Unit 062 Solving System of Linear Differential Equations

Slide 01: In this unit, we will look at two examples where a system of linear differential equations is solved using the technique discussed in a previous unit.

Slide 02: Suppose we have a particle that is moving in a planar force field. The position vector $\boldsymbol{X}$ satisfies the equation $\boldsymbol{X'} = \boldsymbol{AX}$ which we are familiar with from a previous unit.
(#)
The coefficient matrix $\boldsymbol{A}$ of the system is shown here and furthermore, we know that the initial condition of the system is $\boldsymbol{X}(0) = \boldsymbol{X}_0$ given by the matrix as shown.
(#)
We would like to solve this initial value problem.

Slide 03: We proceed with the first step, which is to find all the eigenvalues of $\boldsymbol{A}$.
(#)
The characteristic polynomial of $\boldsymbol{A}$
(#)
can be computed and it
(#)
simplifies to $\lambda^2 - 5\lambda - 6$
(#)
which can be factorised into $(\lambda - 6)(\lambda + 1)$.
(#)
and thus the eigenvalues of $\boldsymbol{A}$ are -1 and 6.
(#)
Recall from a previous unit that if $\boldsymbol{x}$ is an eigenvector associated with the eigenvalue λ_1, then the vector $\boldsymbol{X_1} = e^{\lambda_1 t}\boldsymbol{x}$ will be a solution to the linear system of differential equations $\boldsymbol{X'} = \boldsymbol{AX}$

Slide 04: We will now proceed to find all the linearly independent eigenvectors associated with each eigenvalue λ. In other words, we need to determine a basis for each eigenspace E_λ.
(#)
For the eigenvalue 6,
(#)
we solve the following homogeneous linear system
(#)
and obtain the general solution of the system as shown.
(#)
Thus the eigenspace E_6 is spanned by the single non zero vector $(-5, 2)$.

Slide 05: Similarly, for the eigenvalue -1,
(#)
we solve the following homogeneous linear system
(#)
and obtain the general solution as shown.

(#)
The eigenspace E_{-1} is spanned by the single non zero vector $(1,1)$.

Slide 06: We now construct a linear combination of the solutions $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$ using the two eigenvectors that we have found in the previous step.
(#)
So $\boldsymbol{X_1}$ is the eigenvector $(-5,2)$ multiplied by the scalar e^{6t}.
(#)
While $\boldsymbol{X_2}$ is the eigenvector $(1,1)$ multiplied by the scalar e^{-t}.
(#)
From the discussion in a previous unit, we now know that any linear combination of $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$, that is, for any real numbers k_1 and k_2, the matrix $\boldsymbol{X}$ given by $k_1\boldsymbol{X_1}+k_2\boldsymbol{X_2}$ will always be a solution to our system of differential equations $\boldsymbol{X'} = \boldsymbol{AX}$.

Slide 07: We now make use of the initial condition provided to determine the values of the constants k_1 and k_2.
(#)
Substituting $t = 0$ into the expression for $\boldsymbol{X}$, we have k_1 times $(-5,2)+$ k_2 times $(1,1)$ equal to the initial condition of $(2.9, 2.6)$.
(#)
This is in fact just a linear system involving two variables and two equations
(#)
where upon solving, we have the values of k_1 as $-\frac{3}{70}$ and k_2 equal to $\frac{188}{70}$.

Slide 08: Thus the solution to the initial value problem is given by the expression for $\boldsymbol{X}$ shown here.

Slide 09: Let us consider another example. Suppose we have two tanks of water, tank A and tank B which are connected as shown.
(#)
First, there is a constant in-flow of pure water into tank A at the rate of 15 litres per minute.
(#)
Flowing out of tank A, into tank B, the rate of flow is 20 litres per minute.
(#)
Going in the opposite direction, the rate of flow from tank B into tank A is 5 litres per minute.
(#)
Lastly, there is an out-flow from tank B into an external system at the rate of 15 litres per minute.
(#)
Suppose initially tank A has 200 litres of water and 60 grams of salt completely dissolved,
(#)
while tank B has 200 litres of pure water.
(#)
We would like to determine the amount of salt in each tank at time t.

