NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

Module:	MA1508E Linear Algebra for Engineering
Year/Semester:	2018-2019 (Semester 2)
Practice Problem Set:	6 (Solutions)

Name:

Matriculation Number:

Tutorial Group:

Write down the solutions to the problems below, showing all working involved (imagine it is an examination question). Hand in your answers **together with this question paper** before you leave the classroom. You may refer to any materials while answering the questions. You may also discuss with your friends but do not **copy blindly**.

1. Let λ be an eigenvalue of $\boldsymbol{A}$.

   (a) Show that λ^2 is an eigenvalue of $\boldsymbol{A}^2$. Find an eigenvector of $\boldsymbol{A}^2$ associated with λ^2.

   (b) Show that λ is also an eigenvalue of $\boldsymbol{A}^T$.

   Solution:

   (a) Let $\boldsymbol{x}$ be an eigenvector of $\boldsymbol{A}$ associated with λ. Then

   $$\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x} \Rightarrow \boldsymbol{A}^2\boldsymbol{x} = \boldsymbol{A}(\boldsymbol{A}\boldsymbol{x}) = \boldsymbol{A}(\lambda\boldsymbol{x}) = \lambda(\boldsymbol{A}\boldsymbol{x}) = \lambda(\lambda\boldsymbol{x}) = \lambda^2\boldsymbol{x}.$$

   So λ^2 is an eigenvalue of $\boldsymbol{A}^2$ and $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}^2$ associated with λ^2.

   (b) Since λ is an eigenvalue of $\boldsymbol{A}$, we have $\det(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0$. This implies

   $$\det(\lambda\boldsymbol{I} - \boldsymbol{A})^T = 0 \Rightarrow \det(\lambda\boldsymbol{I}^T - \boldsymbol{A}^T) = 0 \Rightarrow \det(\lambda\boldsymbol{I} - \boldsymbol{A}^T) = 0.$$

   So λ is also an eigenvalue of $\boldsymbol{A}^T$.

2. Follow the steps discussed in this week's tutorial question 5 and solve the following linear recurrence relation

   $$a_n = 3a_{n-1} - 2a_{n-2} \text{ with } a_0 = 0 \text{ and } a_1 = 1.$$

   Hint: Start off by defining $\boldsymbol{x_n} = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$.

   Solution: Let $\boldsymbol{x_n} = \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$ and $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$. Then $\boldsymbol{x_n} = \boldsymbol{A}\boldsymbol{x_{n-1}} = \cdots = \boldsymbol{A}^n\boldsymbol{x_0}$.

We find that $\lambda = 2, 1$ are the two eigenvalues of $\boldsymbol{A}$. Furthermore,
$E_2 = \operatorname{span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$ and $E_1 = \operatorname{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$.
Let $\boldsymbol{P} = \begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$. Thus

$$\begin{aligned}\begin{pmatrix}a_n\\ a_{n+1}\end{pmatrix} = \boldsymbol{x_n} &= \boldsymbol{P}\begin{pmatrix}2^n & 0\\ 0 & 1\end{pmatrix}\boldsymbol{P}^{-1}\boldsymbol{x_0}\\ &= \begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}2^n & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}^{-1}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}2^n - 1\\ 2^{n+1} - 1\end{pmatrix}.\end{aligned}$$

Thus $a_n = 2^n - 1$.