MA1512 AY2018/2019

1 ORDINARY DIFFERENTIAL EQUATIONS

1.1 Introduction

- A **differential equation** is an equation that contains one or more derivatives of a differentiable function.

- The **order** of a DE is the order of the equation's highest order derivative.

- A differential equation is **linear** if it can be put in the form
$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y^{(1)}(x) + a_0 y(x) = F,$$
where F and a_i, $0 \leq i \leq n$, are all functions of x.

For example: $y' = 5y$ and $xy' - \sin x = 0$ are first order, linear; where $(y''')^2 + (y'')^5 - y' = e^x$ is third order, nonlinear.

In general, a DE has many solutions, e.g. $y = \sin x + c$, c an arbitrary constant, is a solution of $y' = \cos x$.

- Solutions containing arbitrary constants are called **general solution** of a given DE.

- Any solution obtained from the general solution by giving specific values to the arbitrary constants is called a **particular solution** of that DE. e.g. $y = \sin x + 1$ is a particular solution of $y' = \cos x$.

Differential equations are solved using integration, and there will be as many integrations as the order of the DE. Therefore, the general solution of an nth-oder DE will have n arbitrary constants.

1.2 Separable equations

A first order DE is called **separable** if it can be written in the form

$$M(x) - N(y)y' = 0$$

or equivalently,

$$M(x)dx = N(y)dy.$$

When we write the DE in this form, we say that we have separated the variables, because everything involving x is on one side, and everything involving y is on the other.

We can solve such a DE by integrating w.r.t. x:

$$\int M(x)dx = \int N(y)dy + c.$$

Example 1. Solve $y' = (1 + y^2)e^x$.

We separate the variables to obtain

$$e^x dx = \frac{1}{1 + y^2}dy.$$

Integrating w.r.t. x gives

$$e^x = \tan^{-1} y + c,$$

or

$$\tan^{-1} y = e^x - c,$$

or

$$y = \tan(e^x - c).$$

Example 2. Experiments show that a radioactive substance decomposes at a rate proportional to the amount present. Starting with 2 mg at certain time, say $t = 0$, what can be said about the amount available at a later time?

Example 3. A copper ball is heated to 100°C. At $t = 0$ it is placed in water which is maintained at 30°C. At the end of 3 mins the temperature of the ball is reduced to 70°C. Find the time at which the temperature of the ball is 31°C.

Physical information: Experiments show that the rate of change dT/dt of the temperature T of the ball w.r.t. time is proportional to the difference between T and the temp T_0 of the surrounding medium. Also, heat flows so rapidly in copper that at any time the temperature is practically the same at all points of the ball.

Example 4. Suppose that a sky diver falls from rest toward the earth and the parachute opens at an instant $t = 0$, when sky diver's speed is $v(0) = v_0 = 10\,\mathrm{m/s}$. Find the speed of the sky diver at any later time t.

Physical assumptions and laws: weight of the man + equipment = 712N, air resistance = bv^2, where $b = 30\,\mathrm{kg/m}$.

Using Newton's second law we obtain

$$m\frac{dv}{dt} = mg - bv^2.$$

Thus $\dfrac{dv}{dt} = -\dfrac{b}{m}(v^2 - k^2),\ k^2 = \dfrac{mg}{b}$

$$\frac{1}{v^2 - k^2}\,dv = -\frac{b}{m}\,dt$$

$$\frac{1}{2k}\left(\frac{1}{v-k} - \frac{1}{v+k}\right)dv = -\frac{b}{m}\,dt.$$

Integrating gives

$$\ell n\left|\frac{v-k}{v+k}\right| = -\frac{2kb}{m}\,t + c_1,$$

or

$$\frac{v-k}{v+k} = ce^{-pt},\ p = \frac{2kb}{m}.$$

($c = \pm e^{c_1}$ according as the ratio on the left is positive or negative).

Solving for v : $v = k\,\dfrac{1 + ce^{-pt}}{1 - ce^{-pt}}$. (Note that $v \to k$ as $t \to \infty$). From $v(0) = v_0$, $c = \dfrac{v_0 - k}{v_0 + k}$. Now $k^2 = \dfrac{mg}{b} = \dfrac{712}{30}$, so $k = 4.87\,\mathrm{m/s}$, $v_0 = 10\,\mathrm{m/s}$, $c = 0.345$, $p = 4.02$.

$$\therefore\ v(t) = 4.87\,\frac{1 + 0.345e^{-4.02t}}{1 - 0.345e^{-4.02t}}.$$

Example 5. The orbit of a planet is the shape it traces out as it moves around the Sun. The best way to describe an orbit is by using plane polar coordinates. These give the position of a point in the plane by specifying its distance r from the origin together with the angle θ made by its position vector with the x axis. A shape or graph in the plane is given by a function of the form $r(\theta)$ (just as, in Cartesian coordinates, a graph is given by a function $y(x)$).

Using his own laws of motion, Isaac Newton discovered that every planet has an orbit which satisfies an equation of the following form:

$$\left(\frac{du}{d\theta}\right)^2 + (u - A)^2 = B^2,$$

where $u(\theta)$ is the reciprocal of $r(\theta)$ and where A, B are positive constants (with the same units as u, namely 1/[length]) with $B/A < 1$. Solve this equation.

This is a separable equation:

$$d\theta = \frac{du/B}{\sqrt{1 - \left(\frac{u-A}{B}\right)^2}},$$

and so we have

$$\theta + C = arcsin\left(\frac{u - A}{B}\right),$$

where C is the constant of integration. Recalling the definition of u we get finally

$$r = \frac{A^{-1}}{1 + \frac{B}{A} sin(\theta + C)}.$$

If you sketch this you will find that it is an egg-shaped curve called an ellipse (Note that it is important here that $B/A < 1$; otherwise you will get other shapes, such as a hyperbola). Thus all of the planets have ellipse-shaped orbits. This is called **Kepler's First Law**.

1.2.1 Reduction to separable form

Certain first order DE are not separable but can be made separable by a simple change of variable. This holds for equations of the form

$$y' = g(\frac{y}{x}) \qquad (\diamondsuit)$$

where g is any function of $\frac{y}{x}$.

We set $\frac{y}{x} = v$, then $y = vx$ and $y' = v + xv'$. Thus $(\diamondsuit)$ becomes $v + xv' = g(v)$, which is separable. Namely, $\frac{dv}{g(v) - v} = \frac{dx}{x}$. We can now solve for v, hence obtain y.

Example 6. Solve $2xyy' - y^2 + x^2 = 0$. $[Ans : x^2 + y^2 = cx]$

1.2.2 Linear Change of Variable

A DE of the form $y' = f(ax + by + c)$, where f is continuous and $b \neq 0$ (if $b = 0$, the equation is separable) can be solved by setting $u = ax + by + c$ and solving the obtained equation.

Example 7. $(2x - 4y + 5)y' + x - 2y + 3 = 0.$

Set $x - 2y = u$, we have

$$(2u + 5)\frac{1}{2}(1 - u') + u + 3 = 0,$$
$$(2u + 5)u' = 4u + 11.$$

Separating variables and integrating :

$$\left(1 - \frac{1}{4u + 11}\right) du = 2dx.$$

Thus $\quad u - \dfrac{1}{4}\ell n|4u + 11| = 2x + c_1,$

or $\quad 4x + 8y + \ell n|4x - 8y + 11| = c.$

1.3 Linear First Order ODEs

A DE which can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{$\clubsuit$}$$

where P and Q are functions of x, is called a linear first order DE. Relation ($\clubsuit$) above is the standard form of such a DE.

To solve ($\clubsuit$), define a new function $R(x)$ by

$$R(x) = e^{\int^x P(s)ds}$$

and note that $R' = RP$ by the chain rule. So $(Ry)' = RPy + Ry'$. Hence if we multiply both sides of ($\clubsuit$) by R we get

$$Ry' + RPy = RQ$$

or

$$(Ry)' = RQ.$$

Now you can integrate both sides and then divide by R to obtain y. The function R is called the **integrating factor** for this equation.

Example 8. Solve

(a) $xy' - 3y = x^2,\ x > 0,$

(b) $y' - y = e^{2x}.$

Example 9. Consider an object of mass m dropped from rest in a medium that offers a resistance proportional to the magnitude of the instantaneous velocity of the object. The goal is to find the position $x(t)$ and velocity $v(t)$ at any time t.

Newton's second law of motion gives $m\dfrac{dv}{dt} = mg - kv$. The initial conditions are $v(0) = 0$ and $x(0) = 0$. The equation is separable.

Solving it directly or by multiplying it by an integrating factor $e^{\frac{k}{m}t}$, we obtain $v = \dfrac{mg}{k}(1 - e^{-\frac{kt}{m}})$, where $v(0) = 0$ has been used.

Set $v = \dfrac{dx}{dt}$ in the above and integrate, and using $x(0) = 0$, we get $x(t) = \dfrac{mg}{k}t - \dfrac{m^2g}{k^2}(1 - e^{-\frac{kt}{m}})$.

Example 10. At time $t = 0$ a tank contains 20 lbs of salt dissolved in 100 gal of water. Assume that water containing 0.25 lb of salt per gallon is entering the tank at a rate of 3 gal/min and the well stirred solution is leaving the tank at the same rate. Find the amount of salt at any time t.

Let the amount of salt at time t be $Q(t)$. The time rate of change $\frac{dQ}{dt}$ equals the inflow minus the outflow.

We obtain $\frac{dQ}{dt} = 3 \times 0.25 - \frac{3Q}{100}$, with $Q(0) = 20$. Solving, $Q(t) = 25 - 5e^{-\frac{3t}{100}}$. Note that $\lim_{t\to\infty} Q(t) = 25$. Thus after sufficiently long time, the salt concentration remains constant at 25 lbs/100 gal.

Example 11. In Example 2 in Section 1.2, we saw that radioactive substances typically decay at a rate proportional to the amount present. Sometimes the product of a radioactive decay is itself a radioactive substance which in turn decays (at a different rate). An interesting example of this is provided by *Uranium-Thorium dating*, which is a method used by palaeontologists to determine how old certain fossils [especially ancient corals] are. Corals filter the sea-water in which they live. Sea-water contains a tiny amount of a certain kind of Uranium [Uranium 234] and the corals absorb this into their bodies. Uranium 234 decays, with a half-life of 245000 years, into Thorium 230, which itself decays with a half-life of 75000 years. Thorium is not found in sea-water; so when the coral dies, it has a certain amount of Uranium in it but no Thorium [because the lifetime of a coral polyp is negligible compared with 245000 years]. It is possible to measure the ratio of the amounts of Uranium and Thorium in any given sample. From this ratio we want to work out the age of the sample [the time when it died]. This is important if we want to know whether global warming is causing corals to die now. [Maybe they die off regularly over long periods of time and the current deaths have nothing to do with global warming.]

Let U(t) be the number of atoms of Uranium in a particular sample of ancient coral and let T(t) be the number of atoms of Thorium. Because each decay of one Uranium atom produces one Thorium atom, Thorium atoms are being born at exactly the same rate at which Uranium atoms die: so we have

$$\frac{dU}{dt} = -k_U U, \tag{1}$$
$$\frac{dT}{dt} = +k_U U - k_T T, \tag{2}$$

where k_U, k_T are constants [related to the half-lives] with $k_U \neq k_T$, and $U(0) = U_0$, $T(0) = 0$. We want to find t given that we know the ratio of T(t) to U(t) at the present time.

Solving (1) with the given data gives $U = U_0\, e^{-k_U t}$. From this we see that $U_0/2 = U_0\, e^{-k_U \times 245000}$ so $k_U = ln(2)/245000$ and similarly $k_T = ln(2)/75000$. So we know these numbers. Notice that k_U is smaller than k_T.

Now (2) becomes

$$\frac{dT}{dt} + k_T T = k_U U_0 e^{-k_U t}.$$

An integrating factor is $e^{k_T t}$. Solving, with $T(0) = 0$, gives

$$T(t) = \frac{k_U}{k_T - k_U} U_0 (e^{-k_U t} - e^{-k_T t}).$$

Unfortunately we don't know U_0 but luckily that goes away when we take the ratio:

$$T/U = \frac{k_U}{k_T - k_U}[1 - e^{(k_U - k_T)t}].$$

[Check that the expression on the right side is positive after t = 0; note also that while both U and T tend to zero, their ratio does not.] So now if we measure the ratio T/U at the present time, we can solve this for t and we have our answer. This method is good for coral fossils up to about half a million years old.

1.3.1 Reduction to linear form

Certain nonlinear DEs can be reduced to a linear form. The most important class of such equations are the **Bernoulli equations** of the form

$$y' + p(x)y = q(x)y^n,$$

where n is any real number.

If $n = 0$ or 1, the equation is linear, otherwise it is nonlinear. To solve, we rewrite this as

$$y^{-n}y' + y^{1-n}p(x) = q(x),$$

and set $y^{1-n} = z$.

Then $(1-n)y^{-n}y' = z'$, and the given DE becomes

$$z' + (1-n)p(x)z = (1-n)q(x),$$

which is a first order linear DE we know how to solve.
Examples. Solve

(a) $y' - Ay = -By^2$, A, B constants.

Observe that $n = 2$. Set $z = y^{1-2} = y^{-1}$. Then $-y^{-2}y' = z'$. The equation becomes $z' + Az = B$, which is separable. Solution $z = \dfrac{B}{A} + ce^{-Ax}$.

$\therefore \quad y = \dfrac{1}{z} = \dfrac{1}{\frac{B}{A} + ce^{-Ax}}$.

