NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

Module: MA1508E Linear Algebra for Engineering
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Tutorial: 8

1. For each of the following matrices $\boldsymbol{A}$, determine a basis for each of the following subspaces (i) row space of $\boldsymbol{A}$; (ii) row space of $\boldsymbol{A}^T$; (iii) nullspace of $\boldsymbol{A}$; (iv) nullspace of $\boldsymbol{A}^T$. State also the dimension of each of these subspaces.

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & -2 \\ 1 & 3 \\ 2 & 1 \\ 3 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 2 & 1 & 4 \end{pmatrix}$

(a) Basis for row space of $\boldsymbol{A} = \{(1,2,0),(0,0,1)\}$ (dimension is 2). Basis for row space of $\boldsymbol{A}^T$ (which is the column space of $\boldsymbol{A}$) $= \{(1,2),(3,0)\}$ (dimension is 2). Basis for the nullspace of $\boldsymbol{A} = \{(-2,1,0)\}$ (dimension is 1). Basis for the nullspace of $\boldsymbol{A}^T = \emptyset$ (nullspace of $\boldsymbol{A}^T$ is the zero space, whose dimension is 0).

(b) Basis for row space of $\boldsymbol{A} = \{(1,0),(0,1)\}$ (dimension is 2). Basis for row space of $\boldsymbol{A}^T = \{(4,1,2,3),(-2,3,1,4)\}$ (dimension is 2). Basis for nullspace of $\boldsymbol{A} = \emptyset$ (dimension is 0). Basis for the nullspace of $\boldsymbol{A}^T = \{(-\frac{5}{14},-\frac{4}{7},1,0),(-\frac{5}{14},-\frac{11}{7},0,1)\}$ (dimension is 2).

(c) Basis for row space of $\boldsymbol{A} = \{(1,0,0,0),(0,1,0,0),(0,0,1,1)\}$ (dimension is 3). Basis for the row space of $\boldsymbol{A}^T = \{(1,0,0,1),(0,1,0,1),(0,1,1,2)\}$ (dimension is 3). Basis for the nullspace of $\boldsymbol{A} = \{(0,0,-1,1)\}$ (dimension is 1). Basis for the nullspace of $\boldsymbol{A}^T = \{(-1,-1,-1,1)\}$ (dimension is 1).

(d) Basis for row space of $\boldsymbol{A} = \{(1,0,1),(0,1,2)\}$ (dimension is 2). Basis for row space of $\boldsymbol{A}^T = \{(1,1,2),(0,1,1)\}$ (dimension is 2). Basis for nullspace of $\boldsymbol{A} = \{(-1,-2,1)\}$ (dimension is 1). Basis for nullspace of $\boldsymbol{A}^T = \{(-1,-1,1)\}$ (dimension is 1).

2. For each of the following $\boldsymbol{A}$ and $\boldsymbol{b}$,

(i) Find a basis for the row space of $\boldsymbol{A}$.

(ii) Find a basis for the column space of $\boldsymbol{A}$.

(iii) Determine nullity($\boldsymbol{A}$). If nullity($\boldsymbol{A}$) > 0, find a basis for the nullspace of $\boldsymbol{A}$.

(iv) Solve the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ and express $\boldsymbol{b}$ as a linear combination of the columns of $\boldsymbol{A}$.

(v) If nullity($\boldsymbol{A}$) > 0, use the basis you obtained in part (iii) to write down the solution set of $\boldsymbol{Ax} = \boldsymbol{b}$.

(a) $\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 5 \\ 0 & 1 & 3 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$.

(b) $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -3 & 3 \\ 3 & 1 & 0 & 4 \\ 1 & 1 & 6 & -2 \\ 3 & 0 & -1 & 1 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 2 \\ 0 \\ -4 \\ -2 \end{pmatrix}$.

(c) $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 0 & 2 \\ -2 & -2 & 1 & -5 \\ 1 & 1 & -1 & 3 \\ 4 & 4 & -1 & 9 \end{pmatrix}$ and $\boldsymbol{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

(a) (i) Basis for row space of $\boldsymbol{A} = \{(1,0,0),(0,1,0),(0,0,1)\}$.

