1. Let $y$ be a solution of the differential equation

$$\frac{dy}{dx} = \frac{xy^2}{x-1},$$

with $x < 1$ and $y(-\frac{1}{2}) = 2$. Find the value of $y(-1)$. Give your answer correct to one decimal place.

(A) 1.8

(B) 1.7

(C) 1.5

(D) 1.4

(E) None of the above

$$\frac{dy}{y^2} = \frac{x}{x-1}dx = \frac{(x-1)+1}{x-1}dx = \left(1+\frac{1}{x-1}\right)dx$$

$$-\frac{1}{y} = x + \ln|x-1| + C$$

$$= x + \ln(1-x) + C \quad (\because x<1)$$

$$x=-\frac{1}{2},\ y=2 \Rightarrow -\frac{1}{2} = -\frac{1}{2} + \ln\frac{3}{2} + C$$

$$\Rightarrow C = -\ln\frac{3}{2}$$

$$\therefore -\frac{1}{y} = x + \ln(1-x) - \ln\frac{3}{2}$$

$$x=-1 \Rightarrow -\frac{1}{y} = -1 + \ln 2 - \ln\frac{3}{2}$$

$$\Rightarrow y = \frac{1}{1-\ln 2+\ln\frac{3}{2}}$$

$$= 1.403\ldots$$

$$\approx \underline{\underline{1.4}}$$

2. Let $a$ and $b$ denote two positive constants. In the radioactive decay example in the notes we have ignored the effect of the temperature on the rate of decay when we made the assumption that the rate of decay is directly proportional to the amount of substance present. However, for a substance which decays fast, we can expect that it will generate a significant amount of heat during the decay and so there will be a noticable rise in the temperature of the substance when it decays. As we know in any chemical process heat will increase the rate of reaction and so it is reasonable to expect that in the case of a substance that decays fast the proportional constant is in fact not a constant but is actually a bounded function that increases with time. You are doing an experiment with one such substance and you find that the rate of decay of this substance can be modelled by the equation $\frac{dQ}{dt} = -\left(\tan^{-1} t\right) Q$, where $Q$ is the amount of the substance measured in gram at time $t$ measured in second. If at time $t = 0$ second you start with $a$ grams of this substance and you observe that you still have $b$ grams of this substance at time $t = 2.5$ second, find the value of $\frac{a}{b}$. Give your answer correct to two decimal places.

(A) 7.28

(B) 6.53

(C) 7.65

(D) 6.72

(E) None of the above

$-\frac{dQ}{Q} = \tan^{-1}t \, dt, \; Q(0)=a, \; Q(2.5)=b$

$-\int_a^b \frac{dQ}{Q} = \int_0^{2.5} \tan^{-1}t \, dt$

$-\ln b + \ln a = \left[t\tan^{-1}t - \frac{1}{2}\ln(t^2+1)\right]_0^{2.5}$

$\frac{a}{b} = e^{2.5\tan^{-1}2.5 - \frac{1}{2}\ln(2.5^2+1)}$

$= 7.280\cdots$

$\approx \underline{\underline{7.28}}$

3. Let $a$ and $b$ denote two positive constants. Let $y$ be a solution of the differential equation

$$\frac{dy}{dx} = \frac{y - \sqrt{x^2 + y^2}}{x},$$

with $0 < x \leq 21a$, $0 \leq y$ and $y((\sqrt{21})a) = 10a$. If $y(b) = a$, find the value of $\frac{b}{a}$. Give your answer correct to two decimal places.

(A) 18.29

(B) 19.06

(C) 18.57

(D) 19.97

(E) None of the above

Let $y = ux \Rightarrow u + x\frac{du}{dx} = \frac{ux - \sqrt{x^2 + u^2x^2}}{x} = u - \sqrt{1+u^2}$

$\int \frac{du}{\sqrt{1+u^2}} = \int -\frac{dx}{x} = -\ln x + C$

Let $\tan\theta = u = y/x$

$\int \frac{du}{\sqrt{1+u^2}} = \int \frac{\sec^2\theta\, d\theta}{\sec\theta} = \ln(\sec\theta + \tan\theta) = \ln\frac{y + \sqrt{x^2+y^2}}{x}$

$\therefore \ln(y + \sqrt{x^2+y^2}) = C \Rightarrow y + \sqrt{x^2+y^2} = D$

$x = \sqrt{21}a,\ y = 10a \Rightarrow 10a + 11a = D \Rightarrow D = 21a$

$\therefore \sqrt{x^2+y^2} = 21a - y \Rightarrow x^2 + y^2 = 441a^2 - 42ay + y^2$

$\Rightarrow x^2 = 441a^2 - 42ay$

$x = b,\ y = a \Rightarrow b^2 = 441a^2 - 42a^2 = 399a^2$

$\Rightarrow b/a = \sqrt{399} = 19.974\ldots$

$\approx \underline{\underline{19.97}}$

4. Let $a$ and $b$ denote two positive constants. Let $y$ denote the solution of the differential equation

$$\frac{dy}{dx} = \frac{2y^2 + ax^4e^x}{2xy}.$$

with $x > 0$, $y > 0$ and $y(1) = \sqrt{a}$. If $y(2) = \sqrt{b}$, find the value of $\frac{b}{a}$. Give your answer correct to two decimal places.

