HOMOGENEOUS LINEAR SYSTEMS

DEFINITION

A linear system is said to be homogeneous if it has the following form:

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & \boxed{0} \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & 0 \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & 0 \end{cases}$$

$a_{11}, a_{12}, \dots, a_{mn}$ are real constants.

SOMETHING SPECIAL

Recall that any linear system behaves in exactly one of the following three ways:

Linear system

no solutions (inconsistent)

exactly one solution

infinitely many solutions

(consistent)

SOMETHING SPECIAL

no solutions (inconsistent)

$\underbrace{\text{exactly one solution} \qquad \text{infinitely many solutions}}_{\text{(consistent)}}$

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & 0 \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & 0 \end{cases}$$

What if we let $x_1 = 0, x_2 = 0, \dots, x_n = 0$?

SOMETHING SPECIAL

~~no solutions (inconsistent)~~

exactly one solution

infinitely many solutions

(consistent)

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & 0 \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & 0 \end{cases}$$

$x_1 = 0, x_2 = 0, \dots, x_n = 0$ is ALWAYS a solution!

DEFINITION

A homogeneous linear system is always consistent.

$x_1 = 0, x_2 = 0, \ldots, x_n = 0$ is called the trivial solution of the homogeneous linear system.

Any other solution (if there exists) is called a non-trivial solution.

~~no solutions (inconsistent)~~

exactly one solution

↓

only trivial solution

infinitely many solutions

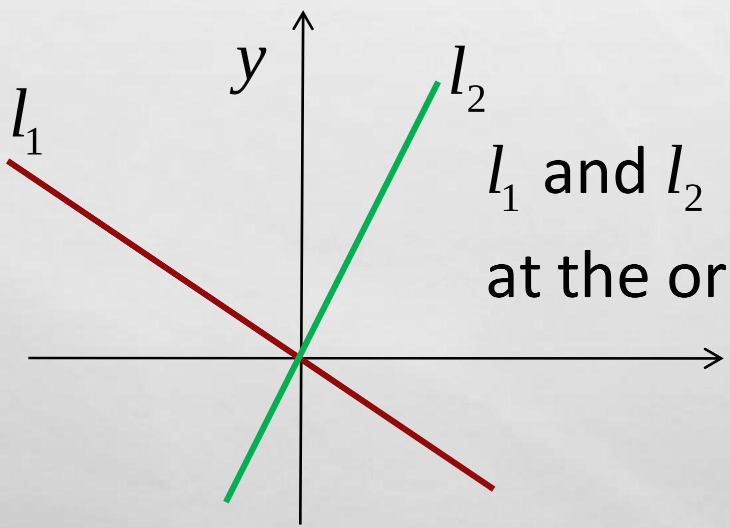
↓

trivial + non-trivial solutions

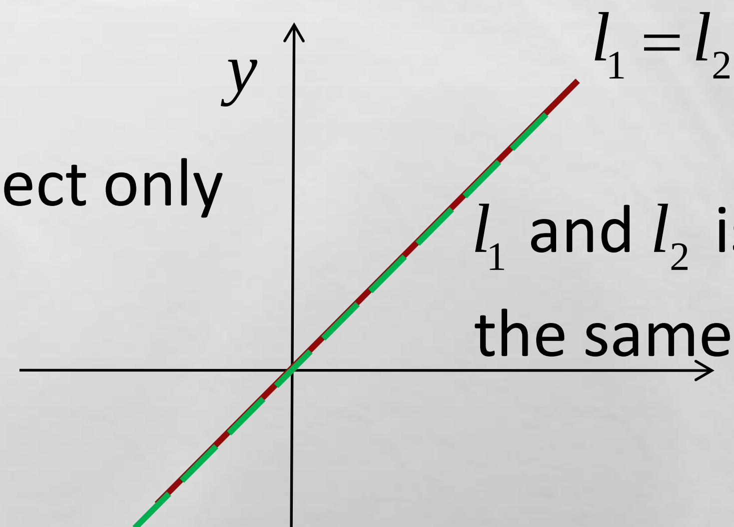
EXAMPLE

l_1 and l_2 are two lines in the xy plane passing through the origin.

$$\begin{cases} a_1x + b_1y = 0 & (l_1) \\ a_2x + b_2y = 0 & (l_2) \end{cases}$$



l_1 and l_2 intersect only at the origin.



l_1 and l_2 is the same line.

HOW MANY LEADING ENTRIES?

What is the maximum number of leading entries in a row-echelon form of each of the following augmented matrices?

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & -3 & 9 & 0 \end{array}\right)$$

2 equations (rows),

3 variables (columns excluding LHS)

Maximum number of leading entries = 2

$$\left(\begin{array}{ccccc|c} 1 & -1 & 2 & -4 & 0 & 0 \\ 2 & 1 & -3 & 0 & 5 & 0 \\ 3 & -2 & 1 & -3 & 1 & 0 \end{array}\right)$$

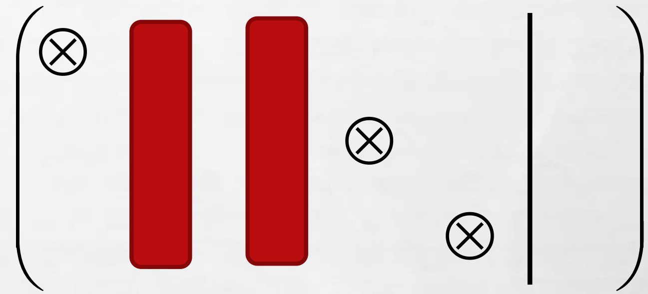
3 equations (rows),

5 variables (columns excluding LHS)

Maximum number of leading entries = 3

HOW MANY LEADING ENTRIES?

$$\left(\begin{array}{ccccc|c} 1 & -1 & 2 & -4 & 0 & 0 \\ 2 & 1 & -3 & 0 & 5 & 0 \\ 3 & -2 & 1 & -3 & 1 & 0 \end{array}\right)$$



If a homogeneous linear system has more unknowns than equations...

... there will ALWAYS be non-pivot columns in a row-echelon form of the augmented matrix (other than the last column).

$\Rightarrow$ infinitely many solutions $\Rightarrow$ non-trivial solutions

UNDERDETERMINED SYSTEMS

$$\begin{cases} x + y - 2z = 0 \\ 2x - 3y + 9z = 0 \end{cases}$$

$$\begin{cases} x_1 - x_2 + 2x_3 - 4x_4 = 0 \\ 2x_1 + x_2 - 3x_3 + 5x_5 = 0 \\ 3x_1 - 2x_2 + x_3 - 3x_4 + x_5 = 0 \end{cases}$$

Homogeneous linear systems with more unknowns than equations always has infinitely many solutions.

SUMMARY

1) What are homogeneous linear systems?

2) Trivial and non trivial solutions

3) How many solutions can a homogeneous system have?

4) Homogeneous systems with more variables than equations (under-determined linear systems)