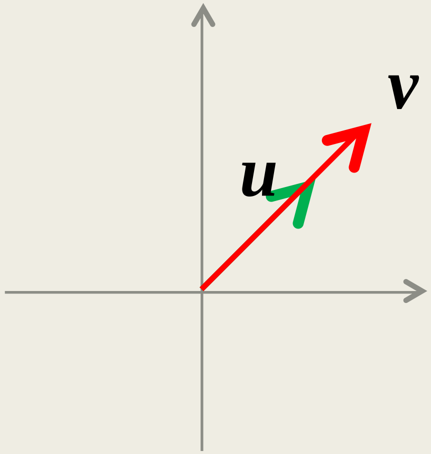
LINEAR INDEPENDENCE IN R^2 AND R^3

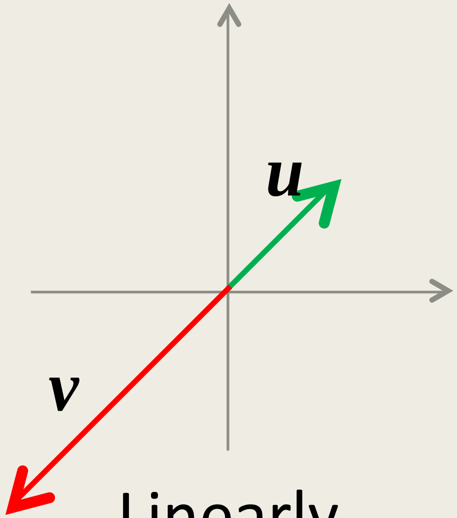
Set with two vectors

For two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$, recall the following:

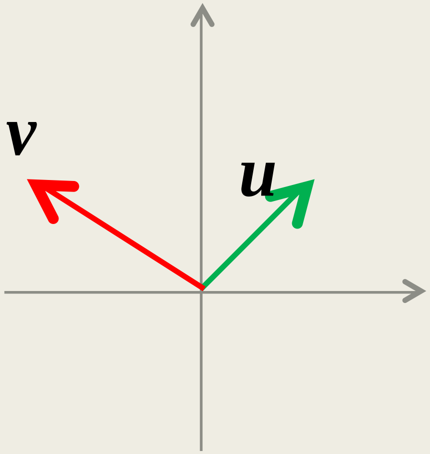
$S = \{\boldsymbol{u}, \boldsymbol{v}\}$ is a linearly dependent set if and only if $\boldsymbol{u}$ and $\boldsymbol{v}$ are scalar multiples of each other (they lie on the same line).



Linearly dependent



Linearly dependent

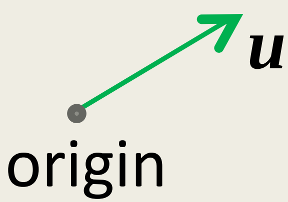


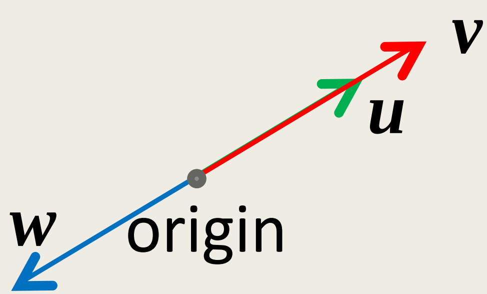
Linearly independent

Set with three vectors

For three vectors in $\mathbb{R}^3$:

$S = \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a linearly dependent set if and only if they lie on the same line or the same plane (when their initial points are placed at the origin).





$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ lie on the same line

$\{\boldsymbol{u}\}$ is a linearly independent set

$\{\boldsymbol{u}, \boldsymbol{v}\}$ is a linearly dependent set

$\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a linearly dependent set

Set with three vectors

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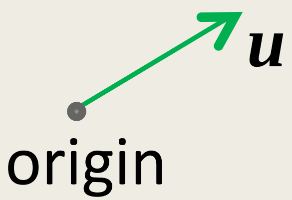
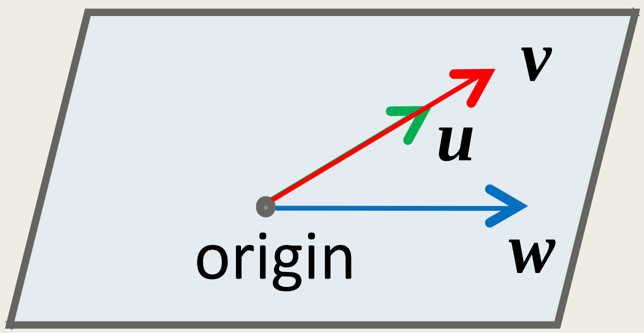