(b) $y' + y = x^2y^2$. $\quad [Ans : y(Ae^x + x^2 + 2x + 2) = 1]$

1.4 Linear second order ODEs

The general form of a second order linear DE is

$$y'' + p(x)y' + q(x)y = F(x). \tag{1}$$

If $F(x) \equiv 0$, the linear DE is called **homogeneous** otherwise it is called **nonhomogeneous**. Note that (1) is **linear** in the sense that it is linear in y and its derivatives.

- $y'' + 4y = e^{-x}\sin x$ is a nonhomogeneous linear second order DE;
- $(1-x^2)y'' - 2xy' + 6y = 0$ or in the above standard form $y'' - \dfrac{2x}{1-x^2}y' + \dfrac{6}{1-x^2}y = 0$ is homogeneous linear second order DE;
- $x(y''y + (y')^2) + 2y'y = 0$ and $y'' = \sqrt{1+(y')^2}$ are nonlinear.

A *solution* of a second order DE on some interval I is a function $y = h(x)$ with derivatives $y' = h'(x)$ and $y'' = h''(x)$ satisfying the DE for all x in I.

1.4.1 Homogeneous DEs

The general solutions of homogeneous equations can be found with the help of the **Superposition** or **linearity** principle, which is contained in the following theorem.

Theorem. For a homogeneous linear DE

$$y'' + p(x)y' + q(x)y = 0, \tag{2}$$

any linear combination of two solutions on an open interval I is also a solution on I. In particular for such an equation, sums and constant multiples of solutions are again solutions.

Proof. See original notes on IVLE.

CAUTION! The above result does not hold for nonhomogeneous or nonlinear DEs.

For example:

- $y = 1 + \cos x$ and $y = 1 + \sin x$ are solutions of the nonhomogeneous linear DE $y'' + y = 1$, but $2(1 + \cos x)$ and $2 + \cos x + \sin x$ are not its solutions,
- $y = 1$ and $y = x^2$ are solutions of the nonlinear DE $yy'' - xy' = 0$. But $-x^2$ and $x^2 + 1$ are not its solutions.

Example 12. Solve the initial value problem $y'' - y = 0$, $y(0) = 5$, $y'(0) = 3$.

It is easy to see that e^x and e^{-x} are solutions of $y'' - y = 0$. Thus $y = c_1 e^x + c_2 e^{-x}$ is also a solution. From $y(0) = 5$ we get $c_1 + c_2 = 5$, and $y'(0) = 3$ gives $c_1 - c_2 = 3$. Solving, $c_1 = 4$, $c_2 = 1$. The required solution is $y = 4e^x + e^{-x}$.

1.4.2 General solution of homogeneous linear second order DE

Let $y_1(x)$ and $y_2(x)$ be defined on some interval I. Then y_1 and y_2 are said to be **linearly dependent** on I if one of them is a constant multiple of the other one. Otherwise they are **linearly independent**.

A **general solution** of

$$y'' + py' + qy = 0$$

on an open interval I is

$$y = c_1 y_1 + c_2 y_2,$$

where y_1 and y_2 are linearly independent solutions of the DE and c_1, c_2 are arbitrary constants.

A **particular solution** of the DE on I is obtained if specific values are assigned to c_1 and c_2.

For example: $y_1 = \cos x$ and $y_2 = \sin x$ are linearly independent solutions of $y'' + y = 0$.

- A general solution is $y = c_1 \cos x + c_2 \sin x$.
- A particular solution is, for example, $y = 2\cos x + \sin x$, (which satisfies $y(0) = 2$ and $y'(0) = 1$).

1.4.3 Homogeneous DE with constant coefficients

Consider

$$y'' + ay' + by = 0, \ a, \ b \ \text{ constants.} \tag{3}$$

Recall that a first order linear DE $y' + ky = 0$, k constant, has $y = e^{-kx}$ as a solution.
We now try the function $y = e^{\lambda x}$ as a solution of (3).
Substituting $y = e^{\lambda x}$ in (3) we obtain $(\lambda^2 + a\lambda + b)e^{\lambda x} = 0$, which implies that $e^{\lambda x}$ is a solution if λ is a solution of

$$\lambda^2 + a\lambda + b = 0. \tag{4}$$

This equation is called the **characteristic equation** (or **auxiliary** equation) of (3).

The roots of (4) are

$$\begin{aligned} \lambda_1 &= \tfrac{1}{2}(-a + \sqrt{a^2 - 4b}\,), \\ \lambda_2 &= \tfrac{1}{2}(-a - \sqrt{a^2 - 4b}\,). \end{aligned}$$

We obtain $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ as solutions of (3).

Depending on the sign of $a^2 - 4b$, equation (4) will have

Case 1: two real roots if $a^2 - 4b > 0$,

Case 2: a real double root (i.e. $\lambda_1 = \lambda_2$) if $a^2 - 4b = 0$,

Case 3: complex conjugate roots if $a^2 - 4b < 0$.

Case 1. Equation (4) has two distinct real roots λ_1 and λ_2.

In this case $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are linearly independent solutions of (3) on any interval. The corresponding general solution of (3) is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$

Example 13. Solve $y'' + y' - 2y = 0$, with $y(0) = 4$, $y'(0) = -5$.

The characteristic equation is $\lambda^2 + \lambda - 2 = 0$ ($a^2 - 4b = 9 > 0$). The roots are $\lambda_1 = 1$ and $\lambda_2 = -2$. The general solution is $y = c_1 e^x + c_2 e^{-2x}$. Initial conditions $y(0) = 4$ and $y'(0) = -5$ give $c_1 + c_2 = 4$ and $c_1 - 2c_2 = -5$. Solving: $c_1 = 1$, $c_2 = 3$. Thus $y = e^x + 3e^{-2x}$ is the solution.

Case 2. Equation (4) has a real double root λ_1 $(= \lambda_2)$.

This occurs when $a^2 - 4b = 0$, and $\lambda_1 = \lambda_2 = -\frac{a}{2}$, from which we get one solution $y_1 = e^{-\frac{a}{2}x}$. To find a second solution y_2, we try $y_2 = xe^{-\frac{ax}{2}}$. This does work: $y_2' = e^{-\frac{ax}{2}} - \frac{a}{2}xe^{-\frac{ax}{2}}$, and

$$\begin{aligned} y_2'' &= -\tfrac{a}{2}e^{-\frac{ax}{2}} - \tfrac{a}{2}e^{-\frac{ax}{2}} + \tfrac{a^2}{4}xe^{-\frac{ax}{2}} \\ &= \left(-a + \tfrac{a^2}{4}x\right) e^{-\frac{ax}{2}} \end{aligned}$$

So

$$\begin{aligned} y_2'' + ay_2' + by_2 &= \left[-a + \tfrac{a^2}{4}x + a - \tfrac{a^2}{2}x + bx\right] e^{-\frac{ax}{2}} \\ &= \left[b - \tfrac{a^2}{4}\right] xe^{-\frac{ax}{2}} = 0 \end{aligned}$$

because $b = \frac{a^2}{4}$.

Thus in this case when $a^2 - 4b = 0$, a linearly independent pair of solutions of $y'' + ay' + by = 0$ on any interval is $e^{-\frac{ax}{2}}$, $xe^{-\frac{ax}{2}}$. The corresponding general solution is

$$y = (c_1 + c_2 x)e^{-\frac{ax}{2}}.$$

Example 14.

(a) Solve $y'' + 8y' + 16y = 0$,

(b) Solve the initial problem $y'' - 4y' + 4y = 0$, $y(0) = 3$, $y'(0) = 1$.

Case 3. Equation (4) has two complex roots λ_1, λ_2.

This happens when $a^2 - 4b < 0$. We set $w = \sqrt{b - \frac{a^2}{4}}$. Then you can easily show that $\lambda_1,\ \lambda_2 = -\frac{a}{2} \pm iw$, where $i^2 = -1$. We try

$$y_1 = e^{-\frac{ax}{2}} \cos wx$$
$$y_2 = e^{-\frac{ax}{2}} \sin wx$$

We leave it to you to show by substitution that these are solutions of $y'' + ay' + by = 0$. The point to remember is that $-\frac{a}{2}$, the real part of λ_1 and λ_2, goes into the exponential part $\left(e^{-\frac{ax}{2}}\right)$ while the imaginary part w goes into the cos and sin part, $\cos(wx)$ and $\sin(wx)$.
For example: If $\lambda_1,\ \lambda_2 = -1 \pm 2i$, then $y_1 = e^{-x}\cos(2x)$ and $y_2 = e^{-x}\sin(2x)$.

To conclude: set

$$\begin{aligned} y_1 &= e^{-\frac{a}{2}x} \cos wx, \\ y_2 &= e^{-\frac{a}{2}x} \sin wx. \end{aligned}$$

Then y_1 and y_2 are linearly independent. The corresponding general solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= e^{-\frac{a}{2}x}(c_1 \cos wx + c_2 \sin wx). \end{aligned}$$

Example 15.

(a) Solve $y'' + 2y' + 5y = 0$,

(b) Solve $y'' + 2y' + 5y = 0$, $y(0) = 1$, $y'(0) = 5$.

1.4.4 Nonhomogeneous equations

We consider

$$y'' + p(x)y' + q(x)y = r(x), \ r(x) \not\equiv 0. \tag{1}$$

The corresponding homogeneous equation is

$$y'' + p(x)y' + q(x)y = 0. \tag{2}$$

Let y_1 and y_2 be any two solutions of (1). Then

$$y_1'' + p(x)y_1' + q(x)y_1 \quad = r(x), \quad (3)$$

and

$$y_2'' + p(x)y_2' + q(x)y_2 \quad = r(x). \quad (4)$$

Subtracting (4) from (3):

$$y_1'' - y_2'' + p(x)(y_1' - y_2') + q(x)(y_1 - y_2) = r(x) - r(x) = 0.$$

Thus $(y_1 - y_2)'' + p(x)(y_1 - y_2)' + q(x)(y_1 - y_2) = 0$, i.e. $y_1 - y_2$ is a solution of (2).

On the other hand, if y_0 is a solution of (2) and y_1 a solution of (1), then clearly $y_1 + y_0$ is again a solution of (1). This suggests the following definition:

Definition. A **general solution** of the nonhomogeneous DE (1) is of the form

$$y(x) = y_h(x) + y_p(x), \tag{$\heartsuit$}$$

where $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$ is a general solution of the homogeneous DE (2) and $y_p(x)$ is any solution of (1) containing no arbitrary constants.

Thus to solve (1), we have to solve the homogeneous equation (2) and find a (particular) solution of (1). The sum of these two is what we want.

Determination of $y_p(x)$

(I) Method of undetermined coefficients

This method applies to equations of the form $y'' + ay' + by = r(x)$, where a and b are constants, and $r(x)$ is a polynomial, exponential function, sine or cosine, or sums or products of such functions.

We denote $y'' + ay' + by$ by $L(y)(x)$, where a and b may be complex numbers. We'll make use of the Principle of super-position:

If $y_1(x)$ is a solution of $L(y)(x) = g_1(x)$ and $y_2(x)$ is a solution of $L(y)(x) = g_2(x)$, then for any constants c_1 and c_2, $y = c_1 y_1(x) + c_2 y_2(x)$ is a solution of $L(y)(x) = c_1 g_1(x) + c_2 g_2(x)$. Using this principle, the problem is reduced to finding a particular solution of $L(y)(x) = p(x)e^{kx}$, where $p(x)$ is a polynomial in x and k is a real or complex constant. Method of undetermined coefficients is adequately described by considering the following three cases, which we illustrate with examples.

1. Polynomial case

In this case $k = 0$. The method begins with "try a polynomial with unknown coefficients".

Example 16.

$$y'' - 4y' + y = x^2 + x + 2.$$

Try $y = Ax^2 + Bx + C$ where A, B, C are constant. Then we have, after substitution,

$$2A - 4(2Ax + B) + Ax^2 + Bx + C = x^2 + x + 2$$

or

$$Ax^2 + (B - 8A)x + 2A - 4B + C = x^2 + x + 2.$$

Comparing coefficients, we have

$$\begin{aligned} A &= 1 \\ B - 8A &= 1 \Rightarrow B = 9 \\ 2A - 4B + C &= 2 \Rightarrow C = 36 \end{aligned}$$

So $Ax^2 + Bx + C = x^2 + 9x + 36$ is a particular solution.

Example 17.

$$y'' - 2y = 2x^3.$$

Try $Ax^3 + Bx^2 + Cx + D$, and we get

$$\begin{aligned} & 6Ax + 2B - 2Ax^3 - 2Bx^2 - 2Cx - 2D \\ = \; & 2x^3 + 0x^2 + 0x + 0. \end{aligned}$$

This means

$$\begin{aligned} -2A &= 2 \Rightarrow A = -1 \\ -2B &= 0 \Rightarrow B = 0 \\ 6A - 2C &= 0 \Rightarrow C = -3 \\ 2B - 2D &= 0 \Rightarrow D = 0 \end{aligned}$$

So $y = -x^3 - 3x$ is a particular solution.

2. Exponential case

Here k is real but not zero. The method begins with "put $y = ue^{kx}$, where $u = u(x)$." This substitution will remove e^{kx} from the equation and reduce the problem to the polynomial case 1 (above).

Example 18.

$$y'' - 4y' + 2y = 2x^3 e^{2x}.$$

Substituting $y = ue^{2x}$, we get

$$\begin{aligned} y' &= u'e^{2x} + 2ue^{2x} \\ y'' &= u''e^{2x} + 4u'e^{2x} + 4ue^{2x}. \end{aligned}$$

Thus the DE in this example becomes:

$$u''e^{2x} - 2ue^{2x} = 2x^3 e^{2x},$$

$$u'' - 2u = 2x^3.$$

Using the result of example 17, $u = -x^3 - 3x$. Thus a particular solution is $y_p(x) = (-x^3 - 3x)e^{2x}$.

