Third Week

MALTHUS MODEL

$$\frac{dN}{dt} = (B - D)N = kN$$

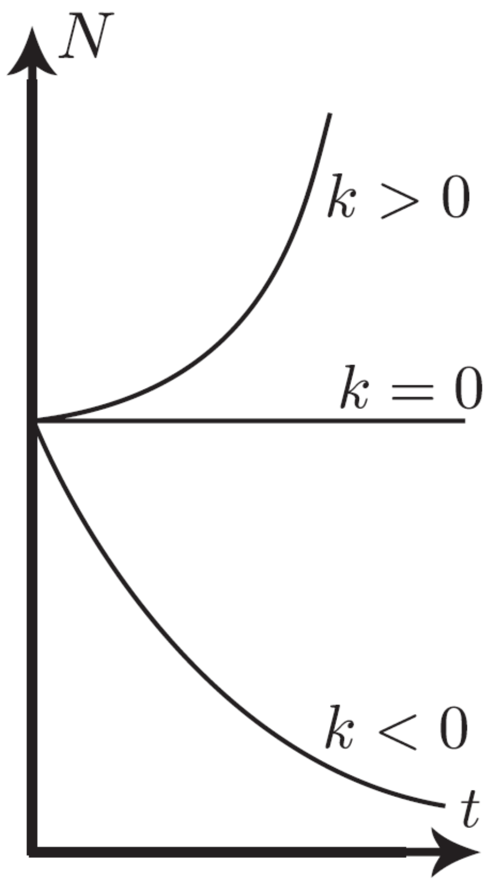
From $\frac{dN}{dt} = kN$ we have $\int \frac{dN}{N} = \int k dt = k \int dt = kt + c$

so $ln(N) = kt + c$ and thus $N(t) = Ae^{kt}$.

Since $\hat{N} = N(0) = A$, we get:

$$N(t) = \hat{N} e^{kt} \tag{2}$$

with graphs as shown on figure 1.


N
$k > 0$
$k = 0$
$k < 0$
t

IMPROVING ON MALTHUS

$$\boxed{D = sN, \quad \begin{array}{c}\text{(LOGISTIC)}\\ \text{ASSUMPTION}\\ s = \text{constant}\end{array}}$$

