Unit 059 Diagonalization Part I

Slide 01: In this unit, we will introduce the concept of diagonalization.

Slide 02: Recall the motivation from the population movement example, before we introduced the concept of eigenvalues and eigenvectors. Given a matrix $\boldsymbol{A}$, we wanted to know if it is possible to find an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{AP}$ is a diagonal matrix $\boldsymbol{D}$.

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Formally, we say that a square matrix $\boldsymbol{A}$ is diagonalizable if such an invertible matrix $\boldsymbol{P}$ can be found such that $\boldsymbol{P}^{-1}\boldsymbol{AP}$ is a diagonal matrix.

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When such a $\boldsymbol{P}$ is found, we say that it diagonalizes $\boldsymbol{A}$.

Slide 03: Let us return to the matrix $\boldsymbol{A}$ from the population movement example one more time. We have found that $\boldsymbol{A}$ has two eigenvalues, 1 and 0.95.

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The eigenspace E_1 is one dimensional and is spanned by the vector $(1, 4)$.

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The eigenspace $E_{0.95}$ is one dimensional and is spanned by the vector $(1, -1)$.

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Now let $\boldsymbol{P}$ be the 2×2 matrix as shown. Notice that the columns of $\boldsymbol{P}$ are precisely the vectors $(1, 4)$ and $(1, -1)$. Now it is easy to check that $\boldsymbol{P}$ is invertible

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and the product of the three matrices $\boldsymbol{P}^{-1}$, $\boldsymbol{A}$ and $\boldsymbol{P}$

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results in the diagonal matrix with 1 and 0.95 as the diagonal entries. Notice that in this diagonal matrix, 1 is in the first column while 0.95 is in the second column. This corresponds to the fact that the first column of $\boldsymbol{P}$ is the eigenvector associated with 1 while the second column of $\boldsymbol{P}$ is the eigenvector associated with 0.95.

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Thus $\boldsymbol{A}$ is diagonalizable and $\boldsymbol{P}$ diagonalizes $\boldsymbol{A}$.

Slide 04: You have also seen this matrix $\boldsymbol{B}$ several times in previous units. We know that $\boldsymbol{B}$ has 2 eigenvalues, namely 3 and 0.

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The eigenspace E_3 is one dimensional and is spanned by the vector $(1, 1, 1)$.

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The eigenspace E_0 is two dimensional and is spanned by the vectors $(1, 0, -1)$ and $(1, -2, 1)$.

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Now let $\boldsymbol{P}$ be the 3×3 matrix as shown. Notice that the columns of $\boldsymbol{P}$ are precisely the vectors $(1, 1, 1)$, $(1, 0, -1)$ and $(1, -2, 1)$. Once again we can check that $\boldsymbol{P}$ is invertible

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and the product of the three matrices $\boldsymbol{P}^{-1}$, $\boldsymbol{B}$ and $\boldsymbol{P}$ results in the diagonal matrix with 3 and 0 as the diagonal entries. Once again, observe that the diagonal entries corresponds to the arrangement of the columns in the matrix $\boldsymbol{P}$.

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Again, we conclude that $\boldsymbol{B}$ is diagonalizable and $\boldsymbol{P}$ diagonalizes $\boldsymbol{B}$.

Slide 05: We have also seen this matrix $\boldsymbol{C}$, that has three distinct eigenvalues 1, $\sqrt{2}$ and $-\sqrt{2}$.
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E_1 is one dimensional and spanned by $(-2, 2, 1)$.
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$E_{\sqrt{2}}$ is one dimensional and spanned by $(-1, \sqrt{2}, 1)$.
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$E_{-\sqrt{2}}$ is also one dimensional and spanned by $(-1, -\sqrt{2}, 1)$.
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We again construct the matrix $\boldsymbol{P}$ using the three basis vectors for the respective eigenspaces as shown. It can be checked that $\boldsymbol{P}$ is invertible
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and $\boldsymbol{P}^{-1}\boldsymbol{C}\boldsymbol{P}$ results in the diagonal matrix with 1, $\sqrt{2}$ and $-\sqrt{2}$ along the diagonal. Once again note the correspondence between the diagonal entries and the columns of the matrix $\boldsymbol{P}$.
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So $\boldsymbol{C}$ is also diagonalizable and we have found a $\boldsymbol{P}$ that diagonalizes it.

