CHAPTER 4

THE LAPLACE TRANSFORM

Definition. Let f be a function defined for all $t \geq 0$. The Laplace transform of f is the function $F(s)$ defined by

$$F(s) = L(f) = \int_0^\infty e^{-st} f(t)dt, \qquad (1)$$

provided the improper integral on the right exists.

The original function $f(t)$ in (1) is called the inverse transform or inverse of $F(s)$ and is denoted by $L^{-1}(F)$; i.e.,

$$f(t) = L^{-1}(F(s)).$$

Notation: Original functions are denoted by lower case letters and their Laplace transforms by the same letters in capitals. Thus $F(s) = L(f(t))$, $Y(s) = L(y(t))$ etc.

Recall that, by definition, for any function h defined on $[0, \infty)$,

$$\int_0^\infty h(t)\, dt = \lim_{b \to \infty} \int_0^b h(t) dt$$

and the integral is said to converge if this limit exists. Because e^{-st} decreases so rapidly with t, the Laplace transform usually does exist [there are exceptions, however], and then we say that the function f HAS A WELL-DEFINED LAPLACE TRANSFORM.

Example 1. Let $f(t) = e^{at}$, when $t \geq 0$.

Find $F(s)$.

Solution.

$$\begin{aligned} F(s) &= L(e^{at}) \\ &= \int_0^\infty e^{-st} e^{at} dt \\ &= \lim_{b\to\infty} \int_0^b e^{-st} e^{at} dt. \end{aligned}$$

Now

$$\int_0^b e^{(a-s)t} dt = \begin{cases} b & \text{if } s = a \\ \frac{e^{b(a-s)}}{a-s} - \frac{1}{a-s} & \text{if } s \neq a. \end{cases}$$

If $s < a$, $a - s > 0$ and $e^{(a-s)b} \to \infty$ as $b \to \infty$. Thus when $s \leq a$, $\int_0^\infty e^{(a-s)t} dt$ diverges. When $s > a$, $a - s < 0$, and $e^{(a-s)b} \to 0$ as $b \to \infty$, and then

$$F(s) = L(e^{at}) = \frac{1}{s-a}, \quad s > a. \tag{2}$$

$\square$

Example 2. Let $f(t) = 1$, $t \geq 0$. Find $F(s)$.

Solution. This function is the same as the one in Example 1 with $a = 0$, thus,

$$L(1) = \frac{1}{s}, \quad s > 0. \tag{3}$$

$\square$

The Laplace transform is a linear operation; i.e., the Laplace transform of a linear combination of functions equals the same linear combination of their Laplace transforms. Thus

Theorem

$$L(af(t) + bg(t)) = aL(f) + bL(g), \tag{4}$$

where a and b are constants.

As a corollary, the inverse Laplace transform also satisfies the linearity property.

$$L^{-1}(aF(s) + bG(s)) = aL^{-1}(F) + bL^{-1}(g). \tag{5}$$

Verification of (4) and (5) is easy.

Example 3. Using (4), we obtain

$$\begin{aligned} L(\cosh at) &= L\left(\frac{1}{2}(e^{at} + e^{-at})\right) \\ &= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) \\ &= \frac{s}{s^2 - a^2}, \quad s > a \geq 0. \end{aligned}$$

$\square$

Example 4. If $F(s) = \frac{3}{s} + \frac{5}{s-7}$,
find $f(t) = L^{-1}(F)$.

Solution. Using (5), we have

$$\begin{aligned} L^{-1}(F) &= L^{-1}(\frac{3}{s}) + 5L^{-1}(\frac{1}{s-7}) \\ &= 3 \cdot 1 + 5 \cdot e^{7t} = 3 + 5e^{7t}. \end{aligned}$$

$\square$

Example 5. Set $a = iw$ in the formula (2)

$$\begin{aligned} L(e^{iwt}) &= L(\cos wt + i\sin wt) \\ &= L(\cos wt) + iL(\sin wt) \\ &= \frac{1}{s - iw} = \frac{s + iw}{s^2 + w^2} \end{aligned}$$

Equating real and imaginary parts, we get

$$L(\cos wt) = \tfrac{s}{s^2+w^2},\ L(\sin wt) = \tfrac{w}{s^2+w^2}. \tag{6,7}$$

$\square$

Example 6. To show that $L(t^n) = \frac{n!}{s^{n+1}}$, $n = 0, 1, 2, \ldots$.

