FINDING A BASIS FOR ROW SPACE

DISCUSSION

Recall the definition of row equivalent.

$$\boldsymbol{A} \xleftrightarrow{\text{Series of e.r.o}} \boldsymbol{B}$$

$\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent

Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ (of the same size) are row equivalent if and only if they have a similar row-echelon form (or they have the same unique reduced row-echelon form).

THEOREM

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be row equivalent matrices. Then the row space of $\boldsymbol{A}$ and the row space of $\boldsymbol{B}$ are identical.

$$\boldsymbol{A} \xleftrightarrow{\text{Series of e.r.o}} \boldsymbol{B}$$

same row space

That is to say, performing elementary row operations on $\boldsymbol{A}$ does not change its row space.

THEOREM

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be row equivalent matrices. Then the row space of $\boldsymbol{A}$ and the row space of $\boldsymbol{B}$ are identical.

$$\boldsymbol{A} \xleftrightarrow{\text{Series of e.r.o}} \boldsymbol{B}$$

same row space

Proof: If we can show that the row space of a matrix does not change after a single elementary row operation (any one of the three types), then it follows that the row space will always be the same after a series of elementary row operations.

THEOREM

Proof: We will use this theorem introduced in an earier unit on linear spans.

Let $S_1 = \{u_1, u_2, \ldots, u_k\}$ and $S_2 = \{v_1, v_2, \ldots, v_m\}$ be subsets of $\mathbb{R}^n$.

Then $\text{span}(S_1) \subseteq \text{span}(S_2) \iff$ each u_i is a linear combination of $v_1, v_2, \ldots, v_m$.

Consider the first type of elementary row operation:

$$\boldsymbol{A} \xrightarrow{cR_i \ (c \neq 0)} \boldsymbol{B}$$

THEOREM

Let $S_1 = \{u_1, u_2, \ldots, u_k\}$ and $S_2 = \{v_1, v_2, \ldots, v_m\}$ be subsets of $\mathbb{R}^n$.

Then $\text{span}(S_1) \subseteq \text{span}(S_2) \iff$ each u_i is a linear combination of $v_1, v_2, \ldots, v_m$.

Rows of $\boldsymbol{A} = \{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_n\}$

Row space of $\boldsymbol{A}$

$= \text{span}\boxed{\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_n\}}$

$\subseteq$?

Rows of $\boldsymbol{B} = \{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, c\boldsymbol{r}_i, \ldots, \boldsymbol{r}_n\}$

Row space of $\boldsymbol{B}$

$= \text{span}\boxed{\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, c\boldsymbol{r}_i, \ldots, \boldsymbol{r}_n\}}$

Is every vector from $\boxed{}$ a linear combination of vectors from $\boxed{}$?

THEOREM

Let $S_1 = \{u_1, u_2, \ldots, u_k\}$ and $S_2 = \{v_1, v_2, \ldots, v_m\}$ be subsets of $\mathbb{R}^n$.

Then $\text{span}(S_1) \subseteq \text{span}(S_2) \iff$ each u_i is a linear combination of $v_1, v_2, \ldots, v_m$.

Rows of $\boldsymbol{A} = \{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_n\}$

Row space of $\boldsymbol{A}$

$= \text{span}\boxed{\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_n\}}$

$\supseteq$?

Rows of $\boldsymbol{B} = \{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, c\boldsymbol{r}_i, \ldots, \boldsymbol{r}_n\}$

Row space of $\boldsymbol{B}$

$= \text{span}\boxed{\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, c\boldsymbol{r}_i, \ldots, \boldsymbol{r}_n\}}$

Is every vector from $\boxed{}$ a linear combination

of vectors from $\boxed{}$?

THEOREM

Proof: We will use this theorem introduced in an earier unit on linear spans.

Let $S_1 = \{u_1, u_2, \ldots, u_k\}$ and $S_2 = \{v_1, v_2, \ldots, v_m\}$ be subsets of $\mathbb{R}^n$.

Then $\text{span}(S_1) \subseteq \text{span}(S_2) \Leftrightarrow$ each u_i is a linear combination of $v_1, v_2, \ldots, v_m$.

Consider the first type of elementary row operation:

$$\boldsymbol{A} \xrightarrow{cR_i \ (c \neq 0)} \boldsymbol{B}$$

Shown: Row space of $\boldsymbol{A} =$ Row space of $\boldsymbol{B}$

THEOREM

Proof: We will use this theorem introduced in an earier unit on linear spans.

