Week 07

MA1508E LINEAR ALGEBRA FOR ENGINEERING

Week 6 Question 1

Number correct	Number wrong	Did not attempt	Total
368 (92%)	34 (8%)		402

Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be vectors in $\mathbb{R}^3$. Which of the following statements is/are definitely true?

(I) If $\boldsymbol{x}$ and $\boldsymbol{y}$ are both linear combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$, then $\boldsymbol{x}+2\boldsymbol{y}$ is also a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

(II) The zero vector $(0, 0, 0)$ is a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

(III) If $\boldsymbol{x}$ is a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$, then $\boldsymbol{x}$ belongs to $\text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$.

(A) (I) and (III) only.

(B) (II) and (III) only.

(C) All 3 statements.

(D) None of the given combinations is correct.

Week 6 Question 2

Number correct	Number wrong	Did not attempt	Total
335 (83%)	67 (17%)		402

Suppose you are given the following, which of the statements below is/are true?

$$\left(\begin{array}{ccc|c} 1 & -1 & -3 & x \\ 1 & 2 & 9 & y \\ -1 & 7 & 27 & z \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 1 & -1 & -3 & x \\ 0 & 3 & 12 & y-x \\ 0 & 0 & 0 & z+3x-2y \end{array}\right).$$

(I) There are vectors in $\mathbb{R}^3$ that **cannot** be written as linear combinations of $(1,1,-1)^T$, $(-1,2,7)^T$ and $(-3,9,27)^T$.

(II) There are vectors in $\mathbb{R}^3$ that can be written as linear combinations of $(1,1,-1)^T$, $(-1,2,7)^T$ and $(-3,9,27)^T$ in more than one way.

(III) $(1,1,-1)^T$, $(-1,2,7)^T$ and $(-3,9,27)^T$ spans $\mathbb{R}^3$.

(A) (I) only.

(B) (III) only.

(C) (II) and (III) only

(D) (I) and (II) only

Week 6 Question 3

How many of the following sets are subspaces?

$$S_1 = \{(x, y, z) \mid x = y = 0\}$$

$$S_2 = \{(a, b, c, d, e) \mid a + b + c + d + e = 1 \quad \text{and} \quad a - b + c - d - e = 0\}$$

$$S_3 = \{(x_1, x_2, x_3, x_4) \mid x_1^2 = x_2^2 \quad \text{and} \quad x_3 = x_4\}$$

(A) None

(B) Exactly one

(C) Excatly two

(D) All three

Number correct	Number wrong	Did not attempt	Total
282 (70%)	119 (30%)	1	402

Week 6 Question 4

Suppose $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are vectors in $\mathbb{R}^3$ such that

$$\boldsymbol{u} = 3\boldsymbol{v} - 2\boldsymbol{w}.$$

Which of the following statements is/are definitely true?

(I) $\text{span}\{\boldsymbol{v}, \boldsymbol{w}\} = \text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$

(II) $\text{span}\{\boldsymbol{u}, \boldsymbol{v}\} = \text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$

(III) $\text{span}\{\boldsymbol{u}\} = \text{span}\{\boldsymbol{v}, \boldsymbol{w}\}$

(IV) $\text{span}\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a line in $\mathbb{R}^3$

(A) All four statements are true.

(B) Only (I) and (IV) are true.

(C) Only (I) and (II) are true.

(D) Only (II), (III) and (IV) are true.

Number correct	Number wrong	Did not attempt	Total
353 (88%)	49 (12%)		402

Week 6 Question 5

Number correct	Number wrong	Did not attempt	Total
355 (88%)	47 (12%)		402

Let S_1 and S_2 be subsets of $\mathbb{R}^4$. Furthermore, we know that $S_1 \subseteq S_2$. Which of the following statements is/are definitely true?

(I) If $\boldsymbol{u} \in S_1$ (that is, $\boldsymbol{u}$ is a vector in S_1), then $\boldsymbol{u} \in S_2$.

(II) If $\boldsymbol{u} \in S_1$, then $\boldsymbol{u} \in \text{span}(S_2)$.

(III) $\text{span}(S_1) \subseteq \text{span}(S_2)$.

(A) All three statements are true.

(B) Only (I) and (II) are true.

(C) Only (II) and (III) are true.

