MA1512 TUTORIAL PROBLEMS

1. First-Order Differential Equations

Problem 1.1. Solve the following differential equations.

(1) $x(x+1)y' = 1$

(2) $\sec(x)y' = \cos 5x$

(3) $y' = e^{x-3y}$

(4) $(1+y)y' + (1-2x)y^2 = 0$

(5) $y' = \dfrac{1-2y-4x}{1+y+2x}$

(6) $y' = \left(\dfrac{x+y+1}{x+y+3}\right)^2$

(7) $xy' + (1+x)y = e^{-x}, \quad x > 0$

(8) $y' - \left(1 + \dfrac{3}{x}\right)y = x+2,\ y(1) = e-1,\ x > 0$

(9) $y' + y + \dfrac{x}{y} = 0$

(10) $2xyy' + (x-1)y^2 = x^2e^x, \quad x > 0$

Problem 1.2. Experiments show that the rate of change of the temperature of a small iron ball is proportional to the difference between its temperature, $T(t)$, and that of its environment, T_{env} (which is constant).

(1) Write down a differential equation describing this situation.
(2) Show that $T = T_{\text{env}}$ is a solution. Does this make sense?
(3) The ball is heated to 300°F and then left to cool in a room at 75°F. Its temperature falls to 200°F in half an hour. Show that its temperature will be 81.6°F after 3 hours of cooling.

Problem 1.3. When a cake is removed from an oven, its temperature is measured at 130°C. Three minutes later, its temperature is 90°C. How long will it take for the cake to cool off to 26°C, with room temperature at 25°C?

Problem 1.4. If we start with an initial concentration, C_i, of radioactive radon-222, what would its concentration be after 5 days? The half-life for radon is 3.8 days.

Problem 1.5. It took the world about 300 years to increase in population from 0.5 billion to 4.0 billion. If we assume exponential growth at a constant rate over that period of time, what would that growth rate be?

Problem 1.6. In very dry regions, the phenomenon called *Virga*[1] is very important because it can endanger airplanes. Virga is rain in the air that is so dry that raindrops evaporate before they can reach the ground.

Suppose that the volume of a raindrop is proportional to the 3/2 power of its surface area. Raindrops are not spherical, but let's assume that they always have the same shape, no matter what their size may be.

(1) Why is the relationship between the raindrop's volume and its surface area reasonable?
(2) Suppose the rate of reduction of the volume of a raindrop is proportional to its surface area. Explain why this is reasonable, and find a formula for the amount of time it takes for a virga raindrop to evaporate completely, expressed in terms of the constants you introduced and the initial surface area of a raindrop. Check that the units of your formula are correct.
(3) Suppose somebody suggests that the rate of reduction of the volume of a raindrop is proportional to the square of the surface area. Argue that this cannot be correct.

Problem 1.7. A student starts a rumor in a school. The number of students who have heard the rumor, $R(t)$, is given by

$$R' = KR[1400 - R],$$

where K is a positive constant, and 1400 is the number of students in that school.

(1) What is the meaning of K? Is this equation reasonable?
(2) By regarding this equation as a Bernoulli equation, find $R(t)$.

Problem 1.8. One theory about the behavior of moths states that they navigate at night by keeping a fixed angle between their velocity vector and the direction of the Moon. A certain moth flies near a candle and mistakes it for the Moon.

(1) Prove that in polar coordinates, (r, θ), with the candle at the origin, the formula for the angle ψ between the radius vector and the velocity vector is given by $\tan\psi = r\, d\theta/dr$.
(2) Use this formula to solve for r as a function of θ and discuss what will happen to the moth.

