NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

Module:	MA1508E Linear Algebra for Engineering
Year/Semester:	2018-2019 (Semester 2)
Tutorial:	4

1. (**$\boldsymbol{LU}$ factorisation**) $\boldsymbol{LU}$ factorisation is a way to solve a given linear system $\boldsymbol{Ax} = \boldsymbol{b}$ efficiently. The discussion below only deals with the special case where $\boldsymbol{A}$ is a square matrix but can be extended to other sizes of $\boldsymbol{A}$ as well.

   (a) Let $\boldsymbol{A} = \begin{pmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{pmatrix}$. Peform exactly **three** elementary row operations on $\boldsymbol{A}$ to reduce $\boldsymbol{A}$ into row-echelon form.

   (b) Let the row-echelon form of $\boldsymbol{A}$ obtained in (a) be $\boldsymbol{U}$. Write down three elementary matrices $\boldsymbol{E_1}$, $\boldsymbol{E_2}$ and $\boldsymbol{E_3}$ such that

$$\boldsymbol{E_3 E_2 E_1 A} = \boldsymbol{U}. \tag{*}$$

   (c) Find the inverses of $\boldsymbol{E_1}$, $\boldsymbol{E_2}$, $\boldsymbol{E_3}$ such that

$$\boldsymbol{A} = \boldsymbol{E_1}^{-1}\boldsymbol{E_2}^{-1}\boldsymbol{E_3}^{-1}\boldsymbol{U}.$$

   (d) Compute the product $\boldsymbol{E_1}^{-1}\boldsymbol{E_2}^{-1}\boldsymbol{E_3}^{-1}$ and check that it is lower triangular. Since it is lower triangular, we have successfully factorised $\boldsymbol{A}$ as $\boldsymbol{LU}$ where $\boldsymbol{U}$ is upper triangular and $\boldsymbol{L}$ is lower triangular. In fact, all the diagonal entries of $\boldsymbol{L}$ are equal to 1. We call such a matrix, a **unit lower triangular** matrix.

(a)

$$\boldsymbol{A} \underset{\boldsymbol{E_1}}{\xrightarrow{R_2 - 2R_1}} \underset{\boldsymbol{E_2}}{\xrightarrow{R_3 + R_1}} \underset{\boldsymbol{E_3}}{\xrightarrow{R_3 + 2R_2}} \begin{pmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{pmatrix} = \boldsymbol{U}.$$

Note that $\boldsymbol{U}$ is a row-echelon form of $\boldsymbol{A}$ and also an upper triangular matrix.

(b)

$$\boldsymbol{E_1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{E_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{E_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}.$$

(c)

$$\boldsymbol{E_1}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{E_2}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{E_3}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}.$$

(d)

$$\boldsymbol{L} = \boldsymbol{E_1}^{-1}\boldsymbol{E_2}^{-1}\boldsymbol{E_3}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix}.$$

Indeed, $\boldsymbol{L}$ is lower triangular and we have $\boldsymbol{A} = \boldsymbol{LU}$ where $\boldsymbol{L}$ is unit lower triangular while $\boldsymbol{U}$ is upper triangular.

2. (**Use of $\boldsymbol{LU}$ factorisation**) To see why $\boldsymbol{LU}$ factorisation is useful, consider a linear system $\boldsymbol{Ax} = \boldsymbol{b}$, where the coefficient matrix $\boldsymbol{A}$ has an $\boldsymbol{LU}$ factorisation. We can rewrite the system $\boldsymbol{Ax} = \boldsymbol{b}$ as $\boldsymbol{L}(\boldsymbol{Ux}) = \boldsymbol{b}$. If we now define $\boldsymbol{y} = \boldsymbol{Ux}$, then we can solve for $\boldsymbol{x}$ in two stages:

   (1) Solve $\boldsymbol{Ly} = \boldsymbol{b}$ for $\boldsymbol{y}$ using *forward substitution*.

   (2) Solve $\boldsymbol{Ux} = \boldsymbol{y}$ for $\boldsymbol{x}$ using *back substitution*.

   Use the $\boldsymbol{LU}$ factorisation to solve the following system:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 2 & 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 4 \end{pmatrix}.$$

   Remark: You will obtain an unique solution for this linear system. Do you think **LU** factorisation can be used if the linear system is inconsistent? Or has infinitely many solutions?

   We factorise the given $\boldsymbol{A}$ as

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   Solving $\boldsymbol{Ly} = \boldsymbol{b}$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 4 \end{pmatrix},$$

   we have $y_1 = 1$, $y_2 = 1$, $y_3 = 2$, $y_4 = 1$. Now we solve $\boldsymbol{Ux} = \boldsymbol{y}$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix},$$

   we have $x_4 = 1$, $x_3 = 0$, $x_2 = -1.5$, $x_1 = 1.5$.

3. Find the determinant for each of the following square matrices by first reducing the matrix into row-echelon form.

(a) $\begin{pmatrix} 3 & -6 & 9 \\ -2 & 7 & -2 \\ 0 & 1 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{pmatrix}$.