(A) 37.15

(B) 36.23

(C) 33.56

(D) 32.38

(E) None of the above

$$\frac{dy}{dx} = \frac{y}{x} + \frac{a}{2}x^3e^xy^{-1} \Rightarrow \frac{dy}{dx} - \frac{y}{x} = \frac{a}{2}x^3e^xy^{-1}$$

$$\text{let } z = y^{1-(-1)} = y^2 \Rightarrow \frac{1}{2y}\frac{dz}{dx} - \frac{y}{x} = \frac{a}{2}x^3e^xy^{-1}$$

$$\Rightarrow \frac{dz}{dx} - \frac{2}{x}z = ax^3e^x$$

$$R = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = \frac{1}{x^2}$$

$$z = x^2\int \frac{1}{x^2}ax^3e^xdx = x^2\{axe^x - ae^x + c\}$$

$$y^2 = ax^3e^x - ax^2e^x + cx^2$$

$$y(1) = \sqrt{a} \Rightarrow a = ae - ae + c \Rightarrow c = a$$

$$\therefore y^2 = ax^3e^x - ax^2e^x + ax^2$$

$$y(2) = \sqrt{b} \Rightarrow b = a(8e^2 - 4e^2 + 4)$$

$$\Rightarrow \frac{b}{a} = 4e^2 + 4 = 33.556\ldots$$

$$\approx \underline{\underline{33.56}}$$

5. A tank contains 100 liters of a salt solution with a concentration of 2 gram/liter. Starting at time $t = 0$ minute, a solution with a salt concentration of 3 gram/liter is added to the tank at 4 liters/min, and the resulting well-stirred mixture is drained out at the same rate. How long does it take for the salt concentration in the tank to reach 2.93 gram/liter? Give your answer in minutes correct to two decimal places.

(A) 76.35 minutes

(B) 62.13 minutes

(C) 73.29 minutes

(D) 66.48 minutes

(E) None of the above

Let $Q$ = amount of salt in gram at time $t$ minute.

$$\frac{dQ}{dt} = 12 - \frac{4Q}{100} = 12 - 0.04Q$$

$$\frac{dQ}{dt} + 0.04Q = 12$$

$$R = e^{\int 0.04\,dt} = e^{0.04t}$$

$$Q = e^{-0.04t}\int 12e^{0.04t}\,dt = e^{-0.04t}\{300e^{0.04t} + c\}$$

$$\therefore Q = 300 + ce^{-0.04t}$$

$$Q(0) = 200 \Rightarrow C = -100$$

$$\therefore Q = 300 - 100e^{-0.04t}$$

$$Q = 293 \Rightarrow 293 = 300 - 100e^{-0.04t}$$

$$\Rightarrow t = 66.481\ldots$$

$$\approx \underline{\underline{66.48}}$$

6. Let $n$ and $m$ denote two positive integers with $1 < m < \frac{n}{2}$. In the "rumour" example studied in Tutorial 2 Question 4, we assume that the rate of spread of the rumour is jointly proportional to the number of people who have heard it and the number of people who have not heard it. However, as time goes on, it is reasonable to expect that there will be more rumours to compete for people's attention and so people's interest in the first rumour will go down with time. Therefore, it may be more accurate to assume that the proportional constant is not a true constant but rather it should be a decreasing function of time which eventually goes to zero. Taking this into consideration, you model the rate of spread with the differential equation $\frac{dR}{dt} = \left(\frac{1}{n}e^{-\frac{1}{10}t}\right) R\,(n-R)$, where $R$ is the number of people who have heard the rumour and $n$ is the population under consideration. Time $t$ being measured in month. Initially at time $t = 0$, it is known that $R = m$. It is also known that at time $t = 10$ month, we have $R = 2m$. Find the value of $\ln \frac{n-m}{\frac{n}{2}-m}$. Give your answer correct to two decimal places.

(A) 6.32

(B) 7.21

(C) 5.95

(D) 8.47

(E) None of the above

$$\frac{n}{R(n-R)}\,dR = e^{-0.1t}\,dt$$

$$\left(\frac{1}{R}+\frac{1}{n-R}\right)dR = e^{-0.01t}\,dt \Rightarrow \ln R - \ln(n-R) = -10e^{-0.1t} + C$$

$$R(0)=m \Rightarrow \ln m - \ln(n-m) = -10 + C$$

$$\therefore \ln R - \ln(n-R) = -10e^{-0.1t} + \ln m - \ln(n-m) + 10$$

$$R(10) = 2m \Rightarrow \ln 2m - \ln(n-2m) = -10e^{-1} + \ln m - \ln(n-m) + 10$$

$$\Rightarrow \ln 2 + \ln(n-m) - \ln(n-2m) = -10e^{-1} + 10$$

$$\Rightarrow \ln \frac{n-m}{n/2-m} = -10e^{-1} + 10$$

$$= 6.321\cdots$$

$$\approx \underline{\underline{6.32}}$$