Example 19.

$$y'' - 4y' + 4y = 20x^3 e^{2x}.$$

Set $y = ue^{2x}$, we get as above $u'' = 20x^3$.
Integrating twice, $u = x^5 + Ax + B$. We may set $A = 0 = B$ and take a particular solution $y_p = x^5 e^{2x}$.

3. Trigonometric case

Here we use complex exponentials. Recall: $e^{s+it} = e^s(\cos t + i \sin t)$. We only need the differentiation property of e^{cx}, c complex.

Recall also: if $y = u(x) + iv(x)$ is a complex valued solution of $L(y)(x) = h_1(x) + ih_2(x)$ (where $u(x)$, $v(x)$, $h_1(x)$ and $h_2(x)$ are real valued functions), then

$$\begin{aligned}
&(u'' + iv'') + a(u' + iv') + b(u + iv) \\
&\qquad = h_1 + ih_2, \\
\Rightarrow\ &(u'' + au' + bu) + i(v'' + av' + bv) \\
&\qquad = h_1 + ih_2.
\end{aligned}$$

Equating real and imaginary parts:

u is a solution of $L(y)(x) = h_1(x)$

v is a solution of $L(y)(x) = h_2(x)$

Example 20. Solve

$$y'' + 4y = 16x \sin 2x. \tag{a}$$

We solve instead

$$z'' + 4z = 16xe^{i2x}. \tag{b}$$

By the above discussion, the imaginary part of a solution of (b) will be a solution of (a). Set $z = ue^{i2x}$, $u = u(x)$. (recall case 2 above) computing z' and z'' and substituting in (b) give

$$u'' + 4iu' = 16x.$$

Solving this, we get $u = -2ix^2 + x$. Thus $z = (-2ix^2 + x)e^{i2x}$, and a particular solution of (a) is

$$y = Imz = x \sin 2x - 2x^2 \cos 2x.$$

Example 21.

$$y'' + 2y' + 5y = 16xe^{-x}\cos 2x. \qquad (c)$$

Solve instead

$$z'' + 2z' + 5z = 16xe^{(-1+2i)x}. \qquad (d)$$

$Re\,(z)$ will be a solution of (c). Put $z = ue^{(-1+2i)x}$, $u = u(x)$.
$[Ans : y = Ae^{(-x)}cos(2x) + Be^{(-x)}sin(2x) + xe^{(-x)}[2xsin(2x) + cos(2x).]$

(II) Method of variation of parameters

We consider

$$y'' + p(x)y' + q(x)y = r(x) \qquad (1)$$

where p, q, r are continuous functions of x on some open interval I.

The continuity of p and q implies that the corresponding homogeneous d.e. $y'' + p(x)y' + q(x)y = 0$ has a general solution $y_h(x) = c_1y_1(x) + c_2y_2(x)$ on I.

The method of variation of parameters involves replacing constants c_1 and c_2 by functions $u(x)$ and $v(x)$ to be determined so that

$$y_p(x) = u(x)y_1(x) + v(x)y_2(x)$$

is a particular solution of (1) on I.

Now $y_p' = u'y_1 + uy_1' + v'y_2 + vy_2'$. We impose

$$u'y_1 + v'y_2 = 0. \qquad (A)$$

Then $y_p' = uy_1' + vy_2'$, and $y_p'' = u'y_1' + uy_1'' + v'y_2' + vy_2''$.

Substituting these into (1) we obtain

$$\begin{aligned}&(u'y_1' + v'y_2' + uy_1'' + vy_2'') + p(uy_1' + vy_2') + q(uy_1 + vy_2)\\ &= u(y_1'' + py_1' + qy_1) + v(y_2'' + py_2' + qy_2) + (u'y_1' + v'y_2')\\ &= r.\end{aligned}$$

Thus

$$u'y_1' + v'y_2' = r. \qquad (B)$$

Solving (A) and (B) for u' and v':

$$u' = -\frac{y_2 r}{y_1 y_2' - y_1' y_2}, \ v' = \frac{y_1 r}{y_1 y_2' - y_1' y_2}. \qquad \text{(C)}$$

Since y_1 and y_2 are linearly independent,
$y_1 y_2' - y_1' y_2 \neq 0$. [Convince yourself that this is true!]

Integrating (C) we obtain

$$\begin{aligned} u &= -\int \frac{y_2 r}{y_1 y_2' - y_1' y_2}\, dx, \\ v &= \int \frac{y_1 r}{y_1 y_2' - y_1' y_2}\, dx. \qquad (D) \end{aligned}$$

We obtain now y_p and hence a general solution of (1).

Note. The term $y_1 y_2' - y_1' y_2$ may be viewed as the determinant $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.
It's called the **Wronskian** of y_1 and y_2.

Caution. When applying the above procedure to solve (1), make sure that the given DE is in standard form (1) where the coefficient of y'' is 1.

Example 22. Solve $y'' + y = \tan x, \ x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

The characteristic equation of $y'' + y = 0$ is $\lambda^2 + 1 = 0$. So $\lambda = \pm i$, and $y_h = c_1 \cos x + c_2 \sin x$. Here $y_1 = \cos x$, $y_2 = \sin x$.

$\therefore y_1 y_2' - y_1' y_2 = \cos x \cos x - (-\sin x) \sin x = 1.$

Using (D) above,

$$\begin{aligned}
u &= -\int \sin x \tan x \, dx \\
&= -\int \frac{\sin^2 x}{\cos x} \, dx \\
&= \int \frac{\cos^2 x - 1}{\cos x} \, dx \\
&= \int (\cos x - \sec x) \, dx \\
&= \sin x - \ell n \, |\sec x + \tan x|, \\
v &= \int \cos x \tan x \, dx \\
&= -\cos x.
\end{aligned}$$

A general solution is

$$\begin{aligned}
y &= y_h + u \cos x + v \sin x \\
&= c_1 \cos x + c_2 \sin x - \cos x \, \ell n \, |\sec x + \tan x|.
\end{aligned}$$

Example 23. Solve $y'' - y = e^{-x} \sin e^{-x} + \cos e^{-x}$.

We have $y_h = c_1 e^x + c_2 e^{-x}$.

We take $y_p = ue^x + ve^{-x}$, and determine u, v as in the above example:

We have $y_1 y_2' - y_1' y_2 = -2$, and evaluating the integrals by means of a change of variable [let $z = e^{-x}$, so $dz = -z dx$] and using integration by parts, we get

$$\begin{aligned}
u &= -\frac{1}{2}(2 \sin e^{-x} - e^{-x} \cos e^{-x}) \\
v &= -\frac{1}{2} e^x \cos e^{-x} \\
y_p &= -e^x \sin e^{-x}.
\end{aligned}$$

The general solution is $y = c_1 e^x + c_2 e^{-x} - e^x \sin e^{-x}$.

2 MORE APPLICATIONS OF ODEs

2.1 The harmonic oscillator

Consider the pendulum shown. The small object, mass m, at the end of the pendulum, is moving on a circle of radius L, so the component of its velocity tangential to the circle is $L\dot{\theta}$. Hence its tangential acceleration is $L\ddot{\theta}$ and so by $\vec{F} = M\vec{a}$ we have

$$mL\ddot{\theta} = -mg\sin\theta.$$

An obvious solution is $\theta = 0$. This is called an **equilibrium** solution, meaning that θ is a constant function. This means that if you set $\theta = 0$ initially, then θ will remain at 0 and the pendulum will not move — which of course we know is correct.

There is another equilibrium solution, $\theta = \pi$. Again, in theory, if you set the pendulum exactly at $\theta = \pi$, then it will remain in that position forever. In reality, of course, it won't! Because the slightest puff of air will knock it over. So this equilibrium is very different from the one at $\theta = 0$. This is a very important distinction!

Equilibrium is said to be **stable** if a small push away from equilibrium remains small. If the small push tends to grow large, then the equilibrium is **unstable**. Obviously this is important for engineers! Especially you want vibrations of structures, engines, etc to remain small.

First equilibrium: Let's look at $\theta = \pi$. By Taylor's theorem, near $\theta = \pi$, we have

$$f(\theta) = f(\pi) + f'(\pi)(\theta - \pi) + \frac{1}{2}f''(\pi)(\theta - \pi)^2 + ...$$

Now $\sin(\pi) = 0$, $\sin'(\pi) = \cos(\pi) = -1$, $\sin''(\pi) = -\sin(\pi) = 0$ etc so

$$\sin(\theta) = 0 - (\theta - \pi) - 0 + \frac{1}{6}(\theta - \pi)^3 \quad \text{etc}$$

For small deviations away from π, $\theta - \pi$ is small, $(\theta - \pi)^3$ is much smaller, etc, so we can approximate

$$\sin(\theta) \approx -(\theta - \pi)$$

so our equation is approximately

$$mL\ddot{\theta} = -mg\sin\theta = mg(\theta - \pi).$$

Let $\phi = \theta - \pi$, so $\ddot{\phi} = \ddot{\theta}$, and now

$$\ddot{\phi} = \frac{g}{L}\phi.$$

The general solution is

$$\phi = Ae^{(\sqrt{g/L})t} + Be^{-(\sqrt{g/L})t}$$

so $\theta = \phi + \pi = Ae^{(\sqrt{g/L})t} + Be^{-(\sqrt{g/L})t} + \pi$.

As you know, the exponential function grows very quickly; so even if θ is close to π initially, it won't stay near to it very long! Very soon, θ will arrive either at $\theta = 0$ or 2π, far away from $\theta = \pi$. The equilibrium is **unstable**!

Second equilibrium: Now what about $\theta = 0$? Here of course we use Taylor's theorem around zero,

$$\begin{aligned} f(\theta) &= f(0) + f'(0)\theta + \frac{1}{2}f''(0)\theta^2 + ... \\ \sin(\theta) &= 0 + \theta - 0 - \frac{1}{6}\theta^3 + ... \end{aligned}$$

so $\sin(\theta) \approx \theta$ and we have approximately

$$\begin{aligned} mL\ddot{\theta} &= -mg\theta \quad \text{or} \\ \ddot{\theta} &= -\frac{g}{L}\theta = -\omega^2\theta \end{aligned}$$

with $\omega^2 = g/L$. That minus sign is crucial!

General solution is $C\cos(\omega t) + D\sin(\omega t)$ where C and D are arbitrary constants.

Now using trigonometric identities you can show that any expression of the form $C\cos(x) + D\sin(x)$ can be written as

$$C\cos(x) + D\sin(x) = \sqrt{C^2 + D^2}\cos(x - \gamma)$$

where $\tan(\gamma) = D/C$. You can see this by taking the scalar product of the vectors $\begin{bmatrix} C \\ D \end{bmatrix}$ and $\begin{bmatrix} \cos(x) \\ \sin(x) \end{bmatrix}$.

So now we can write our general solution as

$$\theta = A\cos(\omega t - \delta)$$

Check: this does satisfy $\ddot{\theta} = -\omega^2\theta$ and it does contain 2 arbitrary constants, A and δ.

In this case, θ is never larger than A, never smaller than $-A$, so if θ was small initially, it remains small! We call A the **amplitude**. So the equilibrium in this case is stable.

This is called **simple harmonic motion**. Clearly θ repeats its values every time ωt increases by 2π (since cos is periodic with period 2π). Now

$$\omega t \to \omega t + 2\pi$$

means

$$t \to t + \frac{2\pi}{\omega}$$

So $\frac{2\pi}{\omega} = 2\pi\sqrt{L/g}$ is the time taken for θ to return to its initial value, the **period**. The number ω is called the **angular frequency**.

2.2 Forced oscillations

Suppose you have a mass m which can move in a horizontal line. It is attached to the end of a spring which exerts a force

$$F_{\text{spring}} = -kx$$

where x is the extension of the spring and k is a constant (called the spring constant). This is **Hooke's Law**. Now we attach an external motor to the mass m. This motor exerts a force $F_0\cos(\alpha t)$, where F_0 is the amplitude of the external force and α is the frequency.

If $F_0 = 0$ we just have, from Newton,

$$m\ddot{x} = -kx,$$

so we get $\ddot{x} = -\omega^2 x, \quad \omega = \sqrt{k/m}$. Here ω is the frequency that the system has if we leave it alone that is, it is the natural frequency. It has nothing to do with α of course $-$ we can choose α to suit ourselves.

If $F_0 \neq 0$, then we have

$$m\ddot{x} + kx = F_0 \cos \alpha t.$$

Let z be a complex function satisfying

$$m\ddot{z} + kz = F_0\ e^{i\alpha t}.$$

Clearly the real part, $Re\ z$, satisfies the above equation, so we can solve for z and them take the real part. We try

$$z = C\ e^{i\alpha t}$$

and get

$$\begin{aligned} mC(i\alpha)^2 e^{i\alpha t} + C\ k\ e^{i\alpha t} &= F_0 e^{i\alpha t} \\ \Rightarrow C = \frac{F_0}{k - m\alpha^2} &= \frac{F_0/m}{\omega^2 - \alpha^2} \end{aligned}$$

So $\quad Re\ z = \frac{F_0/m}{\omega^2-\alpha^2}\cos(\alpha t)$
and the general solution is

$$x = A\cos(\omega t - \delta) + \frac{F_0/m}{\omega^2 - \alpha^2}\cos(\alpha t).$$

Note that

$$\dot{x} = -A\omega \sin(\omega t - \delta) - \frac{\alpha F_0/m}{\omega^2 - \alpha^2}\sin(\alpha t).$$

The arbitrary constants A and δ are fixed by giving $x(0)$ and $\dot{x}(0)$ as usual. For example, suppose $x(0) = \dot{x}(0) = 0$, then

$$\begin{aligned} 0 &= A\cos(\delta) + \frac{F_0/m}{\omega^2 - \alpha^2} \\ 0 &= A\omega \sin(\delta). \end{aligned}$$

Assuming $F_0 \neq 0$, we cannot have $A = 0, \Rightarrow \delta = 0$. So $A = -\frac{F_0/m}{\omega^2 - \alpha^2}$,

$$x = \frac{F_0/m}{\omega^2 - \alpha^2} \left[\cos(\alpha t) - \cos(\omega t) \right].$$

Using the trigonometric identity

$$\cos A - \cos B = -2 \sin\left(\frac{A-B}{2}\right) \sin\left(\frac{A+B}{2}\right)$$

we find

$$x = \frac{2F_0/m}{\alpha^2 - \omega^2} \sin\left[\left(\frac{\alpha - \omega}{2}\right) t\right] \sin\left[\left(\frac{\alpha + \omega}{2}\right) t\right]$$

What happens if we let $\alpha \to \omega$? We have

$$\begin{aligned} A(t) &= \frac{2F_0/m}{\alpha + \omega} \times \frac{\sin\left[\frac{\alpha-\omega}{2} t\right]}{\alpha - \omega} \\ &\to \frac{F_0}{m\omega} \times \frac{t}{2} = \frac{F_0 t}{2m\omega} \end{aligned}$$

by L'Hopital's rule. So in this limit

$$x = \frac{F_0 t}{2m\omega} \sin(\omega t)$$

and we see that the oscillations go completely out of control. This situation is called **resonance**.

