Week 12

MA1508E LINEAR ALGEBRA FOR ENGINEERING

Week 11

IVLE Quiz Discussion

Review of Week 11 (Units 055-059) content

- A real life example on population movement → main question: how to compute high powers of a square matrix?
- (Definition) Eigenvector of a square matrix $\boldsymbol{A}$. Eigenvalue of $\boldsymbol{A}$. Eigenvector associated with an eigenvalue.
- How to find all eigenvalues of a matrix $\boldsymbol{A}$ → characteristic polynomial and characteristic equation of $\boldsymbol{A}$.
- One more equivalent statement to "$\boldsymbol{A}$ is an invertible square matrix".
- Eigenvalues of triangular matrices.
- The collection of all eigenvectors associated with the same eigenvalue λ is a subspace → Eigenspace
- The eigenspace of $\boldsymbol{A}$ is also the nullspace of $(\lambda \boldsymbol{I} - \boldsymbol{A})$.

Review of Week 11 (Units 055-059) content

- Linking back to the population example. Fundamental question is: Given a square matrix $\boldsymbol{A}$, can we find an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{AP}$ is a diagonal matrix $\boldsymbol{D}$?
- (Definition) What is a diagonalizable matrix $\boldsymbol{A}$? When can we say a matrix diagonalizes $\boldsymbol{A}$?
- To prove that a matrix is not diagonalizable by contradiction → tedious!
- A necessary and sufficient condition for a square matrix of order n to be diagonalizable.
- An algorithm to determine (i) if a matrix $\boldsymbol{A}$ is diagonalizable (ii) and if it is, find an invertible matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{A}$.

Week 12 (units 060-064) overview

060 Diagonalization Part II

- A sufficient condition for a square matrix of order n to be diagonalizable.
- Computing powers of $\boldsymbol{A}$ when $\boldsymbol{A}$ is diagonalizable.

061 Systems of linear differential equations

- What is a system of LDE ($\boldsymbol{Y'} = \boldsymbol{AY}$)
- How to construct the solution set S of $\boldsymbol{Y'} = \boldsymbol{AY}$ and how this is related to the eigenvalues and eigenvectors of $\boldsymbol{A}$
- When $\boldsymbol{A}$ is a square matrix of order n, the solution set S of $\boldsymbol{Y'} = \boldsymbol{AY}$ is a n-dimensional vector space of functions.

062 Solving a system of LDE

- Solving a system of LDE with initial conditions

Week 12 (units 060-064) overview

063 Complex vectors – an introduction

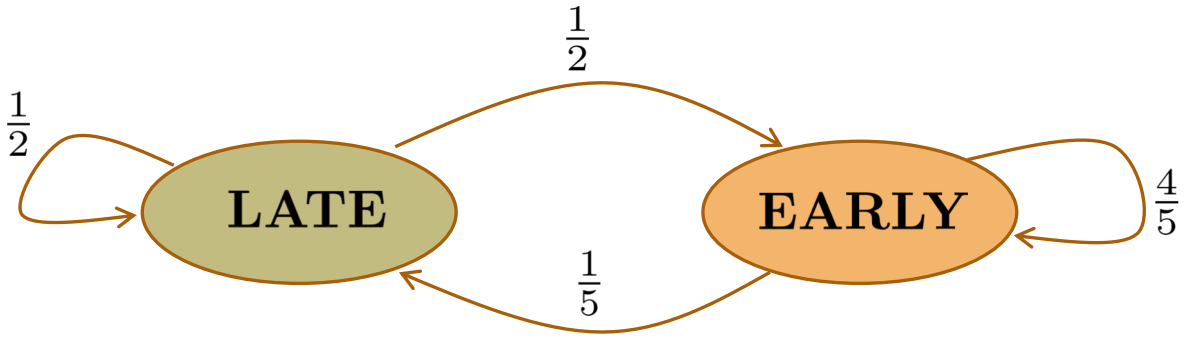
- A quick review on complex numbers (e.g. modulus, complex conjugate)
- Vectors in $\boldsymbol{C}^n$
- Complex dot product and some properties
- Complex eigenvalues always occur in pairs and so do conjugate eigenvectors.

