**W05-01**

**Slide 01:** In this unit, we continue our discussion on subspaces by looking at subspaces in $\mathbb{R}^2$ and $\mathbb{R}^3$.

**Slide 02:** We already know that a subspace is any set that can be expressed as a linear span. Let us consider the linear span of a single vector $\boldsymbol{u}$, where $\boldsymbol{u}$ is a non zero vector in $\mathbb{R}^2$ or $\mathbb{R}^3$. By definition, $\text{span}(\boldsymbol{u})$ is the set of all linear combinations of $\boldsymbol{u}$ with itself, in other words, it is just the set of all scalar multiples of $\boldsymbol{u}$.

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Geometrically, the set $\text{span}(\boldsymbol{u})$ is a straight line in the direction of $\boldsymbol{u}$, and this line certainly passes through the origin, since $\text{span}(\boldsymbol{u})$ contains the zero vector.

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For example, we have the vector $1.5\boldsymbol{u}$

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as well as the vector $-2\boldsymbol{u}$ in this subspace.

**Slide 03:** If $\boldsymbol{u}$ is a vector from $\mathbb{R}^2$ with components $u_1$ and $u_2$, then the subspace $\text{span}(\boldsymbol{u})$ will be the set of all vectors $(cu_1, cu_2)$ where $c$ takes on all possible real numbers.

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We can actually find the equation of this straight line once we know what the two components are.

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We already know that a straight line in $\mathbb{R}^2$ that passes through the origin has the linear equation $ax + by = 0$ so we just need to substitute the components $u_1$ and $u_2$ into the equation.

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For example, if the vector $\boldsymbol{u}$ is $(2, -1)$,

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we will have $2a - b = 0$

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which implies $2a = b$

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so we may let $a = 1$, $b = 2$ and this would be solution to the equation $2a = b$.

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Thus the equation of the line spanned by the vector $(2, -1)$ is $x + 2y = 0$.

**Slide 04:** Next, consider when $\boldsymbol{u}$ is a vector from $\mathbb{R}^3$ with components $u_1$, $u_2$ and $u_3$. Geometrically, $\text{span}(\boldsymbol{u})$ is still a straight line in $\mathbb{R}^3$ that passes through the origin.

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However, unlike in the $\mathbb{R}^2$ case, a single linear equation in three variables describes a plane in $\mathbb{R}^3$ and not a line. In other words, a line cannot be represented by a single linear equation in three variables.

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Thus in this case, we will represent $\text{span}(\boldsymbol{u})$ as the set of all scalar multiples of $(u_1, u_2, u_3)$.

**Slide 05:** Let us now consider the linear span of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. By definition, $\mathrm{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ is the set of all linear combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$,

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and we can write down this set as the set of all $s\boldsymbol{u} + t\boldsymbol{v}$ for all real numbers $s$ and $t$.

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There are two scenarios to consider. Firstly, if $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel,

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then $\boldsymbol{v}$ is just a scalar multiple of $\boldsymbol{u}$

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and we have alreday seen from a previous unit that the linear span of $\boldsymbol{u}$ and $\boldsymbol{v}$ is just the linear span of $\boldsymbol{u}$.

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This has already been considered previously, since the linear span of a single non zero vector is geometrically a straight line that passes through the origin.

**Slide 06:** So now let's consider the case where $\boldsymbol{u}$ and $\boldsymbol{v}$ are not parallel as shown. Notice that by placing both vectors with their initial point at the origin, we are able to have 4 regions partitioned by the dotted lines in the direction of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

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When $s$ and $t$ are both positive, then we will be combining a positive scalar multiple of $\boldsymbol{u}$ with a positive scalar multiple of $\boldsymbol{v}$. It is not difficult to see that in this case, the resulting vector $s\boldsymbol{u} + t\boldsymbol{v}$ will be in the region as shown by the green vector.

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When $s$ is positive but $t$ is negative, then the vector $s\boldsymbol{u} + t\boldsymbol{v}$ will be in the region as shown by the green vector.

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The third case where $s$ is negative and $t$ is positive will result in $s\boldsymbol{u} + t\boldsymbol{v}$ being in the region as shown here.

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Finally, when $s$ and $t$ are both negative, $s\boldsymbol{u} + t\boldsymbol{v}$ will be in this region.

**Slide 07:** In other words, when $\boldsymbol{u}$ and $\boldsymbol{v}$ are not parallel, the subspace $\mathrm{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ will represent a plane geometrically. This plane obviously contains the origin since any linear span always contains the zero vector.

