Unit 007 Gaussian and Gauss-Jordan Elimination

Slide 01: In this unit, we will discuss Gaussian and Gauss-Jordan elimination.

Slide 02: We have seen from a previous unit that a row-echelon form of an augmented matrix allows us to determine how many, if any, solutions the corresponding linear system has.

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By identifying the non pivot columns in a row-echelon form, we will be able to write down a general solution for a consistent linear system.

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The benefits of having an augmented matrix in row-echelon form is apparent. In this unit we will discuss a systematic way of finding an augmented matrix in row-echelon form that is row equivalent to the augmented matrix of the original linear system. We will do this by performning elementary row operations.

Slide 03: Gaussian elimination is a systematic procedure to reduce an augmented matrix to a row-echelon form using elementary row operations. In general, a systematic procedure, which can be described in a number of steps is called an algorithm. Gaussian elimination can also be performed on any matrix, not necessarily augmented matrices.

Slide 04: The first step of Gaussian elimination requires us to locate the leftmost column of the matrix that is not entirely zero. The desired column for some sample matrices is highlighted below.

Slide 05: Look at the topmost entry in the column that you have identified in Step 1. If this entry is non zero, we do nothing in Step 2. For two of the four sample matrices below, we have such a situation. The topmost entry, which is non zero, is now our leading entry, or pivot point.

Slide 06: If the topmost entry in the column identified in Step 1 is zero, we will perform a row swap between the top row and another row so as to bring a non zero entry to the top. Note that a row swap is an elementary row operation. In the two sample matrices below, after performing the row swap, we would now have our leading entry, or pivot point.

Slide 07: It is useful to remember that at the end of Step 2, we would have identified a pivot column and the pivot point in that column.

Slide 08: Recall that a pivot point is also called a leading entry in the row. We will now proceed to eliminate the non zero entries below each pivot point in the pivot column. This is done by adding a suitable multiple of the row with the leading entry to the rows below with the intention of eliminating the entries below the leading entry. For example, in this matrix, we will use the pivot point, which is 1, to eliminate the entry 2 in the third row.

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This is done by adding -2 times of row 1 to row 3 of the matrix.

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Note that the multiple of -2 used here is obtained by taking the negative of the entry we wish to eliminate, which is 2, and divide it by the pivot point, which is 1.

Slide 09: In this example, the pivot point is 2 and there are two entries below it that we need to eliminate. Firstly, we add $\frac{1}{2}$ times of row 1 to row 2 of the matrix. The factor $\frac{1}{2}$ is the negative of -1 divided by 2. The -1 below the pivot point is now eliminated.

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To eliminate the 3, we will add $-\frac{3}{2}$ times of row 1 to row 3.

Slide 10: After performing these two elementary row operations, the resulting matrix is shown as such.

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We now cover the row containing the pivot point that we have just used and consider the remaining submatrix.

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Return to Step 1 and continue until the entire matrix is in row-echelon form. In other words, we will perform Step 1 again, which is to locate the leftmost column of the submatrix that is not entirely made up of zeros.

Slide 11: Gauss-Jordan elimination is an extension of the Gaussian elimination which reduces an augmented matrix to the reduced row-echelon form using elementary row operations.

Slide 12: Not surprisingly, to perform Gauss-Jordan elimination, we first apply the four steps of Gaussian elimination described previously to obtain a row-echelon form for the augmented matrix. For example, the two matrices shown here are already in row-echelon form.

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Continuing with Step 5, we now multiply each non zero row buy a suitable constant so that each leading entry becomes 1. Recall that this is one of the requirements for a matrix to be in reduced row-echelon form.

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For example in this matrix,

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we multiply row 1 by $\frac{1}{2}$ and row 2 by $\frac{2}{3}$,

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which will make each leading entry become 1.

Slide 13: Of course if any of the leading entries are already 1 at row-echelon form, like the one in the first row here, we do not need to do anything to that row.

Slide 14: For the next step, we start from the leading entry in the lowest row and add suitable multiples of this row to rows above it so that entries in the same column above the leading entry becomes 0. Note that this is the additional property that reduced row-echelon forms have compared to row-echelon forms.

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So in this matrix, we start with the leading entry in the lowest row highlighted as shown. We shall call this row the pivot row. To eliminate the 2 immediately above the leading entry, we add -2 times the pivot row to the row immediately above it. The entry immediately above the leading entry now becomes 0.

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It should be noted that the multiple -2 is simply the negative of the entry that we wish to eliminate.

Slide 15: Similarly, for the entry 3 above the leading entry, we will add -3 times of the pivot row to the row containing 3 so that the 3 is eliminated and becomes 0.

Slide 16: In the same way, we add 4 times the pivot row to the top row to eliminate the -4.

Slide 17: We have completed the elimination of all the entries in the same column as the lowest leading entry. We now repeat the same procedure with the next lowest leading entry and eliminate the entries above this next leading entry.

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Continue this way until reduced row-echelon form is attained.

Slide 18: Before we end this unit, let us have a standardised set of notations to represent the elementary row operations that we wish to perform.

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For the first type of elementary row operations, where we multiply a row, say row i by a non zero constant c, we shall write cR_i.

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For exampe, when we multiply row 2 by $-\frac{1}{2}$, we represent that by write $-\frac{1}{2}R_2$. Likewise, $2R_3$ means we multiply row 3 by 2.

Slide 19: For the second type of elementary row operations, where we interchange two rows i and j, we write R_i 'double arrow' R_j.

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So in this matrix, we wish to interchange rows 1 and 2 so we write R_1 'double arrow' R_2.

Slide 20: The third type of elementary row operations is where we add a multiple of one row to another row. Suppose we wish to add k times of row i to row j. In this case, we will write $R_j + kR_i$.

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Remember that when we do this, it is row j that will be changed while row i does not.

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In the example below, we use different multiples of row 1 to add to rows 2 and 3. Firstly, we add -1 times of row 1 to row 2 and this is denoted by $R_2 - R_1$. We also add 2 times of row 1 to row 3, so we write $R_3 + 2R_1$. Once again, remember that rows 2 and 3 will be changed after these two operations but row 1 remains the same.

Slide 21: Let us summarize the main points in this unit.

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We first describe the 4 steps involved in Gaussian elimination, which is a systematic way of performing elementary row operations on a matrix to reduce it to row-echelon form.

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Adding two additional steps to Gaussian elimination gives us Gauss-Jordan elimination, which further reduces a matrix from row-echelon form to reduced row-echelon form.

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Finally, we introduce a set of standardised notations to be used when you are performing elementary row operations on a matrix.