LINEAR COMBINATIONS

DEFINITION

Consider $\boldsymbol{u} = (1, 2, -1)$, $\boldsymbol{v} = (0, 2, 5)$.

$2\boldsymbol{u} + 3\boldsymbol{v} = (2,10,13)$ $\qquad$ $\boldsymbol{u} - 2\boldsymbol{v} = (1, -2, -11)$

$(2,10,13)$ and $(1, -2, -11)$ are both linear combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$.

Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ be vectors in $\mathbb{R}^n$.

For any real numbers $c_1, c_2, \ldots, c_k$, the vector

$$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k$$

is a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$.

EXAMPLE

Consider $\boldsymbol{u} = (1, 2, -1),\ \boldsymbol{v} = (0, 2, 5),\ \boldsymbol{w} = (1, 0, -2).$

Question: Compute the linear combination $2\boldsymbol{u} + 3\boldsymbol{v} - \boldsymbol{w}$

Answer: This is simple.

$$
\begin{aligned}
2\boldsymbol{u} + 3\boldsymbol{v} - \boldsymbol{w} &= 2(1, 2, -1) + 3(0, 2, 5) - (1, 0, -2) \\
&= (1, 10, 15)
\end{aligned}
$$

EXAMPLE

Consider $\boldsymbol{u} = (1,2,-1),\ \boldsymbol{v} = (0,2,5),\ \boldsymbol{w} = (1,0,-2)$.

Question: Is $(0,4,8)$ a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$?

Answer: We need to check whether there are real numbers a, b, c such that

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = (0,4,8)$$

$$a(1,2,-1) + b(0,2,5) + c(1,0,-2) = (0,4,8)$$

How to check?

EXAMPLE

Consider $\boldsymbol{u}=(1,2,-1)$, $\boldsymbol{v}=(0,2,5)$, $\boldsymbol{w}=(1,0,-2)$.

Yes!

Question: Is $(0,4,8)$ a linear combination of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$?

$$a(1,2,-1)+b(0,2,5)+c(1,0,-2)=(0,4,8)$$

$$\begin{cases} a \quad\quad\quad + \ c = 0 \\ 2a + 2b \quad\quad\quad = 4 \\ -a + 5b - 2c = 8 \end{cases}$$

$$a=\tfrac{1}{2},\ b=\tfrac{3}{2},\ c=-\tfrac{1}{2}$$

$$\tfrac{1}{2}\boldsymbol{u}+\tfrac{3}{2}\boldsymbol{v}-\tfrac{1}{2}\boldsymbol{w}=(0,4,8)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 2 & 0 & 4 \\ -1 & 5 & -2 & 8 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gauss-Jordan}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right)$$

EXAMPLE

Consider $\boldsymbol{u} = (2,1,3),\ \boldsymbol{v} = (1,-1,2),\ \boldsymbol{w} = (3,0,5)$.

Question: Is $(3,3,4)$ a linear combination of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$?

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = (3,3,4)$$

$$a(2,1,3) + b(1,-1,2) + c(3,0,5) = (3,3,4)$$

$$\begin{cases} 2a & + & b & + & 3c & = & 3 \\ a & - & b & & & = & 3 \\ 3a & + & 2b & + & 5c & = & 4 \end{cases} \qquad \left(\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -1 & 0 & 3 \\ 3 & 2 & 5 & 4 \end{array}\right)$$

EXAMPLE

Consider $\boldsymbol{u}=(2,1,3),\ \boldsymbol{v}=(1,-1,2),\ \boldsymbol{w}=(3,0,5)$.

Question: Is $(3,3,4)$ a linear combination of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$?

$$a\mathbf{u}+b\mathbf{v}+c\mathbf{w}=(3,3,4) \quad a(2,1,3)+b(1,-1,2)+c(3,0,5)=(3,3,4)$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -1 & 0 & 3 \\ 3 & 2 & 5 & 4 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Is the linear system consistent? Are the solutions unique?

EXAMPLE

Consider $\boldsymbol{u}=(2,1,3),\ \boldsymbol{v}=(1,-1,2),\ \boldsymbol{w}=(3,0,5)$.

Question: Is $(3,3,4)$ a linear combination of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$?

$$a\mathbf{u}+b\mathbf{v}+c\mathbf{w}=(3,3,4) \quad a(2,1,3)+b(1,-1,2)+c(3,0,5)=(3,3,4)$$

$$\begin{cases} a &= 2-t \\ b &= -1-t \\ c &= t \qquad t\in\mathbb{R} \end{cases}$$

$$(a,b,c)=(2,-1,0) \quad (t=0)$$

$$2(2,1,3)-(1,-1,2)+0(3,0,5)=(3,3,4)$$

$$(a,b,c)=(1,-2,1) \quad (t=1)$$

$$(2,1,3)-2(1,-1,2)+(3,0,5)=(3,3,4)$$

EXAMPLE

Consider $\boldsymbol{u} = (2,1,3),\ \boldsymbol{v} = (1,-1,2),\ \boldsymbol{w} = (3,0,5)$.

Question: Is $(1,2,4)$ a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$?

$$a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w} = (1,2,4)$$

$$a(2,1,3) + b(1,-1,2) + c(3,0,5) = (1,2,4)$$

$$\begin{cases} 2a + b + 3c = 1 \\ a - b = 2 \\ 3a + 2b + 5c = 4 \end{cases} \qquad \left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 1 & -1 & 0 & 2 \\ 3 & 2 & 5 & 4 \end{array}\right)$$

EXAMPLE

Consider $\boldsymbol{u} = (2,1,3),\ \boldsymbol{v} = (1,-1,2),\ \boldsymbol{w} = (3,0,5)$.

