W04-05

Slide 01: In this unit, we will introduce the concept of linear span.

Slide 02: We have already discussed linear combinations in a previous unit. For example, we now know what is a linear combination of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

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Certainly, there are many ways of linearly combining these three vectors and it is very likely that different ways of linearly combining these vectors will result in different vectors.

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Suppose we collect all the different linear combinations of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ and put them into a set, intuitively, we would think that such a set would contain a huge number of different vectors.

Slide 03: Indeed this leads us to the definition of linear span. Let S be a set of vectors in $\mathbb{R}^n$, More specifically, let the vectors in S be $\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}$.

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The set of all linear combinations of $\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}$ denoted as follows

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is called the linear span of S. Note that the set notation here can be read as the set of $c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2}$ and so on till $c_k\boldsymbol{u_k}$ such that c_1, c_2 and so on till c_k takes on all possible real number values. Instead of calling this set the linear span of S, we can also say it is the linear span of the vectors $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ till $\boldsymbol{u_k}$.

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This set can be denoted by span(S) or if we wish to write down specifically the vectors in S, we may also do so.

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Intuitively, it may be easier to think of span(S) as the set of all vectors that can be 'generated' by linearly combining the vectors in S. Indeed span(S) would contain a huge number of vectors.

Slide 04: This example was taken from a previous unit, where we have established that the vector $(3, 3, 4)$ is indeed a linear combination of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

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Since the set span$\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ contains all possible linear combinations of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$, the vector $(3, 3, 4)$ belongs to span$\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$.

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Similarly, we have also verified in an earlier unit that the vector $(1, 2, 4)$ is not a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

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Thus $(1, 2, 4)$ does not belong to span$\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$.

Slide 05: Consider the set S containing vectors $(1, 1, 0)$ and $(2, -1, 1)$. Span(S) is the set of all possible linear combinations of these two vectors.

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On the other hand, every vector in span(S) is of the form a times $(1,1,0)$ plus b times $(2,-1,1)$ where a and b are any real numbers.

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So we may write span(S) as the set of all $a(1,1,0)+b(2,-1,1)$ such that a, b belongs to $\mathbb{R}$.

Slide 06: Consider the following set V. Clearly V is a subset of $\mathbb{R}^3$. More precisely, from the way the set V is expressed, we see that it contains vectors with three components, where the second component is any real number a, while the first and third components are expressions involving a and another arbitrary real number b.

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Can the set V be expressed in terms of a linear span?

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An arbitrary vector in V, as shown here, can be written as a times $(2,1,-1)$ plus b times $(1,0,3)$, where a and b are any real number.

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So the set V be now be expressed as the set of all linear combinations of the two vectors $(2,1,-1)$ and $(1,0,3)$. This, by our understanding is precisely the linear span of the two vectors $(2,1,-1)$ and $(1,0,3)$.

Slide 07: We return to an example discussed in a previous unit. Here, we have considered the question on whether every vector in $\mathbb{R}^3$ is a linear combination of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$. Recall that we first set up a vector equation as shown,

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followed by solving the associated linear system, which eventually led to the conclusion that the vector equation can always be solved regardless of the values of x, y and z.

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This meant that every vector in $\mathbb{R}^3$ is a linear combination of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

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In other words, every vector in $\mathbb{R}^3$ belongs to the linear span of $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$. Thus the linear span of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ is the entire Euclidean space $\mathbb{R}^3$.

Slide 08: This is another example discussed from a previous unit. Here, we again wanted to check if evey vector in $\mathbb{R}^3$ is a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

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For this example, we found that the vector equation will be inconsistent for some values of x, y, z,

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which means that not every vector in $\mathbb{R}^3$ is a linear combination of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$.

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Consequently, this also means that the linear span of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ is not the entire $\mathbb{R}^3$. There are vectors, for example, $(1,0,0)$ that does not belong to span$\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$.

Slide 09: Lets look at another example. We would like to show that the linear span of $(1,0,1), (1,1,0)$ and $(0,1,1)$ is $\mathbb{R}^3$.

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In order to do so, we need to show that every vector in $\mathbb{R}^3$ can be written as a linear combination of the three vectors. The way to do this is not new to us.

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We set up a vector equation with an arbitrary vector from $\mathbb{R}^3$ on the right hand side.

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From the vector equation, we can write down the corresponding linear system and then the augmented matrix shown here.

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Proceeding with Gaussian elimination, we find that at row-echelon form, we have three pivot columns on the left side of the vertical line, indicating that the linear system is consistent regardless of the values of x, y, z. Thus the linear span of the three vectors is indeed $\mathbb{R}^3$.

Slide 10: Another similar example here, where we have the linear span of 4 vectors from $\mathbb{R}^3$. However, in this case, we would like to show that the linear span of these 4 vectors is not the entire $\mathbb{R}^3$.

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In order to do this, we need to show that there is some vector in $\mathbb{R}^3$ that is not a linear combination of the 4 vectors.

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As usual, we set up our vector equation, with an arbitrary vector from $\mathbb{R}^3$ on the right hand side.

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Again, from the vector equation, we have a linear system and the associated augmented matrix shown here.

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Performing Gaussian elimination on the matrix yields the following matrix in row-echelon form.

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We see that there is row in the row-echelon form where the left hand side is entirely zeros, while the right side is an expression with x, y and z. So, by choosing, for example $x = 1, y = 0, z = 0$ we can conclude that the vector $(1, 0, 0)$ will not be a linear combination of the 4 vectors and thus $(1, 0, 0)$ does not belong to the linear span of the 4 vectors.

Slide 11: It is interesting to point out the following at this point. In the two examples that we have just seen, the first set of 3 vectors were good enough to span the entire $\mathbb{R}^3$, meaning that these three vectors could 'generate' each and evey vector in $\mathbb{R}^3$.

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While one may think that to have more vectors at our disposal to take linear span with will increase the likelihood of generating more vectors, this is not necessarily true. As seen in the second example, having four vectors, as opposed to having only three in the earlier example, was not enough to span $\mathbb{R}^3$. Thus, having more vectors to take linear span with may not necessarily generate a bigger set of vectors. It very much depends on the vectors that we have to take linear span with.

Slide 12: To summarise this unit.

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We defined the linear span of a set of vectors. This is essentially the set of all possible linear combinations of the vectors and intuitively it should be clear that it is a very big set.

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We then discussed how we can write a set as a linear span when possible.

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The process of checking whether a vector belongs to span(S) or not is not new, as it basically goes back to checking linear combinations.

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Finally, for a set S, we saw how we can check if span(S) is the entire Euclidean n-space.