W03-01

Slide 01: This unit is the first part on the discussion of elementary matrices.

Slide 02: Recall that there are three types of elementary row operations. The first one is to multiply a row, say the i-th row of the matrix by a non zero constant c. Such an operation is denoted by cR_i.

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The second type is to interchange two rows, say the i-th and j-th row. We denote such an operation by $R_i \leftrightarrow R_j$.

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The third and final type, which is perhaps done most frequently during Gaussian elimination is to add k times the i-th row to the j-th row. This is denoted by $R_j + kR_i$.

Let us discuss what happens when each of these operations are peformed on an identity matrix $\boldsymbol{I}_m$.

Slide 03: Consider the first type of elementary row operation performed on $\boldsymbol{I}_m$. Suppose we multiply the i-th row by c.

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Clearly, the resulting matrix will be as follows, which is almost like an identity matrix, except that the (i, i)-entry in the matrix is now c.

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Let's call this matrix $\boldsymbol{E_1}$. What do you think will happen when we pre-multiply $\boldsymbol{E_1}$ to a $m \times n$ matrix $\boldsymbol{B}$? Note that $\boldsymbol{E_1}$ is a square matrix of order m so it makes sense to pre-multiply $\boldsymbol{E_1}$ to a matrix with m rows.

Slide 04: Recall the discussion in an earlier unit on block multiplication. By looking at $\boldsymbol{E_1}$ in terms of its rows, you will notice that pre-multiplying $\boldsymbol{E_1}$, where all except one row in $\boldsymbol{E_1}$ is exactly like what is found in an identity matrix, to $\boldsymbol{B}$, will result in a matrix that is almost identical to $\boldsymbol{B}$, with the exception of only one row.

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This is precisely the i-th row, where

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the i-th row of the resulting matrix will be c times of what the i-th row of $\boldsymbol{B}$ is. Every other row in the resulting matrix will be exactly identical to its counterpart in $\boldsymbol{B}$.

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Thus, you notice immediately that the resulting matrix is exactly what you would get when you perform cR_i on $\boldsymbol{B}$.

Slide 05: So it seems like for this type of elementary row operation, pre-multiplying $\boldsymbol{E_1}$ to $\boldsymbol{B}$ gives the same resulting matrix as performing cR_i on $\boldsymbol{B}$.

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Recall that the matrix $\boldsymbol{E_1}$ was obtained by performing the same operation cR_i on $\boldsymbol{I}_m$. So while we perform cR_i on $\boldsymbol{B}$, resulting in the matrix $\boldsymbol{C_1}$,

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we have discovered that we actually have $\boldsymbol{E_1 B} = \boldsymbol{C_1}$. Let us now consider the second type of elementary row operation.

Slide 06: Once again, let us start off by performing the row swap on $\boldsymbol{I}_m$. Suppose the i-th and j-th rows are swapped.

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The resulting matrix is once again very similar to an identity matrix, except for the following.

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Arising from the row swap, the 1 in the (j, j)-entry is now at the (i, j) position,

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while the 1 previously in the (i, i) position is now in the (j, i) position. Every other entry in the matrix remains the same as before in $\boldsymbol{I}_m$.

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Let's call this matrix $\boldsymbol{E_2}$ and once again, pre-multiply $\boldsymbol{E_2}$ to a $m \times n$ matrix $\boldsymbol{B}$. What do you think will be the resulting matrix?

Slide 07: Once again using the understanding from block multiplication, we see that pre-multiplying $\boldsymbol{E_2}$ to $\boldsymbol{B}$ can be done by considering the matrix $\boldsymbol{E_2}$ row by row.

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Everything will be as per normal as if we are pre-multiplying an identity matrix to $\boldsymbol{B}$, until we reach the i-th row of $\boldsymbol{E_2}$.

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The 1 in the (i, j) entry will be matched with the j-th row of $\boldsymbol{B}$,

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meaning that in the i-th row of the resulting matrix, we will have the j-th row of $\boldsymbol{B}$.

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Similarly, when we reach the j-th row of $\boldsymbol{E_2}$,

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the 1 in the (j, i) entry will be matched with the i-th row of $\boldsymbol{B}$,

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meaning that in the j-th row of the resulting matrix, we will have the i-th row of $\boldsymbol{B}$.

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You now notice immediately that the resulting matrix is exactly what you would get when you perform $R_i \leftrightarrow R_j$ on $\boldsymbol{B}$.

Slide 08: So it seems like for this second type of elementary row operation, pre-multiplying $\boldsymbol{E_2}$ to $\boldsymbol{B}$ gives the same resulting matrix as performing $R_i \leftrightarrow R_j$ on $\boldsymbol{B}$.

