ORTHOGONALITY

EXAMPLE

Let $\boldsymbol{u} = (1,-2,2,-1), \boldsymbol{v} = (1,0,2,0)$.

Compute the following:

$\boldsymbol{u}\cdot\boldsymbol{v}$, $\|\boldsymbol{u}\|$, $\|\boldsymbol{v}\|$, $d(\boldsymbol{u},\boldsymbol{v})$, angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

$$\boldsymbol{u}\cdot\boldsymbol{v} = (1\times 1) + (-2\times 0) + (2\times 2) + (-1\times 0) = 5$$

$$\|\boldsymbol{v}\| = \sqrt{1+4+4+1} = \sqrt{10} \qquad \|\boldsymbol{u}\| = \sqrt{1+0+4+0} = \sqrt{5}$$

$$d(\boldsymbol{u},\boldsymbol{v}) = \|\boldsymbol{u}-\boldsymbol{v}\| = \|(0,-2,0,-1)\| = \sqrt{0+4+0+1} = \sqrt{5}$$

angle between $\boldsymbol{u}$ and $\boldsymbol{v} = \cos^{-1}\left(\dfrac{5}{\sqrt{50}}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{\pi}{4}$

THEOREM

Let c be a scalar and $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be vectors in $\mathbb{R}^n$.

1) $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$

2) $(\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}$

   $\boldsymbol{w} \cdot (\boldsymbol{u} + \boldsymbol{v}) = \boldsymbol{w} \cdot \boldsymbol{u} + \boldsymbol{w} \cdot \boldsymbol{v}$

3) $(c\boldsymbol{u}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot (c\boldsymbol{v}) = c(\boldsymbol{u} \cdot \boldsymbol{v})$

4) $\|c\boldsymbol{u}\| = |c| \|\boldsymbol{u}\|$

5) $\boldsymbol{u} \cdot \boldsymbol{u} \geq 0$ and $\boldsymbol{u} \cdot \boldsymbol{u} = 0$ if and only if $\boldsymbol{u} = \boldsymbol{0}$.

The dot product of any vector with itself is non-negative and the only vector whose dot product with itself is zero is the zero vector.

THEOREM

5) $\boldsymbol{u}\cdot\boldsymbol{u}\geq 0$ and $\boldsymbol{u}\cdot\boldsymbol{u}=0$ if and only if $\boldsymbol{u}=\mathbf{0}$.

Proof: Let $\boldsymbol{u}=(u_1,u_2,\ldots,u_n)$.

$$\boldsymbol{u}\cdot\boldsymbol{u}=(u_1,u_2,\ldots,u_n)\cdot(u_1,u_2,\ldots,u_n)$$

$$=\underbrace{u_1^2}_{\geq 0}+\underbrace{u_2^2}_{\geq 0}+\ldots+\underbrace{u_n^2}_{\geq 0}=0 \Leftrightarrow u_1^2=0,u_2^2=0,\ldots,u_n^2=0$$

$$\Leftrightarrow u_1=0,u_2=0,\ldots,u_n=0$$

$$\Leftrightarrow \boldsymbol{u}=\mathbf{0}$$

DEFINITION (ORTHOGONAL, ORTHONORMAL)

1) Two vectors $\boldsymbol{u},\boldsymbol{v}$ are said to be orthogonal if $\boldsymbol{u}\cdot\boldsymbol{v}=0$.

2) A set S of vectors in $\mathbb{R}^n$ is said to be orthogonal if every pair of distinct vectors in S are orthogonal.

$$S=\{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w},\boldsymbol{x}\}$$

$$\boldsymbol{u}\cdot\boldsymbol{v}=0,\ \boldsymbol{u}\cdot\boldsymbol{w}=0,\ \boldsymbol{u}\cdot\boldsymbol{x}=0$$

$$\boldsymbol{v}\cdot\boldsymbol{w}=0,\ \boldsymbol{v}\cdot\boldsymbol{x}=0,\ \boldsymbol{w}\cdot\boldsymbol{x}=0$$

3) A set S of vectors in $\mathbb{R}^n$ is said to be orthonormal if S is orthogonal and every vector in S is a unit vector.

