Unit 005 Row-echelon forms

Slide 01: The focus of this unit is to introduce row-echelon forms.

Slide 02: A matrix is said to be in row-echelon form if it possesses the following two properties.
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First, if there are any rows consisting entirely of zeros, then these rows must be grouped at the bottom of the matrix. Any row that is not consisting entirely of zeros is called a non zero row.

Slide 03: The second property is that for any two successive non zero rows, the first non zero number in the lower row must occur further to the right than the first non zero number in the higher row.
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For example consider the two successive non zero rows shown. The first non zero number in each of the two rows is indicated.
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As desciribed, the position of the first non zero number in the lower row
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must be positioned further to the right than the first non zero number in the higher row.
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The first non zero number in every non zero row is called the leading entry of that row. Thus, any matrix that has the two properties that we have just described is said to be in row-echelon form.

Slide 04: The leading entry that was defined in the previous slide is also called a pivot point. A column in the matrix that contains a pivot point is called a pivot column.
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Here you see an example of how an augmented matrix in row-echelon form will look like. Notice that the positions of the leading forms something like a diagonal from top left to bottom right. Of course some columns may be skipped in this diagonal arrangement.
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The pivot columns are shown here. A column that is not a pivot column is called a non pivot column.

Slide 05: Note that the concept of row-echelon forms can be applied to any matrix in general, not just for augmented matrices.

Slide 06: Let us go through some observations based on the definition of row-echelon forms. First, by the way row-echelon forms are defined, it is clear that in every pivot column, all entries in the same column below each pivot point must be zero.
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The entries that must be zero are highlighted here.

Slide 07: There is no requirement for the entries above each pivot point to be zero.

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Thus, these entries do not have to be zero.

Slide 08: Obviously, for each non zero row, there is one and only one leading entry.

Slide 09: It is also obvious that every pivot column has one and exactly one pivot point.

Slide 10: We now introduce a special type of row-echelon forms called the reduced row-echelon form. An augmented matrix is said to be in reduced row-echelon form if it is in row-echelon form, meaning that it already has the previously described two properties, and in addition has the following two properties.

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All leading entries must be 1 and not any other non zero number.

Slide 11: In addition, in each pivot column, other than the pivot point, all other entries must be zero.

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Note that this additional requirements means that the entries above each pivot point, which were not required to be zero in the definition of row-echelon forms, must now be zero as well. Thus, an augmented matrix with all the 4 properties described will be said to be in reduced row-echelon form.

Slide 12: Why do we need to study row-echelon forms and how are they useful? Consider the following augmented matrix and the linear system that it corresponds to. It is clear that the augmented matrix is not in row-echelon form. In order to solve the linear system, one would need to do some work.

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What about this augmented matrix? This augmented matrix is in row-echelon form and has three leading entries, one in each row. Looking at the corresponding linear system, you can solve the system easily by first solving for z in the last equation and then substitute the value of z into the second equation to solve for y. Finallly, with the values of y and z, we can solve for x using the first equation. So it seems like an augmented matrix in row-echelon form makes it easier for us to solve its corresponding linear system.

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This augmented matrix is in reduced row-echelon form. It's corresponding linear system, when written out tells us immediately what the solution of the linear system is. We do not need to perform any substitution like we did in the previous case. So it seems like an augmented matrix in reduced row-echelon form makes it even more convenient for us to solve its corresponding linear system.

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Now, what if I tell you that the 3 augmented matrices here are row equivalent? Recall that the linear systems corresponding to augmented matrices that are row equivalent has the same solution set. This is good news for us because we are now able to say that the solution to the linear system at the top is $x = \frac{11}{5}$, $y = -\frac{2}{5}$ and $z = -\frac{3}{5}$. You should now realise that if we can somehow find an augmented matrix that is in row-echelon or reduced row-echelon form and is row equivalent to the first augmented matrix at the top, it will be very helpful for us in solving the linear system.

Slide 13: Let us summarise the main points for this unit.

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We first described the two properties that are required for an augmented matrix to be in row-echelon form.

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For an augmented in row-echelon form, we learnt how to identify the pivot points, pivot columns and non pivot columns.

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We saw that reduced row-echelon forms are row-echelon forms with two additional properties.

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Lastly, we discussed why augmented matrices in row-echelon forms are useful in making the solving of linear systems easier.