EQUIVALENT STATEMENTS (PART I)

A set of equivalent statements

If $\boldsymbol{A}$ is a square matrix, then the following statements are equivalent.

(If one of them is true, so are the rest. If one of them is false, so are the rest.)

1) $\boldsymbol{A}$ is invertible.

2) $\boldsymbol{Ax} = \mathbf{0}$ has only the trivial solution.

3) The reduced row-echelon form of $\boldsymbol{A}$ is $\boldsymbol{I}$.

4) $\boldsymbol{A}$ can be expressed as a product of elementary matrices.

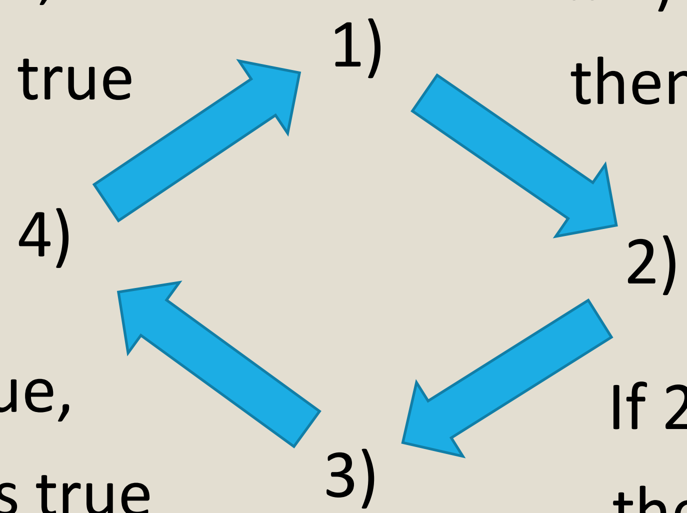
How can we proof this?

If A is a square matrix, then the following statements are equivalent.

(If one of them is true, so are the rest. If one of them is false, so are the rest.)

If 4) is true, then 1) is true

If 1) is true, then 2) is true



If 3) is true, then 4) is true

If 2) is true, then 3) is true

A set of equivalent statements

1) $\boldsymbol{A}$ is invertible $\Rightarrow$ 2) $\boldsymbol{Ax} = \mathbf{0}$ has only the trivial solution.

Assume $\boldsymbol{A}$ is invertible. So $\boldsymbol{A}^{-1}$ exists.

Let $\boldsymbol{u}$ be a solution to $\boldsymbol{Ax} = \mathbf{0}$.

$$\Rightarrow \boldsymbol{Au} = \mathbf{0} \quad \Rightarrow \boldsymbol{A}^{-1}\boldsymbol{Au} = \boldsymbol{A}^{-1}\mathbf{0}$$

$$\Rightarrow \boldsymbol{u} = \mathbf{0}$$

So the only solution to $\boldsymbol{Ax} = \mathbf{0}$ is $\boldsymbol{u} = \mathbf{0}$

That is, $\boldsymbol{Ax} = \mathbf{0}$ has only the trivial solution.

A set of equivalent statements

2) $\boldsymbol{Ax} = \mathbf{0}$ has only the trivial solution $\Rightarrow$

3) The reduced row-echelon form of $\boldsymbol{A}$ is $\boldsymbol{I}$.

$\boldsymbol{A}$ is a square matrix of order n

$\boldsymbol{Ax} = \mathbf{0}$ has only the trivial solution

$\Rightarrow$ Reduced row-echelon form of $\left(\begin{array}{c|c} \boldsymbol{A} & \mathbf{0} \end{array}\right)$

is $\left(\begin{array}{c|c} \boldsymbol{I}_n & \mathbf{0} \end{array}\right)$ $\Rightarrow$ The reduced row-echelon form of $\boldsymbol{A}$ is $\boldsymbol{I}_n$

A set of equivalent statements

3) The reduced row-echelon form of $\boldsymbol{A}$ is $\boldsymbol{I}$ $\Rightarrow$

4) $\boldsymbol{A}$ can be expressed as a product of elementary matrices.

The reduced row-echelon form of $\boldsymbol{A}$ is $\boldsymbol{I}_n$

$\Rightarrow$ There is a sequence of elementary matrices $\boldsymbol{E}_1, \ldots, \boldsymbol{E}_k$ such that

$$\boldsymbol{E}_k \ldots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{I}$$

Recall that all elementary matrices are invertible

$$\boxed{\boldsymbol{E}_k^{-1}} \boldsymbol{E}_k \ldots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A} = \boxed{\boldsymbol{E}_k^{-1}} \boldsymbol{I} \Rightarrow \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{E}_k^{-1}$$

$$\Rightarrow \boxed{\boldsymbol{E}_{k-1}^{-1}} \boldsymbol{E}_{k-1} \ldots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A} = \boxed{\boldsymbol{E}_{k-1}^{-1}} \boldsymbol{E}_k^{-1}$$

$$\Rightarrow \boldsymbol{E}_{k-2} \ldots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{E}_{k-1}^{-1} \boldsymbol{E}_k^{-1}$$

$$\boxed{\boldsymbol{A} = \boldsymbol{E}_1^{-1} \boldsymbol{E}_2^{-1} \ldots \boldsymbol{E}_{k-1}^{-1} \boldsymbol{E}_k^{-1}}$$

$\boldsymbol{A}$ can be expressed as a product of elementary matrices.

A set of equivalent statements

4) $\boldsymbol{A}$ can be expressed as a product of elementary matrices.

$\Rightarrow$ 1) $\boldsymbol{A}$ is invertible

$$\boldsymbol{A} = \boldsymbol{E}_1 \boldsymbol{E}_2 \ldots \boldsymbol{E}_{k-1} \boldsymbol{E}_k$$

Recall that all elementary matrices are invertible

$\Rightarrow$ $\boldsymbol{A}$ is a product of invertible matrices

$\Rightarrow$ $\boldsymbol{A}$ is invertible

Recall that product of invertible matrices is invertible

How can we use this?

What you know
(or have been told)

1) A is invertible.

What you can conclude

2) $Ax = \mathbf{0}$ has only the trivial solution.

3) The reduced row-echelon form of A is I.

4) A can be expressed as a product of elementary matrices.

How can we use this?

What you know

(or have been told)

2) $\boldsymbol{Ax} = \mathbf{0}$ has only the trivial solution.

What you can conclude

1) $\boldsymbol{A}$ is invertible.

3) The reduced row-echelon form of $\boldsymbol{A}$ is $\boldsymbol{I}$.

4) $\boldsymbol{A}$ can be expressed as a product of elementary matrices.

Summary

1) A collection of four equivalent statements, including "A is an invertible square matrix".