# Week 09

MA1508E LINEAR ALGEBRA FOR ENGINEERING

Week 8

# IVLE Quiz Discussion

# Review of Week 08 (Units 040-044) content

- For the same vector space, we know that bases are not unique, but will the different bases have the same number of vectors?
- Too many → linearly dependent, too few → cannot span
- Definition of the dimension of a vector space
- Finding a basis for the solution space of a homogeneous linear system
- If we know a basis for a vector space, determining whether another set is also a basis for the same vector space is easier.
- If $X$ is a subspace of $Y$, then the dimension of $X$ is no larger than the dimension of $Y$.
- Two more equivalent statements to "$\boldsymbol{A}$ is invertible" in terms of basis.

# Review of Week 08 (Units 040-044) content

- What is the row space and column space of a matrix?
- If ***A*** is not a square matrix, then the row space and column space are from entirely different Euclidean spaces.
- How to find a basis for the row space of a matrix? If the rows of ***A*** are already linearly independent (for example if ***A*** is in row-echelon form) then the (non-zero) rows already form a basis for the row space of ***A***.
- If two matrices are row equivalent, then their row spaces are identical.
- How can we use the result above to find a basis for the row space of ***A***?

# Week 09 (units 045-049) overview

## 045 Finding a basis for the column space

- Finding a basis for the column space of a matrix via the row space method (using transpose)
- Another method to find a basis for the column space of a matrix without using transpose

## 046 Rank of a matrix; one more equivalent statement

- The row space and column space of a matrix have the same dimension.
- Definition of the rank of a matrix
- $\boldsymbol{A}$ is invertible $\Leftrightarrow$ $\boldsymbol{A}$ is of full rank

## 047 More on column space and rank

- $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent if and only if $\boldsymbol{b}$ belongs to the column space of $\boldsymbol{A}$
- $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent if and only if $\text{rank}(\boldsymbol{A}) = \text{rank}(\boldsymbol{A}|\boldsymbol{b})$

# Week 09 (units 045-049) overview

## 048 Nullspace of a matrix

- The nullspace and nullity of a matrix
- Dimension Theorem for matrices

## 049 Orthogonal and orthonormal bases

- An orthogonal set of (non-zero) vectors is a linearly independent set
- Orthogonal basis and orthonormal basis
- Writing linear combinations in terms of orthogonal basis vectors

# Example

Let $\boldsymbol{u}_1 = (1,2,0,4)$, $\boldsymbol{u}_2 = (0,1,5,0)$, $\boldsymbol{u}_3 = (-1,3,2,-4)$, $\boldsymbol{u}_4 = (2,1,0,8)$, $\boldsymbol{u}_5 = (3,1,-1,12)$ and $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4, \boldsymbol{u}_5\}$. Find a basis for $V$.

# Method 1

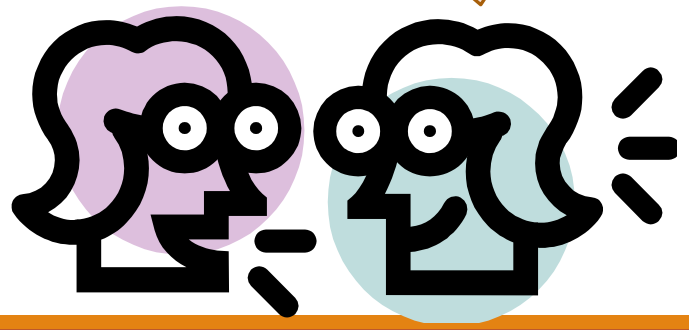
Let $\boldsymbol{u}_1 = (1,2,0,4)$, $\boldsymbol{u}_2 = (0,1,5,0)$, $\boldsymbol{u}_3 = (-1,3,2,-4)$,

$\boldsymbol{u}_4 = (2,1,0,8)$, $\boldsymbol{u}_5 = (3,1,-1,12)$ and $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4, \boldsymbol{u}_5\}$.

Find a basis for $V$ same as "Find a basis for the row space of $\boldsymbol{A}$".

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 5 & 0 \\ -1 & 3 & 2 & -4 \\ 2 & 1 & 0 & 8 \\ 3 & 1 & -1 & 12 \end{pmatrix}$.

Put the vectors as rows. So $V$ is the row space of the matrix $\boldsymbol{A}$.



# Method 1

$\{(1,2,0,4),(0,1,5,0),(0,0,-23,0)\}$ is a basis for $V$.

