W06-02

Slide 01: In this unit, we will discuss a systematic method to find a basis for the row space of a matrix.

Slide 02: Recall that two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be row equivalent if one can be obtained from the other by performing a series of elementary row operations. We can also say that two matrices are row equivalent if and only if they have a similar row-echelon form, or they have the same unique reduced row-echelon form.

Slide 03: We first state and prove a theorem that will be very useful for us to find a basis for the row space of a matrix. Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are row equivalent matrices. Then the row space of $\boldsymbol{A}$ and $\boldsymbol{B}$ will be identical.

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In other words, elementary row operations performed on a matrix do not affect its row space. Or we can say that the row space of a matrix is preserved when elementary row operations are performed on it.

Slide 04: To prove this theorem, first note that if the row space of a matrix did not change after a single elementary row operation, then it would not change after a series of such operations. Thus we just need to show that the row space is not affected after one single elementary row operation.

Slide 05: We will rely heavily on this result which was presented earlier during our discussion on linear span. Basically this result gives us a necessary and sufficient condition for one linear span to be entirely contained inside another linear span. More precisely, the linear span of S_1 is contained inside the linear span of S_2 if and only if each vector in S_1 is a linear combination of the vectors in S_2.

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Let us start off by considering the first type of elementary row operation, which is to multiply a row of the matrix with a non zero constant c. Let $\boldsymbol{B}$ be the resulting matrix after performing such an operation on $\boldsymbol{A}$.

Slide 06: The rows of $\boldsymbol{A}$ can be denoted by $\boldsymbol{r_1}$, $\boldsymbol{r_2}$ until $\boldsymbol{r_n}$. The rows of $\boldsymbol{B}$, in this case, will be almost identical to the rows of $\boldsymbol{A}$, except that the i-th row of $\boldsymbol{B}$ is now $c\boldsymbol{r_i}$ where $\boldsymbol{r_i}$ is the i-th row of $\boldsymbol{A}$.

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By definition, the row spaces of $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively is the linear span of the respective rows in the two matrices.

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Using the necessary and sufficicent condition for one linear span to be contained inside another linear span, how can we show that the row space of $\boldsymbol{A}$ is contained inside the row space of $\boldsymbol{B}$?

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We basically need to answer the question of whether every vector in the yellow set is a linear combination of the vectors in the blue set. The answer to this question is clearly yes, since the only vector in the yellow set that is not in the blue set is the vector $\boldsymbol{r_i}$. But $\boldsymbol{r_i}$ is just a scalar multiple of $c\boldsymbol{r_i}$ which is a vector in the blue set.

Slide 07: Similarly we will show that the row space of $\boldsymbol{B}$ is contained inside the row space of $\boldsymbol{A}$. This can be done by checking whether every vector in the blue set is a linear combination of the vectors in the yellow set. Indeed, since the only vector in the blue set that is not in the yellow set is $c\boldsymbol{r_i}$ but obviously $c\boldsymbol{r_i}$ is a linear combination of the vector $\boldsymbol{r_i}$ which is a vector in the yellow set.

Slide 08: Thus we have shown that $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same row space if $\boldsymbol{B}$ is obtained by performing one elementary row operation of the first type on $\boldsymbol{A}$.

Slide 09: Consider the second type of elementary row operation where $\boldsymbol{A}$ and $\boldsymbol{B}$ differ by exactly one row swap.

Slide 10: If rows i and j in the matrix $\boldsymbol{A}$ are swapped, it should be immediately clear that it is the order in which the rows are written that will be different between matrices $\boldsymbol{A}$ and $\boldsymbol{B}$.

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Again by definition, the row spaces of $\boldsymbol{A}$ and $\boldsymbol{B}$ can be written down easily and it is easy to see that

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they are basically the same.

Slide 11: Thus we have shown that once again, $\boldsymbol{A}$ and $\boldsymbol{B}$ have the same row space.

Slide 12: Let us consider the third type of elementary row operation, where we add c times of the j-th row to the i-th row.