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[1] *See http://en.wikipedia.org/wiki/Virga.*

Problem 1.9. If a cable is held up at two ends at the same height, then it will sag in the middle, making a U-shaped curve called a *catenary*. This is the shape seen in electricity cables suspended between poles, in countries less advanced than Singapore, such as Japan and the US. It can be shown using simple physics that if the shape is given by a function $y(x)$, then this function satisfies

$$\frac{dy}{dx} = \frac{\mu}{T} \int_0^x \sqrt{\left(\frac{dy}{dt}\right)^2 + 1dt},$$

where $x = 0$ at the lowest point of the catenary and $y(0) = 0$, where μ is the weight per unit length of the cable, and where T is the horizontal component of its tension; this horizontal component is a constant along the cable. Find a formula for the shape of the cable. [*Hint:* Use the Fundamental Theorem of Calculus:

Theorem (Fundamental Theorem of Calculus). *Let f be a continuous function defined over $[a, b] \subset \mathbb{R}$, and let*

$$F(x) = \int_a^x f(t)\ dt, \quad x \in [a, b].$$

Then, $F'(x) = f(x)$ for all $x \in (a, b)$.

Then, think of the resulting equation as a first-order ODE.]

Problem 1.10. Psychologists talk about something called a Performance Curve. Suppose an MA1512 student is solving mathematics problems. She starts with ordinary differential equations. Let $P(t)$ be a non-negative function that measures her performance, that is, her success rate at solving differential equations.

(1) Her performance increases rapidly at first, but then the rate of increase slows down as she becomes more expert. Let M, a positive constant, be the best possible performance; then one can suppose that

$$\frac{dP}{dt} = C[M - P],$$

where C is a constant.
   (a) What are the units of this constant? What does this constant measure?
   (b) Solve this equation assuming that she is completely incompetent at $t = 0$; that is, $P(0) = 0$.

(2) Now the student turns to another kind of problem, say in partial differential equations. Again her performance is low at first but gets better in accordance with this equation. As the years go by, her overall ability to solve mathematics problems gradually gets better, so C, instead of being a constant, is really a slowly increasing function of time. Suppose that

$$C(t) = K \tanh\left(\frac{t}{T}\right)$$

for $t \geq 0$, where K and T are positive constants.
   (a) Is this reasonable? Why? What are the meanings of the constants K and T?
   (b) Replacing C with $C(t)$, solve for P, again assuming that $P(0) = 0$.

Problem 1.11. The half-life of Thorium-230 is about 75000 years, while that of Uranium-234 is about 245000 years. A certain sample of ancient coral has a Thorium/Uranium ratio of 10 percent. How old is the coral?

Problem 1.12. A reaction sequence,

$$\underset{\text{organic nitrogen}}{A} \rightarrow \underset{\text{ammonia}}{B} \rightarrow \underset{\text{nitrate}}{C},$$

takes place in wastewater treatment plants as well as in natural aquatic ecosystems in the breakdown of dead or decaying organic matter. In the first step, A is transformed to B through the degradation of organic nitrogen by heterotrophic bacteria. In the second step, B is transformed into C through the nitrification process by nitrifying bacteria. The overall treatment results in the transformation of nitrogen from complex organic forms (organic nitrogen) to simple inorganic forms, which can then either be taken up by plants as nutrients (in aquatic ecosystem), or broken down even further into nitrogen gas by denitrifying bacteria in wastewater treatment plants.

The reactions then satisfy the following system of ordinary differential equations:

$$\begin{cases} \frac{dA}{dt} = -k_1 A \\ \frac{dB}{dt} = k_1 A - k_2 B \\ \frac{dC}{dt} = k_2 B \end{cases}$$

where k_1 and k_2 are positive constants. Suppose that at $t = 0$, we have $A = A_0$ and $B = C = 0$. Find a formula for C in terms of A_0, k_1, and k_2.