Solution.

$$\begin{aligned} L(t^n) &= \int_0^\infty e^{-st} t^n dt \\ &= -\frac{1}{s} e^{-st} t^n \mid_0^\infty \\ &\quad + \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} dt. \end{aligned}$$

First term is zero at $t = 0$ and as $t \to \infty$. Thus, using induction

$$
\begin{aligned}
L(t^n) &= \frac{n}{s} L(t^{n-1}) \\
&= \frac{n(n-1)\dots 1}{s^n}\, L(1) \\
&= \frac{n!}{s^{n+1}}. \qquad (8)
\end{aligned}
$$

$\square$

Example 7. Given $F(s) = \frac{2s+5}{s^2+9}$, find $L^{-1}(F(s))$.

Solution.

$$
\begin{aligned}
L^{-1}\left(\frac{2s+5}{s^2+9}\right) &= L^{-1}\left(\frac{2s}{s^2+9} + \frac{5}{s^2+9}\right) \\
&= 2L^{-1}\left(\frac{s}{s^2+9}\right) + \frac{5}{3}L^{-1}\left(\frac{3}{s^2+9}\right) \\
&= 2\cos 3t + \frac{5}{3}\sin 3t.
\end{aligned}
$$

$\square$

PIECEWISE CONTINUOUS FUNCTIONS

One of the nice things about Laplace transforms is that they are defined by integration, and, unlike differentiation, integration doesn't care whether the function is continuous or not. In fact it is easy to define the Laplace transform of a function whose graph is broken up into pieces. More formally:

A function $f(t)$ defined for $t \geq 0$ has a jump discontinuity at $a \in [0, \infty)$ if the one sided limits

$$\lim_{t \to a^-} f(t) = \ell_- \quad \text{and} \quad \lim_{t \to a^+} f(t) = \ell_+$$

exist but f is not continuous at $t = a$.

By definition, a function $f(t)$ is PIECEWISE

CONTINUOUS on a finite interval $a \le t \le b$ if jump discontinuities are its only discontinuities. Such a function always has a nice Laplace transform (unless it grows extremely quickly, faster than exponentially).

Transform of derivatives and integrals

Theorem Suppose that $f(t)$ is continuous and has a well-defined Laplace transform on $[0, \infty)$ and $f'(t)$ is piecewise continuous on $[0, \infty)$. Then $L(f'(t))$ exists and

$$L(f') = sL(f) - f(0), \quad s > a. \tag{10}$$

Proof. First consider the case when $f'(t)$ is continuous for $[0, \infty)$. Then

$$\begin{aligned} L(f') &= \int_0^\infty e^{-st}\, f'(t)dt \\ &= e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)dt \\ &= \lim_{b\to\infty} e^{-sb} f(b) - f(0) + sL(f). \end{aligned}$$

Since f has a well-defined Laplace transform, the first term on the right is 0 when $s > 0$, showing that $L(f') = sL(f) - f(0)$ for $s > a$. $\square$

If f' is merely piecewise continuous, the proof is quite similar; in this case, the range of integration in the original integral must be split into parts such that f' is continuous in each such part.

Apply (10) to f'' to obtain

$$\begin{aligned} L(f'') &= sL(f') - f'(0) \\ &= s(sL(f) - f(0)) - f'(0) \\ &= s^2 L(f) - sf(0) - f'(0). \end{aligned} \tag{11}$$

Similarly $L(f''') = s^3 L(f) - s^2 f(0) - sf'(0) - f''(0)$, etc. By induction we obtain the following

Theorem Suppose that $f(t)$, $f'(t)$, $f''(t)$, $\ldots$, $f^{(n-1)}(t)$ are continuous and have well-defined Laplace transforms on $[0,\infty)$ and $f^{(n)}(t)$ is piecewise continuous on $[0,\infty)$, then

$$L(f^{(n)}) = s^n L(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0). \tag{12}$$

Example 8. Find $L(\sin^2 t)$.

