Unit 060 Diagonalization Part II

Slide 01: In this unit, we will continue our discussion on diagonalization.

Slide 02: Recall that in a previous unit, we have seen a necessary and sufficient condition for a square matrix of order n to be diagonalizable. The necessary and sufficient condition is for $\boldsymbol{A}$ to have n linearly independent eigenvectors. With this result, we are now able to show that a matrix is not diagonalizable without having to resort to proofs by contradiction.

Slide 03: We had earlier shown that this matrix $\boldsymbol{M}$ was not diagonalizable but it was proven by contradiction which is cumbersome. Let us use the necessary and sufficient condition to help us instead. Notice that $\boldsymbol{M}$ is a 2×2 matrix with only one eigenvalue 2.

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The eigenspace E_2 is one dimensional, as we have seen in an earlier unit.

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Since there are no other eigenspaces to consider, this implies that $\boldsymbol{M}$ has only 1 linearly independent eigenvector and by the necessary and sufficient condition, this means that $\boldsymbol{M}$ is not diagonalizable.

Slide 04: Consider this matrix $\boldsymbol{A}$. We would like to find out if it is diagonalizable and if it is, find an invertible matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{A}$.

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We first notice that $\boldsymbol{A}$ has two distinct eigenvalues, namely 1 and 2. We are able to arrive at this immediately since $\boldsymbol{A}$ is a triangular matrix.

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Since $\boldsymbol{A}$ is a square matrix of order 3, let us investigate the two eigenspaces and see if we can find 3 linearly independent eigenvectors of $\boldsymbol{A}$.

Slide 05: We start with the eigenspace E_1.

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Solving the homogeneous linear system with coefficient matrix $(\boldsymbol{I} - \boldsymbol{A})$

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we start with the augmented matrix as shown

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and then arrive at the reduced row-echelon form.

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This gives us a general solution to the linear system, involving one arbitrary parameter t.

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So E_1 is one dimensional and is spanned by the vector $(1, -1, 8)$.

Slide 06: Moving on to the eigenspace E_2.

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We will solve the homogeneous linear system with coefficient matrix $(2\boldsymbol{I} - \boldsymbol{A})$.

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Starting with the augmented matrix as shown,

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we perform Gauss-Jordan elimination and arrive at the reduced row-echelon form.

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A general solution of the linear sytem is shown and it involves one arbitrary paramter.

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So E_2 is also one dimensional and is spanned by the vector $(0, 0, 1)$.

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Notice that the dimension of E_2 is 1 while in the factorisation of the characteristic polynomial, the exponent of the $(\lambda - 2)$ term was 2. This suggests that the dimension of the eigenspace E_2, which is 1, is smaller than what it is expected to be, if $\boldsymbol{A}$ is to be diagonalizable.

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Our conclusion is therefore that $\boldsymbol{A}$ has only two linearly independent eigenvectors

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while $\boldsymbol{A}$ is a square matrix of order 3. Thus, $\boldsymbol{A}$ is not diagonalizable.

Slide 07: To elaborate slightly on what was observed in the previous example. Recall that Step 1 of the algorithm requires us to solve for the roots of the characteristic equation. Suppose this was done and $\boldsymbol{A}$ has a total of k distinct eigenvalues λ_1, λ_2 to λ_k. It is obvious that $k \leq n$ since the characteristic polynomial is of order n and thus the characteristic equation can have at most n distinct roots.

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Another point to note is that when we solve for the roots of the characteristic equation, it is possible that we will encounter complex roots. This implies that some of the eigenvalues of $\boldsymbol{A}$ may be a complex number.

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When this happens, the vector space that we will need to work with will be the complex vector space $\mathbb{C}^n$ instead of the Euclidean space $\mathbb{R}^n$.

Slide 08: While we have seen a necessary and sufficient condition for a square matrix of order n to be diagonalizable, the following result gives a sufficient condition. If a square matrix $\boldsymbol{A}$ of order n has n distinct eigenvalues, then $\boldsymbol{A}$ will be diagonalizable. The proof of this result will not be discussed here.

Slide 09: Consider this matrix $\boldsymbol{A}$. With the result that we have just seen, it is easy to

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conclude that $\boldsymbol{A}$ is diagonalizable. This is because $\boldsymbol{A}$ is a 3×3 triangular matrix, whose eigenvalues can be easily seen to be 1, 2 and 3. Since $\boldsymbol{A}$ has 3 distinct eigenvalues, it satisfies the sufficient condition to be diagonalizable.

Slide 10: We return to the initial motivation for us to diagonalize a matrix, namely that once a square matrix $\boldsymbol{A}$ can be diagonalized, it would allow us to compute the powers of $\boldsymbol{A}$ efficiently.

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More precisely, if $\boldsymbol{A}$ is diagonalizable and $\boldsymbol{P}$ is an invertible matrix that diagonalizes $\boldsymbol{A}$, then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ is a diagonal matrix $\boldsymbol{D}$, whose diagonal entries are the eigenvalues of $\boldsymbol{A}$. We can therefore write $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$.

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Now $\boldsymbol{A}^k$ will be equal to $\boldsymbol{P}\boldsymbol{D}^k\boldsymbol{P}^{-1}$.

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Since $\boldsymbol{D}$ is a diagonal matrix, $\boldsymbol{D}^k$ can be computed easily as shown.

Slide 11: Consider the following 3×3 matrix $\boldsymbol{A}$. Suppose we wish to compute $\boldsymbol{A}^{10}$. Note that if $\boldsymbol{A}$ is not diagonalizable, then computing $\boldsymbol{A}^{10}$ would be a tedious task.

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Let us start by evaluating the characteristic polynomial of $\boldsymbol{A}$ as shown.

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By cofactor expansion, we have the following

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and after further simplication

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we managed to factorise the characteristic polynomial into 3 factors, namely $(\lambda - 1)$, $(\lambda + 1)$ and $(\lambda - 2)$.

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Thus the eigenvalues of $\boldsymbol{A}$ are -1, 1 and 2. Since $\boldsymbol{A}$ is 3×3 and has three distinct eigenvalues, we now know that it is diagonalizable.

Slide 12: As per usual practice, we need to examine and find a basis for each eigenspace of $\boldsymbol{A}$. Without showing the intermediate steps, we see here, that each eigenspace is one-dimensional and is spanned by a single vector.

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We now let $\boldsymbol{P}$ to be the 3×3 invertible matrix whose columns are the basis vectors from the eigenspaces.

Slide 13: With the matrix $\boldsymbol{P}$ chosen as such, we will have $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ equal to the diagonal matrix with the 3 eigenvalues as the diagonal entries.

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The matrix equation can be rewritten equivalently as $\boldsymbol{A}$ equals $\boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$.

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So now $\boldsymbol{A}^{10}$ is equal to $\boldsymbol{P}\boldsymbol{D}^{10}\boldsymbol{P}^{-1}$. Since $\boldsymbol{D}$ is a diagonal matrix, $\boldsymbol{D}^{10}$ will be just $\boldsymbol{D}$ with its diagonal entries raised to the power of 10.

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The product of the three matrices can be

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evaluated to give us $\boldsymbol{A}^{10}$ as desired.

Slide 14: To summarise this unit.

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We saw a sufficient condition for a $n \times n$ matrix to be diagonalizable.

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Through an example, we saw how $\boldsymbol{A}^n$ can be computed efficiently if $\boldsymbol{A}$ is diagonalizable.