DETERMINANTS AND COFACTOR EXPANSION

A 2x2 matrix

Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $\underline{ad - bc \neq 0}$, let $\boldsymbol{B} = \boxed{\dfrac{1}{ad-bc}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

$$\boldsymbol{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{ad-bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{ad-bc} \begin{pmatrix} ad-bc & -ab+ba \\ cd-dc & -bc+ad \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So if $ad - bc \neq 0$,

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible.

A 2x2 matrix

If $ad - bc \neq 0$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible.

We will now prove:

If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, then $ad - bc \neq 0$.

This would mean

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $ad - bc \neq 0$

A 2x2 matrix

If $\boldsymbol{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, then $ad-bc\neq 0$.

Case 1: If $a=0$, and $c=0$.

$$\boldsymbol{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$$

This matrix will not have $\boldsymbol{I}_2$ as its reduced row-echelon form and so is not invertible.

So we do not need to consider this case, since the hypothesis "If $\boldsymbol{A}$ is invertible" is not satisfied.

A 2x2 matrix

If $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, then $ad - bc \neq 0$.

Case 2: $a \neq 0$ or $c \neq 0$. First suppose $a \neq 0$.

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{R_2 - \frac{c}{a} R_1} \begin{pmatrix} a & b \\ 0 & d - \frac{cb}{a} \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & \frac{ad-bc}{a} \end{pmatrix}$$

So if $\boldsymbol{A}$ is invertible, we must have two leading entries and thus $\frac{ad-bc}{a} \neq 0$ (that is, $ad - bc \neq 0$).

A 2x2 matrix

If $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, then $ad - bc \neq 0$.

Case 2: $a \neq 0$ or $c \neq 0$. Now suppose $a = 0$, $\boxed{c \neq 0.}$

$$\boldsymbol{A} = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{pmatrix} c & d \\ 0 & b \end{pmatrix}$$

So if $\boldsymbol{A}$ is invertible, we must have two leading entries and thus $\boxed{b \neq 0.}$

For this case, this implies $ad - bc \neq 0$.

A 2x2 matrix

$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $ad - bc \neq 0$

The quantity $ad - bc$ is known as the determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Definition

Let $\boldsymbol{A} = (a_{ij})$ be a square matrix of order n.

Let $\boldsymbol{M}_{ij}$ be a square matrix of order $n-1$ obtained by removing the ith row and jth column of $\boldsymbol{A}$.

$$\boldsymbol{A} = \begin{pmatrix} & & & \\ & & 3 & 1 \\ & & & \\ & & 1 & -1 \end{pmatrix}$$

$$\boldsymbol{M}_{11} = \begin{pmatrix} -1 & 3 & 1 \\ -4 & 3 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\boldsymbol{M}_{32} = \begin{pmatrix} 2 & 3 & 0 \\ 4 & 3 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

Definition

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Let $\boldsymbol{M}_{ij}$ be a square matrix of order $n-1$ obtained by removing the ith row and jth column of $\boldsymbol{A}$.

The determinant of $\boldsymbol{A}$ is defined as

$$\det(\boldsymbol{A}) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n} & \text{if } n \geq 2 \end{cases}$$

where $A_{ij} = (-1)^{i+j}\det(\boldsymbol{M}_{ij})$.

$A_{ij} = (-1)^{i+j}\det(\boldsymbol{M}_{ij})$ is called the (i, j)-cofactor of $\boldsymbol{A}$.

Definition

To know the determinant of a $n \times n$ matrix, we need to know the determinants of $(n-1)\times(n-1)$ matrices...

To know the determinant of a $(n-1)\times(n-1)$ matrix, we need to know the determinants of $(n-2)\times(n-2)$ matrices...

$$\det(\boldsymbol{A}) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n} & \text{if } n \geq 2 \end{cases}$$

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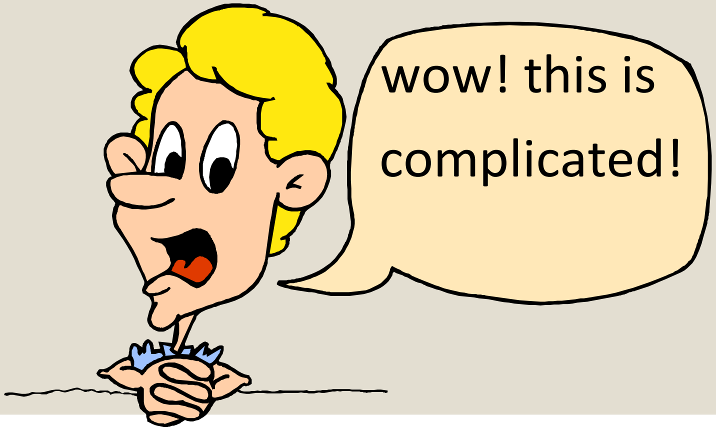

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This is known as cofactor expansion.

Notation

The determinant of $\boldsymbol{A} = (a_{ij})$ is usually written as

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

Example

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\boldsymbol{M}_{11} = (d) \qquad A_{11} = (-1)^{1+1}\det(d) = d$$

$$\boldsymbol{M}_{12} = (c) \qquad A_{12} = (-1)^{1+2}\det(c) = -c$$

$$\det(\boldsymbol{A}) = aA_{11} + bA_{12} = \boxed{ad - bc}$$

We have seen this expression before!

If $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\boldsymbol{A}$ is invertible if and only if $\det(\boldsymbol{A}) \neq 0$.

Summary

1) A necessary and sufficient condition for $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

to be invertible.

$\boxed{\boldsymbol{A} \text{ is invertible if and only if } \det(\boldsymbol{A}) \neq 0.}$

2) Definition of the determinant of a square matrix.

Cofactor expansion.