Unit 003 Elementary row operations

Slide 01: In this unit, we will introduce the concept of elementary row operations.

Slide 02: We have seen that all linear systems can have only three possibilities in terms of how many solutions it has. For linear systems that have no solutions, we say it is inconsistent. For such linear systems, the solution set will be an empty set.

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On the other hand, linear systems that have at least one solution are said to be consistent. We also know that in this case, the system would have either exactly one solution, or infinitely many solutions. The solution set in this case would then be non empty.

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If a linear system has exactly one solution, we say that the linear system has a unique solution.

Slide 03: For any linear system, an effective way of representing the system is to use a rectangular array of numbers. Here you see the rectangular array divided into two portions by a vertical line. The left side of the array contains the coefficients of each variable for each equation in the system. The right side of the array contains the constants found on the right side of each equation.

Slide 04: This rectangular array is called the augmented matrix of the linear system. From the way the matrix is set up, it is clear that a row in the matrix corresponds to an equation in the linear system. For the columns on the left side of the matrix, they correspond to the variables in the linear system. Therefore if the linear system has m equations involving n variables, then the augmented matrix will have m rows and $(n+1)$ columns.

Slides 05: Let us return to linear systems and their solutions. Consider this simple linear system with two equations and two unknowns. How would you go about solving it?

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From your high school days, the following steps may not surprise you. We first multiply equation (2) by 2. This results in a new equation (3).

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We then subtract equation (3) from equation (1). This results in a new equation (4). You would observe that equation (4) now does not contain the variable x.

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Note that by subtracting equation (3) from equation (1), we are in fact adding -1 times of equation (3) to equation (1).

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Solving equation (4) now gives us the value of y, which is $\frac{5}{7}$.

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Finally, substituting the value of $y = \frac{5}{7}$ into equation (3) yields the value of x, which is $\frac{1}{7}$. Note that this linear system is therefore consistent and has a unique solution.

Slide 06: Recall the correspondence between a linear system and its augmented matrix. What we normally do to manipulate the equations in a linear system has its corresponding operations that can be performed on the rows of the augmented matrix.

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Firstly, we may wish to multiply an equation by a non zero constant. This would correspond to multiplying a row of the matrix by the same constant.

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Note that every other row in the augmented matrix remains unchanged after this operation.

Slide 07: We may switch the order in which two equations are written in the linear system, meaning one equation moves higher up, the other moving lower down. Obviously doing this does not change the linear system at all. This would correspond to interchanging two rows of the matrix.

Slide 08: Very often, we would like to add a multiple of one equation in the system to another equation. This would correspond to adding a multiple of one row in the augmented matrix to another row.

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For example, suppose we wish to add 2 times the equation represented by the purple row of the augmented matrix to the other equation represented by the light blue row of the matrix. This would result in a new matrix where the previously light blue row will be changed to a new row representing a new equation in the linear system. Note that all other rows of the matrix, including the purple row which was used, remains unchanged.

Slide 09: This leads us to the definition of elementary row operations. The first type of elementary row operation are those where we multiply a row of the matrix by a non zero constant. It is important to remember that we do not multiply zero to any row of the matrix as this would wipe out the entire row, reducing it to a row of zeros. This would have meant that important information in an equation of the linear system would have been lost.

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The second type of elementary row operation are those where we interchange two rows.

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The third and final type of elementary row operation are those where we add a multiple of one row of the matrix to another row. These three operations that can be performed on an augmented matrix are called elementary row operations.

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Note that elementary row operations can be performed on any matrix in general, not just on augmented matries.

Slide 10: Consider the following example of a linear system with three equations and involving three variables. Its augmented matrix is shown on the right.

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Supose we multiply equation (1) by 2. This would correspond to multiplying row 1 of the matrix by 2.

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Equation (1) now changes to equation (4) and by performing the said elementary row operation on the augmented matrix, we obtain the new augmented matrix as follows.

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Note that it is necesary that this new matrix represents the current linear system with equations (4),(2) and (3).

Slide 11: Suppose we wish to interchange equations (2) and (3). This would correspond to swapping rows 2 and 3 in the augmented matrix.

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After such a swap, the previous equation (3) is now equation (2) and vice versa.

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And correspondingly, the previous row 3 of the matrix is now row 2 of the new matrix and vice versa.

Slide 12: For the third type of elementary row operation, consider adding -2 times of equation (1) to equation (2). In this case, we would need to add -2 times of row 1 to row 2 of the matrix.

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Remember that a multiple of equation (1) was used to add to equation (2) so equation (1) remains unchanged while equation (2) is now changed to equation (4).

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Correspondingly the first row of the augmented matrix remains unchanged while the second row now changes in the new augmented matrix.

Slide 13: In this unit,

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We first defined what are consistent and inconsistent linear systems. What is meant by a unique solution to a linear system.

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We learnt how to use an augmented matrix to represent a linear system effectively.

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And finally, we introduced three types of elementary row operations that can be performed on any augmented matrix. These row operations corresponds to ways that we can manipulate equations in a linear system.