W03-05

Slide 01: In this unit, we first provide a proof to a result that was mentioned in an earlier unit regarding the checking of whether a matrix $\boldsymbol{B}$, having the same size as another matrix $\boldsymbol{A}$, is actually the inverse of $\boldsymbol{A}$. This will be followed by establishing another theorem on singular matrices.

Slide 02: Recall that by definition, if $\boldsymbol{B}$ is a square matrix that is of the same size as $\boldsymbol{A}$, then $\boldsymbol{B}$ is said to be the inverse of $\boldsymbol{A}$ if both $\boldsymbol{AB}$ and $\boldsymbol{BA}$ are equal to the identity matrix. However, it was mentioned in an earlier unit that actually it suffices to check either $\boldsymbol{AB}$ or $\boldsymbol{BA}$ and not necessary to check both.
(#)
We will now explain why.

Slide 03: This result is presented as a Theorem, which states that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are both square matrices of the same size, and if $\boldsymbol{AB} = \boldsymbol{I}$, then it allows us to conclude the following 3 statements. Firstly, we will have $\boldsymbol{BA} = \boldsymbol{I}$, next we know that $\boldsymbol{A}$ is the inverse of $\boldsymbol{B}$ and lastly, $\boldsymbol{B}$ is the inverse of $\boldsymbol{A}$.
(#)
To start off the proof, consider the homogeneous linear system $\boldsymbol{Bx} = \boldsymbol{0}$. Note here that $\boldsymbol{B}$ is the square coefficient matrix of the homogeneous linear system.
(#)
Using the equivalence between the statement that $\boldsymbol{B}$ is invertible and the statement that $\boldsymbol{Bx} = \boldsymbol{0}$ has only the trivial solution, we will attempt to show that $\boldsymbol{Bx} = \boldsymbol{0}$ has only the trivial solution. If this can be done, then we will be able to conclude that $\boldsymbol{B}$ is invertible.

Slide 04: Let $\boldsymbol{u}$ be a solution to the homogeneous linear system.
(#)
Thus $\boldsymbol{Bu}$ must be equal to $\boldsymbol{0}$.
(#)
Pre-multiply both sides of the matrix equation by $\boldsymbol{A}$.
(#)
On the left, since $\boldsymbol{AB} = \boldsymbol{I}$, we have $\boldsymbol{Iu}$. On the right, $\boldsymbol{A0}$ is $\boldsymbol{0}$. This allows us to conlude that $\boldsymbol{u}$ is $\boldsymbol{0}$.
(#)
What we have shown is that if $\boldsymbol{u}$ is a solution to the homogeneous linear system, then $\boldsymbol{u}$ must be the trivial solution. So the linear system has only the trivial solution and thus $\boldsymbol{B}$ is an invertible matrix and we can now safely conclude that $\boldsymbol{B}^{-1}$ exists.

Slide 05: We will now proceed to prove the other two conclusions.
(#)
Starting with something we know, namely $\boldsymbol{AB} = \boldsymbol{I}$,
(#)
the existence of $\boldsymbol{B}^{-1}$ allows us to post-multiply both sides of the equation by $\boldsymbol{B}^{-1}$.
(#)

On the left now we have $\boldsymbol{AI}$, since $\boldsymbol{BB}^{-1}$ is the identity matrix, while on the right we have $\boldsymbol{B}^{-1}$.

(#)

This gives the conclusion $\boldsymbol{A} = \boldsymbol{B}^{-1}$.

(#)

Now that we know $\boldsymbol{A}$ is in fact $\boldsymbol{B}^{-1}$, then $\boldsymbol{A}$ itself is invertible and

(#)

tne inverse of $\boldsymbol{A}$ is the inverse of $\boldsymbol{B}^{-1}$, which is $\boldsymbol{B}$. This establishes the conclusion that $\boldsymbol{B}$ is equal to $\boldsymbol{A}^{-1}$.

(#)

Finally, it is easy to see now that $\boldsymbol{BA}$ is $\boldsymbol{A}^{-1}\boldsymbol{A}$ which is equal to $\boldsymbol{I}$. The proof of the Theorem is thus complete.

