NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

Module: MA1508E Linear Algebra for Engineering
Year/Semester: 2018-2019 (Semester 2)
Tutorial: 9

1. For each of the following linear system $\boldsymbol{Ax} = \boldsymbol{b}$,

    (i) Show that the system is inconsistent;

    (ii) Find a least squares solution $\boldsymbol{x'}$ to the system. Is there a unique least squares solution or infinitely many?

    (iii) Compute the least squares error, defined as $||\boldsymbol{b} - \boldsymbol{Ax'}||$. If there are infinitely many least squares solution and $\boldsymbol{x'_1}$, $\boldsymbol{x'_2}$ are any two of them, would the least squares error $||\boldsymbol{b} - \boldsymbol{Ax'_1}||$ and $||\boldsymbol{b} - \boldsymbol{Ax'_2}||$ be the same?

    (a) $\boldsymbol{A} = \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}.$

    (b) $\boldsymbol{A} = \begin{pmatrix} 3 & 2 & -1 \\ 1 & -4 & 3 \\ 1 & 10 & -7 \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$

2. For each of the following, compute the orthogonal projection of $\boldsymbol{u}$ onto the subspace spanned by $\boldsymbol{v_1}, \cdots, \boldsymbol{v_k}$.

    (a) $\boldsymbol{u} = (1, -6, 1)$, $\boldsymbol{v_1} = (-1, 2, 1)$, $\boldsymbol{v_2} = (2, 2, 4)$.

    (b) $\boldsymbol{u} = (6, 12, 3, 6)$, $\boldsymbol{v_1} = (1, 1, 0, 0)$, $\boldsymbol{v_2} = (1, 0, 1, 0)$, $\boldsymbol{v_3} = (3, 1, 1, 1)$.

3. A series of experiments were performed to investigate the relationship between two physical quantities x and y. The results of the experiments are shown in the table below.

x	0	1	2	3
y	3	2	4	4

    (a) Find a least squares solution $\boldsymbol{x} = (\hat{a}, \hat{b})$ if it is believed that x and y are related linearly, that is, $y = ax + b$.

    (b) Find a least squares solution $\boldsymbol{x} = (\hat{a}, \hat{b}, \hat{c})$ if it is believed that x and y are related by the quadratic polynomial $y = ax^2 + bx + c$.

    (c) Which model (linear or quadratic) would produce a smaller least squares error?

4. Prove that if $\boldsymbol{A}$ has linearly independent column vectors, and if $\boldsymbol{b}$ is orthogonal to the column space of $\boldsymbol{A}$, then the least squares solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is $\boldsymbol{x} = \boldsymbol{0}$.

5. (**$\boldsymbol{QR}$-factorisation**) Let $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $\boldsymbol{u_1} = (1,1,1,0)^T$, $\boldsymbol{u_2} = (-1,0,-1,0)^T$, $\boldsymbol{u_3} = (-1,0,0,-1)^T$.

   (a) Use Gram-Schmidt Process to transform $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ into an orthonormal basis $\{\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3}\}$ for the column space of $\boldsymbol{A}$. (Do not change the order of $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ when applying the Gram-Schmidt Process.)

   (b) Write each of $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ as a linear combination of $\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3}$.

   (c) Hence or otherwise, write $\boldsymbol{A} = \boldsymbol{QR}$ where $\boldsymbol{Q}$ is a 4×3 matrix with orthonormal columns and $\boldsymbol{R}$ is a 3×3 upper triangular matrix with positive entries along its diagonal.

   Remark: $\boldsymbol{QR}$-factorisation is widely used in computer algorithms for various computations concerning matrices.