Fifth Week

Chapter 5. Partial Differential Equations

A **partial differential equation (PDE)** is an equation containing an unknown function $u(x, y, \ldots)$ of *two or more* independent variables $x, y, \ldots$ and its partial derivatives with respect to these variables.

We call u the dependent variable.

Example

$$u_{xy} - 2x + y = 0$$

This is a PDE that involves the function $u(x, y)$ with two independent variables x and y. [Remember that the subscripts mean that you are taking the partial derivative, in this case a second order derivative first with respect to x and then with respect to y.]

Separation of Variables for PDE

This method can be used to solve PDE involving two independent variables, say x and y, that can be 'separated' from each other in the PDE There are similarities between this method and the technique of separating variables for ODE in Chapter 1.We first make an observation:

Suppose $u(x, y) = X(x)Y(y)$.

Then

$$\text{(i)} \quad u_x(x, y) = X'(x)Y(y)$$

$$\text{(ii)} \quad u_y(x, y) = X(x)Y'(y)$$

$$\text{(iii)} \quad u_{xx}(x, y) = X''(x)Y(y)$$

$$\text{(iv)} \quad u_{yy}(x, y) = X(x)Y''(y)$$

$$\text{(v)} \quad u_{xy}(x, y) = X'(x)Y'(y)$$

Notice that each derivative of u remains ‘separated’

as a product of a function of x and a function of y.

We exploit this feature as follows:

5.1.11 Example

Solve $u_x + xu_y = 0$.

Solution: If a solution $u(x, y) = X(x)Y(y)$ exists,

then we obtain

$$X'(x)Y(y) + xX(x)Y'(y) = 0$$

i.e., $$\frac{1}{x} \cdot \frac{X'(x)}{X(x)} = -\frac{Y'(y)}{Y(y)} \tag{9}$$

This gives two o.d.e.'s :

LHS of (9) $= k$ gives $\quad X' = kxX$.

This o.d.e. has general solution

$$X(x) = Ae^{kx^2/2} \tag{a}$$

Similarly, RHS of (9) $= k$ gives $Y' = -kY$.

This o.d.e. has general solution

$$Y(y) \;=\; Be^{-ky} \tag{b}$$

Multiplying (a) and (b), we obtain a general solution of the p.d.e.

$$u(x,y) \;=\; X(x)Y(y) \;=\; Ce^{k(x^2/2\,-\,y)}.$$

The Wave Equation

Suppose you have a very flexible string [meaning that it does not resist bending at all] which lies stretched tightly along the x axis and has its ends fixed at x = 0 and x = π. Then you pull it in the y-direction so that it is stationary and has some specified shape, y = f(x) at time t = 0 [so that f(0)=0 and f(π) = 0].

We can assume that f(x) is continuous and bounded, but we will let it have some sharp corners [but only a finite number of them.]

What will happen if you now let the string go? Clearly the string will start to move. We assume that the only forces acting are those due to the tension in the string, and that the pieces of the string will only move in the y-direction.

Now the y-coordinate of any point on the string will become a function of time as well as a function of x. So it becomes a function y(t,x) of both t and x, and we have to use partial derivatives when we differentiate it. This function satisfies

$$y(t, 0) = 0 \quad y(t, \pi) = 0$$

for all t, because the ends are nailed down, also

$$y(0, x) = f(x)$$

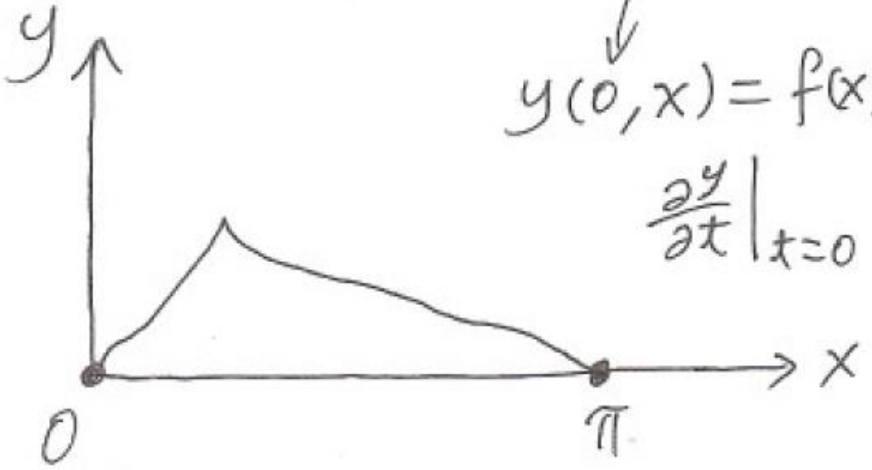
and

$$\frac{\partial y}{\partial t}(0, x) = 0,$$

because the string is initially stationary. Notice that we need *four* pieces of information here, and it is useful to remember that.