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Recall that the matrix $\boldsymbol{E_2}$ was obtained by performing the same operation $R_i \leftrightarrow R_j$ on $\boldsymbol{I}_m$. So while we perform $R_i \leftrightarrow R_j$ on $\boldsymbol{B}$, resulting in the matrix $\boldsymbol{C_2}$,

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we have discovered that we actually have $\boldsymbol{E_2}\boldsymbol{B} = \boldsymbol{C_2}$. Let us now consider the third type of elementary row operation.

Slide 09: As it was done previously, let's add k times the i-th row to the j-th row of $\boldsymbol{I}_m$.

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The resulting matrix may look like this

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depending on whether $j > i$ or not. First consider the case when $j > i$. In this situation, the resulting matrix will again be very similar to the identity matrix, except that

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there is now a k in the (j, i) entry. Because we are assuming $j > i$, this k appears in the bottom left portion of the matrix.

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Let us denote this matrix by $\boldsymbol{E_3}$ and once again pre-multiply $\boldsymbol{E_3}$ to $\boldsymbol{B}$.

Slide 10: Notice that the case where $i > j$ can be considered similarly and for this case, $\boldsymbol{E_3}$ would be like an identity matrix, with the additional k at the (j, i) entry, this time appearing in the top right portion of the matrix.

Slide 11: Let us return to the case where $j > i$ and observe what happens when $\boldsymbol{E_3}$ is pre-multiplied to $\boldsymbol{B}$.

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Once again, using block multiplication ideas and considering $\boldsymbol{E_3}$ row by row, everything will be like pre-multiplying an identity matrix to $\boldsymbol{B}$, until we reach the j-th row. At the j-th row, the k in the (j, i) position will be matched with the i-th row of $\boldsymbol{B}$

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while the 1 in the (j, j) position will be matched with the j-th row of $\boldsymbol{B}$.

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This results in the j-th row of the resulting matrix to be essentially the j-th row of $\boldsymbol{B}$ plus k times the i-th row of $\boldsymbol{B}$.

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Once again, this resulting matrix is the same as if we had performed $R_j + kR_i$ on $\boldsymbol{B}$.

Slide 12: Similar considerations, for the case where $i > j$, leads us to the same observation and conclusion. You may wish to pause briefly to go through this slide before moving on.

Slide 13: Once again, it seems like for this third type of elementary row operation, pre-multiplying $\boldsymbol{E_3}$ to $\boldsymbol{B}$ gives the same resulting matrix as performing $R_j + kR_i$ on $\boldsymbol{B}$.

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Recall that the matrix $\boldsymbol{E_3}$ was obtained by performing the same operation $R_j + kR_i$ on $\boldsymbol{I}_m$. So while we perform $R_j + kR_i$ on $\boldsymbol{B}$, resulting in the matrix $\boldsymbol{C_3}$,

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we have discovered that we actually have $\boldsymbol{E_3 B} = \boldsymbol{C_3}$.

Slide 14: It is now timely to put everything we have discussed together. Suppose $\boldsymbol{B}$ is a $m \times n$ matrix. For each of the three types of elementary row operations that can be performed on $\boldsymbol{B}$, resulting in, respectively, $\boldsymbol{C_1}$, $\boldsymbol{C_2}$ or $\boldsymbol{C_3}$,

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we can construct three corresponding matrices $\boldsymbol{E_1}$, $\boldsymbol{E_2}$ and $\boldsymbol{E_3}$ such that for each $i = 1, 2, 3$, we have $\boldsymbol{E_i B} = \boldsymbol{C_i}$. Note that the matrices $\boldsymbol{E_1}, \boldsymbol{E_2}, \boldsymbol{E_3}$ are constructed by performing exactly the same three types of elementary row operations on an identity matrix of order m. The essence of this discussion is that pre-multiplying a suitably chosen matrix $\boldsymbol{E}$ to $\boldsymbol{B}$ can achieve the same effect as performing an elementary row operation on $\boldsymbol{B}$.

Slide 15: We are now ready to define what are elementary matrices. A square matrix is an elementary matrix if it can be obtained from an identity matrix by performing a single elementary row operation. Thus, the matrices $\boldsymbol{E}$ used in this unit are all elementary matrices.

Slide 16: Let us summarise the main points in this unit.

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We learnt that for each elementary row operation X, there is actually a corresponding square matrix $\boldsymbol{E}$ such that performing the operation X on $\boldsymbol{B}$ produces the same effect as pre-multiplying $\boldsymbol{E}$ to $\boldsymbol{B}$.

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In other words, $\boldsymbol{EB}$ is the result of performing X on $\boldsymbol{B}$.

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The matrix $\boldsymbol{E}$ described above, is defined to be an elementary matrix, which is the result of performing a single elementary row operation X on $\boldsymbol{I}$. In the next unit, we will discuss more properties of elementary matrices.