We see that if a system is forced in a way that agrees with its own natural frequency, it can oscillate uncontrollably. This can be very dangerous!

2.3 Conservation

Newton's 2nd law involves time derivatives, but sometimes it can be expressed in terms of spatial derivatives, by means of the following trick:

$$\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) = \dot{x}\frac{d\dot{x}}{dx} = \frac{dx}{dt}\frac{d\dot{x}}{dx} = \ddot{x} \quad \text{(chain rule)}.$$

For SHM we have

$$m\ddot{x} = -kx$$

so

$$m\frac{d}{dx}(\frac{1}{2}\dot{x}^2) = -kx.$$

But now we can integrate both sides:

$$\frac{1}{2}m\dot{x}^2 = -\frac{1}{2}kx^2 + E$$

where E is a constant of integration. So we have

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2.$$

As you know, $\frac{1}{2}m\dot{x}^2$ is called the **kinetic energy** of the oscillator, and $\frac{1}{2}kx^2$ is called the **potential energy**. We call E the **total energy**.

The fact that E isconstant is called the **conservation of energy**.

3 MATHEMATICAL MODELLING

3.1 Malthus model of population

The total population of a country is clearly a function of time, N(t) (N may be measured in millions, so values of N less than 1 are meaningful). Given the population now, can we predict what it will be in the future?

Suppose that B is a function giving the PER CAPITA BIRTH-RATE in a given society, *ie* B is the number of babies born per second, divided by the total population of the country at that moment. Note that B could be small in a big country and large in a small country - it depends on whether there is a strong social pressure on people to get married and have kids. Now B could depend on time and it could depend on N.

But suppose you don't believe these things: suppose you think that people will always have as many kids as they can, no matter what. Then B is constant. Now just as

$$\text{DISTANCE} = \text{SPEED} \times \text{TIME}$$

when SPEED IS CONSTANT, so also we have

$$\#\text{babies born in time } \delta t = BN\delta t$$

Similarly let D be the DEATH RATE PER CAPITA; again, it could be a function of t (better medicine, fewer smokers) or N (overcrowding leads to famine/disease) but if we assume that it is constant, then

$$\#\text{deaths in time } \delta t = DN\delta t$$

So the change in N, δN, during δt is

$$\delta N = \#birth - \#deaths$$

Provided there is no emigration or immigration. Thus,

$$\delta N = (B - D)N\delta t$$

and so $\frac{\delta N}{\delta t} = (B - D)N$ or in the limit as $\delta t \to 0$,

$$\frac{dN}{dt} = (B - D)N = kN \tag{3}$$

if $k = B - D$.

This model of society was put forward by THOMAS MALTHUS in 1798. Clearly Malthus was assuming a socially STATIC society in which human reproductive behaviour never changes with time or overcrowding, poverty etc. What does Malthus' model predict? Suppose that the population now is $\hat{N}$, and let $t = 0$ now.

From $\frac{dN}{dt} = kN$ we have $\int \frac{dN}{N} = \int kdt = k\int dt = kt + c$ so $ln(N) = kt + c$ and thus $N(t) = Ae^{kt}$. Since $\hat{N} = N(0) = A$, we get:

$$N(t) = \hat{N}e^{kt} \tag{4}$$

with graphs as shown on figure 1.

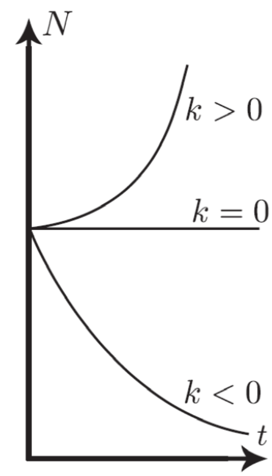


Figure 1: Graphs of $N(t)$, for different values of k

The population collapses if $k < 0$ (more deaths than births per capita), remains stable if (and only if) $k = 0$, and it EXPLODES if $k > 0$ (more births than deaths).

Malthus observed that the population of Europe was increasing, so he predicted a catastrophic POPULATION EXPLOSION; since the food supply could not be expanded so fast, this would be disastrous.

In fact, this didn't happen (in Europe). So Malthus' model is wrong: many millions went to the US, many millions died in wars.

Second, the "static society" assumption has turned out to be wrong in many societies, with B and D both declining as time passed after WW2.

SUMMARY: The Malthus model of population is based on the idea that per capita birth and death rated are independent of time and N. It leads to EXPONENTIAL growth or decay of N.

3.2 Improving on Malthus

Malthus' model is interesting because it shows that static behaviour patterns can lead to disaster. But precisely because e^{kt} grows so quickly, Malthus' assumptions must eventually go wrong - obviously there is a limit to the possible population. Eventually, if we don't control B, then D will have to increase. So we have to assume that D is a function of N.

Clearly, D must be an increasing function of N, but which function? Well, surely the simplest possible choice:

<u>Remember:</u> always go for the simple model before trying a complicated one!

$$\boxed{D = sN, \quad \begin{array}{c}\text{(LOGISTIC)}\\ \text{ASSUMPTION}\\ s = \text{constant}\end{array}} \tag{5}$$

This represents the idea that, in a world with finite resources, large N will eventually cause starvation and disease and so increase D.

<u>Remark:</u> In modelling, it is often useful to take note of $\boxed{\text{units}}$.

Units of D are (#dead people) / second / (total # people) = $(\text{sec})^{-1}$. Units of N are # (ie no units). So if $D = sN$, units of s must be $(\text{sec})^{-1}$.

As before, let $\hat{N}$ be the value of N at $t = 0$. We have to solve

$$\frac{dN}{dt} = BN - DN = BN - sN^2$$

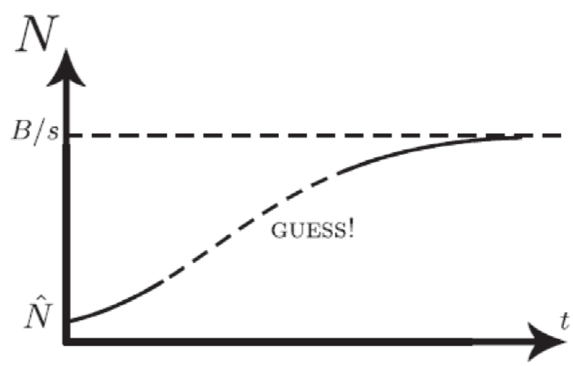
with the condition $N(0) = \hat{N}$

We can and will solve this, but let's try to *guess* what the solution will look like (a useful skill - in many other cases you won't be able to solve exactly). Suppose that $\hat{N}$ is very small. Then (by continuity) $N(t)$ will be very small for t near to zero.

Of course if N is small, N^2 is much smaller and can be neglected (remember that N may be measured in millions or billions, so N can be small). So at early times, our ODE is *almost* linear and so

$$\frac{dN}{dt} \approx BN \to N(t) \approx \hat{N}e^{Bt}$$

So *at first* the population explodes, as Malthus predicted. On the other hand, if N continues to grow, since N^2 grows faster than N, we will reach a point where $sN^2 \approx BN$ ie $N \approx B/s$. At that point , since $\frac{dN}{dt} = BN - sN^2$, the population will stop growing. So B/s should measure the *maximum* population possible. So we *guess* that the solution should look like this:



ie. it starts out exponentially and ends up approaching B/s asymptotically. The dotted part is a reasonable *guess*!

Now that we know what to expect, let's actually solve it.

$$\frac{dN}{dt} = BN - sN^2 \rightarrow t = \int \frac{dN}{N(B-sN)} + c$$

Write $\frac{1}{N(B-sN)} = \frac{\alpha}{N} + \frac{\beta}{B-sN}$

$$\begin{aligned} 1 &= \alpha(B-sN) + \beta N \\ &= \alpha B + (\beta - \alpha s)N \rightarrow 1 = \alpha B \ , \ \beta = \alpha s \\ \alpha &= 1/B \ , \ \beta = s/B, \text{ so} \end{aligned}$$

$$\begin{aligned} \int \frac{dN}{N(B-sN)} &= \frac{1}{B}\int \frac{dN}{N} + \frac{s}{B}\int \frac{dN}{B-sN} \\ &= \frac{1}{B}\ln N - \frac{1}{B}\ln|B-sN| \end{aligned}$$

Now here we begin to feel uneasy - what if $N = B/s$ at some time? (ln(0) is not defined). In fact we should have worried about this when we first wrote $\frac{1}{B-sN}$ - how do we know that we are not dividing by zero? Let's not worry about that just now: let's *assume* (temporarily) that $B - sN$ is never zero. That is, we assume either that N is always either less than B/s or more than B/s. OK, let's take less than first. So $|B-sN| = B - sN$, and we get

$$\begin{aligned} t &= \frac{1}{B}\ln N - \frac{1}{B}\ln(B-sN) + c \\ &= \frac{1}{B}\ln\frac{N}{B-sN} + c \end{aligned}$$

So

$\frac{N}{B-sN} = Ke^{Bt}$. Since $\hat{N} = N(0)$, $\frac{\hat{N}}{B-s\hat{N}} = K$ so

$$\frac{N}{B-sN} = \frac{\hat{N}}{B-s\hat{N}}e^{Bt}$$

Solve for N,

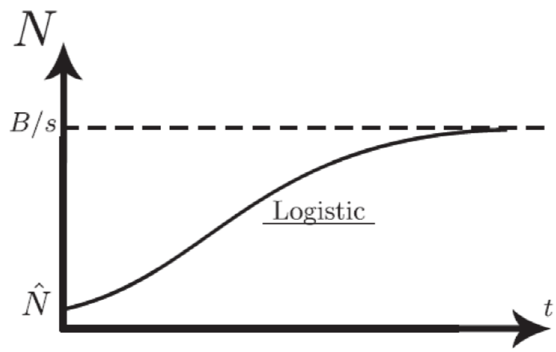
$$N(t) = \frac{B}{s + \left(\frac{B}{\hat{N}} - s\right)e^{-Bt}} \tag{6}$$

Remark: It is a very good habit to check that your solution agrees with your assumptions - to guard against mistakes!

Check: $N(0) = \frac{B}{s+\left(\frac{B}{\hat{N}}-s\right)} = \hat{N}$ correct!

Check: If $B - sN > 0$ is true at $t = 0$ then $B - s\hat{N} > 0$ so $\left(\frac{B}{\hat{N}} - s\right) > 0$ so $\frac{B}{s+\left(\frac{B}{\hat{N}}-s\right)e^{-Bt}} < \frac{B}{s}$ for all t *ie* $N(t) < B/s$ which is consistent.

The graph of (6) is easy to sketch:



This is the famous LOGISTIC CURVE; $N(t)$ given by (6) is called the LOGISTIC FUNCTION; and $\frac{dN}{dt} = BN - sN^2$ is the LOGISTIC EQUATION.

It's easy to see what is happening here. Initially the population is small, plenty of food and space, so we get a Malthusian population explosion. But eventually the death rate rises until it is almost equal to the birth rate, ie. $sN \approx B$ or $N \approx B/s$) and then the population approaches a fixed limit.

This situation is what people usually mean when they use the word "LOGISTIC". But we are not done yet: we assumed that $N(t) < B/s$. What if $N(t) > B/s$?

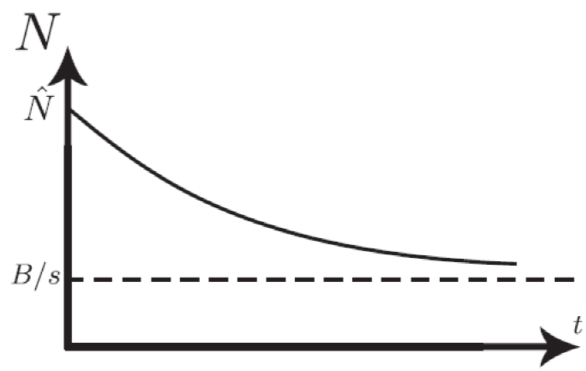
Then $|B - sN| = -(B - sN)$ so:

$$\begin{aligned} t &= \frac{1}{B}\ln N - \frac{1}{B}\ln(sN - B) + c \\ &= \frac{1}{B}\ln\frac{N}{sN - B} + c \Rightarrow \end{aligned}$$

$$N(t) = \frac{B}{s - \left(s - \frac{B}{\hat{N}}\right)e^{-Bt}} \tag{7}$$

Check $N(0) = \hat{N}$ and $N(t) > B/s$.

