NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

Module: MA1508E Linear Algebra for Engineering
Year/Semester: 2018-2019 (Semester 2)
Tutorial: 6

1. For each of the following sets $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$, determine the values of the constant a such that the set S is a linearly dependent set.

   (a) $\boldsymbol{u_1} = (1, 0, 1)$, $\boldsymbol{u_2} = (a, 1, 1)$, $\boldsymbol{u_3} = (1, 1, 3a)$.

   (b) $\boldsymbol{u_1} = (1, 0, 0)$, $\boldsymbol{u_2} = (a, 1, -a)$, $\boldsymbol{u_3} = (1, 2a, 3a + 1)$.

2. Let $\boldsymbol{u_1} = (1, -2, 1, 1, 2)$, $\boldsymbol{u_2} = (-1, 3, 0, 2, -2)$, $\boldsymbol{u_3} = (0, 1, 1, 3, 4)$.

   (a) Show that $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is a linearly independent set.

   (b) Find a vector $\boldsymbol{u_4}$ such that $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$ is a linearly independent set.

   (c) Find a vector $\boldsymbol{u_5}$ such that $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}, \boldsymbol{u_5}\}$ is a basis for $\mathbb{R}^5$.

3. Let $\boldsymbol{v_1} = (1, 2, 3)$, $\boldsymbol{v_2} = (2, 4, 6)$, $\boldsymbol{v_3} = (2, 5, 7)$, $\boldsymbol{v_4} = (3, 5, 9)$, $\boldsymbol{v_5} = (1, 4, 5)$.

   (a) Show that $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}, \boldsymbol{v_5}\}$ is a linearly dependent set.

   (b) Remove one **redundant** vector from S to obtain S' such that $\text{span}(S) = \text{span}(S')$.

   (c) Explain why S' is still a linearly dependent set.

   (d) Remove one more redundant vector from S' to obtain S'' such that $\text{span}(S') = \text{span}(S'')$.

   (e) Determine if S'' is a basis for $\mathbb{R}^3$.

4. Let $V = \{(w+x, w+y, y+z, x+z) \mid w, x, y, z \in \mathbb{R}\}$ and $S = \{(1, 1, 0, 0), (1, 0, -1, 0), (0, -1, 0, 1)\}$.

   (a) Show that V is a subspace of $\mathbb{R}^4$ by writing it as a linear span.

   (b) Show that S is a basis for V.

   (c) Find the coordinate vector of $\boldsymbol{u} = (1, 2, 3, 2)$ relative to S.

   (d) Find a vector $\boldsymbol{v}$ such that $(\boldsymbol{v})_S = (1, 3, -1)$.

5. (All vectors in this question are written as column vectors.) Let $\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}$ be vectors in $\mathbb{R}^n$ and $\boldsymbol{P}$ is a square matrix of order n. Note that $\boldsymbol{Pu_1}$, $\boldsymbol{Pu_2}, \cdots, \boldsymbol{Pu_k}$ are also (column) vectors in $\mathbb{R}^n$.

   (a) Show that if $\boldsymbol{Pu_1}, \boldsymbol{Pu_2}, \cdots, \boldsymbol{Pu_k}$ are linearly independent, then $\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}$ are linearly independent.

(b) Let us investigate the converse of (a). Suppose $\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}$ are linearly independent.

  (i) Show that if $\boldsymbol{P}$ is invertible, then $\boldsymbol{Pu_1}, \boldsymbol{Pu_2}, \cdots, \boldsymbol{Pu_k}$ are linearly independent.

  (ii) If $\boldsymbol{P}$ is singular, are $\boldsymbol{Pu_1}, \boldsymbol{Pu_2}, \cdots, \boldsymbol{Pu_k}$ linearly independent?