Week 10 F2F Example Solutions

1. **Example 9.1**

   (a) A vector $\boldsymbol{v} = (v_1, v_2, v_3, v_4)$ belongs to U if and only if $\boldsymbol{v} \cdot \boldsymbol{u_i} = 0$ for $i = 1, 2, 3$. This is equivalent to

$$\left\{\begin{array}{ccccccccc} v_1 & & & + & v_2 & + & v_3 & = & 0 \\ v_1 & - & v_2 & & & + & 2v_4 & = & 0 \\ v_1 & + & 2v_2 & + & 3v_3 & - & v_4 & = & 0 \end{array}\right. \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 2 & 0 \\ 1 & 2 & 3 & -1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

   So $\boldsymbol{v} \in U$ if and only if $\boldsymbol{v} \in \text{span}\{(-1, -1, 1, 0), (-1, 1, 0, 1)\}$.

   (b) Yes, since U is a linear span, it is a subspace of $\mathbb{R}^4$. It is also true in general that the set U of all vectors orthogonal to a subspace is always a subspace since U is the solution space of some homogeneous linear system.

2. **Example 9.2**

   $\boldsymbol{v_1} = \frac{1}{\sqrt{3}}(1, 1, 1)$, $\boldsymbol{v_2} = \frac{1}{\sqrt{6}}(1, -2, 1)$, $\boldsymbol{v_3} = \frac{1}{\sqrt{2}}(1, 0, -1)$.

3. **Example 9.3**

   (a) Easily shown, as

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

   (b) To use orthogonal projection, we first need to find an orthogonal basis for the column space of the coefficient matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$. Fortunately, the first 2 columns of $\boldsymbol{A}$ are already pairwise orthogonal, so we proceed to find the 3 orthogonal vector

$$\begin{aligned} \boldsymbol{v_3} &= (1, 1, -1, 1) - \frac{(1, 1, -1, 1) \cdot (1, 0, 1, 0)}{(1, 0, 1, 0) \cdot (1, 0, 1, 0)}(1, 0, 1, 0) \\ &\quad - \frac{(1, 1, -1, 1) \cdot (1, 1, -1, 0)}{(1, 1, -1, 0) \cdot (1, 1, -1, 0)}(1, 1, -1, 0) \\ &= (0, 0, 0, 1). \end{aligned}$$

   The projection of $(1, 1, 1, 1)^T$ onto the column space of $\boldsymbol{A}$ is

$$\frac{2}{2}(1, 0, 1, 0) + \frac{1}{3}(1, 1, -1, 0) + \frac{1}{1}(0, 0, 0, 1) = (\frac{4}{3}, \frac{1}{3}, \frac{2}{3}, 1).$$

Solving $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{p}$,

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & \frac{4}{3} \\ 0 & 1 & 1 & \frac{1}{3} \\ 1 & -1 & -1 & \frac{2}{3} \\ 0 & 0 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So a least squares solution is $(x, y, z) = (1, -\frac{2}{3}, 1)$.

(c) Solving $\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A^T}\boldsymbol{b}$:

$$\left(\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & 3 & 3 & 1 \\ 0 & 3 & 4 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & 1 \end{array}\right).$$

We arrive at the same least squares solution as in part (b).