W04-08

Slide 01: In this unit, we will discuss the concept of subspaces.

Slide 02: We start off with a subset V of $\mathbb{R}^n$. The concept of subset is familiar to all of you, so V is just a sub-collection of vectors taken from $\mathbb{R}^n$.

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Suppose there exists a set S of vectors $\boldsymbol{u_1}, \boldsymbol{u_2}$ and so on till $\boldsymbol{u_k}$ from $\mathbb{R}^n$ such that

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the linear span of S is equal to the subset V,

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then we say that the subset V is a subspace of $\mathbb{R}^n$.

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Recall that span(S) is the set of all linear combinations of $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$. When we say span(S) is equal to V, it means that the two sets are exactly the same, so span(S) cannot have vectors not found in V and neither can V have vectors not included inside span(S).

Slide 03: When span(S) is equal to V, we say that V is the subspace spanned by S.

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Or we can say that V is the subspace spanned by $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ till $\boldsymbol{u_k}$.

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Or we can simply say the set S spans V.

Slide 04: Let us consider some simple subspaces. The set containing only the zero vector can be written as the span of the zero vector. Thus by definition, the set containing only the zero vector is a subspace.

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More precisely, we call this the zero subspace of $\mathbb{R}^n$.

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In fact, it is the only subspace that has a finite number of vectors in it. There is only one vector, namely the zero vector in this subspace.

Slide 05: Consider the following vectors from $\mathbb{R}^n$. Notice that $\boldsymbol{e_1}$ has a 1 in the first component and zero everywhere else. $\boldsymbol{e_2}$ has a 1 in the second component and zero everywhere else. Other vectors follow a similar format.

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Now $\mathbb{R}^n$ is the set of all vectors with n components, each of which is a real number. We can represent the set $\mathbb{R}^n$ as shown here.

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An arbitrary vector in $\mathbb{R}^n$ can be written as a linear combination of $\boldsymbol{e_1}$ to $\boldsymbol{e_n}$.

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Since u_1 to u_n can take on all real numbers, the set here is really the set of all linear combinations of $\boldsymbol{e_1}$ to $\boldsymbol{e_n}$.

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This, by definition is the linear span of $\boldsymbol{e_1}$ to $\boldsymbol{e_n}$.

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Thus, we have actually written $\mathbb{R}^n$ as a linear span and thus by definition, $\mathbb{R}^n$ is actually a subspace of itself.

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So to some extent, we can think of $\mathbb{R}^n$ as the largest subspace inside $\mathbb{R}^n$, one that contains every single vector of $\mathbb{R}^n$.

Slide 06: Consider the following set V_1. Clearly, it is a subset of $\mathbb{R}^2$ since it contains vectors with two components.

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Is V_1 a subspace of $\mathbb{R}^2$?

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By definition, we would need to write V_1 as a linear span in order to show that it is indeed a subspace.

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Looking closely at the vectors in V_1, we see that the arbitrary expression for a vector in V_1 can be rewritten as a times $(1, 0)$ plus b times $(-2, 3)$ where a, b are any real numbers.

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This is precisely the linear span of $(1, 0)$ and $(-2, 3)$ since we have all the possible linear combinations of the two vectors in the set V_1.

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We have successfully written V_1 as a linear span and thus shown that is it a subspace of $\mathbb{R}^2$.

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An interesting question to follow up on this is whether V_1 is the entire Euclidean 2-space? In other words, are there vectors in $\mathbb{R}^2$ that is not found inside V_1?

Slide 07: To answer this question is not difficult. In fact we have considered such questions before. We wish to determine if V_1, which is the linear span of $(1, 0)$ and $(-2, 3)$ is the entire $\mathbb{R}^2$.

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To do this, we set up a vector equation with an arbitrary vector in $\mathbb{R}^2$ on the right hand side.

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Write down the associated linear system

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and the augmented matrix

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where in this case, we notice that the matrix is already in row-echelon form with no zero rows.

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From our discussion in a previous unit, we can now conclude that the vector equation will always be consistent for all values of x and y and thus the linear span of the two vectors is the entire $\mathbb{R}^2$.

Slide 08: Let us consider another set V_2, this time V_2 is a subset of $\mathbb{R}^3$.

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Going back much earlier to our study of linear equations, you should be able to describe the set V_2 geometrically.

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Is V_2 a subspace of $\mathbb{R}^3$?

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To answer this, we will again attempt to write V_2 as a linear span.

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An arbitrary vector in V_2 is of the form (x, y, z) where the three components must satisfy the equation $x - 3y + 2z = 0$.

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Solving this linear equation, we obtain the following general solution involving two arbitrary parameters s and t.

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Thus, the set V_2

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can be rewritten in the form as follows. Basically it contains vectors of the form $(3s - 2t, s, t)$ where s and t can take on any real number.

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Now similar to the previous example for V_1, the vector $(3s - 2t, s, t)$ can be written as s times $(3, 1, 0)$ plus t times $(-2, 0, 1)$

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and therefore, V_2 is the linear span of the two vectors $(3, 1, 0)$ and $(-2, 0, 1)$. Thus V_2 is a subspace of $\mathbb{R}^3$ since it can be expressed as a linear span.

Slide 09: Consider this next set V_3, which is also a subset of $\mathbb{R}^3$. Once again, can you describe V_3 geometrically? Is V_3 a subspace of $\mathbb{R}^3$ like V_2?

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A quick observation will reveal that the zero vector does not belong to V_3 since $(0, 0, 0)$ does not satisfy the equation $x - 3y + 2z = 1$.

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Recall from our earlier discussion on linear spans is that any linear span must contain the zero vector. Since V_3 has been shown not to contain the zero vector, we can now safely conclude that V_3 cannot be a linear span.

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Thus V_3 is not a subspace of $\mathbb{R}^3$.

Slide 10: The next set V_4 is also a subset of $\mathbb{R}^3$. We again would like to determine if V_4 is a subspace of $\mathbb{R}^3$. If you recall what happened in the previous example, you may be tempted to check if V_4 contains the zero vector. In this case, V_4 does indeed contain the zero vector since $(0, 0, 0)$ satisfies $x \leq y \leq z$. However, a subset containing the zero vector is not enough for us to conclude that it must be a subspace.

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Now suppose V_4 is indeed a subspace, then by definition, it can be expressed as a linear span. So supppose V_4 is equal to $\text{span}(T)$ for some set T.

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We now use another piece of information that we know of linear spans, established in an earlier unit. Namely, linear spans has this closure property whereby if we take any vectors inside a linear span and combinne these vectors linearly, the resulting vector must still be contained inside the linear span.

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Now note that $(1, 1, 2)$ and $(0, 2, 4)$ are both vectors in V_4 since they satisfy the defining property of $x \leq y \leq z$.

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However, upon linearly combining the two vectors in the form $(1, 1, 2) - 2(0, 2, 4)$, we obtain the vector $(1, -3, -6)$ which is no longer a vector in V_4 since it does not satify the defining property. This means that V_4 does not exhibit closure property and thus is not a linear span. Subsequently, we are able to conclude that V_4 is not a subspace of $\mathbb{R}^3$.

Slide 11: Let us summarise the main points.

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We defined what is meant by a subspace of the Euclidean space. Note that any subset that can be written as a linear span is a subspace.

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We saw two extreme cases of subspaces of $\mathbb{R}^n$. In a way, the zero subspace can be considered the smallest while the entire $\mathbb{R}^n$ is a subspace of itself.

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Using two properties of linear spans established in an earlier unit, we saw two examples on how we can show that a given subset is not a subspace.