EQUIVALENT STATEMENTS PART III

Theorem

Let $\boldsymbol{A}$ be a $n \times n$ matrix. The following statements are equivalent.

1) $\boldsymbol{A}$ is invertible.

2) $\boldsymbol{Ax} = \mathbf{0}$ has only the trivial solution.

3) The rref of $\boldsymbol{A}$ is $\boldsymbol{I}$.

4) $\boldsymbol{A}$ can be expressed as a product of elementary matrices.

5) $\det(\boldsymbol{A}) \neq 0$.

Established in a previous unit

6) The rows of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$.

7) The columns of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$.

Theorem

Proof:

We first prove that $\boldsymbol{Ax} = \mathbf{0}$ has only the trivial solution

$\Leftrightarrow$ The columns of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$.

Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \ldots & \boldsymbol{c}_n \end{pmatrix}$, that is $\boldsymbol{c}_i$ is the i-th column of $\boldsymbol{A}$.

Since $\dim(\mathbb{R}^n) = n$, to show that the columns of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$, it suffices to show that the columns of $\boldsymbol{A}$ are linearly independent.

$$\boldsymbol{Ax} = \mathbf{0} \Leftrightarrow \begin{pmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \ldots & \boldsymbol{c}_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Leftrightarrow x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \ldots + x_n\boldsymbol{c}_n = \mathbf{0}$$

Theorem

Proof:

We first prove that $\boldsymbol{Ax}=\boldsymbol{0}$ has only the trivial solution

$\Leftrightarrow$ The columns of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$.

Done!

$$\boxed{\boldsymbol{Ax}=\boldsymbol{0}} \Leftrightarrow \begin{pmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \dots & \boldsymbol{c}_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Leftrightarrow \boxed{x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \dots + x_n\boldsymbol{c}_n = \boldsymbol{0}}$$

$\boxed{\boldsymbol{Ax}=\boldsymbol{0}}$ has only the trivial solution

$\Leftrightarrow \boxed{x_1\boldsymbol{c}_1 + x_2\boldsymbol{c}_2 + \dots + x_n\boldsymbol{c}_n = \boldsymbol{0}}$ has only the trivial solution

$\Leftrightarrow \boldsymbol{c}_1, \boldsymbol{c}_2, \dots, \boldsymbol{c}_n$ are linearly independent

$\Leftrightarrow$ The columns of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$.

Theorem

Proof:

We first prove that $\boldsymbol{Ax} = \mathbf{0}$ has only the trivial solution

$\Leftrightarrow$ The columns of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$.

Done!

We now prove that $\det(\boldsymbol{A}) \neq 0$

Done!

$\Leftrightarrow$ The rows of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$.

$\boxed{\det(\boldsymbol{A}) \neq 0 \Leftrightarrow \text{The columns of } \boldsymbol{A} \text{ forms a basis for } \mathbb{R}^n.}$

$\det(\boldsymbol{A}^T) \neq 0 \Leftrightarrow$ The columns of $\boldsymbol{A}^T$ forms a basis for $\mathbb{R}^n$.

$\Leftrightarrow$

$\det(\boldsymbol{A}) \neq 0 \Leftrightarrow$ The rows of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$.

Theorem

Let $\boldsymbol{A}$ be a $n \times n$ matrix. The following statements are equivalent.

1) $\boldsymbol{A}$ is invertible.

2) $\boldsymbol{Ax} = \boldsymbol{0}$ has only the trivial solution.

3) The rref of $\boldsymbol{A}$ is $\boldsymbol{I}$.

4) $\boldsymbol{A}$ can be expressed as a product of elementary matrices.

5) $\det(\boldsymbol{A}) \neq 0$.

6) The rows of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$.

7) The columns of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^n$.

Example

Is $\{(1,1,1),(-1,0,2),(3,1,3)\}$ a basis for $\mathbb{R}^3$?

Yes!

Compute $\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} -1 & 2 \\ 3 & 3 \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ 3 & 1 \end{vmatrix} \neq 0$

Is $\{(1,1,1,1),(-1,1,-1,1),(0,1,-1,0),(2,1,1,0)\}$ a basis for $\mathbb{R}^4$?

No!

Compute $\begin{vmatrix} 1 & -1 & 0 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 0 \end{vmatrix} = 0$

Summary

1) Two more equivalent statements to "$\boldsymbol{A}$ is invertible".