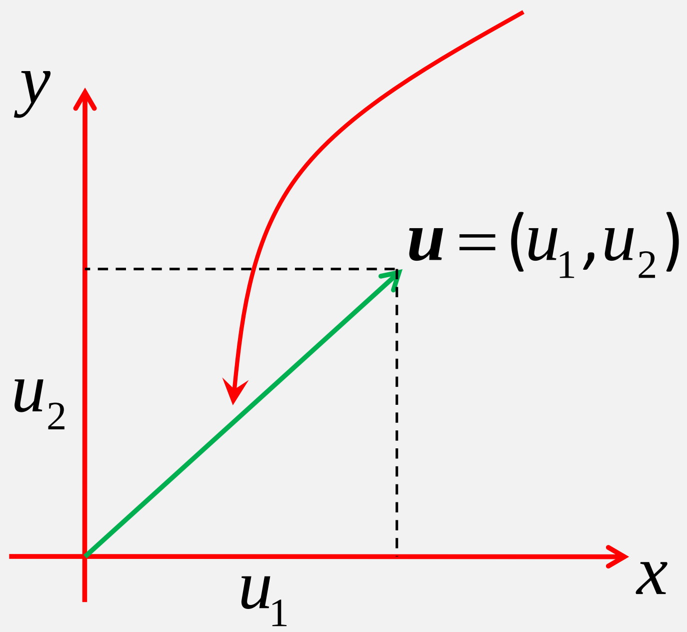
MORE ON EUCLIDEAN VECTORS

LENGTH OF A VECTOR

If $\boldsymbol{u} = (u_1, u_2)$ is a vector in $\mathbb{R}^2$, the length of $\boldsymbol{u}$ is defined to be

$$\|\boldsymbol{u}\| = \sqrt{u_1^2 + u_2^2}$$



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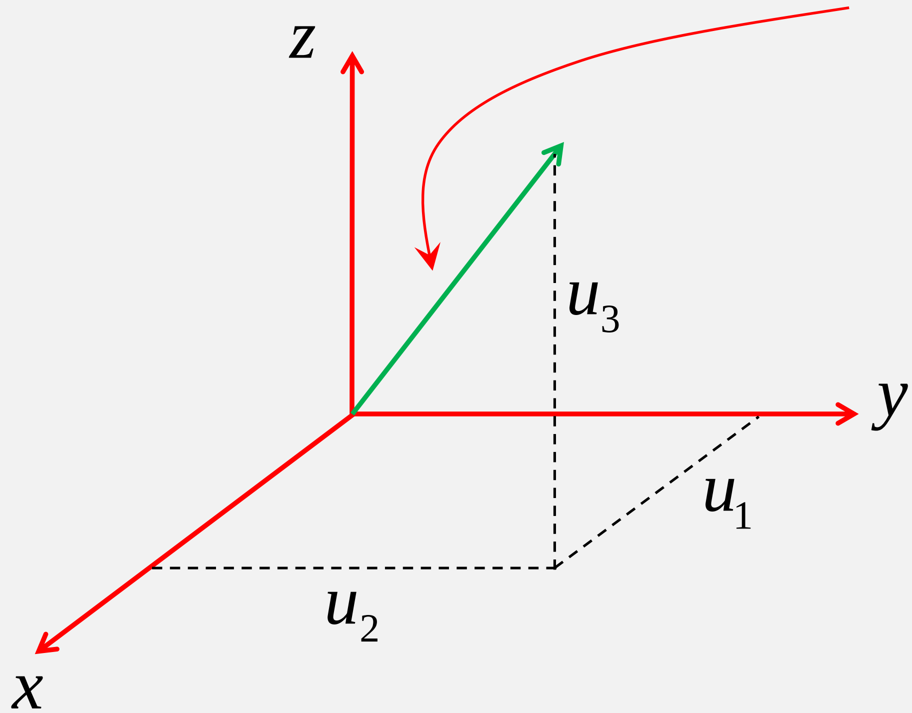

How do you think we should define the length of a vector $\boldsymbol{u}$ in $\mathbb{R}^n$?

LENGTH OF A VECTOR

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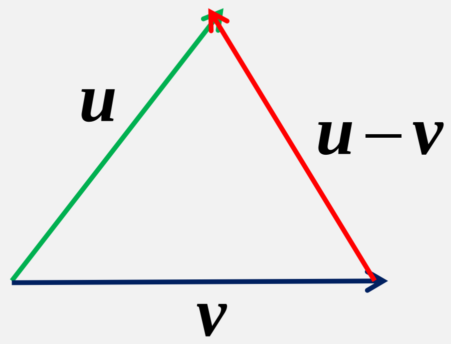
DISTANCE BETWEEN TWO VECTORS

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, the distance between $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined to be the length of the vector $\boldsymbol{u}-\boldsymbol{v}$.



distance between $\boldsymbol{u}$ and $\boldsymbol{v}$

$$= d(\boldsymbol{u},\boldsymbol{v}) = \|\boldsymbol{u}-\boldsymbol{v}\|$$

The definition is similar when $\boldsymbol{u},\boldsymbol{v}$ are vectors in $\mathbb{R}^n$.

