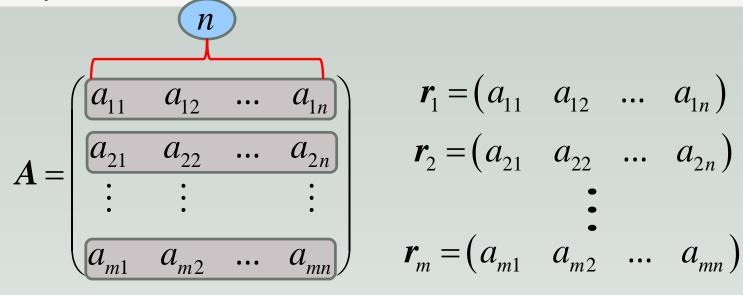
ROW SPACE AND COLUMN SPACE

EFINITION

Given any $m \times n$ matrix A,



$$r_1 = (a_{11} \quad a_{12} \quad \dots \quad a_{1n})$$

$$\mathbf{r}_2 = (a_{21} \quad a_{22} \quad \dots \quad a_{2n})$$

$$\mathbf{r}_{m} = \begin{pmatrix} a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

The rows of A can be considered as vectors in \mathbb{R}^n .

 \Rightarrow span $\{r_1,r_2,...,r_m\}$ is a subspace of \mathbb{R}^n ,

This subspace is called the row space of A.

DEFINITION

Given any $m \times n$ matrix A,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \mathbf{A}$$

The columns of A can be considered as vectors in \mathbb{R}^m .

 \Rightarrow span $\{c_1, c_2, ..., c_n\}$ is a subspace of \mathbb{R}^m ,

$$\boldsymbol{c}_{1} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \quad \boldsymbol{c}_{2} = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \quad \boldsymbol{\cdot} \quad \boldsymbol{c}_{n} = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

This subspace is called the column space of A.

REMARK

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{bmatrix}$$

$$A^{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 4 & -2 \\ 1 & 2 & 0 \\ 4 & 1 & 1 \end{bmatrix}$$

The row space of A is the column space of A^T

The column space of A is the row space of A^T

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

The row space of A is a subspace of \mathbb{R}^3 .

The column space of A is a subspace of \mathbb{R}^4 .

Note that if A is not a square matrix, then the row space and column space of A contains totally 'different type' of vectors.

$$A = \begin{bmatrix} 2 & -1 & 0 & r_1 \\ 1 & -1 & 3 & r_2 & \text{We write } r_1 = (2, -1, 0) \text{ (as a vector)} \\ -5 & 1 & 0 & r_3 & \text{rather than a row matrix } (2 & 1 & 0). \\ \hline 1 & 0 & 1 & r_4 & & \end{bmatrix}$$

The row space of A is a subspace of \mathbb{R}^3 .

$$= \operatorname{span}\{r_1, r_2, r_3, r_4\}$$

$$= \{a(2,1,0) + b(1,-1,3) + c(-5,1,0) + d(1,0,1) \mid a,b,c,d \in \mathbb{R}\}$$

$$= \{(2a+b-5c+d, -a-b+c, 3b+d) \mid a,b,c,d \in \mathbb{R}\}$$

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 3 \\ -5 & 1 & 0 \\ 1 & c_1 & c_2 & c_3 \end{pmatrix}$$

The column space of A is a subspace of \mathbb{R}^4 .

= span
$$\{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3\}$$

$$= \left\{ a \begin{pmatrix} 2 \\ 1 \\ -5 \\ 1 \end{pmatrix} + b \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} \middle| a,b,c \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} 2a-b \\ a-b+3c \\ -5a+b \\ a+c \end{pmatrix} \middle| a,b,c \in \mathbb{R} \right\}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

How to find a basis for the row space of *A*?

The row space of A is a subspace of \mathbb{R}^5 .

The column space of A is a subspace of \mathbb{R}^3 .

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

How to find a basis for the row space of A?

row space of $A = \text{span}\{(1,0,-1,1,4),(0,1,4,2,1),(0,0,-2,0,1)\}$

If (1,0,-1,1,4), (0,1,4,2,1), (0,0,-2,0,1) (that is, the rows of A) are linearly independent, then obviously they will form a basis for the row space of A.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 & 1 & 4 \\ 0 & 1 & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix}$$

How to find a basis for the row space of *A*?

row space of $A = \text{span}\{(1,0,-1,1,4),(0,1,4,2,1),(0,0,-2,0,1)\}$

$$(0,0,0,0) = a(1,0,-1,1,4) + b(0,1,4,2,1) + c(0,0,-2,0,1)$$

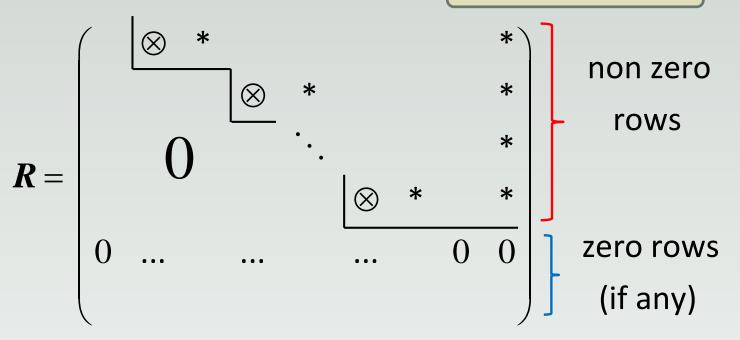
$$\Rightarrow a = 0, b = 0, c = 0$$

So the three rows of A are linearly independent and thus form a basis for the row space of A.

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & 0 & -1 & 1 & 4 \\ 0 & \mathbf{1} & 4 & 2 & 1 \\ 0 & 0 & \mathbf{-2} & 0 & 1 \end{bmatrix}$$

Note that A is in row echelon form.

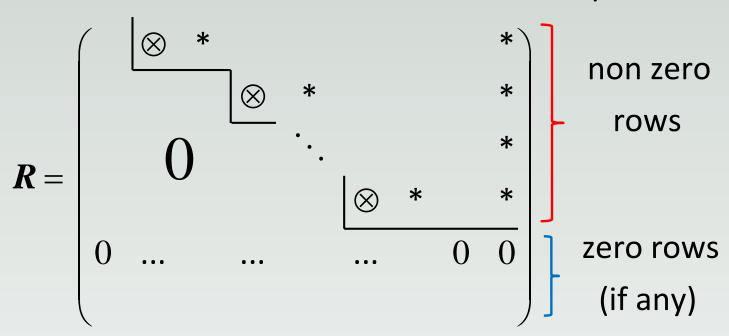
What if we want to find a basis for the row space of a matrix R that is in row echelon form?



$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & 0 & -1 & 1 & 4 \\ 0 & \mathbf{1} & 4 & 2 & 1 \\ 0 & 0 & -2 & 0 & 1 \end{bmatrix}$$

Note that *A* is in row echelon form.

The non zero rows of R are always linearly independent and thus forms a basis for the row space of R.



SUMMARY

- 1) Definition of the row space and column space of a $m \times n$ matrix A.
- 2) If the rows of A are linearly independent, then they form a basis for the row space of A.
- 3) If R is a matrix in row-echelon form, then the non-zero rows of R form a basis for the row space of R.