W03-03

Slide 01: In this unit we will discuss a set of equivalent statements. This first part of the discussion will be continued subsequently in future units.

Slide 02: Consider a square matrix $\boldsymbol{A}$. We will now present a set of 4 logically equivalent statements. What logically equivalent means that if one of the statements in the set is known to be true, then the rest of the statements in the set will also be true. Similarly, if one of the statements is found to be false, then the rest of the statements are also false. The first statement in the set is the simple statement that $\boldsymbol{A}$ is invertible.

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The second statement states that the homogeneous linear system $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ has only the trivial solution. Note that the square matrix $\boldsymbol{A}$ is the coefficient matrix of the linear system. Recall that all homogeneous linear systems either has only the trivial solution or has non trivial solutions in addition to the trivial one.

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The third statement asserts that the unique reduced row-echelon form of $\boldsymbol{A}$ is the identity matrix $\boldsymbol{I}$.

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The fourth statement says that the matrix $\boldsymbol{A}$ can be written as a product of elementary matrices. Recall that elementary matrices are those square matrices that are obtained when a single elementary row operation is performed on an identity matrix.

Slide 03: So how do we prove that the four statements are indeed equivalent?

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Our strategy is to first show that statement number 1 implies statement number 2. This means that if we assume that statement 1 is true, we will show that this implies that statement 2 must be true.

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This is followed by showing that if statement 2 is true, then it would imply that statement 3 must be true.

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We then continue to show that if statement 3 is true, then it would imply that statement 4 must be true.

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To complete the proof that the four statements are equivalent, we will show that if statement 4 is true, then it would imply that statement 1 must be true. It should be clear now that if the four implications are shown, then once any one of the four statements is true, the other three statements must also be true.

Slide 04: Let us start with proving the first implication, which is if $\boldsymbol{A}$ is a invertible square matrix, then $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ will have only the trivial solution.

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Since we assume that $\boldsymbol{A}$ is invertible, then we know that $\boldsymbol{A}^{-1}$ exists.

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Suppose $\boldsymbol{u}$ is a solution to $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$, so $\boldsymbol{A}\boldsymbol{u}$ must be $\mathbf{0}$.

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Pre-multiplying $\boldsymbol{A}^{-1}$ on both sides of the matrix equation, we have

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$\boldsymbol{u} = \boldsymbol{0}$.

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This means that the only solution $\boldsymbol{u}$ to the equation $\boldsymbol{Ax} = \boldsymbol{0}$ is the trivial solution $\boldsymbol{u} = \boldsymbol{0}$.

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In other words, $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution, which is statement 2.

Slide 05: Next, let us prove that statement 2 implies statement 3.

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Note that $\boldsymbol{A}$ is a square matrix of order n.

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So if statement 2 is true, meaning that $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution, if we were to solve the homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$,

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what do you think would be the reduced row-echelon form of the augmented matrix?

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It should be easy to see that the reduced row-echelon form of the augmented matrix would have an identity matrix on the left side of the vertical line.

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Clearly, this would mean that performing the same set of elementary row operations on $\boldsymbol{A}$ would reduce it to the identity matrix of order n and thus we have arrived at statement 3.

Slide 06: Moving on to the next implication.

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Suppose now that we assume that the reduced row-echelon form of $\boldsymbol{A}$ is the identity matrix of order n.

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Similar to the previous implication, we know there is a sequence of elementary row operations that can be performed on $\boldsymbol{A}$ to reduce it to $\boldsymbol{I}_n$. This means that there is a sequence of elementary matrices $\boldsymbol{E_1}$ to $\boldsymbol{E_k}$ such that $\boldsymbol{E_k E_{k-1}} \cdots \boldsymbol{E_2 E_1 A}$ is

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equal to the reduced row-echelon form of $\boldsymbol{A}$, which is $\boldsymbol{I}$.

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Recall from an earlier unit that all elementary matrices are invertible,

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thus in particular, we have $\boldsymbol{E_k}^{-1}$ as the inverse of $\boldsymbol{E_k}$ and suppose we pre-multiply $\boldsymbol{E_k}^{-1}$ to both sides of the matrix equation,

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we now have the following matrix equation. We can now repeat the same procedure by multiplying the inverse of $\boldsymbol{E_{k-1}}$

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on both sides of the equation

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which will result in the following.
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Continuing this way, we see that we have effectively moved all the elementary matrices, well, actually the inverses of the elementary matrices to the right hand side of the equation. The matrix equation now reads $\boldsymbol{A} = \boldsymbol{E_1}^{-1}\boldsymbol{E_2}^{-1}\cdots\boldsymbol{E_k}^{-1}$. It is important to remember at this point that the inverse of an elementary matrix is itself an elementary matrix, thus $\boldsymbol{A}$ is now shown to be a product of elementary matrices and thus statement 4 is established.

Slide 07: To complete the proof, we now show that statement 4 will imply statement 1.
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This is actually easy to see because since $\boldsymbol{A}$ is a product of elementary matrices as shown here, and also the fact that all elementary matrices are invertible,
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we now know that $\boldsymbol{A}$ is a product of invertible matrices.
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However, from a result established in an earlier unit, we know that the product of invertible matrices result in an invertible matrix, this means that $\boldsymbol{A}$ itself, as a product of invertible matrices, must be invertible too. This establishes statement 1.

Slide 08: How can we use this knowledge of the equivalence between these statements? For example, suppose we know, or have been told that $\boldsymbol{A}$ is an invertible square matrix.
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This would allow us to conclude that
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the other 3 statements in the collection is also correct, so for example, we can safely conclude that $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution; or that the reduced row-echelon form of $\boldsymbol{A}$ is $\boldsymbol{I}$.

Slide 09: Similarly, if we already know that $\boldsymbol{A}$ is a square matrix such that $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution, then
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we can conclude that
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$\boldsymbol{A}$ is invertible; or $\boldsymbol{A}$ can be expressed as a product of elementary matrices.

Slide 10: In summary,
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this unit gives a collection of four equivalent statements, including one that says that $\boldsymbol{A}$ is an invertible square matrix. In future units, we will be expanding this collection of equivalent statements by adding in new ones along the way.