BASES I

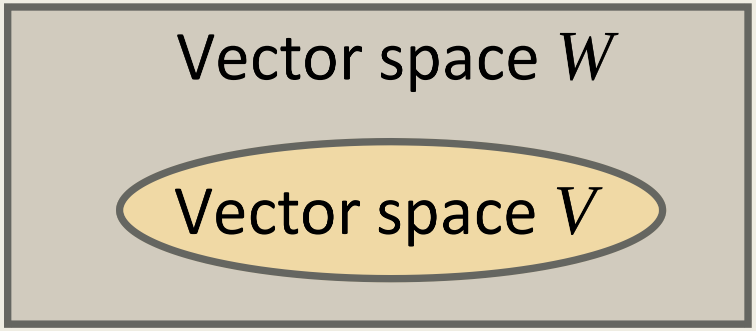
Vector spaces

1) A set V is called a vector space if either $V = \mathbb{R}^n$ or V is a subspace of $\mathbb{R}^n$ for some positive integer n.

Vector space V

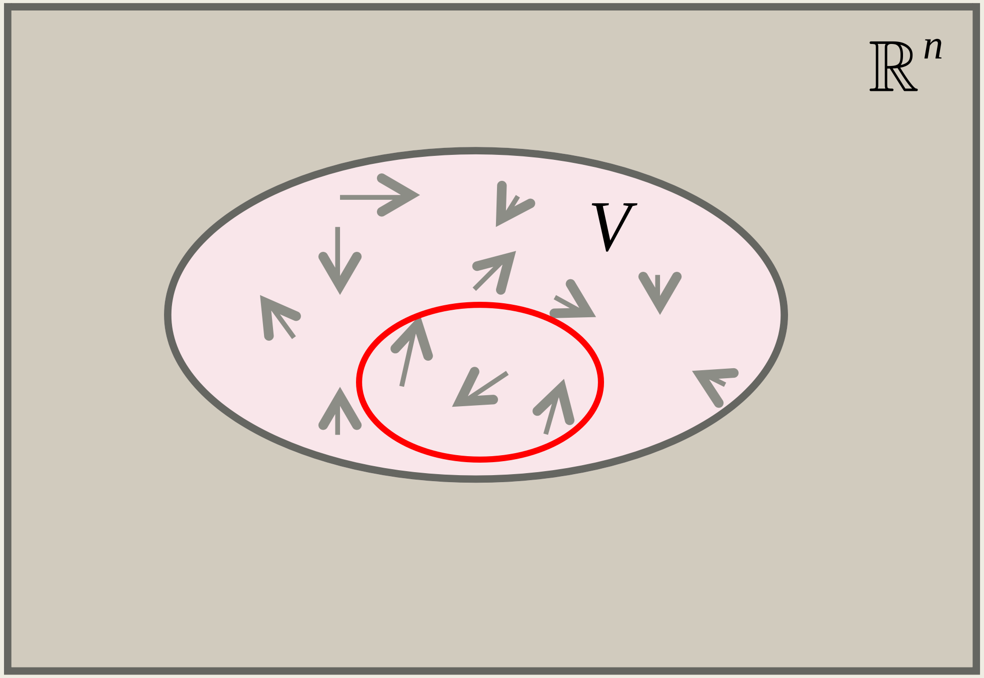
$\mathbb{R}^n$ or Subspace of $\mathbb{R}^n$

2) Let W be a vector space. A set V is called a subspace of W if V is a vector space contained in W.



Finding a small set

Consider a vector space V.



Such a set can then be used to build a 'coordinate system' for V.

Question:

Find a subset S of V, containing as few vectors as possible, so that every vector in V is a linear combination of the vectors in S (that is, $\text{span}(S) = V$).

Definition

Let $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k\}$ be a subset of a vector space V.

Then S is called a basis (plural bases) for V if

1. S is linearly independent and
2. S spans V.

Example

Show that $S = \{(2,4),(1,0)\}$ is a basis for $\mathbb{R}^2$. DONE!

1) S contains only two vectors, which are not multiples of each other, so S is a linearly independent set.

2) Does S span $\mathbb{R}^2$? YES!

Is $a(2,4) + b(1,0) = (x, y)$ always consistent for all (x, y)?

$$\left(\begin{array}{cc|c} 2 & 1 & x \\ 4 & 0 & y \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 2 & 1 & x \\ 0 & -2 & y-2x \end{array}\right)$$

Example

Show that $S=\{(1,2,1),(2,9,0),(3,3,4)\}$ is a basis for $\mathbb{R}^3$.

