Week 11 F2F Example Solutions

1. **Example 10.1**

   (a) Let $\boldsymbol{P} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

   (b) $\boldsymbol{A}^{10} = \begin{pmatrix} 1 & 0 & 4^{10}-1 \\ 0 & 4^{10} & 0 \\ 0 & 0 & 4^{10} \end{pmatrix}$

   (c) For example, let $\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ and $\boldsymbol{B} = \boldsymbol{P}\boldsymbol{C}\boldsymbol{P}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Then $\boldsymbol{B}^2 = \boldsymbol{A}$.

2. **Example 10.2**

   (a) The eigenvalues are 2, 0, 1 and -1.

   (b) $\boldsymbol{u_1}$ is an eigenvector associated with 2.
   $\boldsymbol{u_2}$ is an eigenvector associated with 0.
   $\boldsymbol{u_3} + \boldsymbol{u_4}$ is an eigenvector associated with 1.
   $\boldsymbol{u_3} - \boldsymbol{u_4}$ is an eigenvector associated with -1.

   (c) Note that $\boldsymbol{u_1}$, $\boldsymbol{u_2}$, $\boldsymbol{u_3}$, $\boldsymbol{u_3}+\boldsymbol{u_4}$, $\boldsymbol{u_3}+\boldsymbol{u_4}$ are linearly independent eigenvectors. Since $\boldsymbol{B}$ is 4×4 and has 4 linearly independent eigenvectors, it is diagonalizable.

3. **Example 10.3**

   Use cofactor expansion along the first row:

$$d_n = \begin{vmatrix} 3 & 1 & & & & 0 \\ 1 & 3 & 1 & & & \\ & 1 & 3 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 3 & 1 \\ 0 & & & & 1 & 3 \end{vmatrix}_{n\times n}$$

$$= 3\begin{vmatrix} 3 & 1 & & & 0 \\ 1 & 3 & \ddots & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 3 & 1 \\ 0 & & & 1 & 3 \end{vmatrix}_{(n-1)\times(n-1)} - \begin{vmatrix} 1 & 1 & & & 0 \\ 0 & 3 & \ddots & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 3 & 1 \\ 0 & & & 1 & 3 \end{vmatrix}_{(n-1)\times(n-1)}.$$

The first determinant above is d_{n-1}. By using cofactor expansion along the first column, we find that the second determinant is d_{n-2}. So

$$d_n = 3d_{n-1} - d_{n-2}.$$

Note that $d_1 = 3$ and $d_2 = \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 8$.

By the procedure discussed in F2F10, we obtain

$$d_n = \left(\frac{5+3\sqrt{5}}{10}\right)\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{5-3\sqrt{5}}{10}\right)\left(\frac{3-\sqrt{5}}{2}\right)^n. \qquad (*)$$

Working for $(*)$ Let $\lambda_1 = \left(\frac{3+\sqrt{5}}{2}\right)$ and $\lambda_2 = \left(\frac{3-\sqrt{5}}{2}\right)$.

Then $d_n = A\lambda_1^2 + B\lambda_2^n$ for some $A, B \in \mathbb{R}$. When $n = 1, 2$, we have

$$\left\{\begin{array}{rcl} 3 & = & A\lambda_1 + B\lambda_2 \\ 8 & = & A\lambda_1^2 + B\lambda_2^2 \end{array}\right.$$

Subtracting the second equation from λ_1 times the first equation, we have

$$12\lambda_1 - 32 = B(6\sqrt{5} - 10) \Rightarrow B = \frac{14 - 6\sqrt{5}}{10 - 6\sqrt{5}} \times \frac{10 + 6\sqrt{5}}{10 + 6\sqrt{5}} \Rightarrow B = \left(\frac{5 - 3\sqrt{5}}{10}\right).$$

Solving for A, we now have

$$A = \left(\frac{5 + 3\sqrt{5}}{10}\right).$$