064 More on system of LDE

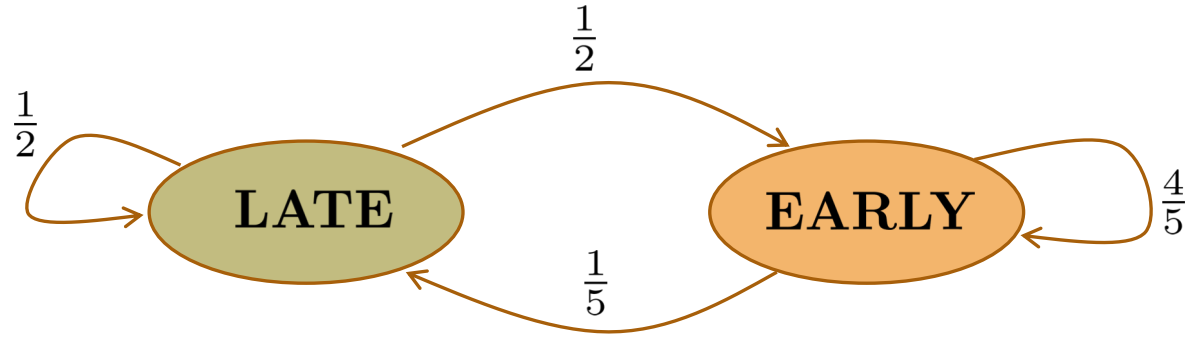
- System of LDE with 0 as an eigenvalue
- System of LDE with complex eigenvalues (and eigenvectors)

Example

A utility company finds that, in general, if a customer pays a bill late one month, there is about $\frac{1}{2}$ of the time that person will pay before the due date next month; and if a customer pays early one month, there are about $\frac{8}{10}$ of the time that person pays early again the following month. In January, the company finds that all 10,000 customers pay their bills on time. Estimate the number of customers that will pay on time in April. Will the number of customers that pay on time stabilize in the long run? If so, estimate the number of customers that pay on time each month in the long run.



Example



Let a_n (respectively, b_n) be the number of customers who pay late (respectively, early) in month n.

Then for $n = 1, 2, \ldots,$ $\quad a_n = \frac{1}{2}a_{n-1} + \frac{1}{5}b_{n-1} \qquad b_n = \frac{1}{2}a_{n-1} + \frac{4}{5}b_{n-1}$

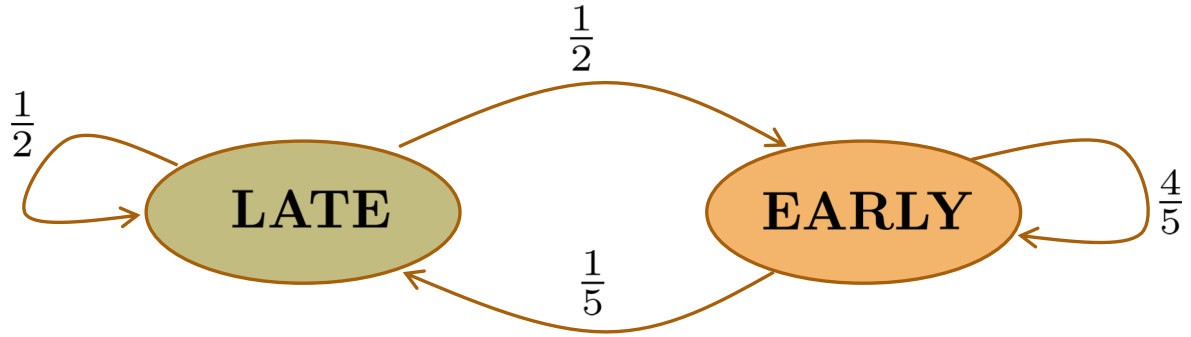
$$\begin{cases} a_n = \frac{1}{2}a_{n-1} + \frac{1}{5}b_{n-1} \\ b_n = \frac{1}{2}a_{n-1} + \frac{4}{5}b_{n-1}. \end{cases}$$

Let $\boldsymbol{x}_n = \begin{pmatrix} a_n \\ b_n \end{pmatrix}$ and $\boldsymbol{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{2} & \frac{4}{5} \end{pmatrix}$.

