Computing orthogonal projection

Theorem (orthogonal projection)

Let V be a subspace of $\mathbb{R}^n$ and $\boldsymbol{w}$ a vector in $\mathbb{R}^n$.

If $\boxed{\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\} \text{ is an orthogonal basis for } V}$, then the projection of $\boldsymbol{w}$ onto V is

$$\left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|^2}\right)\boldsymbol{u}_1+\left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|^2}\right)\boldsymbol{u}_2+\ldots+\left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_k}{\|\boldsymbol{u}_k\|^2}\right)\boldsymbol{u}_k$$

Looks familiar?

If $\boxed{\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\} \text{ is an orthonormal basis for } V}$, then the projection of $\boldsymbol{w}$ onto V is

$$(\boldsymbol{w}\cdot\boldsymbol{v}_1)\boldsymbol{v}_1+(\boldsymbol{w}\cdot\boldsymbol{v}_2)\boldsymbol{v}_2+\ldots+(\boldsymbol{w}\cdot\boldsymbol{v}_k)\boldsymbol{v}_k$$

An expression we have seen before

If $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ is an orthogonal basis for a vector space V, then for any vector $\boldsymbol{w} \in V$,

$$\boldsymbol{w} = \left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|^2}\right)\boldsymbol{u}_1 + \left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|^2}\right)\boldsymbol{u}_2 + \ldots + \left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_k}{\|\boldsymbol{u}_k\|^2}\right)\boldsymbol{u}_k$$

From earlier unit

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Why does this make sense?

An expression we have seen before

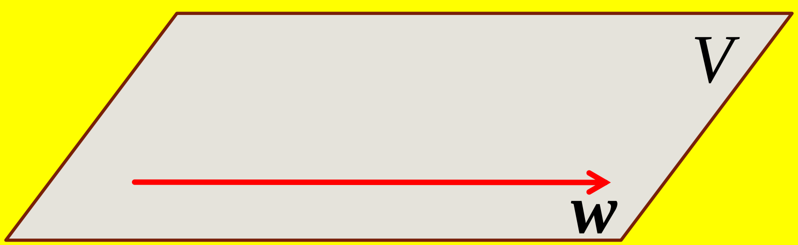
$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ is an orthogonal basis for V (subspace of $\mathbb{R}^n$)

...then for any vector $\boldsymbol{w} \in V$,

$$\left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|^2}\right)\boldsymbol{u}_1 + \ldots + \left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_k}{\|\boldsymbol{u}_k\|^2}\right)\boldsymbol{u}_k$$

$$= \boldsymbol{w}$$

$$\boldsymbol{w} = \boldsymbol{w} + \boldsymbol{0}$$

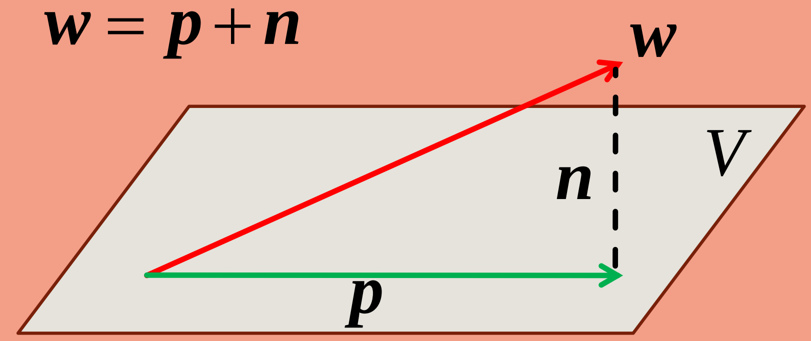


...and $\boldsymbol{w}$ a vector in $\mathbb{R}^n$.

$$\left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|^2}\right)\boldsymbol{u}_1 + \ldots + \left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_k}{\|\boldsymbol{u}_k\|^2}\right)\boldsymbol{u}_k$$

is the projection of $\boldsymbol{w}$ onto V.

$$\boldsymbol{w} = \boldsymbol{p} + \boldsymbol{n}$$



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If $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\}$ is an orthogonal basis for V, then the projection of $\boldsymbol{w}$ onto V is

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Proof: Let

$$\boldsymbol{p} = \left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|^2}\right)\boldsymbol{u}_1 + \left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|^2}\right)\boldsymbol{u}_2 + \dots + \left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_k}{\|\boldsymbol{u}_k\|^2}\right)\boldsymbol{u}_k$$

($\boldsymbol{p}$ belongs to V)

and

$$\boldsymbol{n} = \boldsymbol{w} - \boldsymbol{p}$$

(so that $\boldsymbol{w} = \boldsymbol{n} + \boldsymbol{p}$)

Theorem (orthogonal projection)

Proof: Let

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($\boldsymbol{p}$ belongs to V)

and

$$\boldsymbol{n}=\boldsymbol{w}-\boldsymbol{p}$$

(so that $\boldsymbol{w}=\boldsymbol{n}+\boldsymbol{p}$)

To prove the theorem, we just need to show that $\boldsymbol{n}$ is orthogonal to V.

