W05-05

Slide 01: In this unit, we introduce the concept of a basis.

Slide 02: We first define what is known as a vector space. A set of vectors V is said to be a vector space if V is either the Euclidean n-space for some positive integer n, or V is a subspace of $\mathbb{R}^n$.

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We have already understood the self-contained feature of subspaces. So essentially, a vector space is a collection of vectors that only interact among themselves. Of course, the entire $\mathbb{R}^n$ is one such vector space,

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and another possibility is that V does not need to be the entire $\mathbb{R}^n$ but instead a subspace of $\mathbb{R}^n$.

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We can have subspaces within a vector space, so the concept of subspaces we have seen earlier is not restricted to subspaces of $\mathbb{R}^n$ but any vector space V contained inside another vector space W is called a subspace of W.

Slide 03: We now consider a vector space V. Note that V can be the entire $\mathbb{R}^n$ or a subspace of $\mathbb{R}^n$. We would like to find a subset S of V, containing as few vectors as possible, so that every vector in V is a linear combination of the vectors in S. In other words, we would like to find a small set of vectors S, such that S spans the vector space V.

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Once such a such set is found, we can use it to build a coordinate system for the vector space V.

Slide 04: This leads us to the definition of a basis. Let S be a subset of a vector space V, where S contains vectors $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$. We say that S is a basis for V if S satisfies two properties. Firstly, S must be a linearly independent set. Secondly S must span the vector space V. Note that the two concepts of linear independence and linear span have been discussed previously.

Slide 05: Let us look at a few examples. For this example, we would like to show that S is a basis for $\mathbb{R}^2$.

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We first check if S is a linearly independent set. Indeed it is easy to see that since S contains only two vectors which are not multiples of each other, S is definitely a linearly independent set.

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Next, does S span $\mathbb{R}^2$?

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Recall that we need to check whether every vector in $\mathbb{R}^2$ can be written as a linear combination of the two vectors in S. In other words, is the vector equation shown here always consistent for all x, y?

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You should be familiar with the method of checking this. We write down a linear system and the corresponding augemented matrix,

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whose row-echelon form is shown here. This row-echelon form tells us that regardless of the values taken by x and y, the vector equation is always consistent. Thus S spans $\mathbb{R}^2$.

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Since we have checked that S satisfies both properties of a basis for $\mathbb{R}^2$, we have shown the required.

Slide 06: Another similar example is seen here, where we would show that S is a basis for $\mathbb{R}^3$.

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We first check whether S spans $\mathbb{R}^3$.

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Once again, we consider the following vector equation and determine if it is always consistent for all vectors (x, y, z).

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This is the augmented matrix of the linear system

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and upon performing Gaussian elimination, we arrive at the following row-echelon form. Note that the three entries on the right hand side of the vertical line are some expressions involving x, y and z.

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What is important in this case is that we observe the three columns on the left of the vertical line are all pivot columns. Thus, once again, the linear system, as well as the vector equation, will always be consisten for all values of x, y and z. Therefore, S spans $\mathbb{R}^3$.

Slide 07: We now need to check whether S is a linearly independent set.

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By definition, we need to check if the following vector equation has only the trivial solution. In this case, we actually do not need to do much further work before obtaining the conclusion.

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Recall the following step involved when we determine if S spans $\mathbb{R}^3$. Note that the vector equation we are considering now is simply one where we let $x = y = z = 0$. Thus

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we can simply infer that at row-echelon form, the homogeneous linear system we are investigating would give us the following matrix as shown. Once again, the three pivot columns on the left side will enable us to conclude that S is indeed a linearly independent set.

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We have now checked both conditions and thus shown that S is a basis for $\mathbb{R}^3$.

Slide 08: What about this set S? Is S a basis for $\mathbb{R}^4$?

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The answer is immediately known to be no, because we already know from a previous unit that 3 vectors will never be able to span $\mathbb{R}^4$. Thus S can never be a basis for $\mathbb{R}^4$.

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What about this set S? It does have 4 vectors but can it be a basis for $\mathbb{R}^4$?

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Once again, the answer is no. A quick observation will reveal that the 4 vectors in S all have 0 in the second component. Thus, for example, the vector $(0, 1, 0, 0)$ will not be a linear combination of the vectors in S. In other words, S does not span $\mathbb{R}^4$ and thus cannot be a basis for it.

Slide 09: Let us look at a few remarks on bases.

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A basis is the answer to our question of finding the smallest possible number of vectors to span a vector space.

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For the extreme case of the zero space, we will define the empty set as the basis for the zero space. Note that this is for convenience and we normally are not concerned with the zero space.

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Other than the zero space, any vector space has infinitely many different bases. In other words, bases are not unique.

Slide 10: We will end this unit with a theorem. If S is a basis for a vector space V, then every vector $\boldsymbol{v}$ in the vector space can be expressed as a linear combination of these basis vectors. This is obvious, since the basis vectors $\boldsymbol{u_1}, \boldsymbol{u_2}$ to $\boldsymbol{u_k}$ are supposed to span the vector space V. However, this theorem asserts that the $\boldsymbol{v}$ can be expressed as a linear combination of these basis vectors in exactly one way. In other words, the expression is unique.

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To prove this result, suppose it can be done in two ways, meaning that there are two sets of coefficients, a_1 to a_k and b_1 to b_k such that $\boldsymbol{v}$ can be written in terms of the basis vectors. We now have two vector equations (1) and (2).

Slide 11: We are now going to take the difference between equations (1) and (2)

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and when we do so, we have this vector equation (*) where the zero vector is on the left hand side. On the right hand side we have a linear combination of the basis vectors $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$.

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Now note that S is a basis, which means that it has to be a linearly independent set.

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So the vector equation (*) must have only the trivial solution.

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This implies that the coefficents must all be zero

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which in turn implies that $a_1 = b_1$, $a_2 = b_2$ and so on till $a_k = b_k$.

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Since the two sets of coefficients are actually the same, this means that there is only just one way to write $\boldsymbol{v}$ in terms of the basis vectors. We have thus concluded the proof.

Slide 12: Let us summarise this unit.

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We defined what is a vector space and also a subspace of a vector space. This is a generalisation of our earlier definition of a subspace for $\mathbb{R}^n$.

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We defined what is a basis for a vector space.

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Lastly, we saw that there is a unique way of representing any vector in a vector space in terms of a fixed set of basis vectors.