# COMPLEX VECTORS – AN INTRODUCTION

# REVIEW OF COMPLEX NUMBERS

Recall that if $z = a + bi$ is a complex number, then:

1) $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$ are called the real part of $z$ and imaginary part of $z$, respectively.

2) $|z| = \sqrt{a^2 + b^2}$ is called the modulus (or absolute value) of $z$.

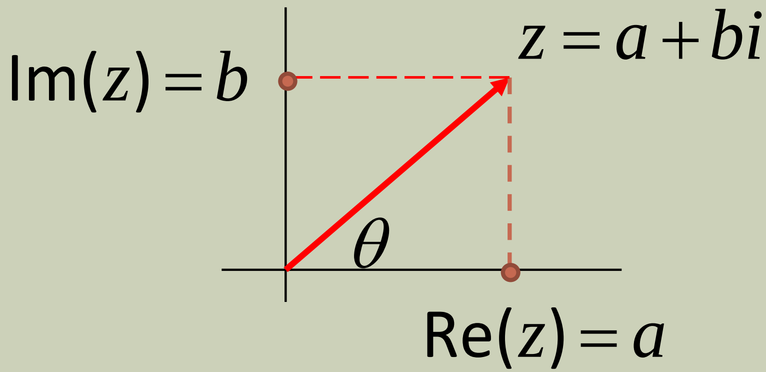
3) $\bar{z} = a - bi$ is called the complex conjugate of $z$.

4) $\bar{z}z = (a - bi)(a + bi) = a^2 + b^2 = |z|^2$.

# REVIEW OF COMPLEX NUMBERS

Recall tht if $z = a + bi$ is a complex number, then:

5) The angle $\theta$ in the figure below is called the $\boxed{\text{argument}}$ of $z$.




6) $\text{Re}(z) = |z|\cos\theta$, $\text{Im}(z) = |z|\sin\theta$.

7) $z = |z|(\cos\theta + i\sin\theta)$ is called the $\boxed{\text{polar form}}$ of $z$.

# VECTORS IN $\mathbb{C}^n$

So far, we have dealt with vectors in $\mathbb{R}^n$, where each of the $n$ component in the vector $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ is a real number.

For example, $\boldsymbol{u} = (1, \pi, -0.5) \in \mathbb{R}^3$.

We are aware that complex numbers can be represented as $a + bi$ where $a, b$ are real numbers.

$i = \sqrt{-1}$

This gives a natural extension to define $\mathbb{C}^n$ as follows:

$\boldsymbol{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{C}^n$ if and only if for each $i$, $v_i \in \mathbb{C}$.

# VECTORS IN $\mathbb{C}^n$

We are aware that complex numbers can be represented as $a+bi$ where $a,b$ are real numbers. This gives a natural extension to define $\mathbb{C}^n$ as follows:

$\boldsymbol{v}=(v_1,v_2,\ldots,v_n)\in\mathbb{C}^n$ if and only if for each $i$, $v_i\in\mathbb{C}$.

For example,

$$\boldsymbol{v}=(2-5i,3,-4+2i)\in\mathbb{C}^3.$$

# VECTORS IN $C^n$

A vector $\boldsymbol{v} \in \mathbb{C}^n$ can be split into real and imaginary parts:

$$\boldsymbol{v} = (a_1 + b_1 i, a_2 + b_2 i, \ldots, a_n + b_n i)$$

$$= (a_1, a_2, \ldots, a_n) + i(b_1, b_2, \ldots, b_n) = \mathrm{Re}(\boldsymbol{v}) + i\,\mathrm{Im}(\boldsymbol{v})$$

$$\overline{\boldsymbol{v}} = (a_1 - b_1 i, a_2 - b_2 i, \ldots, a_n - b_n i)$$

$$= (a_1, a_2, \ldots, a_n) - i(b_1, b_2, \ldots, b_n) = \mathrm{Re}(\boldsymbol{v}) - i\,\mathrm{Im}(\boldsymbol{v})$$

# ALGEBRAIC PROPERTIES OF COMPLEX CONJUGATE

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in $\mathbb{C}^n$ and if $k$ is a scalar, then:

1) $\overline{\overline{\boldsymbol{u}}} = \boldsymbol{u}$  2) $\overline{k\boldsymbol{u}} = \overline{k}(\overline{\boldsymbol{u}})$  3) $\overline{\boldsymbol{u}+\boldsymbol{v}} = \overline{\boldsymbol{u}} + \overline{\boldsymbol{v}}$  4) $\overline{\boldsymbol{u}-\boldsymbol{v}} = \overline{\boldsymbol{u}} - \overline{\boldsymbol{v}}$

# COMPLEX DOT PRODUCT

Let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ be vectors in $\mathbb{C}^n$.

The complex dot product of $\boldsymbol{u}$ with $\boldsymbol{v}$ is:

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 \overline{v_1} + u_2 \overline{v_2} + \ldots + u_n \overline{v_n}$$

The Euclidean norm on $\mathbb{C}^n$ is:

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{|v_1^2| + |v_2^2| + \ldots + |v_n^2|}$$

