DETERMINANTS AND ELEMENTARY ROW OPERATIONS

Elementary row operations (1st type)

$$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

We already know that

$$\det(\boldsymbol{A}) = aei + bfg + cdh - ceg - afh - bdi$$

$$\big\downarrow kR_3$$

$$\boldsymbol{B}_1 = \begin{pmatrix} a & b & c \\ d & e & f \\ kg & kh & ki \end{pmatrix}$$

Recall that $\boldsymbol{E}_1\boldsymbol{A} = \boldsymbol{B}_1$ where

$$\boldsymbol{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{pmatrix}$$

$$\begin{aligned} \det(\boldsymbol{B}_1) &= aeki + bfkg + cdkh - cekg - afkh - bdki \\ &= k(aei + bfg + cdh - ceg - afh - bdi) = k\det(\boldsymbol{A}) \end{aligned}$$

Elementary row operations (1st type)

$$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \xrightarrow{kR_3} \begin{pmatrix} a & b & c \\ d & e & f \\ kg & kh & ki \end{pmatrix} = \boldsymbol{B}_1$$

$\boxed{\boldsymbol{E}_1\boldsymbol{A} = \boldsymbol{B}_1}$ where $\boldsymbol{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{pmatrix}$ $\det(\boldsymbol{E}_1) = k$

$$\boxed{\begin{aligned} \det(\boldsymbol{B}_1) &= k\det(\boldsymbol{A}) \\ &= \det(\boldsymbol{E}_1)\det(\boldsymbol{A}) \end{aligned}}$$

So it seems like elementary row operations of this type changes the determinant of $\boldsymbol{A}$ by a factor of k and we have 'determinant of product equals to product of determinants' (to some extent).

Elementary row operations (2nd type)

$$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

We already know that

$$\det(\boldsymbol{A}) = aei + bfg + cdh - ceg - afh - bdi$$

$$R_1 \leftrightarrow R_3 \Big\downarrow$$

$$\boldsymbol{B}_2 = \begin{pmatrix} g & h & i \\ d & e & f \\ a & b & c \end{pmatrix}$$

Recall that $\boldsymbol{E}_2\boldsymbol{A} = \boldsymbol{B}_2$ where

$$\boldsymbol{E}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \det(\boldsymbol{B}_2) &= gec + hfa + idb - iea - gfb - hdc \\ &= -(iea + gfb + hdc - gec - hfa - idb) = -\det(\boldsymbol{A}) \end{aligned}$$

Elementary row operations (2nd type)

$$\boldsymbol{A}=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} g & h & i \\ d & e & f \\ a & b & c \end{pmatrix}=\boldsymbol{B}_2$$

$\boxed{\boldsymbol{E}_2\boldsymbol{A}=\boldsymbol{B}_2}$ where $\boldsymbol{E}_2=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\det(\boldsymbol{E}_2)=-1$

$$\boxed{\begin{aligned}\det(\boldsymbol{B}_2)&=-\det(\boldsymbol{A})\\ &=\det(\boldsymbol{E}_2)\det(\boldsymbol{A})\end{aligned}}$$

So it seems like elementary row operations of this type changes the determinant of $\boldsymbol{A}$ by a factor of -1 and we have 'determinant of product equals to product of determinants' (to some extent).

Elementary row operations (3rd type)

$$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

We already know that

$$\det(\boldsymbol{A}) = aei + bfg + cdh - ceg - afh - bdi$$

$$\boldsymbol{A} \xrightarrow{R_2 + kR_3} \boldsymbol{B}_3$$

Recall that $\boldsymbol{E}_3\boldsymbol{A} = \boldsymbol{B}_3$ where

$$\boldsymbol{B}_3 = \begin{pmatrix} a & b & c \\ d+kg & e+kh & f+ki \\ g & h & i \end{pmatrix} \qquad \boldsymbol{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det(\boldsymbol{B}_3) = ai(e+kh) + bg(f+ki) + ch(d+kg)$$
$$-cg(e+kh) - ah(f+ki) - bi(d+kg) \; = \det(\boldsymbol{A})$$

