W06-01

Slide 01: In this unit, we will introduce two fundamental subspaces associated with a matrix, namely the row space and the column space.

Slide 02: Given any $m \times n$ matrix $\boldsymbol{A}$,
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we will denote the rows of $\boldsymbol{A}$ by $\boldsymbol{r_1}$, $\boldsymbol{r_2}$ and so on till $\boldsymbol{r_m}$.
(#)
Each row of $\boldsymbol{A}$ can be considered as a vector. Since the matrix $\boldsymbol{A}$ has n columns, these vectors have n components and thus are vectors in $\mathbb{R}^n$.
(#)
The linear span of these row vectors of $\boldsymbol{A}$ will thus be a subspace of $\mathbb{R}^n$.
(#)
We call this subspace the row space of the matrix $\boldsymbol{A}$.

Slide 03: We can also do the same for the columns of $\boldsymbol{A}$.
(#)
Let $\boldsymbol{c_1}$, $\boldsymbol{c_2}$ until $\boldsymbol{c_n}$ be the columns of $\boldsymbol{A}$.
(#)
We can consider these columns as vectors and since each of them has m components, these are vectors in $\mathbb{R}^m$.
(#)
The linear span of these column vectors of $\boldsymbol{A}$ will be a subspace of $\mathbb{R}^m$.
(#)
We call this subspace the column space of the matrix $\boldsymbol{A}$.

Slide 04: Recall that when we transpose a matrix $\boldsymbol{A}$, the rows of $\boldsymbol{A}$ becomes the columns of $\boldsymbol{A}^T$ while the columns of $\boldsymbol{A}$ becomes the rows of $\boldsymbol{A}^T$.
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Thus, the row space of $\boldsymbol{A}$ is the column space of $\boldsymbol{A}^T$.
(#)
While the column space of $\boldsymbol{A}$ is the row space of $\boldsymbol{A}^T$.

Slide 05: Let us consider an example. The matrix $\boldsymbol{A}$ shown here is a 4×3 matrix.
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Thus the row space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^3$
(#)
and the column space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^4$.
(#)
Note that if $\boldsymbol{A}$ is not a square matrix, then the row space and column space of the matrix would contain totally different type of vectors, meaning that they would come from different Euclidean spaces.

Slide 06: Let us return to the matrix $\boldsymbol{A}$ shown earlier, where the row space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^3$.
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Let $\boldsymbol{r_1}$ to $\boldsymbol{r_4}$ denote the rows of $\boldsymbol{A}$, then the row space of $\boldsymbol{A}$ is the linear span of $\boldsymbol{r_1}$ to $\boldsymbol{r_4}$.

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More precisely, the row space of $\boldsymbol{A}$ contains all the possible linear combinations of $(2,1,0)$, $(1,-1,3)$, $(-5,1,0)$ and $(1,0,1)$. Notice that when we write $\boldsymbol{r_1}$ as a vector, we use commas to indicate that this is a vector and not a row matrix.

(#)

We can rewrite the set to obtain an expression for an arbitrary vector in the row space of $\boldsymbol{A}$. Here, a,b,c,d can take on any real numbers.

Slide 07: For the same matrix $\boldsymbol{A}$, the column space is a subspace of $\mathbb{R}^4$.

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Once again, if we represent the columns of $\boldsymbol{A}$ by $\boldsymbol{c_1}$, $\boldsymbol{c_2}$ and $\boldsymbol{c_3}$, then the column space of $\boldsymbol{A}$ will be the linear span of $\boldsymbol{c_1}$, $\boldsymbol{c_2}$ and $\boldsymbol{c_3}$.

(#)

This is essentially the set of all linear combinations of $(2,1,-5,1)$, $(-1,-1,1,0)$ and $(0,3,0,1)$.

(#)

The set can be rewritten to obtain an expression for an arbitrary vector in the column space of $\boldsymbol{A}$. Here, a,b,c can take on any real numbers.

Slide 08: Let us consider another matrix $\boldsymbol{A}$.

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It is easy to see that the row space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^5$,

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and the column space of $\boldsymbol{A}$ is a subspace of $\mathbb{R}^3$.

(#)

We are now interested in finding a basis for the row space of $\boldsymbol{A}$.

Slide 09: The row space of $\boldsymbol{A}$ is the linear span of the three row vectors of $\boldsymbol{A}$.

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In other words, we already know that these three row vectors span the row space. If these three row vectors are also linearly independent, then they will form a basis for the row space of $\boldsymbol{A}$.

Slide 10: To see if they are indeed linearly independent, we set up the vector equation as shown.

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By observation, looking at the first component leads to the conclusion that $a = 0$.

(#)

Looking at the second component tells us that $b = 0$.

(#)

And lastly looking at the third component tells us that $c = 0$.

(#)

Since the vector equation has only the trivial solution, the three rows of $\boldsymbol{A}$ are linearly independent and thus form a basis for the row space of $\boldsymbol{A}$.

Slide 11: Observe that for the matrix $\boldsymbol{A}$ which we have just seen, it was easy for us to conclude that the row vectors are linearly independent by making some quick observations. This was possible because $\boldsymbol{A}$ is actually in row-echelon form. The position of the leading entry in each row is such that we can quickly conclude that the vector equation has only the trivial solution.

(#)

This is not a coincidence. For any matrix $\boldsymbol{R}$ that is in row-echelon form, if we would like to find a basis for the row space of $\boldsymbol{R}$,

Slide 12: The non zero rows of $\boldsymbol{R}$ will always be linearly independent and thus forms a basis for the row space of $\boldsymbol{R}$. Notice that the zero rows found at the bottom of the matrix are zero vectors which can be disregarded.

Slide 13: To summarise the main points.

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We defined what is the row space and column space of a $m \times n$ matrix $\boldsymbol{A}$.

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We also saw that if the rows of $\boldsymbol{A}$ are linearly independent, since they span the row space of $\boldsymbol{A}$, they would form a basis for the row space.

(#)

In the event that a matrix $\boldsymbol{R}$ is in row-echelon form, then to find a basis for the row space of $\boldsymbol{R}$ is easy. We simply take the non zero rows of $\boldsymbol{R}$ in this case.