

RANK OF A MATRIX; ONE MORE EQUIVALENT STATEMENT

THEOREM

The row space and the column space of a matrix have the same dimension.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

The diagram illustrates a matrix \mathbf{A} with dimensions $m \times n$. A red bracket above the matrix indicates the number of columns is n . A red bracket to the right of the matrix indicates the number of rows is m . A yellow box highlights the second row, representing a row vector in the row space. An orange box highlights the second column, representing a column vector in the column space. The intersection of these two boxes is the element a_{22} .

Note that the **row space** of \mathbf{A} is a subspace of \mathbb{R}^n while the **column space** of \mathbf{A} is a subspace of \mathbb{R}^m .

These two subspaces may have nothing to do with each other (if $m \neq n$) but the theorem states that they have the same dimension.

PROOF

Strategy: We will use what we learnt in the previous section to find bases for both subspaces.

Let \mathbf{R} be a row echelon form of \mathbf{A} .

$$\begin{array}{c} \mathbf{A} \\ \left(\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array} \right) \end{array} \xrightarrow{\text{Gaussian Elimination}} \begin{array}{c} \mathbf{R} \\ \left(\begin{array}{cccccc} \begin{array}{|c|} \hline \otimes \\ \hline \end{array} & * & & & & * \\ & \begin{array}{|c|} \hline \otimes \\ \hline \end{array} & * & & & * \\ & & \ddots & & & * \\ 0 & \dots & & \begin{array}{|c|} \hline \otimes \\ \hline \end{array} & * & * \\ & & & \begin{array}{|c|} \hline \otimes \\ \hline \end{array} & * & * \\ 0 & \dots & \dots & \dots & 0 & 0 \end{array} \right) \end{array}$$

PROOF

Question: What is a basis for the row space of A ?

Answer: The non-zero rows of R .

Question: What is a dimension of the row space of A ?

Answer: The number of non-zero rows of R .

= number of leading entries in R .

$$\begin{pmatrix} & 11 & & 12 & & & 1n \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Gaussian
Elimination \rightarrow

$$\begin{matrix} & & & R \\ \left(\begin{array}{ccccccc} \otimes & * & & & & & * \\ & \otimes & * & & & & * \\ & & \ddots & & & & * \\ 0 & & & & \otimes & * & * \\ & & & & & & 0 & 0 \end{array} \right) \end{matrix}$$

PROOF

Question: What is a basis for the column space of A ?

Answer: The columns of A corresponding to the pivot columns of R .

Question: What is a dimension of the column space of A ?

$$\begin{array}{c}
 \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}
 \end{array}
 \xrightarrow{\text{Gaussian Elimination}}
 \begin{array}{c}
 \begin{pmatrix} \boxed{\otimes} & * & & & * \\ & \boxed{\otimes} & * & & * \\ & 0 & & \ddots & * \\ & & & \boxed{\otimes} & * \\ 0 & \dots & \dots & \dots & 0 & 0 \end{pmatrix}
 \end{array}$$

A R

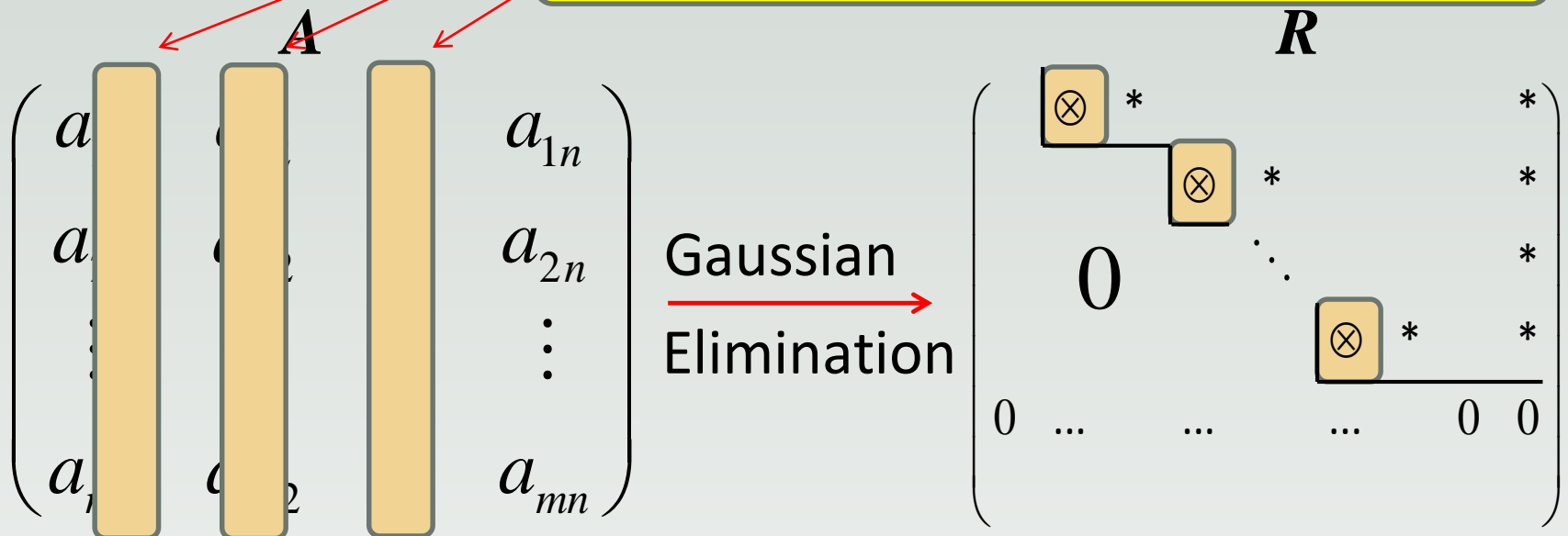
PROOF

Question: What is a dimension of the column space of A ?

