WRITING SOLUTIONS FROM ROW-ECHELON FORMS

WHAT CAN ROW-ECHELON FORM TELL US?

Remember that any linear system has either

(i) no solution (that is, inconsistent); or

(ii) exactly one solution (that is, an unique solution); or

(iii) infinitely many solutions.

By looking at a row-echelon form of the augmented matrix of the linear system, we can determine which of the above holds for the linear system.

INCONSISTENT LINEAR SYSTEMS

If the augmented matrix of a linear system has a row-echelon form whose last column is a pivot column, then the linear system is inconsistent.

$$\left(\begin{array}{cccc|c} \otimes & & & & \\ & \otimes & & & \\ & & & \otimes & \\ 0 & 0 & \dots & 0 & \otimes \end{array}\right)$$

last column

$\otimes$: leading entry

In other words, there is a row where every entry is zero except the last entry, which is non zero.

INCONSISTENT LINEAR SYSTEMS

If the augmented matrix of a linear system has a row-echelon form whose last column is a pivot column, then the linear system is inconsistent.

$$\begin{cases} x - y = 2 \\ y = 0 \\ \boxed{0x + 0y + 0z = 2} \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{2} \end{array}\right)$$

CONSISTENT LINEAR SYSTEMS

If the augmented matrix of a linear system has a row-echelon form whose last column is NOT a pivot column, then the linear system is consistent.

$$\left(\begin{array}{cccc|c} \otimes & & & & \\ & \otimes & & & \\ & & & \otimes & \\ & & & & \\ 0 & 0 & \ldots & 0 & 0 \end{array}\right)$$

Consistent = Only one (unique) solution

OR

Infinitely many solutions?

CONSISTENT LINEAR SYSTEMS (UNIQUE SOLUTION)

If the augmented matrix of a consistent linear system has a row-echelon form where every column (except the last) is a pivot column, then the linear system has a unique (that is, exactly one) solution.

$$\left(\begin{array}{ccccc|c} \otimes & & & & & \\ & \otimes & & & & \\ & & \otimes & & & \\ & & & \ddots & & \\ & & & & \otimes & \\ 0 & 0 & 0 & \dots & 0 & 0 \end{array}\right)$$

Every column here is a pivot column

CONSISTENT LINEAR SYSTEMS (UNIQUE SOLUTION)

If the augmented matrix of a consistent linear system has a row-echelon form where every column (except the last) is a pivot column, then the linear system has a unique (that is, exactly one) solution.

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 4 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 3 & 2 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \left\{\begin{array}{rcrcrcrcr} -w & & & + & y & + & 4z & = & 0 \\ & & x & - & y & + & 3z & = & 1 \\ & & & & 3y & + & 2z & = & -1 \\ & & & & & & z & = & 0 \end{array}\right.$$

Back substitution

$z = 0 \rightarrow$ solve for $y \rightarrow$ solve for $x \rightarrow$ solve for w

CONSISTENT LINEAR SYSTEMS (INFINITELY MANY SOLUTIONS)

If the augmented matrix of a consistent linear system has a row-echelon form where some column (other than the last) is NOT a pivot column, then the linear system has infinitely many solutions.

$$\left(\begin{array}{ccccc|c} \otimes & * & & & & \\ & * & & & & \\ & 0 & \otimes & & & \\ & \vdots & & \ddots & & \\ & 0 & & & \otimes & \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{array}\right)$$

Some column here is NOT a pivot column

CONSISTENT LINEAR SYSTEMS (INFINITELY MANY SOLUTIONS)

If the augmented matrix of a consistent linear system has a row-echelon form where some column (other than the last) is NOT a pivot column, then the linear system has infinitely many solutions.

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 4 & 0 \\ 0 & 1 & -1 & 3 & 1 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \left\{\begin{array}{rcrcrcrcr} -w & & & + & y & + & 4z & = & 0 \\ & & x & - & y & + & 3z & = & 1 \\ & & & & & & 2z & = & -1 \end{array}\right.$$

How to write down the general solution?

WRITING A GENERAL SOLUTION

5 variables: x_1, x_2, x_3, x_4, x_5

$$\left(\begin{array}{ccccc|c} 0 & -1 & 4 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 3 & 3 & 6 \end{array}\right)$$

reduced row-echelon form?

