BLOCK MULTIPLICATION

NOTATION

Let $\boldsymbol{A}$ be a $m \times p$ matrix, $\boldsymbol{B}$ be a $p \times n$ matrix.

Note that $\boldsymbol{AB}$ will be a $m \times n$ matrix.

Write $\boldsymbol{A}$ as

$$\boldsymbol{A} = \begin{pmatrix} \boxed{\quad \boldsymbol{a}_1 \quad} \\ \boxed{\quad \boldsymbol{a}_2 \quad} \\ \vdots \\ \vdots \\ \boxed{\quad \boldsymbol{a}_m \quad} \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} \boxed{\boldsymbol{b}_1} & \boxed{\boldsymbol{b}_2} & \cdots & \cdots & \boxed{\boldsymbol{b}_n} \end{pmatrix}$$

$$\boxed{\boldsymbol{b}_j = j\text{th column of } \boldsymbol{B}} = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ \vdots \\ b_{pj} \end{pmatrix}$$

$$\boxed{\boldsymbol{a}_i = i\text{th row of } \boldsymbol{A}} = \begin{pmatrix} a_{i1} & a_{i2} & \cdots & \cdots & a_{ip} \end{pmatrix}$$

NOTATION

Let $\boldsymbol{A}$ be a $m \times p$ matrix, $\boldsymbol{B}$ be a $p \times n$ matrix.

First way of computing $\boldsymbol{AB}$: entry by entry

$$\boldsymbol{AB} = \begin{pmatrix} \boldsymbol{a}_1\boldsymbol{b}_1 & \boldsymbol{a}_1\boldsymbol{b}_2 & \dots & \boldsymbol{a}_1\boldsymbol{b}_n \\ \boldsymbol{a}_2\boldsymbol{b}_1 & \boldsymbol{a}_2\boldsymbol{b}_2 & \dots & \boldsymbol{a}_2\boldsymbol{b}_n \\ \vdots & \vdots & & \vdots \\ \boldsymbol{a}_m\boldsymbol{b}_1 & \boldsymbol{a}_m\boldsymbol{b}_2 & \dots & \boldsymbol{a}_m\boldsymbol{b}_n \end{pmatrix}$$

$$\boldsymbol{a}_i\boldsymbol{b}_j = (i, j)\text{-entry of } \boldsymbol{AB} = \begin{pmatrix} a_{i1} & a_{i2} & \dots & a_{ip} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{pj} \end{pmatrix}$$

$$= \sum_{k=1}^{p} a_{ik} b_{kj}$$

NOTATION

Let $\boldsymbol{A}$ be a $m \times p$ matrix, $\boldsymbol{B}$ be a $p \times n$ matrix.

Second way of computing $\boldsymbol{AB}$: row by row

$$\boldsymbol{AB} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \vdots \\ \boldsymbol{a}_m \end{pmatrix} \boldsymbol{B} = \begin{pmatrix} \boldsymbol{a}_1\boldsymbol{B} \\ \boldsymbol{a}_2\boldsymbol{B} \\ \vdots \\ \vdots \\ \boldsymbol{a}_m\boldsymbol{B} \end{pmatrix}$$

$\boldsymbol{a}_i\boldsymbol{B} = i$th row of $\boldsymbol{AB}$ $= \begin{pmatrix} a_{i1} & a_{i2} & \dots & a_{ip} \end{pmatrix} \boldsymbol{B}$

NOTATION

Let $\boldsymbol{A}$ be a $m \times p$ matrix, $\boldsymbol{B}$ be a $p \times n$ matrix.