Unit vector: $\|\boldsymbol{v}\| = 1$

Orthogonal vectors: $\boldsymbol{u} \cdot \boldsymbol{v} = 0$

# EXAMPLE

Find $\boldsymbol{u}\cdot\boldsymbol{v}$, $\boldsymbol{v}\cdot\boldsymbol{u}$, $\|\boldsymbol{u}\|$ and $\|\boldsymbol{v}\|$ for the vectors

$$\boldsymbol{u}=(1+i,i,3-i) \text{ and } \boldsymbol{v}=(1+i,2,4i).$$

$$\boldsymbol{u}\cdot\boldsymbol{v}=(1+i)(\overline{1+i})+i(\overline{2})+(3-i)(\overline{4i})$$

$$=(1+i)(1-i)+i(2)+(3-i)(-4i) =1-i^2+2i-12i+4i^2$$

$$=1-3-10i=\underline{-2-10i}$$

$$\boldsymbol{v}\cdot\boldsymbol{u}=(1+i)(\overline{1+i})+2(\overline{i})+(4i)(\overline{3-i})$$

$$=(1+i)(1-i)+2(-i)+(4i)(3+i) =1-i^2-2i+12i+4i^2$$

$$=1-3+10i =\underline{-2+10i}$$

# EXAMPLE

Find $\boldsymbol{u}\cdot\boldsymbol{v}$, $\boldsymbol{v}\cdot\boldsymbol{u}$, $\|\boldsymbol{u}\|$ and $\|\boldsymbol{v}\|$ for the vectors

$\boldsymbol{u}=(1+i,i,3-i)$ and $\boldsymbol{v}=(1+i,2,4i)$.

$$\|\boldsymbol{u}\|=\sqrt{\boldsymbol{u}\cdot\boldsymbol{u}}=\sqrt{|1+i|^2+|i|^2+|3-i|^2}=\sqrt{2+1+10}=\sqrt{13}$$

$$\|\boldsymbol{v}\|=\sqrt{\boldsymbol{v}\cdot\boldsymbol{v}}=\sqrt{|1+i|^2+|2|^2+|4i|^2}=\sqrt{2+4+16}=\sqrt{22}$$

# THEOREM

Let $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ be vectors in $\mathbb{C}^n$ and if $k$ is a scalar, then:

(a) $\boldsymbol{u} \cdot \boldsymbol{v} = \overline{\boldsymbol{v} \cdot \boldsymbol{u}}$ (antisymmetry property)

(b) $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$ (distributive property)

(c) $k(\boldsymbol{u} \cdot \boldsymbol{v}) = (k\boldsymbol{u}) \cdot \boldsymbol{v}$

(d) $\boldsymbol{u} \cdot k\boldsymbol{v} = \overline{k}(\boldsymbol{u} \cdot \boldsymbol{v})$

(e) $\boldsymbol{v} \cdot \boldsymbol{v} \geq 0$ and $\boldsymbol{v} \cdot \boldsymbol{v} = 0$ if and only if $\boldsymbol{v} = \boldsymbol{0}$

# THEOREM (COMPLEX EIGENVALUES)

If $\lambda$ is an eigenvalue of a (real) matrix $\boldsymbol{A}$ of order $n$, and if $\boldsymbol{x}$ is a corresponding eigenvector, then $\overline{\lambda}$ is also an eigenvalue of $\boldsymbol{A}$ and $\overline{\boldsymbol{x}}$ is a corresponding eigenvector.

# EXAMPLE

Show that $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$ is diagonalizable by finding a matrix $\boldsymbol{P}$ with complex entries that diagonalizes $\boldsymbol{A}$.

Solution:

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda - 1 & -1 \\ 2 & \lambda - 3 \end{vmatrix} = (\lambda - 1)(\lambda - 3) + 2 = \lambda^2 - 4\lambda + 5$$

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0 \Leftrightarrow \lambda = \frac{4 \pm \sqrt{16 - 4(1)(5)}}{2} \Rightarrow \lambda = 2 + i \text{ or } 2 - i$$

# EXAMPLE

Solution:

$$\lambda_1 = 2+i: \quad \begin{pmatrix} 2+i-1 & -1 \\ 2 & 2+i-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1+i & -1 & 0 \\ 2 & -1+i & 0 \end{array}\right) \xrightarrow{R_2 - \frac{2}{1+i}R_1} \left(\begin{array}{cc|c} 1+i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x = \dfrac{s}{2}(1-i) \\ y = s \end{cases} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = s\begin{pmatrix} 1-i \\ 2 \end{pmatrix} \Rightarrow E_{\lambda_1} = \text{span}\left\{\begin{pmatrix} 1-i \\ 2 \end{pmatrix}\right\}$$

# EXAMPLE

Solution:

$$\lambda_2 = 2-i: \quad \begin{pmatrix} 2-i-1 & -1 \\ 2 & 2-i-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1-i & -1 & 0 \\ 2 & -1-i & 0 \end{array}\right) \xrightarrow{R_2 - \frac{2}{1-i}R_1} \left(\begin{array}{cc|c} 1-i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x = \dfrac{s}{2}(1+i) \\ y = s \end{cases} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = s\begin{pmatrix} 1+i \\ 2 \end{pmatrix} \Rightarrow E_{\lambda_2} = \text{span}\left\{\begin{pmatrix} 1+i \\ 2 \end{pmatrix}\right\}$$

# EXAMPLE

Solution:

$\lambda_1 = \boxed{2+i}$:

$E_{\lambda_1} = \text{span}\left\{ \boxed{\begin{pmatrix} 1-i \\ 2 \end{pmatrix}} \right\}$

$\lambda_2 = \boxed{2-i}$:

$E_{\lambda_2} = \text{span}\left\{ \boxed{\begin{pmatrix} 1+i \\ 2 \end{pmatrix}} \right\}$

Let $\boldsymbol{P} = \begin{pmatrix} 1-i & 1+i \\ 2 & 2 \end{pmatrix}$, then $\boldsymbol{A} = \boldsymbol{PDP}^{-1}$ where

$$\boldsymbol{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 2+i & 0 \\ 0 & 2-i \end{pmatrix}$$

# SUMMARY

1) A quick review of complex numbers
   (e.g modulus, conjugate)

2) Vectors in $\mathbb{C}^n$.

3) Complex dot product and some properties.

4) Complex eigenvalues always occur in pairs and so do
   the conjugate eigenvectors.