<u>Solution.</u> Here $f(t) = \sin^2 t$, $f'(t) = 2\sin t \cos t = \sin 2t$, $f(0) = 0$. In view of (7) we obtain

$$\begin{aligned} L(f') &= L(\sin 2t) \\ &= \frac{2}{s^2+4} \\ &= sL(f) - f(0) = sL(f) \end{aligned}$$

$\therefore \quad L(\sin^2 t) = \frac{2}{s(s^2+4)}$. $\square$

Example 9. Find $L(t \sin \alpha t)$.

Solution. Here $f(t) = t \sin \alpha t$ and $f(0) = 0$. Also

$$f'(t) = \sin \alpha t + \alpha t \cos \alpha t, \quad f'(0) = 0$$

$$f''(t) = 2\alpha \cos \alpha t - \alpha^2 t \sin \alpha t$$

$$= 2\alpha \cos \alpha t - \alpha^2 f(t)$$

$\therefore$ by (11), $L(f'') = 2\alpha L(\cos \alpha t) - \alpha^2 L(f) = s^2 L(f)$.

$\therefore \quad (s^2 + \alpha^2) L(f) = 2\alpha L(\cos \alpha t) = \frac{2\alpha s}{s^2+\alpha^2}$.

Hence $L(t \sin \alpha t) = \frac{2\alpha s}{(s^2+\alpha^2)^2}$. □

Solution of Initial value problems

Consider the initial value problem

$$y'' + ay' + by = r(t)$$

$$y(0) = k_0, \quad y'(0) = k_1 \tag{13}$$

with a, b, k_0 and k_1 constants. The function $r(t)$ has the Laplace transform $R(s)$.

Step 1. Take the Laplace transform of both sides of the d.e. using the linearity property of the Laplace transform.

$$s^2L(y)-sy(0)-y'(0)+a(sL(y)-y(0))+bL(y) = L(r).$$

Step 2. Use the given initial conditions to arrive at the subsidiary equation

$$s^2L(y) - sk_0 - k_1 + a(sL(y) - k_0) + bL(y) = L(r).$$

Step 3. Solve this for $L(y)$:

$$L(y) = \frac{(s+a)k_0 + k_1 + R(s)}{s^2 + as + b}.$$

Step 4. Reduce the above to a sum of terms whose inverses can be found, so that the solution $y(t)$ of (13) is obtained.

Example 10. Solve $y'' + y = e^{2t}$, $y(0) = 0$, $y'(0) = 1$.

Solution. Taking the Laplace transform of both sides of the d.e. and using initial conditions we arrive at

$$s^2 L(y) - sy(0) - y'(0) + L(y) = \frac{1}{s-2}.$$

$\therefore \quad L(y) = \frac{1}{s^2+1}\left(1 + \frac{1}{s-2}\right) = \frac{s-1}{(s-2)(s^2+1)}.$

Now

$$\frac{s-1}{(s-2)(s^2+1)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+1}$$

for appropriate constants A, B and C. Multiply both sides of this equation by $(s-2)(s^2+1)$ and equate the coefficients of s^0, s, s^2 to obtain $A - 2C = -1$, $-2B + C = 1$, $A + B = 0$. Thus $A = \frac{1}{5}$, $B = -\frac{1}{5}$, and $C = \frac{3}{5}$.

