CHARACTERISTIC EQUATION OF A MATRIX

DEFINITION - RECALL

Let $\boldsymbol{A}$ be a square matrix of order n.

A nonzero column vector $\boldsymbol{u} \in \mathbb{R}^n$ is called an eigenvector of $\boldsymbol{A}$ if

$$\boldsymbol{Au} = \lambda \boldsymbol{u} \quad \text{for some scalar } \lambda.$$

'multiplying $\boldsymbol{A}$ to $\boldsymbol{u}$ results in some scalar multiple of $\boldsymbol{u}$.'

The scalar λ is called an eigenvalue of $\boldsymbol{A}$

and $\boldsymbol{u}$ is said to be an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue λ.

EXAMPLES (EIGENVALUES/EIGENVECTORS)

$$\boldsymbol{A}=\begin{pmatrix}0.96 & 0.01\\ 0.04 & 0.99\end{pmatrix} \qquad \boldsymbol{x}=\begin{pmatrix}1\\ 4\end{pmatrix} \qquad \boldsymbol{y}=\begin{pmatrix}1\\ -1\end{pmatrix}$$

$$\boldsymbol{A}\boldsymbol{x}=\begin{pmatrix}0.96 & 0.01\\ 0.04 & 0.99\end{pmatrix}\begin{pmatrix}1\\ 4\end{pmatrix}=\begin{pmatrix}1\\ 4\end{pmatrix}=1\boldsymbol{x}$$

1 is an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ associated with eigenvalue 1.

$$\boldsymbol{A}\boldsymbol{y}=\begin{pmatrix}0.96 & 0.01\\ 0.04 & 0.99\end{pmatrix}\begin{pmatrix}1\\ -1\end{pmatrix}=\begin{pmatrix}0.95\\ -0.95\end{pmatrix}=0.95\boldsymbol{y}$$

0.95 is an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{y}$ is an eigenvector of $\boldsymbol{A}$ associated with eigenvalue 0.95.

EXAMPLES (EIGENVALUES/EIGENVECTORS)

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \boldsymbol{y} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \boldsymbol{z} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\boldsymbol{Bx} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = 3\boldsymbol{x}$$

3 is an eigenvalue of $\boldsymbol{B}$ and $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{B}$ associated with eigenvalue 3.

EXAMPLES (EIGENVALUES/EIGENVECTORS)

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \boldsymbol{y} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \boldsymbol{z} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\boldsymbol{By} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\boldsymbol{y}$$

0 is an eigenvalue of $\boldsymbol{B}$ and $\boldsymbol{y}$ is an eigenvector of $\boldsymbol{B}$ associated with eigenvalue 0.

$$\boldsymbol{Bz} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\boldsymbol{z}$$

$\boldsymbol{z}$ is an eigenvector of $\boldsymbol{B}$ associated with eigenvalue 0.

EXAMPLES (EIGENVALUES/EIGENVECTORS)

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \boldsymbol{y} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad \boldsymbol{z} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

associated with eigenvalue 3.

associated with eigenvalue 0.

Observe that

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

HOW TO FIND ALL EIGENVALUES OF A MATRIX

Let $\boldsymbol{A}$ be a square matrix of order n.

λ is an eigenvalue of $\boldsymbol{A}$

$\Leftrightarrow \boldsymbol{Au} = \lambda\boldsymbol{u}$ for some non zero column vector $\boldsymbol{u}$ $(\in \mathbb{R}^n)$

$\Leftrightarrow \lambda\boldsymbol{u} - \boldsymbol{Au} = \boldsymbol{0}$ for some non zero column vector $\boldsymbol{u}$ $(\in \mathbb{R}^n)$

$\Leftrightarrow (\lambda\boldsymbol{I} - \boldsymbol{A})\boldsymbol{u} = \boldsymbol{0}$ for some $\boxed{\text{non zero column vector } \boldsymbol{u}}$ $(\in \mathbb{R}^n)$

$\Leftrightarrow (\lambda\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$ has $\boxed{\text{non trivial solutions}}$

$\Leftrightarrow \det(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0$ (that is, $\lambda\boldsymbol{I} - \boldsymbol{A}$ is singular)

HOW TO FIND ALL EIGENVALUES OF A MATRIX

Let $\boldsymbol{A}$ be a square matrix of order n.

