Chapter 5. Partial Differential Equations

5.1 Partial Differential Equations

A **partial differential equation (PDE)** is an equation containing an unknown function $u(x, y, \ldots)$ of *two or more* independent variables $x, y, \ldots$ and its partial derivatives with respect to these variables. We call u the dependent variable.

5.1.1 Example

(i) $u_{xy} - 2x + y \; = \; 0$

This is a PDE that involves the function $u(x, y)$ with two independent variables x and y. [Remember that the subscripts mean that you are taking the partial derivative, in this case a second order derivative first with respect to x and then with respect to y.]

(ii) $w_{xy} + x(w_z)^2 = yz$

This is a PDE that involves the function $w(x, y, z)$ with three independent variables x, y and z.

5.1.2 Example

The function

$$u(x, y) = x^2 y - \frac{1}{2} x y^2 + F(x) + G(y) \qquad (1)$$

is a solution of the PDE in example 5.1.1 (i). Here F and G can be any (arbitrary) single variable functions.

Indeed, by taking partial derivatives of (1):

$$u_x = 2xy - \frac{1}{2} y^2 + F'(x) \text{ and}$$

$$u_{xy} \;=\; 2x - y,$$

we see that the function (1) satisfies the PDE. NOTICE that, just as the solution of an ordinary differential equation involves arbitrary CONSTANTS, the solution of a PDE will involve arbitrary FUNCTIONS!

Suppose we require the PDE to also satisfy the conditions

$$u(x,0) = x^3 \text{ and } u(0,y) = \sin(3y).$$

Then using (1), we have

$$x^3 = u(x,0) = F(x) + G(0)$$

and

$$\sin(3y) = u(0,y) = F(0) + G(y).$$

By putting $x = 0$ in the first of these equations [or $y = 0$ into the second] we see that $0 = F(0) + G(0)$. So adding the two equations together we get

$$F(x)+G(y)+F(0)+G(0) = F(x)+G(y) = x^3+sin(3y).$$

Substituting this back into the solution we have finally

$$u(x, y) = x^2y - \frac{1}{2}xy^2 + x^3 + \sin(3y)$$

which satisfies the additional conditions.

5.1.3 Example

In general, the totality of solutions of a PDE is very large.

The Laplace equation $u_{xx} + u_{yy} = 0$ has the following solutions

$$u(x,y) = x^2 - y^2, \quad u(x,y) = e^x \cos y,$$

$$u(x,y) = \ln(x^2 + y^2), \quad \text{etc}$$

which are entirely different from each other.

5.1.4 Order of Differential Equations

The **order** of the PDE is the order of the highest derivative present.

Example 5.1.1 (i) is a PDE of order 2 and (ii) is also a PDE of order 2.

5.1.5 Linearity and Homogeneity

An order 1 *linear* PDE has the form

$$Au_x + Bu_y + Cu = Z$$

and an order 2 *linear* PDE has the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = Z$$

where A, B, C, D, E, F, Z are constants or functions of x and y but not functions of u.

An order 1 or 2 linear PDE is said to be *homogeneous* if the Z term in the above form is 0.

5.1.6 Example

PDE	**order**	**linear**	**homogeneous**
$4u_{xx} - u_t = 0$	2	yes	yes
$x^2 R_{yyy} = y^3 R_{xx}$	3	yes	yes
$tu_{tx} + 2u_x = x^2$	2	yes	no
$4u_{xx} - uu_t = 0$	2	no	n.a.
$(u_x)^2 + (u_y)^2 = 2$	1	no	n.a.

5.1.7 Superposition Principle

If u_1 and u_2 are any solutions of a linear homogeneous differential equation, then

$$u = c_1u_1 + c_2u_2,$$

where c_1 and c_2 are any constants, is also a solution of that equation.

5.1.8 Example

Referring to the particular solutions of Laplace equation $u_{xx}+u_{yy} = 0$ in Example 5.1.3, by superposition principle,

$$u(x, y) = 3(x^2 - y^2) - 7e^x \cos y + 10 \ln (x^2 + y^2)$$

is again a solution of the Laplace equation.

5.1.9 Separation of Variables for PDE

This method can be used to solve PDE involving two independent variables, say x and y, that can be 'separated' from each other in the PDE There are similarities between this method and the technique of separating variables for ODE in Chapter 1.We first make an observation:

Suppose $u(x, y) = X(x)Y(y)$.

Then

(i) $u_x(x, y) = X'(x)Y(y)$

(ii) $u_y(x, y) = X(x)Y'(y)$

(iii) $u_{xx}(x, y) = X''(x)Y(y)$

(iv) $u_{yy}(x, y) = X(x)Y''(y)$

(v) $u_{xy}(x, y) = X'(x)Y'(y)$

Notice that each derivative of u remains 'separated' as a product of a function of x and a function of y.

