**W05-03**

**Slide 01:** In this unit, we will continue with our discussion on linearly independence.

**Slide 02:** We once again recall the notion of redundancy introduced several units ago.

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When $\boldsymbol{u_k}$ is a linearly combination of $\boldsymbol{u_1}$ to $\boldsymbol{u_{k-1}}$, we say it is redundant in the span of $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$. While it was mentioned previously that this concept of redundancy is closely related to linear independence, the formal definition of linear independence was stated in terms of whether a vector equation has only trivial solutions or not. There seems to be little relation to the concept of redundancy.

**Slide 03:** We will now present a theorem to state this relation.

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Let $S$ be a set containing vectors $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ to $\boldsymbol{u_k}$ from $\mathbb{R}^n$, where $k \geq 2$.

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$S$ is a linearly dependent set if and only if there is at least one vector, say $\boldsymbol{u_i}$ in $S$ that can be written as a linear combination of the other vectors in $S$.

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Algebraically, it means that there are constants $a_1$, $a_2$ and so on such that we can write $\boldsymbol{u_i}$ in terms of the other vectors in the set.

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In relation to the concept of redundancy, this would imply that the linear span of all the vectors $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ to $\boldsymbol{u_k}$ is the same as the linear span of the $k-1$ vectors after removing $\boldsymbol{u_i}$.

**Slide 04:** On the flip side, $S$ will be linearly independent set if and only if no vector in $S$ can be written as a linear combination of the others.

**Slide 05:** With this theorem, it is now clear that a set of vectors is linearly dependent if and only if there exists at least one redundant vector in the set.

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This is the same as saying that a set of vectors is linearly independent if and only if there is no redundant vector in the set.

**Slide 06:** Is this set $S$ a linearly independent set? Of course, we can go back to the formal definition of linearly independence and check for non trivial solutions to the vector equation.

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However, using the notion of redundancy, we observe that one of the vectors in $S$, namely $(2,4)$ is actually a linear combination of the other two vectors. Thus $S$ is not a linearly independent set.

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What about this set $S$?

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It is easy to see that $(-1, 0, 0)$ cannot be written as a linear combination of $(0, 3, 0)$ and $(0, 0, 7)$ since these two vectors both have zero at the first component.

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Taking linear combinations of these two vectors will never result in a vector with $-1$ at the first component.

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Again $(0, 3, 0)$ is clearly not a linearly combination of the other two vectors

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by looking at the second component.

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And lastly, $(0, 0, 7)$ is not a linear combination of the other two vectors

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by looking at the third component. Thus this set has no redundant vectors and is therefore a linearly independent set. Note that such a method may not be easy to use if the vectors are not in such a form that makes such observations easy.

**Slide 07:** For this theorem, let $S$ be a set of vectors in $\mathbb{R}^n$. In particular, observe that $S$ has exactly $k$ vectors, namely $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ to $\boldsymbol{u_k}$. The theorem states that if $k$ is strictly larger than $n$, then the set $S$ is immediately known to be linearly dependent.

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Notice that to use this theorem, we just need to compare the two integers $k$ and $n$. If $k > n$, the conclusion follows. Note also that the theorem does not tell us anything about the set $S$ if $k \leq n$.

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To prove this result, let us write down the vectors $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$ specifically, each with their respective components as shown.

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We will use the formal definition of linearly indepdence and set up the vector equation as shown, with the zero vector on the right hand side.

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We write the vectors $\boldsymbol{u_1}$ to $\boldsymbol{u_k}$ explicitly in terms of their components.

**Slide 08:** From the vector equation, we are now able to write down the corresponding homogeneous linear system.

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Notice that we have one equation written down for each component.

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And one unknown for each vector on the left hand side of the vector equation.

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Thus, we have a total of $n$ equations involving $k$ unknowns in this homogeneous linear system.

**Slide 09:** This is the augmented matrix representing the homogeneous linear system. We have a total of $n$ rows and $k$ columns on the left hand side.

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Suppose we perform Gaussian elimination on the augmented matrix and arrive at a row-echelon form.

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Under the condition of the theorem, if $k > n$, this would mean that there are more columns on the left than rows in the augmented matrix. What can we say about the highlighted portion of the augmented matrix in this case?

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Since there are more columns than rows, we are guaranteed to observe non pivot columns on the left hand side at row-echelon form.

**Slide 10:** This allows us to conclude that the homogeneous linear system will have non trivial solutions.

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Equivalently, this means that the vector equation has non trivial solutions too.

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And thus the conclusion that $S$ is a linearly dependent set follows.

**Slide 11:** We are now able to say that a set with three or more vectors in $\mathbb{R}^2$ will always be linearly dependent.

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Likewise, any set of four or more vectors in $\mathbb{R}^3$ will always be linearly dependent.

**Slide 12:** To summarise the main points in this unit.

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We drew the connection between the notion of redundancy and the formal definition of linear independence.

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We also presented a theorem which states that a set of $k$ vectors from $\mathbb{R}^n$ will always be linearly dependent if $k$ is strictly larger than $n$.