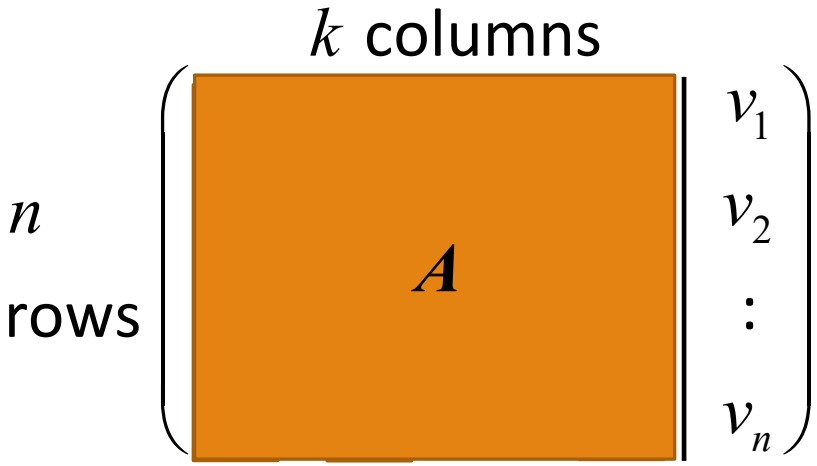
(D) None of the given combination given is correct.

Review of Week 06 (Units 029-034) content

- What is a linear combination?
- To compute a linear combination → easy. To check if a given vector is a linear combination of some other vectors → solving linear system (arising from a vector equation).
- Is every vector in $\boldsymbol{R}^n$ a linear combination of a given set of vectors? How to check?
- If S is a set of vectors in $\boldsymbol{R}^n$, what is span(S) (or the linear span of S)?
- How to check whether a given vector belongs to span(S)?
- Can a subset of $\boldsymbol{R}^n$ be written as a linear span?
- How to check span(S)=$\boldsymbol{R}^n$? How to check span(S) is not equal to $\boldsymbol{R}^n$?

Review of Week 06 (Units 029-034) content

- A general discussion: $S=\{\boldsymbol{u}_1, \boldsymbol{u}_2,\ldots, \boldsymbol{u}_k\}$ is a subset of $\boldsymbol{R}^n$. We want to determine if span(S)=$\boldsymbol{R}^n$.



If a row-echelon form of $\boldsymbol{A}$ does not have a zero row,

span(S)=$\boldsymbol{R}^n$

- If k is strictly less than n, then S cannot span $\boldsymbol{R}^n$.

Review of Week 06 (Units 029-034) content

- A necessary and sufficient condition for span(S_1) to be a subset of span(S_2) →solving linear systems
- "Redundant vector" theorem (removing a vector from a set without changing linear span).
- What is a subspace (of $\boldsymbol{R}^n$)? How is it different from a subset?
- Two extreme cases of subspaces – zero space and entire $\boldsymbol{R}^n$.
- How to determine if a given subset is a subspace.
- How to show that a given subset is NOT a subspace (using property of linear span).

Review of Week 06 (Units 029-034) content

- Linear span of one vector in $\boldsymbol{R}^2$ or $\boldsymbol{R}^3$ (geometrical interpretation) – equation of line?
- Linear span of two vectors in $\boldsymbol{R}^2$ or $\boldsymbol{R}^3$ (geometrical interpretation) – equation of plane?
- Characterising all the subspaces in $\boldsymbol{R}^2$ or $\boldsymbol{R}^3$.

Week 07 (units 035-039) overview

035 Linear Independence Part I

- Definition of linear independence
- By checking whether a homogeneous linear system has only the trivial solution, we determine if a given set of vectors are linearly independent.
- Linear independence for sets with one or two vectors.

036 Linear Independence Part II

- Linear independence and redundancy
- 'Guaranteed' linear dependence

037 Linear Independence in $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$

- Linear independence for two or three vectors in $\boldsymbol{R}^2$ and $\boldsymbol{R}^3$
- When can we add vectors into a linearly independent set and preserve the linear independence property?

Week 07 (units 035-039) overview

038 Bases Part I

- What is a vector space? A subspace of a vector space.
- Definition of a basis (for a vector space)
- Uniqueness in representing a vector in terms of a set of basis vectors

039 Bases Part II and coordinate vectors

- Definition of coordinate vectors (relative to a basis)
- The standard basis for $\boldsymbol{R}^n$

The motivation for a basis (for a vector space)

Consider $\mathbb{R}^{20}$ (just as an example):

Let $S \subseteq \mathbb{R}^{20}$. Consider the number of vectors in S.

