FINDING A BASIS FOR COLUMN SPACE

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

How to find a basis for the column space of $\boldsymbol{A}$?

$$\text{column space of } \boldsymbol{A} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} \right\}$$

column space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^3$,

$\Rightarrow$ the dimension of this subspace is at most 3

So if we can identify 3 linearly independent vectors (out of the 5) from the set above...

EXAMPLE

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

How to find a basis for the column space of A?

$$\text{column space of } A = \text{span}\{ \underbrace{\boxed{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \boxed{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}, \boxed{\begin{pmatrix} -1 \\ 4 \\ -2 \end{pmatrix}}}_{}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} \}$$

linearly independent

column space of A is a subspace of $\mathbb{R}^3$,

$\Rightarrow$ the dimension of this subspace is at most 3

Column space of A is the entire $\mathbb{R}^3$.

FINDING BASIS FOR COLUMN SPACES

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

How to find a basis for the column space of $\boldsymbol{A}$?

That was based on observations...

Yes, you are right. A more systematic approach is needed.

FINDING BASIS FOR COLUMN SPACES

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

column space of $\boldsymbol{A}$

$=$ row space of $\boldsymbol{A}^T$

$$\boldsymbol{A}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 4 & -2 \\ 1 & 2 & 0 \\ 4 & 1 & 1 \end{pmatrix}$$

So we already know how to do it!

Note the relationship between column space of $\boldsymbol{A}$ and row space of $\boldsymbol{A}^T$ ...

FINDING BASIS FOR COLUMN SPACES

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

column space of $\boldsymbol{A}$

$=$ row space of $\boldsymbol{A}^T$

$$\boldsymbol{A}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 4 & -2 \\ 1 & 2 & 0 \\ 4 & 1 & 1 \end{pmatrix}$$

So to find a basis for the column space of $\boldsymbol{A}$, we can use the previous method to find a basis for the row space of $\boldsymbol{A}^T$.

In what follows, we will discuss another method.

IMPORTANT TO NOTE

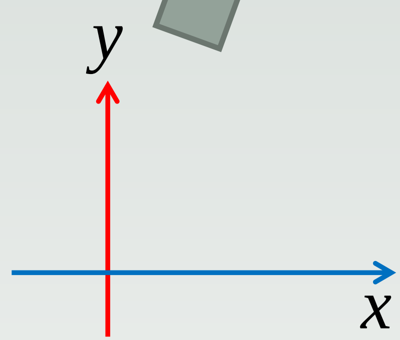
Elementary row operations preserve the row space of a matrix but NOT the column space.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \boldsymbol{B}$$

column space of $\boldsymbol{A}$ $\neq$ column space of $\boldsymbol{B}$

$= \text{span}\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$ $\qquad$ $= \text{span}\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$

$= \boxed{x - \text{axis}}$ $\qquad$ $= \boxed{y - \text{axis}}$



THEOREM

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be row equivalent matrices. Then the following statements hold:

$$\boldsymbol{A} = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & * & * \end{pmatrix} \xleftrightarrow[\text{e.r.o}]{\text{Series of}} \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & * & * \end{pmatrix} = \boldsymbol{B}$$

A given set of columns of $\boldsymbol{A}$ is linearly independent if and only if the corresponding columns of $\boldsymbol{B}$ is linearly independent.

THEOREM

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be row equivalent matrices. Then the following statements hold:

$$\boldsymbol{A} = \begin{pmatrix} \boxed{*} & * & \boxed{*} & * & \boxed{*} \\ * & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & * & * \end{pmatrix} \xrightarrow{\text{e.r.o}} \begin{pmatrix} \boxed{*} & * & \boxed{*} & * & \boxed{*} \\ * & * & * & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & * & * \end{pmatrix} = \boldsymbol{B}$$

Remember: Column space of $\boldsymbol{A} \neq$ Column space of $\boldsymbol{B}$

A given set of columns of $\boldsymbol{A}$ forms a basis for the column space of $\boldsymbol{A}$ if and only if the corresponding columns of $\boldsymbol{B}$ forms a basis for the column space of $\boldsymbol{B}$.

OBSERVATION

If $\boldsymbol{R}$ is a matrix in row echelon form, the pivot columns of $\boldsymbol{R}$ always form a basis for the column space of $\boldsymbol{R}$.

$$\boldsymbol{R} = \begin{pmatrix} & \otimes & * & & & & & * \\ & & & \otimes & * & & & * \\ & & \mathbf{0} & & \ddots & & & * \\ & & & & & \otimes & * & * \\ 0 & \cdots & & \cdots & & \cdots & 0 & 0 \end{pmatrix}$$

Question : How to find a basis for the column space of a matrix $\boldsymbol{A}$?

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be row equivalent matrices.

A given set of columns of $\boldsymbol{A}$ forms a basis for the column space of $\boldsymbol{A}$ if and only if the corresponding columns of $\boldsymbol{B}$ forms a basis for the column space of $\boldsymbol{B}$.

FINDING A BASIS FOR COLUMN SPACE

If $\boldsymbol{R}$ is a matrix in row echelon form, the pivot columns of $\boldsymbol{R}$ always form a basis for the column space of $\boldsymbol{R}$.

$$\boldsymbol{R} = \begin{pmatrix} & \otimes & * & & & & & * \\ & & & \otimes & * & & & * \\ & & 0 & & \ddots & & & * \\ & & & & & \otimes & * & * \\ 0 & \cdots & & \cdots & & \cdots & 0 & 0 \end{pmatrix}$$

Question: How to find a basis for the column space of a matrix $\boldsymbol{A}$?

Answer

Let $\boldsymbol{R}$ be a row echelon form of $\boldsymbol{A}$.

Remember NOT to take the columns of $\boldsymbol{R}$ as your answer!

A basis for the column space of $\boldsymbol{A}$ can be obtained by taking the columns of $\boldsymbol{A}$ that correspond to the pivot columns in $\boldsymbol{R}$.

EXAMPLE

Find a basis for the column space of the following matrix.

$$\boldsymbol{A} = \begin{pmatrix} \boxed{2} & 2 & \boxed{-1} & \boxed{0} & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ 0 & 0 & \frac{3}{2} & -3 & \frac{3}{2} \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Performing Gaussian Elimination on $\boldsymbol{A}$:

A basis for the column space of $\boldsymbol{A}$ is $\left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ -3 \\ 1 \\ 0 \end{pmatrix} \right\}$

SUMMARY

1) Finding a basis for the column space of a matrix via the row space method (by considering $\boldsymbol{A}^T$).

2) Another method to find a basis for the column space of a matrix without having to transpose $\boldsymbol{A}$.