NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

SEMESTER II, 2018/2019
MA1508E MID-TERM TEST

Full Name : ________________________________

Matric/Student Number : _______________

Tutorial Group : ___________

INSTRUCTIONS
PLEASE READ CAREFULLY

- Write your **full name, matric number and tutorial group** clearly above on this cover page.
- There are **3** questions printed on **2** pages. Answer **all** questions.
- You must show all your working clearly, failure to do so will result in marks deducted.
- Use pen for this test.
- All answers and working have to be written on the answer book provided.
- Start on a new page for each question.
- Tie this cover page (and question paper) together with your answer book before submission.

Question 1

(a) Consider the following linear system where k is a constant.

$$\begin{cases} x & - & 3y & & & = & 6 \\ x & & & + & 3z & = & -3 \\ 2x & + & ky & + & (3-k)z & = & 1 \end{cases}$$

Find all values of k such that the linear system

(i) has exactly one solution.

(ii) has infinitely many solutions.

(iii) has no solution.

$$\left(\begin{array}{ccc|c} 1 & -3 & 0 & 6 \\ 1 & 0 & 3 & -3 \\ 2 & k & 3-k & 1 \end{array}\right) \xrightarrow[R_3-2R_1]{R_2-R_1} \left(\begin{array}{ccc|c} 1 & -3 & 0 & 6 \\ 0 & 3 & 3 & -9 \\ 0 & k+6 & 3-k & -11 \end{array}\right)$$

$$\xrightarrow{\frac{1}{3}R_2} \left(\begin{array}{ccc|c} 1 & -3 & 0 & 6 \\ 0 & 1 & 1 & -3 \\ 0 & k+6 & 3-k & -11 \end{array}\right) \xrightarrow{R_3-(k+6)R_2} \left(\begin{array}{ccc|c} 1 & -3 & 0 & 6 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & -2k-3 & 3k+7 \end{array}\right)$$

So the linear system

(i) exactly one solution when $k \neq -\frac{3}{2}$.

(ii) will not have infinitely many solutions for any value of k.

(iii) has no solution when $k = -\frac{3}{2}$.

(b) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be a 4×4 matrices such that $\det(\boldsymbol{A}) = 1508$ and $\det(\boldsymbol{B}) = -1508$. Determine if the following statements are true. If a statement is true, provide a proof. If a statement is false, provide an example of $\boldsymbol{A}$ and/or $\boldsymbol{B}$ (with the desired determinant) and $\boldsymbol{c}$ such that the statement is violated.

(i) For any 4×1 matrix $\boldsymbol{c}$, the equation $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{x} = \boldsymbol{c}$ has exactly one solution.

(ii) The homogeneous linear system $(\boldsymbol{A}-\boldsymbol{B})\boldsymbol{x} = \boldsymbol{0}$ always has infinitely many solutions.

(iii) For any 4×1 matrix $\boldsymbol{c}$, the equation $(\boldsymbol{AB})\boldsymbol{x} = \boldsymbol{c}$ is always consistent.

(i) The statement is false. For example, we may let

$$\boldsymbol{A}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1508\end{pmatrix}\quad \boldsymbol{B}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1508\end{pmatrix}\quad \boldsymbol{c}=\begin{pmatrix}2\\2\\2\\0\end{pmatrix}.$$

Then the linear system $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{x}=\boldsymbol{c}$,

$$\begin{pmatrix}2&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}2\\2\\2\\0\end{pmatrix}$$

has infinitely many solutions $x_1=1, x_2=1, x_3=1, x_4=t$, $t\in\mathbb{R}$.

(ii) The statement is false. For example, we may let

$$\boldsymbol{A}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1508\end{pmatrix}\quad \boldsymbol{B}=\begin{pmatrix}-1&0&0&0\\0&-1&0&0\\0&0&2&0\\0&0&0&-754\end{pmatrix}.$$

In this case, the linear system $(\boldsymbol{A}-\boldsymbol{B})\boldsymbol{x}=\boldsymbol{0}$,

$$\begin{pmatrix}2&0&0&0\\0&2&0&0\\0&0&-1&0\\0&0&0&2262\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

has only the trivial solution.

