# MATRIX INVERSE LAWS

# CANCELLATION LAW

If $\boldsymbol{A}$ is an invertible square matrix and

$$\boldsymbol{AB}_1 = \boldsymbol{AB}_2 \;\Rightarrow \boldsymbol{A}^{-1}\boldsymbol{AB}_1 = \boldsymbol{A}^{-1}\boldsymbol{AB}_2$$

$$\Rightarrow \boldsymbol{IB}_1 = \boldsymbol{IB}_2$$

then $\boldsymbol{B}_1 = \boldsymbol{B}_2$.

If $\boldsymbol{A}$ is an invertible square matrix and

$$\boldsymbol{C}_1\boldsymbol{A} = \boldsymbol{C}_2\boldsymbol{A} \;\Rightarrow \boldsymbol{C}_1\boldsymbol{AA}^{-1} = \boldsymbol{C}_2\boldsymbol{AA}^{-1}$$

$$\Rightarrow \boldsymbol{C}_1\boldsymbol{I} = \boldsymbol{C}_2\boldsymbol{I}$$

then $\boldsymbol{C}_1 = \boldsymbol{C}_2$.

# CANCELLATION LAW

If $\boldsymbol{A}$ is not an invertible matrix, the cancellation law may not hold:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$ (known to be singular by previous example)

$$\boldsymbol{B}_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Check that $\boldsymbol{AB}_1 = \boldsymbol{AB}_2$, but $\boldsymbol{B}_1 \neq \boldsymbol{B}_2$.

$$\boldsymbol{B}_2 = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$$

# TRANSPOSE OF A MATRIX

Let $A = (a_{ij})_{m\times n}$ be a $\boxed{m\times n}$ matrix.

The **transpose** of $A$, denoted by $A^T$, is a $\boxed{n\times m}$ matrix whose <u>$(i, j)$-entry is $a_{ji}$</u>.

$$
\begin{array}{cc}
 & \overbrace{\hphantom{\quad\quad\quad\quad\quad\quad}}^{n \text{ columns}} \\
m \text{ rows} \left\{ \vphantom{\begin{matrix} \\ \\ \\ \\ \\ \end{matrix}} \right. & \left( \begin{matrix} \quad\quad\quad\quad\quad\quad \\ \\ \\ \\ \\ \end{matrix} \right) \\
 & A
\end{array}
\qquad
\begin{array}{cc}
 & \overbrace{\hphantom{\quad\quad\quad\quad\quad\quad}}^{m \text{ columns}} \\
n \text{ rows} \left\{ \vphantom{\begin{matrix} \\ \\ \\ \\ \\ \end{matrix}} \right. & \left( \begin{matrix} \quad\quad\quad\quad\quad\quad \\ \\ \\ \\ \\ \end{matrix} \right) \\
 & A^T
\end{array}
$$

# EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 3 & 0 & -1 & 2 & 0 \\ 2 & 1 & 0 & 4 & 6 \\ 1 & -1 & 1 & 1 & 0 \end{pmatrix}$$

$$\boldsymbol{B} = \begin{pmatrix} 1 & -1 & 0 & 3 \\ -1 & 2 & 3 & 4 \\ 0 & 3 & -3 & 1 \\ 3 & 4 & 1 & 0 \end{pmatrix}$$

$$\boldsymbol{A}^T = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 2 & 4 & 1 \\ 0 & 6 & 0 \end{pmatrix}$$

$$\boldsymbol{B}^T = \begin{pmatrix} 1 & -1 & 0 & 3 \\ -1 & 2 & 3 & 4 \\ 0 & 3 & -3 & 1 \\ 3 & 4 & 1 & 0 \end{pmatrix}$$

# SYMMETRIC IN TERMS OF TRANSPOSE

A matrix $\boldsymbol{A}$ is symmetric if and only if

$$\boxed{\boldsymbol{A} = \boldsymbol{A}^T}$$

$$\begin{pmatrix} & a_{ij} \\ a_{ji} & \end{pmatrix}$$

$a_{ij} = a_{ji}$ for all $i$, $j$.

# SOME RESULTS ON TRANSPOSE

Let $\boldsymbol{A}$ be a $m \times n$ matrix.

1) $(\boldsymbol{A}^T)^T = \boldsymbol{A}$

2) $(\boldsymbol{A}+\boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$ (transpose of sum equal sum of transpose).

3) $(a\boldsymbol{A})^T = a\boldsymbol{A}^T$

4) If $\boldsymbol{B}$ is a $n \times p$ matrix, then $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$

(transpose of product equal to product of transpose, but with order reversed).

