COMPUTING INVERSE USING GAUSSIAN ELIMINATION

From a previous unit

Recall that for any matrix $\boldsymbol{A}$, there exists elementary matrices $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_k$ such that

$$\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1 \boldsymbol{A}$$

is the reduced row-echelon form of $\boldsymbol{A}$.

If $\boldsymbol{A}$ is invertible, we know that

$$\boxed{\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{I}_n}$$

From a previous unit

If $\boldsymbol{A}$ is invertible, we know that

$$\boxed{\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{I}_n}$$

To check whether a given square matrix $\boldsymbol{B}$ is the inverse of $\boldsymbol{A}$, we only need to check

either

$$\boldsymbol{AB} = \boldsymbol{I} \qquad \text{OR} \qquad \boldsymbol{BA} = \boldsymbol{I}$$

Since $(\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1)$ and $\boldsymbol{A}$ are both square matrices of the same size, we can conclude that

$$\boxed{(\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1) = \boldsymbol{A}^{-1}}$$

Finding the inverse of A

$$\boxed{\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{I}_n}$$

$$\boxed{(\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1) = \boldsymbol{A}^{-1}}$$

If $\boldsymbol{A}$ is a square matrix of order n, consider the following $n \times 2n$ matrix:

$$\left(\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{I}_n \end{array} \right)$$

Finding the inverse of A

$$\boldsymbol{E}_k\boldsymbol{E}_{k-1}\ldots\boldsymbol{E}_1\boldsymbol{A}=\boldsymbol{I}_n$$

$$(\boldsymbol{E}_k\boldsymbol{E}_{k-1}\ldots\boldsymbol{E}_1)=\boldsymbol{A}^{-1}$$

What if we premultiply $(\boldsymbol{E}_k\boldsymbol{E}_{k-1}\ldots\boldsymbol{E}_1)$ to this $n\times 2n$ matrix?

$$\left(\begin{array}{c|c}\boldsymbol{A} & \boldsymbol{I}_n\end{array}\right)$$

$$\begin{aligned}
&(\boldsymbol{E}_k\boldsymbol{E}_{k-1}\ldots\boldsymbol{E}_1)\left(\boldsymbol{A}\mid\boldsymbol{I}_n\right)\\
&=\left(\boldsymbol{E}_k\boldsymbol{E}_{k-1}\ldots\boldsymbol{E}_1\boldsymbol{A}\mid\boldsymbol{E}_k\boldsymbol{E}_{k-1}\ldots\boldsymbol{E}_1\boldsymbol{I}_n\right)\\
&=\left(\boldsymbol{I}_n\mid\boldsymbol{E}_k\boldsymbol{E}_{k-1}\ldots\boldsymbol{E}_1\right)\\
&=\left(\boldsymbol{I}_n\mid\boldsymbol{A}^{-1}\right)
\end{aligned}$$

Finding the inverse of A

$$
\begin{aligned}
&(\boldsymbol{E}_k\boldsymbol{E}_{k-1}\ldots\boldsymbol{E}_1)\left(\boldsymbol{A} \mid \boldsymbol{I}_n\right)\\
&=\left(\boldsymbol{E}_k\boldsymbol{E}_{k-1}\ldots\boldsymbol{E}_1\boldsymbol{A} \mid \boldsymbol{E}_k\boldsymbol{E}_{k-1}\ldots\boldsymbol{E}_1\boldsymbol{I}_n\right)\\
&=\left(\boldsymbol{I}_n \mid \boldsymbol{E}_k\boldsymbol{E}_{k-1}\ldots\boldsymbol{E}_1\right)\\
&=\left(\boldsymbol{I}_n \mid \boldsymbol{A}^{-1}\right)
\end{aligned}
$$

This provides us with a way to find the inverse of an invertible matrix $\boldsymbol{A}$.

Question:

What happens if the matrix $\boldsymbol{A}$ is not invertible?

Finding the inverse of A

$$
\begin{aligned}
&(\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1)\left(\boldsymbol{A} \mid \boldsymbol{I}_n\right) \\
&= \left(\boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1 \boldsymbol{A} \mid \boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1 \boldsymbol{I}_n\right) \\
&= \left(\boldsymbol{I}_n \mid \boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1\right) \\
&= \left(\boldsymbol{I}_n \mid \boldsymbol{A}^{-1}\right)
\end{aligned}
$$

when $\boldsymbol{A}$ is singular

$$= \left(\boldsymbol{R} \mid \boldsymbol{E}_k \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_1\right)$$

Question:

What happens if the matrix $\boldsymbol{A}$ is not invertible?

Answer:

If $\boldsymbol{A}$ is singular, then its reduced row-echelon form will not be the identity matrix.

Example

Determine if the following matrix is invertible and if so, find its inverse.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right)$$

Example

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 + 2R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{-R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

Example

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right) \xrightarrow[R_2+3R_3]{R_1-3R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$$\xrightarrow{R_1-2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right)$$

$\boldsymbol{A}$ is invertible and

$$\boldsymbol{A}^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

Example

Show that the following matrix is singular.

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 2 & 6 & 3 \\ 1 & -2 & -6 & -4 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 2 & 6 & 3 & 0 & 1 & 0 & 0 \\ 1 & -2 & -6 & -4 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{c|c} \boldsymbol{R} & * \end{array}\right)$$

$\boldsymbol{R} =$ reduced row-echelon of $\boldsymbol{A} \neq \boldsymbol{I}_4$

Thus the matrix $\boldsymbol{A}$ is singular.

Summary

1) A method to find the inverse of an invertible matrix.

2) A method to show that a matrix is singular.