W06-03

Slide 01: In this unit, we will discuss a systematic method to find a basis for the column space of a matrix.

Slide 02: Let us return to the same matrix $\boldsymbol{A}$ where we have already found a basis for the row space. How do we find a basis for the column space now? Recall that the column space of $\boldsymbol{A}$ is the subspace of $\mathbb{R}^3$ that is spanned by the five columns of $\boldsymbol{A}$.

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As mentioned, for this matrix $\boldsymbol{A}$, the column space is a subspace of $\mathbb{R}^3$.

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This means that the dimension of this subspace is at most 3.

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However, since there are 5 vectors that spans this subspace, if 3 out of the 5 vectors can be shown to be linearly independent,

Slide 03: which is indeed the case, as it is easy to check that the first 3 columns of $\boldsymbol{A}$ are linearly independent,

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then this subspace of $\mathbb{R}^3$ is in fact the entire $\mathbb{R}^3$. Thus, the first three columns of $\boldsymbol{A}$ will form a basis for the column space of $\boldsymbol{A}$. In fact, we can also say that the standard basis of $\mathbb{R}^3$ is also a basis for the column space of $\boldsymbol{A}$ since the column space of $\boldsymbol{A}$ is the entire $\mathbb{R}^3$.

Slide 04: Well, we were able to find a basis for the column space of $\boldsymbol{A}$ in this example because of our observation. This may not work for matrices in general. So we do need to find a systematic way of doing this.

Slide 05: Actually, if we recall the relationship between the column space of $\boldsymbol{A}$ with the row space of $\boldsymbol{A}^T$,

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as the columns of $\boldsymbol{A}$, that spans the column space of $\boldsymbol{A}$ are actually

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the rows of $\boldsymbol{A}^T$, which spans the row space of $\boldsymbol{A}^T$.

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Thus, to find a basis for the column space of $\boldsymbol{A}$ is the same as finding a basis for the row space of $\boldsymbol{A}^T$.

Slide 06: From a previous unit, we have already found a systematic method to find a basis for the row space of any matrix. Thus, using the transpose method, we already have a way to find a basis for the column space of a matrix.

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In what follows, we will discuss another method to find a basis for the column space of a matrix.

Slide 07: It is important to note that while we have already shown in a previous unit that elementary row operations does not affect the row space of a matrix, the same cannot be said for column spaces. In other words, the column space of a matrix is almost certain to change once we perform elementary row operations to it.

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For example, consider the simple row swap performed on the matrix $\boldsymbol{A}$. The column space of $\boldsymbol{A}$ is simply the linear span of $(1,0)$, which we know is the x-axis.

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After the row swap was performed on $\boldsymbol{A}$, the column space of the resulting matrix $\boldsymbol{B}$ is now the linear span of $(0,1)$, which is the y-axis.

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Clearly, the two column spaces are different. This simple example has shown that column spaces are not preserved by elementary row operations.

Slide 08: Before we present the method to find a basis for the column space of a matrix, we need to state two important results that relates the columns of row equivalent matrices. These results will be stated without proof. The first result states that for row equivalent matrices $\boldsymbol{A}$ and $\boldsymbol{B}$,

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a given set of columns of $\boldsymbol{A}$

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is linearly independent if and only if

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the corresponding set of columns of $\boldsymbol{B}$

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is linearly independent.

Slide 09: While the previous result gives the correspondence between the columns in terms of linear independence, the second result here takes it one step further to state the correspondence in terms of being a basis. More precisely the result states that for row equivalent matrices $\boldsymbol{A}$ and $\boldsymbol{B}$,

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a given set of columns of $\boldsymbol{A}$

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forms a basis for the column space of $\boldsymbol{A}$ if and only if

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the corresponding set of columns of $\boldsymbol{B}$

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forms a basis for the column space of $\boldsymbol{B}$.

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One should always remember that while this theorem gives the correspondence between the columns that forms a basis for the column spaces of the two matrices, the column spaces of the two row equivalent matrices are not the same. The theorem simply gives the correspondence between the columns to be bases for the matrices' **respective** column spaces.

Slide 10: We are now ready to present the method to find a basis for the column space of a matrix $\boldsymbol{A}$. The result that we have just seen in the previous slide will be utilised.

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Observe that if $\boldsymbol{R}$ is a matrix in row-echelon form, as shown here,

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the pivot columns of $\boldsymbol{R}$ will always form a basis for the column space of $\boldsymbol{R}$.

Slide 11: Thus to find a basis for the column space of $\boldsymbol{A}$, we first obtain $\boldsymbol{R}$, a row-echelon form of $\boldsymbol{A}$. Then using the correspondence result between the columns of $\boldsymbol{A}$ and $\boldsymbol{R}$, a basis for the column space of $\boldsymbol{A}$ can be obtained by taking the columns of $\boldsymbol{A}$ that correspond to the pivot columns of $\boldsymbol{R}$.

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A word of caution is to remember not to take the columns of $\boldsymbol{R}$ as your answer. The pivot of columns of $\boldsymbol{R}$ will be a basis for the column space of $\boldsymbol{R}$, but not for the column space of $\boldsymbol{A}$.

Slide 12: Let us look at one example. To find a basis for the column space of this matrix,

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we will first find a row-echelon form of it. Once we have obtained a row-echelon form,

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we identify the pivot columns in the row-echelon form. For this example, we see that the pivot columns are the first, third and fourth columns.

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This means that the first, third and fourth columns of $\boldsymbol{A}$

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will form a basis for the column space of $\boldsymbol{A}$.

Slide 13: To summarise this unit,

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we first observed that using the row space method, we actually already have a way to find a basis for the column space of a matrix. This is done by considering the transpose of the matrix.

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We then went on to discuss another way to find a basis for the column space of a matrix without having to transpose it. This is done by using the correspondence between the columns of two row equivalent matrices.