ANSWERS TO MA1506 TUTORIAL 5

Question 1

Following the standard equations for the Malthus Model [Chapter 3]:

$$
\begin{aligned}
&N = \hat{N}e^{kt}; N(0) = 10000 = \hat{N}\\
&N(2.5) = 10000e^{2.5k} = 11000\\
&\Rightarrow e^{2.5k} = 1.1 \Rightarrow k = \frac{1}{2.5}\ell n(1.1)\\
&\qquad\qquad\qquad\quad = 0.0381\\
&N(10) = 10000e^{10k} = 10000e^{10(0.0381)} \approx 14600\\
&20000 = 10000e^{kt} \rightarrow t = \frac{1}{k}\ell n(2)\\
&\qquad\qquad\qquad\qquad\quad = 18.18 \text{ hours}
\end{aligned}
$$

Question 2

The logistic equation has 3 kinds of solution, one increasing, one constant, and one decreasing. Since the number of bugs in this problem clearly increases, the relevant solution of the logistic equation is

$$N = \frac{N_\infty}{1 + \left(\frac{N_\infty}{\hat{N}} - 1\right)e^{-Bt}}$$

Here $\hat{N} = 200$, $B = 1.5$, so at $t = 2$ we have

$$
\begin{aligned}
360 &= \frac{N_\infty}{1 + \left(\frac{N_\infty}{200} - 1\right)e^{-1.5\times 2}}\\
&\Rightarrow 360 + \frac{360}{200}e^{-3}N_\infty - 360e^{-3} = N_\infty\\
N_\infty &= \frac{360(1-e^{-3})}{1 - \frac{360}{200}e^{-3}} \approx 376\\
N(3) &= \frac{N_\infty}{1 + \left(\frac{N_\infty}{200} - 1\right)e^{-4.5}} \approx 372
\end{aligned}
$$

Question 3 First compare 80 with $\dfrac{B^2}{4s}$.

From Question 2 we know $B = 1.5$ and and $N_\infty = 376$, so $N_\infty = B/s \Rightarrow s = \frac{1.5}{376} \Rightarrow \frac{B^2}{4s} =$ 141. This is the maximum number we can kill without causing extinction.

Setting $E = 80$,

$$\begin{matrix}\beta_1\\ \beta_2\end{matrix} = \frac{B \mp \sqrt{B^2 - 4Es}}{2s} = \begin{matrix}64\\ 312\end{matrix}.$$

Since the initial number of bugs was 200, which is between these two values, we see that the limiting number is $\beta_2 = 312$, since this is the stable equilibrium.

Question 4

We have $B_\infty = \frac{B}{s} = 194600$ so since $B = 0.09866$, $s = \frac{B}{N_\infty} = \frac{0.09866}{194600}$.
Maximum hunting rate is

$$\frac{B^2}{4s} = \frac{(0.09866)^2}{4 \times \frac{0.09866}{194600}} = 4800$$

Since $10000 > 4800$, birds are doomed.

Question 5

For the fish to survive a 10% downward fluctuation, we must have (in the extreme case)

$\beta_1 = 90\% \ \ \beta_2$ i.e.

$$\begin{aligned}
\frac{B - \sqrt{B^2 - 4Es}}{2s} &= 0.9\left[\frac{B + \sqrt{B^2 - 4Es}}{2s}\right] \\
B - \sqrt{} &= 0.9B + 0.9\sqrt{}
\end{aligned}$$

$$\begin{aligned}
0.1B &= 1.9\sqrt{} \\
0.01B^2 &= 3.61(B^2 - 4Es) = 3.61B^2 - 14.44Es \\
14.44E &= 3.6B^2/s \\
E &= 0.2493074\frac{B^2}{s} \\
&= 0.997 \times \left(\frac{B^2}{4s}\right)
\end{aligned}$$

So a less than 1% drop in the catch below E* will give a 10% margin of safety.

Question 6.

(a) We shall use the following s-Shifting property:

$L(f(t)) = F(s) \Rightarrow L(e^{ct}f(t)) = F(s-c)$

$\therefore \quad L(t^2) = \dfrac{2}{s^3} \Rightarrow$ use $L(t^n) = \dfrac{n!}{s^{n+1}}$

$\therefore \quad L(t^2e^{-3t}) = L(e^{-3t}t^2) = \dfrac{2}{(s+3)^3}$

(b) Here u denotes the Unit Step Function given by

$u(t-a)\begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t > a \end{cases}$

We shall use the following t-Shifting property:

$L(f(t)) = F(s) \Rightarrow L\{f(t-a)u(t-a)\} = e^{-as}F(s)$

Let$f(t-2) = t$

$\therefore \quad f(t) = t+2$

$\therefore \quad L(f(t)) = L(t+2) = L(t) + 2L(1)$

$= \dfrac{1}{s^2} + \dfrac{2}{s}$

$\therefore \quad L(tu(t-2)) = L\{f(t-2)u(t-2)\}$

$= e^{-2s}(\dfrac{1}{s^2} + \dfrac{2}{s})$

Question 7. (a)