And now the graph is:



Again, the meaning is clear: the initial population was so big that the death rate exceeded the birth rate, so of course the population declines until it gets near to the long-term sustainable value.

The number B/s is called the CARRYING CAPACITY or the SUSTAINABLE POPULATION - in all cases, it is the value approached by $N(t)$ as $t \to \infty$. If we set

$$N_\infty = B/s \tag{8}$$

then our solutions are:

$$N(t) = \frac{N_\infty}{1 + \left(\frac{N_\infty}{\hat{N}} - 1\right) e^{-Bt}} (\hat{N} < N_\infty) \tag{9}$$

$$N(t) = \frac{N_\infty}{1 - \left(1 - \frac{N_\infty}{\hat{N}}\right) e^{-Bt}} (\hat{N} > N_\infty) \tag{10}$$

(obtained by dividing numerator and denominator by s in (6) and (7).

But we aren't finished yet! We had to assume that N is never equal to B/s, ie. to N_∞. So we have to think about this.

First, let's ask what happens if $\hat{N} = N_\infty$, ie. $N(t)$ is initially N_∞.

Intuitively, since N_∞ is the sustainable population, you would expect that $N(t) = \text{constant} = \hat{N} = N_\infty$ should be possible!

Indeed, substitute $N = N_\infty$ into $\frac{dN}{dt} = BN - sN^2$ and the left side is zero while the right side is $BN_\infty - sN_\infty^2 = N_\infty[B - sN_\infty] = N_\infty[B - B] = 0$. So we have

$$N(t) = N_\infty \quad (\hat{N} = N_\infty) \tag{11}$$

Clearly (9) (10) (11) cover all possible values of $\hat{N}$.

SUMMARY: A simple way to improve on Malthus is to replace his assumption $D = \text{constant}$ by the LOGISTIC ASSUMPTION $D = sN$, $s = \text{constant}$. If $\hat{N} < B/s = N_\infty$, then the graph of $N(t)$ is the "S-shape" on page 41.

3.3 Harvesting

A major application of modelling is in dealing with populations of animals e.g. fish. We want to know how many we can eat without wiping them out (gross). Let's build on our logistic model, ie. *assume* that the fish population would follow that model if we didn't catch any. Next, *assume* that we catch

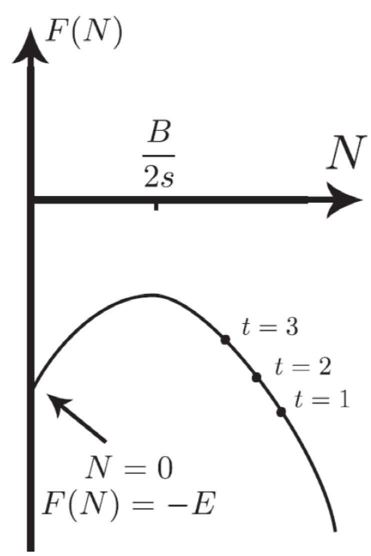
E (constant) fish per year. Then we have:

$$\frac{dN}{dt} = (B - sN)N - E \qquad \boxed{\text{Basic Harvesting Model}}$$

We'll now try to guess what the solutions should look like. We are particularly interested in the long term - will the harvesting eventually exterminate the fish? Consider the function:

$$F(N) = (B - sN)N - E.$$

The graph of $F(N)$ can take one of 3 forms.



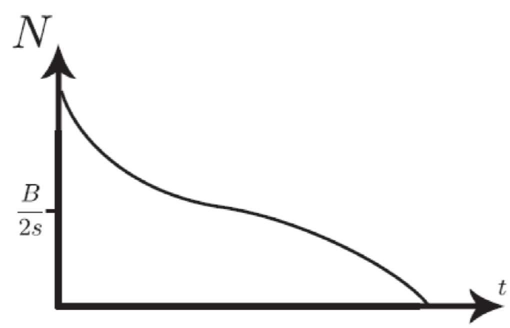
In the first case, the quadratic

$$-sN^2 + BN - E$$

has no solutions, ie.

$$B^2 - 4sE < 0 \text{ or } E > \frac{B^2}{4s}.$$

Since $\frac{dN}{dt} = F(N)$, we see that in this case the population always declines. Note that there is no t-axis in this picture, but you should imagine time passing by thinking of a moving spot on the graph. Notice that as we move through $t = 1, 2, 3$ we have to move to the left since $\frac{dN}{dt} < 0$ always. But $|\frac{dN}{dt}|$ is DECREASING, so the rate of decline slows down as time goes on, until we pass $N = \frac{B}{2s}$. After that, $|\frac{dN}{dt}|$ increases and the value of N decreases rapidly to zero. Congratulations - you have wiped out your fish!



In drawing the previous picture, we assumed that $\hat{N}$, the initial value of N, was greater than $\frac{B}{2s}$, the maximum point on the graph of $F(N)$. Note that $\frac{dN}{dt} = F(N)$ implies

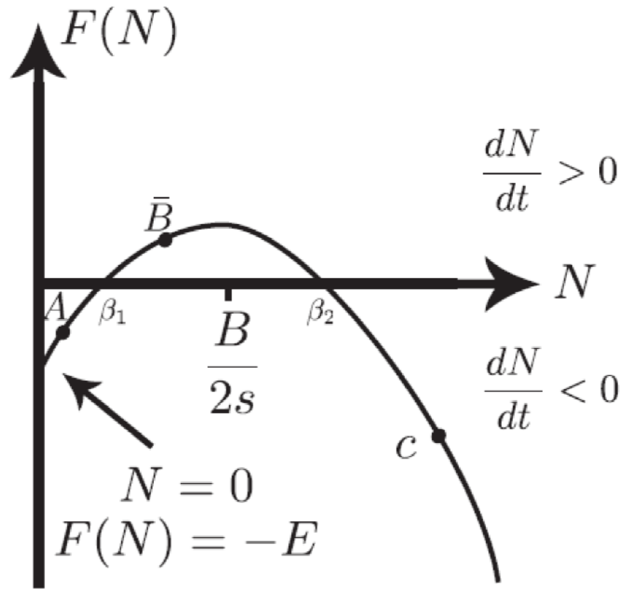
$$\frac{d^2N}{dt^2} = F'(N)\frac{dN}{dt} = F'(N)F(N)$$

so watch for "points of inflection" on the graph of $N(t)$ at values of N where $F'(N)$ or $F(N)$ vanish.

Clearly it is **not** a good idea to harvest at a rate $E > \frac{B^2}{4s}$ (Check units). So let's assume <u>second case</u> - that our fishermen ease off and harvest at a rate $E < \frac{B^2}{4s}$. The special case E EXACTLY EQUALS $\frac{B^2}{4s}$ is clearly impossible in reality, but we will come back to it later anyway!

Now the graph of $F(N)$ is as shown in the next diagram.

Again, remembering that $\frac{dN}{dt} = F(N)$, we see that

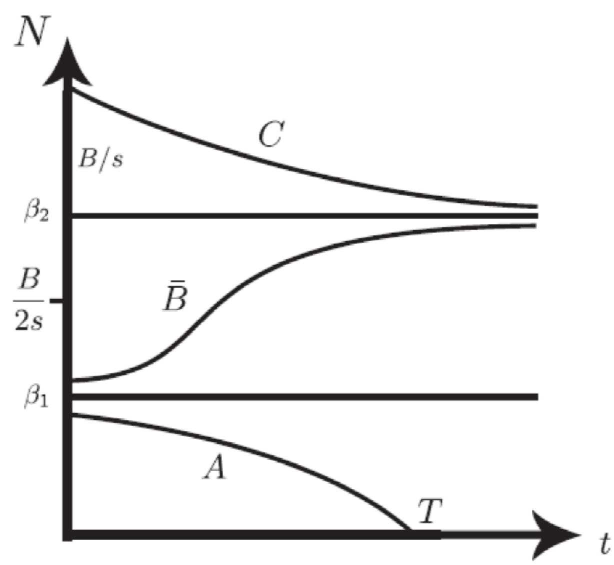


$$\begin{aligned}
\frac{dN}{dt} &< 0 \text{ if } 0 < N < \beta_1 \\
\frac{dN}{dt} &> 0 \text{ if } \beta_1 < N < \beta_1 \\
\frac{dN}{dt} &< 0 \text{ if } N > \beta_2
\end{aligned}$$

and of course $\frac{dN}{dt} = 0$ at $N = \beta_1$ and β_2, where

$$\begin{matrix}\beta_1 \\ \beta_2\end{matrix} = \frac{-B \pm \sqrt{B^2 - 4Es}}{-2s} = \frac{B \mp \sqrt{B^2 - 4Es}}{2s}$$

Now suppose $\hat{N} = N(0)$ is large, so we start at point C on the diagram. Then $\frac{dN}{dt} < 0$ so the fish stocks decline toward β_2. Suppose on the other hand that $\hat{N}$ is small, but still more than β_1. Then we might start at $\bar{B}$ and the number of fish will INCREASE until we reach β_2. If $\hat{N}$ is very small, however, then we are at a point like A, and the fish population will collapse to zero. So we get a picture like this:



Of course, if $\hat{N} = \beta_1$ or β_2, then since $F(\beta_1) = 0$ and $F(\beta_2) = 0$, $\frac{dN}{dt} = F(N)$ has solutions $N(t) = \beta_1$ and $N(t) = \beta_2$, the constant solutions. We call β_1 and β_2 the EQUILIBRIUM POPULATIONS: given a fixed harvesting rate, if the initial population is either β_1 or β_2, then IN THEORY the population remains steady - which is good!

BUT there is a vast DIFFERENCE between β_1 and β_2! Look at the diagram and suppose you have exactly β_2 fish. Now suppose a SMALL number of new fish arrive from somewhere else. Then the diagram shows that the population will decline back to β_2. If some fish go away, the population will INCREASE back to β_2. Of course, such things happen all the time, so it's VERY GOOD to have this kind of behaviour! We say that β_2 is a STABLE EQUILIBRIUM POPULATION.

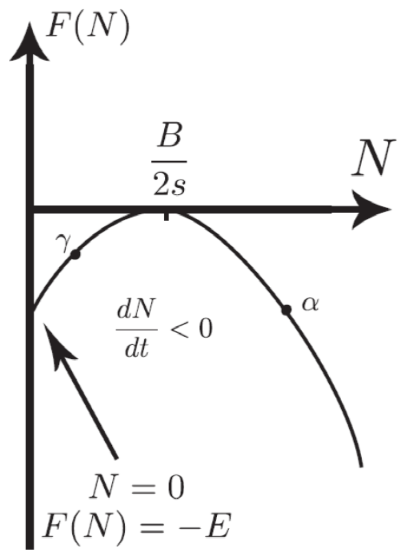
BUT NOW LOOK AT β_1! If a few more fish arrive, also fine - in fact the population INCREASES (to β_2) BUT suppose a few fish decide to move on. THEN YOUR FISH STOCKS BECOME EXTINCT! Note that this can happen though E is relatively small (we are assuming $E < \frac{B^2}{4s}$. We say that β_1 is an UNSTABLE EQUILIBRIUM POPULATION.

The time required to reach $N = 0$ is called the EXTINCTION TIME. It can be computed: since $\frac{dN}{dt} = N(B - sN) - E$, we have:

$$\int_0^T dt = T = \int_{\hat{N}}^0 \frac{dN}{N(B - sN) - E}$$

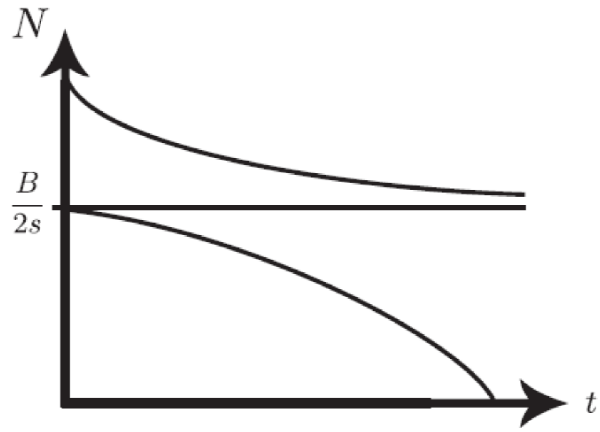
Of course it is very important to know T - it is the amount of time you have to save the situation!

Finally, let's consider the <u>third</u> possible graph for $F(N)$:



Clearly $\hat{N} = \frac{B}{2s}$ is a solution; it is the only equilibrium population. If $\hat{N} > \frac{B}{2s}$ (for example, at the point α) then the population declines to $\frac{B}{2s}$, asymptotically. But if we start at the point γ, then (since $\frac{dN}{dt} < 0$ everywhere below the axis - always remember that $\frac{dN}{dt} = F(N)$) the population will collapse to zero. So we have an UNSTABLE equilibrium at $\hat{N} = \frac{B}{2s}$. We only call it stable if it is stable to perturbations in BOTH directions!.

The graph of N is as shown.



Clearly, the first and third cases are bad! The first case was $E > \frac{B^2}{4s}$. So we want the second case, with $E < \frac{B^2}{4s}$, and we want STABLE equilibrium, that is, a population which fluctuates around $\beta_2 = \frac{B+\sqrt{B^2-4Es}}{2s}$.

4 THE LAPLACE TRANSFORM

4.1 Introduction

Definition. Let f be a function defined for all $t \geq 0$. The **Laplace transform** of f is the function $F(s)$ defined by

$$F(s) = L(f) = \int_0^{\infty} e^{-st} f(t)dt, \tag{1}$$

provided the improper integral on the right exists.

