1. You started an experiment with 25 grams of a radioactive substance and after 25 hours you found that you still have 21.3 grams left. If the half life of that radioactive substance is T hours, find the value of T. Give your answer correct to the nearest integer.

$$\frac{dy}{dt} = -ky, \quad y(0) = 25$$

$$\Rightarrow \quad y = 25e^{-kt}$$

$$21.3 = 25e^{-25k} \Rightarrow \ln\frac{21.3}{25} = -25k$$

$$12.5 = 25e^{-kT} \Rightarrow \ln\frac{1}{2} = -kT$$

$$\therefore \quad \frac{\ln 21.3 - \ln 25}{-\ln 2} = \frac{25}{T}$$

$$T = \frac{25\ln 2}{\ln 25 - \ln 21.3}$$

$$= 108.190\cdots$$

$$\approx \underline{\underline{108}}$$

2. Let a and b denote two positive constants and let y denote a solution of $x\frac{dy}{dx} + 2y = 3ax + 2b$, $x > 0$. It is known that $y\left(\frac{1}{2}\right) = 20$, $y(1) = 9$ and $y(2) = 8$. Find the value of $y(5)$. Give your answer correct to two decimal places.

$$\frac{dy}{dx} + \frac{2}{x}y = 3a + \frac{2b}{x}, \quad x>0$$

$$R = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$y = \frac{1}{x^2}\int x^2\left(3a + \frac{2b}{x}\right)dx$$

$$= \frac{1}{x^2}\left\{ax^3 + bx^2 + c\right\} = ax + b + \frac{c}{x^2}$$

$$\begin{cases} \frac{1}{2}a + b + 4c = 20 \\ a + b + c = 9 \\ 2a + b + \frac{c}{4} = 8 \end{cases} \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 & 4 \\ 1 & 1 & 1 \\ 2 & 1 & \frac{1}{4} \end{pmatrix}^{-1} \begin{pmatrix} 20 \\ 9 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\therefore y = 2x + 3 + \frac{4}{x^2}$$

$$y(5) = 10 + 3 + \frac{4}{25} = \underline{\underline{13.16}}$$

3. Let a denote a positive constant and let y denote a solution of $\frac{dy}{dx} = \frac{3ax^3-2y^3}{3xy^2}$, $x > 0$. It is known that $[y(1)]^3 = 4$ and $[y(2)]^3 = \frac{97}{4}$. Find the value of $y(3)$. Give your answer correct to two decimal places.

$$\frac{dy}{dx} = \frac{ax^2}{y^2} - \frac{2}{3x}y \Rightarrow \frac{dy}{dx} + \frac{2}{3x}y = ax^2y^{-2}$$

Let $z = y^{1-(-2)} = y^3 \Rightarrow dz = 3y^2dy$

$$\therefore \frac{1}{3y^2}\frac{dz}{dx} + \frac{2}{3x}y = ax^2y^{-2}$$

$$\Rightarrow \frac{dz}{dx} + \frac{2}{x}z = 3ax^2$$

$$R = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$z = \frac{1}{x^2}\int x^2(3ax^2)\,dx = \frac{1}{x^2}\left\{\frac{3}{5}ax^5 + c\right\}$$

$$y^3 = \frac{3}{5}ax^3 + \frac{c}{x^2}$$

$$\begin{cases} \frac{3}{5}a + c = 4 \\ \frac{24}{5}a + \frac{c}{4} = \frac{97}{4} \end{cases} \Rightarrow \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & 1 \\ \frac{24}{5} & \frac{1}{4} \end{pmatrix}^{-1} \begin{pmatrix} 4 \\ \frac{97}{4} \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$\therefore y = \left(3x^3 + \frac{1}{x^2}\right)^{1/3}$$

$$y(3) = \left(81 + \frac{1}{9}\right)^{1/3} = 4.328\ldots \approx \underline{\underline{4.33}}$$

4. Let b and T denote two positive constants. At time $t = 0$ a cup of coffee which had a temperature of 80 ^oC was brought into a place that was kept at a constant temperature of T ^oC. Subsequently the temperature of the coffee was found to be 70 ^oC, 60 ^oC and 50 ^oC at time $t = b$ minutes, $t = 10$ minutes and $t = 20$ minutes respectively. Find the value of b. Give your answer correct to two decimal places.

$$\frac{dy}{dt} = -k(y-T), \quad y(0)=80$$

$$\Rightarrow (y-T) = (80-T)e^{-kt}$$

$$\therefore \begin{cases} 60-T=(80-T)e^{-10k} \\ 50-T=(80-T)e^{-20k} \end{cases}$$

$$\therefore \frac{60-T}{50-T} = e^{10k}$$

$$\therefore (60-T)^2 = (80-T)(50-T) \Rightarrow T=40$$

$$\therefore 20 = 40e^{-10k} \Rightarrow k = \frac{\ln 2}{10}$$

$$\therefore y-40 = 40e^{-\frac{\ln 2}{10}t}$$

$$30 = 40e^{-\frac{\ln 2}{10}b}$$

$$\therefore b = \frac{10(\ln 4 - \ln 3)}{\ln 2} = 4.150\cdots$$

$$\approx \underline{\underline{4.15}}$$