We exploit this feature as follows:

5.1.10 Illustration of Separation of Variables

Consider a PDE of the form

$$u_x = f(x)g(y)u_y.$$

If a solution of the form $u(x, y) = X(x)Y(y)$ exists, then we obtain

$$X'(x)Y(y) = f(x)g(y)X(x)Y'(y)$$

$$\text{i.e.,} \qquad \frac{1}{f(x)} \cdot \frac{X'(x)}{X(x)} = g(y)\frac{Y'(y)}{Y(y)}.$$

LHS is a function of x only while RHS is a function of y only. We conclude that

$$\text{LHS} \;=\; \text{RHS} \;=\; \text{some constant } k.$$

Thus, we obtain two ODEs

$$\frac{1}{f(x)} \cdot \frac{X'(x)}{X(x)} \;=\; k \Rightarrow X'(x) = kf(x)X(x) \qquad (2)$$

$$g(y)\frac{Y'(y)}{Y(y)} \;=\; k \Rightarrow Y'(y) = \frac{k}{g(y)}Y(y) \qquad (3)$$

Note that (2) is an ODE with independent variable x and dependent variable X while (3) is an ODE with independent variable y and dependent variable Y.

By solving (2) and (3) respectively for $X(x)$ and $Y(y)$, we obtain the solution $u(x, y) = X(x)Y(y)$.

5.1.11 Example

Solve $u_x + xu_y = 0.$

Solution: If a solution $u(x,y) = X(x)Y(y)$ exists,

then we obtain

$$X'(x)Y(y) + xX(x)Y'(y) = 0$$

$$\text{i.e.,} \qquad \frac{1}{x} \cdot \frac{X'(x)}{X(x)} = -\frac{Y'(y)}{Y(y)} \tag{4}$$

This gives two ODEs :

LHS of (4) $= k$ gives $X' = kxX$.

This ODE has general solution

$$X(x) = Ae^{kx^2/2} \tag{a}$$

Similarly, RHS of (4) $= k$ gives $Y' = -kY$.

This ODE has general solution

$$Y(y) \;=\; Be^{-ky} \qquad \text{(b)}$$

Multiplying (a) and (b), we obtain a solution of the PDE

$$u(x, y) \;=\; X(x)Y(y) \;=\; Ce^{k(x^2/2 \;-\; y)}.$$

5.2 The Wave Equation

Suppose you have a very flexible string [meaning that it does not resist bending at all] which lies stretched tightly along the x axis and has its ends fixed at x = 0 and x = π. Then you pull it in the y-direction so that it is stationary and has some specified shape, y = f(x) at time t = 0 [so that f(0)=0 and f(π) = 0]. We can assume that f(x) is continuous and bounded, but we will let it have some sharp corners [but only a finite number of them.]

What will happen if you now let the string go? Clearly the string will start to move. We assume that the only forces acting are those due to the tension in the string, and that the pieces of the string will only move in the y-direction.

Now the y-coordinate of any point on the string will become a function of time as well as a function of x. So it becomes a function y(t,x) of both t and x, and we have to use partial derivatives when we differentiate it. This function satisfies

$$y(t,0) = 0 \quad y(t,\pi) = 0$$

for all t, because the ends are nailed down, also

$$y(0, x) = f(x)$$

and

$$\frac{\partial y}{\partial t}(0, x) = 0,$$

because the string is initially stationary. Notice that we need *four* pieces of information here, and it is useful to remember that.

Suppose that the mass per unit length of the string is constant and equal to μ. Then the mass of a small piece of the string is μdx. [We assume that we don't pull the string too far, so it never bends much, hence the length of the small piece can be approximated by dx.] So the mass times the acceleration of the small piece is $\mu\text{dx}\frac{\partial^2 y}{\partial t^2}$. The force acting in the y direction is just the difference between the y-components of the tension at the two ends of the piece, and from physics this turns out to be $\text{d}(\text{K}\frac{\partial y}{\partial x})$ where K is a certain positive constant. Using Newton's second law we get

$$d\left(K\frac{\partial y}{\partial x}\right) = \mu dx\frac{\partial^2 y}{\partial t^2},$$

or

$$c^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2},$$

where c^2 is a positive constant. This is the famous WAVE EQUATION. Notice that it involves FOUR derivatives altogether, two involving x, and two involving t. That matches up with the fact we mentioned earlier, that we needed FOUR pieces of information to nail down a solution [the endpoints, the initial position, the initial velocity].

Note that the following function solves the wave equation [with those four conditions]:

$$y(t,x) = \frac{1}{2}\Big[f(x+ct) + f(x-ct)\Big].$$

Here f(x) gives the initial shape of the string, as above. [Verify that this does satisfy the PDE, that y(0,x) = f(x), and that $\frac{\partial y}{\partial t} = 0$ for all x when t = 0, as it should be. Initially f(x) was only defined between 0 and π, but we can extend it to be an odd, periodic function of period 2π, and then you can also verify that y(t,0) = y(t,π) = 0. See Tutorial 6.] This is called d'Alembert's solution of the wave equation.