We have to solve

$$\frac{dN}{dt} = BN - DN = BN - sN^2$$

with the condition $N(0) = \hat{N}$

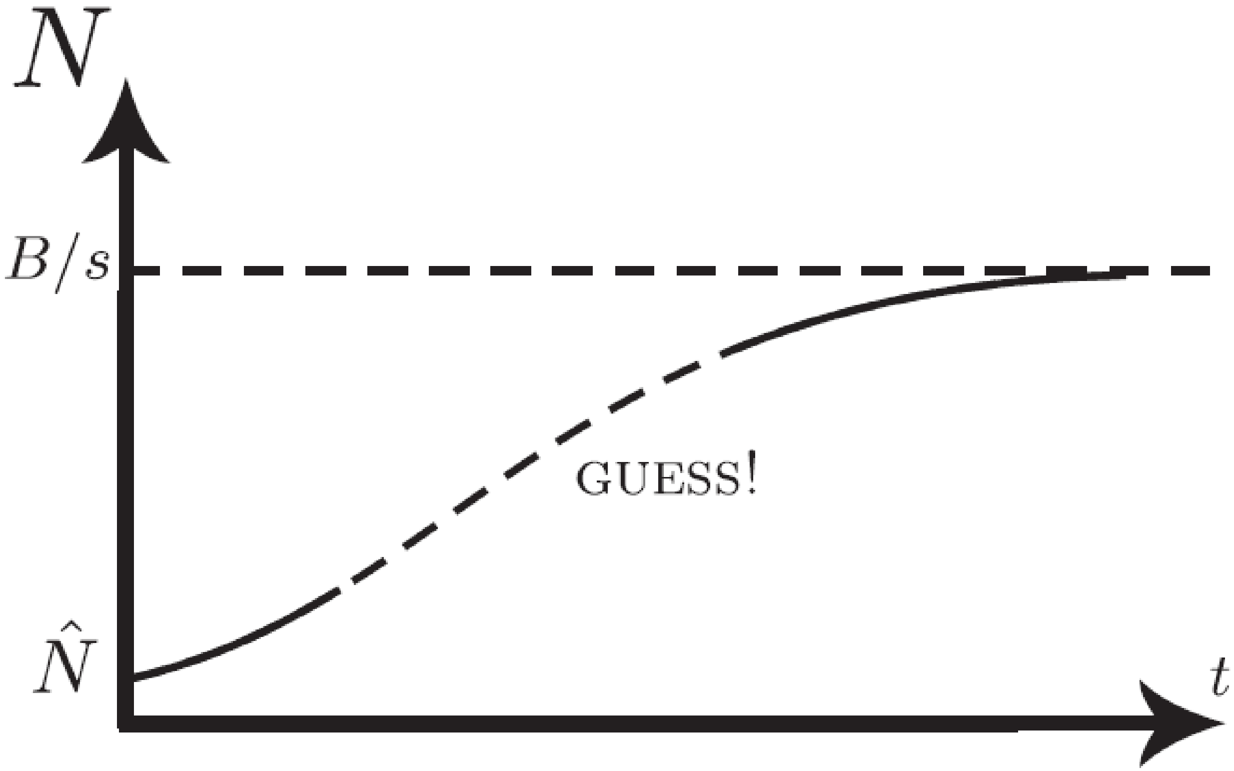
We can and will solve this, but let's try to GUESS what the solution will look like (a useful skill - in many other cases you won't be able to solve exactly!). Suppose that $\hat{N}$ is very small. Then (by continuity) $N(t)$ will be very small for t near to zero.

Of course if N is small, N^2 is much smaller and can be neglected. [Remember that N may be measured in millions or billions, so N can be small.] So at early times, our ODE is ALMOST linear and so

$$\frac{dN}{dt} \approx BN \to N(t) \approx \hat{N}e^{Bt}$$

So AT FIRST the population explodes, as Malthus predicted. On the other hand, if N continues to grow, since N^2 grows faster than N, we will reach a point where $sN^2 \approx BN$ ie $N \approx B/S$.

At that point , since $\frac{dN}{dt} = BN - SN^2$, the population will stop growing. So B/S should measure the MAXIMUM population possible. So we GUESS that the solution should look like this:



N
B/s
$\hat{N}$
GUESS!
t

The number B/s is called the CARRYING CAPACITY or the SUSTAINABLE POPULATION

$$\frac{dN}{dt} = BN - sN^2 \to t = \int \frac{dN}{N(B - sN)} + c$$

Write $\frac{1}{N(B-sN)} = \frac{\alpha}{N} + \frac{\beta}{B-sN}$

$$\begin{aligned} 1 &= \alpha(B - sN) + \beta N \\ &= \alpha B + (\beta - \alpha s)N \to 1 = \alpha B \ , \ \beta = \alpha s \\ \alpha &= 1/B \ , \ \beta = s/B, \text{ so} \end{aligned}$$

$$\int \frac{dN}{N(B - sN)} = \frac{1}{B}\int \frac{dN}{N} + \frac{s}{B}\int \frac{dN}{B - sN}$$
$$= \frac{1}{B}\ln N - \frac{1}{B}\ln|B - sN|$$

Now here we begin to feel uneasy - what if $N = B/s$ at some time? (ln(0) is not defined).

