NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

Module: MA1508E Linear Algebra for Engineering
Year/Semester: 2018-2019 (Semester 2)
Tutorial: 3

1. For each of the following matrices $\boldsymbol{A}$, use elementary row operations to determine if $\boldsymbol{A}$ is invertible, and if so, find $\boldsymbol{A}^{-1}$. For the matrices that are invertible, express them as a product of elementary matrices.

   (a) $\begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{pmatrix}$

   (c) $\begin{pmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & -4 & 0 & 0 \\ 1 & 2 & 12 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & -4 & -5 \end{pmatrix}$

2. For each of the following matrices $\boldsymbol{B}$, find all values of k such that $\boldsymbol{B}$ is invertible and find the matrix $\boldsymbol{B}^{-1}$ (in terms of k).

   (a) $\begin{pmatrix} k & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{pmatrix}$ (c) $\begin{pmatrix} k & k & k \\ 1 & k & k \\ 1 & k & k \end{pmatrix}$

3. For each of the following matrices $\boldsymbol{C}$, find $\det(\boldsymbol{C})$ by cofactor expansion.

   (a) $\begin{pmatrix} 1 & 3 & 2 \\ 4 & 1 & -2 \\ 2 & 1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & -1 & 2 \\ 1 & 3 & 2 \\ 5 & 1 & 6 \end{pmatrix}$

   (c) $\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 0 \\ 1 & 1 & -2 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 1 & 2 & 1 \\ 3 & 0 & 1 & 1 \\ -1 & 2 & -2 & 1 \\ -3 & 2 & 3 & 1 \end{pmatrix}$.

4. Let $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$ be 2×2 matrices and let

$$\boldsymbol{C} = \begin{pmatrix} a_{11} & a_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad \boldsymbol{D} = \begin{pmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \boldsymbol{E} = \begin{pmatrix} 0 & \gamma_1 \\ \gamma_2 & 0 \end{pmatrix},$$

   where $\gamma_1, \gamma_2 \in \mathbb{R}$.

   (a) Show that $\det(\boldsymbol{A}+\boldsymbol{B}) = \det(\boldsymbol{A}) + \det(\boldsymbol{B}) + \det(\boldsymbol{C}) + \det(\boldsymbol{D})$.

(b) Show that if $\boldsymbol{B} = \boldsymbol{E}\boldsymbol{A}$, then $\det(\boldsymbol{A}+\boldsymbol{B}) = \det(\boldsymbol{A}) + \det(\boldsymbol{B})$.

5. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be square matrices of order n and $\boldsymbol{M}$ be the square matrix of order $2n$ defined as
$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0}_n \\ \boldsymbol{0}_n & \boldsymbol{B} \end{pmatrix}.$$
Use the result in Unit 18 (Equivalent Statements Part I), show that if either $\boldsymbol{A}$ or $\boldsymbol{B}$ is singular, then $\boldsymbol{M}$ must be singular.

6. Let $\boldsymbol{A}, \boldsymbol{C}, \boldsymbol{D}$ be square matrices of order n, and let $\boldsymbol{I}$ and $\boldsymbol{0}$ denote the identity and zero matrices of order n. Let $|\boldsymbol{X}|$ denote the determinant of $\boldsymbol{X}$. Show that

   (a) $\begin{vmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \end{vmatrix} = |\boldsymbol{A}|$.

   (Hint: Start by performing cofactor expansion along last row.)

   (b) $\begin{vmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{D} \end{vmatrix} = |\boldsymbol{D}|$.

   (Hint: Start by performing cofactor expansion along first row.)

   (c) $\begin{vmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{D} \end{vmatrix} = |\boldsymbol{A}|\,|\boldsymbol{D}|$.

   (Hint: Write the matrix as a product of two partitioned (block) matrices.)

   (d) $\begin{vmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{0} & \boldsymbol{D} \end{vmatrix} = |\boldsymbol{A}|\,|\boldsymbol{D}|$.

   (Hint: Consider the transpose of the matrix in part (c).)

(**Remark:** Once we have established part (d), the result in Question 5 can be obtained immediately.)