Question 1 (a) [10 marks]

(i) Let $y(x)$ denote the solution of the differential equation

$$\frac{dy}{dx} = e^{x-y},$$

with $y(1) = 2$. Find the value of $y(4)$. Give your answer correct to two decimal places.

(ii) Let $y(x)$ denote the solution of the differential equation

$$y\frac{dy}{dx} - y^2 = x,$$

with $x > 0.6$, $y > 0$ and $y(1) = 1$. Find the value of $y(3)$. Give your answer correct to two decimal places.

Answer 1(a)(i)	4.08	**Answer 1(a)(ii)**	11.53

(Show your working below and on the next page.)

(i) $e^y dy = e^x dx$

$e^y = e^x + C$

$y(1)=2 \Rightarrow e^2 = e + C$

$\Rightarrow C = e^2 - e$

$e^y = e^x + e^2 - e$

$y = \ln(e^x + e^2 - e)$

$y(4) = \ln(e^4 + e^2 - e)$

$= 4.082\cdots$

$\approx \underline{\underline{4.08}}$

(ii) $y' - y = xy^{-1}$

Let $z = y^{1-(-1)} = y^2 \Rightarrow z' = 2yy'$

$\frac{z'}{2y} - y = xy^{-1} \Rightarrow z' - 2y^2 = 2x$

$\Rightarrow z' - 2z = 2x$

$R = e^{\int -2dx} = e^{-2x}$

$z = e^{2x}\int 2xe^{-2x}dx$

$= e^{2x}\{-xe^{-2x} - \frac{1}{2}e^{-2x} + c\}$

$y^2 = -x - \frac{1}{2} + ce^{2x}$

$y(1)=1 \Rightarrow 1 = -1 - \frac{1}{2} + ce^2 \Rightarrow c = \frac{5}{2e^2}$

$x=3 \Rightarrow y = \sqrt{-3 - \frac{1}{2} + \frac{5}{2e^2}e^6}$

$= 11.532\cdots \approx \underline{\underline{11.53}}$

Question 1 (b) [10 marks]

(i) At time $t = 0$ a particle with mass 0.3 kg is projected vertically upwards at a velocity u metre per second towards the sky. It is observed that at time $t = 0.38$ second the particle reaches the highest point of its trajectory. If the gravitational constant is $g = 9.8$ metre per second square and the air resistance is equal to $0.3v^2$ when the velocity of the particle is v metre per second, find the value of u. Give your answer correct to two decimal places.

(ii) Let $y(x)$ be a solution of $y'' - 2y' + 10y = 0$, such that $y(0) = 1$ and $y'(0) = 7$. Find the value of $y\left(\frac{\pi}{4}\right)$. Give your answer correct to two decimal places. (Note: It is alright if you want to use Laplace Transform to solve this problem.)

Answer 1(b)(i)	7.81	**Answer 1(b)(ii)**	1.55

(Show your working below and on the next page.)

(i) $0.3\frac{dv}{dt} = -0.3g - 0.3v^2$

$\frac{dv}{9.8+v^2} = -dt$

$\int_u^0 \frac{dv}{9.8+v^2} = -\int_0^{0.38} dt$

$\frac{1}{\sqrt{9.8}}\tan^{-1}\frac{u}{\sqrt{9.8}} = 0.38$

$u = \sqrt{9.8}\tan(0.38\times\sqrt{9.8})$

$= 7.810\ldots$

$\approx \underline{\underline{7.81}}$

(ii) $\lambda^2 - 2\lambda + 10 = 0 \Rightarrow \lambda = 1 \pm 3i$

$y = C_1 e^x \cos 3x + C_2 e^x \sin 3x$

$y(0) = 1 \Rightarrow C_1 = 1$

$\Rightarrow y = e^x\cos 3x + C_2 e^x \sin 3x$

$y' = e^x\cos 3x - 3e^x\sin 3x + C_2 e^x \sin 3x + 3C_2 e^x\cos 3x$

$y'(0) = 7 \Rightarrow 7 = 1 + 3C_2$

$\Rightarrow C_2 = 2$

$\therefore y = e^x\cos 3x + 2e^x \sin 3x$

$y\left(\frac{\pi}{4}\right) = e^{\frac{\pi}{4}}\cos\frac{3\pi}{4} + 2e^{\frac{\pi}{4}}\sin\frac{3\pi}{4}$

$= 1.550\ldots$

$\approx \underline{\underline{1.55}}$

Question 2 (a) [10 marks]

(i) Let $y(x)$ be a solution of $y''-y'-2y=3e^{2x}$, such that $y(0)=2$ and $y'(0)=2$. Find the value of $y(1)$. Give your answer correct to two decimal places. (Note: It is alright if you want to use Laplace Transform to solve this problem.)