(iii) The statement is true. Since $\det(\boldsymbol{AB})=\det(\boldsymbol{A})\det(\boldsymbol{B})\neq 0$, the matrix $\boldsymbol{AB}$ is invertible. Thus for any $\boldsymbol{c}$, $\boldsymbol{ABx}=\boldsymbol{c}$ has a (unique) solution $\boldsymbol{x}=(\boldsymbol{AB})^{-1}\boldsymbol{c}$.

Question 2

(a) Let $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 3 \\ 1 & -1 & 2 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$.

(i) Find elementary matrices $\boldsymbol{E_1}, \boldsymbol{E_2}, \boldsymbol{E_3}$ such that

$$\boldsymbol{E_2}\boldsymbol{E_1}\boldsymbol{A} = \boldsymbol{E_3}\boldsymbol{B}.$$

(ii) Using the equation $\boldsymbol{E_2}\boldsymbol{E_1}\boldsymbol{A} = \boldsymbol{E_3}\boldsymbol{B}$, compute the determinant of $\boldsymbol{A}$. (You are not allowed to use other methods.) Explain why $\boldsymbol{A}$ is invertible.

(iii) Express $\boldsymbol{A}^{-1}$ as a product of **exactly six** elementary matrices. (**Hint:** $\boldsymbol{B}$ is a row-echelon form of $\boldsymbol{A}$.)

(i)

$$\boldsymbol{A} \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{pmatrix} \xrightarrow{\frac{1}{2}R_3} \boldsymbol{B}.$$

So the three elementary matrices are

$$\boldsymbol{E_1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \boldsymbol{E_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \boldsymbol{E_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(ii) Since $\det(\boldsymbol{B}) = 2$,

$$\frac{1}{2} \cdot 1 \cdot 1 \cdot \det(\boldsymbol{A}) = \det(\boldsymbol{B}) = 2.$$

Thus $\det(\boldsymbol{A}) = 4$.

(iii) We perform 3 more elementary row operations on $\boldsymbol{B}$:

$$\boldsymbol{B} \xrightarrow{\frac{1}{2}R_2} \xrightarrow{R_2 - \frac{3}{2}R_3} \xrightarrow{R_1 + R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So $\boldsymbol{A}^{-1} = \boldsymbol{E_6}\boldsymbol{E_5}\boldsymbol{E_4}\boldsymbol{E_3}^{-1}\boldsymbol{E_2}\boldsymbol{E_1}$ where $\boldsymbol{E_1}, \boldsymbol{E_2}, \boldsymbol{E_3}$ are given in part (i) and

$$\boldsymbol{E_3}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \quad \boldsymbol{E_4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \boldsymbol{E_5} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix} \quad \boldsymbol{E_6} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) Let $\boldsymbol{A}$ be a 3×3 matrix where the columns of $\boldsymbol{A}$ are $\boldsymbol{a_1}, \boldsymbol{a_2}$ and $\boldsymbol{a_3}$ respectively. That is,

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a_1} & \boldsymbol{a_2} & \boldsymbol{a_3} \end{pmatrix}.$$

Let $\boldsymbol{B}$ be another 3×3 matrix whose second and third column are the same as the second and third column of $\boldsymbol{A}$, that is

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{b_1} & \boldsymbol{a_2} & \boldsymbol{a_3} \end{pmatrix}.$$

Let $\boldsymbol{C}$ be the following 3×3 matrix:

$$\boldsymbol{C} = \begin{pmatrix} m\boldsymbol{a_1} + n\boldsymbol{b_1} & \boldsymbol{a_2} & \boldsymbol{a_3} \end{pmatrix}, \quad m, n \in \mathbb{R}.$$