# MORE RESULTS FOR INVERSE

1) Let $\boldsymbol{A}$ be an invertible matrix (so $\boldsymbol{A}^{-1}$ exists) and $c$ a non zero scalar. Then $c\boldsymbol{A}$ is invertible and

$$(c\boldsymbol{A})^{-1} = \tfrac{1}{c}\boldsymbol{A}^{-1}.$$

$$(\tfrac{1}{c}\boldsymbol{A}^{-1})(c\boldsymbol{A})$$

$$= (\tfrac{1}{c} \times c)\boldsymbol{A}^{-1}\boldsymbol{A}$$

$$= \boldsymbol{I}$$

$$(c\boldsymbol{A})(\tfrac{1}{c}\boldsymbol{A}^{-1})$$

$$= (c \times \tfrac{1}{c})\boldsymbol{A}\boldsymbol{A}^{-1}$$

$$= \boldsymbol{I}$$

# MORE RESULTS FOR INVERSE

2) Let $\boldsymbol{A}$ be an invertible matrix (so $\boldsymbol{A}^{-1}$ exists) then

$\boldsymbol{A}^T$ is invertible and

$$(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T.$$

Inverse of transpose equal transpose of inverse

$$(\boldsymbol{A}^{-1})^T \boldsymbol{A}^T \qquad\qquad \boldsymbol{A}^T (\boldsymbol{A}^{-1})^T$$

$$= (\boldsymbol{A}\boldsymbol{A}^{-1})^T \qquad\qquad = (\boldsymbol{A}^{-1}\boldsymbol{A})^T$$

$$= \boldsymbol{I} \qquad\qquad = \boldsymbol{I}$$

# MORE RESULTS FOR INVERSE

3) Let $\boldsymbol{A}$ be an invertible matrix (so $\boldsymbol{A}^{-1}$ exists) then

$\boldsymbol{A}^{-1}$ is invertible and

$$(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}.$$

$$\boldsymbol{A}\boldsymbol{A}^{-1} \qquad \boldsymbol{A}^{-1}\boldsymbol{A}$$

$$= \boldsymbol{I} \qquad = \boldsymbol{I}$$

# MORE RESULTS FOR INVERSE

4) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two invertible matrices of the same size (so $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}^{-1}$ exists) then $\boxed{\boldsymbol{AB}}$ is invertible and

$$(\boldsymbol{AB})^{-1} = \boxed{\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}}.$$

Inverse of product equal product of inverse*

$$\boxed{\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}}(\boxed{\boldsymbol{AB}})$$

$$= \boldsymbol{B}^{-1}(\boldsymbol{A}^{-1}\boldsymbol{A})\boldsymbol{B}$$

$$= \boldsymbol{B}^{-1}(\boldsymbol{I})\boldsymbol{B}$$

$$= \boldsymbol{B}^{-1}\boldsymbol{B} = \boldsymbol{I}$$

$$(\boxed{\boldsymbol{AB}})\boxed{\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}}$$

$$= \boldsymbol{A}(\boldsymbol{BB}^{-1})\boldsymbol{A}^{-1}$$

$$= \boldsymbol{A}(\boldsymbol{I})\boldsymbol{A}^{-1}$$

$$= \boldsymbol{AA}^{-1} = \boldsymbol{I}$$

# MORE RESULTS FOR INVERSE

4) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two invertible matrices of the same size (so $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}^{-1}$ exists) then $\boldsymbol{AB}$ is invertible and

$$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}.$$

Inverse of product equal product of inverse*

As an extension to the above, if $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_k$ are invertible matrices of the same size, then $(\boldsymbol{A}_1\boldsymbol{A}_2\ldots\boldsymbol{A}_k)$ is invertible and

$$\boxed{(\boldsymbol{A}_1\boldsymbol{A}_2\ldots\boldsymbol{A}_k)^{-1} = \boldsymbol{A}_k{}^{-1}\ldots\boldsymbol{A}_2{}^{-1}\boldsymbol{A}_1{}^{-1}.}$$

# DEFINITION

Let $\boldsymbol{A}$ be a square matrix and $n$ be a non negative integer, then

$$\boldsymbol{A}^n = \begin{cases} \boldsymbol{I} & \text{if } n = 0; \\ \underbrace{\boldsymbol{A}\boldsymbol{A}\ldots\boldsymbol{A}}_{n \text{ times}} & \text{if } n \geq 1. \end{cases}$$

If $\boldsymbol{A}$ is an invertible square matrix, then $\boldsymbol{A}^n$ is invertible and we define

$$\boldsymbol{A}^{-n} = (\boldsymbol{A}^{-1})^n = \underbrace{\boldsymbol{A}^{-1}\boldsymbol{A}^{-1}\ldots\boldsymbol{A}^{-1}}_{n \text{ times}} \quad \text{and } (\boldsymbol{A}^n)^{-1} = \boldsymbol{A}^{-n}$$

# EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \qquad \boldsymbol{A}^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

$$\boldsymbol{A}^2 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 8 \\ 4 & 11 \end{pmatrix}$$

$$\boldsymbol{A}^{-2} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 11 & -8 \\ -4 & 3 \end{pmatrix}$$

$$\boldsymbol{A}^2\boldsymbol{A}^{-2} = \begin{pmatrix} 3 & 8 \\ 4 & 11 \end{pmatrix}\begin{pmatrix} 11 & -8 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{A}^{-2}\boldsymbol{A}^2$$

# SUMMARY

1) Some laws involving the inverse of a matrix.

2) Transpose of a matrix and some laws.

3) Inverse of the powers of an invertible matrix.