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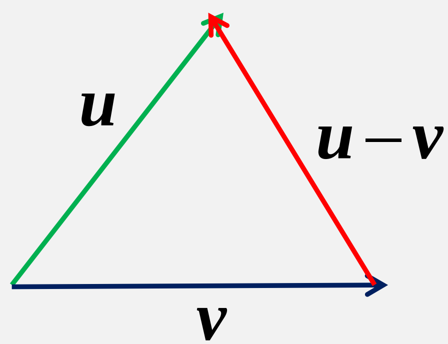

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If $\boldsymbol{u}=(u_1,u_2)$ and $\boldsymbol{v}=(v_1,v_2)$ are vectors in $\mathbb{R}^2$,

$$d(\boldsymbol{u},\boldsymbol{v}) = \|\boldsymbol{u}-\boldsymbol{v}\| = \sqrt{(u_1-v_1)^2+(u_2-v_2)^2}$$

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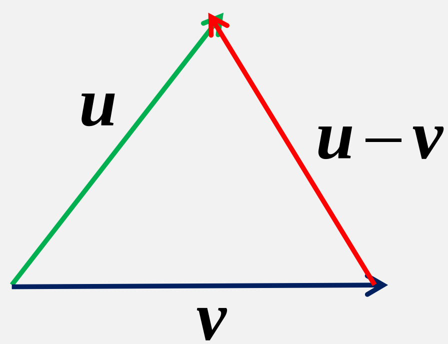

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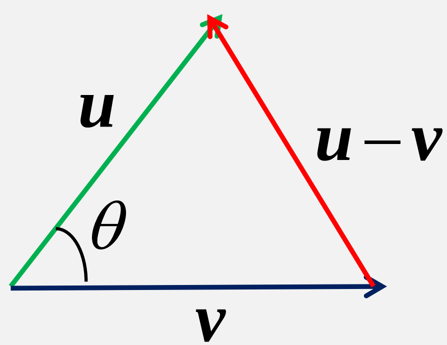
ANGLE BETWEEN TWO VECTORS

If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, let the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ be θ.



By cosine rule,

$$\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

$$\Rightarrow \cos\theta = \frac{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2}{2\|\mathbf{u}\|\|\mathbf{v}\|}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2}{2\|\mathbf{u}\|\|\mathbf{v}\|}\right)$$

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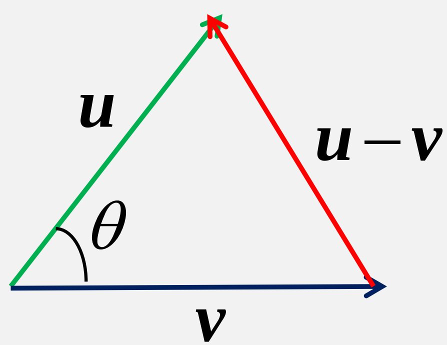

$$\theta = \cos^{-1}\left(\frac{\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - \|\boldsymbol{u}-\boldsymbol{v}\|^2}{2\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\right)$$

If $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$ belong to $\mathbb{R}^2$,

$$\theta = \cos^{-1}\left(\frac{u_1^2 + u_2^2 + v_1^2 + v_2^2 - (u_1 - v_1)^2 - (u_2 - v_2)^2}{2\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\right)$$

$$= \cos^{-1}\left(\frac{u_1 v_1 + u_2 v_2}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\right)$$

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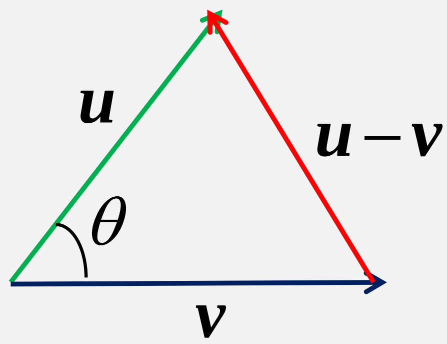

$$\boxed{\theta = \cos^{-1}\left(\frac{\|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2 - \|\boldsymbol{u}-\boldsymbol{v}\|^2}{2\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\right)}$$

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DOT PRODUCT, NORM, DISTANCE, ANGLE

Let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ be two vectors in $\mathbb{R}^n$.

1) The dot product (or inner product) of $\boldsymbol{u}$ and $\boldsymbol{v}$ is the value

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

2) The norm (or length) of $\boldsymbol{u}$ is

$$\|\boldsymbol{u}\| = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$$

Vectors of norm 1 are called unit vectors.

DOT PRODUCT, NORM, DISTANCE, ANGLE

Let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ be two vectors in $\mathbb{R}^n$.

3) The distance between $\boldsymbol{u}$ and $\boldsymbol{v}$ is

$$d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}$$

4) The angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is

$$\cos^{-1}\left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\right)$$

DOT PRODUCT AND MATRIX PRODUCT

Let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ be two vectors in $\mathbb{R}^n$ (here $\boldsymbol{u}$ and $\boldsymbol{v}$ are written as row vectors).

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

matrix product

dot product

$$= \begin{pmatrix} u_1 & u_2 & \ldots & u_n \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \boldsymbol{u}\boldsymbol{v}^T$$

DOT PRODUCT AND MATRIX PRODUCT

Let $\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ be two vectors in $\mathbb{R}^n$

(here $\boldsymbol{u}$ and $\boldsymbol{v}$ are written as column vectors).

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

$$= \begin{pmatrix} u_1 & u_2 & \ldots & u_n \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \boldsymbol{u}^T \boldsymbol{v}$$

matrix product

dot product

SUMMARY

1) Definitions of:

(a) Norm (or length) of a vector;

(b) Distance between two vectors;

(c) Angle between two vectors;

(d) Dot product between two vectors;

2) Dot product and matrix product.