a_1 = 2 \quad \cdots$

That's easy enough...what is a_{256} ?

I need to find a_{254} and a_{255}..

I need to find a_{253} and a_{254}..

Example (Fibonacci numbers)

Fibonacci numbers: For $n \geq 2,\ a_n = a_{n-1} + a_{n-2}$

$$a_0 = 0 \qquad a_1 = 1 \qquad a_2 = a_1 + a_0 = 1 \qquad a_3 = a_2 + a_1 = 2 \quad \cdots$$

Let $\boldsymbol{x}_n = \begin{pmatrix} \boxed{a_n} \\ \boxed{a_{n+1}} \end{pmatrix}$ and so $\boldsymbol{x}_{n-1} = \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix}$.

How do we relate $\boldsymbol{x}_n$ and $\boldsymbol{x}_{n-1}$?

Can we express a_n in terms of a_{n-1} and a_n? $\boxed{a_n = 0a_{n-1} + 1a_n}$

Can we express a_{n+1} in terms of a_{n-1} and a_n? $\boxed{a_{n+1} = 1a_{n-1} + 1a_n}$

Example (Fibonacci numbers)

Fibonacci numbers: For $n \geq 2$, $a_n = a_{n-1} + a_{n-2}$

$$a_0 = 0 \qquad a_1 = 1 \qquad a_2 = a_1 + a_0 = 1 \qquad a_3 = a_2 + a_1 = 2 \quad \cdots$$

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How do we relate $\boldsymbol{x}_n$ and $\boldsymbol{x}_{n-1}$?

$$\begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix}$$

$$\boxed{\boldsymbol{x}_0 = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$\boxed{a_n = 0a_{n-1} + 1a_n}$$

$$\boxed{a_{n+1} = 1a_{n-1} + 1a_n}$$

$$\boldsymbol{x}_n = \boldsymbol{A}\boldsymbol{x}_{n-1} = \boldsymbol{A}^2\boldsymbol{x}_{n-2} = \boldsymbol{A}^3\boldsymbol{x}_{n-3} = \ldots = \boldsymbol{A}^n\boldsymbol{x}_0$$

Example (Fibonacci numbers)

$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad \boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$ $\quad$ Let's see if we can diagonalize $\boldsymbol{A}$.

$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda - 1 \end{vmatrix} = \lambda^2 - \lambda - 1$

Remember $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= (\lambda - (\frac{1+\sqrt{5}}{2}))(\lambda - (\frac{1-\sqrt{5}}{2}))$$

So $\boldsymbol{A}$ has two eigenvalues: $\lambda_1 = \dfrac{1+\sqrt{5}}{2}$ and $\lambda_2 = \dfrac{1-\sqrt{5}}{2}$.

Example (Fibonacci numbers)

$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $\boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$ Let's see if we can diagonalize $\boldsymbol{A}$.

Proceeding as before, we find that

$$E_{\lambda_1} = \text{span}\left\{\begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{pmatrix}\right\} \quad E_{\lambda_2} = \text{span}\left\{\begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{pmatrix}\right\}$$

So $\boldsymbol{A}$ has two linearly independent eigenvectors and can be diagonalized.

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Example (Fibonacci numbers)

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad \boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$$

Let's see if we can diagonalize $\boldsymbol{A}$.

Let $\boldsymbol{P} = \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix}$ and we have

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \Leftrightarrow \boldsymbol{A} = \boldsymbol{P}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\boldsymbol{P}^{-1}$$

So $\boldsymbol{A}$ has two linearly independent eigenvectors and can be diagonalized.

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Example (Fibonacci numbers)

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad \boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$$

Now you can compute a_{256}!

$$\begin{pmatrix} \boxed{a_n} \\ \boxed{a_{n+1}} \end{pmatrix} = \boldsymbol{x}_n = \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \boxed{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n} \\ \boxed{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}} \end{pmatrix}$$

Example (Fibonacci numbers – part II)

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad \boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$$

I know that $\boldsymbol{A}$ can definitely be diagonalized (why?)

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda - 1 \end{vmatrix} = \lambda^2 - \lambda - 1$$

$$= (\lambda - (\frac{1+\sqrt{5}}{2}))(\lambda - (\frac{1-\sqrt{5}}{2}))$$

We will now solve for a_n without explicitly finding the invertible matrix $\boldsymbol{P}$.

So $\boldsymbol{A}$ has two eigenvalues: $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$.