The original function $f(t)$ in (1) is called the **inverse transform** or inverse of $F(s)$ and is denoted by $L^{-1}(F)$; i.e.,

$$f(t) = L^{-1}(F(s)).$$

Notation: Original functions are denoted by lower case letters and their Laplace transforms by the same letters in capitals. Thus $F(s) = L(f(t))$, $Y(s) = L(y(t))$ etc.

Recall that, by definition, for any function h defined on $[0, \infty)$,

$$\int_0^{\infty} h(t)\, dt = \lim_{b \to \infty} \int_0^{b} h(t)dt$$

and the integral is said to converge if this limit exists. Because e^{-st} decreases so rapidly with t, the Laplace transform usually does exist (there are exceptions, however), and then we say that the function f **has a well-defined Laplace transform**.

Example 1. Let $f(t) = e^{at}$, when $t \geq 0$. Find $F(s)$.

Solution.

$$\begin{aligned} F(s) &= L(e^{at}) \\ &= \int_0^\infty e^{-st} e^{at} dt \\ &= \lim_{b\to\infty} \int_0^b e^{-st} e^{at} dt. \end{aligned}$$

Now

$$\int_0^b e^{(a-s)t} dt = \begin{cases} b & \text{if } s = a \\ \frac{e^{b(a-s)}}{a-s} - \frac{1}{a-s} & \text{if } s \neq a. \end{cases}$$

If $s < a$, $a - s > 0$ and $e^{(a-s)b} \to \infty$ as $b \to \infty$. Thus when $s \leq a$, $\int_0^\infty e^{(a-s)t} dt$ diverges. When $s > a$, $a - s < 0$, and $e^{(a-s)b} \to 0$ as $b \to \infty$, and then

$$F(s) = L(e^{at}) = \frac{1}{s-a}, \quad s > a. \tag{2}$$

□

Example 2. Let $f(t) = 1$, $t \geq 0$. Find $F(s)$.

Solution. This function is the same as the one in Example 1 with $a = 0$, thus,

$$L(1) = \frac{1}{s}, \quad s > 0. \tag{3}$$

□

Note: The Laplace transform is a *linear operation*; i.e., the Laplace transform of a linear combination of functions equals the same linear combination of their Laplace transforms.

Theorem

$$L(af(t) + bg(t)) = aL(f) + bL(g), \tag{4}$$

where a and b are constants.

Corollary The inverse Laplace transform also satisfies the linearity property.

$$L^{-1}(aF(s) + bG(s)) = aL^{-1}(F) + bL^{-1}(g). \tag{5}$$

Verification of (4) and (5) is easy.

Example 3. Using (4), we obtain:

$$\begin{aligned} L(\cosh at) &= L\left(\frac{1}{2}(e^{at} + e^{-at})\right) \\ &= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) \\ &= \frac{s}{s^2 - a^2}, \quad s > a \geq 0. \end{aligned}$$

□

Example 4. If $F(s) = \frac{3}{s} + \frac{5}{s-7}$, find $f(t) = L^{-1}(F)$.

<u>Solution.</u> Using (5), we have

$$\begin{aligned} L^{-1}(F) &= L^{-1}(\frac{3}{s}) + 5L^{-1}(\frac{1}{s-7}) \\ &= 3 \cdot 1 + 5 \cdot e^{7t} = 3 + 5e^{7t}. \end{aligned}$$

□

Example 5. Set $a = iw$ in the formula (2):

$$\begin{aligned} L(e^{iwt}) &= L(\cos wt + i\sin wt) \\ &= L(\cos wt) + iL(\sin wt) \\ &= \frac{1}{s - iw} = \frac{s + iw}{s^2 + w^2} \end{aligned}$$

Equating real and imaginary parts, we get

$$\begin{aligned} L(\cos wt) &= \frac{s}{s^2 + w^2}, \\ L(\sin wt) &= \frac{w}{s^2 + w^2}. \end{aligned} \tag{6,7}$$

□

Example 6. To show that $L(t^n) = \frac{n!}{s^{n+1}}$, $n = 0, 1, 2, \ldots$.

Solution.

$$\begin{aligned} L(t^n) &= \int_0^\infty e^{-st} t^n dt \\ &= -\frac{1}{s} e^{-st} t^n \mid_0^\infty \\ &\quad + \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} dt. \end{aligned}$$

First term is zero at $t = 0$ and as $t \to \infty$. Thus, using induction

$$L(t^n) \qquad = \tfrac{n}{s} L(t^{n-1})$$

$$= \tfrac{n(n-1)\ldots 1}{s^n}\, L(1)$$
$$= \tfrac{n!}{s^{n+1}}. \qquad (8)$$

$\square$

Example 7. Given $\frac{F(s)=2s+5}{s^2+9}$, find $L^{-1}(F(s))$.

Solution.

$$L^{-1}\left(\frac{2s+5}{s^2+9}\right) = L^{-1}\left(\tfrac{2s}{s^2+9} + \tfrac{5}{s^2+9}\right)$$
$$= 2L^{-1}\left(\tfrac{s}{s^2+9}\right) + \tfrac{5}{3}L^{-1}\left(\tfrac{3}{s^2+9}\right)$$
$$= 2\cos 3t + \tfrac{5}{3}\sin 3t. \qquad \square$$

4.1.1 Piecewise continuous functions

One of the nice things about Laplace transforms is that they are defined by integration, and, unlike differentiation, integration doesn't care whether the function is continuous or not. In fact it is easy to define the Laplace transform of a function whose graph is broken up into pieces.

More formally: A function $f(t)$ defined for $t \geq 0$ has a **jump discontinuity** at $a \in [0, \infty)$ if the one sided limits

$$\lim_{t \to a^-} f(t) = \ell_- \text{ and } \lim_{t \to a^+} f(t) = \ell_+$$

exist but f is not continuous at $t = a$.

By definition, a function $f(t)$ is **piecewise continuous** on a finite interval $a \leq t \leq b$ if jump discontinuities are its only discontinuities. Such a function always has a nice Laplace transform (unless it grows extremely quickly, faster than exponentially).

4.2 Transform of derivatives and integrals

Theorem Suppose that $f(t)$ is continuous and has a well-defined Laplace transform on $[0, \infty)$ and $f'(t)$ is piecewise continuous on $[0, \infty)$. Then $L(f'(t))$ exists and

$$L(f') = sL(f) - f(0), \quad s > a. \tag{10}$$

Proof. First consider the case when $f'(t)$ is continuous for $[0, \infty)$. Then

$$\begin{aligned} L(f') &= \int_0^\infty e^{-st}\, f'(t)dt \\ &= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)dt \\ &= \lim_{b\to\infty} e^{-sb} f(b) - f(0) + sL(f). \end{aligned}$$

Since f has a well-defined Laplace transform, the first term on the right is 0 when $s > 0$, showing that $L(f') = sL(f) - f(0)$ for $s > a$. □

If f' is merely piecewise continuous, the proof is quite similar; in this case, the range of integration in the original integral must be split into parts such that f' is continuous in each such part.

Apply (10) to f'' to obtain

$$\begin{aligned} L(f'') &= sL(f') - f'(0) \\ &= s(sL(f) - f(0)) - f'(0) \\ &= s^2L(f) - sf(0) - f'(0). \quad (11) \end{aligned}$$

Similarly $L(f''') = s^3L(f) - s^2f(0) - sf'(0) - f''(0)$, etc. By induction we obtain the following:

Theorem Suppose that $f(t)$, $f'(t)$, $f''(t)$, $\ldots$, $f^{(n-1)}(t)$ are continuous and have well-defined Laplace transforms on $[0, \infty)$ and $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\begin{aligned} L(f^{(n)}) &= s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) \\ &\qquad - \cdots - f^{(n-1)}(0). \qquad (12) \end{aligned}$$

Example 8. Find $L(\sin^2 t)$.

Solution. Here $f(t) = \sin^2 t$, $f'(t) = 2 \sin t \cos t = \sin 2t$, $f(0) = 0$. In view of (7) we obtain

$$\begin{aligned} L(f') &= L(\sin 2t) \\ &= \frac{2}{s^2 + 4} \\ &= sL(f) - f(0) = sL(f) \end{aligned}$$

$\therefore \quad L(\sin^2 t) = \frac{2}{s(s^2+4)}$. $\square$

Example 9. Find $L(t \sin \alpha t)$.

Solution. Here $f(t) = t \sin \alpha t$ and $f(0) = 0$. Also

$$\begin{aligned} f'(t) &= \sin \alpha t + \alpha t \cos \alpha t, \quad f'(0) = 0 \\ f''(t) &= 2\alpha \cos \alpha t - \alpha^2 t \sin \alpha t \\ &= 2\alpha \cos \alpha t - \alpha^2 f(t) \end{aligned}$$

$\therefore$ by (11), $\quad L(f'') = 2\alpha L(\cos \alpha t) - \alpha^2 L(f) = s^2 L(f)$.
$\therefore \quad (s^2 + \alpha^2) L(f) = 2\alpha L(\cos \alpha t) = \frac{2\alpha s}{s^2+\alpha^2}$.

Hence $L(t \sin \alpha t) = \frac{2\alpha s}{(s^2+\alpha^2)^2}$. $\square$

4.3 Solution of Initial value problems

Consider the initial value problem

$$\begin{aligned} y'' + ay' + by &= r(t) \\ y(0) = k_0, \qquad y'(0) &= k_1 \end{aligned} \qquad (13)$$

with a, b, k_0 and k_1 constants. The function $r(t)$ has the Laplace transform $R(s)$.

Step 1. Take the Laplace transform of both sides of the d.e. using the linearity property of the Laplace transform.

$$s^2L(y) - sy(0) - y'(0) + a(sL(y) - y(0)) + bL(y) = L(r).$$

Step 2. Use the given initial conditions to arrive at the subsidiary equation

$$s^2L(y) - sk_0 - k_1 + a(sL(y) - k_0) + bL(y) = L(r).$$

Step 3. Solve this for $L(y)$:

$$L(y) = \frac{(s+a)k_0 + k_1 + R(s)}{s^2 + as + b}.$$

Step 4. Reduce the above to a sum of terms whose inverses can be found, so that the solution $y(t)$ of (13) is obtained.

Example 10. Solve $y'' + y = e^{2t}$, $y(0) = 0$, $y'(0) = 1$.

Solution. Taking the Laplace transform of both sides of the d.e. and using initial conditions we arrive at

$$s^2L(y) - sy(0) - y'(0) + L(y) = \frac{1}{s-2}.$$

$\therefore \quad L(y) = \frac{1}{s^2+1}\left(1 + \frac{1}{s-2}\right) = \frac{s-1}{(s-2)(s^2+1)}.$

Now

$$\frac{s-1}{(s-2)(s^2+1)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+1}$$

for appropriate constants A, B and C. Multiply both sides of this equation by $(s-2)(s^2+1)$ and equate the coefficients of s^0, s, s^2 to obtain $A-2C=-1$, $-2B+C=1$, $A+B=0$. Thus $A=\frac{1}{5}$, $B=-\frac{1}{5}$, and $C=\frac{3}{5}$.

$$\begin{aligned}\therefore \quad L(y) &= \tfrac{1}{5}\cdot\tfrac{1}{s-2} - \tfrac{s-3}{5(s^2+1)}\\ &= \tfrac{1}{5(s-2)} - \tfrac{s}{5(s^2+1)} + \tfrac{3}{5(s^2+1)}.\end{aligned}$$

Taking the inverse transform of both sides we get (using (2), (6), and (7))

$$y(t) = \frac{1}{5}e^{2t} - \frac{1}{5}\cos t + \frac{3}{5}\sin t.$$

□

4.4 Transform of the integral of a function

Theorem. If $f(t)$ is piecewise continuous and has a well-defined Laplace transform on $[0,\infty)$, then

$$L\left(\int_0^t f(\tau)d\tau\right) = \frac{1}{s}L(f) \qquad (s>0, s>a). \tag{14}$$

Example 11. Find $f(t)$ if $L(f) = \frac{1}{s^2(s^2+w^2)}$.

Solution.

$$\because \; L\left(\frac{1}{w}\sin wt\right) = \frac{1}{s^2+w^2}$$

we use (14) to get

$$\begin{aligned}L\left(\frac{1}{w}\int_0^t \sin w\tau d\tau\right) &= L\left(\frac{1-\cos wt}{w^2}\right)\\ &= \frac{1}{s(s^2+w^2)}\end{aligned}$$

and

$$L\left(\frac{1}{w^2}\int_0^t (1-\cos w\tau)d\tau\right) = L\left(\frac{1}{w^2}(t-\frac{\sin wt}{w})\right)$$
$$= \frac{1}{s^2(s^2+w^2)}.$$

$\therefore \quad f(t) = \frac{1}{w^2}\left(t - \frac{\sin wt}{w}\right).$ $\square$

Theorem. (s-Shifting)

If $f(t)$ has the transform $F(s)$, $s > a$, then

$$L(e^{ct}f(t)) = F(s-c), \quad s-c > a. \tag{15}$$

Thus using the formulae (8), (6), and (7) we obtain

$$L(e^{ct}t^n) = \frac{n!}{(s-c)^{n+1}}$$
$$L(e^{ct}\cos wt) = \frac{s-c}{(s-c)^2+w^2}$$
$$L(e^{ct}\sin wt) = \frac{w}{(s-c)^2+w^2}.$$

Example 12. Solve $y'' + 2y' + 5y = 0$,

$y(0) = 2$, $y'(0) = -4$.

Solution. Following the procedure outlined earlier we obtain

$$L(y) = \frac{2s}{(s+1)^2+2^2}$$
$$= \frac{2(s+1)}{(s+1)^2+2^2} - \frac{2}{(s+1)^2+2^2}$$

$\therefore \quad y(t) = e^{-t}(2\cos 2t - \sin 2t).$ $\square$

Example 13. Solve $y'' - 2y' + y = e^t + t$,$y(0) = 1$, $y'(0) = 0$.

