Week 12 F2F Example Solutions

1. **Example 11.1** Note that $e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$ for $x \in \mathbb{R}$.

   (a) Since $\boldsymbol{A}^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}$ for $n = 1, 2, \ldots,$

   $$e^{\boldsymbol{A}} = \begin{pmatrix} 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots & 0 & 0 \\ 0 & 1 + \frac{1}{1!}2 + \frac{1}{2!}2^2 + \cdots & 0 \\ 0 & 0 & 1 + \frac{1}{1!}3 + \frac{1}{2!}3^2 + \cdots \end{pmatrix} = \begin{pmatrix} e & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^3 \end{pmatrix}.$$

   (b) Let $\boldsymbol{P} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$. Since $\boldsymbol{A}^n = \boldsymbol{P}\begin{pmatrix} 2^n & 0 \\ 0 & 4^n \end{pmatrix}\boldsymbol{P}^{-1}$ for $n = 1, 2, \ldots,$

   $$e^{\boldsymbol{A}} = \boldsymbol{P}\begin{pmatrix} 1 + \frac{1}{1!}2 + \frac{1}{2!}2^2 + \cdots & 0 \\ 0 & 1 + \frac{1}{1!}4 + \frac{1}{2!}4^2 + \cdots \end{pmatrix}\boldsymbol{P}^{-1} = \frac{1}{2}\begin{pmatrix} e^4 + e^2 & e^4 - e^2 \\ e^4 - e^2 & e^4 + e^2 \end{pmatrix}.$$

2. **Example 11.2** Let a_n, b_n and c_n be the percentage of customers choosing brand A, B and C, respectively, after n months. Then for $n = 1, 2, \ldots,$

   $$\begin{cases} a_n = 0.97a_{n-1} + 0.01b_{n-1} + 0.02c_{n-1} \\ b_n = 0.01a_{n-1} + 0.97b_{n-1} + 0.02c_{n-1} \\ c_n = 0.02a_{n-1} + 0.02b_{n-1} + 0.96c_{n-1}. \end{cases}$$

   Let $\boldsymbol{x_n} = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$ and $\boldsymbol{A} = \begin{pmatrix} 0.97 & 0.01 & 0.02 \\ 0.01 & 0.97 & 0.02 \\ 0.02 & 0.02 & 0.96 \end{pmatrix}$.

   Then $\boldsymbol{x_n} = \boldsymbol{A}\boldsymbol{x}_{n-1} = \cdots = \boldsymbol{A}^n\boldsymbol{x_0}$ where $\boldsymbol{x_0} = \begin{pmatrix} 100 \\ 0 \\ 0 \end{pmatrix}$.

   By Algorithm 6.2.4, we find $\boldsymbol{P} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & 0 & 2 \end{pmatrix}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.96 & 0 \\ 0 & 0 & 0.94 \end{pmatrix}$.

   Then

   $$\boldsymbol{x_n} = \boldsymbol{P}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.96^n & 0 \\ 0 & 0 & 0.94^n \end{pmatrix}\boldsymbol{P}^{-1}\boldsymbol{x_0} = \frac{50}{3}\begin{pmatrix} 2 + 3 \cdot 0.96^n + 0.94^n \\ 2 - 3 \cdot 0.96^n + 0.94^n \\ 2 - 2 \cdot 0.94^n \end{pmatrix}.$$

   The present market shares are $\frac{50}{3}[2 + 3 \cdot 0.96^4 + 0.94^4]\% \approx 88.8\%$, $\frac{50}{3}[2 - 3 \cdot 0.96^4 + 0.94^4]\% \approx 3.9\%$ and $\frac{50}{3}[2 - 2 \cdot 0.94^4]\% \approx 7.3\%$ for brand A, B and C, respectively.

   The market shares will stabilize after a long run and $\lim_{n\to\infty} \boldsymbol{x_n} = \begin{pmatrix} \frac{100}{3} \\ \frac{100}{3} \\ \frac{100}{3} \end{pmatrix}$.

3. **Example 11.3** Set $y_3 = y_1'$ and $y_4 = y_2'$. This gives the first-order system

$$\begin{cases} y_1' &= & & & & & y_3 & & \\ y_2' &= & & & & & & & y_4 \\ y_3' &= & 2y_1 &+& y_2 &+& y_3 &+& y_4 \\ y_4' &= & -5y_1 &+& 2y_2 &+& 5y_3 &-& y_4 \end{cases}$$

The coefficient matrix for this system is

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 1 \\ -5 & 2 & 5 & -1 \end{pmatrix}.$$

Solving for the eigenvalues of $\boldsymbol{A}$, we find that $\boldsymbol{A}$ has 4 distinct eigenvalues $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 3$, $\lambda_4 = -3$ and the corresponding eigenvectors

$$\boldsymbol{x_1} = (1, -1, 1, -1)^T \qquad \boldsymbol{x_2} = (1, 5, -1, -5)^T$$

$$\boldsymbol{x_3} = (1, 1, 3, 3)^T \qquad \boldsymbol{x_4} = (1, -5, -3, 15)^T.$$

Thus, the general solution to the first-order system is of the form

$$c_1\boldsymbol{x_1}e^t + c_2\boldsymbol{x_2}e^{-t} + c_3\boldsymbol{x_3}e^{3t} + c_4\boldsymbol{x_4}e^{-3t}.$$

Now we use the initial condition provided to find c_1, c_2, c_3, c_4. When $t = 0$, we have

$$c_1\boldsymbol{x_1} + c_2\boldsymbol{x_2} + c_3\boldsymbol{x_3} + c_4\boldsymbol{x_4} = (4, 4, 4, -4)$$

or equivalently

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 5 & 1 & -5 \\ 1 & -1 & 3 & -3 \\ -1 & -5 & 3 & 15 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \\ -4 \end{pmatrix}.$$

The above system can be solved to give the unique solution $c_1 = 2, c_2 = 1, c_3 = 1, c_4 = 0$. Thus the solution to the initial value problem is

$$\boldsymbol{Y} = 2\boldsymbol{x_1}e^t + \boldsymbol{x_2}e^{-t} + \boldsymbol{x_3}e^{3t}.$$

Thus

$$\begin{pmatrix} y_1 \\ y_2 \\ y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2e^t + e^{-t} + e^{3t} \\ -2e^t + 5e^{-t} + e^{3t} \\ 2e^t - e^{-t} + 3e^{3t} \\ -2e^t - 5e^{-t} + 3e^{3t} \end{pmatrix}.$$