NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

Module: MA1508E Linear Algebra for Engineering
Year/Semester: 2018-2019 (Semester 2)
Practice Problem Set: 5

Name:

Matriculation Number:

Tutorial Group:

Write down the solutions to the problems below, showing all working involved (imagine it is an examination question). Hand in your answers **together with this question paper** before you leave the classroom. You may refer to any materials while answering the questions. You may also discuss with your friends but do not **copy blindly**.

(More on $\boldsymbol{QR}$ Factorisation:) $\boldsymbol{QR}$ factorisation can be useful when we are finding least squares solution to a linear system $\boldsymbol{Ax} = \boldsymbol{b}$.
Recall that if $\boldsymbol{x'}$ is a least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$, then $\boldsymbol{x'}$ is a solution to the equation

$$\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}.$$

Suppose $\boldsymbol{A}$ is factored into $\boldsymbol{QR}$, where $\boldsymbol{Q}$ is a matrix with orthonormal columns and $\boldsymbol{R}$ is an upper triangular square matrix with positive diagonal entries. Then the above equation becomes

$$(\boldsymbol{QR})^T(\boldsymbol{QR})\boldsymbol{x} = (\boldsymbol{QR})^T\boldsymbol{b} \Rightarrow \boldsymbol{R}^T(\boldsymbol{Q}^T\boldsymbol{Q})\boldsymbol{Rx} = \boldsymbol{R}^T\boldsymbol{Q}^T\boldsymbol{b}.$$

Note that since $\boldsymbol{Q}$ has orthonormal columns, $\boldsymbol{Q}^T\boldsymbol{Q}$ will be an identity matrix (convince yourself of this fact). Thus we now have

$$\boldsymbol{R}^T\boldsymbol{Rx} = \boldsymbol{R}^T\boldsymbol{Q}^T\boldsymbol{b}.$$

As $\boldsymbol{R}$ is invertible (why?), so is $\boldsymbol{R}^T$ and thus

$$\boldsymbol{R}^T\boldsymbol{Rx} = \boldsymbol{R}^T\boldsymbol{Q}^T\boldsymbol{b} \Rightarrow \boldsymbol{Rx} = \boldsymbol{Q}^T\boldsymbol{b} \Rightarrow \boldsymbol{x} = \boldsymbol{R}^{-1}\boldsymbol{Q}^T\boldsymbol{b}.$$

This implies that in order to find a least squares solution of $\boldsymbol{Ax} = \boldsymbol{b}$, if we have $\boldsymbol{A} = \boldsymbol{QR}$, then a least squares solution $\boldsymbol{x'}$ can be found by computing $\boldsymbol{R}^{-1}\boldsymbol{Q}^T\boldsymbol{b}$.

Please turn over for the Questions 1 and 2 ...

1. Let $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Use the Gram-Schmidt Process to find an orthonormal basis for the column space of $\boldsymbol{A}$.

Solution:

$$\begin{aligned} \boldsymbol{v_1} &= (1,1,1,0)^T \\ \boldsymbol{v_2} &= (1,1,1,1)^T - \frac{3}{3}(1,1,1,0)^T \\ &= (0,0,0,1)^T \\ \boldsymbol{v_3} &= (0,0,1,1)^T - \frac{1}{3}(1,1,1,0)^T - \frac{1}{1}(0,0,0,1)^T \\ &= (-\frac{1}{3},\ -\frac{1}{3},\ \frac{2}{3},\ 0)^T \end{aligned}$$

So an orthonormal basis for the column space of $\boldsymbol{A}$ is $\{\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3}\}$ where

$$\boldsymbol{w_1} = \frac{1}{\sqrt{3}}\boldsymbol{v_1} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \boldsymbol{w_2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \boldsymbol{w_3} = \frac{1}{\sqrt{6}}\boldsymbol{v_3} = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \\ 0 \end{pmatrix}.$$

2. Let $\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 10 \\ 4 \\ -6 \end{pmatrix}$.

   (a) Use the Gram-Schmidt Process to find an orthonormal basis for the column space of $\boldsymbol{A}$.

   (b) Factorise $\boldsymbol{A}$ into $\boldsymbol{QR}$ where $\boldsymbol{Q}$ has orthonormal columns and $\boldsymbol{R}$ is upper triangular.

   (c) Find a least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

Solution:

(a) Let $\boldsymbol{v_1} = (2,1,2)^T$.

$$\boldsymbol{v_2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{5}{9}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}.$$

Normalizing, we have $\{\boldsymbol{w_1}, \boldsymbol{w_2}\}$ as a basis for the column space of $\boldsymbol{A}$, where

$$\boldsymbol{w_1} = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{w_2} = \frac{1}{3\sqrt{2}}\begin{pmatrix} -1 \\ 4 \\ -1 \end{pmatrix}.$$

(b) Let the two columns of $\boldsymbol{A}$ be $\boldsymbol{u_1}, \boldsymbol{u_2}$. We now write $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ as a linear combination of $\boldsymbol{w_1}$ and $\boldsymbol{w_2}$.

$$\begin{aligned}
\boldsymbol{u_1} &= (\boldsymbol{u_1} \cdot \boldsymbol{w_1})\boldsymbol{w_1} + (\boldsymbol{u_1} \cdot \boldsymbol{w_2})\boldsymbol{w_2} \\
&= 3\boldsymbol{w_1} + 0\boldsymbol{w_2} \\
\boldsymbol{u_2} &= (\boldsymbol{u_2} \cdot \boldsymbol{w_1})\boldsymbol{w_1} + (\boldsymbol{u_2} \cdot \boldsymbol{w_2})\boldsymbol{w_2} \\
&= \frac{5}{3}\boldsymbol{w_1} + \frac{2}{3\sqrt{2}}\boldsymbol{w_2}
\end{aligned}$$

So

$$\begin{pmatrix} \boldsymbol{w_1} & \boldsymbol{w_2} \end{pmatrix} \begin{pmatrix} 3 & \frac{5}{3} \\ 0 & \frac{2}{3\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{u_1} & \boldsymbol{u_2} \end{pmatrix} \Leftrightarrow \boldsymbol{QR} = \boldsymbol{A}.$$

(c) By the discussion above, a least square solution is

$$\boldsymbol{x} = \boldsymbol{R}^{-1}\boldsymbol{Q}^T\boldsymbol{b} = \begin{pmatrix} \frac{1}{3} & -\frac{5}{3\sqrt{2}} \\ 0 & \frac{3}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3\sqrt{2}} & \frac{4}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \end{pmatrix} \begin{pmatrix} 10 \\ 4 \\ -6 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ -\frac{1}{2} & 2 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 10 \\ 4 \\ -6 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 6 \end{pmatrix}.$$