MA1512 TUTORIAL 2

1. Solve the following differential equations:

(a) $xy' + (1+x)y = e^{-x}, \qquad x > 0$

(b) $y' - (1+\frac{3}{x})y = x+2, \qquad y(1) = e-1, \qquad x > 0$

(c) $y' + y + \frac{x}{y} = 0$ (d) $2xyy' + (x-1)y^2 = x^2e^x, \qquad x > 0$

2. If a cable is held up at two ends at the same height, then it will sag in the middle, making a U-shaped curve called a **catenary**. This is the shape seen in electricity cables suspended between poles, in countries less advanced than Singapore, such as Japan and the US. It can be shown using simple physics that if the shape is given by a function y(x), then this function satisfies

$$\frac{dy}{dx} = \frac{\mu}{T}\int_0^x \sqrt{\left(\frac{dy}{dt}\right)^2 + 1}\, dt,$$

where x = 0 at the lowest point of the catenary and y(0) = 0, where μ is the weight per unit length of the cable, and where T is the horizontal component of its tension; this horizontal component is a constant along the cable. Find a formula for the shape of the cable. [Hint: use the Fundamental Theorem of Calculus, and think of the resulting equation as a **first-order** ODE.]

3. Psychologists talk about something called a **Performance Curve**. Suppose an MA1512 student is solving mathematics problems. She starts with ordinary differential equations. Let P(t) be a non-negative function that measures her performance, that is, her success rate at solving DEs. Her performance increases rapidly at first, but then the rate of increase slows down as she becomes more expert. Let M, a positive constant, be the best possible performance; then one can suppose that P satisfies

$$\frac{dP}{dt} = C[M - P],$$

where C is a constant. What are the units of this constant? What does this constant measure? Solve this equation assuming that she is completely incompetent at t = 0 [that is, P(0) = 0].

Now the student turns to another kind of problem, say in partial differential equations. Again her performance is low at first but gets better in accordance with this equation. Now as the years go by, her overall ability to solve mathematics problems gradually gets better, so C, instead of being a constant, is really a slowly increasing function of time. Suppose that C(t) = K tanh(t/T), t $\geq$ 0, where K and T are positive constants. [Is this reasonable? Why? What are the meanings of the constants K and T?] Replacing C with C(t), solve for P, again assuming that P(0) = 0.

4. A student starts a rumour in a school. The number of students who have heard the rumour, R(t), is given by

$$\frac{dR}{dt} = KR[1400 - R],$$

where K is a positive constant, and 1400 is the number of students in that school. What is the meaning of K? Is this equation reasonable? By regarding this equation as a Bernoulli equation, find R(t).

5. The half-life of Thorium 230 is about 75000 years, while that of Uranium 234 is about 245000 years. A certain sample of ancient coral has a Thorium/Uranium ratio of 10 percent. How old is the coral?

Question 6 (suggested by FoE)

A reaction sequence, A $\to$ B $\to$ C takes place in wastewater treatment plants as well as in natural aquatic ecosystems in the breakdown of dead or decaying organic matter. In the first step, A (organic nitrogen) is transformed to B (ammonia) through the degradation of organic nitrogen by heterotrophic bacteria. In the second step, B (ammonia) is transformed into C (nitrate) through the nitrification process by nitrifying bacteria. The overall treatment results in the transformation of nitrogen from complex organic forms (organic nitrogen) to simple inorganic forms, which can then either be taken up by plants as nutrients (in aquatic ecosystem), or broken down even further into nitrogen gas by denitrifying bacteria in wastewater treatment plants.

The reactions then satisfy the following system of ODE, where k_1 and k_2 are two positive constants:

$$\begin{cases} \frac{dA}{dt} = -k_1 A \\ \frac{dB}{dt} = k_1 A - k_2 B \\ \frac{dC}{dt} = k_2 B \end{cases} .$$

Suppose at $t = 0$, we have $A = A_0$ and $B = C = 0$. Find a formula for C in terms of A_0, k_1 and k_2.

(Hint: Observe that the first two equations are the same as those in the Uranium-Thorium dating example.)