Unit 002 Geometrical interpretations

Slide 01: In this unit, we look at some geometrical interpretations of linear equations and their solutions.

Slide 02: Consider the linear equation $x + 2y = 2$. We have seen earlier that a general solution for the equation can be written in the following manner where s is an arbitrary parameter. As x takes on all possible values of s, we obtain infinitely many solutions $(s, \frac{1}{2}(2 - s))$, each being a point on the xy plane represented by the red dots.
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Thus the solution set of the equation contains all such points and these points together form the line $x + 2y = 2$.

Slide 03: What about the following linear equation in three variables? Note once again that in this equation, the coefficent of z is 0. We have also seen earlier that a general solution for the equation involves two arbitrary parameters s and t. What geometrical object do the solutions of this equation form?
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Before that, note the correspondence between the linear equation $x+2y+0z = 2$ which involves three variables x, y, z and the linear equation $x + 2y = 2$ which only involves two variables. While linear equation in three varaibles may sometimes be written by omitting the z term but we should never confuse that with a linear equation in two variables x and y only. As seen previously, the solutions of $x + 2y = 2$ form a straight line in the xy plane.
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To visualise the solutions to the linear equation $x + 2y + 0z = 2$, consider the plane $z = 0$ in the xyz space. You can think of this as the horizontal plane at the center of the z axis. Every point on this plane must have the z coordinate equal to 0.
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The points on this plane where $y = s$ and $x = 2 - 2s$ are represented by the red dots and as s takes on all possible real number values, the infinitely many such red dots will form a straight line that lies on the plane $z = 0$. Note that all the points on this line satisfy the equation $x + 2y + 0z = 2$, and furthermore, the value of z is equal to 0.

Slide 04: Similar considerations on another plane, say the plane $z = 1$ would yield another straight line on the plane, where all the points on this straight line satisfy the equation $x + 2y + 0z = 2$, and furthermore, the value of z is equal to 1.

Slide 05: Likewise, we can get another line on the plane $z = -1$.

Slide 06: The collection of all the straight lines described previously, one for each real number value that z can take, will give us a plane in the xyz space. This is precisely the plane with equation $x + 2y + 0z = 2$ which is made up of all the solutions to the linear equation.

Slide 07: Moving from one single linear equation to this example of a linear system with two equations in two variables x and y. We have already seen previously that each linear equation in two variables is in fact a line in the xy plane. Let us call the two lines l_1 and l_2. Since a solution to the equation representing l_1 is a point on the line, a solution to the linear system would be a point that lies on both l_1 and l_2.

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One possibility is that the two lines intersect in exactly one point. In this case, there is exactly one solution to the linear system.

Slide 08: It is also possible that the two lines are parallel and do not intersect. In this case, the linear system has no solution because there isn't any point that lies on both the lines l_1 and l_2.

Slide 09: The third and last possibility is that the two lines are exactly the same. In this case, there are infinitely many points on the same line which means there are infinitely many solutions to the linear system.

Slide 10: For the linear system in the previous example, there were three possibilities in terms of the number of solutions the system can have. It turns out that every linear system, regardless of how many equations there are and how many variables it involves, always has only three possibilities in terms of the number of solutions it has.

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In other words, it is not possible to have a linear system that has exactly three solutions.

Slide 11: So if we are told that a linear system has at least three solutions, it means that the system will have infinitely many solutions.

Slide 12: We have seen how two lines in the xy plane, represented by a linear system with two equations involving two variables can result in one of the three possibilities in terms of the number of solutions the system has. Let us consider the following linear system consisting of three linear equations involving three variables x, y and z. We have already seen that each of these equations represent a plane in the xyz space. Thus we have three planes in the xyz space, namely p_1, p_2 and p_3.

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Similarly, a solution to this linear system would be a point (x, y, z) that lies on all the three planes simultaneously.

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Now that we know that there are only three possibilities in terms of how many solutions the system has, can you think about how the three planes can be oriented in the xyz space which will result in each of the three possibilities? You are encouraged to discuss various scenarios with your friends and then with your lecturer or tutor when you meet him in class.

Slide 13: Let us recap the main points in this unit.

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We saw that the solutions of a linear equation in two variables form a line in the xy plane.

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The solutions of a linear equation in three variables, on the other hand, form a plane in the xyz space. Note that lines and planes are geometrical objects that we can visualise.

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The solutions to a linear system in two variables are essentially points, if any, that lies on every line whose equations are in the linear system. There are only three possibilities in terms of the number of solutions any linear system can have.

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Similarly, for a linear system in three variables, we are looking at the intersection of planes and solutions to the system would be points that lies on all the planes simultaneously.