DETERMINANTS AND ELEMENTARY ROW OPERATIONS

Elementary row operations (1st type)

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$
 We already know that
$$\det(\mathbf{A}) = aei + bfg + cdh - ceg - afh - bdi$$

$$\det(A) = aei + bfg + cdh - ceg - afh - bdi$$



Recall that
$$E_1A = B_1$$
 where

$$\boldsymbol{B}_{1} = \begin{pmatrix} a & b & c \\ d & e & f \\ kg & kh & ki \end{pmatrix}$$

$$\boldsymbol{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{pmatrix}$$

$$det(\mathbf{B}_1) = aeki + bfkg + cdkh - cekg - afkh - bdki$$
$$= k(aei + bfg + cdh - ceg - afh - bdi) = kdet(\mathbf{A})$$

Elementary row operations (1st type)

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \xrightarrow{kR_3} \begin{pmatrix} a & b & c \\ d & e & f \\ kg & kh & ki \end{pmatrix} = \mathbf{B}_1$$

$$\begin{bmatrix} \boldsymbol{E}_{1}\boldsymbol{A} = \boldsymbol{B}_{1} \end{bmatrix} \text{ where } \boldsymbol{E}_{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{pmatrix} \det(\boldsymbol{E}_{1}) = k$$
So it seems like elementary

So it seems like elementary row operations of this type

$$\det(\boldsymbol{E}_1) = k$$

$$\mathsf{det}(\pmb{B}_1) = k \mathsf{det}(\pmb{A})$$
 $= \mathsf{det}(\pmb{E}_1) \mathsf{det}(\pmb{A})$

changes the determinant of A by a factor of k and we have 'determinant of product equals to product of determinants' (to some extent).

Elementary row operations (2nd type)

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$
 We already know that
$$\det(\mathbf{A}) = aei + bfg + cdh - ceg - afh - bdi$$

$$det(A) = aei + bfg + cdh - ceg - afh - bdi$$

$$R_1 \leftrightarrow R_3$$

Recall that
$$\boldsymbol{E}_{2}\boldsymbol{A} = \boldsymbol{B}_{2}$$
 where

$$\boldsymbol{B}_2 = \begin{pmatrix} g & h & i \\ d & e & f \\ a & b & c \end{pmatrix}$$

$$\boldsymbol{E}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$det(\mathbf{B}_2) = gec + hfa + idb - iea - gfb - hdc$$
$$= -(iea + gfb + hdc - gec - hfa - idb) = -det(\mathbf{A})$$

Elementary row operations (2nd type)

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \xrightarrow{\mathbf{R}_1 \longleftrightarrow \mathbf{R}_3} \begin{pmatrix} g & h & i \\ d & e & f \\ a & b & c \end{pmatrix} = \mathbf{B}_2$$

$$(\mathbf{g} \ n \ t)$$
 $(\mathbf{a} \ b \ c)$

$$\mathbf{E}_2 \mathbf{A} = \mathbf{B}_2 \text{ where } \mathbf{E}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \det(\mathbf{E}_2) = -1$$
So it seems like elementary
$$\det(\mathbf{B}_2) = -\det(\mathbf{A})$$

So it seems like elementary row operations of this type

$$\det(\boldsymbol{E}_2) = -1$$

$$\det(\boldsymbol{B}_{2}) = -\det(\boldsymbol{A})$$

$$= \det(\boldsymbol{E}_{2})\det(\boldsymbol{A})$$

changes the determinant of A by a factor of -1 and we have 'determinant of product equals to product of determinants' (to some extent).

Elementary row operations (3rd type)

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$
 We already know that
$$\det(\mathbf{A}) = aei + bfg + cdh - ceg - afh - bdi$$

$$det(A) = aei + bfg + cdh - ceg - afh - bdi$$

$$R_2 + kR_3$$

Recall that $E_3A = B_3$ where

$$\mathbf{B}_{3} = \begin{pmatrix} a & b & c \\ d+kg & e+kh & f+ki \\ g & h & i \end{pmatrix} \qquad \mathbf{E}_{3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$$

$$det(\mathbf{B}_3) = ai(e+kh) + bg(f+ki) + ch(d+kg)$$
$$-cg(e+kh) - ah(f+ki) - bi(d+kg) = det(\mathbf{A})$$

Elementary row operations (3rd type)