We will attempt to diagonalize $\boldsymbol{A}$.

Then $\boldsymbol{x_n} = \boldsymbol{A}\boldsymbol{x}_{n-1} = \cdots = \boldsymbol{A}^{n-1}\boldsymbol{x_1}$ where $\boldsymbol{x_1} = \begin{pmatrix} 0 \\ 10000 \end{pmatrix}$.

Example



$$\boldsymbol{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{2} & \frac{4}{5} \end{pmatrix}$$

Let $\boldsymbol{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\boldsymbol{P}^{-1} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$.

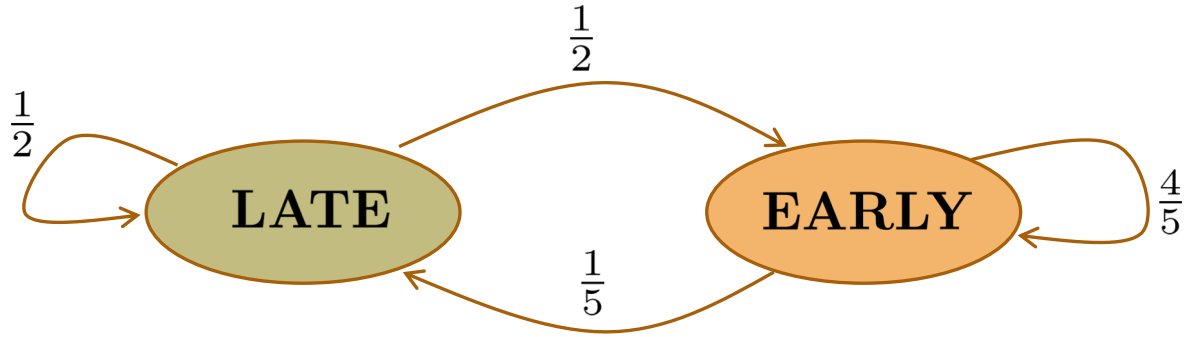
$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda - \frac{1}{2} & -\frac{1}{5} \\ -\frac{1}{2} & \lambda - \frac{4}{5} \end{vmatrix} = \lambda^2 - 1.3\lambda + 0.4 - 0.1$$

$$= \lambda^2 - 1.3\lambda + 0.3 = (\lambda - 1)(\lambda - 0.3)$$

So $\boldsymbol{A}$ has two distinct eigenvalues, $\lambda_1 = 1$ and $\lambda_2 = 0.3$.

Since $\boldsymbol{A}$ is 2×2 and has 2 distinct eigenvalues, we know that $\boldsymbol{A}$ is diagonalizable.
Let $\boldsymbol{P}$ be an invertible matrix of order 2 that diagonalizes $\boldsymbol{A}$.

Example



$$\boldsymbol{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{2} & \frac{4}{5} \end{pmatrix}$$ So $\boldsymbol{A}$ has two distinct eigenvalues, $\lambda_1 = 1$ and $\lambda_2 = 0.3$.

From earlier discussion, we have seen that: $\boldsymbol{A}^{n-1} = \boldsymbol{P}\boldsymbol{D}^{n-1}\boldsymbol{P}^{-1} = \boldsymbol{P}\begin{pmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{pmatrix}\boldsymbol{P}^{-1}$

$$\Rightarrow \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \boldsymbol{x_n} = \boldsymbol{A}^{n-1}\boldsymbol{x_1} = \begin{pmatrix} A\lambda_1^{n-1} + B\lambda_2^{n-1} \\ C\lambda_1^{n-1} + D\lambda_2^{n-1} \end{pmatrix} \text{ for some } A, B, C, D \in \mathbb{R}.$$