Let $S_1 = \{u_1, u_2, \ldots, u_k\}$ and $S_2 = \{v_1, v_2, \ldots, v_m\}$ be subsets of $\mathbb{R}^n$.

Then span$(S_1) \subseteq$ span(S_2) $\Leftrightarrow$ each u_i is a linear combination of $v_1, v_2, \ldots, v_m$.

Consider the second type of elementary row operation:

$$\boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{B}$$

THEOREM

Let $S_1 = \{u_1, u_2, \ldots, u_k\}$ and $S_2 = \{v_1, v_2, \ldots, v_m\}$ be subsets of $\mathbb{R}^n$.

Then $\text{span}(S_1) \subseteq \text{span}(S_2) \iff$ each u_i is a linear combination of $v_1, v_2, \ldots, v_m$.

Rows of $\boldsymbol{A} = \{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_n\}$

Rows of $\boldsymbol{B} = \{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_n\}$

Row space of $\boldsymbol{A} = \text{span}\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_n\}$

$=$

Row space of $\boldsymbol{B} = \text{span}\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_n\}$

THEOREM

Proof: We will use this theorem introduced in an earier unit on linear spans.

Let $S_1 = \{u_1, u_2, \ldots, u_k\}$ and $S_2 = \{v_1, v_2, \ldots, v_m\}$ be subsets of $\mathbb{R}^n$.

Then $\text{span}(S_1) \subseteq \text{span}(S_2) \;\Leftrightarrow\;$ each u_i is a linear combination of $v_1, v_2, \ldots, v_m$.

Consider the second type of elementary row operation:

$$\boldsymbol{A} \xrightarrow{R_i \leftrightarrow R_j} \boldsymbol{B}$$

Shown: Row space of $\boldsymbol{A} =$ Row space of $\boldsymbol{B}$

THEOREM

Proof: We will use this theorem introduced in an earier unit on linear spans.

Let $S_1 = \{u_1, u_2, \ldots, u_k\}$ and $S_2 = \{v_1, v_2, \ldots, v_m\}$ be subsets of $\mathbb{R}^n$.

Then $\text{span}(S_1) \subseteq \text{span}(S_2) \;\Leftrightarrow\;$ each u_i is a linear combination of $v_1, v_2, \ldots, v_m$.

Consider the third type of elementary row operation:

$$\boldsymbol{A} \xrightarrow{R_i + cR_j} \boldsymbol{B}$$

THEOREM

Let $S_1 = \{u_1, u_2, \ldots, u_k\}$ and $S_2 = \{v_1, v_2, \ldots, v_m\}$ be subsets of $\mathbb{R}^n$.

Then span$(S_1) \subseteq$ span(S_2) $\Leftrightarrow$ each u_i is a linear combination of $v_1, v_2, \ldots, v_m$.

Rows of $\boldsymbol{A} = \{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_n\}$

Rows of $\boldsymbol{B} = \{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_i + c\boldsymbol{r}_j, \ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_n\}$

Row space of $\boldsymbol{A} =$ span$\boxed{\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_n\}}$

$\subseteq$?

Row space of $\boldsymbol{B} =$ span$\boxed{\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_i + c\boldsymbol{r}_j, \ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_n\}}$

Every vector from $\boxed{}$ a linear combination of vectors from $\boxed{}$?

THEOREM

Let $S_1 = \{u_1, u_2, \ldots, u_k\}$ and $S_2 = \{v_1, v_2, \ldots, v_m\}$ be subsets of $\mathbb{R}^n$.

Then $\text{span}(S_1) \subseteq \text{span}(S_2) \Leftrightarrow$ each u_i is a linear combination of $v_1, v_2, \ldots, v_m$.

YES!

$$\boxed{\boldsymbol{r}_i} = \boxed{(\boldsymbol{r}_i + c\boldsymbol{r}_j)} - c\boxed{\boldsymbol{r}_j}$$

Every vector from $\boxed{}$ a linear combination of vectors from $\boxed{}$?

Row space of $\boldsymbol{A} = \text{span}\boxed{\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_n\}}$

$\subseteq$?

Row space of $\boldsymbol{B} = \text{span}\boxed{\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_i + c\boldsymbol{r}_j, \ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_n\}}$

THEOREM

Let $S_1 = \{u_1, u_2, \ldots, u_k\}$ and $S_2 = \{v_1, v_2, \ldots, v_m\}$ be subsets of $\mathbb{R}^n$.