(ii) Basis for column space of $\boldsymbol{A} = \{(1,2,0),(-2,1,1),(0,5,3)\}$.

(iii) Nullity($\boldsymbol{A}$) $= 0$.

(iv) Let $\boldsymbol{a_i}$, $i = 1,2,3$ be the first, second and third columns of $\boldsymbol{A}$. Solving $\boldsymbol{b} = c_1\boldsymbol{a_1} + c_2\boldsymbol{a_2} + c_3\boldsymbol{a_3}$, we have $\boldsymbol{b} = -7\boldsymbol{a_1} - 4\boldsymbol{a_2} + 3\boldsymbol{a_3}$.

(b) (i) Basis for row space of $\boldsymbol{A} = \{(1,0,0,0),(0,1,0,4),(0,0,1,-1)\}$.

(ii) Basis for column space of $\boldsymbol{A} = \{(1,3,1,3),(0,1,1,0),(-3,0,6,-1)\}$.

(iii) Nullity($\boldsymbol{A}$) $= 1$. A basis for the nullspace of $\boldsymbol{A} = \{(0,-4,1,1)\}$.

(iv) There are infinitely many answers. One of them is $\boldsymbol{b} = -\boldsymbol{a_1} + 3\boldsymbol{a_2} - \boldsymbol{a_3} + 0\boldsymbol{a_4}$.

(v) The solution set of $\boldsymbol{Ax} = \boldsymbol{b}$ is

$$\left\{ \begin{pmatrix} -1 \\ 3 \\ -1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ -4 \\ 1 \\ 1 \end{pmatrix} \middle| s \in \mathbb{R} \right\}$$

(c) (i) Basis for row space of $\boldsymbol{A} = \{(1,1,0,2),(0,0,1,-1)\}$.

(ii) Basis for column space of $\boldsymbol{A} = \{(1,-2,1,4),(0,1,-1,-1)\}$.

(iii) Nullity($\boldsymbol{A}$) $= 2$. A basis for the nullspace of $\boldsymbol{A} = \{(-1,1,0,0),(-2,0,1,1)\}$.

(iv) There are infinitely many answers. One of them is $\boldsymbol{b} = -3\boldsymbol{a_1} + \boldsymbol{a_2} + \boldsymbol{a_3} + \boldsymbol{a_4}$.

(v) The solution set of $\boldsymbol{Ax} = \boldsymbol{b}$ is

$$\left\{ \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \middle| s, t \in \mathbb{R} \right\}.$$

3. Let $\boldsymbol{A}$ be a 3×4 matrix. Suppose $x_1 = 1$, $x_2 = 0$, $x_3 = -1$, $x_4 = 0$ is a solution to a non-homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{b}$ and that the homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$ has a general solution $x_1 = t - 2s$, $x_2 = s + t$, $x_3 = s$, $x_4 = t$, where s, t are arbitrary parameters.

   (a) Find a basis for the nullspace of $\boldsymbol{A}$ and determine the nullity of $\boldsymbol{A}$.

   (b) Find a general solution for the system $\boldsymbol{Ax} = \boldsymbol{b}$.

   (c) Write down the reduced row-echelon form of $\boldsymbol{A}$.

   (d) Find a basis for the row space of $\boldsymbol{A}$ and determine the rank of $\boldsymbol{A}$.

   (e) Do we have enough information for us to find the column space of $\boldsymbol{A}^T$?

   (a) Since $(x_1, x_2, x_3, x_4)^T = (t - 2s,\ s + t,\ s,\ t)^T = s(-2, 1, 1, 0)^T + t(1, 1, 0, 1)^T$, $\{(-2, 1, 1, 0)^T,\ (1, 1, 0, 1)^T\}$ is a basis for the nullspace of $\boldsymbol{A}$. The nullity of $\boldsymbol{A}$ is 2.