Slide 06: We have also seen this matrix $\boldsymbol{M}$ from a previous unit. This 2×2 matrix has only one eigenvalue 2 and E_2 is a one dimensional subspace of $\mathbb{R}^2$ spanned by $(0, 1)$.
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We will now proceed to show that $\boldsymbol{M}$ is not diagonalizable.
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Suppose $\boldsymbol{M}$ is actually diagonalizable. We will show that this assumption will lead to a contradiction. If $\boldsymbol{M}$ is diagonalizable, then there exists an invertible 2×2 matrix with entries, say, a, b, c, d such that the product of the three matrices as shown will result in a diagonal matrix, with entries say, λ and μ.

Slide 07: Now the matrix equation
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can be rewritten by premultiplying the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on both sides of the equation.
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Computing the products on both sides of the equation, and then equating the 4 corresponding entries of both matrices, we arrive at the following linear system that must be satisfied by a, b, c and d. In particular, look at the first equation $2a = \lambda a$ and the other equation $a + 2c = \lambda c$.
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If $a \neq 0$, then $2a = \lambda a$ will imply that $\lambda = 2$.
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But now if $\lambda = 2$ then equation (2) will impy that $a = 0$ which is a contradiction since we said $a \neq 0$.
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Thus a must be zero and from equation (2) we have $2c = \lambda c$. Clearly c cannot be zero too because if it is then the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ will be singular. Thus $c \neq 0$ and equation (2) will imply that λ must be 2.

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We can go through exactly the same argument when considering the other two equations not highlighted in the linear system and arrive at the conclusion that $b = 0$ and $\mu = 2$.

Slide 08: We now realise there is a problem

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because with both a and b equal to 0, we again arrive at the conclusion that the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is singular, again a contradiction.

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So our initial assumption that such a 2×2 invertible matrix exists that can diagonalize $\boldsymbol{M}$ is wrong and thus we have shown that $\boldsymbol{M}$ is not diagonalizable.

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While we have managed to show that $\boldsymbol{M}$ is not diagonalizable, this approach to do so by contradiction is obviously not comprehensive nor efficient. We would need to find another approach to determine whether any given matrix $\boldsymbol{A}$ is diagonalizable or not.

Slide 09: The following theorem gives us precisely what we are looking for, a necessary and sufficient condition that can be checked, that will tell us whether a square matrix of order n is diagonalizable. The theorem states that $\boldsymbol{A}$, which is a square matrix of order n, is diagonalizable if and only if $\boldsymbol{A}$ has n linearly independent eigenvectors.

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It is extremely important that one does not misunderstand the statment of the theorem. The n eigenvectors as required in the theorem needs to be linearly independent. Without insisting that the eigenvectors are linearly independent, it would be trivial to find n eigenvectors of $\boldsymbol{A}$ since we already know that the elements in the eigenspace E_λ are all eigenvectors of $\boldsymbol{A}$, and there are already infinitely many of them in one eigenspace. The proof of this theorem will not be discussed here.

Slide 10: We are now ready to present a step by step algorithm that will allow us to determine whether $\boldsymbol{A}$ is diagonalizable. In addition, if $\boldsymbol{A}$ is indeed diagonalizable, this procedure will also find an invertible matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{A}$.

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The first step involves finding all the eigenvalues of $\boldsymbol{A}$. From a previous unit, we know that this can be done by solving for all the roots of the characteristic equation of $\boldsymbol{A}$. Suppose we have done that and found that $\boldsymbol{A}$ has k distinct eigenvalues λ_1, λ_2 to λ_k.

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For step two, we will consider each of the eigenspaces E_{λ_i} and find a basis S_{λ_i} for the eigenspace E_{λ_i}. Recall that this require us to solve a homogeneous linear system.

Slide 11: After we have obtained each of the basis S_{λ_i},

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we take the union of all the k different bases to form the set S. We will look at the number of vectors in this set S.

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If the number of vectors in S is less than n, then our conclusion is that $\boldsymbol{A}$ will not be diagonalizable.

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If the number of vectors in S is exactly n, say the vectors in S are $\boldsymbol{u_1}, \boldsymbol{u_2}$ and so on till $\boldsymbol{u_n}$, then $\boldsymbol{A}$ will be diagonalizable and the matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{A}$ is formed by using the vectors $\boldsymbol{u_1}$ to $\boldsymbol{u_n}$ as the columns of $\boldsymbol{P}$. Now some of you may wonder what happens if the number of vectors in S is more than n?

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Well, the fact is that if you have done all the working and computation carefully, this will never happen. Therefore, if it does, head back to your working from the first step and check again!

Slide 12: Let us summarise this unit.

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We defined what is a diagonalizable matrix.

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We presented a necessary and sufficient condition for a $n \times n$ matrix to be diagonalizable.

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Finally, we have an algorithm to determine if a square matrix $\boldsymbol{A}$ is diagonalizable and if it is, the algorithm allows us to contruct a matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{A}$.