Question: Is $(1,2,4)$ a linear combination of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$?

No!

$$a\boldsymbol{u} + b\boldsymbol{v} + c\boldsymbol{w} = (1,2,4) \quad a(2,1,3) + b(1,-1,2) + c(3,0,5) = (1,2,4)$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 1 & -1 & 0 & 2 \\ 3 & 2 & 5 & 4 \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 3 \end{array}\right)$$

Is the linear system consistent?

EXAMPLE

Consider $\boldsymbol{e}_1=(1,0,0,0)$, $\boldsymbol{e}_2=(0,1,0,0)$, $\boldsymbol{e}_3=(0,0,1,0)$, $\boldsymbol{e}_4=(0,0,0,1)$

$$(1,2,3,4)=1\boldsymbol{e}_1+2\boldsymbol{e}_2+3\boldsymbol{e}_3+4\boldsymbol{e}_4$$

$$(-3,\tfrac{1}{3},0,2)=-3\boldsymbol{e}_1+\tfrac{1}{3}\boldsymbol{e}_2+0\boldsymbol{e}_3+2\boldsymbol{e}_4$$

Any (w,x,y,z) in $\mathbb{R}^4$:

$$(w,x,y,z)=w\boldsymbol{e}_1+x\boldsymbol{e}_2+y\boldsymbol{e}_3+z\boldsymbol{e}_4$$

Every vector $\boldsymbol{u}=(w,x,y,z)$ in $\mathbb{R}^4$ is a linear combination of $\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3,\boldsymbol{e}_4$.

EXAMPLE

Consider $\boldsymbol{u} = (1,2,-1),\ \boldsymbol{v} = (0,2,5),\ \boldsymbol{w} = (1,0,-2).$

Question: Is every vector in $\mathbb{R}^3$ a linear combination of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$?

arbitrary vector in $\mathbb{R}^3$

$$a(1,2,-1) + b(0,2,5) + c(1,0,-2) = (x,y,z)$$

$$\begin{cases} a & & & + & c & = & x \\ 2a & + & 2b & & & = & y \\ -a & + & 5b & - & 2c & = & z \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 0 & 1 & x \\ 2 & 2 & 0 & y \\ -1 & 5 & -2 & z \end{array}\right)$$

EXAMPLE

Consider $\boldsymbol{u}=(1,2,-1),\ \boldsymbol{v}=(0,2,5),\ \boldsymbol{w}=(1,0,-2)$.

Yes!

Question: Is every vector in $\mathbb{R}^3$ a linear combination of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$?

arbitrary vector in $\mathbb{R}^3$

$$a(1,2,-1)+b(0,2,5)+c(1,0,-2)=(x,y,z)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & x \\ 2 & 2 & 0 & y \\ -1 & 5 & -2 & z \end{array}\right) \xrightarrow[\text{Elimination}]{\text{Gaussian}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & x \\ 0 & 2 & -2 & y-2x \\ 0 & 0 & -4 & z-\frac{5y}{2}+6x \end{array}\right)$$

always consistent, regardless of the values of x,y,z.

EXAMPLE

Consider $\boldsymbol{u}=(3,6,2),\ \boldsymbol{v}=(-1,0,1),\ \boldsymbol{w}=(3,12,7)$.

Question: Is every vector in $\mathbb{R}^3$ a linear combination of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$?

arbitrary vector in $\mathbb{R}^3$

$$a(3,6,2)+b(-1,0,1)+c(3,12,7)=(x,y,z)$$

$$\begin{cases} 3a & - & b & + & 3c & = & x \\ 6a & & & + & 12 & = & y \\ 2a & + & b & + & 7c & = & z \end{cases} \qquad \left(\begin{array}{ccc|c} 3 & -1 & 3 & x \\ 6 & 0 & 12 & y \\ 2 & 1 & 7 & z \end{array}\right)$$

EXAMPLE

No!

Consider $\boldsymbol{u} = (3,6,2)$, $\boldsymbol{v} = (-1,0,1)$, $\boldsymbol{w} = (3,12,7)$.

Question: Is every vector in $\mathbb{R}^3$ a linear combination of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$?

arbitrary vector in $\mathbb{R}^3$

$$a(3,6,2) + b(-1,0,1) + c(3,12,7) = (x,y,z)$$

$$\left(\begin{array}{ccc|c} 3 & -1 & 3 & x \\ 6 & 0 & 12 & y \\ 2 & 1 & 7 & z \end{array}\right) \xrightarrow{\text{Gaussian Elimination}} \left(\begin{array}{ccc|c} 3 & -1 & 3 & x \\ 0 & 2 & 6 & y-2x \\ 0 & 0 & 0 & \boxed{z - \frac{5y}{6} + x} \end{array}\right)$$

will be inconsistent, for some values of x,y,z.

SUMMARY

1) What is a linear combination of vectors.

2) How to check whether a given vector is a linear combination of some other vectors.

Vector equation $\rightarrow$ Linear system $\rightarrow$ check consistency

3) How to check whether every vector in $\mathbb{R}^n$ is a linear combination of some (collection of) vectors.