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \xrightarrow{\mathbf{R}_2 + k\mathbf{R}_3} \begin{pmatrix} a & b & c \\ d + kg & e + kh & f + ki \\ g & h & i \end{pmatrix} = \mathbf{B}_3$$

$$\mathbf{E}_{3}\mathbf{A} = \mathbf{B}_{3} \text{ where } \mathbf{E}_{3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \text{ det}(\mathbf{E}_{3}) = 1 \text{ (why?)}$$
So it seems like elementary

So it seems like elementary row operations of this type

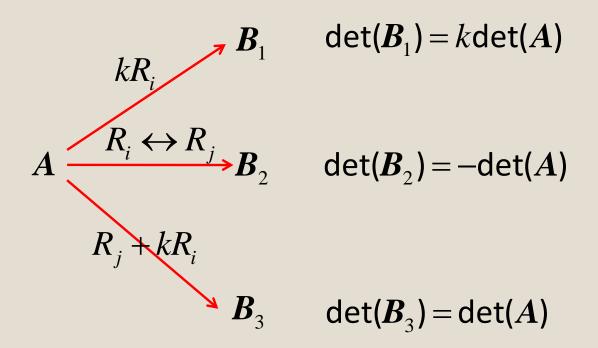
$$det(\mathbf{B}_3) = det(\mathbf{A})$$

$$= det(\mathbf{E}_3) det(\mathbf{A})$$

does not change the determinant of $oldsymbol{A}$ and we have 'determinant of product equals to product of determinants' (to some extent).

Theorem

Let A be a square matrix. Then



Furthermore, if E is an elementary matrix of the same size as A, then det(EA) = det(E)det(A).

wow, this is quite something, but how can we use such a result?

suppose you want to find the determinant of A... ... you first find the determinant of its rowechelon form.



 $A \longrightarrow R$

oh I know, then we keep track of what e.r.o. have been performed on A...

Yes! R is a triangular matrix whose determinant is easy to evaluate... we can now 'backtrack' to find the determinant of A.



 $A \longrightarrow R$

Example

Find the determinant of the following matrix using elementary row operations.

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 3 & 4 \\ 1 & -2 & 3 \end{pmatrix}$$

Example

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 3 & 4 \\ 1 & -2 & 3 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & -2 & 3 \\ 2 & 3 & 4 \\ 0 & 1 & -1 \end{pmatrix} = A_1$$

$$\det(A_1) = -\det(A)$$

$$-\det(A) = -5$$
, so $\det(A) = 5$.

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 7 & -2 \\ 0 & 0 & -\frac{5}{7} \end{pmatrix} \leftarrow \begin{pmatrix} R_3 - \frac{1}{7}R_2 \\ 0 & 7 & -2 \\ 0 & 1 & -1 \end{pmatrix} = A_2 \det(A_1)$$

$$= A_3$$

$$\det(A_3) = \det(A_2)$$

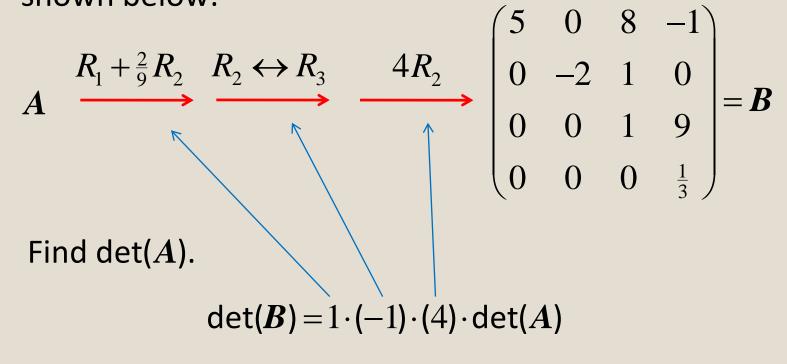
$$\det(A_3) = 1 \times 7 \times (-\frac{5}{7}) = -5$$

$$= \det(A_2) = \det(A_1)$$

$$= -\det(A)$$

Example

Suppose A and B are row equivalent matrices as shown below:



$$5 \cdot (-2) \cdot 1 \cdot \frac{1}{3} = -4 \operatorname{det}(A)$$

$$\frac{5}{6} = \operatorname{det}(A)$$

Can you find A?

Summary

- 1) How the 3 types of elementary row operations changes the determinant of a square matrix.
- 2) Computing the determinant of a matrix using elementary row operations.