Week 08 IVLE Quiz

1. Which of the following statements below is/are correct?

   (I) A non zero vector space V can have infinitely many different bases.

   (II) If a set of k vectors can span V, then the dimension of V is at most k.

   (III) If a set of k vectors cannot span V, then the dimension of V is at least k.

   (A) (I) only.

   (B) (I) and (II) only.

   (C) All three statements are correct.

   (D) None of the given combinations is correct.

   Answer: (B). (I) is correct since bases (for the same vector space V) is not unique. (II) is correct since the set of k vectors can possibly be 'trimmed' down to a linearly independent set by removing redundant vectors. This gives us a basis for V, which has at most k vectors. Thus the dimension of V is at most k. (III) is not correct, since for example, $\{(1,0),(2,0),(3,0)\}$ is not able to span $\mathbb{R}^2$ does not mean that the dimension of $\mathbb{R}^2$ is at least 3.

2. Suppose $\boldsymbol{A}$ is a matrix with 7 columns and a row-echelon form of $\boldsymbol{A}$ has 3 non pivot columns. How many of the statements below is/are correct?

   (I) The solution space of $\boldsymbol{Ax} = \boldsymbol{0}$ is a subspace of $\mathbb{R}^4$.

   (II) The dimension of the solution space of $\boldsymbol{Ax} = \boldsymbol{0}$ is 3.

   (III) Three non zero vectors $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ such that $\boldsymbol{Au_1} = \boldsymbol{Au_2} = \boldsymbol{Au_3} = \boldsymbol{0}$ will form a basis for the solution space of $\boldsymbol{Ax} = \boldsymbol{0}$.

   (A) None.

   (B) Exactly one.

   (C) Exactly two.

   (D) All three.

   Answer: (B). (I) is incorrect since the solution space of $\boldsymbol{Ax} = \boldsymbol{0}$ is a subspace of $\mathbb{R}^7$. (II) is correct since there are 3 non pivot columns in a row-echelon form of $\boldsymbol{A}$, which implies that a general solution of $\boldsymbol{Ax} = \boldsymbol{0}$ would have 3 arbitrary parameters, giving rise to 3 vectors in a basis for the solution space. (III) is not correct, since for example, if $\boldsymbol{u_1} = 2\boldsymbol{u_2}$. Then $\boldsymbol{Au_1} = \boldsymbol{Au_2} = \boldsymbol{0}$ but $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is not a linearly independent set.

3. Let V be a subspace of $\mathbb{R}^5$. Suppose we know that the dimension of V is 3. How many of the following statements below is/are correct?

   (I) Any 3 linearly independent vectors in $\mathbb{R}^5$ will form a basis for V.

(II) Any basis for V will contain 3 linearly independent vectors.

(III) If S is a basis for V and $\boldsymbol{u} \in \mathbb{R}^5$ does not belong to V, then the dimension of $\text{span}(S \cup \{\boldsymbol{u}\}) = 4$.

(A) None

(B) Exactly one

(C) Exactly two

(D) All three

Answer: (C). (I) is not necessarily correct since the 3 vectors may not be all from V. (II) is correct since we are told that the dimension of V is 3. (III) is correct since $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is already a linearly independent set and $\boldsymbol{u}$ is not a linearly combination of $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$. So $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u}\}$ is a linearly independent set of 4 vectors, so the dimension is 4.

4. Let $\boldsymbol{A}$ be a square matrix of order 4. Suppose $\det(\boldsymbol{A}) = 3$. Which of the following statements is/are correct?

(I) The 4 rows of $\boldsymbol{A}$ forms a basis for the row space of $\boldsymbol{A}$.

(II) The row space of $\boldsymbol{A}$ has dimension 4.

(III) The 4 columns of $\boldsymbol{A}$ are linearly independent vectors.

(A) (I) and (II) only.

(B) (II) and (III) only

(C) (I) and (III) only

(D) None of the given combinations is correct.

Answer: (D). Since $\boldsymbol{A}$ is invertible, we can say that the rows and columns of $\boldsymbol{A}$ are all bases for $\mathbb{R}^4$, so (I) is correct. (II) is also correct since the row space of $\boldsymbol{A}$ is $\mathbb{R}^4$ itself. (III) is also correct since the columns of $\boldsymbol{A}$ forms a basis for $\mathbb{R}^4$.

5. Let $\boldsymbol{u}$ be a non zero vector in $\mathbb{R}^4$. If V contains all vectors in $\mathbb{R}^4$ that are orthogonal to $\boldsymbol{u}$, which of the statements is correct?

(A) V is a subspace of $\mathbb{R}^4$ with dimension 2.

(B) V is a subspace of $\mathbb{R}^4$ with dimension 3.

(C) V is not a subspace.

(D) None of the given statements is correct.

Answer: (B). Suppose $\boldsymbol{u} = (w, x, y, z)$. Then $V = \{(v_1, v_2, v_3, v_4) \mid wv_1 + xv_2 + yv_3 + zv_4 = 0\}$, which is the solution space of a homogeneous linear system with 1 equation and 4 unknowns. The general solution of this system has 3 arbitrary parameters, so the solution space has dimension 3.