Example (Fibonacci numbers – part II)

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \boldsymbol{x}_n = \boldsymbol{A}^n \boldsymbol{x}_0$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \lambda_2 = \frac{1-\sqrt{5}}{2}$$

Let $\boldsymbol{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\boldsymbol{P}^{-1} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$$\boxed{\begin{pmatrix} \lambda_1{}^n & 0 \\ 0 & \lambda_2{}^n \end{pmatrix}}$$

$$\begin{pmatrix} \boxed{a_n} \\ a_{n+1} \end{pmatrix} = \boldsymbol{x}_n = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \boxed{\begin{pmatrix} (\frac{1+\sqrt{5}}{2})^n & 0 \\ 0 & (\frac{1-\sqrt{5}}{2})^n \end{pmatrix}} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} a(\frac{1+\sqrt{5}}{2})^n & b(\frac{1-\sqrt{5}}{2})^n \\ c(\frac{1+\sqrt{5}}{2})^n & d(\frac{1-\sqrt{5}}{2})^n \end{pmatrix} \begin{pmatrix} f \\ h \end{pmatrix} = \begin{pmatrix} \boxed{A(\frac{1+\sqrt{5}}{2})^n + B(\frac{1-\sqrt{5}}{2})^n} \\ C(\frac{1+\sqrt{5}}{2})^n + D(\frac{1-\sqrt{5}}{2})^n \end{pmatrix} \quad A, B, C, D \in \mathbb{R}$$

Example (Fibonacci numbers – part II)

$$\boldsymbol{A}=\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix} \qquad \boldsymbol{x}_n=\boldsymbol{A}^n\boldsymbol{x}_0$$

$a_0=0 \qquad a_1=1$

(Initial conditions)

$$a_n=A(\tfrac{1+\sqrt{5}}{2})^n+B(\tfrac{1-\sqrt{5}}{2})^n \qquad a_0=A(\tfrac{1+\sqrt{5}}{2})^0+B(\tfrac{1-\sqrt{5}}{2})^0=A+B=0$$

$$a_1=A(\tfrac{1+\sqrt{5}}{2})^1+B(\tfrac{1-\sqrt{5}}{2})^1=A(\tfrac{1+\sqrt{5}}{2})+B(\tfrac{1-\sqrt{5}}{2})=1$$

Solving $\begin{cases} A + B = 0 \\ \left(\frac{1+\sqrt{5}}{2}\right)A + \left(\frac{1-\sqrt{5}}{2}\right)B = 1 \end{cases}$ we have $A=\dfrac{1}{\sqrt{5}}$ and $B=\dfrac{-1}{\sqrt{5}}$,

$$\Rightarrow \boxed{a_n=\tfrac{1}{\sqrt{5}}\left(\tfrac{1+\sqrt{5}}{2}\right)^n-\tfrac{1}{\sqrt{5}}\left(\tfrac{1-\sqrt{5}}{2}\right)^n}$$

Example 10.1

Let $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

(a) Find a matrix P that diagonalizes A.

(b) Compute A^{10}.

(c) Find a matrix B such that $B^2 = A$.

Example 10.2

Let $\boldsymbol{B}$ be a 4×4 matrix and $\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3,\boldsymbol{u}_4\}$ a basis for $\mathbb{R}^4$. Suppose

$$\boldsymbol{B}\boldsymbol{u}_1 = 2\boldsymbol{u}_1, \qquad \boldsymbol{B}\boldsymbol{u}_2 = \boldsymbol{0}, \qquad \boldsymbol{B}\boldsymbol{u}_3 = \boldsymbol{u}_4, \qquad \boldsymbol{B}\boldsymbol{u}_4 = \boldsymbol{u}_3$$

(a) Write down all the eigenvalues of $\boldsymbol{B}$.

(b) For each eigenvalue of $\boldsymbol{B}$, write down one eigenvector associated with it.

(c) Is $\boldsymbol{B}$ a diagonalizable matrix? Justify your answer.

Example 10.3

Let d_n be the determinant of the following square matrix of order n.

$$\begin{pmatrix} 3 & 1 & & & & \\ 1 & 3 & 1 & & \Large 0 & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & \Large 0 & & \ddots & 3 & 1 \\ & & & & 1 & 3 \end{pmatrix}$$

Show that $d_n = 3d_{n-1} - d_{n-2}$. Hence, or otherwise, find d_n.

Finally…

THE END