Unit 006 Writing solutions from row-echelon forms

Slide 01: In this unit, we will discuss how row-echelon forms can help us determine and write down solutions of linear systems.

Slide 02: We have seen from a previous unit that any linear system can have only three possibilities in terms of how many solutions it has. It can either be inconsistent, consistent with a unique solution or consistent with infinitely many solutions.

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It turns out that if a linear system's augmented matrix is in row-echelon form, we can determine which of the above holds for the system. This is done by looking at the row-echelon form.

Slide 03: We first consider inconsistent linear systems. If a linear system's augmented matrix has a row-echelon form such that the last column is a pivot column, this will mean that the system is inconsistent.

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When we say the last column of the augmented matrix, we are referring to the column on the right side of the vertical line.

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In other words, there is a row in the matrix where every entry is zero, except the last entry, which is non zero.

Slide 04: Consider this augmented matrix in row-echelon form, where the last column is a pivot column.

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If we write out the corresponding linear system, it is easy to see why there is an equation that cannot be satisfied by any values of the variables.

Slide 05: We now turn our attention to consistent linear systems. As opposed to inconsistent linear systems, if a linear system's augmented matrix has a row-echelon form such that the last column is not a pivot column, then the system will be consistent.

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What remains for us to discuss is when will the system have an unique solution and when will it have infinitely many solutions?

Slide 06: We now focus our attention on the left side of the vertical line. If a linear system's augmented matrix has a row-echelon where every column, other than the last column is a pivot column, then the linear system will have a unique solution.

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Note that it is possible to have zero rows at the bottom of the matrix. What is important is that each column on the left side of the vertical line is a pivot column.

Slide 07: Consider this augmented matrix which is in row-echelon form. Notice that every column other than the last column is a pivot column.

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Suppose the 4 variables are w, x, y, z. This would be the linear system corresponding to the augmented matrix.

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It is easy to see that starting with the last equation, we have $z = 0$. This will allow us to solve for y in the second last equation. This is followed by solving for x using the second equation and then for w in the first equation.

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What we are doing is known as back substitution since we started from the bottom equation and then substituted the solutions to the variables obtained into equations higher up.

Slide 08: If a linear system's augmented matrix has a row-echelon form where there is some column, other than the lat column, that is not a pivot column, then the linear system will have infinitely many solutions.

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Here the purple column is an example of such a non pivot column on the left side of the vertical line. The linear system corresponding to this augmented matrix will be consistent and has infinitely many solutions.

Slide 09: Consider this augmented matrix which is in row-echelon form. The purple column is a non pivot column and since the last column is also a non pivot column, the linear system corresponding to this augmented matrix will be consistent and has infinitely many solutions.

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Suppose the four variables for the linear system are w, x, y and z. This would be the linear system corresponding to the augmented matrix. How should we go about writing down the general solution for this linear system?

Slide 10: Consider the following example of an augmented matrix, which is in row-echelon form. Is it in reduced row-echelon form?

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Not, since it does not possess the additional properties that reduced row-echelon forms must have.

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If we write out the linear system with five variables x_1 to x_5, we notice that the first variable x_1 is not featured. This is expected because the entire first column of the augmented matrix, which corresponds to the variable x_1 is made up of zeros.

Slide 11: To write down the general solution for the linear system, we identify all the non pivot columns on the left side of the vertical line and assign an arbitrary parameter to each of the variables corresponding to the non pivot columns.

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So in this case, there are three pivot columns

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and two non pivot columns, corresponding to variables x_1 and x_5.

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As mentioned previously, since x_1 does not appear in any of the equations, none of the other variables will depend on the value of x_1.

Slide 12: Now after we have assigned x_1 and x_5 to be arbitrary parameters s and t respectively,

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we start from the last equation $3x_4 + 3x_5 = 6$ which would imply $3x_4 + 3t = 6$, in other words $x_4 = 2 - t$.

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Substitute $x_5 = t$ and $x_4 = 2-t$ into the next higher equation, namely, $x_3+x_4+x_5 = 3$. This gives us $x_3 = 1$. Note that it is coincidental that x_3 is a constant and not an expression involving the arbitrary parameter t.

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We next substitute what we know about x_5, x_4 and x_3 into the next higher equation, namely, $-x_2 + 4x_3 - x_5 = 0$. This would lead us to $x_2 = 4 - t$.

Slide 13: We now have all that we need to know about all the variables in the linear system.

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This is again the process of back substitution which we had seen earlier.

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We are now able to write down a general solution for the linear system as shown here.

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As we have observed earlier, this is a linear system that has infinitely many solutions.

Slide 14: Consider this augmented matrix which is in reduced row-echelon form. Once again, it represents a consistent linear system that has infinitely many solutions.

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Similar to the previous example, we identify the two non pivot variable columns. In this case, the two non pivot variable columns correspond to w and y.

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We now write down the linear system and as before, assign arbitrary parameters to the variables corresponding to the non pivot columns. Thus, we let $w = s$ and $y = t$.

Slide 15: To obtain a general solution, we once again start from the last equation,

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which tells us that $z = 4$.

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Moving on to the next equation, since $x + y = -2$ and $y = t$, we have $x = -2 - t$.

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Finally the next equation tells us that v is equal to $-s$.

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We are now able to write down a general solution of the linear system as follows. Note that in this example, we did not need to perform any back substitution as we moved from one equation to the next higher one. The reason for this is because our augmented matrix was in reduced row-echelon form and not just row-echelon form like in the previous example.

Slide 16: Let us summarise the points in this unit.

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We saw how row-echelon forms can be useful in telling us whether a linear system has none, uniquely one, or infinitely many solutions.

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In particular, when an augmented matrix in row-echelon form has the last column as a pivot column, the linear system is inconsistent.

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On the other hand, if the augmented matrix in row-echelon form is such that the last column is not a pivot column, then the linear system is consistent.

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Further to that, we also discussed what we need to check in a row-echelon form in order to determine if the linear system has uniquely one or infinitely many solutions.

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Lastly, we saw a few examples on how to write down a general solution of a consistent linear system with infinitely many solutions. It should be noted that if the augmented matrix is in reduced row-echelon form, writing down a general solution is easy and does not require back substitution.