Solution. Taking Laplace transform of the d.e.

$$\begin{aligned} & s^2 L(y) - sy(0) - y'(0) - 2(sL(y) - y(0)) + L(y) \\ & = \frac{1}{s-1} + \frac{1}{s^2} \end{aligned}$$

or

$$(s^2 - 2s + 1)L(y) = s - 2 + \frac{1}{s-1} + \frac{1}{s^2}$$

or

$$\begin{aligned} L(y) &= \frac{s-2}{(s-1)^2} + \frac{1}{(s-1)^3} + \frac{1}{s^2(s-1)^2} \\ &= \frac{1}{s-1} - \frac{1}{(s-1)^2} + \frac{1}{(s-1)^3} \\ &\quad + \frac{1}{(s-1)^2} - \frac{2}{s-1} + \frac{1}{s^2} + \frac{2}{s} \end{aligned}$$

$\therefore \quad y(t) = \frac{t^2}{2}e^t - e^t + t + 2 = (\frac{t^2}{2} - 1)e^t + t + 2.$ $\square$

4.5 Unit Step (Heaviside) function

Definition.

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases} \tag{16}$$

Example 14. Graph $f(t) = u(t-1) - u(t-3)$.

Clearly $f(t) = 1$ when $1 < t < 3$ and 0 otherwise. $\square$

Observe that if $0 < a < b$

$$u(t-a) - u(t-b) = \begin{cases} 0, & t < a \\ 1, & a < t < b \\ 0, & t > b. \end{cases}$$

Let $g(t)$ be some function of t. Then, if $0 < a < b$

$$g(t)(u(t-a) - u(t-b)) = \begin{cases} 0 & \text{if } t < a \\ g(t) & \text{if } a < t < b \\ 0 & t > b. \end{cases}$$

You can think of this in the following way: this function is "OFF" until t = a. Then it suddenly turns "ON" the function g(t). It remains "ON" until t = b, where it suddenly switches "OFF" again. You can easily imagine lots of engineering situations where this might be useful, for example in circuit theory. Notice that this function is usually *discontinuous* but this won't be a problem for the Laplace transform, because the transform is defined by an integral, and discontinuous functions can often be integrated. In fact, the main advantage of the Laplace transform is that it allows us to solve ODEs with discontinuous right-hand-sides. Such ODEs often come up in engineering applications.

Example 15. Express

$$f(t) = \begin{cases} t, & 0 < t < 1 \\ 2 - t, & 1 < t < 2 \\ 0, & 2 < t < 3 \\ 1, & t > 3 \end{cases}$$

Solution.

$$\begin{aligned} f(t) = t(u(t) - u(t-1)) \\ + (2-t)(u(t-1) \\ - u(t-2)) + u(t-3). \end{aligned}$$

Example 16. Sketch

$$\begin{aligned} g(t) = 2u(t) + tu(t-1) + (3-t)u(t-2) \\ - 3u(t-4), \quad t > 0. \end{aligned}$$

Solution.
When

$$\begin{aligned} 0 < t < 1 \quad & g(t) = 2 \cdot 1 + t \cdot 0 + (3-t) \cdot 0 - 3 \cdot 0 = 2 \\ 1 < t < 2 \quad & g(t) = 2 \cdot 1 + t \cdot 1 + (3-t) \cdot 0 - 3 \cdot 0 \\ & \quad\;\; = 2 + t \\ 2 < t < 4 \quad & g(t) = 2 \cdot 1 + t \cdot 1 + (3-t) \cdot 1 - 3 \cdot 0 = 5 \\ t > 4 \quad & g(t) = 2 \cdot 1 + t \cdot 1 + (3-t) \cdot 1 - 3 \cdot 1 = 2 \end{aligned}$$

Theorem. (t-Shifting)
If $L(f(t)) = F(s)$ then

$$L(f(t-a)u(t-a)) = e^{-as}F(s). \tag{17}$$

Example 17. Setting $f(t-a) = 1$ in (17) we get

$$L(u(t-a)) = \frac{e^{-as}}{s}. \tag{18}$$

Example 18. Compute $L(t^2u(t-1))$.

Solution.

$$\begin{aligned} L(t^2u(t-1)) &= L((t-1+1)^2u(t-1)) \\ &= L(((t-1)^2 + 2(t-1) + 1)u(t-1)) \\ &= e^{-s}(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}). \end{aligned}$$

□

Example 19. Compute $L((e^t+1)u(t-2))$.

Solution.

$$\begin{aligned} L((e^t+1)u(t-2)) &= L((e^{t-2}e^2+1)u(t-2)) \\ &= e^{-2s}\left(\frac{e^2}{s-1} + \frac{1}{s}\right). \end{aligned}$$

$\square$

The next and final problem in this section is rather complicated, but it really just involves assembling a lot of small bits and pieces. Notice that *none* of the methods we learned in earlier chapters would have allowed us to solve this problem, so it should convince you that the Laplace Transform is really useful!

Example 20. Solve the initial value problem

$$y'' + 3y' + 2y = g(t), \quad y(0) = 0, \quad y'(0) = 1,$$

with

$$g(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & t > 1 \end{cases} .$$

Solution.

$$g(t) = u(t) - u(t-1)$$

.

Taking the Laplace transform of both sides of the d.e. we obtain, using (18),

$$\begin{aligned} & s^2 L(y) - sy(0) - y'(0) + 3(sL(y) - y(0)) + 2L(y) \\ & = \frac{1}{s} - \frac{e^{-s}}{s} \end{aligned}$$

so that,

$$L(y) = \frac{s+1}{s(s^2+3s+2)} - e^{-s}\left[\frac{1}{s(s^2+3s+2)}\right].$$

Now

$$\begin{aligned}\frac{s+1}{s(s^2+3s+2)} &= \frac{s+1}{s(s+1)(s+2)} \\ &= \frac{1}{s(s+2)} \\ &= \frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+2}\right)\end{aligned}$$

$\therefore \quad L^{-1}(\frac{1}{s(s+2)}) = \frac{1}{2}(1-e^{-2t})$.

Also

$$\begin{aligned}\frac{1}{s(s^2+3s+2)} &= \frac{1}{s(s+1)(s+2)} \\ &= \frac{1}{2}\left(\frac{1}{s} + \frac{1}{s+2}\right) - \frac{1}{s+1}\end{aligned}$$

$\therefore \quad L^{-1}\left(\frac{1}{s(s+1)(s+2)}\right) = \frac{1}{2}(1+e^{-2t}) - e^{-t}$

$$\therefore \quad L^{-1}\left(\frac{e^{-s}}{s(s+1)(s+2)}\right) = \left\{\tfrac{1}{2}(1+e^{-2(t-1)}) - e^{-(t-1)}\right\} u(t-1).$$

[using (17)]

Finally

$$\begin{aligned}y(t) &= \frac{1}{2}(1-e^{-2t}) \\ &\quad - \left[\frac{1}{2}(1+e^{-2(t-1)}) - e^{-(t-1)}\right] u(t-1).\end{aligned}$$

□

4.6 The Dirac delta function

Definition. Let $f_h(t)$ be a family of functions of t defined by

$$f_h(t) = \begin{cases} 0 & t \lneq 0 \\ 1/h & 0 \leq t \leq h \\ 0 & t \geq h \end{cases}$$

for all $h > 0$. Notice that, for any h,

$$\int_0^\infty f_h(t)dt = \int_0^h \frac{1}{h}dt = 1.$$

Thus for example $f_{10^{-100}}(t)$ is a function with maximum value 10^{100} and yet the area under its graph is still 1. The graph is an extremely tall but sharp and narrow spike next to $t = 0$. We define

$$\text{“}\delta(t) \equiv \lim_{h\to 0} f_h(t)\text{”}.$$

Of course this doesn't really make sense mathematically, but you can think of $\delta(t)$ as an extremely tall and narrow spike at $t = 0$. Similarly, you can think of δ(t - a) as an infinitely tall and narrow spike at t = a. Note

$$\int_0^\infty \delta(t)dt = 1, \quad \delta(t) = 0 \text{ everywhere} \quad \text{except } t = 0.$$

Now let $g(t)$ be any function, and consider

$$\int_0^\infty f_h(t)g(t)dt = \frac{1}{h}\int_0^h g(t)dt.$$

If h is very small, $\int_0^h g(t)dt \approx g(0)h$, so

$$\int_0^\infty f_h(t)g(t)dt \approx g(0).$$

Since the approximation gets better and better as $h \to 0$, we have

$$\int_0^\infty \delta(t)g(t)dt = g(0).$$

In a similar way,

$$\int_0^\infty \delta(t-a)g(t)dt = g(a).$$

Hence the Laplace transform of $\delta(t-a)$ is

$$L\left[\delta(t-a)\right] = \int_0^\infty e^{-st}\delta(t-a)dt = e^{-as}.$$

So $L^{-1}\left[e^{-as}\right] = \delta(t-a)$.
Note that by setting $a = 0$ we have

$$L^{-1}[1] = \delta(t).$$

Example: injections
Suppose that a doctor injects, almost instantly, 100 mg of morphine into a patient. He does it again 24 hours later. Suppose that the HALF-LIFE of morphine in the patient's body is 18 hours. Find the amount of morphine in the patient at any time.

Solution: Half-life refers to the exponential function e^{-kt}. "Half-life 18 hours" = 0.75 days means $\frac{1}{2} = e^{-k\times 0.75}$, that is, k = $\dfrac{\ell n(2)}{0.75} = 0.924$. So without the injections,

$$\frac{dy}{dt} = -ky, \quad k = 0.924.$$

The injections are at a rate of 100 mg per day, but concentrated in delta-function spikes at $t = 0$ and $t = 1$ [time unit is DAYS]. So we have
$\frac{dy}{dt} = -ky + 100\delta(t) + 100\delta(t-1)$
$\rightarrow$
$sL[y] - y(0) = -0.924L[y] + 100 \times 1 + 100e^{-s}$

By the way, notice that we have to think of the delta function as something which itself has *units*. In this case you should think of the delta function as something that has units of [1/time], so that this equation has consistent units. In many problems, it is actually very helpful to work out what units the delta function has — it can have different units in different problems! This is particularly useful in physics problems where something gets hit suddenly and gains some momentum instantly — you can use this in some of the tutorial problems.

Since $y(0) = 0$,

$$
\begin{aligned}
L(y) &= 100 \times \frac{1 + e^{-s}}{s + 0.924} \\
&= \frac{100}{s + 0.924} + \frac{100e^{-s}}{s + 0.924}
\end{aligned}
$$

So, using the t-shifting theorem, we get

$$
\begin{aligned}
y &= 100e^{-0.924t} + 100e^{-0.924(t-1)}u(t-1) \\
&= 100e^{-0.924t} \quad 0 < t < 1 \\
&= 100(1 + e^{0.924})e^{-0.924t} \quad t > 1.
\end{aligned}
$$

Example: parameter reconstruction
Sometimes it happens, in Engineering applications, that you have a system whose nature you understand [for example, you may know that it is a damped harmonic oscillator] but you don't know the values of the parameters [the spring constant, the mass, the friction coefficient]. In such a case, what you can do is to "poke" the system with a sudden, sharp force, and watch how it behaves. Then you can **reconstruct the parameters of the system** as follows.

Suppose for example that you have a damped harmonic oscillator — say, a spring with a mass attached — which is initially at rest, that is, x(0) = $\dot{x}(0)$ = 0. and you poke it with a unit impulse [change of momentum = 1 in MKS units] at time t = 1; in other words, the applied force is just $\delta(t-1)$. You observe that the displacement of the mass is

$$x(t) = u(t-1)e^{-(t-1)}sin(t-1).$$

[You can use Graphmatica to look at the graph by putting in

$$y = exp(-(x-1)) * sin(x-1) * step(x-1);$$

note that this oscillator is underdamped, despite the shape of the graph!]

Question: what are the values of the mass, the spring constant, the frictional coefficient?

Solution: Newton's Second Law in this case says:

$$M\ddot{x} = -kx - b\dot{x} + \delta(t-1).$$

With the given initial data, taking the Laplace transform gives

$$Ms^2X(s) = -kX(s) - bsX(s) + e^{-s},$$

and so

$$X(s) = \frac{e^{-s}}{Ms^2 + bs + k}.$$

But the Laplace transform of the given response function is [using both t-shifting and s-shifting!]

$$X(s) = \frac{e^{-s}}{(s+1)^2 + 1} = \frac{e^{-s}}{s^2 + 2s + 2},$$

so by inspection we see that M = 1, b = 2, k = 2 in MKS units, and we have successfully reconstructed the parameters of this system, just by hitting it with a delta function impulse! Similar ideas work for electrical circuits etc etc etc. Useful idea.

If you look at the graph of the solution you will see that the derivative jumps suddenly at t = 1; that is, there is a sharp corner there. This is typical in problems involving delta-function impulses, basically because such impulses cause the momentum, and therefore the velocity, to change suddenly.

5 PARTIAL DIFFERENTIAL EQUATIONS

5.1 Introduction

A **partial differential equation (PDE)** is an equation containing an unknown function $u(x, y, \ldots)$ of *two or more* independent variables $x, y, \ldots$ and its partial derivatives with respect to these variables. We call u the dependent variable.

Example (i) $u_{xy} - 2x + y \;=\; 0$

This is a PDE that involves the function $u(x, y)$ with two independent variables x and y. [Remember that the subscripts mean that you are taking the partial derivative, in this case a second order derivative first with respect to x and then with respect to y.]

Example (ii) $w_{xy} + x(w_z)^2 \;=\; yz$

This is a PDE that involves the function $w(x, y, z)$ with three independent variables x, y and z.