Slide 13: Once again, we write down the rows of $\boldsymbol{A}$.

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The rows of $\boldsymbol{B}$ differ in only one row which is the i-th row. Note that the i-th row of $\boldsymbol{B}$ is now $\boldsymbol{r_i} + c\boldsymbol{r_j}$. Every other row in $\boldsymbol{B}$ is exactly the same as before.

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By definition, the row spaces of $\boldsymbol{A}$ and $\boldsymbol{B}$ are as shown here.

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Is the row space of $\boldsymbol{A}$ a subset of the row space of $\boldsymbol{B}$?

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Using the necessary and sufficient condition, we need to check if every vector from the yellow set is a linear combination of the vectors in the blue set.

Slide 14: Since the only vector in the yellow set that is not in the blue set is $\boldsymbol{r_i}$, we just need to check if $\boldsymbol{r_i}$ can be written as a linear combination of the vectors in the blue set.

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Indeed it can be done easily, as $\boldsymbol{r_i}$ is $(\boldsymbol{r_i} + c\boldsymbol{r_j}) - c\boldsymbol{r_j}$.

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Thus we have established the first subset inclusion.

Slide 15: We will now consider the other subset inclusion. In order to show that the row space of $\boldsymbol{B}$ is contained inside the row space of $\boldsymbol{A}$, we need to check if every vector in the blue set is a linear combination of the vectors in the yellow set.

Slide 16: Since the only vector in the blue set that is not in the yellow set is $\boldsymbol{r_i} + c\boldsymbol{r_j}$, we just need to check that this vector can be written as a linear combination of the vectors in the yellow set.

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Indeed it can be easily done, since both $\boldsymbol{r_i}$ and $\boldsymbol{r_j}$ are vectors in the yellow set.

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Thus we have establisehd the second subset inclusion.

Slide 17: We have therefore shown that for the third type of elementary row operations, matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ will have the same row space. The proof of the theorem is now complete and we know that row equivalent matrices have the same row space.

Slide 18: Let us consider an example. The four matrices shown here are all row equivalent, since one can be obtained from another via a series of elementary row operations.

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With the preceding theorem which we have proven, the row space of the first matrix,

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will be equal to the row space of the second matrix,

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which will be equal to the row space of the third matrix

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and also the fourth.

Slide 19: Let us return to the problem of how to find a basis for the row space of any given matrix $\boldsymbol{A}$.

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As expected, we will rely on the theorem that we have seen in this unit, namely that row equivalent matrices have the same row space.

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We also had the observation from a previous unit that if a matrix $\boldsymbol{R}$ is in row-echelon form, then the non zero rows of $\boldsymbol{R}$ will always be linearly independent and thus forming a basis for the row space of $\boldsymbol{R}$.

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The strategy is now clear. From the given matrix $\boldsymbol{A}$, we just need to find a row-echelon form $\boldsymbol{R}$ of $\boldsymbol{A}$. Since $\boldsymbol{A}$ and $\boldsymbol{R}$ are row equivalent, their row spaces are identical. Thus, a basis for the row space of $\boldsymbol{R}$ will also be a basis for the row space of $\boldsymbol{A}$. In other words, we will just take the non zero rows in $\boldsymbol{R}$ to be a basis for the row space of $\boldsymbol{A}$.

Slide 20: For this example, we would like to find a basis for the row space of the matrix $\boldsymbol{A}$.

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As described, we will proceed to find a row-echelon form of $\boldsymbol{A}$, which is shown here.

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A basis for the row space of $\boldsymbol{A}$ would thus consists of the three non zero rows of this row-echelon form. Namely, the vectors $(2, 2, -1, 0, 1)$, $(0, 0, \frac{3}{2}, -3, \frac{3}{2})$ and $(0, 0, 0, 3, 0)$.

Slide 21: To summarise this unit,

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we first proved that row equivalent matrices have the same row space. In other words, elementary row operations preserve the row space of a matrix and keeps it unchanged.

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Using this result, we now have a systematic method of finding a basis for the row space of a matrix.