$\therefore \quad L(y) = \frac{1}{5} \cdot \frac{1}{s-2} - \frac{s-3}{5(s^2+1)}$

$$= \frac{1}{5(s-2)} - \frac{s}{5(s^2+1)} + \frac{3}{5(s^2+1)}.$$

Taking the inverse transform of both sides we get (using (2), (6), and (7))

$$y(t) = \frac{1}{5}e^{2t} - \frac{1}{5}\cos t + \frac{3}{5}\sin t.$$

$\square$

Transform of the integral of a function

Theorem. If $f(t)$ is piecewise continuous and has a well-defined Laplace transform on $[0, \infty)$, then

$$L\left(\int_0^t f(\tau)d\tau\right) = \frac{1}{s}L(f) \qquad (s > 0, s > a). \quad (14)$$

Example 11. Find $f(t)$ if $L(f) = \frac{1}{s^2(s^2+w^2)}$.

<u>Solution.</u>

$$\because \ L\left(\frac{1}{w}\sin wt\right) = \frac{1}{s^2 + w^2}$$

we use (14) to get

$$\begin{aligned} L\left(\frac{1}{w}\int_0^t \sin w\tau d\tau\right) &= L\left(\frac{1 - \cos wt}{w^2}\right) \\ &= \frac{1}{s(s^2 + w^2)} \end{aligned}$$

and

$$L\left(\frac{1}{w^2}\int_0^t (1-\cos w\tau)d\tau\right) = L\left(\frac{1}{w^2}(t-\frac{\sin wt}{w})\right)$$
$$= \frac{1}{s^2(s^2+w^2)}.$$

$\therefore \quad f(t) = \frac{1}{w^2}\left(t - \frac{\sin wt}{w}\right).$ $\square$

Theorem. (s-Shifting)

If $f(t)$ has the transform $F(s)$, $s > a$, then

$$L(e^{ct}f(t)) = F(s-c), \quad s-c > a. \tag{15}$$

Thus using the formulae (8), (6), and (7) we obtain

$$L(e^{ct}t^n) = \frac{n!}{(s-c)^{n+1}}$$
$$L(e^{ct}\cos wt) = \frac{s-c}{(s-c)^2+w^2}$$
$$L(e^{ct}\sin wt) = \frac{w}{(s-c)^2+w^2}.$$

Example 12. Solve $y'' + 2y' + 5y = 0$,

$y(0) = 2,\ y'(0) = -4.$

Solution. Following the procedure outlined earlier we obtain

$$\begin{aligned} L(y) &= \frac{2s}{(s+1)^2 + 2^2} \\ &= \frac{2(s+1)}{(s+1)^2 + 2^2} - \frac{2}{(s+1)^2 + 2^2} \end{aligned}$$

$\therefore \quad y(t) = e^{-t}(2\cos 2t - \sin 2t).$ $\square$

Example 13. Solve $y'' - 2y' + y = e^t + t$, $y(0) = 1,\ y'(0) = 0.$

Solution. Taking Laplace transform of the d.e.

$$s^2 L(y) - sy(0) - y'(0) - 2(sL(y) - y(0)) + L(y)$$
$$= \frac{1}{s-1} + \frac{1}{s^2}$$

or

$$(s^2 - 2s + 1)L(y) = s - 2 + \frac{1}{s-1} + \frac{1}{s^2}$$

or

$$
\begin{aligned}
L(y) &= \frac{s-2}{(s-1)^2} + \frac{1}{(s-1)^3} + \frac{1}{s^2(s-1)^2} \\
&= \frac{1}{s-1} - \frac{1}{(s-1)^2} + \frac{1}{(s-1)^3} \\
&\quad + \frac{1}{(s-1)^2} - \frac{2}{s-1} + \frac{1}{s^2} + \frac{2}{s}
\end{aligned}
$$

$\therefore \quad y(t) = \frac{t^2}{2}e^t - e^t + t + 2 = (\frac{t^2}{2} - 1)e^t + t + 2.$ $\square$

Unit Step (Heaviside) function

Definition.

$$
u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases} \tag{16}
$$

Example 14. Graph $f(t) = u(t-1) - u(t-3)$.