2. Second Order Differential Equations

Problem 2.1. (1) Solve the following differential equations:
   (a) $y'' + 6y' + 9y = 0, \quad y(0) = 1, \quad y'(0) = -1$
   (b) $y'' - 2y' + (1 + 4\pi^2)y = 0, \quad y(0) = -2, \quad y'(0) = 6\pi - 2$

(2) Solve the following differential equations.
   (a) $y'' + 2y' + 10y = 25x^2 + 3$
   (b) $y'' - 6y' + 8y = x^2 e^{3x}$
   (c) $y'' - y = 2x \sin x$
   (d) $y'' + 4y = \sin^2 x$
   (e) $y'' + y = \sec x$

Problem 2.2. Solve the differential equation $y''+4y = \sin^2 x$ using both the method of undetermined coefficients and variation of parameters, and verify that the two solutions you obtain are indeed identical.

Problem 2.3. A simple pendulum of length 1 meter is at rest at its stable equilibrium position. At time $t = 0$, it is given an initial angular velocity of 1 radian per second. Neglecting friction, determine the angular displacement of the pendulum at time $t = 0.8$ seconds. Take the value of the gravitational constant to be $g = 9.8$ meters per second squared, and give your answer correct to two decimal places.

Problem 2.4. A fully loaded large oil tanker can be modeled as a solid object with perfectly vertical sides and a perfectly horizontal bottom, so all horizontal cross-sections have the same area, A. Archimedes' principle[2] states that the upward force exerted on a ship by the sea is equal to the weight of the water pushed aside by the ship. Let ρ be the mass density of seawater, and let M be the mass of the ship, so that its weight is Mg, where $g = 9.8\,\mathrm{m\ sec}^{-2}$.

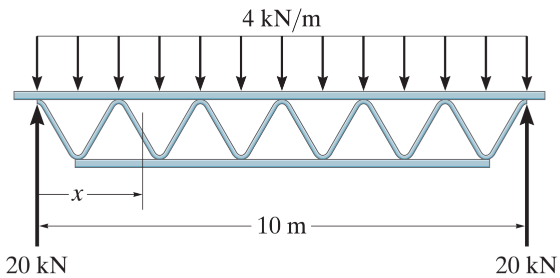
(1) Find the distance d from sea level to the bottom of the ship when the ship is at rest. This is called the *draught* of the ship.
(2) Suppose, now, that the ship is not at rest; instead, it is moving in the vertical direction. Let $d + x(t)$ be the distance from sea level to the bottom of the ship, where d is the draught as above. Show that if gravity and buoyancy are the only forces acting on the ship, it will bob up and down with an angular frequency given by $\omega = \sqrt{\rho A g / M}$.
(3) Next, suppose that waves from a storm strike the ship [which is initially at rest with $x(0) = 0$] and exert a vertical force $F_0 \cos(\omega t)$ on the ship, where F_0 is the amplitude of the wave force. Let H be the height of the deck of the ship above sea level when the ship is at rest. [We assume that the ship is heavily loaded, so H is muss less than d.] Write down a formula which allows you to compute when the ship sinks. [That is, find an equation satisfied by t_{sink}, the time at which the ship's deck first goes under water.] You don't need to solve this equation.

Problem 2.5. In a given beam set-up, the amount of deflection can be mathematically related to the load acting on the beam through an equation called the *Moment Equation.* The moment equation can be expressed as

$$EI\frac{d^2v}{dx^2} = M,$$

where v is the deflection, M is the moment function for a given beam set-up, E is Young's Modulus, and I is the beam's moment of inertia about the neutral axis. EI, taken together, represents the beam flexural rigidity or resistance to bending, assuming a constant for the length of the beam, while x is the distance from the origin (selected arbitrarily); v' is the slope of the bend at any point along the beam—used as a boundary condition.

Each simply supported floor joist, as shown below, is subjected to a uniform design loading of 4 kN/m.



Find the maximum deflection of the joist. EI is constant, and the moment function is given by $M = 20x - 2x^2$.