Third way of computing $\boldsymbol{AB}$: column by column

$$\boldsymbol{AB} = \boldsymbol{A}\begin{pmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \dots & \dots & \boldsymbol{b}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{Ab}_1 & \boldsymbol{Ab}_2 & \dots & \dots & \boldsymbol{Ab}_n \end{pmatrix}$$

$$\boldsymbol{Ab}_j = j\text{th column of } \boldsymbol{AB} = \boldsymbol{A}\begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ \vdots \\ b_{pj} \end{pmatrix}$$

EXAMPLE

Let $\boldsymbol{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & 2 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \\ 3 & 2 \end{pmatrix}$

$$\boldsymbol{A}\boldsymbol{b}_1 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}}$$

$$\boldsymbol{A}\boldsymbol{b}_2 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} 0 \\ -1 \\ 10 \end{pmatrix}}$$

$$\boldsymbol{A}\boldsymbol{B} = \begin{pmatrix} 1 & 0 \\ 5 & -1 \\ 3 & 10 \end{pmatrix}$$

EXAMPLE

Let $\boldsymbol{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & 2 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \\ 3 & 2 \end{pmatrix}$

$$\boldsymbol{a}_1\boldsymbol{B} = \begin{pmatrix} 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \\ 3 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 0 \end{pmatrix}}$$

$$\boldsymbol{a}_2\boldsymbol{B} = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \\ 3 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} 5 & -1 \end{pmatrix}}$$

$$\boldsymbol{a}_3\boldsymbol{B} = \begin{pmatrix} 0 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \\ 3 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} 3 & 10 \end{pmatrix}}$$

$$\boldsymbol{AB} = \begin{pmatrix} \boxed{1 \quad 0} \\ \boxed{5 \quad -1} \\ \boxed{3 \quad 10} \end{pmatrix}$$

DISCUSSION

The way the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are 'partitioned' does not have to be row by row, column by column.

$$\boldsymbol{A} = \begin{pmatrix} \boxed{\boldsymbol{a}_1} \\ \boxed{\boldsymbol{a}_2} \\ \vdots \\ \vdots \\ \boxed{\boldsymbol{a}_m} \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} \boxed{\boldsymbol{b}_1} & \boxed{\boldsymbol{b}_2} & \cdots & \cdots & \boxed{\boldsymbol{b}_n} \end{pmatrix}$$

$$\boxed{\boldsymbol{b}_j = j\text{th column of } \boldsymbol{B}} = \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ \vdots \\ b_{pj} \end{pmatrix}$$

$$\boxed{\boldsymbol{a}_i = i\text{th row of } \boldsymbol{A}} = \begin{pmatrix} a_{i1} & a_{i2} & \cdots & \cdots & a_{ip} \end{pmatrix}$$

BLOCK PARTITION

Consider the following 3×6 matrix $\boldsymbol{A}$ and its 6 submatrices.

$$\boldsymbol{A} = \left(\begin{array}{ccc|cc|c} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ \hline -8 & -6 & 3 & 1 & 7 & -4 \end{array}\right) = \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \boldsymbol{A}_{13} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \boldsymbol{A}_{23} \end{pmatrix}$$

$$\boldsymbol{A}_{11} = \begin{pmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \end{pmatrix} \qquad \boldsymbol{A}_{12} = \begin{pmatrix} 5 & 9 \\ 0 & -3 \end{pmatrix} \qquad \boldsymbol{A}_{13} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\boldsymbol{A}_{21} = \begin{pmatrix} -8 & -6 & 3 \end{pmatrix} \qquad \boldsymbol{A}_{22} = \begin{pmatrix} 1 & 7 \end{pmatrix} \qquad \boldsymbol{A}_{23} = \begin{pmatrix} -4 \end{pmatrix}$$

BLOCK PARTITION

$$\boldsymbol{A}=\left(\begin{array}{ccc|cc|c} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ \hline -8 & -6 & 3 & 1 & 7 & -4 \end{array}\right)=\begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \boldsymbol{A}_{13} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \boldsymbol{A}_{23} \end{pmatrix}$$

$$\boldsymbol{A}_{11}=\begin{pmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \end{pmatrix} \qquad \boldsymbol{A}_{12}=\begin{pmatrix} 5 & 9 \\ 0 & -3 \end{pmatrix} \qquad \boldsymbol{A}_{13}=\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

$$\boldsymbol{A}_{21}=\begin{pmatrix} -8 & -6 & 3 \end{pmatrix} \qquad \boldsymbol{A}_{22}=\begin{pmatrix} 1 & 7 \end{pmatrix} \qquad \boldsymbol{A}_{23}=\begin{pmatrix} -4 \end{pmatrix}$$