(a) 33 (b) 7 (c) 39

4. Suppose we know that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6.$$

Evaluate the determinant of the following matrices.

(a) $\begin{pmatrix} d & e & f \\ g & h & i \\ a & b & c \end{pmatrix}$ (b) $\begin{pmatrix} 3a & 3b & 3c \\ -d & -e & -f \\ 4g & 4h & 4i \end{pmatrix}$ (c) $\begin{pmatrix} a & b & c \\ d & e & f \\ 2a & 2b & 2c \end{pmatrix}$

(d) $\begin{pmatrix} a+d & b+e & c+f \\ -d & -e & -f \\ g & h & i \end{pmatrix}$ (e) $\begin{pmatrix} a+g & b+h & c+i \\ d & e & f \\ g & h & i \end{pmatrix}$

(f) $\begin{pmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{pmatrix}$

(a) -6 ($R_1 \leftrightarrow R_2$ and $R_2 \leftrightarrow R_3$)
(b) 72 ($3R_1$, $-R_2$ and $4R_3$)
(c) 0 ($R_3 - 2R_1$)
(d) 6 ($R_1 + R_2$, $-R_2$)
(e) -6 ($R_1 + R_3$)
(f) 18 ($-3R_1$, $R_3 - 4R_2$)

5. Determine whether the following subsets of $\mathbb{R}^4$ are equal to each other.

$$S = \{(p,q,p,q) \mid p,q \in \mathbb{R}\},$$

$$T = \{(x,y,z,w) \mid x+y-z-w = 0\},$$

$$V = \left\{ (a,b,c,d) \;\middle|\; \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ a & b & c & d \end{vmatrix} = 0 \right\}.$$

Briefly explain why one subset is equal (or not equal) to another subset.

Since

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ a & b & c & d \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ b & c & d \end{vmatrix} + \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ a & b & d \end{vmatrix} = a+b-d-c.$$

$V = \{(a,b,c,d) \mid a+b-d-c=0\} = \{(x,y,z,w) \mid x+y-z-w=0\} = T$. On the other hand, $S \neq T$ because $(1,-1,0,0)$ belongs to T but $(1,-1,0,0)$ does not belong to S.

6. Consider a triangle in $\mathbb{R}^4$ with vertices $A = (1,1,0,0)$, $B = (1,-1,0,0)$ and $C = (2,0,0,1)$.

   (a) Find the lengths of the sides of the triangle.

   (b) Find the angle between AB and AC.

   (c) Verify the cosie rule: $2|AB||AC|\cos\theta = |AB|^2 + |AC|^2 - |BC|^2$, where θ is the angle between AB and AC.

   (a) $(1,1,0,0) - (1,-1,0,0) = (0,2,0,0)$ so $|AB| = \sqrt{(0^2+2^2+0^2+0^2)} = 2$. Likewise $|BC| = \sqrt{3}$ and $|AC| = \sqrt{3}$.

   (b) $\boldsymbol{u} = AB = (1,-1,0,0) - (1,1,0,0) = (0,-2,0,0)$, $\boldsymbol{v} = AC = (2,0,0,1) - (1,1,0,0) = (1,-1,0,1)$. So the angle between AB and AC is $\cos^{-1}\left(\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{|AB||AC|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.7$ degrees.

   (c) It is easy to verify that $2 \cdot 2 \cdot \sqrt{3} \cdot \frac{1}{\sqrt{3}} = 4 + 3 - 3$.

7. Let $\boldsymbol{u_1} = (1,3,-2,0,2,0)$, $\boldsymbol{u_2} = (2,6,-5,-2,4,-3)$, $\boldsymbol{u_3} = (0,0,5,10,0,15)$, $\boldsymbol{u_4} = (2,6,0,8,4,18)$ and $\boldsymbol{v} = (-3,-1,-2,1,1,0)$.

   (a) Verify that $\boldsymbol{v}$ is orthogonal to $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ and $\boldsymbol{u_4}$.

   (b) Construct a 4×6 matrix $\boldsymbol{A}$ with the vectors $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}$ as the rows of $\boldsymbol{A}$. Furthermore, write the vector $\boldsymbol{v}$ as a column matrix $\boldsymbol{v}$.

   (c) What do you think is the matrix product $\boldsymbol{Av}$?

   (d) Generalise this observation in terms of any homogeneous linear system $\boldsymbol{Ax} = \boldsymbol{0}$ and its solutions. (**Note:** This idea will be discussed in greater detail later in the course.)

   (a) Easy to verify that $\boldsymbol{v} \cdot \boldsymbol{u_i} = 0$ for $i = 1,2,3,4$.

   (b)
$$\boldsymbol{A} = \begin{pmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{pmatrix} \qquad \boldsymbol{v} = \begin{pmatrix} -3 \\ -1 \\ -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

   (c) $\boldsymbol{Av} = \boldsymbol{0}$.

   (d) Any solution $\boldsymbol{v}$ to $\boldsymbol{Ax} = \boldsymbol{0}$ (involving n unknowns), writen as a vector in $\mathbb{R}^n$ is always orthogonal to each row of $\boldsymbol{A}$, each also being a vector in $\mathbb{R}^n$.