Slide 10: Let us introduce the following notations.
(#)
First, let $y_1(t)$ represent the amount of salt in tank A at time t.
(#)
Similarly, let $y_2(t)$ represent the amount of salt in tank B at time t.
(#)
Then the matrix $\boldsymbol{Y}(0)$ comprising of $y_1(0)$ and $y_2(0)$ would be the initial amount of salt in each tank.

Slide 11: A key observation to make at this point is that due to the in-flow and out-flow into and out of each tank, the amount of water in each tank will remain at 200 litres.
(#)
Now the rate of change in the amount of salt in each tank must be equal to the rate of salt flowing in minus the rate of salt flowing out.

Slide 12: First considering tank A,
(#)
the rate of salt flowing in will be the rate of salt flowing in from tank B. This is given by the concentration of salt in tank B's water at time t, which is $y_2(t)$ divide by 200, multiplied by the flow rate into tank A, which is 5 litres per minute.
(#)
This gives us $y_2(t)$ divide by 40 grams per minute.
(#)
The rate of salt flowing out of tank A will be the concentration of salt in tank A's water at time t, which is $y_1(t)$ divide by 200, multiplied by the flow rate into tank B, which is 20 litres per minute.
(#)
This gives us $y_1(t)$ divide by 10 grams per minute.
(#)
Thus, we have the rate of change in the amount of salt in tank A as shown here.

Slide 13: Moving on to tank B,
(#)
the rate of salt flowing in will be the rate of salt flowing in from tank A. This is given by the concentration of salt in tank A's water at time t, which is $y_1(t)$ divide by 200, multiplied by the flow rate into tank B, which is 20 litres per minute.
(#)
This gives us $y_1(t)$ divide by 10 grams per minute.
(#)
The rate of salt flowing out of tank B will be the concentration of salt in tank B's water at time t, which is $y_2(t)$ divide by 200, multiplied by the flow rate into tank A plus the flow rate into the external system, which adds up to 20 litres per minute.
(#)
This gives us $y_2(t)$ divide by 10 grams per minute.
(#)

We now have the rate of change in the amount of salt in tank B as shown here.

Slide 14: We are now able to write down two differential equations that describes the rate of change of the amount of salt in both tanks.
(#)
This allows us to set up a system of linear differential equations as shown here
(#)
which can be represented by $\boldsymbol{Y'} = \boldsymbol{AY}$
(#)
where $\boldsymbol{A}$ is a 2×2 matrix as shown here
(#)
and the initial condition is as discussed earlier

Slide 15: We will proceed similar to the first example, where we begin by finding all the eigenvalues of the matrix $\boldsymbol{A}$.
(#)
The characteristic polynomial can be factorised as shown here
(#)
which tells us that the two eigenvalues of $\boldsymbol{A}$ are $-\frac{3}{20}$ and $-\frac{1}{20}$.
(#)
We next find all the linearly independent eigenvectors associated with each of the two eigenvalue λ.
(#)
Without going into the details, we see that both eigenspaces are 1-dimensional and we have the vectors that forms a basis for each of the eigenspaces shown here.

Slide 16: This will enable us to construct a linear combination of the solutions $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$
(#)
which results in a general solution to the system of linear differential equations as
(#)
$\boldsymbol{Y} = k_1$ times the vector $(1, -2)$ times the scalar $e^{-3t/20}$ plus k_2 times the vector $(1, 2)$ times the scalar $e^{-t/20}$.
(#)
We will now use the initial condition provided to solve for the constants k_1 and k_2.
(#)
This is essentially solving a linear system with two equations and two unknowns,
(#)
giving us the value of k_1 and k_2 to be both equal to 30.

Slide 17: We have now obtained the solution to the initial value problem as shown.
(#)
Writing this solution as a matrix
(#)
will tell us the amount of salt in both tanks A and B at time t as required.

Slide 18: To summarise this unit,

(#)

we saw two examples on how to solve a system of linear differential equations $\boldsymbol{Y'} = \boldsymbol{AY}$ with given initial conditions.