MATRIX INVERSE LAWS

CANCELLATION LAW

If A is an invertible square matrix and

$$AB_1 = AB_2 \implies A^{-1}AB_1 = A^{-1}AB_2$$
$$\implies IB_1 = IB_2$$

then $\boldsymbol{B}_1 = \boldsymbol{B}_2$.

If A is an invertible square matrix and

$$C_1A=C_2A\quad \Rightarrow C_1AA^{-1}=C_2AA^{-1}$$
 then $C_1=C_2$.
$$\Rightarrow C_1I=C_2I$$

CANCELLATION LAW

If \boldsymbol{A} is not an invertible matrix, the cancellation law may not hold:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$
 (known to be singular by previous example)

$$\boldsymbol{B}_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

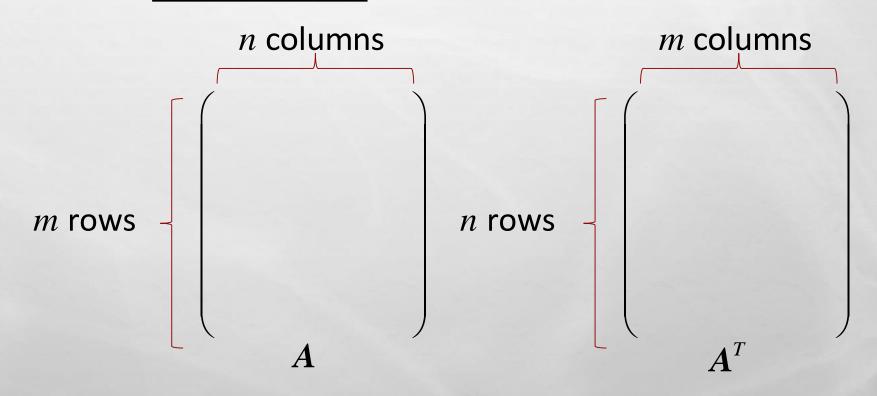
Check that $AB_1 = AB_2$, but $B_1 \neq B_2$.

$$\boldsymbol{B}_2 = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$$

TRANSPOSE OF A MATRIX

Let $A = (a_{ij})_{m \times n}$ be a $m \times n$ matrix.

The transpose of A, denoted by A^T , is a $n \times m$ matrix whose (i, j)-entry is a_{ji} .



EXAMPLE

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & -1 & 2 & 0 \\ 2 & 1 & 0 & 4 & 6 \\ 1 & -1 & 1 & 1 & 0 \end{pmatrix}$$

$$\mathbf{A}^{T} = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 2 & 4 & 1 \\ 0 & 6 & 0 \end{pmatrix}$$

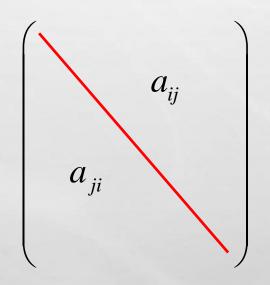
$$\boldsymbol{B} = \begin{pmatrix} 1 & -1 & 0 & 3 \\ -1 & 2 & 3 & 4 \\ 0 & 3 & -3 & 1 \\ 3 & 4 & 1 & 0 \end{pmatrix}$$

$$\boldsymbol{B}^{T} = \begin{pmatrix} 1 & -1 & 0 & 3 \\ -1 & 2 & 3 & 4 \\ 0 & 3 & -3 & 1 \\ 3 & 4 & 1 & 0 \end{pmatrix}$$

SYMMETRIC IN TERMS OF TRANSPOSE

A matrix A is symmetric if and only if

$$\boxed{\boldsymbol{A} = \boldsymbol{A}^T}$$



$$a_{ij} = a_{ji}$$
 for all i, j .

SOME RESULTS ON TRANSPOSE

Let A be a $m \times n$ matrix.

$$1) (A^T)^T = A$$

- 2) $(A + B)^T = A^T + B^T$ (transpose of sum equal sum of transpose).
- $3) (aA)^T = aA^T$
- 4) If \mathbf{B} is a $n \times p$ matrix, then $(\mathbf{A}\mathbf{B})^T = \mathbf{B}^T \mathbf{A}^T$ (transpose of product equal to product of transpose, but with order reversed).

1) Let A be an invertible matrix (so A^{-1} exists) and c a non zero scalar. Then cA is invertible and

$$(c\mathbf{A})^{-1} = \frac{1}{c}\mathbf{A}^{-1}$$

2) Let A be an invertible matrix (so A^{-1} exists) then

 A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$
.

Inverse of transpose equal transpose of inverse

$$(A^{-1})^T A^T$$

$$A^T(A^{-1})^T$$

$$= (AA^{-1})^T$$

$$= (\boldsymbol{A}^{-1}\boldsymbol{A})^T$$

$$=I$$

$$=I$$

3) Let A be an invertible matrix (so A^{-1} exists) then

 A^{-1} is invertible and

$$(A^{-1})^{-1} = A.$$

$$AA^{-1}$$

$$A^{-1}$$

$$=I$$

$$A^{-1}A$$

$$=I$$

4) Let A and B be two invertible matrices of the same size (so A^{-1} and B^{-1} exists) then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Inverse of product equal product of inverse*

$$B^{-1}A^{-1}$$

$$\mathbf{A}\mathbf{B}\mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$= \boldsymbol{B}^{-1}(\boldsymbol{A}^{-1}\boldsymbol{A})\boldsymbol{B}$$

$$= \mathbf{A}(\mathbf{B}\mathbf{B}^{-1})\mathbf{A}^{-1}$$

$$= \boldsymbol{B}^{-1}(\boldsymbol{I})\boldsymbol{B}$$

$$= \boldsymbol{A(I)}\boldsymbol{A}^{-1}$$

$$= \boldsymbol{B}^{-1} \boldsymbol{B} = \boldsymbol{I}$$

$$= \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$$

4) Let A and B be two invertible matrices of the same size (so A^{-1} and B^{-1} exists) then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Inverse of product equal product of inverse*

As an extension to the above, if $A_1, A_2, ..., A_k$ are invertible matrices of the same size, then $(A_1, A_2, ..., A_k)$ is invertible and

$$(A_1A_2...A_k)^{-1} = A_k^{-1}...A_2^{-1}A_1^{-1}.$$

DEFINITION

Let A be a square matrix and n be a non negative integer, then

$$A^{n} = \begin{cases} I & \text{if } n = 0; \\ \underbrace{AA...A}_{n \text{ times}} & \text{if } n \ge 1. \end{cases}$$

If A is an invertible square matrix, then A^n is invertible and we define

$$A^{-n} = (A^{-1})^n = \underbrace{A^{-1}A^{-1}...A^{-1}}_{n \text{ times}}$$
 and $(A^n)^{-1} = A^{-n}$

EXAMPLE

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \qquad \mathbf{A}^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 8 \\ 4 & 11 \end{pmatrix}$$

$$\mathbf{A}^{-2} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 11 & -8 \\ -4 & 3 \end{pmatrix}$$

$$\mathbf{A}^{2}\mathbf{A}^{-2} = \begin{pmatrix} 3 & 8 \\ 4 & 11 \end{pmatrix} \begin{pmatrix} 11 & -8 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{A}^{-2}\mathbf{A}^{2}$$

SUMMARY

1) Some laws involving the inverse of a matrix.

2) Transpose of a matrix and some laws.

3) Inverse of the powers of an invertible matrix.