Clearly $f(t) = 1$ when $1 < t < 3$ and 0 otherwise. $\square$

Observe that if $0 < a < b$

$$
u(t-a) - u(t-b) = \begin{cases} 0, & t < a \\ 1, & a < t < b \\ 0, & t > b. \end{cases}
$$

Let $g(t)$ be some function of t. Then, if $0 < a < b$

$$g(t)(u(t-a) - u(t-b)) = \begin{cases} 0 & \text{if } t < a \\ g(t) & \text{if } a < t < b \\ 0 & t > b. \end{cases}$$

You can think of this in the following way: this function is "OFF" until t = a. Then it suddenly turns "ON" the function g(t). It remains "ON" until t = b, where it suddenly switches "OFF" again. You can easily imagine lots of engineering situations where this might be useful, for example in circuit theory. Notice that this function is usually DISCONTINUOUS, but this won't be a problem for the Laplace transform, because the transform is defined by an integral, and discontinuous functions can often be integrated. In fact, the main advantage of the Laplace transform is that it allows us to solve ODEs with discontinuous right-hand-sides. Such ODEs often come up in engineering applications.

Example 15. Express

$$f(t) = \begin{cases} t, & 0 < t < 1 \\ 2 - t, & 1 < t < 2 \\ 0, & 2 < t < 3 \\ 1, & t > 3 \end{cases}$$

Solution.

$$\begin{aligned} f(t) = t(u(t) - u(t-1)) \\ &+ (2-t)(u(t-1) \\ &- u(t-2)) + u(t-3). \end{aligned}$$

Example 16. Sketch

$$\begin{aligned} g(t) = 2u(t) + tu(t-1) + (3-t)u(t-2) \\ - 3u(t-4), \quad t > 0. \end{aligned}$$

Solution.

When

$$
\begin{aligned}
0 < t < 1 \quad & g(t) = 2 \cdot 1 + t \cdot 0 + (3 - t) \cdot 0 - 3 \cdot 0 = 2 \\
1 < t < 2 \quad & g(t) = 2 \cdot 1 + t \cdot 1 + (3 - t) \cdot 0 - 3 \cdot 0 \\
& \quad\;\;\, = 2 + t \\
2 < t < 4 \quad & g(t) = 2 \cdot 1 + t \cdot 1 + (3 - t) \cdot 1 - 3 \cdot 0 = 5 \\
t > 4 \quad & g(t) = 2 \cdot 1 + t \cdot 1 + (3 - t) \cdot 1 - 3 \cdot 1 = 2
\end{aligned}
$$

Theorem. (t-Shifting)

If $L(f(t)) = F(s)$ then

$$L(f(t-a)u(t-a)) = e^{-as}F(s). \tag{17}$$

Example 17. Setting $f(t-a) = 1$ in (17) we get

$$L(u(t-a)) = \frac{e^{-as}}{s}. \tag{18}$$

Example 18. Compute $L(t^2 u(t-1))$.

Solution.

$$\begin{aligned}
L(t^2u(t-1)) &= L((t-1+1)^2u(t-1)) \\
&= L(((t-1)^2+2(t-1)+1)u(t-1)) \\
&= e^{-s}(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}).
\end{aligned}$$

□

Example 19. Compute $L((e^t+1)u(t-2))$.

Solution.

$$\begin{aligned}
L((e^t+1)u(t-2)) &= L((e^{t-2}e^2+1)u(t-2)) \\
&= e^{-2s}\left(\frac{e^2}{s-1}+\frac{1}{s}\right).
\end{aligned}$$

□

The next and final problem in this section is rather complicated, but it really just involves assembling a lot of small bits and pieces. Notice that

NONE of the methods we learned in earlier chapters would have allowed us to solve this problem, so it should convince you that the Laplace Transform is really useful!

Example 20. Solve the initial value problem

$$y'' + 3y' + 2y = g(t), \;\; y(0) = 0, \;\; y'(0) = 1,$$

with
$$g(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & t > 1 \end{cases} .$$

<u>Solution.</u> $g(t) = u(t) - u(t-1)$

.

