W04-03

Slide 01: In this unit, we will introduce the concept of orthogonality.

Slide 02: Let us begin with an example on some of the concepts introduced in the previous unit. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vectors in $\mathbb{R}^4$ as shown. We would like to compute the following quantities, namely $\boldsymbol{u} \cdot \boldsymbol{v}$, the length of $\boldsymbol{u}$ and $\boldsymbol{v}$, the distance between $\boldsymbol{u}$ and $\boldsymbol{v}$ as well as the angle between them.

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We start with the dot product between $\boldsymbol{u}$ and $\boldsymbol{v}$. Recall this is computed by multiplying the corresponding components of the two vectors before summing them up. This gives us the value of 5.

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The length, or the norm of $\boldsymbol{v}$ is the square root of the sum of the squares of $\boldsymbol{v}$'s components. Thus, we take the square root of 1+4+4+1, which gives the answer as follows.

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The norm of $\boldsymbol{u}$ can be computed similarly, giving us the square root of 5.

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The distance between $\boldsymbol{u}$ and $\boldsymbol{v}$ is the length of the vector $\boldsymbol{u}-\boldsymbol{v}$ and upon computation, we have the square root of 5.

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Finally the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is the arccosine of the expression $\boldsymbol{u} \cdot \boldsymbol{v}$ divided by the length of $\boldsymbol{u}$ times the length of $\boldsymbol{v}$. Thus we have the arccosine of $\frac{1}{\sqrt{2}}$ which gives us $\frac{\pi}{4}$.

Slide 03: We will now present a theorem that contains several results involving Euclidean vectors. Let c be a scalar and $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors in $\mathbb{R}^n$.

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First, the dot product $\boldsymbol{u} \cdot \boldsymbol{v}$ is the same as $\boldsymbol{v} \cdot \boldsymbol{u}$.

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Next, we have the following distributive law of dot product over addition.

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When we multiply a scalar to a vector, the effect on the dot product between two vectors is the following, which is expected. In other words, the scalar can be taken out first and only multiplied to the result after we have computed the dot product between the two vectors.

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The norm of a scalar multiple of a vector is simply the norm of the vector multiplied with the absolute value of the scalar.

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Lastly, the dot product of a vector with itself is always non negative. In fact the dot product of a vector with itself is zero if and only if the vector is the zero vector.

Slide 04: Let us prove the last result. Suppose the vector $\boldsymbol{u}$ has components u_1, u_2 and so on.

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Then $\boldsymbol{u} \cdot \boldsymbol{u}$ can be computed as such

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giving us $u_1^2 + u_2^2$ and so on until u_n^2.

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Since the components of $\boldsymbol{u}$ are all real numbers, it is immediate that each term in the summation is a non negative real number.

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This means that the summation is equal to zero if and only if each individual term in the summation is equal to zero.

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This is in turn equivalent to saying the each component in the vector $\boldsymbol{u}$ is zero.

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And thus $\boldsymbol{u}$ is the zero vector.

Slide 05: We will now define two related concepts.

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Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are said to be orthogonal if the dot product between them is 0. Obviously, the two vectors in question must be from the same Euclidean space.

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We can also use the term orthogonal to describe a set. We say a set S of vectors in $\mathbb{R}^n$ is orthogonal if evey pair of distinct vectors in S are orthogonal.

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To illustrate this, consider the set S with vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ and $\boldsymbol{x}$. For S to be an orthogonal set,

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we must have $\boldsymbol{u} \cdot \boldsymbol{v}$, $\boldsymbol{u} \cdot \boldsymbol{w}$, $\boldsymbol{u} \cdot \boldsymbol{x}$ equal to zero,

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and also $\boldsymbol{v} \cdot \boldsymbol{w}$, $\boldsymbol{v} \cdot \boldsymbol{x}$ and $\boldsymbol{w} \cdot \boldsymbol{x}$ equal to zero. All the 6 possible pairings of the 4 vectors in S must be orthgonal to each other before we can say that S is an orthgonal set.

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As an extension to an orthogonal set, we say that a set S is orthonormal if it is not only orthogonal but in addition, every vector in S must be a unit vector. Recall that a unit vector is one with length equal to 1.

Slide 06: The concept of orthogonality is actually not new.

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Recall the definition of the angle between two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. If the two vectors are orthogonal, meaning that $\boldsymbol{u} \cdot \boldsymbol{v}$ is zero, then the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$, which is the arccosine of $\boldsymbol{u} \cdot \boldsymbol{v}$ divided by the product of the lengths of the two vectors,

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will be the arccoisne of zero, which we know is $\frac{\pi}{2}$, which is 90 degrees, in the case where vectors are in $\mathbb{R}^2$ or $\mathbb{R}^3$ which we can visualise.

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Thus, the concept of orthogonality is a generalisation of perpendicularity that we are familiar with.

Slide 07: Consider the following example of the set S containing three vectors in $\mathbb{R}^3$. The vectors $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ and $\boldsymbol{u_3}$ are shown geometrically with the appropriate colors to make it clear.

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Computing the dot products $\boldsymbol{u_1} \cdot \boldsymbol{u_2}$, $\boldsymbol{u_1} \cdot \boldsymbol{u_3}$ and $\boldsymbol{u_2} \cdot \boldsymbol{u_3}$, we obtain zero in each case.

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Thus S is an orthogonal set.

Slide 08: Once we have an orthogonal set, it is easy to convert it into an orthonormal set.

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Let $\boldsymbol{v_1}$ be the vector $\boldsymbol{u_1}$ divided by its length. Note that $\boldsymbol{v_1}$ is essentially in the same direction as $\boldsymbol{u_1}$, only the magnitude is different.

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Likewise, we can define $\boldsymbol{v_2}$ and $\boldsymbol{v_3}$ to be the scaled version of their counterparts $\boldsymbol{u_2}$ and $\boldsymbol{u_3}$.

Slide 09: By doing so, we have maintained the pairwise orthogonality property of the set, but at the same time, made each of the vectors $\boldsymbol{v_1}$, $\boldsymbol{v_2}$ and $\boldsymbol{v_3}$ a unit vector.

Slide 10: It is now immediately clear that the set S', containing vectors $\boldsymbol{v_1}, \boldsymbol{v_2}$ and $\boldsymbol{v_3}$ is an orthonormal set.

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The process that we have just described here where we convert and orthogonal set into an orthonormal set is known as normalizing.

Slide 11: Let us summarise the main points in this unit.

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We had a theorem that gives some properties on dot products. In particular, we saw that a vector dot with itself is always non negative and is zero if and only if the vector is the zero vector.

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We introduced the term orthogonal to describe the relationship between two vectors. In $\mathbb{R}^2$ or $\mathbb{R}^3$ this is essentially the same as perpendicularity.

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We also defined what is an orthogonal set.

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Orthonormal set is one that is orthogonal and the vectors in the set are unit vectors. We can always normalize an orthogonal set to obtain an orthonormal set.