So the eigenvalues of $\boldsymbol{A}$ are all the numbers λ that makes the matrix $(\lambda \boldsymbol{I} - \boldsymbol{A})$ singular.

λ is an eigenvalue of $\boldsymbol{A}$

$\Leftrightarrow \det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$ (that is, $\lambda \boldsymbol{I} - \boldsymbol{A}$ is singular)

Note that if expanded (by cofactor expansion), $\det(\lambda \boldsymbol{I} - \boldsymbol{A})$ is a polynomial in λ of degree n.

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = c_n \lambda^n + c_{n-1} \lambda^{n-1} + \ldots + c_1 \lambda + c_0$$

DEFINITION

Let $\boldsymbol{A}$ be a square matrix of order n.

The polynomial $\det(\lambda \boldsymbol{I} - \boldsymbol{A})$ is called the $\boxed{\text{characteristic polynomial}}$ of $\boldsymbol{A}$; and

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = c_n \lambda^n + c_{n-1} \lambda^{n-1} + \ldots + c_1 \lambda + c_0 \longleftarrow \text{characteristic polynomial}$$

the equation $\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$ is called the $\boxed{\text{characteristic equation}}$ of $\boldsymbol{A}$.

$$c_n \lambda^n + c_{n-1} \lambda^{n-1} + \ldots + c_1 \lambda + c_0 = 0 \longleftarrow \text{characteristic equation}$$

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix} \qquad \lambda \boldsymbol{I} - \boldsymbol{A} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda - 0.96 & -0.01 \\ -0.04 & \lambda - 0.99 \end{pmatrix}$$

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = (\lambda - 0.96)(\lambda - 0.99) - (-0.01)(-0.04)$$

$$= \lambda^2 - 1.95\lambda + 0.95$$

$$= (\lambda - 1)(\lambda - 0.95)$$

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0 \Leftrightarrow \lambda = 1 \text{ or } \lambda = 0.95.$$

The eigenvalues of $\boldsymbol{A}$ are 1 and 0.95.

EXAMPLE

$$\boldsymbol{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \lambda \boldsymbol{I} - \boldsymbol{B} = \begin{pmatrix} \lambda - 1 & -1 & -1 \\ -1 & \lambda - 1 & -1 \\ -1 & -1 & \lambda - 1 \end{pmatrix}$$

$\det(\lambda \boldsymbol{I} - \boldsymbol{B}) = \lambda^3 - 3\lambda^2$ (some hardwork required!)

$= \lambda^2(\lambda - 3)$

$\det(\lambda \boldsymbol{I} - \boldsymbol{B}) = 0 \Leftrightarrow \lambda = 0$ or λ=3.

The eigenvalues of $\boldsymbol{B}$ are 0 and 3.

EXAMPLE

$$\boldsymbol{C} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} \qquad \lambda\boldsymbol{I} - \boldsymbol{C} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & -2 \\ -1 & -1 & \lambda - 1 \end{pmatrix}$$

$\det(\lambda\boldsymbol{I} - \boldsymbol{C}) = \lambda^3 - \lambda^2 - 2\lambda + 2$ (some hardwork required!)

What are the roots?

Try $\lambda = -2, -1, 0, 1, 2$

$\lambda = 1{:}\ \ 1^3 - 1^2 - 2(1) + 2 = 0$

So $\lambda = 1$ is a root of the characteristic equation

EXAMPLE

$$\boldsymbol{C} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} \qquad \lambda \boldsymbol{I} - \boldsymbol{C} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & -2 \\ -1 & -1 & \lambda - 1 \end{pmatrix}$$

$$\det(\lambda \boldsymbol{I} - \boldsymbol{C}) = \lambda^3 - \lambda^2 - 2\lambda + 2$$

(some hardwork required!)

$$= (\lambda - 1)\boxed{(\lambda^2 - 2)}$$

obtained from long division

$$= (\lambda - 1)(\lambda - \sqrt{2})(\lambda + \sqrt{2})$$

$$\det(\lambda \boldsymbol{I} - \boldsymbol{C}) = 0 \Leftrightarrow \lambda = 1,\ \sqrt{2} \text{ or } -\sqrt{2}.$$

The eigenvalues of $\boldsymbol{C}$ are 1, $\sqrt{2}$ and $-\sqrt{2}$.

SUMMARY

1) Characteristic polynomial and characteristic equation of a square matrix.

2) How to find all the eigenvalues of a square matrix using the characteristic equation.