In fact we should have worried about this when we first wrote $\frac{1}{B-sN}$ - how do we know that we are not dividing by zero?? Let's not worry about that just now: let's ASSUME (temporarily) THAT $B - sN$ IS NEVER ZERO. That is, we assume either that N is always either LESS THAN B/s or MORE THAN B/s. OK, let's take LESS THAN first. So $|B - sN| = B - sN$, and we get

$$\begin{aligned} t &= \frac{1}{B}\ln N - \frac{1}{B}\ln(B - sN) + c \\ &= \frac{1}{B}\ln\frac{N}{B - sN} + c \end{aligned}$$

So

$\frac{N}{B-sN} = Ke^{Bt}$. Since $\hat{N} = N(0)$, $\frac{\hat{N}}{B-s\hat{N}} = K$

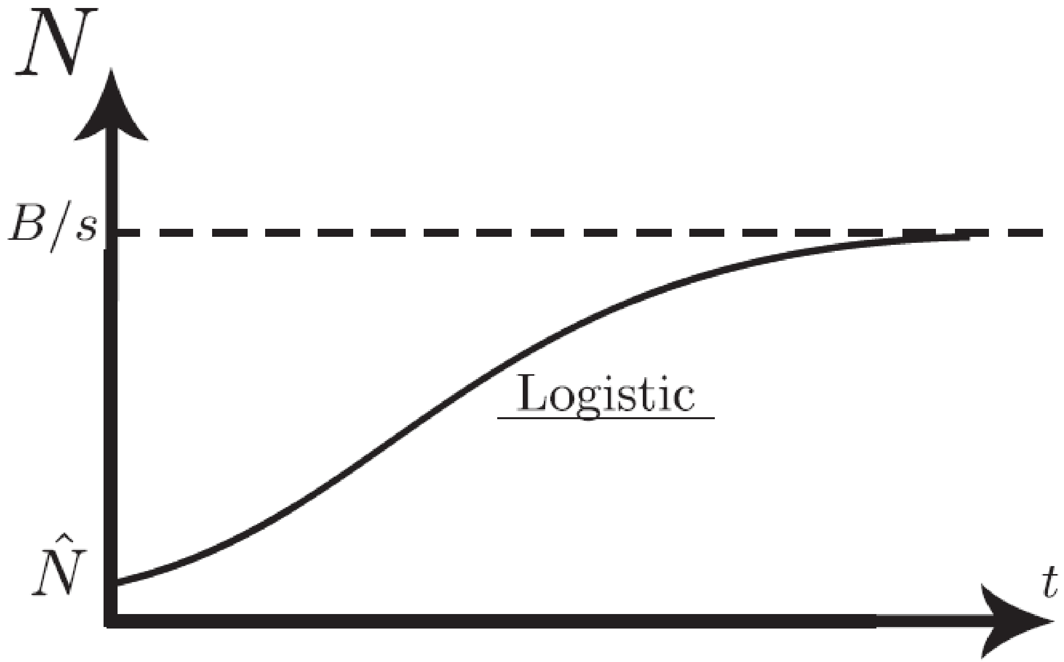
so

$$\frac{N}{B-sN} = \frac{\hat{N}}{B-s\hat{N}}e^{Bt}$$

Solve for N,

$$N(t) = \frac{B}{s + \left(\frac{B}{\hat{N}} - s\right)e^{-Bt}} \tag{4}$$

The graph of (4) is easy to sketch:



This is the famous LOGISTIC CURVE; $N(t)$ given by (4) is called the LOGISTIC FUNCTION; and $\frac{dN}{dt} = BN - sN^2$ is the LOGISTIC EQUATION.

It's easy to see what is happening here. Initially the population is small, plenty of food and space, so we get a Malthusian population explosion. But eventually the death rate rises until it is almost equal to the birth rate (*ie* $sN \approx B$ or $N \approx B/s$) and then the population approaches a fixed limit.