$\boldsymbol{u_1} \neq \boldsymbol{0}$: $\{\boldsymbol{u_1}\}$ linearly independent

Theorem

Let $\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}$ be linearly independent vectors in $\mathbb{R}^n$.

Suppose $\boldsymbol{u_{k+1}} \in \mathbb{R}^n$ is NOT a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}$,
then $\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}, \boldsymbol{u_{k+1}}$ are linearly independent.

The motivation for a basis (for a vector space)

Consider $\mathbb{R}^{20}$ (just as an example):

Let $S \subseteq \mathbb{R}^{20}$. Consider the number of vectors in S.

$\{\boldsymbol{u_1}\}\{\boldsymbol{u_1}, \boldsymbol{u_2}\}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$

$\boldsymbol{u_1} \neq \boldsymbol{0}$: $\{\boldsymbol{u_1}\}$ linearly independent

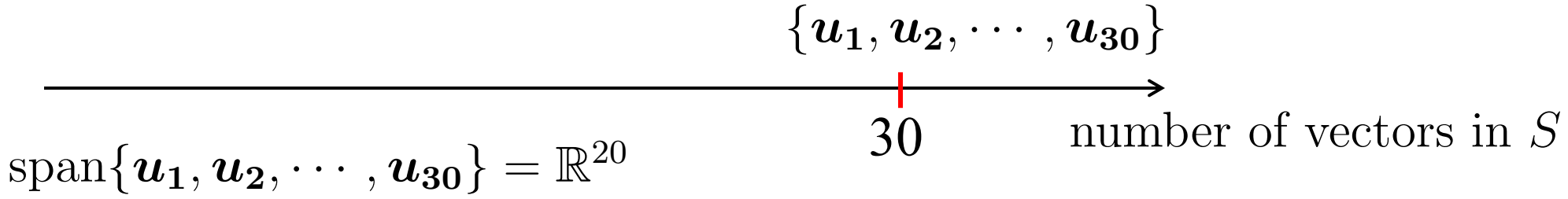
$\boldsymbol{u_2}$ not a linear combination (scalar multiple) of $\boldsymbol{u_1}$: $\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$ linearly independent

$\boldsymbol{u_3}$ not a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}$: $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ linearly independent

To preserve linear independence, how many $\boldsymbol{u_4}, \boldsymbol{u_5}, \cdots$ can be added?

The motivation for a basis (for a vector space)

Consider $\mathbb{R}^{20}$ (just as an example): Let $S \subseteq \mathbb{R}^{20}$. Consider the number of vectors in S.



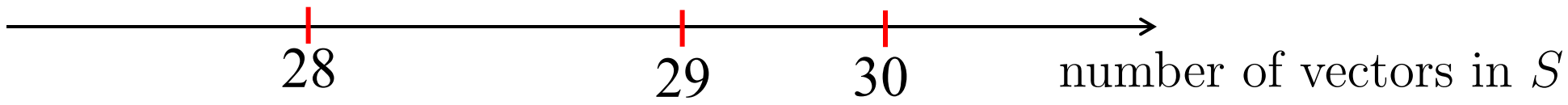
Theorem

If $\boldsymbol{u_k}$ is a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_{k-1}}$, then
$\text{span}\ \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_{k-1}}\} = \text{span}\ \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_{k-1}}, \boldsymbol{u_k}\}$

The motivation for a basis (for a vector space)

Consider $\mathbb{R}^{20}$ (just as an example): Let $S \subseteq \mathbb{R}^{20}$. Consider the number of vectors in S.

$\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_{28}}\}$ $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_{29}}\}$ $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_{30}}\}$



$\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_{30}}\} = \mathbb{R}^{20}$