(ii) Let $y(x)$ be the solution of the differential equation

$$y''=y^2$$

such that

$$x<\sqrt{2},\quad y>0,\quad y'>0,\quad y(0)=3,\quad y'(0)=3\sqrt{2}.$$

Find the value of $y(1)$. Give your answer correct to two decimal places.

Answer 2(a)(i)	15.15	Answer 2(a)(ii)	34.97

(Show your working below and on the next page.)

(i) Let $Y=L(y)$

$\therefore s^2Y-2s-2-sY+2-2Y=\frac{3}{s-2}$

$Y=\frac{2s^2-4s+3}{(s-2)^2(s+1)}$

$=\frac{1}{(s-2)^2}+\frac{1}{s-2}+\frac{1}{s+1}$

$y=xe^{2x}+e^{2x}+e^{-x}$

$y(1)=e^2+e^2+e^{-1}$

$=15.145\cdots$

$\approx \underline{\underline{15.15}}$

(ii) Using $y''=\frac{d}{dy}\left(\frac{1}{2}y'^2\right)$

$\Rightarrow \frac{1}{2}y'^2=\frac{1}{3}y^3+C$

$y(0)=3,\ y'(0)=3\sqrt{2}\Rightarrow 9=9+C\Rightarrow C=0$

$\therefore y'=\sqrt{\frac{2}{3}}\,y^{3/2}$

$-2y^{-\frac{1}{2}}=\sqrt{\frac{2}{3}}x+D$

$y(0)=3\Rightarrow D=-\frac{2}{\sqrt{3}}$

$\therefore -2y^{-1/2}=\sqrt{\frac{2}{3}}x-\frac{2}{\sqrt{3}}$

$y=\frac{1}{\left(\frac{1}{\sqrt{3}}-\sqrt{\frac{1}{6}}x\right)^2}$

$x=1\Rightarrow y=\frac{1}{\left(\frac{1}{\sqrt{3}}-\sqrt{\frac{1}{6}}\right)^2}=34.970\cdots$

$\approx \underline{\underline{34.97}}$

Question 2 (b) [10 marks]

(i) A particle moves along the x-axis in forced oscillation without friction such that the displacement x (measured in metre) of the particle from the origin at any time t (measured in second) satisfies the differential equation

$$\ddot{x} + 32x = 16\sqrt{2}\cos\alpha t.$$

Initially at time $t = 0$, the particle is at rest at the origin. It is known that α is the resonant frequency. Find the distance of the particle from the origin at time $t = 7$ second. Give your answer in metre correct to two decimal places.

(ii) The monkey population at the Bukit Timah Nature Reserve follows a logistic model with a birth rate per capita of 10% per year. Initially at time $t = 0$ there were 2000 monkeys at the Reserve. After a very long time, the population settled down to an equilibrium value of M monkeys. If there were 1200 monkeys when time $t = 10$ year, find the value of M. Give your answer correct to the nearest integer.

Answer 2(b)(i)	13.25	**Answer 2(b)(ii)**	973

(Show your working below and on the next page.)

(i) Resonance $\Rightarrow \alpha = \sqrt{32}$

From notes: $x = At\sin\sqrt{32}t$

$\therefore \dot{x} = A\sin\sqrt{32}t + \sqrt{32}At\cos\sqrt{32}t$

$\ddot{x} = 2\sqrt{32}A\sin\sqrt{32}t - 32At\sin\sqrt{32}t$

$\Rightarrow 2\sqrt{32}A = 16\sqrt{2} \Rightarrow A = 2$

$\therefore x = 2t\sin\sqrt{32}t$

$t = 7 \Rightarrow x = 14\sin[7(\sqrt{32})]$

$= 13.253\cdots$

$\approx \underline{13.25}$

(ii) $B = 0.1$, $\hat{N} = 2000$, $N_\infty = M$

From notes

$$N = \frac{N_\infty}{1+\left(\frac{N_\infty}{\hat{N}}-1\right)e^{-Bt}} = \frac{M}{1+\left(\frac{M}{2000}-1\right)e^{-t/10}}$$

$$1200 = \frac{M}{1+\left(\frac{M}{2000}-1\right)e^{-1}}$$

$\therefore M = 973.4\cdots$

$\approx \underline{973}$

Question 3 (a) [10 marks]

(i) The growth of a type of bacteria follows a Malthus model with a birth rate per capita of 1.23 per bacteria per hour and a death rate per capita of D per bacteria per hour. If the number of bacteria doubles every two hours, find the value of D. Give your answer correct to two decimal places.

