ROW EQUIVALENT MATRICES

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Two augmented matrices are said to be row equivalent if one can be obtained from the other by a series of elementary row operations.

Remark: The concept of row equivalent matrices can be used for any matrix in general (not just augmented matrices).

EXAMPLE

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

add -2 times of row 1 to row 2

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

add -3 times of row 1 to row 3

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 6 & -9 & 3 \end{array}\right)$$

add $\frac{6}{4}$ times of row 2 to row 3

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 0 & -15 & 9 \end{array}\right)$$

all row equivalent

REMARK

If we perform one elementary row operation on augmented matrix $\boldsymbol{A}$ to obtain augemented matrix $\boldsymbol{B}$, we can perform another elementary row operation on $\boldsymbol{B}$ to obtain $\boldsymbol{A}$.

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right) \underset{\substack{\text{add } +2 \text{ times of} \\ \text{row 1 to row 2}}}{\overset{\substack{\text{add } -2 \text{ times of} \\ \text{row 1 to row 2}}}{\rightleftarrows}} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

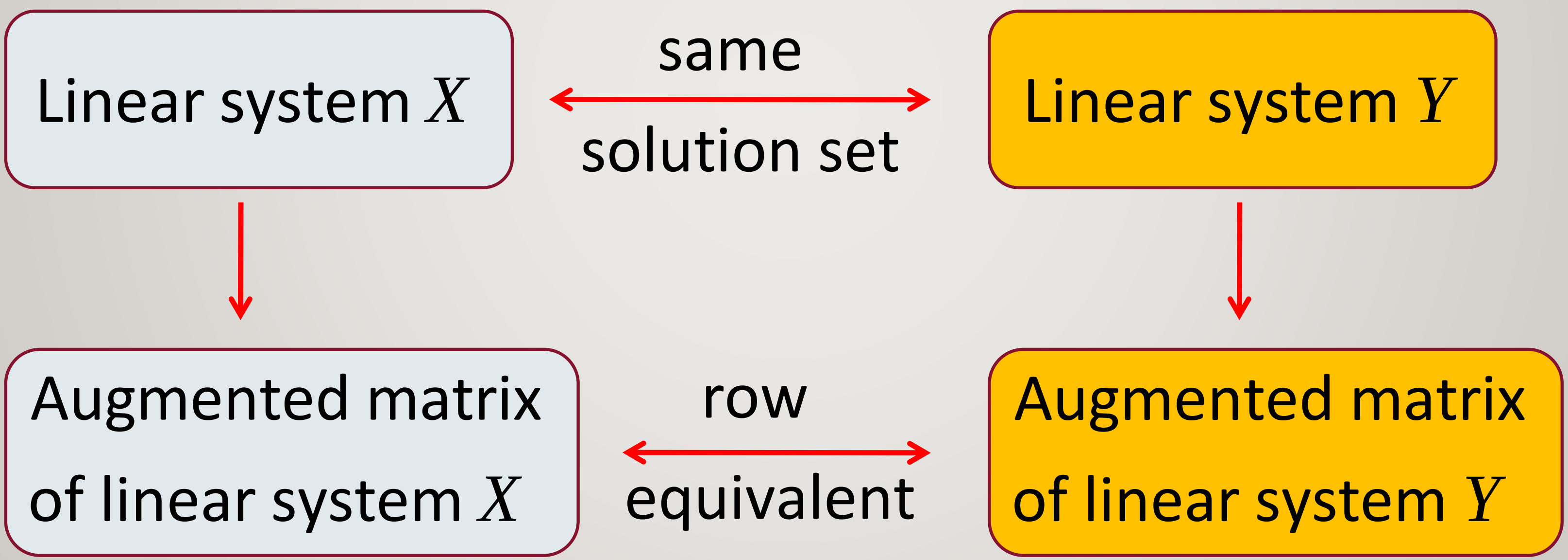
REMARK

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right) \underset{\text{Multiply row 1 by } \frac{1}{3}}{\overset{\text{Multiply row 1 by 3}}{\rightleftarrows}} \left(\begin{array}{ccc|c} 3 & 3 & 9 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right) \underset{\text{swap rows 2 and 3}}{\overset{\text{swap rows 2 and 3}}{\rightleftarrows}} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 3 & 9 & 0 & 3 \\ 2 & -2 & 2 & 4 \end{array}\right)$$

THEOREM

If augmented matrices of two linear systems are row equivalent, then the two linear systems have the same solution set.



all row equivalent

EXAMPLE

$$\begin{cases} x + y + 3z = 0 & (1) \\ 2x - 2y + 2z = 4 & (2) \\ 3x + 9y = 3 & (3) \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

$$\begin{cases} x + y + 3z = 0 & (1) \\ -4y - 4z = 4 & (4) \\ 3x + 9y = 3 & (3) \end{cases}$$

$$\begin{cases} x + y + 3z = 0 & (1) \\ -4y - 4z = 4 & (4) \\ +6y - 9z = 3 & (5) \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 6 & -9 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 0 & -15 & 9 \end{array}\right)$$

$$\begin{cases} x + y + 3z = 0 & (1) \\ -4y - 4z = 4 & (4) \\ -15z = 9 & (6) \end{cases}$$

EXAMPLE

All have the same solution set.

$$\begin{cases} x + y + 3z = 0 & (1) \\ 2x - 2y + 2z = 4 & (2) \\ 3x + 9y = 3 & (3) \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

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$$\begin{cases} x + y + 3z = 0 & (1) \\ - 4y - 4z = 4 & (4) \\ - 15z = 9 & (6) \end{cases}$$

WHICH IS EASIER TO SOLVE?

$$\begin{cases} x + y + 3z = 0 & (1) \\ 2x - 2y + 2z = 4 & (2) \\ 3x + 9y = 3 & (3) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & -2 & 2 & 4 \\ 3 & 9 & 0 & 3 \end{array}\right)$$

$\updownarrow$ row equivalent

$$\begin{cases} x + y + 3z = 0 & (1) \\ \quad - 4y - 4z = 4 & (4) \\ \quad\quad - 15z = 9 & (6) \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -4 & -4 & 4 \\ 0 & 0 & -15 & 9 \end{array}\right)$$

"nice form"?

OUR STRATEGY

Since row equivalent augmented matrices corresponds to linear systems having the same solution set...

... our strategy is to perform elementary row operations on the (starting) augemented matrix until it changes into a 'nice' form.

Given linear system we want to solve

augmented matrix in 'nice' form

$$\left(\begin{array}{ccc|c} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array}\right) \xrightarrow[\text{operations}]{\text{elementary row}} \left(\begin{array}{ccc|c} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array}\right)$$

SUMMARY

1) Definition of row equivalent matrices.

2) The 'reverse' of an elementary row operation is also an elementary row operation.

3) If augmented matrices of two linear systems are row equivalent, then the two linear systems have the same solution set.