Answer: The number of columns here.

= The number of pivot columns in R .

= The number of leading entries in R .



PROOF

Question: What is a dimension of the row space of A ?

Answer: The number of leading entries in R .

Question: What is a dimension of the column space of A ?

Answer: The number of leading entries in R .

$$\begin{array}{c}
 \mathbf{A} \\
 \left(\begin{array}{cccc}
 a_{11} & a_{12} & \cdots & a_{1n} \\
 a_{21} & a_{22} & \cdots & a_{2n} \\
 \vdots & \vdots & & \vdots \\
 a_{m1} & a_{m2} & \cdots & a_{mn}
 \end{array} \right)
 \end{array}
 \xrightarrow{\text{Gaussian Elimination}}
 \begin{array}{c}
 \mathbf{R} \\
 \left(\begin{array}{ccccccc}
 \boxed{\otimes} & * & & & & & * \\
 & \boxed{\otimes} & * & & & & * \\
 & & \ddots & & & & * \\
 0 & & & \boxed{\otimes} & * & & * \\
 0 & \cdots & \cdots & \cdots & 0 & 0
 \end{array} \right)
 \end{array}$$

EXAMPLE

$$\mathbf{C} = \begin{pmatrix} 2 & 0 & 3 & -1 & 8 \\ 2 & 1 & 1 & -2 & 5 \\ -4 & -3 & 0 & 5 & -7 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 2 & 0 & 3 & -1 & 8 \\ 0 & 1 & -2 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A basis for the row space of \mathbf{C} is:

$$\{(2, 0, 3, -1, 8), (0, 1, -2, -1, -3)\}$$

A basis for the column space of \mathbf{C} is:

$$\left\{ \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}$$

The dimension of both the row space and column space of \mathbf{C} is 2.

DEFINITION

The **rank** of a matrix is the dimension of its row space (or column space).

The rank of A is denoted by $\text{rank}(A)$.

By our earlier discussion,

$\text{rank}(A)$ = number of non zero rows in a row echelon form of A

= number of pivot columns in a row echelon form of A

EXAMPLES (RANK)

1) $\text{rank}(\mathbf{0}) = 0$

3) $\text{rank}(\mathbf{C}) = 2$

2) $\text{rank}(\mathbf{I}_n) = n$

$$\mathbf{C} = \begin{pmatrix} 2 & 0 & 3 & -1 & 8 \\ 2 & 1 & 1 & -2 & 5 \\ -4 & -3 & 0 & 5 & -7 \end{pmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{pmatrix} 2 & 0 & 3 & -1 & 8 \\ 0 & 1 & -2 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

EXAMPLES (RANK)

$$A = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

$m \geq n$

What is the largest possible value for $\text{rank}(A)$?

n

$$A = \begin{pmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \vdots & \vdots & \vdots \\ \dots & \dots & \dots \end{pmatrix}$$

$n \geq m$

What is the largest possible value for $\text{rank}(A)$?

m

REMARKS

1) For a $m \times n$ matrix A , $\text{rank}(A) \leq \min\{m, n\}$.

2) For a $m \times n$ matrix A , if $\text{rank}(A) = \min\{m, n\}$,

we say that A is of **full rank**.

3) **$\text{rank}(A)$** = dimension of **row space of A**



= dimension of **column space of A^T**

= $\text{rank}(A^T)$

REMARKS

4) A square matrix A (of order n) is of full rank if and only if $\det(A) \neq 0$.

Proof: $\text{rank}(A) = n$

\Leftrightarrow dimension of row space of A is n

$\Leftrightarrow \{e_1, e_2, \dots, e_n\}$ is a basis for the row space of A

$\Leftrightarrow A$ is row equivalent to I_n

\Leftrightarrow RREF of A is $I_n \Leftrightarrow \det(A) \neq 0$

ONE MORE STATEMENT

Let A be an $n \times n$ matrix. The following statements are equivalent.

- 1) A is invertible
- 2) $A\mathbf{x} = \mathbf{0}$ has only trivial solution
- 3) RREF of A is I
- 4) A can be written as produce of elementary matrices
- 5) $\det(A) \neq 0$
- 6) Rows of A forms a basis for \mathbb{R}^n
- 7) Columns of A forms a basis for \mathbb{R}^n
- 8) $\text{rank}(A) = n$

SUMMARY

- 1) The row space and column space of a matrix have the same dimension.
- 2) Definition of the rank of a matrix.
- 3) One more equivalent statement added in terms of the rank of A .