Week 09 IVLE Quiz

1. If $\boldsymbol{Ax} = \boldsymbol{b}$ ($\boldsymbol{b} \neq \boldsymbol{0}$) is a consistent linear system, which of the following statements are definitely true?

   (I) The column space of $\boldsymbol{A}$ has infinitely many vectors (that is, it is not the zero subspace).

   (II) The reduced row-echelon form of $\boldsymbol{A}$ has no zero rows.

   (III) The column space of $\boldsymbol{A}$ is the same as the column space of the augmented matrix $(\boldsymbol{A} \mid \boldsymbol{b})$.

   (A) (I) only.

   (B) (I) and (III) only.

   (C) All three statements are correct.

   (D) None of the given combinations is correct.

   Answer: (B). (I) is true since $\boldsymbol{b}$ is a non zero vector that belongs to the column space of $\boldsymbol{A}$ (since $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent) then all scalar multiples of $\boldsymbol{b}$ will also be in the column space of $\boldsymbol{A}$. It is not possible to determine whether (II) is true or not as the reduced row-echelon form of $\boldsymbol{A}$ may or may not have zero rows (we only know that $\boldsymbol{Ax} = \boldsymbol{b}$ is consistent). (III) is true since $\boldsymbol{b}$ is a linear combination of the columns of $\boldsymbol{A}$ means that in any row-echlon form of $(\boldsymbol{A} \mid \boldsymbol{b})$ the last column will not be a pivot column, so whatever columns of $\boldsymbol{A}$ that forms a basis for the column space of $\boldsymbol{A}$ will also be a basis for the column space of $(\boldsymbol{A} \mid \boldsymbol{b})$.

2. $\boldsymbol{R}$ is the reduced row-echelon form of $\boldsymbol{A}$. Suppose $\boldsymbol{R}$ has k leading entries. How many of the following statements is/are correct?

   (I) The dimension of the row space of $\boldsymbol{A}$ is k.

   (II) The dimension of the column space of $\boldsymbol{A}$ is k.

   (III) The row space of $\boldsymbol{A}$ is the same as the column space of $\boldsymbol{A}$.

   (IV) The row space of $\boldsymbol{A}$ is a subspace of the column space of $\boldsymbol{A}$.

   (A) None.

   (B) Exactly one.

   (C) Exactly two.

   (D) Three or more.

   Answer: (C). (I) and (II) are both correct as the k is the rank of $\boldsymbol{A}$. (III) is not necessarily correct, for example, when $\boldsymbol{A}$ is not a square matrix, then the column space and row space of $\boldsymbol{A}$ would be subspaces of different Euclidean spaces so they are not comparable. (IV) is incorrect for the same reason as (III).

3. Which of the statements below is/are correct about the nullspace of a matrix $\boldsymbol{A}$?

(I) The nullspace of $\boldsymbol{A}$ is equal to the solution space of $\boldsymbol{Ax} = \boldsymbol{0}$.

(II) The nullspace of $\boldsymbol{A}$ and the nullspace of $\boldsymbol{A}^T$ have the same dimension.

(III) The dimension of the nullspace of $\boldsymbol{A}$ is the number of non pivot columns in a row-echelon form of $\boldsymbol{A}$.

(A) (II) and (III) only

(B) (I) and (II) only

(C) (I) and (III) only

(D) None of the given combinations is correct

Answer: (C). (I) is correct by definition. (II) is incorrect, say, for example, when $\boldsymbol{A}$ is a 4×6 matrix of rank 3. Then the dimension of the nullspace of $\boldsymbol{A}$ (that is the nullity of $\boldsymbol{A}$) would be 3 while the dimension of the nullspace of $\boldsymbol{A}^T$ would be 1. (III) is correct since the number of non pivot columns in a row-echelon form of $\boldsymbol{A}$ would give rise to the number of arbitrary parameters in a general solution of $\boldsymbol{Ax} = \boldsymbol{0}$. Each of these arbitrary parameters would give rise to one vector in a basis for the nullspace of $\boldsymbol{A}$.

4. Let $\boldsymbol{A}$ be a 4×6 matrix with rank 3. How many of the statements below are correct?

(I) The nullity of $\boldsymbol{A}$ is 1.

(II) The rank of $\boldsymbol{A}^T$ is 3.

(III) The nullity of $\boldsymbol{A}^T\boldsymbol{A}$ is less than or equal to the nullity of $\boldsymbol{A}$.

(A) Exactly one.

(B) Exactly two.

(C) All three.

(D) None.

Answer: (B). (I) is incorrect since the nullity of $\boldsymbol{A}$ is $6 - 3 = 3$. (II) is correct since the ranks of $\boldsymbol{A}$ and $\boldsymbol{A}^T$ are the same. (III) is correct since the nullspace of $\boldsymbol{A}^T\boldsymbol{A}$ and the nullspace of $\boldsymbol{A}$ are always the same (see Tutorial 8).

5. Let $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ be an orthonormal basis for a subspace V of $\mathbb{R}^n$. If $\boldsymbol{w}$ is a vector in V such that $\boldsymbol{w} \cdot \boldsymbol{u_1} > 0$ and

$$\boldsymbol{w} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k},$$

How many statements below regarding c_1 is/are definitely correct?

(I) c_1 is always positive.

(II) c_1 is always an integer.

(III) c_1 is always negative.

(A) Exactly one.

(B) Exactly two.

(C) None.

Answer: (A). (I) is correct since c_1 is precisely $\boldsymbol{w} \cdot \boldsymbol{u_1}$. (II) is not necessary correct since $c_1 = \boldsymbol{w} \cdot \boldsymbol{u_1}$ may not be an integer. (III) is incorrect since we have already established that (I) is correct.