Week 07 F2F Example Solutions

1. **Example 6.1**

$$c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + c_3\boldsymbol{u_3} = \mathbf{0} \quad \Leftrightarrow \quad \begin{cases} ac_1 - \ \ c_2 + \ \ c_3 = 0 \\ \ \ c_1 + ac_2 - \ \ c_3 = 0 \\ -c_1 + \ \ c_2 + ac_3 = 0 \end{cases}$$

   Solving the system, we find that the system has exactly one solution if and only if $a \neq 0$. Thus $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ are linearly independent if and only if $a \neq 0$.

2. **Example 6.2**

   (a) Yes, any nonempty subset of a linearly independent set is linearly independent.

   (b) Since $(\boldsymbol{u} - \boldsymbol{v}) + (\boldsymbol{v} - \boldsymbol{w}) + (\boldsymbol{w} - \boldsymbol{u}) = \mathbf{0}$, S_2 is linearly dependent.

   (c) $a(\boldsymbol{u}-\boldsymbol{v})+b(\boldsymbol{v}-\boldsymbol{w})+c(\boldsymbol{w}+\boldsymbol{u}) = \mathbf{0} \quad \Leftrightarrow \quad (a+c)\boldsymbol{u}+(-a+b)\boldsymbol{v}+(-b+c)\boldsymbol{w} = \mathbf{0}$. Since $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent, we have

$$\begin{cases} a \qquad + c = 0 \\ -a + b \qquad = 0 \\ \qquad - b + c = 0. \end{cases}$$

   The system has only the trivial solution $a = 0$, $b = 0$, $c = 0$. Thus S_3 is linearly independent.

   (d) Similar to S_3, we can show that S_4 is linearly independent.

   (e) S_5 is linearly dependent since $(\boldsymbol{u}+\boldsymbol{v})+(\boldsymbol{v}+\boldsymbol{w})+(\boldsymbol{u}+\boldsymbol{w})-2(\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w}) = \mathbf{0}$.

3. **Example 6.3**

   (a) If $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent, then the two planes V and W intersect at the line spanned by $\boldsymbol{u}$ and hence $V \cap W = \text{span}\{\boldsymbol{u}\}$.

   (b) V and W are planes in $\mathbb{R}^3$. So $\boldsymbol{u}, \boldsymbol{v}$ are linearly independent and $\boldsymbol{u}, \boldsymbol{w}$ are linearly independent. If $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly dependent, then $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ must lie on the same plane and hence $V = W = V \cap W$.

4. **Example 6.4**

   (a) $\{ (1, 0, 0, 0),\ (0, 0, 1, 0) \}$ is a basis.

   (b) $\{ (1, 0, 0, 1),\ (0, 1, 1, 0) \}$ is a basis.

   (c) $\{ (1, \frac{1}{2}, \frac{1}{3}, 0),\ (0, 0, 0, 1) \}$ is a basis.

   (d) A general solution is

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = s \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{where } s, t \in \mathbb{R}.$$

   $\{(1, -1, 1, 0),\ (-2, 1, 0, 1)\}$ is a basis for the solution space.

5. **Example 6.5**

(a)

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 2 & 2 & -1 & y \\ -1 & 1 & 3 & z \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 2 & -1 & y-2x \\ 0 & 0 & \frac{7}{2} & z+2x-\frac{1}{2}y \end{array}\right)$$

Thus S spans $\mathbb{R}^3$. When $x = y = z = 0$, it is also clear that S is a linearly indepdent set. So S is a basis for $\mathbb{R}^3$.

(b) Substitute $x = 1, y = 0, z = 1$ into part (a), we have

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & \frac{7}{2} & 3 \end{array}\right)$$

This gives us the solution $a = 1, b = -\frac{4}{7}, c = \frac{6}{7}$. Thus the coordinate vector is $(\boldsymbol{v})_S = (1, -\frac{4}{7}, \frac{6}{7})$.