$\dfrac{s}{s^2+10s+26} = \dfrac{s}{(s+5)^2+1} = \dfrac{(s+5)-5}{(s+5)^2+1}$

Let$F(s) = \dfrac{s-5}{s^2+1}$

$\therefore \quad L^{-1}(\dfrac{s}{s^2+10s+26}) = L^{-1}(F(s+5))$

$= L^{-1}(F(s-(-5))$

$= e^{-5t}L^{-1}(F(s)) \rightarrow$ use s-shifting

$= e^{-5t}L^{-1}(\dfrac{s}{s^2+1} - \dfrac{5}{s^2+1})$

$= e^{-5t}\{L^{-1}(\dfrac{s}{s^2+1}) - 5L^{-1}(\dfrac{1}{s^2+1})\}$

$= e^{-5t}(\text{cost} - 5\sin t)$

(b) Let $F(s) = \frac{1+2s}{s^3}$

$= \frac{1}{s^3} + \frac{2}{s^2}$

$\therefore \quad L^{-1}(F(s)) = \frac{t^2}{2} + 2t \quad \rightarrow \quad (\text{use } L(t^n) = \frac{n!}{s^{n+1}})$

Let $f(t) = \frac{t^2}{2} + 2t$

Using t-shifting,

$$L^{-1}(e^{-2s}\frac{1+2s}{s^3}) = L^{-1}(e^{-2s}F(s))$$
$$= f(t-2)u(t-2)$$
$$= \{\frac{(t-2)^2}{2} + 2(t-2)\}u(t-2)$$
$$= \frac{1}{2}(t^2-4)u(t-2)$$
$$= (\frac{1}{2}t^2-2)u(t-2)$$

Question 8. (a)

Let $L(y(t)) = Y(s)$

We shall use $L(y'(t)) = sY(s) - y(0)$.

We have

$$L(y') = L(tu(t-2))$$
$$\Rightarrow sY(s) - 4 = e^{-2s}(\frac{1}{s^2} + \frac{2}{s})$$
$$\Rightarrow Y(s) = e^{-2s}(\frac{1+2s}{s^3}) + \frac{4}{s}$$
$$\therefore \quad y(t) = L^{-1}(Y(s))$$
$$= L^{-1}\{e^{-2s}(\frac{1+2s}{s^3})\} + 4L^{-1}(\frac{1}{s})$$
$$= (\frac{1}{2}t^2-2)u(t-2) + 4$$

(by a previous question.)

(b) We shall use

$$L(y'') = s^2Y - sy(0) - y'(0)$$

We have

$$L(y'' - 2y') = L(4)$$

$$\Rightarrow s^2Y - sy(0) - y'(0) - 2\{sY - y(0)\} = \frac{4}{s}$$

$$\Rightarrow s^2Y - s - 2sY + 2 = \frac{4}{s}$$

$$\Rightarrow (s^2 - 2s)Y = \frac{4}{s} + s - 2 = \frac{4 + s^2 - 2s}{s}$$

$$\Rightarrow Y = \frac{s^2 - 2(s-2)}{s^2(s-2)}$$

$$= \frac{1}{s-2} - \frac{2}{s^2}$$

$$\therefore \quad y = L^{-1}(\frac{1}{s-2} - \frac{2}{s^2})$$

$$= e^{2t} - 2t$$

Question 9 Solution

Dividing the equation by $m_0 - \alpha t$ yields

$$\frac{dv}{dt} = -g + \frac{\alpha\beta}{m_0 - \alpha t}$$

$$\Rightarrow v(t) = \int_0^t \left(-g + \frac{\alpha\beta}{m_0 - \alpha s} \right) ds + v(0)$$

Thus,

$$v(t) = [-gs - \beta \ln(m_0 - \alpha s)]\,|_{s=0}^{s=t} = -gt + \beta \ln \frac{m_0}{m_0 - \alpha t}$$

where we used the condition $0 \le t < m_0/\alpha$ so that $m_0 - \alpha t > 0$.
Since the height h(t) of the rocket satisfies h(0) = 0, we find

$$\begin{aligned} h(t) &= \int_0^t v(s) ds = \int_0^t \left(-gs + \beta \ln \frac{m_0}{m_0 - \alpha s} \right) ds \\ &= \left[-\frac{gs^2}{2} + \beta s \ln m_0 + \frac{\beta}{\alpha}(m_0 - \alpha s) \ln \frac{m_0 - \alpha s}{e} \right]_{s=0}^{s=t} \\ &= \beta t - \frac{gt^2}{2} - \frac{\beta}{\alpha}(m_0 - \alpha t) \ln \frac{m_0}{m_0 - \alpha t} \end{aligned}$$