NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

Module: MA1508E Linear Algebra for Engineering
Year/Semester: 2018-2019 (Semester 2)
Tutorial: 5

1. Let $\boldsymbol{u_1} = (1, 2, -1)$, $\boldsymbol{u_2} = (6, 4, 2)$, $\boldsymbol{u_3} = (9, 2, 7)$, $\boldsymbol{u_4} = (4, -1, 8)$, $\boldsymbol{u_5} = (1, 2, 3)$.

   (a) Is $\boldsymbol{u_3}$ a linear combination of $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$? Is $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\} = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$? Is either $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$ or $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ equals to $\mathbb{R}^3$?

   (b) Is $\boldsymbol{u_4}$ a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}$ and $\boldsymbol{u_3}$? Is $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\} = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\}$? Is $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\} = \mathbb{R}^3$?

   (c) Is $\boldsymbol{u_5}$ a linear combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ and $\boldsymbol{u_4}$? Is $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\} = \text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}, \boldsymbol{u_5}\}$? Is $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}, \boldsymbol{u_5}\} = \mathbb{R}^3$?

2. For each of the following matrices $\boldsymbol{A}$, express the solution space of $\boldsymbol{Ax} = \boldsymbol{0}$ as a linear span. Give a geometrical interpretation of the solution space (in other words, describe the geometrical object represented by the linear span).

   (a) $\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{pmatrix}$ (b) $\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & 6 \end{pmatrix}$

   (c) $\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 3 \\ -3 & -3 & 1 \\ 4 & 1 & 2 \end{pmatrix}$ (d) $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

3. Let $V = \{(x, y, z) \mid 2x - y + 3z = 0\}$ be a subset of $\mathbb{R}^3$.

   (a) Is V a subspace of $\mathbb{R}^3$? If so, describe the subspace geometrically.

   (b) Let $S = \{(1, -1, -1), (1, 2, 0)\}$. Show that $\text{span}(S) = V$.

   (c) Let $\boldsymbol{u} = (0, 3, a)$, where a is a real number. Suppose $T = S \cup \{\boldsymbol{u}\}$. Find all values of a such that

      (i) $\text{span}(T) = \mathbb{R}^3$.

      (ii) $\text{span}(T) = V$.

4. Let $\boldsymbol{u_1} = (2, 0, 2, -4)$, $\boldsymbol{u_2} = (1, 0, 2, 5)$, $\boldsymbol{u_3} = (0, 3, 6, 9)$, $\boldsymbol{u_4} = (1, 1, 2, -1)$, $\boldsymbol{v_1} = (-1, 2, 1, 0)$, $\boldsymbol{v_2} = (3, 1, 4, 0)$, $\boldsymbol{v_3} = (0, 1, 1, 3)$, $\boldsymbol{v_4} = (-4, 3, -1, 6)$. Determine if the following are true.

   (a) $\text{span}\{\boldsymbol{u_1},\ \boldsymbol{u_2},\ \boldsymbol{u_3},\ \boldsymbol{u_4}\} \subseteq \text{span}\{\boldsymbol{v_1},\ \boldsymbol{v_2},\ \boldsymbol{v_3},\ \boldsymbol{v_4}\}$.

   (b) $\text{span}\{\boldsymbol{v_1},\ \boldsymbol{v_2},\ \boldsymbol{v_3},\ \boldsymbol{v_4}\} \subseteq \text{span}\{\boldsymbol{u_1},\ \boldsymbol{u_2},\ \boldsymbol{u_3},\ \boldsymbol{u_4}\}$.

(c) $\text{span}\{\boldsymbol{u_1},\ \boldsymbol{u_2},\ \boldsymbol{u_3},\ \boldsymbol{u_4}\} = \mathbb{R}^4$.

(d) $\text{span}\{\boldsymbol{v_1},\ \boldsymbol{v_2},\ \boldsymbol{v_3},\ \boldsymbol{v_4}\} = \mathbb{R}^4$.

5. For each of the following subsets S of $\mathbb{R}^3$ (or $\mathbb{R}^4$), determine if S is a subspace of $\mathbb{R}^3$ (or $\mathbb{R}^4$) and for those which are, write S as a linear span.

   (a) $S = \{(a, b, c) \mid abc = 0\}$.

   (b) $S = \{(x, y, z) \mid 4y = z\}$.

   (c) $S = \{(a, b, c) \mid a \le b \le c\}$

   (d) $S = \{(w, x, y, z) \mid 2x + 3y - z = 0 \text{ and } x + 2y - z = 0\}$.

   (e) $S = \{\boldsymbol{Au} \mid \boldsymbol{u} \in \mathbb{R}^3\}$ where $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}$ (here $\boldsymbol{u}$ is written as a column vector).

   (f) $S = \{\boldsymbol{u} \in \mathbb{R}^4 \mid \boldsymbol{Au} = \boldsymbol{u}\}$ where $\begin{pmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (here $\boldsymbol{u}$ is written as a column vector).

6. Determine which of the following statements are true. Justify your answer.

   (a) If $\boldsymbol{u}$ is a nonzero vector in $\mathbb{R}^1$, then $\text{span}\{\boldsymbol{u}\} = \mathbb{R}^1$.

   (b) If $\boldsymbol{u}, \boldsymbol{v}$ are nonzero vectors in $\mathbb{R}^2$ such that $\boldsymbol{u} \neq \boldsymbol{v}$, then $\text{span}\{\boldsymbol{u}, \boldsymbol{v}\} = \mathbb{R}^2$.

   (c) If S_1 and S_2 are finite subsets of $\mathbb{R}^n$, then $\text{span}(S_1 \cap S_2) = \text{span}\ (S_1) \cap \text{span}(S_2)$.

   (d) If S_1 and S_2 are finite subsets of $\mathbb{R}^n$, then $\text{span}(S_1 \cup S_2) = \text{span}(S_1) \cup \text{span}(S_2)$.