Unit 061 System of Linear Differential Equations

Slide 01: In this unit, we will apply what we have studied on the eigenvalues of a matrix to solving a system of linear differential equations.

Slide 02: There are many applied problems where several quantities that we are interested in studying are varying continuously in time. These quantities are inter-related and these relationships can be represented by a system of linear differential equations like the one shown here. The quantities are represented by y_1, y_2 and so on. Here you see that the first equation relates the first order derivative of y_1 as a function of y_1, y_2 and so on till y_n.

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We can represent the first derivatives of y_1, y_2 and so on till y_n as a matrix,

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while y_1, y_2 and so on can also be represented as a matrix.

Slide 03: These variables y_1 to y_n are assumed to be differentiable functions of t.

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It is easy to see that the system of differential equations that we have can be represented efficiently using matrix multiplication. We first have the matrix of derivatives on the left

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which is equal to a square matrix containing the coefficients of the system pre-multiplied to the column matrix containing the variables y_1 to y_n.

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This matrix equation can be represented as $\boldsymbol{Y}' = \boldsymbol{AY}$. How do we go about solving such a system?

Slide 04: Let us consider the simplest case where $n = 1$. This means that there is only one variable y, whose first derivative with respect to t is some scalar multiple of $y(t)$ itself.

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A simple result from Calculus tells us that $y(t) = ce^{at}$ where c is a constant will be a solution to this differential equation since

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differentiating y with respect to t gives us $y'(t) = cae^{at}$ which is equal to $ay(t)$.

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If we extend this to the case where we have more than just one variable, suppose we let $\boldsymbol{Y}$ to be the following column matrix, where each component is some constant multiplied to $e^{\lambda t}$,

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we can represent $\boldsymbol{Y}$ simply as $e^{\lambda t}$ times of a column matrix $\boldsymbol{x}$.

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Let us see if the column matrix $\boldsymbol{Y}$ is a solution to the matrix equation $\boldsymbol{Y}' = \boldsymbol{AY}$.

Slide 05: With our choice of $\boldsymbol{Y}$, since $\boldsymbol{x}$ is a column matrix of constants,

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it is easy to see that $\boldsymbol{Y'}$ is just λ times $e^{\lambda t}$ times of $\boldsymbol{x}$.

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Now if we choose λ to be an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{x}$ be an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue λ, then we will have $\boldsymbol{Ax} = \lambda\boldsymbol{x}$.

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Now if we multiply $\boldsymbol{A}$ on both sides of the equation $\boldsymbol{Y} = e^{\lambda t}\boldsymbol{x}$, we have $\boldsymbol{AY} = e^{\lambda t}\boldsymbol{Ax}$

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which in turn is equal to $e^{\lambda t}\lambda\boldsymbol{x}$

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which can be easily seen to be

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the derivative $\boldsymbol{Y'}$.

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Thus our choice of $\boldsymbol{Y} = e^{\lambda t}\boldsymbol{x}$ is indeed a solution to the system of linear differential equation $\boldsymbol{Y'} = \boldsymbol{AY}$.

Slide 06: So what happens when we have more than one solution arising from different eigenvalues of $\boldsymbol{A}$? For example, if $\boldsymbol{x_1}$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue λ_1, then we have seen that $\boldsymbol{Y_1} = e^{\lambda_1 t}\boldsymbol{x_1}$ will be a solution to the system $\boldsymbol{Y'} = \boldsymbol{AY}$.

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Similarly, if $\boldsymbol{x_2}$ is another eigenvector of $\boldsymbol{A}$ associated with another eigenvalue λ_2, then we will have another solution $\boldsymbol{y_2} = e^{\lambda_2 t}\boldsymbol{x_2}$.

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What about any linear combination of $\boldsymbol{Y_1}$ and $\boldsymbol{Y_2}$?

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For any real numbers k_1 and k_2, will $k_1\boldsymbol{Y_1} + k_2\boldsymbol{Y_2}$ also be a solution to the system $\boldsymbol{Y'} = \boldsymbol{AY}$?

Slide 07: Let's differentiate this linear combination of $\boldsymbol{Y_1}$ and $\boldsymbol{Y_2}$.

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We know that this gives us k_1 times the derivative of $\boldsymbol{Y_1}$ plus k_2 times the derivative of $\boldsymbol{Y_2}$.

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Now since $\boldsymbol{Y_1}$ and $\boldsymbol{Y_2}$ are both solutions to the system, we have $\boldsymbol{Y_1}'$ equal to $\boldsymbol{AY_1}$ and $\boldsymbol{Y_2}'$ equal to $\boldsymbol{AY_2}$.

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Factorising $\boldsymbol{A}$ from the two terms, we have the following.

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Notice that we have arrived at the conclusion that the derivative of $k_1\boldsymbol{Y_1} + k_2\boldsymbol{Y_2}$ is equal to the pre-multiplication of $\boldsymbol{A}$ to this linear combination.

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This implies that $k_1\boldsymbol{Y_1} + k_2\boldsymbol{Y_2}$ is also a solution to the system $\boldsymbol{Y'} = \boldsymbol{AY}$.

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Generalising this beyond taking linear combinations of two solutions, we now see that if $\boldsymbol{Y_1}$ to $\boldsymbol{Y_n}$ are n solutions to the system, then any linear combination of these n solutions will also be a solution.

Slide 08: We have seen that the solution set to the system $\boldsymbol{Y'} = \boldsymbol{AY}$ has the same closure property as subspaces that we are familiar with. Indeed, this solution set is a subspace of the vector space that contains all continuous vector-valued functions.

Slide 09: Let us conclude this unit with a few remarks. Recall that we wish to solve a system of linear differential equations $\boldsymbol{Y'} = \boldsymbol{AY}$. Let S be the solution set of this system.

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It can shown that there always exists a fundamental set of solutions to the system.

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If $\boldsymbol{A}$ is a square matrix of order n, then this fundamental set will contain n linearly independent functions. Note that in this case, the vectors are functions and not Euclidean vectors that we have mostly seen so far.

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Every solution in the set S is expressed uniquely as a linear combination of these n functions in the fundamental set.

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Thus, this fundamental set of solutions is precisely a basis for the solution space of the system of linear differential equations.

Slide 10: As we have a basis containing n linearly independent functions, the dimension of the solution space S is n.

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When we are given a specific vector $\boldsymbol{Y}_0$, this will allow us to identify the unique solution $\boldsymbol{Y}$ from the solution space S that satisfies $\boldsymbol{Y}' = \boldsymbol{AY}$ and the initial value $\boldsymbol{Y}(0)$ equal to the specified vector. This class of problems are known as initial value problems and we will see some examples in subsequent units.

Slide 11: To summarise this unit,

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We introduced a simple system of linear differential equations that is represented by the matrix equation $\boldsymbol{Y}' = \boldsymbol{AY}$.

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Using our understanding of eigenvalues and eigenvectors, we saw how the solution set of the system can be constructed.

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When $\boldsymbol{A}$ is a square matrix of order n, this solution set is a n-dimensional vector space of functions.