W03-04

Slide 01: In this unit we will learn how to compute the inverse of an invertible square matrix. Through the same process, we will also learn how to conclude that a square matrix is singular.

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Slide 02: Recall the following discussion from a previous unit.

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We saw that for any matrix $\boldsymbol{A}$, performing a series of elementary row operations on $\boldsymbol{A}$, can be represented by pre-multiplying a series of elementary matrices to $\boldsymbol{A}$. Such a series of elementary matrices $\boldsymbol{E_1}, \boldsymbol{E_2}, ..., \boldsymbol{E_k}$ can be chosen such that $\boldsymbol{E_k E_{k-1} ... E_1 A}$ is the unique reduced row-echelon form of $\boldsymbol{A}$.

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We also know now that if $\boldsymbol{A}$ is invertible, this reduced row-echelon form of $\boldsymbol{A}$ will be the identity matrix.

Slide 03: Also in an earlier unit when we introduced the concept of invertible matrices, we stated that given a square matrix $\boldsymbol{B}$ of the same size as $\boldsymbol{A}$, in order to check whether $\boldsymbol{B}$ is the inverse of $\boldsymbol{A}$, it is sufficient to just check whether $\boldsymbol{AB}$ or $\boldsymbol{BA}$ is equal to $\boldsymbol{I}$ and once this has been established to be true, then the two matrices will be inverses of each other.

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Thus, from the matrix equation highlighted in blue, since $\boldsymbol{A}$ and the product of elementary matrices $\boldsymbol{E_k E_{k-1} ... E_1}$ are both square matrices of the same size, the matrix equation would imply that $\boldsymbol{A}^{-1}$ must be the product of these elementary matrices.

Slide 04: We now put what we already know at the top of this slide.

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Suppose $\boldsymbol{A}$ is a square matrix of order n. Consider the following $n \times 2n$ matrix with the left hand side being $\boldsymbol{A}$ and the right hand side being the identity matrix of order n.

Slide 05: We will now pre-multiply the entire sequence of elementary matrices $\boldsymbol{E_k E_{k-1} ... E_1}$

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to this $n \times 2n$ matrix.

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Note that pre-multiplying the sequence of elementary matrices to the $n \times 2n$ matrix can be considered as pre-multiplying to each of the $n \times n$ matrix on either side of the vertical line.

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We now use the first fact that pre-multiplying the sequence of elementary matrices to $\boldsymbol{A}$, which is equivalent to performing the sequence of elementary row operations on $\boldsymbol{A}$ would reduce the matrix $\boldsymbol{A}$ to the identity matrix. This is of course, assuming that $\boldsymbol{A}$ is an invertible matrix.

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Next, we recognise that the product of these elementary matrices is actually the inverse of $\boldsymbol{A}$, which is what you would observe on the right side of the vertical line at this point.

Slide 06: The above discussion clearly provides us with a way to find the inverse of an invertible matrix $\boldsymbol{A}$. To recap, to find $\boldsymbol{A}^{-1}$, we will construct a $n \times 2n$ matrix with $\boldsymbol{A}$ on the left side and $\boldsymbol{I}$ on the right side. Proceed to perform elementary row operations on this matrix until the left side of the matrix is reduced from $\boldsymbol{A}$ to $\boldsymbol{I}$. What results on the right side of the matrix is our desired $\boldsymbol{A}^{-1}$.

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A natural question that one may ask is, what happens if the matrix $\boldsymbol{A}$ that we are investigating turns out to be singular?

Slide 07: Well, we already know that $\boldsymbol{A}$ is invertible is equivalent to the fact that the reduced row-echelon form of $\boldsymbol{A}$ will be the identity matrix. Therefore, if $\boldsymbol{A}$ turns out to be singular, after we have pre-multiplied the sequence of elementary matrices to the $n \times 2n$ matrix, the left side of the matrix will result in the reduced row-echelon form of $\boldsymbol{A}$ which is not the identity matrix. This will then indicate to us that $\boldsymbol{A}$ is indeed singular.

Slide 08: Let us go through an example. Determine if the following matrix $\boldsymbol{A}$ is invertible, and if so, find its inverse.

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As discussed previously, we start by constructing a 3×6 matrix with $\boldsymbol{A}$ on the left and $\boldsymbol{I_3}$ on the right.

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We will proceed with performing elementary row operations on this matrix with the intention of reducing the matrix $\boldsymbol{A}$ on the left side of the vertical line to its reduced row-echelon form. In accordance to the Gauss-Jordan elimination, the first two elementary row operations performed are shown here.

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The current state of the 3×6 matrix is shown and we will continue with the elimination steps.

Slide 09: The next elementary row operation performed will be $R_3 + 2R_2$.

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And at this point you will observe that the left hand side is already in row-echelon form. At this point, we should be able to conclude that because a row-echelon form of the 3×3 matrix $\boldsymbol{A}$ has 3 leading entries, it is certain that $\boldsymbol{A}$ will have the identity matrix $\boldsymbol{I_3}$ as its reduced row-echelon form, meaning that it is indeed invertible. Since we would need to find $\boldsymbol{A}^{-1}$ anyway, we will need to proceed with more elimination steps.

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We first multiply row 3 by -1,

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resulting in this matrix.

Slide 10: Following three more elementary row operations, we arrive at the following and it is observed now that the matrix $\boldsymbol{A}$ on the left has been successfully reduced to the identity matrix $\boldsymbol{I_3}$.

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In accordance to our earlier discussion, we can conclude that $\boldsymbol{A}$ is invertible,

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and the 3×3 matrix found on the right hand side of the final matrix is precisely our $\boldsymbol{A}^{-1}$.

Slide 11: We will go through the same procedure to show that this 4×4 matrix is singular.

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As usual, we set up a 4×8 matrix as shown. While I will not show the details of the elimination performed on the matrix, I would urge you to try and perform Gaussian elimination to see the outcome.

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For this matrix, you should observe that after some elementary row operations, the left hand side is reduced to the reduced row-echelon form of $\boldsymbol{A}$,

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denoted by $\boldsymbol{R}$,

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which is not the identity matrix of order 4. Thus we are able to conclude that $\boldsymbol{A}$ is singular.

Slide 12: In summary,

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we have derived a method to find the inverse of an invertible matrix.

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Using the same method, we have also a way to show that a matrix is singular.