Let $\boldsymbol{u}_1=(1,2,0,4)$, $\boldsymbol{u}_2=(0,1,5,0)$, $\boldsymbol{u}_3=(-1,3,2,-4)$, $\boldsymbol{u}_4=(2,1,0,8)$, $\boldsymbol{u}_5=(3,1,-1,12)$ and $V=\text{span}\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3,\boldsymbol{u}_4,\boldsymbol{u}_5\}$.

Find a basis for $V$ same as "Find a basis for the row space of $\boldsymbol{A}$".

$$\boldsymbol{A}=\begin{pmatrix}1&2&0&4\\0&1&5&0\\-1&3&2&-4\\2&1&0&8\\3&1&-1&12\end{pmatrix}\xrightarrow[\text{Elimination}]{\text{Gaussian}}\begin{pmatrix}1&2&0&4\\0&1&5&0\\0&0&-23&0\\0&0&0&0\\0&0&0&0\end{pmatrix}=\boldsymbol{R}$$

# Method 2

Let $\boldsymbol{u}_1 = (1,2,0,4)$, $\boldsymbol{u}_2 = (0,1,5,0)$, $\boldsymbol{u}_3 = (-1,3,2,-4)$,

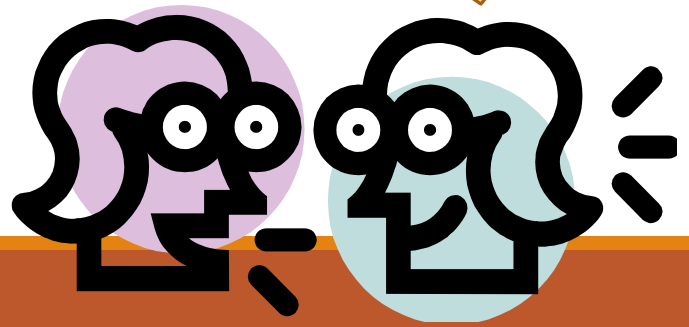
$\boldsymbol{u}_4 = (2,1,0,8)$, $\boldsymbol{u}_5 = (3,1,-1,12)$ and $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4, \boldsymbol{u}_5\}$.

Find a basis for $V$ same as "Find a basis for the column space of $\boldsymbol{B}$".

Let $\boldsymbol{B} = \begin{pmatrix} 1 & 0 & -1 & 2 & 3 \\ 2 & 1 & 3 & 1 & 1 \\ 0 & 5 & 2 & 0 & -1 \\ 4 & 0 & -4 & 8 & 12 \end{pmatrix}$.

Note that $\boldsymbol{B} = \boldsymbol{A}^T$ ($\boldsymbol{A}$ from Method 1)

Put the vectors as columns. So $V$ is the columns space of the matrix $\boldsymbol{B}$.



# Method 2

Let $\boldsymbol{u}_1 = (1,2,0,4)$, $\boldsymbol{u}_2 = (0,1,5,0)$, $\boldsymbol{u}_3 = (-1,3,2,-4)$,

$\boldsymbol{u}_4 = (2,1,0,8)$, $\boldsymbol{u}_5 = (3,1,-1,12)$ and $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4, \boldsymbol{u}_5\}$.

Find a basis for $V$ same as "Find a basis for the column space of $\boldsymbol{B}$".

$\{(1,2,0,4),(0,1,5,0),(-1,3,2,-4)\}$ is a basis for $V$.

$$\boldsymbol{B} = \begin{pmatrix} 1 & 0 & -1 & 2 & 3 \\ 2 & 1 & 3 & 1 & 1 \\ 0 & 5 & 2 & 0 & -1 \\ 4 & 0 & -4 & 8 & 12 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 1 & 0 & -1 & 2 & 3 \\ 0 & 1 & 5 & -3 & -5 \\ 0 & 0 & -23 & 15 & 24 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \boldsymbol{R}$$

# Method 1 vs Method 2

Let $\boldsymbol{u}_1 = (1,2,0,4)$, $\boldsymbol{u}_2 = (0,1,5,0)$, $\boldsymbol{u}_3 = (-1,3,2,-4)$,

$\boldsymbol{u}_4 = (2,1,0,8)$, $\boldsymbol{u}_5 = (3,1,-1,12)$ and $V = \text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4, \boldsymbol{u}_5\}$. Find a basis for $V$.

Method 1

$\{(1,2,0,4),(0,1,5,0),(0,0,-23,0)\}$ is a basis for $V$.

Method 2

$\{(1,2,0,4),(0,1,5,0),(-1,3,2,-4)\}$ is a basis for $V$.