Taking the Laplace transform of both sides of the d.e. we obtain, using (18),

$$s^2L(y) - sy(0) - y'(0) + 3(sL(y) - y(0)) + 2L(y)$$
$$= \frac{1}{s} - \frac{e^{-s}}{s}$$

so that,

$$L(y) = \frac{s+1}{s(s^2+3s+2)} - e^{-s}\left[\frac{1}{s(s^2+3s+2)}\right].$$

Now

$$\begin{aligned}
\frac{s+1}{s(s^2+3s+2)} &= \frac{s+1}{s(s+1)(s+2)} \\
&= \frac{1}{s(s+2)} \\
&= \frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+2}\right)
\end{aligned}$$

$$\therefore \quad L^{-1}\left(\tfrac{1}{s(s+2)}\right) = \tfrac{1}{2}(1 - e^{-2t}).$$

Also

$$\begin{aligned}
\frac{1}{s(s^2+3s+2)} &= \frac{1}{s(s+1)(s+2)} \\
&= \frac{1}{2}\left(\frac{1}{s} + \frac{1}{s+2}\right) - \frac{1}{s+1}
\end{aligned}$$

$$\therefore \quad L^{-1}\left(\tfrac{1}{s(s+1)(s+2)}\right) = \tfrac{1}{2}(1 + e^{-2t}) - e^{-t}$$

$$\begin{aligned}
\therefore \quad & L^{-1}\left(\frac{e^{-s}}{s(s+1)(s+2)}\right) \\
&= \left\{\frac{1}{2}(1 + e^{-2(t-1)}) - e^{-(t-1)}\right\} u(t-1).
\end{aligned}$$

[using (17)]

Finally

$$
\begin{aligned}
y(t) &= \frac{1}{2}(1 - e^{-2t}) \\
&\quad - \left[\frac{1}{2}(1 + e^{-2(t-1)}) - e^{-(t-1)}\right] u(t-1).
\end{aligned}
$$

$\square$

THE DIRAC DELTA FUNCTION

1. **Definition**. Let $f_h(t)$ be a family of functions of t defined by

$$
f_h(t) = \begin{cases} 0 & t < 0 \\ 1/h & 0 \leq t \leq h \\ 0 & t \geq h \end{cases}
$$

for all $h > 0$. Notice that, for any h,

$$
\int_0^\infty f_h(t)dt = \int_0^h \frac{1}{h} dt = 1.
$$

Thus for example $f_{10^{-100}}(t)$ is a function with maximum value 10^{100} and yet the area under its graph is still 1. The graph is an extremely tall but sharp and narrow spike next to $t = 0$. We define

$$\text{“}\delta(t) \equiv \lim_{h\to 0} f_h(t)\text{”}.$$

Of course this doesn't really make sense mathematically, but you can think of $\delta(t)$ as an extremely tall and narrow spike at $t = 0$. Similarly, you can think of δ(t - a) as an infinitely tall and narrow spike at t = a. Note

$$\int_0^\infty \delta(t)dt = 1, \quad \delta(t) = 0 \text{ everywhere} \quad \text{EXCEPT } t = 0.$$

Now let $g(t)$ be any function, and consider

$$\int_0^\infty f_h(t)g(t)dt = \frac{1}{h}\int_0^h g(t)dt.$$

If h is very small, $\int_0^h g(t)dt \approx g(0)h$, so

$$\int_0^\infty f_h(t)g(t)dt \approx g(0).$$

Since the approximation gets better and better as $h \to 0$, we have

$$\int_0^\infty \delta(t)g(t)dt = g(0).$$

In a similar way,

$$\int_0^\infty \delta(t-a)g(t)dt = g(a).$$

Hence the Laplace transform of $\delta(t-a)$ is

$$L\left[\delta(t-a)\right] = \int_0^\infty e^{-st}\delta(t-a)dt = e^{-as}.$$

So $L^{-1}\left[e^{-as}\right] = \delta(t-a)$.

Note that by setting $a = 0$ we have

$$L^{-1}[1] = \delta(t).$$

EXAMPLE: INJECTIONS!