Slide 06: The following example illustrates how the preceding theorem can be used. Suppose we know that $\boldsymbol{A}$ is a square matrix such that $\boldsymbol{A}^2 - 6\boldsymbol{A} + 8\boldsymbol{I}$ is the zero matrix. We wish to prove that $\boldsymbol{A}$ is invertible.

(#)

Starting with the given equation,

(#)

we move $8\boldsymbol{I}$ to the right hand side. We are left with $\boldsymbol{A}^2 - 6\boldsymbol{A}$ on the left.

(#)

To factorise $\boldsymbol{A}^2 - 6\boldsymbol{A}$, note that we cannot write it as $\boldsymbol{A}$ into $(\boldsymbol{A} - 6)$ since $\boldsymbol{A}$ is a matrix while 6 is a scalar.

(#)

Instead, we factorise the left side as $\boldsymbol{A}$ into $(\boldsymbol{A} - 6\boldsymbol{I})$.

(#)

Dividing both sides by -8 results in the following equation. Notice that $-\frac{1}{8}$ is placed together with $(\boldsymbol{A} - 6\boldsymbol{I})$ which is possible since $-\frac{1}{8}$ is a scalar.

(#)

Now is the time to use the previously proven theorem. Since $-\frac{1}{8}$ of the matrix $(\boldsymbol{A} - 6\boldsymbol{I})$ has the same size as the square matrix $\boldsymbol{A}$, we have shown the existence of another square matrix of similar size to $\boldsymbol{A}$ such that the product of the two matrices results in $\boldsymbol{I}$. The previous theorem allows us to conclude $\boldsymbol{A}$ is invertible and that $-\frac{1}{8}$ of $(\boldsymbol{A} - 6\boldsymbol{I})$ is precisely the inverse of $\boldsymbol{A}$.

Slide 07: This next theorem states that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are two square matrices of the same size and if one of them, say $\boldsymbol{A}$ is singular, then both products $\boldsymbol{AB}$ and $\boldsymbol{BA}$ will be singular as well.

(#)

To begin, we will first show that $\boldsymbol{AB}$ is singular. To do this, we will use the method of contradiction.

(#)

This is done by first supposing that the opposite of what we want to establish is actually true. Thus we assume $\boldsymbol{AB}$ is actually invertible. For the method of contradiction to work, we need to show that this assumption is wrong and will lead to some statement that we know for sure is false.

(#)

Our assumption that $\boldsymbol{AB}$ is invertible means that there is another square matrix $\boldsymbol{C}$ of the same size as $\boldsymbol{A}$ and $\boldsymbol{B}$ such that post-multiplying $\boldsymbol{C}$ to $\boldsymbol{AB}$ results in $\boldsymbol{I}$.

(#)

Since the multiplication of matrices follows the associative law, we can rewrite the left hand side as $\boldsymbol{A}$ pre-multiplied to $\boldsymbol{BC}$.

Slide 08: However, notice now that $\boldsymbol{BC}$ is also a square matrix with the same size as $\boldsymbol{A}$. So $\boldsymbol{A}$ pre-multiplied to $\boldsymbol{BC}$ resulting in the identity matrix would imply that

(#)

$\boldsymbol{A}$ must be an invertible matrix with $\boldsymbol{BC}$ as its inverse.

(#)

This is the contradiction that we want, since we know that $\boldsymbol{A}$ is a singular matrix. The proof by method of contradiction is thus complete.

(#)

It is left as an exercise for you to show in a similar manner, that $\boldsymbol{BA}$ is also a singular matrix, once we know that $\boldsymbol{A}$ is singular.

Slide 09: In this unit, we established

(#)

the result that in order to check whether $\boldsymbol{B}$ is the inverse of $\boldsymbol{A}$, it suffices to check either $\boldsymbol{AB} = \boldsymbol{I}$ or $\boldsymbol{BA} = \boldsymbol{I}$ and not necessary for both. This of course, is for the case where $\boldsymbol{A}$ and $\boldsymbol{B}$ are both square matrices of the same size.

(#)

We also established a result that if $\boldsymbol{A}$ is a singular square matrix, then both $\boldsymbol{AB}$ and $\boldsymbol{BA}$ will be singular. Here $\boldsymbol{B}$ is a square matrix with the same size as $\boldsymbol{A}$.