$\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is a basis for $V$.

The basis found using Method 2 will always be a subset of $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4, \boldsymbol{u}_5\}$.

# Another relationship

We have already seen how the consistency of a linear system $\boldsymbol{Ax} = \boldsymbol{b}$ is related to the column space of $\boldsymbol{A}$.

It turns out that the entire solution set for $\boldsymbol{Ax} = \boldsymbol{b}$ can be obtained if we have the nullspace of $\boldsymbol{A}$ and a 'particular' solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

Nullspace of $\boldsymbol{A}$
(solution set/space for $\boldsymbol{Ax} = \boldsymbol{0}$)

A particular solution (say $\boldsymbol{v}$)
to $\boldsymbol{Ax} = \boldsymbol{b}$

Solution set for
$\boldsymbol{Ax} = \boldsymbol{b}$

# Theorem

Suppose the system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution $\boldsymbol{v}$. Then the solution set of the system is given by

$$M = \{\boldsymbol{u} + \boldsymbol{v} \mid \boldsymbol{u} \text{ is an element of the nullspace of } \boldsymbol{A}\}$$

That is, $\boldsymbol{Ax} = \boldsymbol{b}$ has a general solution

$$\boldsymbol{x} = (\text{a general solution for } \boldsymbol{Ax} = \boldsymbol{0})$$

$$+(\text{a particular solution to } \boldsymbol{Ax} = \boldsymbol{b})$$

# Example

Consider the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ where

$$\boldsymbol{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -3 \end{pmatrix}$$

It can be shown that the nullspace of $\boldsymbol{A}$ is $\left\{ s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \middle| \; s,t \in \mathbb{R} \right\}$

# Example

Check that

$$\boldsymbol{A}=\begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}=\begin{pmatrix} 2-2-1+1 \\ -1+1+2-3+1 \\ 1+1+1 \\ 1-1-2-1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \\ 3 \\ -3 \end{pmatrix}=\boldsymbol{b}$$

So is a particular solution to $\boldsymbol{Ax}=\boldsymbol{b}$.

# Example

nullspace of $\boldsymbol{A} = \left\{ s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad s,t \in \mathbb{R} \right\}$

$$\boldsymbol{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2-2-1+1 \\ -1+1+2-3+1 \\ 1+1+1 \\ 1-1-2-1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -3 \end{pmatrix} = \boldsymbol{b}$$

So the solution set of $\boldsymbol{Ax} = \boldsymbol{b}$ can be written as:

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \middle| \quad s,t \in \mathbb{R} \right\}$$

# Example

So the solution set of $\boldsymbol{Ax} = \boldsymbol{b}$ can be written as:

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \middle| \; s,t \in \mathbb{R} \right\}$$

A general solution to $\boldsymbol{Ax} = \boldsymbol{b}$ is

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad s,t \in \mathbb{R}$$

# Example 8.1

Let $W$ be the subspace of $\mathbb{R}^5$ spanned by the following vectors

$\boldsymbol{u}_1 = (1,-2,0,0,3)$ $\quad$ $\boldsymbol{u}_2 = (2,-5,-3,-2,6)$

$\boldsymbol{u}_3 = (0,5,15,10,0)$ $\quad$ $\boldsymbol{u}_4 = (2,1,15,8,6)$.

(a) Find a basis for $W$.

(b) What is the dimension for $W$?

(c) Extend the basis for $W$ found in (a) to a basis for $\mathbb{R}^5$.

# Example 8.2

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 & -1 & 1 \\ 1 & 1 & -1 & 2 & 0 \\ 3 & 1 & 0 & 0 & -1 \end{pmatrix}$.

(a) Find a basis for the row space of $\boldsymbol{A}$.

(b) Find a basis for the nullspace of $\boldsymbol{A}$.

(c) If $\boldsymbol{b} = \begin{pmatrix} -1 \\ 9 \\ 4 \end{pmatrix}$, show that $\boldsymbol{x} = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \\ 2 \end{pmatrix}$ is a solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

(d) Write down a general solution for $\boldsymbol{Ax} = \boldsymbol{b}$.

# Example 8.3

Let $\boldsymbol{u}_1 = (-2,-4,1)$, $\boldsymbol{u}_2 = (3,-1,2)$ and $\boldsymbol{u}_3 = (1,-1,-2)$.

(a) Show that $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

(b) Express $\boldsymbol{w} = (1,1,0)$ as a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$.

# Finally…

THE END