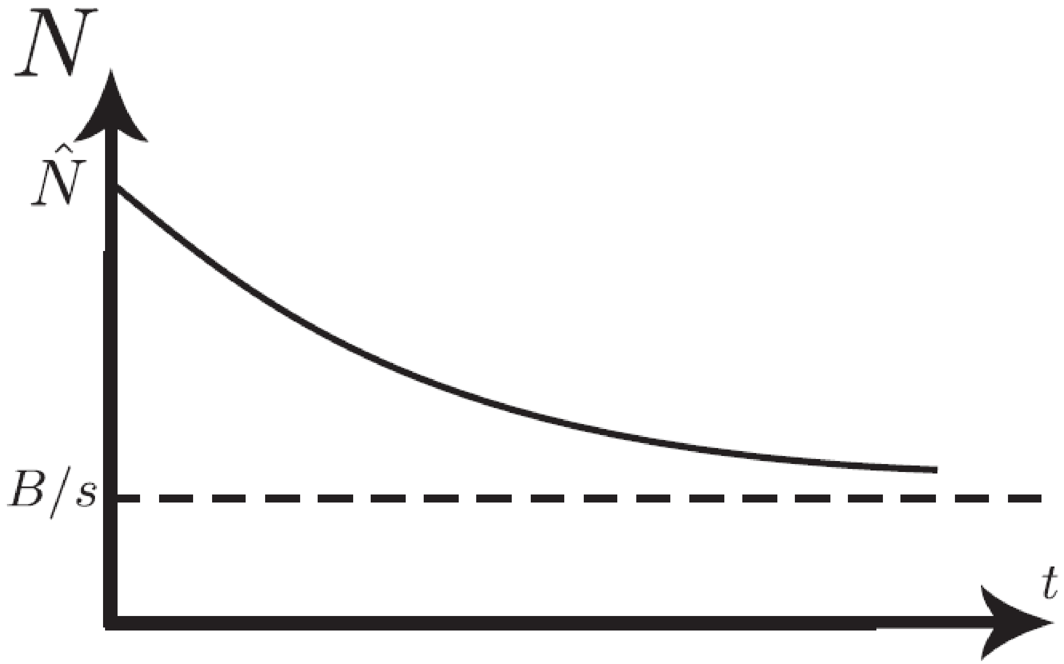
What if $N(t) > B/s$?

Then $|B - sN| = -(B - sN)$ so:

$$\begin{aligned} t &= \frac{1}{B}\ln N - \frac{1}{B}\ln(sN - B) + c \\ &= \frac{1}{B}\ln\frac{N}{sN - B} + c \Rightarrow \end{aligned}$$

$$N(t) = \frac{B}{s - \left(s - \frac{B}{\hat{N}}\right) e^{-Bt}} \tag{5}$$

And now the graph is



The number B/s is called the CARRYING CAPACITY or the SUSTAINABLE POPULATION - in all cases, it is the value approached by $N(t)$ as $t \to \infty$. If we set

$$N_\infty = B/s \tag{6}$$

then our solutions are:

$$N(t) = \frac{N_\infty}{1 + \left(\frac{N_\infty}{\hat{N}} - 1\right) e^{-Bt}} (\hat{N} < N_\infty) \quad (7)$$

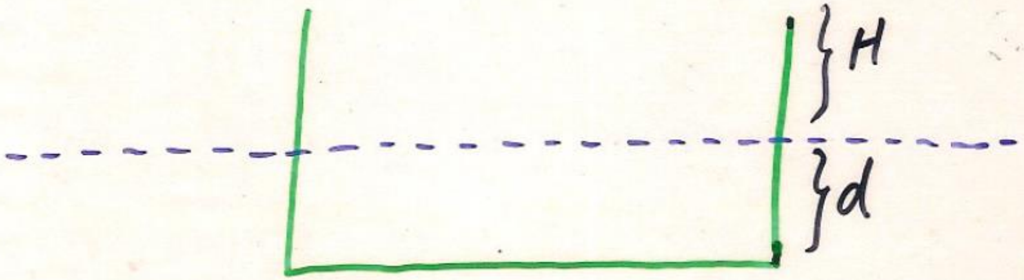
$$N(t) = \frac{N_\infty}{1 - \left(1 - \frac{N_\infty}{\hat{N}}\right) e^{-Bt}} (\hat{N} > N_\infty) \quad (8)$$

