W03-10

Slide 01: In this unit, we will discuss some properties of determinants.

Slide 02: Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two square matrices of order n and c is a constant. The first result here states that the determinant of $c\boldsymbol{A}$ will be c^n times the determinant of $\boldsymbol{A}$.

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In order to see this, notice that if we multiply the constant c to each of the n rows in $\boldsymbol{A}$, we will obtain the matrix $c\boldsymbol{A}$.

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Since each one of these elementary row operation changes the determinant by a factor of c, the determinant of $c\boldsymbol{A}$ would naturally be c^n times the determinant of $\boldsymbol{A}$.

Slide 03: The next result states that the determinant of $\boldsymbol{AB}$ is the determinant of $\boldsymbol{A}$ times the determinant of $\boldsymbol{B}$. Essentially, this means that the determinant of the product of two matrices is equal to the product of the two matrices' determinants.

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It should be noted that this result generalises an earlier result where we have an elementary matrix $\boldsymbol{E}$ premultiplied to $\boldsymbol{A}$, then the determinant of $\boldsymbol{EA}$ is the determinant of $\boldsymbol{E}$ multiplied by the determinant of $\boldsymbol{A}$. For the more general result we see here, we do not require either of the matrices to be an elementary matrix.

Slide 04: First consider the case where $\boldsymbol{A}$ is singular. In an earlier unit, we have already shown that if $\boldsymbol{A}$ is singular, then $\boldsymbol{AB}$ will be singular.

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SInce a matrix being singular is equivalent to its determinant being zero, we can say that both $\boldsymbol{A}$ and $\boldsymbol{AB}$ has determinant zero. This means that the determinant of $\boldsymbol{AB}$ and the determinant of $\boldsymbol{A}$ multiplied by the determinant of $\boldsymbol{B}$ are both equal to zero. This establishes the statement for this case.

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We next consider the case when $\boldsymbol{A}$ is invertible.

Slide 05: Since $\boldsymbol{A}$ is invertible is equivalent to the fact that $\boldsymbol{A}$ can be written as a product of elementary matrices, we write $\boldsymbol{A}$ as a product of elementary matrices $\boldsymbol{E_1}$, $\boldsymbol{E_2}$ and so on until $\boldsymbol{E_k}$.

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Post-multiplying both sides of the equation by $\boldsymbol{B}$, we have the following.

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Thus the determinant of $\boldsymbol{AB}$ is equal to the determinant of the product of matrices on the right hand side.

Slide 06: As we have mentioned earlier, we already know that the determinant of $\boldsymbol{EA}$ is equal to the determinant of $\boldsymbol{E}$ multiplied by the determinant of $\boldsymbol{A}$. Let us apply this result repeatedly on the expression on the right hand side.

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We first rewrite the right hand side as the determinant of $\boldsymbol{E_k}$ multiplied by the determinant of the product of the remaining matrices.

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Continuing with this, we see that the right hand side is now the product of the determinants of the elementary matrices as well as that of $\boldsymbol{B}$.

Slide 07: We now apply the same result to combine the product of the determinants of the elementary matrices back into one determinant.

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Starting with combining the determinant of $\boldsymbol{E_2}$ and determinant of $\boldsymbol{E_1}$

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and proceeding similarly, we eventually have the determinant of the product of all elementary matrices.

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Since the product of the elementary matrices is $\boldsymbol{A}$, the right hand side now is simply the determinant of $\boldsymbol{A}$ multiplied by the determinant of $\boldsymbol{B}$ and we are done with this case. The statement has thus been proven.

Slide 08: The third result states that if $\boldsymbol{A}$ is invertible, then the determinant of $\boldsymbol{A}^{-1}$ is 1 divided by the determinant of $\boldsymbol{A}$.

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To prove this, notice that since $\boldsymbol{A}$ is invertible, $\boldsymbol{A}^{-1}$ exists and $\boldsymbol{A}^{-1}\boldsymbol{A}$ is equals to $\boldsymbol{I}$. Thus the determinant of $\boldsymbol{A}^{-1}\boldsymbol{A}$ is equal to the determinant of $\boldsymbol{I}$.

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By applying the result we have just proven before this, we can write the determinant of $\boldsymbol{A}^{-1}\boldsymbol{A}$ as the determinant of $\boldsymbol{A}^{-1}$ times the determinant of $\boldsymbol{A}$.

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Furthermore, as $\boldsymbol{I}$ is a triangular matrix, it is easy to see that the determinant of $\boldsymbol{I}$ is 1.

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The desired result now follows immediately.

Slide 09: As an example, consider the following matrix $\boldsymbol{A}$. It is easy to verify, using cofactor expansion, that determinant of $\boldsymbol{A}$ is 34. The determinant of $4\boldsymbol{A}$, will then be 4^3 times the determinant of $\boldsymbol{A}$, since $\boldsymbol{A}$ is a 3×3 matrix. This evaluates to 2176.

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Since the determinant of $\boldsymbol{A}$ is non zero, we can conclude that it is invertible. Furthermore by the result we have proven in this unit, the determinant of $\boldsymbol{A}^{-1}$ would be 1 divided by the determinant of $\boldsymbol{A}$.

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Given another 3×3 matrix $\boldsymbol{B}$ as shown, we can again compute the determinant of $\boldsymbol{B}$ to be -1.

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This would allow us to conclude that the determinant of $\boldsymbol{AB}$ to be the product of the determinants of the two matrices. Thus determinant of $\boldsymbol{AB}$ is -34.

Slide 10: To summarise this unit,

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We established several results on determinants.

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The first relates the determinant of $c\boldsymbol{A}$ to the determinant of $\boldsymbol{A}$.

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The second relates to the determinant of a product of matrices.

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The third result relates, for an invertible matrix $\boldsymbol{A}$, the determinant of $\boldsymbol{A}$ with the determinant of $\boldsymbol{A}^{-1}$.