(ii) Let B, s and E denote three positive constants with $E < \frac{B^2}{4s}$. It is known that the differential equation $\frac{dx}{dt} = Bx - sx^2 - E$ has a stable equilibrium solution $x = \lambda$ and an unstable equilibrium solution $x = \alpha$. If $\frac{B^2}{sE} = \frac{17}{4}$, find the value of $\frac{\lambda}{\alpha}$. Give your answer correct to two decimal places.

Answer 3(a)(i)	0.88	**Answer 3(a)(ii)**	1.64

(Show your working below and on the next page.)

(i) $\frac{dN}{dt} = (1.23 - D)N$

$N = \hat{N} e^{(1.23-D)t}$

$2\hat{N} = \hat{N} e^{(1.23-D)\times 2}$

$\ln 2 = 2 \times (1.23 - D)$

$\therefore D = 0.883\ldots$

$\approx \underline{\underline{0.88}}$

(ii) $Bx - sx^2 - E = 0 \Rightarrow sx^2 - Bx + E = 0$

$\Rightarrow x = \frac{B \pm \sqrt{B^2 - 4sE}}{2s}$

$\therefore \lambda = \frac{B + \sqrt{B^2-4sE}}{2s}, \quad \alpha = \frac{B - \sqrt{B^2 - 4sE}}{2s}$

$\frac{\lambda}{\alpha} = \frac{B + \sqrt{B^2-4sE}}{B - \sqrt{B^2-4sE}}$

$= \frac{\frac{B}{\sqrt{sE}} + \sqrt{\frac{B^2}{sE} - 4}}{\frac{B}{\sqrt{sE}} - \sqrt{\frac{B^2}{sE} - 4}}$

$= \frac{\sqrt{\frac{17}{4}} + \sqrt{\frac{1}{4}}}{\sqrt{\frac{17}{4}} - \sqrt{\frac{1}{4}}} = 1.640\ldots$

$\approx \underline{\underline{1.64}}$

Question 3 (b) [10 marks]

(i) Let $F(s) = L((te^t)\,u(t-1))$, where L denotes the Laplace transform and u denotes the unit step function. Find the value of $F(1.8)$. Give your answer correct to two decimal places.

(ii) At time $t = 0$ a doctor injected 150 mg of morphine into a patient. At time $t = 2$ day the doctor injected 100 mg of morphine into the same patient. If the half-life of morphine in the patient's body is 0.5 day, find the amount of morphine in the patient's body at time $t = 3$ days. Give your answer in mg correct to two decimal places.

Answer 3(b)(i)	1.26	Answer 3(b)(ii)	27.34

(Show your working below and on the next page.)

(i) $F(s) = L\{(t-1+1)e^{(t-1)+1}u(t-1)\}$

$= L\{e(t-1)e^{t-1}u(t-1) + ee^{t-1}u(t-1)\}$

$= e\left\{\frac{1}{(s-1)^2} + \frac{1}{s-1}\right\}e^{-s}$

$F(1.8) = \left\{\frac{1}{0.8^2} + \frac{1}{0.8}\right\}e^{-0.8}$

$= 1.263\ldots$

$\approx \underline{\underline{1.26}}$

(ii) $k = \ln 2/0.5 = 2\ln 2$

$\begin{cases} \frac{dy}{dt} = -2\ln 2\, y + 150\delta(t) + 100\delta(t-2) \\ y(0) = 0 \end{cases}$

$sY = -2\ln 2\, Y + 150 + 100e^{-2s}$

$Y = \frac{150}{s+2\ln 2} + \frac{100}{s+2\ln 2}e^{-2s}$

$y = 150e^{(-2\ln 2)t} + 100e^{(-2\ln 2)(t-2)}u(t-2)$

$t=3 \Rightarrow y = 150e^{-6\ln 2} + 100e^{-2\ln 2}$

$= 27.343\ldots$

$\approx \underline{\underline{27.34}}$

Question 4 (a) [10 marks]

(i) Let $f(t) = L^{-1}\left(\frac{1}{(s-1)^2(s-2)^2}\right)$, where L^{-1} denotes the inverse Laplace transform. Find the value of $f(1.5)$. Give your answer correct to two decimal places.