THE CONCEPT OF ORTHOGONALITY

Two vectors $\boldsymbol{u}, \boldsymbol{v}$ are said to be orthogonal if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

$$\boldsymbol{u} \cdot \boldsymbol{v} = 0$$

$$\Rightarrow \text{angle between } \boldsymbol{u} \text{ and } \boldsymbol{v} = \cos^{-1}\left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\right)$$

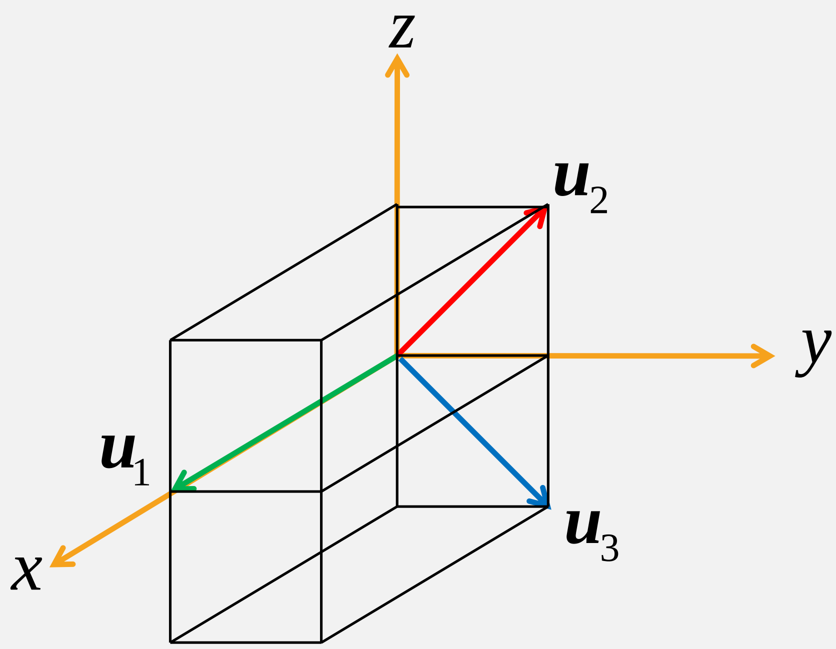
$$= \cos^{-1}(0) = \frac{\pi}{2}.$$

Thus the concept of orthogonality is a generalization of perpendicularity that we are familiar in $\mathbb{R}^2$ and $\mathbb{R}^3$.

EXAMPLE

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ where

$\boldsymbol{u}_1 = (2,0,0)$; $\boldsymbol{u}_2 = (0,1,1)$; $\boldsymbol{u}_3 = (0,1,-1)$.



$$\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = 0$$

$$\boldsymbol{u}_1 \cdot \boldsymbol{u}_3 = 0$$

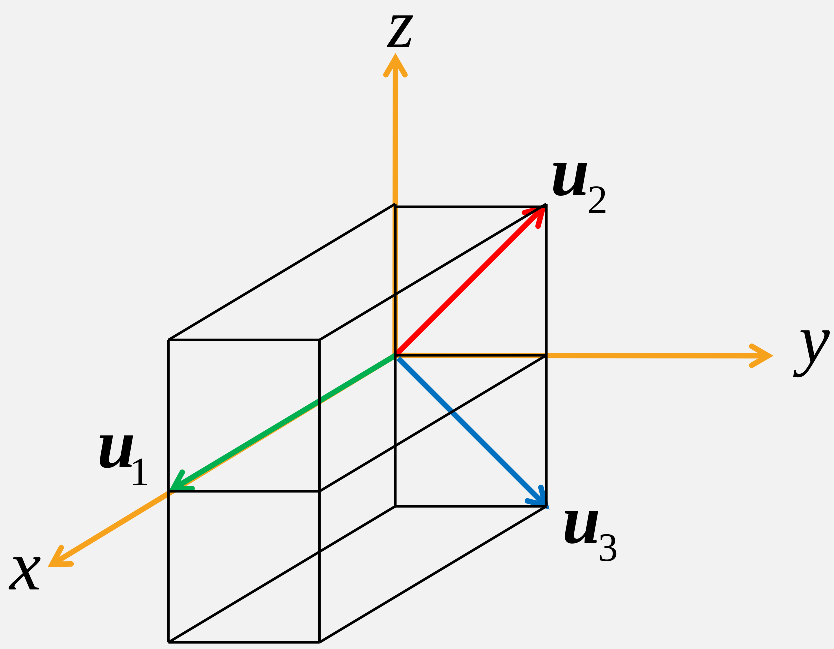
$$\boldsymbol{u}_2 \cdot \boldsymbol{u}_3 = 0$$