Suppose that a doctor injects, almost instantly, 100 mg of morphine into a patient. He does it again 24 hours later. Suppose that the HALF-LIFE of morphine in the patient's body is 18 hours. Find the amount of morphine in the patient at any time.

Solution: Half-life refers to the exponential function e^{-kt}. "Half-life 18 hours" = 0.75 days means $\frac{1}{2} = e^{-k\times 0.75}$, that is, k = $\dfrac{\ell n(2)}{0.75} = 0.924$. So without the injections,

$$\frac{dy}{dt} = -ky, \quad k = 0.924.$$

The injections are at a rate of 100 mg per day, but concentrated in delta-function spikes at $t = 0$ and $t = 1$ [time unit is DAYS]. So we have

$\frac{dy}{dt} = -ky + 100\delta(t) + 100\delta(t-1)$

$\rightarrow$

$$sL[y] - y(0) = -0.924L[y] + 100 \times 1 + 100e^{-s}$$

By the way, notice that we have to think of the delta function as something which itself has UNITS. In this case you should think of the delta function as something that has units of [1/time], so that this equation has consistent units. In many problems, it is actually very helpful to work out what units the delta function has — it can have different units in different problems! This is particularly useful in physics problems where something gets hit suddenly and gains some momentum instantly — you can use this in some of the tutorial problems.

Since $y(0) = 0$,

$$\begin{aligned} L(y) &= 100 \times \frac{1 + e^{-s}}{s + 0.924} \\ &= \frac{100}{s + 0.924} + \frac{100e^{-s}}{s + 0.924} \end{aligned}$$

So, using the t-shifting theorem, we get

$$\begin{aligned} y &= 100e^{-0.924t} + 100e^{-0.924(t-1)}u(t-1) \\ &= 100e^{-0.924t} \quad 0 < t < 1 \\ &= 100(1 + e^{0.924})e^{-0.924t} \quad t > 1. \end{aligned}$$

EXAMPLE: PARAMETER RECONSTRUCTION!

Sometimes it happens, in Engineering applications, that you have a system whose nature you understand [for example, you may know that it is a damped harmonic oscillator] but you don't know the values of the parameters [the spring constant, the mass, the friction coefficient]. In such a case, what you can do is to "poke" the system with a sudden, sharp force, and watch how it behaves. Then you can **reconstruct the parameters of the system** as follows.

Suppose for example that you have a damped harmonic oscillator — say, a spring with a mass attached — which is initially at rest, that is, x(0) $= \dot{x}(0) = 0$. and you poke it with a unit impulse [change of momentum = 1 in MKS units] at time t = 1; in other words, the applied force is just $\delta(t-1)$.

You observe that the displacement of the mass is

$$x(t) = u(t-1)e^{-(t-1)}sin(t-1).$$

[You can use Graphmatica to look at the graph by putting in

$$y = exp(-(x-1)) * sin(x-1) * step(x-1);$$

note that this oscillator is underdamped, despite the shape of the graph!] Question: what are the values of the mass, the spring constant, the frictional coefficient?

Solution: Newton's Second Law in this case says:

$$M\ddot{x} = -kx - b\dot{x} + \delta(t-1).$$

With the given initial data, taking the Laplace transform gives

$$Ms^2X(s) = -kX(s) - bsX(s) + e^{-s},$$

and so

$$X(s) = \frac{e^{-s}}{Ms^2 + bs + k}.$$

But the Laplace transform of the given response function is [using both t-shifting and s-shifting!]

$$X(s) = \frac{e^{-s}}{(s+1)^2 + 1} = \frac{e^{-s}}{s^2 + 2s + 2},$$

so by inspection we see that M = 1, b = 2, k = 2 in MKS units, and we have successfully reconstructed the parameters of this system, just by hitting it with a delta function impulse! Similar ideas work for electrical circuits etc etc etc. Useful idea.

If you look at the graph of the solution you will see that the derivative jumps suddenly at t = 1; that is, there is a sharp corner there. This is typical in problems involving delta-function impulses, basically because such impulses cause the momentum, and therefore the velocity, to change suddenly.