Then $\text{span}(S_1) \subseteq \text{span}(S_2) \iff$ each u_i is a linear combination of $v_1, v_2, \ldots, v_m$.

Rows of $\boldsymbol{A} = \{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_n\}$

Rows of $\boldsymbol{B} = \{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_i + c\boldsymbol{r}_j, \ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_n\}$

Row space of $\boldsymbol{A} = \text{span}\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_n\}$

$\supseteq$?

Row space of $\boldsymbol{B} = \text{span}\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_i + c\boldsymbol{r}_j, \ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_n\}$

Every vector from $\boxed{}$ a linear combination of vectors from $\boxed{}$?

THEOREM

Let $S_1 = \{u_1, u_2, \ldots, u_k\}$ and $S_2 = \{v_1, v_2, \ldots, v_m\}$ be subsets of $\mathbb{R}^n$.

Then $\text{span}(S_1) \subseteq \text{span}(S_2) \iff$ each u_i is a linear combination of $v_1, v_2, \ldots, v_m$.

YES!

$$\boxed{\boldsymbol{r}_i + c\boldsymbol{r}_j} = \boxed{\boldsymbol{r}_i} + c\boxed{\boldsymbol{r}_j}$$

Every vector from $\boxed{}$ a linear combination of vectors from $\boxed{}$?

Row space of $\boldsymbol{A} = \text{span}\boxed{\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_n\}}$

$\supseteq$?

Row space of $\boldsymbol{B} = \text{span}\boxed{\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_i + c\boldsymbol{r}_j, \ldots, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_n\}}$

THEOREM

Proof: We will use this theorem introduced in an earier unit on linear spans.

Let $S_1 = \{u_1, u_2, \ldots, u_k\}$ and $S_2 = \{v_1, v_2, \ldots, v_m\}$ be subsets of $\mathbb{R}^n$.

Then $\text{span}(S_1) \subseteq \text{span}(S_2) \Leftrightarrow$ each u_i is a linear combination of $v_1, v_2, \ldots, v_m$.

Consider the third type of elementary row operation:

$$\boldsymbol{A} \xrightarrow{R_i + cR_j} \boldsymbol{B}$$

Shown: Row space of $\boldsymbol{A} =$ Row space of $\boldsymbol{B}$

EXAMPLE

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 4 \\ \frac{1}{2} & 1 & 2 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} \frac{1}{2} & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{2R_1} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_1 - R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} &\operatorname{span}\{(0,0,1),(0,2,4),(\tfrac{1}{2},1,2)\} \\ &= \operatorname{span}\{(\tfrac{1}{2},1,2),(0,2,4),(0,0,1)\} \\ &= \operatorname{span}\{(1,2,4),(0,2,4),(0,0,1)\} \\ &= \operatorname{span}\{(1,0,0),(0,2,4),(0,0,1)\} \end{aligned}$$

BACK TO THIS QUESTION

Question : How to find a basis for the row space of a matrix $\boldsymbol{A}$?

$$\boldsymbol{R} = \begin{pmatrix} \otimes & * & & & & * \\ & & \otimes & * & & * \\ & 0 & & \ddots & & * \\ & & & & \otimes & * \quad * \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{pmatrix}$$

The non zero rows of $\boldsymbol{R}$ are always linearly independent and thus forms a basis for the row space of $\boldsymbol{R}$.

Answer :

Find a row-echelon form $\boldsymbol{R}$ of $\boldsymbol{A}$.

A basis for the row space of $\boldsymbol{R}$ is also a basis for the row space of $\boldsymbol{A}$.

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be row equivalent matrices. Then the row space of $\boldsymbol{A}$ and the row space of $\boldsymbol{B}$ are identical.

EXAMPLE

Find a basis for the row space of the following matrix.

$$\boldsymbol{A} = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} \boxed{2 \quad 2 \quad -1 \quad 0 \quad 1} \\ \boxed{0 \quad 0 \quad \tfrac{3}{2} \quad -3 \quad \tfrac{3}{2}} \\ \boxed{0 \quad 0 \quad 0 \quad 3 \quad 0} \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \end{pmatrix}$$

Performing Gaussian Elimination on $\boldsymbol{A}$:

A basis for the row space of $\boldsymbol{A}$ is

$$\{(2,2,-1,0,1),(0,0,\tfrac{3}{2},-3,\tfrac{3}{2}),(0,0,0,3,0)\}$$

SUMMARY

1) Row equivalent matrices have the same row space.

2) A method to find a basis for the row space of a matrix.