$\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ lie on the same plane

$\{\boldsymbol{u}\}$ is a linearly independent set

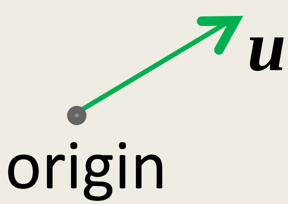
$\{\boldsymbol{u}, \boldsymbol{v}\}$ is a linearly dependent set

$\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a linearly dependent set

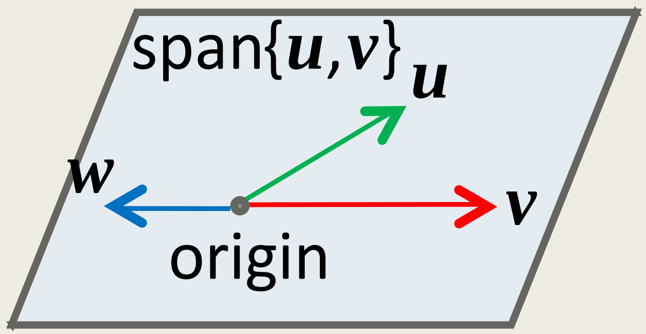
Set with three vectors

For three vectors in $\mathbb{R}^3$:

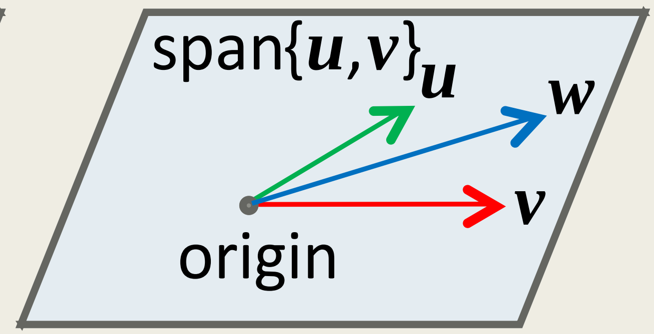
$S = \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a linearly dependent set if and only if they lie on the same line or the same plane (when their initial points are placed at the origin).



$\{\boldsymbol{u}\}$ is a linearly independent set



$\{\boldsymbol{u}, \boldsymbol{v}\}$ is a linearly independent set

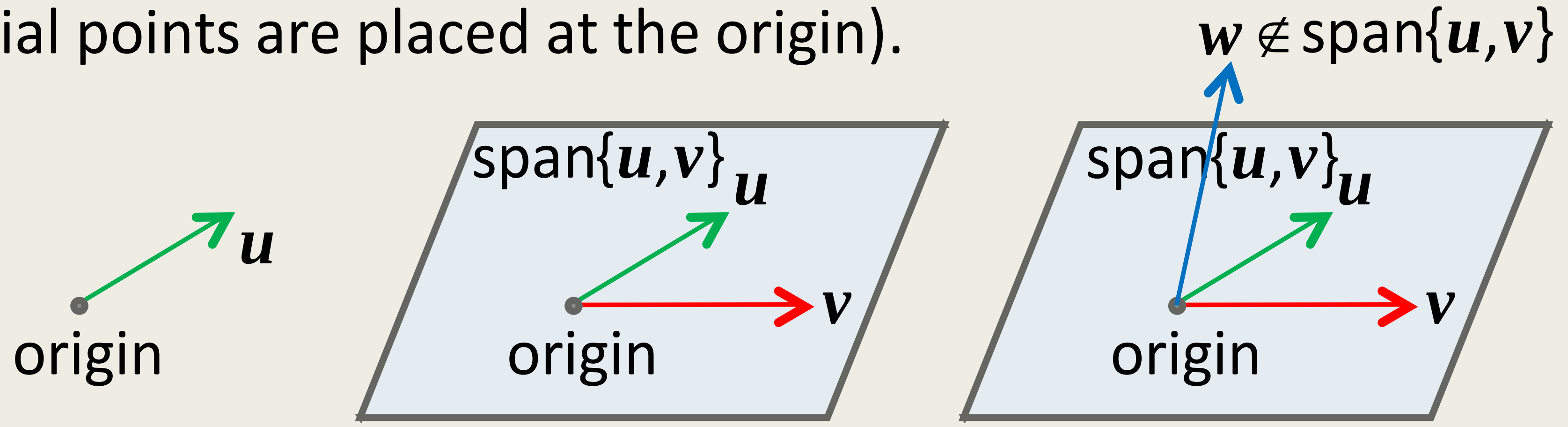


$\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a linearly dependent set

Set with three vectors

For three vectors in $\mathbb{R}^3$:

$S = \{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a linearly dependent set if and only if they lie on the same line or the same plane (when their initial points are placed at the origin).



