SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS

LINEAR DIFFERENTIAL EQUATIONS

In many applied problems, several quantities are varying continuously in time, and they are related by a system of differential equations.

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_n'(t) \end{pmatrix} \quad \begin{cases} y_1' & = & a_{11}y_1 & + & a_{12}y_2 & + & \dots & + & a_{1n}y_n \\ y_2' & = & a_{21}y_1 & + & a_{22}y_2 & + & \dots & + & a_{2n}y_n \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ y_n' & = & a_{1n}y_1 & + & a_{2n}y_2 & + & \dots & + & a_{nn}y_n \end{cases} \quad \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}$$

LINEAR DIFFERENTIAL EQUATIONS

We assume $y_1, \dots, y_n$ are differentiable functions of t.

$$\begin{cases} y_1' &= a_{11}y_1 &+ a_{12}y_2 &+ \dots &+ a_{1n}y_n \\ y_2' &= a_{21}y_1 &+ a_{22}y_2 &+ \dots &+ a_{2n}y_n \\ &\vdots &\vdots &\vdots &\vdots \\ y_n' &= a_{1n}y_1 &+ a_{2n}y_2 &+ \dots &+ a_{nn}y_n \end{cases}$$

$$\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}$$

$$\left(\boxed{\boldsymbol{Y}'}\right) = \left(\boxed{\boldsymbol{A}}\right)\left(\boxed{\boldsymbol{Y}}\right)$$

A SOLUTION TO THE SYSTEM

When $n=1$: $y'(t)=ay(t)$

Then $y(t)=ce^{at}$ where c is a constant is a solution since

$$y(t)=ce^{at} \Rightarrow y'(t)=cae^{at}=a(ce^{at})=ay(t)$$

Let

$$\boldsymbol{Y}=\begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}=\begin{pmatrix} x_1e^{\lambda t} \\ x_2e^{\lambda t} \\ \vdots \\ x_ne^{\lambda t} \end{pmatrix}=e^{\lambda t}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}=e^{\lambda t}\boldsymbol{x}$$

Let's check if $e^{\lambda t}\boldsymbol{x}$ really works!

A SOLUTION TO THE SYSTEM

$$\boldsymbol{Y} = e^{\lambda t}\boldsymbol{x} \Rightarrow \boldsymbol{Y}' = \lambda e^{\lambda t}\boldsymbol{x}$$

$$\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}$$

So if we choose λ to be an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{x}$ be an eigenvector of $\boldsymbol{A}$ in the eigenspace E_{λ}, we have $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$

$$\boldsymbol{Y} = e^{\lambda t}\boldsymbol{x} \Rightarrow \boldsymbol{A}\boldsymbol{Y} = e^{\lambda t}\boldsymbol{A}\boldsymbol{x}$$

$$\Rightarrow \boldsymbol{A}\boldsymbol{Y} = e^{\lambda t}\lambda\boldsymbol{x} \quad (\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x})$$

$$\Rightarrow \boldsymbol{A}\boldsymbol{Y} = \lambda(e^{\lambda t}\boldsymbol{x})$$

$$\Rightarrow \boldsymbol{A}\boldsymbol{Y} = \boldsymbol{Y}'$$

Let's check if $e^{\lambda t}\boldsymbol{x}$ really works!

YES!

COMBINING SOLUTIONS

We have seen that if $\boldsymbol{x}_1$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue λ_1, then $\underline{\boldsymbol{Y}_1 = e^{\lambda_1 t}\boldsymbol{x}_1}$ is a solution to $\boxed{\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}.}$

Similarly, if $\boldsymbol{x}_2$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue λ_2, then $\underline{\boldsymbol{Y}_2 = e^{\lambda_2 t}\boldsymbol{x}_2}$ is also a solution to $\boxed{\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}.}$

What about

$$k_1\boldsymbol{Y}_1 + k_2\boldsymbol{Y}_2 \qquad \text{where } k_1, k_2 \in \mathbb{R}?$$

(a linear combination of $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$)

Will $k_1\boldsymbol{Y}_1 + k_2\boldsymbol{Y}_2$ also be a solution to $\boxed{\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}}$?

COMBINING SOLUTIONS

$$(\boxed{k_1\boldsymbol{Y}_1 + k_2\boldsymbol{Y}_2})'$$

$$= k_1\boldsymbol{Y}_1' + k_2\boldsymbol{Y}_2'$$

$$= k_1\boldsymbol{A}\boldsymbol{Y}_1 + k_2\boldsymbol{A}\boldsymbol{Y}_2$$

$$= \boldsymbol{A}(\boxed{k_1\boldsymbol{Y}_1 + k_2\boldsymbol{Y}_2})$$

$\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$ are solutions to $\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}$?

$$\boldsymbol{Y}_1' = \boldsymbol{A}\boldsymbol{Y}_1 \qquad \boldsymbol{Y}_2' = \boldsymbol{A}\boldsymbol{Y}_2$$

Thus, $k_1\boldsymbol{Y}_1 + k_2\boldsymbol{Y}_2$ is also a solution to $\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}$.

Generalising, if $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_n$ are solutions of $\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}$, then any linear combination $k_1\boldsymbol{Y}_1 + k_2\boldsymbol{Y}_2 + \ldots + k_n\boldsymbol{Y}_n$ will also be a solution.

SET OF ALL SOLUTIONS

In general, the solutions of the system of linear differential equations:

$$\begin{cases} y_1' = a_{11}y_1 + a_{12}y_2 + \dots + a_{1n}y_n \\ y_2' = a_{21}y_1 + a_{22}y_2 + \dots + a_{2n}y_n \\ \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ y_n' = a_{1n}y_1 + a_{2n}y_2 + \dots + a_{nn}y_n \end{cases}$$

or simply represented by $\boldsymbol{Y}' = \boldsymbol{AY}$, will form a subspace of the vector space of all continuous vector-valued functions.

REMARKS

$$\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y} \qquad (*)$$

$S =$ solution set of (*)

1) There always exists a fundamental set of solutions to (*).

2) If $\boldsymbol{A}$ is a square matrix of order n, then there are n linearly independent functions in a fundamental set.

3) Each solution in the set S is a unique linear combination of these n functions in the fundamental set.

4) Thus, a fundamental set of solutions is a basis for the set of all solutions of (*).

REMARKS

$$\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y} \qquad (*)$$

$S =$ solution set of (*)

5) S is an $n-$dimensional vector space of functions.

6) If a vector $\boldsymbol{Y}_0$ is specified, then the initial value problem is to construct the unique $\boldsymbol{Y}$ (in the set S) such that $\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}$ and $\boldsymbol{Y}(0) = \boldsymbol{Y}_0$.

SUMMARY

1) What is a system of linear differential equations ($\boldsymbol{Y}' = \boldsymbol{AY}$).

2) How to construct the solution set S of $\boldsymbol{Y}' = \boldsymbol{AY}$ and how this is related to the eigenvalues/eigenvectors of $\boldsymbol{A}$.

3) When $\boldsymbol{A}$ is $n \times n$, the solution set S of $\boldsymbol{Y}' = \boldsymbol{AY}$ is a $n-$dimensional vector space of functions.