$\boldsymbol{A}_{11}, \boldsymbol{A}_{12}, \ldots, \boldsymbol{A}_{23}$ are called the blocks or submatrices of the matrix $\boldsymbol{A}$.

REMARK

If two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same size are partitioned in the same way, then the submatrices can be added or subtracted in the same way.

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \boldsymbol{A}_{13} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \boldsymbol{A}_{23} \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} & \boldsymbol{B}_{13} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} & \boldsymbol{B}_{23} \end{pmatrix}$$

$$\boldsymbol{A} + \boldsymbol{B} = \begin{pmatrix} \boldsymbol{A}_{11} + \boldsymbol{B}_{11} & \boldsymbol{A}_{12} + \boldsymbol{B}_{12} & \boldsymbol{A}_{13} + \boldsymbol{B}_{13} \\ \boldsymbol{A}_{21} + \boldsymbol{B}_{21} & \boldsymbol{A}_{22} + \boldsymbol{B}_{22} & \boldsymbol{A}_{23} + \boldsymbol{B}_{23} \end{pmatrix}$$

BLOCK MULTIPLICATION

Partitioned matrices can be multiplied by the usual row-column (matching) rule provided that for a product $\boldsymbol{AB}$, the column partition of $\boldsymbol{A}$ matches the row partition of $\boldsymbol{B}$.

Number of partitions the columns of $\boldsymbol{A}$ are partitioned into.

Number of partitions the rows of $\boldsymbol{B}$ are partitioned into.

EXAMPLE

$$\boldsymbol{A} = \left(\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & -4 & -2 & 7 & -1 \end{array}\right) = \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{pmatrix}$$

$$\boldsymbol{B} = \left(\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hline -1 & 3 \\ 5 & 2 \end{array}\right) = \begin{pmatrix} \boldsymbol{B}_{11} \\ \boldsymbol{B}_{21} \end{pmatrix}$$

$$\boldsymbol{AB} = \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{B}_{11} \\ \boldsymbol{B}_{21} \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{A}_{11}\boldsymbol{B}_{11} + \boldsymbol{A}_{12}\boldsymbol{B}_{21} \\ \boldsymbol{A}_{21}\boldsymbol{B}_{11} + \boldsymbol{A}_{22}\boldsymbol{B}_{21} \end{pmatrix}$$

EXAMPLE

$$\boldsymbol{AB} = \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{B}_{11} \\ \boldsymbol{B}_{21} \end{pmatrix}$$

$$= \begin{pmatrix} \boxed{\boldsymbol{A}_{11}\boldsymbol{B}_{11} + \boldsymbol{A}_{12}\boldsymbol{B}_{21}} \\ \boxed{\boldsymbol{A}_{21}\boldsymbol{B}_{11} + \boldsymbol{A}_{22}\boldsymbol{B}_{21}} \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 4 \\ -6 & 2 \\ ? & ? \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 4 \\ -6 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\boldsymbol{A}_{11}\boldsymbol{B}_{11} = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 5 & -2 \end{pmatrix} \begin{pmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 15 & 12 \\ 2 & -5 \end{pmatrix}}$$

$$\boldsymbol{A}_{12}\boldsymbol{B}_{21} = \begin{pmatrix} 0 & -4 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 5 & 2 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -20 & -8 \\ -8 & 7 \end{pmatrix}}$$

SUMMARY

1) How to consider a matrix as blocks or submatrices.

2) Matrix multiplication via rows or columns.

3) More generally, matrix multiplication by blocks.