   (b) A general solution of $\boldsymbol{Ax} = \boldsymbol{b}$ is $x_1 = t - 2s + 1$, $x_2 = s + t$, $x_3 = s - 1$, $x_4 = t$ where s, t are arbitrary parameters.

   (c) The reduced row-echelon form of $\boldsymbol{A}$ is $\begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

   (d) $\{(1, 0, 2, -1),\ (0, 1, -1, -1)\}$ is a basis for the row space of $\boldsymbol{A}$. The rank of $\boldsymbol{A}$ is 2.

   (e) We know everything about the column space of $\boldsymbol{A}^T$, since that is just the row space of $\boldsymbol{A}$. (If we are asked about the column space of $\boldsymbol{A}$, however, we wouldn't be able to find the column space of $\boldsymbol{A}$ with the given information.)

4. Let $\boldsymbol{A}$ be a $m \times n$ matrix. Show that

   (a) If $\boldsymbol{x}$ belongs to the nullspace of $\boldsymbol{A}^T\boldsymbol{A}$, then $\boldsymbol{Ax}$ belongs to both the column space of $\boldsymbol{A}$ and the nullspace of $\boldsymbol{A}^T$.

   (b) Nullspace of $\boldsymbol{A}^T\boldsymbol{A}$ is equal to the nullspace of $\boldsymbol{A}$.

   (c) Rank of $\boldsymbol{A}$ is equal to the rank of $\boldsymbol{A}^T\boldsymbol{A}$.

   (d) If $\boldsymbol{A}$ has linearly independent columns, then $\boldsymbol{A}^T\boldsymbol{A}$ is invertible.

   (a) $\boldsymbol{Ax}$ is always a vector in the column space of $\boldsymbol{A}$. If $\boldsymbol{x}$ belongs to the nullspace of $\boldsymbol{A}^T\boldsymbol{A}$, then $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{0}$ which implies $\boldsymbol{A}^T(\boldsymbol{Ax}) = \boldsymbol{0}$. So $\boldsymbol{Ax}$ belongs toe the nullspace of $\boldsymbol{A}^T$.

(b) Let $\boldsymbol{u}$ be any vector in the nullspace of $\boldsymbol{A}$, i.e. $\boldsymbol{A}\boldsymbol{u} = \mathbf{0}$. Then $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{u} = \boldsymbol{A}^T\mathbf{0} = \mathbf{0}$. So $\boldsymbol{u}$ is also a vector in the nullspace of $\boldsymbol{A}^T\boldsymbol{A}$.
We have shown that the nullspace of $\boldsymbol{A}$ is a subspace of the nullspace of $\boldsymbol{A}^T\boldsymbol{A}$.

Let $\boldsymbol{v}$ be any vector in the nullspace of $\boldsymbol{A}^T\boldsymbol{A}$, i.e. $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{v} = \mathbf{0}$. Suppose $\boldsymbol{A}\boldsymbol{v} = (b_1, b_2, \ldots, b_m)^T$. Then

$$\begin{array}{rc} & (\boldsymbol{A}\boldsymbol{v})^T(\boldsymbol{A}\boldsymbol{v}) = \boldsymbol{v}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{v} = \boldsymbol{v}^T\mathbf{0} = \mathbf{0} \\ \Rightarrow & b_1^2 + b_2^2 + \cdots + b_m^2 = 0 \\ \Rightarrow & b_1 = b_2 = \cdots = b_m = 0. \end{array}$$

That is, $\boldsymbol{A}\boldsymbol{v} = \mathbf{0}$. So $\boldsymbol{v}$ is also a vector in the nullspace of $\boldsymbol{A}$.
We have shown that the nullspace of $\boldsymbol{A}^T\boldsymbol{A}$ is a subspace of the nullspace of $\boldsymbol{A}$.

Hence the nullspace of $\boldsymbol{A}$ is equal to the nullspace of $\boldsymbol{A}^T\boldsymbol{A}$.

(c) By (b), $\text{nullity}(\boldsymbol{A}) = \text{nullity}(\boldsymbol{A}^T\boldsymbol{A})$.