S is an orthogonal set

EXAMPLE

Converting an orthogonal set to an orthonormal set:

$$\boldsymbol{u}_1 = (2,0,0); \qquad \boldsymbol{u}_2 = (0,1,1); \qquad \boldsymbol{u}_3 = (0,1,-1).$$



$$\boldsymbol{v}_1 = \frac{\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|} = (1,0,0)$$

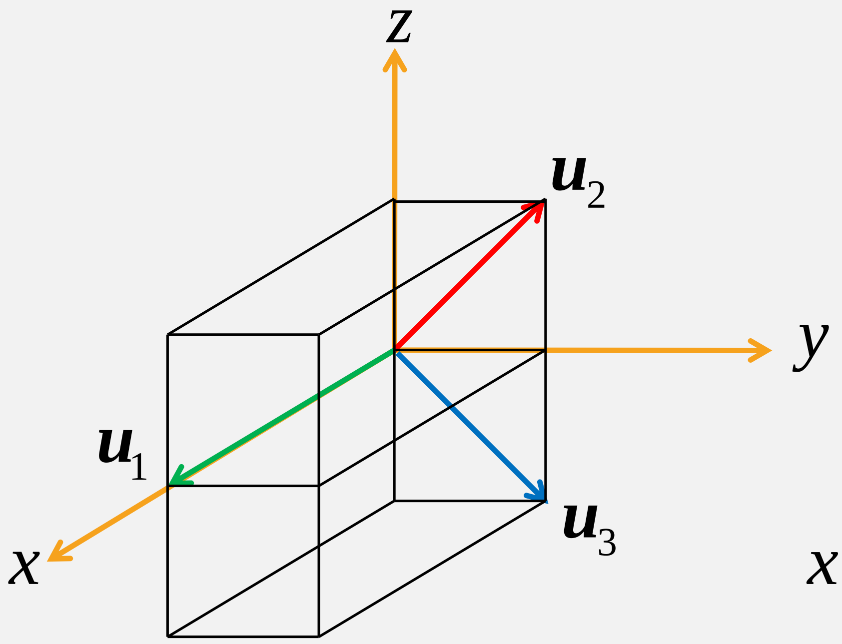
$$\boldsymbol{v}_2 = \frac{\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|} = \frac{1}{\sqrt{2}}(0,1,1)$$