Elementary row operations (3rd type)

$$\boldsymbol{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \xrightarrow{R_2 + kR_3} \begin{pmatrix} a & b & c \\ d+kg & e+kh & f+ki \\ g & h & i \end{pmatrix} = \boldsymbol{B}_3$$

$\boxed{\boldsymbol{E}_3 \boldsymbol{A} = \boldsymbol{B}_3}$ where $\boldsymbol{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$ $\det(\boldsymbol{E}_3) = 1$ (why?)

$$\boxed{\begin{aligned} \det(\boldsymbol{B}_3) &= \det(\boldsymbol{A}) \\ &= \det(\boldsymbol{E}_3)\det(\boldsymbol{A}) \end{aligned}}$$

So it seems like elementary row operations of this type does not change the determinant of $\boldsymbol{A}$ and we have 'determinant of product equals to product of determinants' (to some extent).

Theorem

Let $\boldsymbol{A}$ be a square matrix. Then

$$\boldsymbol{A} \xrightarrow{kR_i} \boldsymbol{B}_1 \qquad \det(\boldsymbol{B}_1) = k\det(\boldsymbol{A})$$

$$\boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{B}_2 \qquad \det(\boldsymbol{B}_2) = -\det(\boldsymbol{A})$$

$$\boldsymbol{A} \xrightarrow{R_j + kR_i} \boldsymbol{B}_3 \qquad \det(\boldsymbol{B}_3) = \det(\boldsymbol{A})$$

Furthermore, if $\boldsymbol{E}$ is an elementary matrix of the same size as $\boldsymbol{A}$, then $\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$.

wow, this is quite something, but how can we use such a result?

suppose you want to find the determinant of $\boldsymbol{A}$...
... you first find the determinant of its row-echelon form.

$$\boldsymbol{A} \longrightarrow \boldsymbol{R}$$

oh I know, then we keep track of what e.r.o. have been performed on $\boldsymbol{A}$...

Yes! $\boldsymbol{R}$ is a triangular matrix whose determinant is easy to evaluate... we can now 'backtrack' to find the determinant of $\boldsymbol{A}$.

$$\boldsymbol{A} \longrightarrow \boldsymbol{R}$$

Example

Find the determinant of the following matrix using elementary row operations.

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 3 & 4 \\ 1 & -2 & 3 \end{pmatrix}$$

Example

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 3 & 4 \\ 1 & -2 & 3 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & -2 & 3 \\ 2 & 3 & 4 \\ 0 & 1 & -1 \end{pmatrix}$$

$= A_1$

$\det(A_1) = -\det(A)$

$-\det(A) = -5$, so $\det(A) = 5$.

$$\xrightarrow{R_2 - 2R_1} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 7 & -2 \\ 0 & 1 & -1 \end{pmatrix}$$

$= A_2$

$\det(A_2) = \det(A_1)$

$$\xrightarrow{R_3 - \frac{1}{7}R_2} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 7 & -2 \\ 0 & 0 & -\frac{5}{7} \end{pmatrix}$$

$= A_3$

$\det(A_3) = \det(A_2)$

$$\begin{aligned} \det(A_3) &= 1 \times 7 \times (-\tfrac{5}{7}) = -5 \\ &= \det(A_2) = \det(A_1) \\ &= -\det(A) \end{aligned}$$

Example

Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent matrices as shown below:

$$\boldsymbol{A} \xrightarrow{R_1 + \frac{2}{9}R_2} \xrightarrow{R_2 \leftrightarrow R_3} \xrightarrow{4R_2} \begin{pmatrix} 5 & 0 & 8 & -1 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} = \boldsymbol{B}$$

Find $\det(\boldsymbol{A})$.

$$\det(\boldsymbol{B}) = 1 \cdot (-1) \cdot (4) \cdot \det(\boldsymbol{A})$$

$$5 \cdot (-2) \cdot 1 \cdot \tfrac{1}{3} = -4\det(\boldsymbol{A})$$

$$\tfrac{5}{6} = \det(\boldsymbol{A})$$

Can you find $\boldsymbol{A}$?

Summary

1) How the 3 types of elementary row operations changes the determinant of a square matrix.

2) Computing the determinant of a matrix using elementary row operations.