No. Row-echelon form.

Writing out the linear system:

$$\begin{cases} -x_2 + 4x_3 - x_5 = 0 \\ x_3 + x_4 + x_5 = 3 \\ 3x_4 + 3x_5 = 6 \end{cases}$$

What happened to x_1?

WRITING A GENERAL SOLUTION

For any 'variable' column (left side of vertical line) that is a non pivot column, we will assign an arbitrary parameter to that variable.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & & & & x_5 & \end{array} \\ \left(\begin{array}{ccccc|c} 0 & -1 & 4 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 3 & 3 & 6 \end{array}\right) \end{array}$$

In this case, since x_1 does not appear in any of the equations, none of the variables will 'depend' on the value of x_1.

$$\begin{cases} -x_2 + 4x_3 - x_5 = 0 \\ x_3 + x_4 + x_5 = 3 \\ 3x_4 + 3x_5 = 6 \end{cases}$$

WRITING A GENERAL SOLUTION

Let $x_1 = s, x_5 = t$, where $s, t \in \mathbb{R}$.

$$\begin{cases} -x_2 + 4x_3 - x_5 = 0 \\ x_3 + x_4 + x_5 = 3 \\ 3x_4 + 3x_5 = 6 \end{cases}$$

Starting from the last equation:

$3x_4 + 3t = 6 \Leftrightarrow 3x_4 = 6 - 3t \Leftrightarrow x_4 = 2 - t.$

Substitute into next 'higher' equation:

$x_3 + (2 - t) + t = 3 \Leftrightarrow x_3 = 1.$

Substitute into next 'higher' equation:

$-x_2 + 4(1) - t = 0 \Leftrightarrow x_2 = 4 - t.$

WRITING A GENERAL SOLUTION

Let $\underline{x_1 = s, x_5 = t}$, where $s, t \in \mathbb{R}$.

$3x_4 + 3t = 6 \Leftrightarrow 3x_4 = 6 - 3t \Leftrightarrow \underline{x_4 = 2 - t}$.

$x_3 + (2-t) + t = 3 \Leftrightarrow \underline{x_3 = 1}$.

$-x_2 + 4(1) - t = 0 \Leftrightarrow \underline{x_2 = 4 - t}$.

Back substitution

A general solution is:

$$\begin{cases} x_1 &= s \\ x_2 &= 4-t \\ x_3 &= 1 \\ x_4 &= 2-t \\ x_5 &= t, \quad s, t \in \mathbb{R}. \end{cases}$$

Linear system has infinitely many solutions.

WRITING A GENERAL SOLUTION

5 variables: v, w, x, y, z

$$\begin{array}{cccccc} & w & & y & & \\ \end{array}$$
$$\left(\begin{array}{ccccc|c} 1 & \boxed{1} & 0 & \boxed{0} & 0 & 0 \\ 0 & \boxed{0} & 1 & \boxed{1} & 0 & -2 \\ 0 & \boxed{0} & 0 & \boxed{0} & 1 & 4 \end{array}\right)$$

reduced row-echelon form?

Yes

Writing out the linear system:

$$\begin{cases} v + w = 0 \\ x + y = -2 \\ z = 4 \end{cases}$$

Let $w = s, y = t,\ s, t \in \mathbb{R}$.

WRITING A GENERAL SOLUTION

$$\begin{cases} v + w & & = 0 \\ & x + y & = -2 \\ & & z = 4 \end{cases}$$

No back substitution!

Let $w = s, y = t,\ s,t \in \mathbb{R}$.

Starting from the last equation:

$z = 4$.

Next equation:

$x + t = -2 \Leftrightarrow x = -2 - t$.

Next equation:

$v + s = 0 \Leftrightarrow v = -s$.

A general solution is:

$$\begin{cases} v = -s \\ w = s \\ x = -2 - t \\ y = t \\ z = 4, \quad s,t \in \mathbb{R}. \end{cases}$$

SUMMARY

1) What row-echelon forms can tell us.

2) Last column is a pivot column - inconsistent

3) Last column is non-pivot - consistent.

4) Unique solution vs. Infinitely many solutions.

5) How to write down a general solution based on a (reduced) row-echelon form.