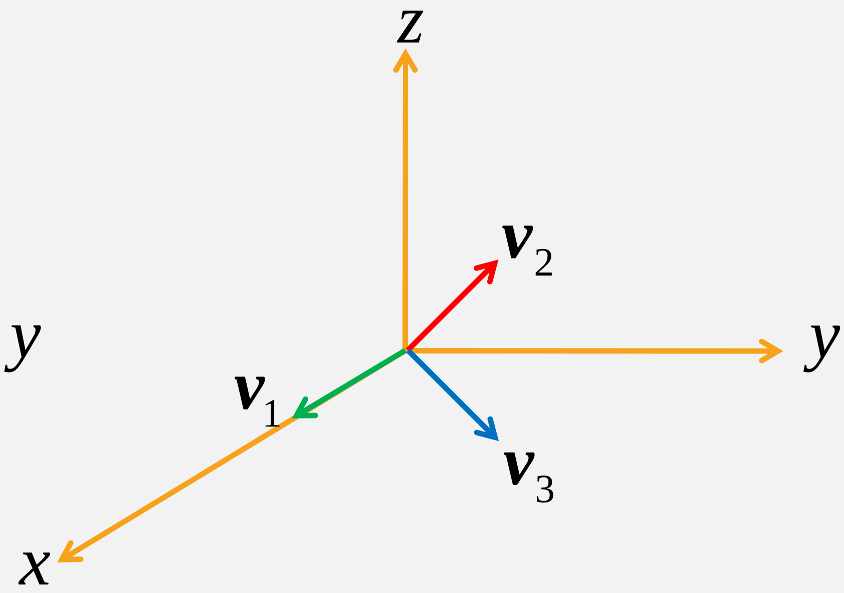
$$\boldsymbol{v}_3 = \frac{\boldsymbol{u}_3}{\|\boldsymbol{u}_3\|} = \frac{1}{\sqrt{2}}(0,1,-1)$$

EXAMPLE

Converting an orthogonal set to an orthonormal set:

$$\mathbf{v}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = (1,0,0) \qquad \mathbf{v}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{1}{\sqrt{2}}(0,1,1) \qquad \mathbf{v}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \frac{1}{\sqrt{2}}(0,1,-1)$$





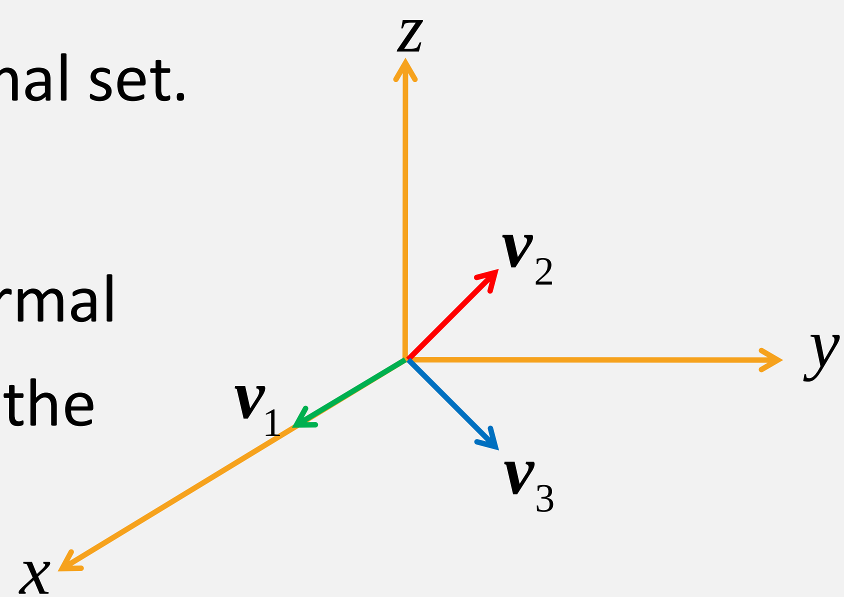
EXAMPLE

Converting an orthogonal set to an orthonormal set:

$$\boldsymbol{v}_1 = \frac{\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|} = (1,0,0) \qquad \boldsymbol{v}_2 = \frac{\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|} = \frac{1}{\sqrt{2}}(0,1,1) \qquad \boldsymbol{v}_3 = \frac{\boldsymbol{u}_3}{\|\boldsymbol{u}_3\|} = \frac{1}{\sqrt{2}}(0,1,-1)$$

$S' = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ is an orthonormal set.

This process of converting an orthogonal set to an orthonormal set by dividing each vector in the set by its length is called normalizing.



SUMMARY

1) A theorem with some results on lengths and dot product for Euclidean vectors.

2) When do we say two vectors are orthogonal.

3) When do we say that a set is orthogonal.

4) Orthonormal sets. Normalizing an orthogonal set.