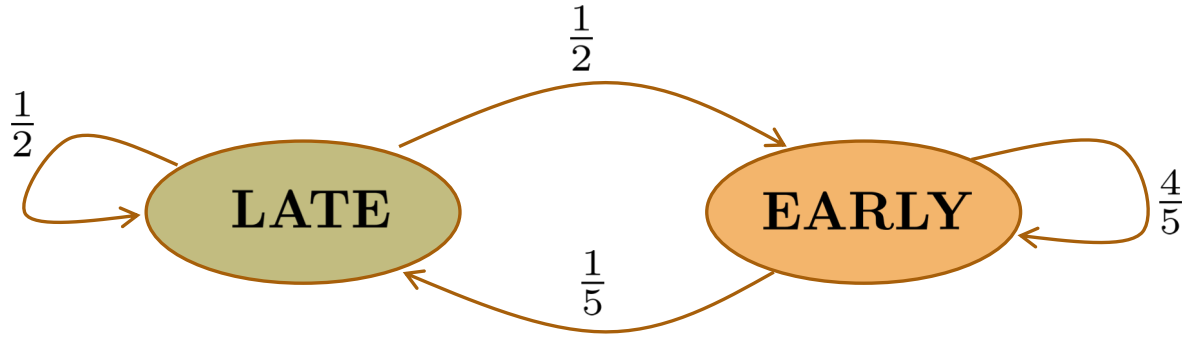
So $b_n = C(1)^{n-1} + D(0.3)^{n-1}$

$b_1 = 10000$, $b_2 = \frac{1}{2}(0) + \frac{4}{5}(10000) = 8000$

$$\begin{cases} 10000 = C + D \\ 8000 = C + 0.3D \end{cases}$$

$$\Rightarrow C = \frac{50000}{7}, \; D = \frac{20000}{7}$$

Example



$\boldsymbol{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{5} \\ \frac{1}{2} & \frac{4}{5} \end{pmatrix}$ So $\boldsymbol{A}$ has two distinct eigenvalues, $\lambda_1 = 1$ and $\lambda_2 = 0.3$.

From earlier discussion, we have seen that: $\boldsymbol{A}^{n-1} = \boldsymbol{P}\boldsymbol{D}^{n-1}\boldsymbol{P}^{-1} = \boldsymbol{P}\begin{pmatrix} \lambda_1^{n-1} & 0 \\ 0 & \lambda_2^{n-1} \end{pmatrix}\boldsymbol{P}^{-1}$

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$C = \frac{50000}{7}, \ D = \frac{20000}{7}$

$b_n = \frac{50000}{7}(1)^{n-1} + \frac{20000}{7}(0.3)^{n-1} = \frac{50000}{7} + \frac{20000}{7}(0.3)^{n-1}$

$b_4 \approx 7220$

As $n \to \infty$, $b_n \to \frac{50000}{7} \approx 7143$.

Higher Order Systems

Instead of a first-order system of linear differential equations $\boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}$, what happens if we have a second-order system of the form $\boldsymbol{Y}'' = \boldsymbol{A}_1\boldsymbol{Y} + \boldsymbol{A}_2\boldsymbol{Y}'$?

We can actually translate the second-order system into a first-order system as follows by defining

$$\begin{aligned} y_{n+1}(t) &= y_1'(t) \\ y_{n+2}(t) &= y_2'(t) \\ \vdots \quad & \quad \vdots \\ y_{2n}(t) &= y_n'(t) \end{aligned}$$

Higher Order Systems

$$\boxed{\boldsymbol{Y}'' = \boldsymbol{A}_1\boldsymbol{Y} + \boldsymbol{A}_2\boldsymbol{Y}'}$$

Let

$$\boldsymbol{Y}_1 = \boldsymbol{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad \text{and} \quad \boxed{\boldsymbol{Y}_2 = \boldsymbol{Y}'} = \begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ \vdots \\ y_{2n} \end{pmatrix}$$

$$\begin{aligned} y_{n+1}(t) &= y_1'(t) \\ y_{n+2}(t) &= y_2'(t) \\ &\vdots \\ y_{2n}(t) &= y_n'(t) \end{aligned}$$