time $t<0$

time $t=0$

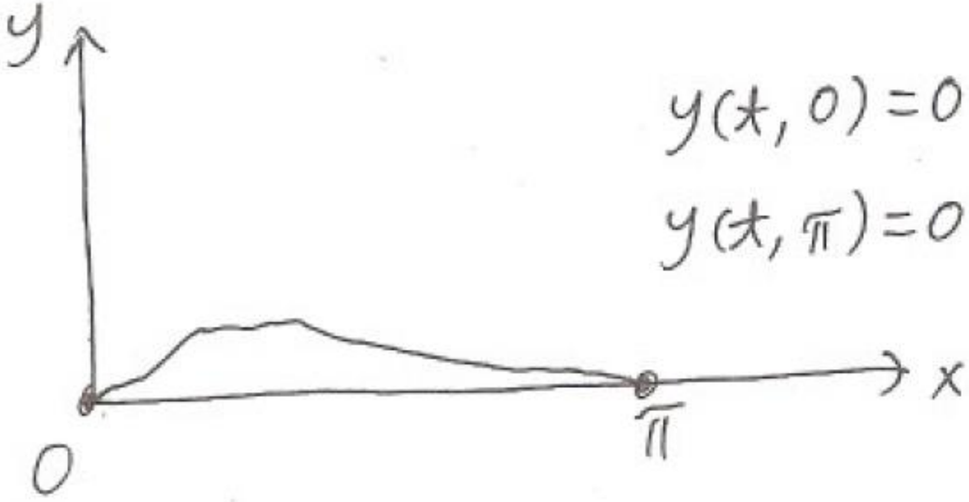


t-coordinate

$$y(0,x)=f(x)$$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0}=0$$

time $t > 0$



$$y(t, 0) = 0$$

$$y(t, \pi) = 0$$

$$c^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2},$$

where c^2 is a positive constant. This is the famous

WAVE EQUATION.

Notice that it involves FOUR derivatives altogether, two involving x, and two involving t. That matches up with the fact we mentioned earlier, that we needed FOUR pieces of information to nail down a solution [the endpoints, the initial position, the initial velocity].

d'Alembert's solution of the wave equation

Note that the following function solves the wave equation [with those four conditions]:

$$y(t, x) = \frac{1}{2}\Big[f(x + ct) + f(x - ct)\Big].$$

Tutorial 5

1. The oil tanker in Tutorial 3 is at rest in an almost calm sea. Suddenly, at time $t = T > 0$, it is hit by a single rogue wave [http://en.wikipedia.org/wiki/Rogue_wave] which imparts to it a vertical [upward] momentum P, doing so almost instantaneously. Neglecting friction, solve for $x(t)$, the downward displacement of the ship. How far down does the ship go [if it doesn't sink!]? [Hint: according to Newton's second law, momentum is the time integral of force. So to get the force as a function of time in this problem, you have to find a function which is zero except at $t = T$, and which has an integral equal to P. Note that the delta function has units of $1/\text{time}$.]

$$F = m\,(\text{acceleration})$$

$$= m\frac{dv}{dt}$$

$$= \frac{d}{dt}(mv) \qquad (\text{assume mass } m = \text{constant})$$

Now F is applied at $t = T$ and assume that it lasted for a small time h to $t = T + h$.

It produces a change of momentum from 0 to P in the time interval $T \le t \le T + h$.

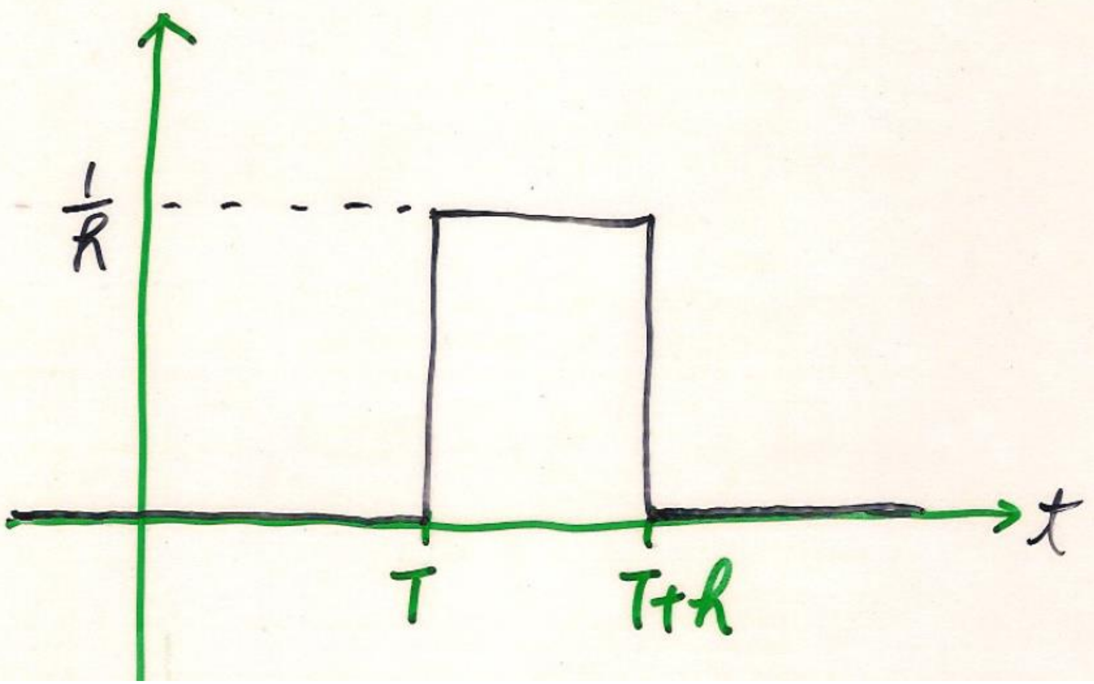
$\therefore$ Average rate of change of momentum