Tutorial 3

Question 5

A fully loaded large oil tanker can be modelled as a solid object with perfectly vertical sides and a perfectly horizontal bottom, so all horizontal cross-sections have the same area, equal to A. Archimedes' principle [http://en.wikipedia.org/wiki/Buoyancy] states that the upward force exerted on a ship by the sea is equal to the weight of the water pushed aside by the ship. Let ρ be the mass density of seawater, and let M be the mass of the ship, so that its weight is Mg, where g is 9.8 m/sec^2. When the ship is at rest, find the distance d from sea level to the bottom of the ship. This is called the **draught** of the ship.

1° At rest position :



Vol of sea water displaced = Ad

Weight of sea water displaced = $\rho A d g$

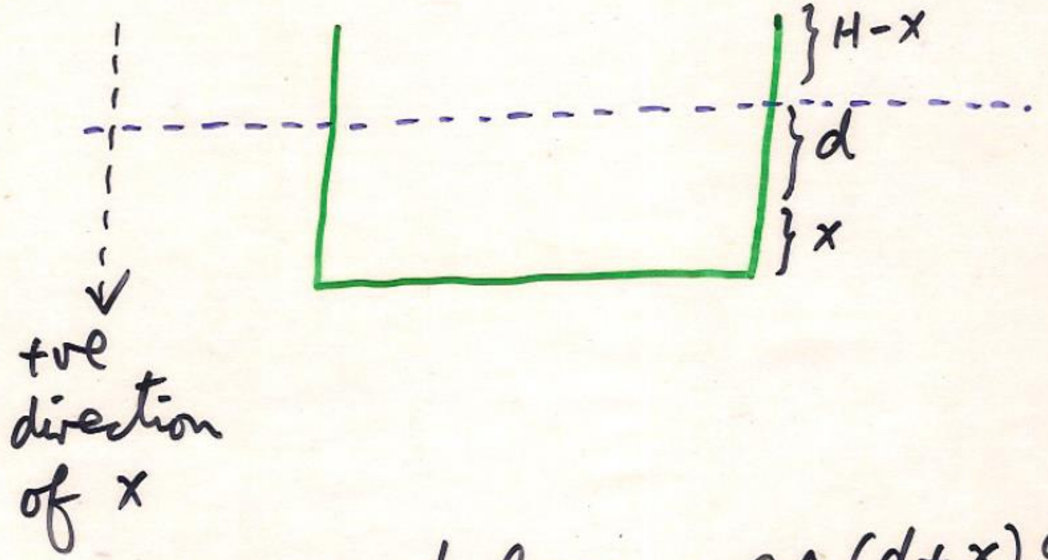
$\because$ The ship is at rest,

$$\therefore \quad Mg = \rho A d g \quad \text{----} \quad (1)$$

$$\therefore \quad d = \frac{M}{\rho A}$$

Suppose now that the ship is *not* at rest; instead it is moving in the vertical direction. Let d + x(t) be the distance from sea level to the bottom of the ship, where d is the draught as above. Show that, if gravity and buoyancy are the only forces acting on the ship, it will bob up and down with an angular frequency given by $\omega = \sqrt{\rho\,\mathrm{A}\,\mathrm{g}/\mathrm{M}}$.

2° Small displacement x:



upward force $= \rho A (d + x) g$

$$\therefore M\ddot{x} = Mg - \rho A(d+x)g$$

$$= \rho A d g - \rho A(d+x)g \quad \text{(by ①)}$$

$$= -\rho A g x$$

$$\therefore \ddot{x} = -\frac{\rho A g}{M}x = -\left(\sqrt{\frac{\rho A g}{M}}\right)^2 x$$

This is a simple harmonic motion with frequency $\sqrt{\frac{\rho A g}{M}}$. //