That is, we need to show that $\boldsymbol{n}$ is orthogonal to each of the basis vectors $\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k$.

Theorem (orthogonal projection)

Proof: For $i = 1, 2, \ldots, k$,

$$\boxed{\boldsymbol{p} = \left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|^2}\right)\boldsymbol{u}_1 + \left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|^2}\right)\boldsymbol{u}_2 + \ldots + \left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_k}{\|\boldsymbol{u}_k\|^2}\right)\boldsymbol{u}_k}$$

$$\begin{aligned}
\boldsymbol{n}\cdot\boldsymbol{u}_i &= (\boldsymbol{w}-\boldsymbol{p})\cdot\boldsymbol{u}_i \\
&= \boldsymbol{w}\cdot\boldsymbol{u}_i - \boldsymbol{p}\cdot\boldsymbol{u}_i \\
&= \boldsymbol{w}\cdot\boldsymbol{u}_i - \boxed{\left[\left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|^2}\right)\boldsymbol{u}_1 + \left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|^2}\right)\boldsymbol{u}_2 + \ldots + \left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_k}{\|\boldsymbol{u}_k\|^2}\right)\boldsymbol{u}_k\right]\cdot\boldsymbol{u}_i} \\
&= \boldsymbol{w}\cdot\boldsymbol{u}_i - \left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_i}{\|\boldsymbol{u}_i\|^2}\right)\boldsymbol{u}_i\cdot\boldsymbol{u}_i \qquad \text{since } \left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_j}{\|\boldsymbol{u}_j\|^2}\right)\boldsymbol{u}_j\cdot\boldsymbol{u}_i = 0 \quad \text{if } i \neq j \\
&= \boldsymbol{w}\cdot\boldsymbol{u}_i - \left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_i}{\|\boldsymbol{u}_i\|^2}\right)\|\boldsymbol{u}_i\|^2 \quad = \boldsymbol{w}\cdot\boldsymbol{u}_i - \boldsymbol{w}\cdot\boldsymbol{u} = 0
\end{aligned}$$

Theorem (orthogonal projection)

Proof: Let

$$\boldsymbol{p}=\left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|^2}\right)\boldsymbol{u}_1+\left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|^2}\right)\boldsymbol{u}_2+\ldots+\left(\frac{\boldsymbol{w}\cdot\boldsymbol{u}_k}{\|\boldsymbol{u}_k\|^2}\right)\boldsymbol{u}_k$$

($\boldsymbol{p}$ belongs to V)

and

$$\boldsymbol{n}=\boldsymbol{w}-\boldsymbol{p}$$

(so that $\underline{\boldsymbol{w}=\boldsymbol{n}+\boldsymbol{p}}$)

To prove the theorem, we just need to show that $\boldsymbol{n}$ is orthogonal to V.

If $\{\boldsymbol{v}_1,\boldsymbol{v}_2,\ldots,\boldsymbol{v}_k\}$ is an orthonormal basis for V, then the projection of $\boldsymbol{w}$ onto V is

$$(\boldsymbol{w}\cdot\boldsymbol{v}_1)\boldsymbol{v}_1+(\boldsymbol{w}\cdot\boldsymbol{v}_2)\boldsymbol{v}_2+\ldots+(\boldsymbol{w}\cdot\boldsymbol{v}_k)\boldsymbol{v}_k$$

Example (orthogonal projection)

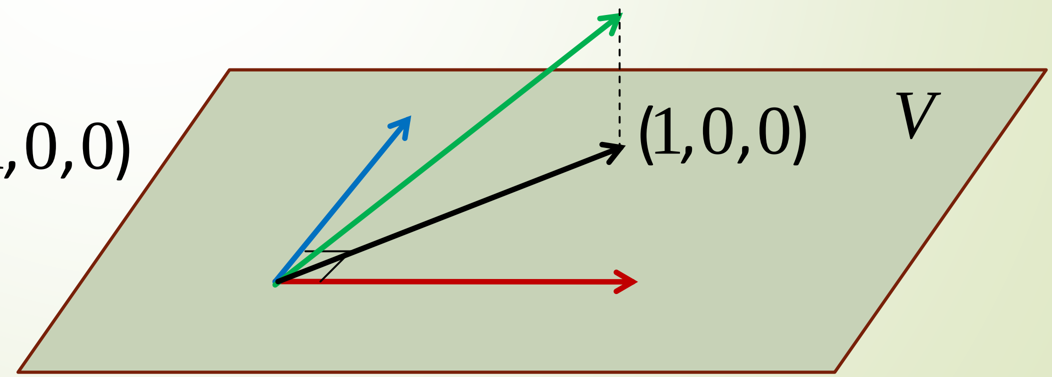
Let $V = \text{span}\{(1,0,1),(1,0,-1)\}$ be a subspace of $\mathbb{R}^3$ (V is a plane).

$$(1,0,1)\cdot(1,0,-1) = 1+0-1 = 0$$

$\Rightarrow \{(1,0,1),(1,0,-1)\}$ is an orthogonal basis for V.