$$= \frac{p}{h} \quad \text{for } T \leq t \leq T+h$$

$$\therefore \quad F = \begin{cases} \dfrac{p}{h} & T \leq t \leq T+h \\ 0 & t < T \text{ or } T+h < t \end{cases}$$

Recall that:

define $g_h(t-T)$ by

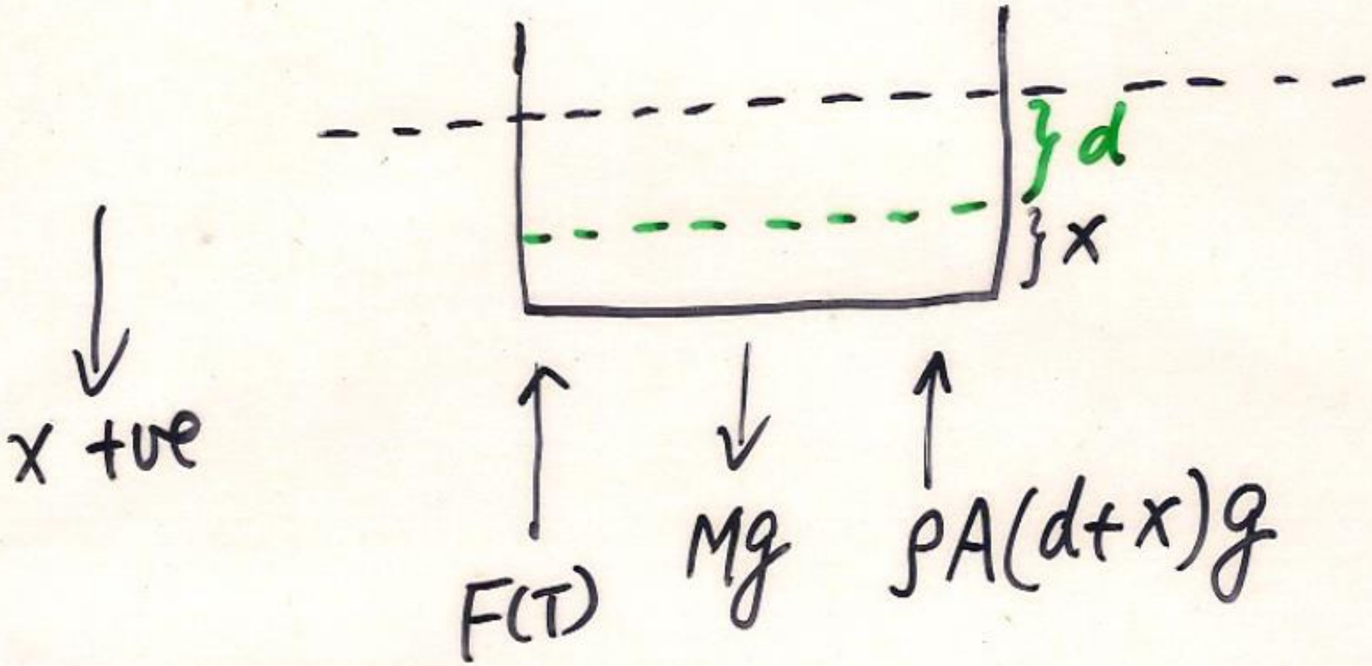


then $\lim\limits_{h \to 0^+} g_h(t-T) = \delta(t-T)$

$$\therefore \quad F = P\, g_h(t-T)$$

$$h \to 0^+ \Rightarrow F(T) = P\,\delta(t-T)$$

Now return to the oil tanker:



$$M\ddot{x} = Mg - \rho A(d+x)g - F(T)$$

$$= Mg - \rho A(d+x)g - P\delta(t-T)$$

Initially the ship is at rest

$$\Rightarrow x(0)=0, \quad \dot{x}(0)=0, \quad Mg = \rho A d g$$

$$\therefore \boxed{M\ddot{x} = -\rho A g x - P\delta(t-T)}$$