1) Does S span $\mathbb{R}^3$? YES!

Is $a(1,2,1)+b(2,9,0)+c(3,3,4)=(x,y,z)$ always consistent for all (x,y,z)?

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & x \\ 2 & 9 & 3 & y \\ 1 & 0 & 3 & z \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & * \\ 0 & 5 & -3 & * \\ 0 & 0 & -\frac{1}{5} & * \end{array}\right)$$

Some expression involving x, y, z

Example

Show that $S = \{(1,2,1),(2,9,0),(3,3,4)\}$ is a basis for $\mathbb{R}^3$.

2) Is S a linearly independent set? YES!

DONE!

Does $a(1,2,1)+b(2,9,0)+c(3,3,4)=(0,0,0)$ have only the trivial solution?

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & x \\ 2 & 9 & 3 & y \\ 1 & 0 & 3 & z \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & * \\ 0 & 5 & -3 & * \\ 0 & 0 & -\frac{1}{5} & * \end{array}\right)$$

Some expression involving x, y, z

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 9 & 3 & 0 \\ 1 & 0 & 3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 5 & -3 & 0 \\ 0 & 0 & -\frac{1}{5} & 0 \end{array}\right)$$

Example

Is $S = \{(1,1,0,1),(2,1,0,3),(3,-1,0,3)\}$ a basis for $\mathbb{R}^4$?

No. We know (from a previous unit) that 3 vectors cannot span $\mathbb{R}^4$.

Is $S = \{(1,0,1,1),(0,0,1,2),(-1,0,0,1),(2,0,3,3)\}$ a basis for $\mathbb{R}^4$?

No. $(0,1,0,0)$ is not a linear combination of vectors in S.

Remarks

1) A basis for a vector space V contains the smallest possible number of vectors that can span V.

2) For convenience, we say that the empty set $\varnothing$ is the basis for the zero space.

3) Except the zero space, any vector space has infinitely many different bases.

Theorem

If $S=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k\}$ is a basis for a vector space V, then every vector $\boldsymbol{v}\in V$ can be expressed in the form (as a linear combination of $\boldsymbol{u}_1,\boldsymbol{u}_2,\ldots,\boldsymbol{u}_k$)

$$\boldsymbol{v}=c_1\boldsymbol{u}_1+c_2\boldsymbol{u}_2+\ldots+c_k\boldsymbol{u}_k$$

in exactly one way, where $c_1,c_2,\ldots,c_k\in\mathbb{R}$.

Proof: Suppose it can be done in two ways.

$$\boldsymbol{v}=a_1\boldsymbol{u}_1+a_2\boldsymbol{u}_2+\ldots+a_k\boldsymbol{u}_k \qquad (1)$$

$$\boldsymbol{v}=b_1\boldsymbol{u}_1+b_2\boldsymbol{u}_2+\ldots+b_k\boldsymbol{u}_k \qquad (2)$$

Theorem

It's actually just ONE way!

Proof: Suppose it can be done in two ways.

$$\boldsymbol{v} = a_1\boldsymbol{u}_1 + a_2\boldsymbol{u}_2 + \ldots + a_k\boldsymbol{u}_k \qquad (1)$$

$$\boldsymbol{v} = b_1\boldsymbol{u}_1 + b_2\boldsymbol{u}_2 + \ldots + b_k\boldsymbol{u}_k \qquad (2)$$

(1) − (2) $\Rightarrow$

$$\mathbf{0} = (a_1 - b_1)\boldsymbol{u}_1 + (a_2 - b_2)\boldsymbol{u}_2 + \ldots + (a_k - b_k)\boldsymbol{u}_k \qquad (*)$$

But S is a basis means it is a linearly independent set.

$\Rightarrow$ (*) has only the trivial solution.

$$\Rightarrow a_1 - b_1 = 0, a_2 - b_2 = 0, \ldots, a_k - b_k = 0$$

$$\Rightarrow a_1 = b_1, a_2 = b_2, \ldots, a_k = b_k$$

Summary

1) What is a vector space? A subspace of a vector space.

2) Definition of a basis (for a vector space).

3) Uniqueness in representing a vector in terms of a set of basis vectors.