You can think about f(x-ct) in the following way. First, think about f(x): it is a function with some definite shape. Now what is f(x-1)? It is *exactly the same* shape, but shifted to the right by one unit. Similarly f(x-2) is the *same* shape shifted 2 units to the right, and similarly f(x-ct) is f(x) shifted to the right by ct. But if t is time, then ct is something which increases linearly with time, at a rate controlled by c. *In other words, f(x-ct) represents the shape f(x) moving to the right at a constant speed c.* In other words, it represents a WAVE [of arbitrary shape] moving to the right at constant speed c. Similarly f(x+ct) represents a wave [with the same shape as f(x)] moving to the left at constant speed c. So d'Alembert's solution says that the solutions of the wave equation [with the given boundary and initial conditions] can be found by superimposing two waves of those forms.

5.3 Heat Equation

Consider the temperature in a long thin bar or wire of constant cross section and homogeneous material, which is oriented along the x-axis and is *perfectly insulated laterally*, so that heat only flows in x-direction. Then the temperature u depends only on x and t and is given by the one-dimensional heat equation

$$u_t = c^2 u_{xx}, \tag{5}$$

where c^2 is a positive constant called the thermal diffusivity [units length^2/time]. It measures how quickly heat moves through the bar, and depends on what it is made of.

5.3.1 Zero temperature at ends of rod

Let's assume that the ends $x = 0$ and $x = L$ of the bar are kept at temperature zero, so that we have the boundary conditions

$$u(0, t) = 0, \quad u(L, t) = 0 \quad \text{for all} \;\; t \tag{6}$$

and the initial temperature of the bar is $f(x)$, so that we have the initial condition

$$u(x, 0) = f(x). \tag{7}$$

We call the p.d.e (5) together with the conditions (6) and (7) a *boundary value heat equation problem.* Notice that, unlike the wave equation, which needs *four* pieces of data, here we only need *three*, which matches the fact that the heat equation only involves a total of three derivatives [two in the spatial direction, but only one in the time direction].

5.3.2 Example

Solve

$$u_t = 2u_{xx}, \qquad 0 < x < 3, \quad t > 0,$$

given boundary conditions $u(0,t) = 0, \quad u(3,t) = 0,$

and initial condition $u(x,0) = 5\sin 4\pi x.$

Solution:

We use the method of separation of variables. Let

$u(x,t) = X(x)T(t)$. Then $u_t = 2u_{xx}$ gives

$$XT' = 2X''T,$$

or equivalently

$$\frac{X''}{X} = \frac{T'}{2T}.$$

Each side must be a constant k. So

$$X'' - kX = 0 \qquad \text{(A)}$$

$$T' - 2kT = 0 \qquad \text{(B)}$$

The solutions of (A) are of three types; but clearly

we want $X(x)$ to vanish at TWO values of x [the two ends of the bar]. Of course, exponential functions [k positive] and straight-line [k = 0] functions cannot do that. So we have to choose k to be negative, so as to get trigonometric functions which *can* vanish at two values of x. So we have

$$X(x) = a \cos \sqrt{-k}x + b \sin \sqrt{-k}x. \tag{8}$$

The solutions of (B) are

$$T(t) = de^{2kt}$$

for the same negative value of k as in (A).

So now we have a simple solution of the heat equation, by just multiplying $X(x)$ into $T(t)$.

Now we just have to use the boundary conditions.

They can be written as

$$X(0)T(t) = 0 \quad \text{and} \quad X(3)T(t) = 0.$$

Now since $T(t) = de^{2kt} \neq 0$ for any t, we conclude that $X(0) = 0$ and $X(3) = 0$.

Substituting $x = 0$ and 3 separately into our expression for $X(x)$, we get

$$X(0) = a = 0$$

$$X(3) = a\cos 3\sqrt{-k} + b\sin 3\sqrt{-k} = 0$$

Solving these two equations, we get

$a = 0$ and $b\sin 3\sqrt{-k} = 0$.

Since we do not want a and b both zero, this implies

$\sin 3\sqrt{-k} = 0$ which gives

$$\sqrt{-k} = \frac{n\pi}{3} \text{ or } k = \frac{-n^2\pi^2}{9} \qquad \text{where } n = 0, 1, 2, \ldots$$

Putting this back into our general solution and absorbing d into b [since the product of arbitrary constants is another arbitrary constant] we have

$$u_n(x,t) = b_n e^{-2n^2\pi^2 t/9} \sin \frac{n\pi x}{3} \qquad (HS_n)$$

is a solution for each $n = 1, 2, 3, \ldots$.

We have used up two of our pieces of information. But one remains: we have to satisfy

$$u(x,0) = 5\sin 4\pi x.$$

We want to construct a solution from among (HS_n) that satisfies this initial condition.

Now substituting $t = 0$ into (HS_n) for any n,

$u_n(x, 0) = \ b_n sin\dfrac{n\pi x}{3}$.

If we take $n = 12$ and $\mathrm{b}_{12} = 5$, we have

$$u_{12}(x, 0) = \ 5sin\frac{12\pi x}{3} = \ 5sin4\pi x$$

.

Hence, the particular solution that will also satisfy the initial condition is

$$u_{12}(x, t) = 5e^{-32\pi^2 t} \sin 4\pi x.$$