Second Week

1.4 Second order linear differential equations

Homogeneous d.e. with constant coefficients

Consider

$$y'' + ay' + by = 0, \;\; a, \;\; b \;\text{ constants.} \tag{1}$$

Example 15.

(i) Solve $y'' + 2y' + 5y = 0$.

$$y'' + 2y' + 5y = 0$$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$$

$$\therefore \underline{\underline{y = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x}}$$

Example 17. $y'' - 2y = 2x^3$.

Step 1

$y'' - 2y = 0 \quad \text{-----} \quad 1$

$\lambda^2 - 2 = 0$

$\lambda = \pm\sqrt{2}$

Step 2 Try $y = Ax^3 + Bx^2 + Cx + D$

$$\therefore y' = 3Ax^2 + 2Bx + C$$

$$y'' = 6Ax + 2B$$

$$\therefore y'' - 2y = -2Ax^3 - 2Bx^2 + (6A - 2C)x + (2B - 2D)$$

Compare coefficients

$$\begin{cases} -2A=2 \\ -2B=0 \\ 6A-2C=0 \\ 2B-2D=0 \end{cases}$$

$\therefore A=-1,\ B=0,\ C=-3,\ D=0$

$\therefore y=-x^3-3x$

Step 3 General solution of 1 is

$$y = c_1 e^{\sqrt{2}x} + c_2 e^{-\sqrt{2}x} - x^3 - 3x$$

(II) Method of variation of parameters

Variation of parameters

$$y'' + py' + qy = r \quad \text{----- (1)}$$

Suppose y_1, y_2 are linearly independent solution of

$$y'' + py' + qy = 0 \quad \text{----- (2)}$$

A particular solution is given by

$$y = u y_1 + v y_2$$

$$u = -\int \frac{y_2 r}{y_1 y_2' - y_1' y_2} \, dx,$$

$$v = \int \frac{y_1 r}{y_1 y_2' - y_1' y_2} \, dx.$$

Note. The term $y_1y_2' - y_1'y_2$ may be viewed as the determinant $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. It's called the Wronskian of y_1 and y_2.

Caution. When applying the above procedure to solve (1), make sure that the given d.e. is in standard form (1) where the coefficient of y'' is 1.

Example 22. Solve $y'' + y = \tan x$, $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

$$y'' + y = \tan x \quad \text{---} \; (*)$$

Step 1. $y'' + y = 0 \quad \text{---} \; (1)$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$y_1 = \cos x, \quad y_2 = \sin x$

are lin. indep. sol. of (1)

Step 2 Try $y = u\cos x + v\sin x$ in (*)

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$u = -\int \sin x \tan x \, dx$$

$$v = \int \cos x \tan x \, dx$$

$$u = -\int \frac{\sin^2 x}{\cos x}\,dx$$

$$= -\int \frac{1-\cos^2 x}{\cos x}\,dx$$

$$= \int (\cos x - \sec x)\,dx$$

$$= \sin x - \ln|\sec x + \tan x|$$

$$v = \int \cos x \tan x \, dx$$

$$= \int \sin x \, dx$$

$$= -\cos x$$

$$y = u y_1 + v y_2$$

$$= (\sin x - \ln|\sec x + \tan x|)\cos x$$

$$+ (-\cos x)\sin x$$

$$= -\cos x \ln|\sec x + \tan x|$$

Step 3 Gen. sol. of (*) is

$$y = c_1 \cos x + c_2 \sin x$$
$$- \cos x \ln|\sec x + \tan x|$$

Tutorial 2

2. If a cable is held up at two ends at the same height, then it will sag in the middle, making a U-shaped curve called a **catenary**. This is the shape seen in electricity cables suspended between poles, in countries less advanced than Singapore, such as Japan and the US. It can be shown using simple physics that if the shape is given by a function y(x), then this function satisfies

$$\frac{dy}{dx} = \frac{\mu}{T}\int_0^x \sqrt{\left(\frac{dy}{dt}\right)^2 + 1}\, dt,$$

where x = 0 at the lowest point of the catenary and y(0) = 0, where μ is the weight per unit length of the cable, and where T is the horizontal component of its tension; this horizontal component is a constant along the cable. Find a formula for the shape of the cable. [Hint: use the Fundamental Theorem of Calculus, and think of the resulting equation as a **first-order** ODE.]

Some notes on hyperbolic functions

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Note: (1) $\cosh^2 x - \sinh^2 x = 1$

(2) $\cosh^2 x + \sinh^2 x = \cosh 2x$

Proof: $\cosh^2 x = \frac{1}{4}\{e^{2x} + 2 + e^{-2x}\}$

$\sinh^2 x = \frac{1}{4}\{e^{2x} - 2 + e^{-2x}\}$ //

(3) $\frac{d}{dx} \cosh x = \sinh x$

(4) $\frac{d}{dx} \sinh x = \cosh x$

Other hyperbolic functions:

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}, \quad \operatorname{csch} x = \frac{1}{\sinh x}$$

To find $\frac{d}{dx} \sinh^{-1} x$

Let $y = \sinh^{-1} x$

$\therefore \sinh y = x$

$$\frac{d}{dx} \Rightarrow (\cosh y)\frac{dy}{dx} = 1$$

$\because \cosh^2 y - \sinh^2 y = 1$

and $\cosh y \geq 0$

$$\therefore \cosh y = \sqrt{1+\sinh^2 y} = \sqrt{1+x^2}$$

$$\therefore \frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}} \quad //$$