[2] *See http://en.wikipedia.org/wiki/Buoyancy.*

3. Mathematical Modeling with Differential Equations

Problem 3.1. A model rocket having initial mass m_0 kg is launched vertically from the ground. The rocket expels gas at a constant rate of α kg/sec and at a constant velocity of β m/sec relative to the rocket. Assume that the magnitude of the gravitational force is proportional to the mass with proportionality constant g. Because the mass is not constant, Newton's second law leads to the equation

$$(m_0 - \alpha t)\,\frac{dv}{dt} - \alpha\beta = -g\,(m_0 - \alpha t)\,,$$

where $v = dx/dt$ is the velocity of the rocket, x is its height above the ground, and $m_0 - \alpha t$ is the mass of the rocket at t seconds after launch. If the initial velocity is zero, solve the above equation to determine the velocity of the rocket and its height above ground for $0 \le t < m_0/\alpha$.

Problem 3.2. The bacteria in a certain culture number 10000 initially. Two and a half hours later, there are 11000 of them. Assume a Malthus model for the growth of bacteria.

(1) How many bacteria will there be 10 hours after the start of the experiment?
(2) How long will it take for the number to reach 20000?

Problem 3.3. You have 200 bugs in a bottle. Every day, you supply them with food and count them. After two days, you have 360 bugs. It is known that the birth rate for this kind of bug is 150% per day.

(1) Is this a sensible way of stating a birth rate per capita? Why?
(2) Assuming that the population is given by a logistic model, find the number of bugs after 3 days. Predict how many bugs you will have eventually.
(3) Now, suppose that you are keeping the bugs to develop a new insecticide. Suppose you remove 80 bugs per day from the bottle, and that all of these bugs die as a result of being sprayed with this insecticide. What is the limiting population in this case, and what is the maximum number of bugs you can put to death per day without causing the population to die out?

Problem 3.4. The sandhill crane is a beautiful Canadian bird with an unfortunate liking for farm crops. For many years, the cranes were protected by law, and eventually they settled down to a logistic equilibrium population of 194600 with birth rate per capita 9.866% per year. Eventually the patience of the farmers was exhausted, and they managed to have the hunting ban lifted. The farmers shot 10000 cranes per year, which they argued was reasonable since it only represents 5% of the original population. Show that the sandhill crane is doomed.

Problem 3.5. Suppose that Peruvian fishermen take a fixed number of anchovies per year from an anchovy stock which would otherwise behave logistically, apart from occasional natural disasters. As we've discussed, any fishing rate greater than or equal to $B^2/4s$ will be disastrous. Let's call this number E^*.

The fishermen want to take as many anchovies as they safely can; that is, they want the fish to be able to bounce back from a natural disaster that pushes their population down by 10%. Advise them—tell them the maximum number of fish they can take, expressed as a percentage of E^*.

Problem 3.6. The harvesting model considered during the lectures predicts that a population will rebound if all harvesting is stopped. Unfortunately, this is not always true: if you drive the population of some species down too low, the animals may have trouble finding mates, or they may be forced to breed with relatively close kin, which reduces genetic variability and hence their ability to resist disease. For such animals, such as certain rare species of tigers, extinction will result if the population falls too low, even if all harvesting is forbidden. Biologists call this phenomena *depensation.*

(1) Show that this situation can be modeled by the ordinary differential equation

$$\frac{dN}{dt} = -aN^3 + bN^2 - cN,$$

where N is the population, and a, b, c are positive constants such that $b^2 > 4ac$.

(2) Using the differential equation above, find the population size below which extinction will occur.

4. The Laplace Transform

For the following problems, let $u(t-c)$ denote the unit step function and $\delta(t-c)$ the Dirac delta function.