Then $\boxed{\boldsymbol{Y}_1' = \boldsymbol{0}\boldsymbol{Y}_1 + \boldsymbol{I}_n\boldsymbol{Y}_2}$ and $\boxed{\boldsymbol{Y}_2' = \boldsymbol{Y}_1'' = \boldsymbol{A}_1\boldsymbol{Y}_1 + \boldsymbol{A}_2\boldsymbol{Y}_2}$

Higher Order Systems

The two equations $\boxed{\boldsymbol{Y}_1' = \mathbf{0}\boldsymbol{Y}_1 + \boldsymbol{I}_n\boldsymbol{Y}_2}$ and $\boxed{\boldsymbol{Y}_2' = \boldsymbol{A}_1\boldsymbol{Y}_1 + \boldsymbol{A}_2\boldsymbol{Y}_2}$ can be combined to give the first-order system with a $2n \times 2n$ matrix as shown:

$$\begin{pmatrix} \boldsymbol{Y}_1' \\ \boldsymbol{Y}_2' \end{pmatrix} = \begin{pmatrix} \boxed{\mathbf{0}_n \quad \boldsymbol{I}_n} \\ \boxed{\boldsymbol{A}_1 \quad \boldsymbol{A}_2} \end{pmatrix} \begin{pmatrix} \boldsymbol{Y}_1 \\ \boldsymbol{Y}_2 \end{pmatrix}$$

$\boldsymbol{Y}_1$(= the original $\boldsymbol{Y}$) and $\boldsymbol{Y}_2$(= the first derivative of $\boldsymbol{Y}$) can now be solved as per normal.

Example 11.1

Let $\boldsymbol{A}$ be a square matrix. The exponential of $\boldsymbol{A}$ is defined to be the matrix

$$e^{\boldsymbol{A}} = \boldsymbol{I} + \boldsymbol{A} + \frac{1}{2!}\boldsymbol{A}^2 + \frac{1}{3!}\boldsymbol{A}^3 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}\boldsymbol{A}^n$$

For each of the following, compute $e^{\boldsymbol{A}}$:

(a) $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

(b) $\boldsymbol{A} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

Example 11.2

In a large city, the soft-drink market was 100% dominated by brand A. Four months ago, two new brands B and C were introduced to the market. According to the market research, for each month, about 1% and 2% of the customers of brand A switch to brands B and C respectively; about 1% and 2% of the customers of brand B switch to brands A and C respectively.; and about 2% and 2% of the customers of brand C switch to brands A and B respectively. Compute the present market shares of the three brands of soft drink. Will the market shares stabilise in the long run if the trend continues? If so, estimatethe market shares in the long run.

Example 11.3

Solve the initial value problem:

$$\begin{aligned} y_1'' &= 2y_1 + y_2 + y_1' + y_2' \\ y_2'' &= -5y_1 + 2y_2 + 5y_1' - y_2' \end{aligned}$$

given the initial condition $y_1(0) = y_2(0) = y_1'(0) = 4$ and $y_2'(0) = -4$.

Finally…

THE END

Final examination

- Date: 30th April, Tuesday, EVENING (5-7pm)
- Venue: MPSH 1 (Section A)
- Duration: 2 hours.
- Number of questions: 8
- Closed Book Examination.
- One help sheet (same as mid term test).
- No programmable calculators.
- Write your answers in the answer books provided. You are allowed to use pencil to write your answers.
- Do not write your name on the answer books.
- Show all your working.
- Everything is included

Final administrative information

- Remember to verify your scores in IVLE for:
  - (10%) Combined IVLE quiz total (maximum score = 55)
  - (10%) Combined practice problem set total (maximum score = 60)
  - (10%) Combined F2F quiz total (maximum score = 58)
  - (20%) Mid Term Test score (maximum score = 40)
- Remember to bring your identification card for the examination.
- Stay healthy for the examination period – no make up examination