1. Muinaru is a radio-active element found on the planet Krypton. It decays at a rate proportional to the square of the amount present. Starting with 5 gm of Muinaru at time $t = 0$ minute, the amount becomes 4 gm at time $t = 1$ minute. If the amount becomes 0.01319 gm at time $t = T$ minutes, find the value of T. Give your answer correct to the nearst integer.

$$\frac{dy}{dt} = -ky^2$$

$$-\frac{dy}{y^2} = k\,dt$$

$$\frac{1}{y} = kt + C$$

$$y(0) = 5 \Rightarrow C = 0.2$$

$$y(1) = 4 \Rightarrow 0.25 = k + 0.2 \Rightarrow k = 0.05$$

$$\therefore y = \frac{1}{0.05t + 0.2}$$

$$0.01319 = \frac{1}{0.05T + 0.2}$$

$$\Rightarrow T = 1512.30\cdots$$

$$\approx \underline{\underline{1512}}$$

2. Let $y(x)$ denote the solution to the equation

$$y' - y = \left((297)^{1512}\right) x^2,$$

with $y'(1) = 0$. Find the value of $\ln(y(5))$. Give your answer correct to the nearest integer.

$$R = e^{\int -dx} = e^{-x}$$

$$y = e^{x} \int (297)^{1512} x^2 e^{-x} dx$$

$$= (297)^{1512} e^{x} \int x^2 e^{-x} dx$$

$$= (297)^{1512} e^{x} \{-x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + c\}$$

$$= (297)^{1512} \{-x^2 - 2x - 2 + ce^{x}\}$$

$$y' = (297)^{1512} \{-2x - 2 + ce^{x}\}$$

$$y'(1) = 0 \Rightarrow c = \frac{4}{e}$$

$$\therefore y = (297)^{1512} \{-x^2 - 2x - 2 + 4e^{x-1}\}$$

$$y(5) = (297)^{1512} (4e^4 - 37)$$

$$\ln(y(5)) = 1512 \ln(297) + \ln(4e^4 - 37)$$

$$= 8614.12\ldots$$

$$\approx \underline{\underline{8614}}$$

3. Let $y(x)$ denote the solution to the equation

$$(x+y)\,y' = x+y+1,$$

with $y(0)=1$ and $y(1)=k$, where k denotes a constant. Find the value of $\frac{e^k}{\sqrt{|2k+3|}}$. Give your answer correct to two decimal places.

Let $u = x+y \Rightarrow u' = 1+y'$

$u(u'-1) = u+1$

$u'-1 = (u+1)/u$

$u' = \dfrac{2u+1}{u}$

$\dfrac{u}{2u+1}\,du = dx \Rightarrow \dfrac{2u}{2u+1}\cdot du = 2dx$

$\Rightarrow \left(1 - \dfrac{1}{2u+1}\right) du = 2dx$

$u - \frac{1}{2}\ln|2u+1| = 2x + C$

$x + y - \frac{1}{2}\ln|2x+2y+1| = 2x + C$

$y(0)=1 \Rightarrow 1 - \frac{1}{2}\ln 3 = C$

$\therefore x+y - \frac{1}{2}\ln|2x+2y+1| = 2x + 1 - \frac{1}{2}\ln 3$

$y(1)=k \Rightarrow 1+k - \frac{1}{2}\ln|3+2k| = 3 - \frac{1}{2}\ln 3$

$\Rightarrow \dfrac{e(e^k)}{\sqrt{|2k+3|}} = \dfrac{e^3}{\sqrt{3}}$

$\Rightarrow \dfrac{e^k}{\sqrt{|2k+3|}} = \dfrac{e^2}{\sqrt{3}} = 4.266\cdots \approx \underline{4.27}$

4. Let a denote a positive constant. Let $y(x)$ denote the solution to the equation

$$y' = \frac{ax^2 + xy + y^2}{xy},$$

with $y(1) = a$ and $y(2) = 6a$. Find the value of a. Give your answer correct to two decimal places.

Let $y = ux \Rightarrow y' = u'x + u$

$$u'x + u = \frac{ax^2 + ux^2 + u^2x^2}{ux^2} = \frac{a + u + u^2}{u}$$

$$\Rightarrow u'x = \frac{a+u}{u} \Rightarrow \frac{u}{u+a}\,du = \frac{dx}{x} \Rightarrow \left(1 - \frac{a}{u+a}\right)du = \frac{dx}{x}$$

$$\therefore u - a\ln|u+a| = \ln x + C$$

$$\therefore \frac{y}{x} - a\ln\left|\frac{y}{x} + a\right| = \ln x + C$$

$$y(1) = a \Rightarrow a - a\ln 2a = C$$

$$\Rightarrow \frac{y}{x} - a\ln\left|\frac{y}{x} + a\right| = \ln x + a - a\ln 2a$$

$$y(2) = 6a \Rightarrow 3a - a\ln 4a = \ln 2 + a - a\ln 2a$$

$$3a - a\ln 4 - a\ln a = \ln 2 + a - a\ln 2 - a\ln a$$

$$3a - 2a\ln 2 = \ln 2 + a - a\ln 2$$

$$2a = \ln 2 + a\ln 2$$

$$a(2 - \ln 2) = \ln 2$$

$$a = \frac{\ln 2}{2 - \ln 2} = 0.530\cdots \approx \underline{\underline{0.53}}$$