That is, the first column of $\boldsymbol{C}$ is $m\boldsymbol{a_1} + n\boldsymbol{b_1}$ and the second and third columns are $\boldsymbol{a_2}$ and $\boldsymbol{a_3}$ respectively. Prove that

$$\det(\boldsymbol{C}) = m\det(\boldsymbol{A}) + n\det(\boldsymbol{B}).$$

First, note that $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ differ only in the first column. So when we compute the determinant of these 3 matrices by cofactor expansion (along the first column), the first column cofactors of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are all equal, denoted by A_{11}, A_{21} and A_{31}. Let a_{11}, a_{21} and a_{31} be the first column entries of $\boldsymbol{A}$. So

$$\det(\boldsymbol{A}) = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}.$$

Similarly, if b_{11}, b_{21} and b_{31} are the first column entries of $\boldsymbol{B}$, we have

$$\det(\boldsymbol{B}) = b_{11}A_{11} + b_{21}A_{21} + b_{31}A_{31}.$$

Now the first column entries of $\boldsymbol{C}$ are $ma_{11} + nb_{11}$, $ma_{21} + nb_{21}$ and $ma_{31} + nb_{31}$. So

$$\begin{aligned} \det(\boldsymbol{C}) &= (ma_{11} + nb_{11})A_{11} + (ma_{21} + nb_{21})A_{21} + (ma_{31} + nb_{31})A_{31} \\ &= m(a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}) + n(b_{11}A_{11} + b_{21}A_{21} + b_{31}A_{31}) \\ &= m\det(\boldsymbol{A}) + n\det(\boldsymbol{B}) \end{aligned}$$

Question 3

Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ where

$$\boldsymbol{u_1} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \quad \boldsymbol{u_2} = \begin{pmatrix} a \\ a \\ b \\ b \end{pmatrix}, \quad \boldsymbol{u_3} = \begin{pmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix},$$

where a, b are real numbers.

(i) If $a = 1, b = -1$, compute $\cos(\theta)$ where θ is the angle between $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$.

(ii) If $a = -\frac{1}{\sqrt{2}}$, $b = \frac{1}{\sqrt{2}}$, compute the distance between $\boldsymbol{u_2}$ and $\boldsymbol{u_3}$.

(iii) Find all values of a and b such that S is an orthonormal set.

(i) When $a = 1$, $b = -1$,

$$\boldsymbol{u_2} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \Rightarrow cos(\theta) = \frac{\boldsymbol{u_1} \cdot \boldsymbol{u_2}}{||\boldsymbol{u_1}||||\boldsymbol{u_2}||} = \frac{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}}{1 \cdot 2} = \frac{1}{\sqrt{2}}.$$

(ii) When $a = -\frac{1}{\sqrt{2}}$, $b = \frac{1}{\sqrt{2}}$,

$$\boldsymbol{u_2} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \Rightarrow \boldsymbol{u_2} - \boldsymbol{u_3} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad d(\boldsymbol{u_2}, \boldsymbol{u_3}) = ||\boldsymbol{u_2} - \boldsymbol{u_3}|| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1.$$

(iii) For S to be an orthogonal set,

$$\boldsymbol{u_2} \cdot \boldsymbol{u_1} = 0 \Rightarrow \frac{a}{\sqrt{2}} - \frac{b}{\sqrt{2}} = 0 \Rightarrow a = b.$$

It should be noted that $\boldsymbol{u_2} \cdot \boldsymbol{u_3} = 0$ would yield the same $a = b$ and $\boldsymbol{u_1} \cdot \boldsymbol{u_3} = 0$. For S to be an orthonormal set, the length of $\boldsymbol{u_2}$ must be 1.

$$||\boldsymbol{u_2}|| = 1 \Rightarrow \sqrt{a^2 + a^2 + b^2 + b^2} = 1 \Rightarrow \sqrt{a^2 + a^2 + a^2 + a^2} = 1 \Rightarrow \pm 2a = 1 \Rightarrow a = \pm\frac{1}{2}.$$

So the values of a, b such that S is an orthonormal set is $(a, b) = (\frac{1}{2}, \frac{1}{2})$ and $(a, b) = (-\frac{1}{2}, -\frac{1}{2})$.

END OF TEST