Since $\boldsymbol{A}$ is an $m \times n$ matrix, $\boldsymbol{A}^T\boldsymbol{A}$ is an $n \times n$ matrix. By the Dimension Theorem for Matrices,

$$\text{rank}(\boldsymbol{A}) = n - \text{nullity}(\boldsymbol{A}) = n - \text{nullity}(\boldsymbol{A}^T\boldsymbol{A}) = \text{rank}(\boldsymbol{A}^T\boldsymbol{A}).$$

(d) If $\boldsymbol{A}$ has linearly independent columns, then in any row-echelon form of $\boldsymbol{A}$, every column will be a pivot column. This implies that $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ has only the trivial solution, that is, the nullspace of $\boldsymbol{A}$ is the zero space. By part (b), the nullspace of $\boldsymbol{A}^T\boldsymbol{A}$ is also the zero space. Since $\boldsymbol{A}^T\boldsymbol{A}$ is a square matrix, this implies that $\boldsymbol{A}^T\boldsymbol{A}$ must be invertible.

5. Let $\boldsymbol{w} = (0, 1, 2, 3)$.

(a) Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$ where $\boldsymbol{u_1} = (2, 1, 0, 0)$, $\boldsymbol{u_2} = (-1, 0, 0, 1)$, $\boldsymbol{u_3} = (2, 0, -1, 1)$, $\boldsymbol{u_4} = (0, 0, 1, 1)$. Show that S is a basis for $\mathbb{R}^4$. Is S an orthogonal basis for $\mathbb{R}^4$? Compute $(\boldsymbol{w})_S$.

(b) Let $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$ where $\boldsymbol{v_1} = (1, 2, 2, -1)$, $\boldsymbol{v_2} = (1, 1, -1, 1)$, $\boldsymbol{v_3} = (-1, 1, -1, -1)$, $\boldsymbol{v_4} = (-2, 1, 1, 2)$. Show that T is a basis for $\mathbb{R}^4$. Is T an orthogonal basis for $\mathbb{R}^4$? Compute $(\boldsymbol{w})_T$.

(a) We can show that S is a basis for $\mathbb{R}^4$ by checking that the following matrix is invertible:

$$\begin{vmatrix} 2 & -1 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix} = -4 \neq 0.$$

S is not an orthogonal basis for $\mathbb{R}^4$ since, for example, $\boldsymbol{u_1} \cdot \boldsymbol{u_2} \neq 0$. To compute $(\boldsymbol{w})_S$, we solve the system $\boldsymbol{w} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + c_3\boldsymbol{u_3} + c_4\boldsymbol{u_4}$:

$$\left(\begin{array}{cccc|c} 2 & -1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 1 & 1 & 1 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & 0 & -\frac{1}{4} \\ 0 & 0 & 0 & 1 & \frac{7}{4} \end{array}\right)$$

So $(\boldsymbol{w})_S = (1, \frac{3}{2}, -\frac{1}{4}, \frac{7}{4})$.

(b) T is an orthogonal basis for $\mathbb{R}^4$ since it is a set of 4 orthogonal non zero vectors in $\mathbb{R}^4$. To see that T is an orthogonal set, we check $\boldsymbol{v_i} \cdot \boldsymbol{v_j} = 0$ for all $i \neq j$. To compute $(\boldsymbol{w})T$, we have $\boldsymbol{w} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + c_3\boldsymbol{v_3} + c_4\boldsymbol{v_4}$ where $c_i = \frac{\boldsymbol{w} \cdot \boldsymbol{v_i}}{\boldsymbol{v_i} \cdot \boldsymbol{v_i}}$ for $i = 1, 2, 3, 4$. We find that $c_1 = \frac{3}{10}$, $c_2 = \frac{1}{2}$, $c_3 = -1$, $c_4 = \frac{9}{10}$. Thus $(\boldsymbol{w})_T = (\frac{3}{10}, \frac{1}{2}, -1, \frac{9}{10})$.