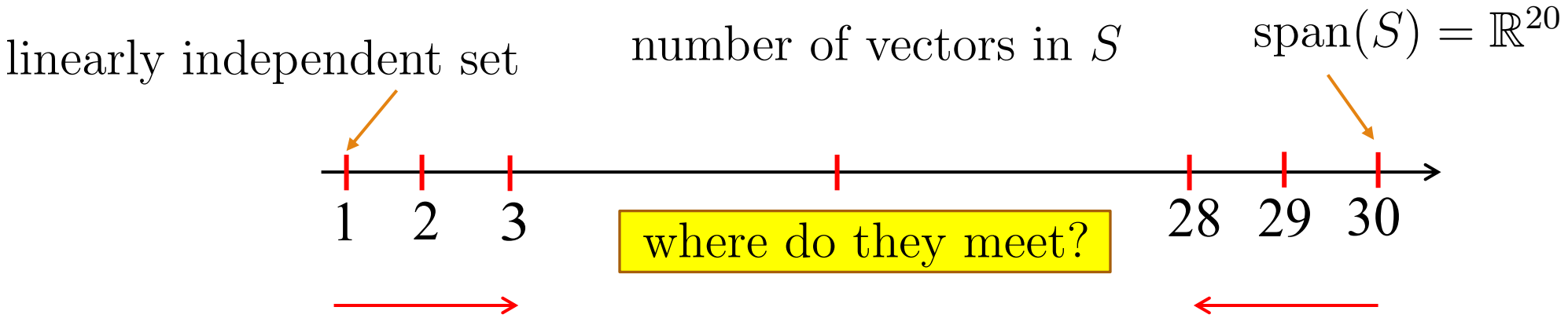
$\boldsymbol{u_{30}}$ is a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_{29}}$: span $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_{29}}\} = \mathbb{R}^{20}$

$\boldsymbol{u_{29}}$ is a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_{28}}$: span $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_{28}}\} = \mathbb{R}^{20}$

To preserve $\text{span}\{\} = \mathbb{R}^{20}$, how many vectors can be removed?

The motivation for a basis (for a vector space)

Consider $\mathbb{R}^{20}$ (just as an example): Let $S \subseteq \mathbb{R}^{20}$. Consider the number of vectors in S.



increase number of vectors in S, maintain linear independence property of S

decrease number of vectors in S, maintain property of $\text{span}(S) = \mathbb{R}^{20}$

Example 6.1

Let $\boldsymbol{u}_1 = (a,1,-1)$, $\boldsymbol{u}_2 = (-1,a,1)$, $\boldsymbol{u}_3 = (1,-1,a)$. For what values of a are $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ linearly independent?

Example 6.2

Suppose $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ are linearly independent vectors in $\mathbb{R}^n$. Determine which of the following sets are linearly independent.

(a) $S_1 = \{\boldsymbol{u},\boldsymbol{v}\}$

(b) $S_2 = \{\boldsymbol{u}-\boldsymbol{v},\boldsymbol{v}-\boldsymbol{w},\boldsymbol{w}-\boldsymbol{u}\}$

(c) $S_3 = \{\boldsymbol{u}-\boldsymbol{v},\boldsymbol{v}-\boldsymbol{w},\boldsymbol{u}+\boldsymbol{w}\}$

(d) $S_4 = \{\boldsymbol{u},\boldsymbol{u}+\boldsymbol{v},\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}\}$

(e) $S_5 = \{\boldsymbol{u}+\boldsymbol{v},\boldsymbol{v}+\boldsymbol{w},\boldsymbol{u}+\boldsymbol{w},\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}\}$

Example 6.3

Let $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ be vectors in $\mathbb{R}^3$ such that $V = \text{span}\{\boldsymbol{u},\boldsymbol{v}\}$ and $W = \text{span}\{\boldsymbol{u},\boldsymbol{w}\}$ are planes in $\mathbb{R}^3$. Find $V \cap W$ if

(a) $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ are linearly independent

(b) $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ are not linearly independent

Example 6.4

Find a basis for each of the following subspace of $\mathbb{R}^4$:

(a) the subspace containing all vectors of the form $(w,0,y,0)$

(b) the subspace containing all vectors of the form (w,x,x,w)

(c) the subspace containing all vectors of the form (w,x,y,z) with $w = 2x = 3y$

(d) the solution space of

$$\begin{cases} 2w + 3x + y + z = 0 \\ -3w + x + 4y - 7z = 0 \\ w + 2x + y = 0 \end{cases}$$

Example 6.5

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ where $\boldsymbol{u}_1 = (1,2,-1), \boldsymbol{u}_2 = (0,2,1), \boldsymbol{u}_3 = (0,-1,3)$.

(a) Show that S is a basis for $\mathbb{R}^3$.

(c) Find the coordinate vector of $\boldsymbol{v} = (1,0,1)$ relative to S.

Finally…

THE END