What is the projection of $\boldsymbol{w} = (1,1,0)$ onto V?

$$\frac{(1,1,0)\cdot\boxed{(1,0,1)}}{\|(1,0,1)\|^2}(1,0,1) + \frac{(1,1,0)\cdot\boxed{(1,0,-1)}}{\|(1,0,-1)\|^2}(1,0,-1) = (1,0,0)$$



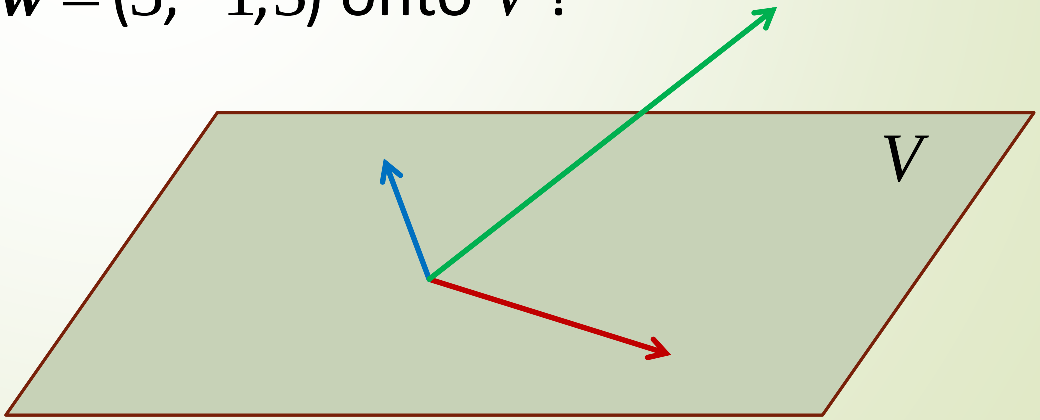
Example (orthogonal projection)

Let $V = \text{span}\{(1,1,1),(1,3,-1)\}$ be a subspace of $\mathbb{R}^3$ (V is a plane).

$(1,1,1)\cdot(1,3,-1) = 3 \neq 0$

$\Rightarrow \{(1,1,1),(1,3,-1)\}$ is a basis, but not an orthogonal basis for V.

What is the projection of $\boldsymbol{w} = (3,-1,3)$ onto V?

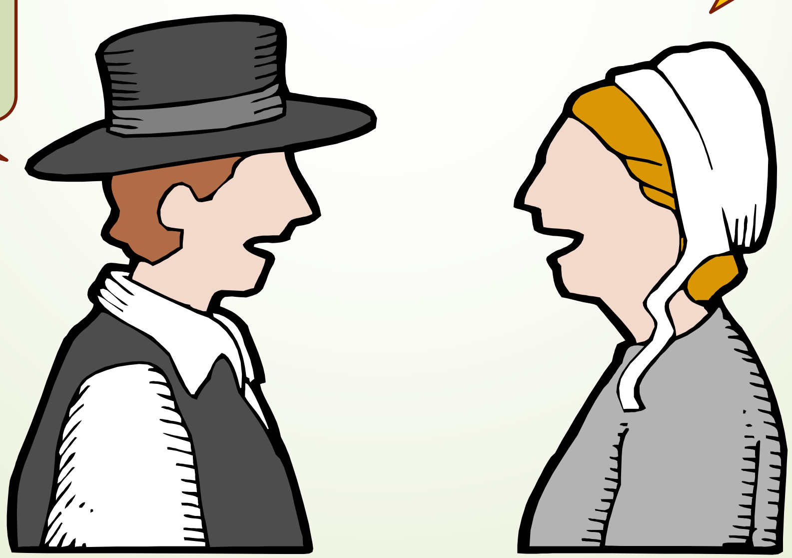


Finding orthogonal bases

We can only use the orthogonal projection theorem if we have an orthogonal basis for a vector space...

yes, so what if we don't have one?

We find one!



Summary

1) How to compute orthogonal projection onto a vector space V provided we have an orthogonal / orthonormal basis for V.