Example The function

$$u(x, y) \;=\; x^2y - \frac{1}{2}xy^2 + F(x) + G(y) \tag{12}$$

is a solution of the PDE in example 5.1 (i). Here F and G can be any (arbitrary) single variable functions.

Indeed, by taking partial derivatives of (12):

$$u_x \;=\; 2xy - \frac{1}{2}y^2 + F'(x) \text{ and}$$

$$u_{xy} \;=\; 2x - y,$$

we see that the function (1) satisfies the PDE.

Notice that, just as the solution of an ordinary differential equation involves arbitrary *constants*, the solution of a PDE will involve arbitrary *functions*!

Suppose we require the PDE to also satisfy the conditions

$$u(x,0) = x^3 \text{ and } u(0,y) = \sin(3y).$$

Then using (12), we have

$$x^3 = u(x,0) = F(x) + G(0)$$

and

$$\sin(3y) = u(0,y) = F(0) + G(y).$$

By putting $x = 0$ in the first of these equations [or $y = 0$ into the second] we see that $0 = F(0) + G(0)$. So adding the two equations together we get

$$F(x) + G(y) + F(0) + G(0) = F(x) + G(y) = x^3 + sin(3y).$$

Substituting this back into the solution we have finally

$$u(x,y) \;=\; x^2y - \frac{1}{2}xy^2 + x^3 + \sin(3y)$$

which satisfies the additional conditions.

Example In general, the totality of solutions of a PDE is very large. The Laplace equation $u_{xx} + u_{yy} \;=\; 0$ has the following solutions

$$u(x,y) = x^2 - y^2, \quad u(x,y) = e^x \cos y,$$

$$u(x,y) = \ln\left(x^2 + y^2\right), \quad \text{etc}$$

which are entirely different from each other.

5.1.1 Order of Differential Equations

The **order** of the PDE is the order of the highest derivative present.

Note: Example 5.1 (i) is a PDE of order 2 and (ii) is also a PDE of order 2.

5.1.2 Linearity and Homogeneity

An order 1 *linear* PDE has the form

$$Au_x + Bu_y + Cu = Z$$

and an order 2 *linear* PDE has the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = Z$$

where A, B, C, D, E, F, Z are constants or functions of x and y but not functions of u.

An order 1 or 2 linear PDE is said to be *homogeneous* if the Z term in the above form is 0.

Example

PDE	**order**	**linear**	**homogeneous**
$4u_{xx} - u_t = 0$	2	yes	yes
$x^2 R_{yyy} = y^3 R_{xx}$	3	yes	yes
$tu_{tx} + 2u_x = x^2$	2	yes	no
$4u_{xx} - uu_t = 0$	2	no	n.a.
$(u_x)^2 + (u_y)^2 = 2$	1	no	n.a.

5.1.3 Superposition Principle

If u_1 and u_2 are any solutions of a linear homogeneous differential equation, then

$$u \;=\; c_1 u_1 + c_2 u_2,$$

where c_1 and c_2 are any constants, is also a solution of that equation.

Example Referring to the particular solutions of Laplace equation $u_{xx} + u_{yy} = 0$ in Example 5.1, by superposition principle,

$$u(x, y) = 3(x^2 - y^2) - 7e^x \cos y + 10 \ln (x^2 + y^2)$$

is again a solution of the Laplace equation.

5.2 Separation of Variables for PDE

This method can be used to solve PDE involving two independent variables, say x and y, that can be 'separated' from each other in the PDE.

There are similarities between this method and the technique of separating variables for ODE in Chapter 1.We first make an observation:

Suppose $u(x,y) = X(x)Y(y)$. Then:

(i) $u_x(x,y) = X'(x)Y(y)$

(ii) $u_y(x,y) = X(x)Y'(y)$

(iii) $u_{xx}(x,y) = X''(x)Y(y)$

(iv) $u_{yy}(x,y) = X(x)Y''(y)$

(v) $u_{xy}(x,y) = X'(x)Y'(y)$

Notice that each derivative of u remains 'separated' as a product of a function of x and a function of y. We exploit this feature as follows:

5.2.1 Illustration of Separation of Variables

Consider a PDE of the form

$$u_x = f(x)g(y)u_y.$$

If a solution of the form $u(x,y) = X(x)Y(y)$ exists, then we obtain

$$\begin{aligned} X'(x)Y(y) &= f(x)g(y)X(x)Y'(y) \\ \text{i.e.,} \qquad \frac{1}{f(x)} \cdot \frac{X'(x)}{X(x)} &= g(y)\frac{Y'(y)}{Y(y)}. \end{aligned}$$

LHS is a function of x only while RHS is a function of y only. We conclude that

$$\text{LHS} = \text{RHS} = \text{some constant } k.$$

Thus, we obtain two ODEs

$$\frac{1}{f(x)} \cdot \frac{X'(x)}{X(x)} = k \Rightarrow X'(x) = kf(x)X(x) \tag{13}$$

$$g(y)\frac{Y'(y)}{Y(y)} = k \Rightarrow Y'(y) = \frac{k}{g(y)}Y(y) \tag{14}$$

Note that (13) is an ODE with independent variable x and dependent variable X while (14) is an ODE with independent variable y and dependent variable Y.

By solving (13) and (14) respectively for $X(x)$ and $Y(y)$, we obtain the solution $u(x,y) = X(x)Y(y)$.

Example Solve $u_x + xu_y = 0$.

Solution: If a solution $u(x,y) = X(x)Y(y)$ exists, then we obtain

$$X'(x)Y(y) + xX(x)Y'(y) = 0$$

$$\text{i.e.,} \qquad \frac{1}{x} \cdot \frac{X'(x)}{X(x)} = -\frac{Y'(y)}{Y(y)} \tag{15}$$

This gives two ODEs: LHS of (15) $= k$ gives $X' = kxX$.

This ODE has general solution

$$X(x) = Ae^{kx^2/2} \tag{a}$$

Similarly, RHS of (15) $= k$ gives $Y' = -kY$.

This ODE has general solution

$$Y(y) = Be^{-ky} \tag{b}$$

Multiplying (a) and (b), we obtain a solution of the PDE

$$u(x,y) = X(x)Y(y) = Ce^{k(x^2/2 - y)}.$$

5.3 The Wave Equation

Suppose you have a very flexible string [meaning that it does not resist bending at all] which lies stretched tightly along the x axis and has its ends fixed at x = 0 and x = π. Then you pull it in the y-direction so that it is stationary and has some specified shape, y = f(x) at time t = 0 [so that f(0)=0 and f(π) = 0]. We can assume that f(x) is continuous and bounded, but we will let it have some sharp corners [but only a finite number of them.]

What will happen if you now let the string go? Clearly the string will start to move. We assume that the only forces acting are those due to the tension in the string, and that the pieces of the string will only move in the y-direction.

Now the y-coordinate of any point on the string will become a function of time as well as a function of x. So it becomes a function y(t,x) of both t and x, and we have to use partial derivatives when we differentiate it. This function satisfies

$$y(t,0) = 0 \quad y(t,\pi) = 0$$

for all t, because the ends are nailed down, also

$$y(0,x) = f(x)$$

and

$$\frac{\partial y}{\partial t}(0,x) = 0,$$

because the string is initially stationary. Notice that we need *four* pieces of information here, and it is useful to remember that.

Suppose that the mass per unit length of the string is constant and equal to μ. Then the mass of a small piece of the string is μdx. [We assume that we don't pull the string too far, so it never bends much, hence the length of the small piece can be approximated by dx.] So the mass times the acceleration of the small piece is μdx$\frac{\partial^2 y}{\partial t^2}$. The force acting in the y direction is just the difference between the y-components of the tension at the two ends of the piece, and from physics this turns out to be d(K$\frac{\partial y}{\partial x}$) where K is a certain positive constant. Using Newton's second law we get

$$d\Big(K\frac{\partial y}{\partial x}\Big) = \mu dx\frac{\partial^2 y}{\partial t^2},$$

or

$$c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2},$$

where c^2 is a positive constant. This is the famous **wave equation**. Notice that it involves *four* derivatives altogether, two involving x, and two involving t. That matches up with the fact we mentioned earlier, that we needed *four* pieces of information to nail down a solution [the endpoints, the initial position, the initial velocity].

Note that the following function solves the wave equation [with those four conditions]:

$$y(t, x) = \frac{1}{2}\Big[f(x + ct) + f(x - ct)\Big].$$

Here f(x) gives the initial shape of the string, as above. [Verify that this does satisfy the PDE, that y(0,x) = f(x), and that $\frac{\partial y}{\partial t} = 0$ for all x when t = 0, as it should be. Initially f(x) was only defined between 0 and π, but we can extend it to be an odd, periodic function of period 2π, and then you can also verify that y(t,0) = y(t,π) = 0. See Tutorial 6.] This is called d'Alembert's solution of the wave equation.

You can think about f(x-ct) in the following way. First, think about f(x): it is a function with some definite shape. Now what is f(x-1)? It is *exactly the same* shape, but shifted to the right by one unit. Similarly f(x-2) is the *same* shape shifted 2 units to the right, and similarly f(x-ct) is f(x) shifted to the right by ct. But if t is time, then ct is something which increases linearly with time, at a rate controlled by c. *In other words, f(x-ct) represents the shape f(x) moving to the right at a constant speed* c. In other words, it represents a WAVE [of arbitrary shape] moving to the right at constant speed c. Similarly f(x+ct) represents a wave [with the same shape as f(x)] moving to the left at constant speed c. So d'Alembert's solution says that the solutions of the wave equation [with the given boundary and initial conditions] can be found by superimposing two waves of those forms.

5.4 Heat Equation

Consider the temperature in a long thin bar or wire of constant cross section and homogeneous material, which is oriented along the x-axis and is *perfectly insulated laterally*, so that heat only flows in x-direction. Then the temperature u depends only on x and t and is given by the one-dimensional heat equation

$$u_t = c^2 u_{xx}, \tag{16}$$

where c^2 is a positive constant called the thermal diffusivity [units length^2/time]. It measures how quickly heat moves through the bar, and depends on what it is made of.

5.4.1 Zero temperature at ends of rod

Let's assume that the ends $x = 0$ and $x = L$ of the bar are kept at temperature zero, so that we have the boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0 \quad \text{for all} \;\; t \tag{17}$$

and the initial temperature of the bar is $f(x)$, so that we have the initial condition

$$u(x,0) = f(x). \tag{18}$$

We call the PDE (16) together with the conditions (17) and (18) a **boundary value heat equation problem**.

Notice that, unlike the wave equation, which needs *four* pieces of data, here we only need *three*, which matches the fact that the heat equation only involves a total of three derivatives [two in the spatial direction, but only one in the time direction].

Example Solve

$$u_t = 2u_{xx}, \qquad 0 < x < 3, \quad t > 0,$$

given boundary conditions $u(0,t) = 0, \quad u(3,t) = 0$, and initial condition $u(x,0) = 5\sin 4\pi x$.

Solution:

We use the method of separation of variables. Let $u(x,t) = X(x)T(t)$. Then $u_t = 2u_{xx}$ gives

$$XT' = 2X''T,$$

or equivalently

$$\frac{X''}{X} = \frac{T'}{2T}.$$

Each side must be a constant k. So

$$X'' - kX = 0 \tag{A}$$

$$T' - 2kT = 0 \tag{B}$$

The solutions of (A) are of three types; but clearly we want $X(x)$ to vanish at TWO values of x [the two ends of the bar]. Of course, exponential functions [k positive] and straight-line [k = 0] functions cannot do that. So we have to choose k to be negative, so as to get trigonometric functions which *can* vanish at two values of x. So we have

$$X(x) = a\cos\sqrt{-k}x + b\sin\sqrt{-k}x. \tag{19}$$

The solutions of (B) are

$$T(t) = de^{2kt}$$

for the same negative value of k as in (A).

So now we have a simple solution of the heat equation, by just multiplying $X(x)$ into $T(t)$.

Now we just have to use the boundary conditions. They can be written as

$$X(0)T(t) = 0 \quad \text{and} \quad X(3)T(t) = 0.$$

Now since $T(t) = de^{2kt} \neq 0$ for any t, we conclude that $X(0) = 0$ and $X(3) = 0$.

Substituting $x = 0$ and 3 separately into our expression for $X(x)$, we get

$$X(0) = a = 0$$
$$X(3) = a\cos 3\sqrt{-k} + b\sin 3\sqrt{-k} = 0$$

Solving these two equations, we get
$a = 0$ and $b\sin 3\sqrt{-k} = 0$.

Since we do not want a and b both zero, this implies $\sin 3\sqrt{-k} = 0$ which gives

$$\sqrt{-k} = \frac{n\pi}{3} \text{ or } k = \frac{-n^2\pi^2}{9} \qquad \text{where } n = 0,\ 1,\ 2,\ \ldots$$

Putting this back into our general solution and absorbing d into b [since the product of arbitrary constants is another arbitrary constant] we have

$$u_n(x,t) = b_n e^{-2n^2\pi^2 t/9} \sin\frac{n\pi x}{3} \qquad (HS_n)$$

is a solution for each $n = 1,\ 2,\ 3,\ \ldots$.

We have used up two of our pieces of information. But one remains: we have to satisfy

$$u(x,0) = 5\sin 4\pi x.$$

We want to construct a solution from among (HS_n) that satisfies this initial condition.

Now substituting $t = 0$ into (HS_n) for any n,

$$u_n(x,0) = b_n sin\frac{n\pi x}{3}.$$

If we take $n = 12$ and $b_{12} = 5$, we have

$$u_{12}(x,0) = 5sin\frac{12\pi x}{3} = 5sin 4\pi x.$$

Hence, the particular solution that will also satisfy the initial condition is

$$u_{12}(x,t) = 5e^{-32\pi^2 t}\sin 4\pi x.$$