**Slide 08:** When $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{R}^2$, note that $\mathbb{R}^2$ itself is a plane, so in this case, $\mathrm{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ must be the entire $\mathbb{R}^2$ itself.

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If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{R}^3$, then this subspace can be written as the set of all $s\boldsymbol{u}+t\boldsymbol{v}$ for all real numbers $s$ and $t$.

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If the components of $\boldsymbol{u}$ and $\boldsymbol{v}$ are as shown,

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we may write this subspace in a more explicit manner.

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Since we already know that geometrically, this is a plane in $\mathbb{R}^3$ that contains the origin, can we then proceed to find the equation of this plane? Recall that any plane in $\mathbb{R}^3$ that contains the origin can be represented by the single linear equation $ax + by + cz = 0$.

**Slide 09:** Let us consider the example where, say, $\boldsymbol{u} = (1, 0, -2)$ and $\boldsymbol{v} = (-1, 1, 0)$.

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To find the equation of the plane, we will substitute the two vectors into the linear equation and obtain two equations involving the three unknowns $a, b$ and $c$.

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To solve this linear system, we write down the augmented matrix

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and proceed with Gauss-Jordan elimination. The reduced row-echelon form of the augmented matrix is shown here.

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It is easy to write down a general solution for the system

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and subsequently obtain a particular solution to $a, b$ and $c$. With this, we have found the equation of the plane spanned by the two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$.

**Slide 10:** The preceding discussion has actually characterised all the different subspaces in $\mathbb{R}^2$ and $\mathbb{R}^3$. For $\mathbb{R}^2$, there are only three types of subspaces.

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Firstly, we already know that the zero subspace is a subspace of any $\mathbb{R}^n$. This subspace is the linear span of the zero vector and geoemetrically, it is the point represented by the origin.

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Next, the span of a single non zero vector $\boldsymbol{u}$ is also a subspace of $\mathbb{R}^2$. Geometrically, this type of subspace are the lines in $\mathbb{R}^2$ that passes through the origin.

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Lastly, the linear span of two non zero and non parallel vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ would be a plane in $\mathbb{R}^2$ and this is in fact the entire $\mathbb{R}^2$.

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Thus, the three types of subspaces of $\mathbb{R}^2$ are as characterised here.

**Slide 11:** For subspaces of $\mathbb{R}^3$,

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We again have the zero subspace, which is the origin.

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We have the linear span of a single non zero vector $\boldsymbol{u}$ and this would be a straight line in $\mathbb{R}^3$ that passes through the origin.

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If $\boldsymbol{u}$ and $\boldsymbol{v}$ are non zero and non parallel vectors, then the subspace span$\{\boldsymbol{u}, \boldsymbol{v}\}$ will be a plane in $\mathbb{R}^3$ that contains the origin.

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Lastly, although we have not discussed the linear span of three vectors in this unit, you should note that when certain conditions are met, the linear span of three vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ will be the entire $\mathbb{R}^3$.

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Thus, there are four types of subspaces of $\mathbb{R}^3$ and they are characterised here.

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Let us briefly describe what are the conditions that $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ need to meet before they can span $\mathbb{R}^3$. Firstly, we need to ensure that $\boldsymbol{u}$ is not a linear combination of $\boldsymbol{v}$ and $\boldsymbol{w}$. The reason being that if $\boldsymbol{u}$ is a linear combination of $\boldsymbol{v}$ and $\boldsymbol{w}$, then the linear span of the three vectors is in fact the same as the linear span of just $\boldsymbol{v}$ and $\boldsymbol{w}$ and we already know that this would be a plane if $\boldsymbol{v}$ and $\boldsymbol{w}$ are not multiples of each other.

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For the same reason, we also require $\boldsymbol{v}$ and $\boldsymbol{w}$ not to be linear combinations of the other two vectors, respectively,

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This concept of a set of vectors where none of the vectors in the set is a linear combination of the other vectors in the set will be introduced and discussed extensively in the following units.

**Slide 12:** Let us summarise the main points in this unit.

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We discussed the geometrical interpretation of the linear span of a single non zero vector. Remember that for $\mathbb{R}^2$ and $\mathbb{R}^3$, these subspaces are lines passing through the origin.

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Next, the linear span of two vectors is discussed. Again, we examined the geometrical object in this case and when the two vectors are not multiples of each other, these subspaces are planes containing the origin.

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We then characterised all the subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$.