$\{\boldsymbol{u}\}$ is a linearly independent set

$\{\boldsymbol{u}, \boldsymbol{v}\}$ is a linearly independent set

$\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is a linearly independent set

Theorem

Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ be linearly independent vectors in $\mathbb{R}^n$.

Suppose $\boldsymbol{u}_{k+1} \in \mathbb{R}^n$ is NOT a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$, then $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k, \boldsymbol{u}_{k+1}$ are linearly independent.

Proof: Suppose (for a contradiction) that $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k, \boldsymbol{u}_{k+1}$ are linearly dependent.

$\Rightarrow \; c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k + c_{k+1}\boldsymbol{u}_{k+1} = \mathbf{0}$ has non trivial solutions

Let $d_1, d_2, \ldots, d_k, d_{k+1}$ be a non trivial solution to the vector equation. Note that $d_1, d_2, \ldots, d_k, d_{k+1}$ are not all zero.

$$\Rightarrow \; d_1\boldsymbol{u}_1 + d_2\boldsymbol{u}_2 + \ldots + d_k\boldsymbol{u}_k + d_{k+1}\boldsymbol{u}_{k+1} = \mathbf{0}$$

Theorem

Proof: Suppose (for a contradiction) that $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k, \boldsymbol{u}_{k+1}$ are linearly dependent.

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$$d_1\boldsymbol{u}_1 + d_2\boldsymbol{u}_2 + \dots + d_k\boldsymbol{u}_k + d_{k+1}\boldsymbol{u}_{k+1} = \mathbf{0}$$

Case 1: $d_{k+1} = 0$. This means that not all $d_1, \dots, d_k$ are zero.

$$d_1\boldsymbol{u}_1 + d_2\boldsymbol{u}_2 + \dots + d_k\boldsymbol{u}_k + d_{k+1}\boldsymbol{u}_{k+1} = \mathbf{0}$$

$$\Rightarrow d_1\boldsymbol{u}_1 + d_2\boldsymbol{u}_2 + \dots + d_k\boldsymbol{u}_k = \mathbf{0}$$

$\Rightarrow$ Contradiction, since $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k$ are linearly independent.

Theorem

Proof: Suppose (for a contradiction) that $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k, \boldsymbol{u}_{k+1}$ are linearly dependent.

$\Rightarrow\ c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \ldots + c_k\boldsymbol{u}_k + c_{k+1}\boldsymbol{u}_{k+1} = \mathbf{0}$ has non trivial solutions

Let $d_1, d_2, \ldots, d_k, d_{k+1}$ be a non trivial solution to the vector equation. Note that $d_1, d_2, \ldots, d_k, d_{k+1}$ are not all zero.

$$d_1\boldsymbol{u}_1 + d_2\boldsymbol{u}_2 + \ldots + d_k\boldsymbol{u}_k + d_{k+1}\boldsymbol{u}_{k+1} = \mathbf{0}$$

Case 2: $d_{k+1} \neq 0.\ \ d_1\boldsymbol{u}_1 + d_2\boldsymbol{u}_2 + \ldots + d_k\boldsymbol{u}_k + d_{k+1}\boldsymbol{u}_{k+1} = \mathbf{0}$

$$\Rightarrow \boldsymbol{u}_{k+1} = -\left(\tfrac{d_1}{d_{k+1}}\right)\boldsymbol{u}_1 - \left(\tfrac{d_2}{d_{k+1}}\right)\boldsymbol{u}_2 + \ldots - \left(\tfrac{d_k}{d_{k+1}}\right)\boldsymbol{u}_k$$

$\Rightarrow$ Contradiction, since $\boldsymbol{u}_{k+1}$ is not a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$.

Theorem

Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ be linearly independent vectors in $\mathbb{R}^n$.

Suppose $\boldsymbol{u}_{k+1} \in \mathbb{R}^n$ is NOT a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$,

then $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k, \boldsymbol{u}_{k+1}$ are linearly independent.

Question: What do you think is the largest value for k for this statement to be true?

In other words, when will we no longer be able to find a vector $\boldsymbol{u}_{k+1}$ that is NOT a linear combination of $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$?

Summary

1) Linear independence for two or three vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

2) When can we add vectors into a linearly independent set and preserve the linear independence property?