MORE ON COLUMN SPACE AND RANK

COLUMN SPACE AND LINEAR SYSTEMS

$$\begin{cases} 2x - y = -1 \\ x - y + 3z = 4 \\ -5x + y = -2 \\ x + z = 3 \end{cases} \Leftrightarrow \boldsymbol{Ax} = \boldsymbol{b} \text{ where}$$

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\Leftrightarrow x\begin{pmatrix} 2 \\ 1 \\ -5 \\ 1 \end{pmatrix} + y\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ -2 \\ 3 \end{pmatrix}$$

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} -1 \\ 4 \\ -2 \\ 3 \end{pmatrix}$$

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(*)

$\boldsymbol{Ax} = \boldsymbol{b}$ is consistent means x, y, z can be found to satisfy (*)

$\Rightarrow$ $\boldsymbol{b}$ is a linear combination of the columns of $\boldsymbol{A}$.

That is, $\boldsymbol{b}$ belongs to the column space of $\boldsymbol{A}$.

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THEOREM

Let $\boldsymbol{A}$ be a $m \times n$ matrix. Then the column space of $\boldsymbol{A}$ is

$$\left\{ \boldsymbol{A}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \middle| \, u_1, u_2, \ldots, u_n \in \mathbb{R} \right\} = \left\{ \boldsymbol{A}\boldsymbol{u} \mid \boldsymbol{u} \in \mathbb{R}^n \right\}$$

A system of linear equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b}$ lies in the column space of $\boldsymbol{A}$.

EXAMPLE

Consider the following linear system

$$\begin{cases} 2x - y = 1 \\ x - y + 3z = 0 \\ -5x + y = 0 \\ x + z = 0 \end{cases}$$

Rewritting as $\boldsymbol{Ax} = \boldsymbol{b}$ where

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

EXAMPLE

Solving $\boldsymbol{Ax} = \boldsymbol{b}$ using Gaussian Elimination,

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 1 & -1 & 3 & 0 \\ -5 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

The linear system is inconsistent since the last column of a row echelon form of $(\boldsymbol{A} \mid \boldsymbol{b})$ is a pivot column.

is a row echelon form of $\boldsymbol{A}$, so $\operatorname{rank}(\boldsymbol{A}) = 3$.

EXAMPLE

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The linear system is inconsistent since the last column of a row echelon form of $(\boldsymbol{A} | \boldsymbol{b})$ is a pivot column.

Since the last column is a pivot column, $\text{rank}(\boldsymbol{A} | \boldsymbol{b}) = \text{rank}(\boldsymbol{A}) + 1 = 4$

REMARK

A linear system $\boldsymbol{Ax}=\boldsymbol{b}$ is consistent if and only if $\boldsymbol{A}$ and the augmented matrix $(\boldsymbol{A}\,|\,\boldsymbol{b})$ have the same rank.

SUMMARY

1) $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b}$ belongs to the column space of $\boldsymbol{A}$.

2) $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent if and only if $\text{rank}(\boldsymbol{A}) = \text{rank}(\boldsymbol{A} \mid \boldsymbol{b})$.