NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

Module: MA1508E Linear Algebra for Engineering
Year/Semester: 2018-2019 (Semester 2)
Tutorial: 11

1. Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ a & 1 \end{pmatrix}$, where $a \in \mathbb{R}$. Find all values of a such that

   (a) $\boldsymbol{A}$ has only one eigenvalue.

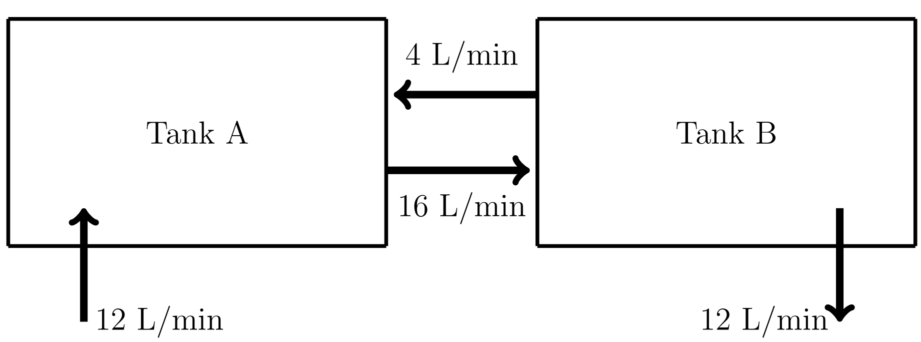
   (b) $\boldsymbol{A}$ has two eigenvalues -1 and 2. In this case, compute $\boldsymbol{A}^{-10}$ using diagonalisation.

   (c) $\boldsymbol{A}$ has a pair of complex eigenvalues.

2. Each matrix $\boldsymbol{A}$ below has complex eigenvalues. Find a matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{A}$ and determine $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$.

   (a) $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; (b) $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}$; (c) $\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{pmatrix}$.

3. Consider two large tanks that are connected as shown in the figure below.



   Tank A is initially filled with 100 L (litres) of water and 40 g (grams) of salt was dissolved in it. Tank B is initially filled with 100 L of water and 20 g of salt was dissolved in it. The well-mixed solution from Tank A is constantly pumped into Tank B at the rate of 16 L per minute while the solution in Tank B is pumped back into Tank A at the rate of 4 L per minute. Pure water is constantly pumped into Tank A at the rate of 12 L per minute while water exits the system from Tank B at the rate of 12 L per minute.

   At t minutes after the start of the mixing, let $a(t)$ and $b(t)$ be the amount of salt in Tanks A and B respectively. Construct a system of linear first order differential equations to evaluate $a(t)$ and $b(t)$ for each t.

Hence deduce that the amount of salt in Tank B will always be less than twice the amount of salt in Tank A.

4. Two species of fish, species A and species B, live in the same ecosystem (e.g. a pond) and compete with each other for food, water and space. Let the population of species A and B at time t years be given by $a(t)$ and $b(t)$ respectively.

   In the absence of species B, species A's growth rate is $4a(t)$ but when species B are present, the competition slows the growth of species A to $a'(t) = 4a(t) - 2b(t)$. In a similar manner, when species A is absent, species B's growth rate is $3b(t)$ but in the presence of species A, the growth rate reduces to $b'(t) = 3b(t) - a(t)$.

   (i) Write down a system of linear differential equations involving $a(t), b(t), a'(t)$ and $b'(t)$.

   (ii) Represent the system in (i) as $\boldsymbol{x'}(t) = \boldsymbol{A}\boldsymbol{x}(t)$ where

   $$\boldsymbol{A} \text{ is a } 2 \times 2 \text{ matrix and } \boldsymbol{x}(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}, \quad \boldsymbol{x'}(t) = \begin{pmatrix} a'(t) \\ b'(t) \end{pmatrix}.$$

   (iii) Solve the system using the initial condition $a(0) = 60$, $b(0) = 120$.

5. **(Repeated eigenvalues)** This question illustrates what we should do if a system of linear differential equations $\boldsymbol{Y'} = \boldsymbol{AY}$ (where $\boldsymbol{A}$ is a 2×2 matrix) is such that $\boldsymbol{A}$ has only 1 eigenvalue λ and $\dim(E_\lambda) = 1$.

   Suppose $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue λ. Let $\boldsymbol{u}$ be a non zero vector in $\mathbb{R}^2$ such that

   $$(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{u} = \boldsymbol{v}.$$

   Prove that

   $$\boldsymbol{Y}(t) = c_1 e^{\lambda t}\boldsymbol{v} + c_2 e^{\lambda t}(t\boldsymbol{v} + \boldsymbol{u}), \qquad c_1, c_2 \in \mathbb{R}$$

   satisfies $\boldsymbol{Y'} = \boldsymbol{AY}$ and is thus a solution to the system of linear differential equations. We call this solution a **generalised** eigenvector of $\boldsymbol{A}$ associated with λ.

   Use the technique above to solve the system of linear differential equations $\boldsymbol{Y'} = \boldsymbol{AY}$ where $\boldsymbol{A} = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}$ and the system has the initial condition $y_1(0) = 1$ and $y_2(0) = 3$.

6. Solve the following systems of second order linear differential equations.

   (a) $y'' + 2y' + 5y = 0$;

   (b)

   $$\begin{cases} y_1'' = -2y_2 + y_1' + 2y_2' \\ y_2'' = 2y_1 + 2y_1' - y_2' \end{cases}$$

   with initial conditions $y_1(0) = 1$, $y_2(0) = 0$, $y_1'(0) = -3$, $y_2'(0) = 2$.