Problem 4.1. (1) Find the Laplace transform of the following functions:
   (a) $t^2 e^{-3t}$
   (b) $tu(t-2)$
(2) Find the inverse Laplace transforms of the following functions.
   (a) $s(s^2+10s+26)^{-1}$
   (b) $e^{-2s}(1+2s)s^{-3}$
(3) Solve the following initial value problems using Laplace transforms.
   (a) $y' = tu(t-2), \quad y(0)=4$
   (b) $y'' - 2y' = 4, \quad y(0)=1,\ y'(0)=0$

Problem 4.2. Consider a solid object with mass m immersed in water. A pulse horizontal force is applied initially. The dynamic equations is

$$m\frac{d^2x}{dt^2} = -A\frac{dx}{dt} + B\delta(t),$$

where the first term on the right is the hydrodynamic resistance force, and A, B are constants. The initial conditions are $x(0)=0$ and $x'(0)=0$. Apply the Laplace Transform to solve the above ordinary differential equation to determine the location x as a function of time t.

Problem 4.3. Recall the oil tanker in Problem 2.4, which is at rest in an almost calm sea. Suddenly, at time $t = T > 0$, it is hit by a single rogue wave,[3] which instantaneously imparts an upward vertical momentum P.
(1) Neglecting friction, solve for $x(t)$, the downward displacement of the ship.
(2) How far down does the ship go (if it doesn't sink)? [*Hint:* According to Newton's second law, momentum is the time integral of force. So, to get force as a function of time, you have to find a function which is zero except at $t = T$, and which has an integral equal to P. Note that the delta function has units time^{-1}.]

Problem 4.4. Billionaire engineer Tan Ah Lian attributes her enormous success to the fact that she never talked in class when she was an engineering student at NUS. During lecture one day, the professor announces that a certain gadget contains an electrical circuit with resistance R, capacitance C, and inductance L. Sadly, Ah Lian could not hear all of the numerical values corresponding to these variables due to the incessant babbling of a talkative minority; all she could hear was that the resistance R is 2 ohms.

Undeterred, she steals back into the room after class and quickly switches the gadget on and off at $t = 2$, thus firing a short burst of voltage into it, and finds that the current is

$$I(t) = u(t-2)\left[e^{-(t-2)}\cos(t-2) - e^{-(t-2)}\sin(t-2)\right] \text{ amperes.}$$

Ah Lian knows that the current in an electrical circuit is a function of time, $I(t)$, which satisfies the equation

$$V(t) = RI + L\dot{I} + \frac{1}{C}\int_0^t I\,dt.$$

The voltage $V(t)$ is a given function of time, and R, L, and C are constants. $\dot{I}$ denotes the time derivative of I. She then deduces what the professor must have said about the values of the inductance L and the capacitance C. What are her answers? [*Hint:* Recall the formula for the Laplace transform of an integral.]

[3] *See https://en.wikipedia.org/wiki/Rogue_wave.*

5. Partial Differential Equations

Problem 5.1. Using the method of separation of variables, solve the following partial differential equations.

(1) $yu_x - xu_y = 0$
(2) $u_x = yu_y, \quad y > 0$
(3) $u_{xy} = u$
(4) $xu_{xy} + 2yu = 0, \quad x > 0$

Problem 5.2. The wave equation is given by

$$c^2 y_{xx} = y_{tt},$$

subject to the conditions $y(t,0) = y(t,\pi) = 0$, $y(0,x) = f(x)$, and $y_t(0,x) = 0$. Show that the d'Alembert solution of the wave equation,

$$y(t,x) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right],$$

where $f(x)$ is an odd periodic function with period 2π, indeed satisfies the wave equation, with the given boundary and initial conditions.

Problem 5.3. Solve

$$u_t = 2u_{xx}, \quad 0 < x < 3, \quad t > 0,$$

given the boundary conditions $u(0,t) = 0$, $u(3,t) = 0$, and initial condition

$$u(x,0) = \sin^5 \pi x.$$

[*Hint:* Use the trigonometric identity $\sin^5 \pi x = \frac{5}{8}\sin \pi x - \frac{5}{16}\sin 3\pi x + \frac{1}{16}\sin 5\pi x$.]