(ii) Let $y(t)$ be the solution of the differential equation

$$y'' + 3y' + 2y = 2\{u(t-2) - u(t-4)\}$$

such that

$$y(0) = 0 \text{ and } y'(0) = 0,$$

where u denotes the unit step function. Find the value of $y(4.1)$. Give your answer correct to two decimal places.

Answer 4(a)(i)	5.64	Answer 4(a)(ii)	0.76

(Show your working below and on the next page.)

(i) $f(t) = L^{-1}\left(\frac{1}{(s-1)^2(s-2)^2}\right)$

$= L^{-1}\left\{\frac{1}{(s-1)^2} + \frac{2}{s-1} + \frac{1}{(s-2)^2} - \frac{2}{s-2}\right\}$

$= te^t + 2e^t + te^{2t} - 2e^{2t}$

$f(1.5) = 3.5e^{1.5} - 0.5e^3$

$= 5.643\cdots$

$\approx \underline{\underline{5.64}}$

(ii) Let $Y = L(y)$

$s^2Y + 3sY + 2Y = 2\left\{\frac{e^{-2s}}{s} - \frac{e^{-4s}}{s}\right\}$

$Y = 2\left\{\frac{e^{-2s}}{s(s+1)(s+2)} - \frac{e^{-4s}}{s(s+1)(s+2)}\right\}$

$= \left(\frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}\right)e^{-2s}$

$- \left(\frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}\right)e^{-4s}$

$y = \left\{1 - 2e^{-(t-2)} + e^{-2(t-2)}\right\}u(t-2)$

$- \left\{1 - 2e^{-(t-4)} + e^{-2(t-4)}\right\}u(t-4)$

$y(4.1) = \left\{1 - 2e^{-2.1} + e^{-4.2}\right\}$

$- \left\{1 - 2e^{-0.1} + e^{-0.2}\right\}$

$= 0.761\cdots$

$\approx \underline{\underline{0.76}}$

Question 4 (b) [10 marks]

(i) Let $w = w(x, y)$ denote a function of two variables x and y. If $w(x, y)$ is the answer that you get by applying the method of separation of variables to solve the partial differential equation $x^2(\frac{\partial w}{\partial x}) = w + y\frac{\partial w}{\partial y}$, with $x > 0$, $y > 0$ and $w(1, 1) = \frac{3}{e^2}$, find the value of $w(3, 3)$. Give your answer correct to two decimal places.

(ii) Let $y(t, x)$ be the solution of the wave equation

$$y_{tt} = y_{xx} \quad , \quad 0 \le t, \; 0 \le x \le \pi,$$

with $y(t, 0) = y(t, \pi) = 0$, $y(0, x) = \sin^3 x$, $y_t(0, x) = 0$.
Find the value of $y\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$. Give your answer correct to two decimal places.
(Suggestion: You may want to use d'Alembert's solution to the wave equation.)

Answer 4(b)(i)	4.62	Answer 4(b)(ii)	0.56

(Show your working below and on the next page.)

(i) Let $W = XY$

$x^2 X'Y = XY + yXY'$

$x^2 \frac{X'}{X} = 1 + y\frac{Y'}{Y} = k$

$x^2 \frac{X'}{X} = k \Rightarrow X = Ae^{-\frac{k}{x}}$

$1 + y\frac{Y'}{Y} = k \Rightarrow Y = By^{k-1}$

$\therefore W = Ce^{-\frac{k}{x}} y^{k-1}$

$\frac{3}{e^2} = Ce^{-k} \Rightarrow C = 3, k = 2$

$\therefore W = 3e^{-\frac{2}{x}} y$

$W(3,3) = 9e^{-\frac{2}{3}}$

$= 4.620\ldots$

$\approx \underline{4.62}$

(ii) $y(t, x) = \frac{1}{2}\{\sin^3(x+t) + \sin^3(x-t)\}$

$\therefore y(\frac{\pi}{6}, \frac{\pi}{3})$

$= \frac{1}{2}\{\sin^3(\frac{\pi}{3} + \frac{\pi}{6}) + \sin^3(\frac{\pi}{3} - \frac{\pi}{6})\}$

$= 0.562\ldots$

$\approx \underline{0.56}$