Unit 015 Matrix inverse laws

Slide 01: In this unit, we will see some laws involving the inverse of a matrix.

Slide 02: The first law is commonly known as the cancellation law, which you may find familiar with real numbers operation. If $\boldsymbol{A}$ is an invertible square matrix and $\boldsymbol{AB_1}$ is equal to $\boldsymbol{AB_2}$, then we can conclude that $\boldsymbol{B_1}$ is equal to $\boldsymbol{B_2}$.

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To see this, note that since $\boldsymbol{A}$ is invertible, we can write down the matrix $\boldsymbol{A}^{-1}$ and by premultiplying the inverse of $\boldsymbol{A}$ on both sides of the equation,

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we have $\boldsymbol{IB_1}$ on the left and $\boldsymbol{IB_2}$ on the right. This is essentially the conclusion that we want, which is $\boldsymbol{B_1} = \boldsymbol{B_2}$.

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In a similar manner, if $\boldsymbol{A}$ is an invertible matrix such that $\boldsymbol{C_1A}$ is equal to $\boldsymbol{C_2A}$, then we must have $\boldsymbol{C_1} = \boldsymbol{C_2}$.

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Again, this conclusion can be obtained by post-multiplying $\boldsymbol{A}^{-1}$ on both sides of the equation,

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which gives $\boldsymbol{C_1I} = \boldsymbol{C_2I}$ and the result follows.

Slide 03: Note that this ability to 'cancel $\boldsymbol{A}$ on both sides of the equation is usually not valid when $\boldsymbol{A}$ is not invertible.

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For example, this simple 2×2 matrix has already been shown in an earlier unit, that it is singular.

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With these two matrices $\boldsymbol{B_1}$ and $\boldsymbol{B_2}$, we can check that even though $\boldsymbol{AB_1}$ is equal to $\boldsymbol{AB_2}$, $\boldsymbol{B_1}$ is clearly different from $\boldsymbol{B_2}$.

Slide 04: Let us define what is known as the transpose of a matrix. Suppose $\boldsymbol{A}$ is a $m \times n$ matrix with entries a_{ij}.

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The transpose of $\boldsymbol{A}$, denoted by $\boldsymbol{A}^T$ is a $n \times m$ matrix whose (i, j)-entry is actually a_{ji}. In other words, the (i, j)-entry in $\boldsymbol{A}^T$ is the (j, i)-entry in $\boldsymbol{A}$.

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Thus, if $\boldsymbol{A}$ is a matrix with m rows and n columns, then $\boldsymbol{A}^T$ will be a matrix with n rows and m columns.

Slide 05: For example, this matrix $\boldsymbol{A}$ is 3×5, its transpose would be 5×3. You should note that to write down $\boldsymbol{A}^T$, you just need to write the rows of $\boldsymbol{A}$ as columns. So the first row of $\boldsymbol{A}$ becomes the first column of $\boldsymbol{A}^T$ and so on.

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What about this matrix $\boldsymbol{B}$? Note that $\boldsymbol{B}$ is a square matrix of order 5 thus its transpose will also be a square matrix of order 5.

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In fact, when you write down the matrix $\boldsymbol{B}^T$, you will have the following matrix and careful observation will reveal that $\boldsymbol{B}$ and $\boldsymbol{B}^T$ are actually the same matrix. You may recall that we have defined matrices with such a property in an earlier unit.

Slide 06: Symmetric matrices are square matrices with the property that every (i,j)-entry is the same as the (j,i)-entry. Now that we have learnt the definition of the transpose of matrix, we can now define a square matrix $\boldsymbol{A}$ as symmetric if and only if $\boldsymbol{A}$ and $\boldsymbol{A}^T$ are the same.

Slide 07: Let us discuss some results on transpose. Suppose $\boldsymbol{A}$ is a $m \times n$ matrix.
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It is clear that if we transpose $\boldsymbol{A}^T$, we obtain back the matrix $\boldsymbol{A}$.
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The transpose of the sum of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is the sum of their respective transposes.
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If we have a scalar multiple of $\boldsymbol{A}$, the tranpose can be done before or after multiplying a to the matrix. The result would be exactly identical.
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If $\boldsymbol{B}$ is a $n \times p$ matrix, then we can compute the product $\boldsymbol{AB}$. The transpose of $\boldsymbol{AB}$ is the product $\boldsymbol{B}^T\boldsymbol{A}^T$. So in other words, the transpose of a product is equal to the product of the respective transposes, but you must remember to reverse the order of writing the matrices. Thus when $\boldsymbol{B}$ is postmultiplied to $\boldsymbol{A}$ and the transpose is taken, the result is to have $\boldsymbol{B}^T$ premultiplied to $\boldsymbol{A}^T$.

Slide 08: Let us see a few more results. The first one here states that if $\boldsymbol{A}$ is invertible, then for any non zero scalar c, $c\boldsymbol{A}$ is also invertible. In fact, the inverse of $c\boldsymbol{A}$ is $\frac{1}{c}\boldsymbol{A}^{-1}$. To prove this result
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We have a candidate for the inverse of $c\boldsymbol{A}$ which we will use to test by pre and post multiplying it to $c\boldsymbol{A}$. On the left, we pre-multiply $\frac{1}{c}\boldsymbol{A}^{-1}$ to $c\boldsymbol{A}$ while on the right, we post-multiply.
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Since $\frac{1}{c}$ and c are both constants, we can take them out and compute their product seperately.
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It is easy to see that in both cases, we are left with $\boldsymbol{I}$, since both $\boldsymbol{A}^{-1}\boldsymbol{A}$ and $\boldsymbol{A}\boldsymbol{A}^{-1}$ are both equal to $\boldsymbol{I}$.

Slide 09: The next result states that if $\boldsymbol{A}$ is invertible, then $\boldsymbol{A}^T$ would be invertible and the inverse of $\boldsymbol{A}^T$ is the transpose of $\boldsymbol{A}^{-1}$.
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Again, we have a candidate to be tested, which is the transpose of $\boldsymbol{A}^{-1}$. Let us pre and post multiply the transpose of $\boldsymbol{A}^{-1}$ to $\boldsymbol{A}^T$, as shown here.
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Using a property we saw earlier in this unit, the product of two transposes can be rewritten as the transpose of the product, with the order reversed. So we have $\boldsymbol{A}\boldsymbol{A}^{-1}$ on the left side and $\boldsymbol{A}^{-1}\boldsymbol{A}$ on the right.

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In both cases, this results in $\boldsymbol{I}$ and we are done.

Slide 10: The next result states that if $\boldsymbol{A}$ is invertible, then $\boldsymbol{A}^{-1}$ is also invertible. In fact the pair of matrices are inverses of each other.

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Once again, we can pre and post multiply the candidate, which is $\boldsymbol{A}$ to $\boldsymbol{A}^{-1}$.

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It is noted immediately that we have $\boldsymbol{I}$ for both cases.

Slide 11: Next, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are both invertible matrices of the same size, then the product $\boldsymbol{AB}$ will also be invertible and the inverse of $\boldsymbol{AB}$ is $\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$. Note that this result means that the product of invertible matrices will result in another invertible matrix.

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The candidate to be tested here is $\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$. We will pre and post multiply this candidate to $\boldsymbol{AB}$.

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Due to the commutative law for matrix multiplication, we can group the matrix products as follows. On the left, we have $\boldsymbol{A}^{-1}\boldsymbol{A}$ and on the right we have $\boldsymbol{B}\boldsymbol{B}^{-1}$.

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This simplifies the expression on the left to be $\boldsymbol{B}^{-1}\boldsymbol{I}\boldsymbol{B}$ and the expression on the right to be $\boldsymbol{A}\boldsymbol{I}\boldsymbol{A}^{-1}$.

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In both cases, it is easy to see that we have $\boldsymbol{I}$ as the result.

Slide 12: As an extension to the previous result, if we have a collection of invertible matrices $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, and so on till $\boldsymbol{A}_k$, all of the same size, then the product of these matrices will be invertible whose inverse is precisely the product of their inverses, only with the order reversed.

Slide 13: Let us now define the powers of invertible matrices. We have seen how $\boldsymbol{A}^n$ is defined in an earlier unit.

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If $\boldsymbol{A}$ is an invertible square matrix, then by what we have established earlier, the matrix $\boldsymbol{A}^n$ will also be invertible, whose inverse is simply the product of n copies of $\boldsymbol{A}^{-1}$. We thus define $\boldsymbol{A}^{-n}$ to be the product of n copies of $\boldsymbol{A}^{-1}$.

Slide 14: Consider the following example where $\boldsymbol{A}$ is a 2×2 matrix which is invertible. You can verify that the inverse of $\boldsymbol{A}$, given here, is indeed the correct one.

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The square of $\boldsymbol{A}$ is computed as such.

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By our definition $\boldsymbol{A}^{-2}$ is the product of $\boldsymbol{A}^{-1}$ with $\boldsymbol{A}^{-1}$, which gives this matrix.

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We can now verify that $\boldsymbol{A}^{-2}$ is indeed the inverse of $\boldsymbol{A}^2$.

Slide 15: Let us summarise this unit.
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We established some laws involving the inverse of a matrix.
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We defined what is called the transpose of a matrix and some related laws.
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Lastly, we defined the inverse of the powers of an invertible matrix.