# Week 05

MA1508E LINEAR ALGEBRA FOR ENGINEERING

# Week 3

# IVLE Quiz Discussion

# Week 4

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# Review of Week 03 (Units 013-017) content

- Block multiplication: entry by entry, row by row, column by column, block by block.
- What is an inverse of a matrix? Invertible and singular matrices.
- Checking if a given 'candidate' ***B*** is an inverse of ***A***
- Uniqueness of inverse
- Checking ***BA***=***I*** and ***AB***=***I***
- Cancellation law for invertible matrices
- Definition of the transpose of a matrix; symmetric matrices in terms of transpose
- Some results on transpose
- Some results on inverse

# Review of Week 03 (Units 013-017) content

- Powers of an invertible matrix
- Linear system and matrix equation
- What happens when we apply on ERO on an identity matrix?
- A square matrix is an elementary matrix if it can be obtained from an identity matrix by performing a single elementary row operation.
- All elementary matrices are invertible and their inverse is also an elementary matrix.
- If an elementary matrix represents a single elementary row operation *X,* then the inverse represents the elementary row operation that does the opposite of *X*.

# Review of Week 04 (Units 018-022) content

- A set of four equivalent statements (on the invertibility of a matrix)
- Building on one part of the proof of the preceding theorem, we develop a method to find the inverse of an invertible matrix.
- This method also allows us to know when a matrix is invertible.
- If ***A*** and ***B*** are both square matrices of the same size, then to check whether ***B*** is the inverse of ***A***, we only need to check ***AB*=*I*** or ***BA*=*I*** (and not both)
- If ***A*** and ***B*** are both square matrices of the same size and one of them is singular, then both ***AB*** and ***BA*** will be singular.
- The term *ad-bc* we saw in the 2x2 matrix example – is there a general expression for bigger matrices?
- Determinant of a matrix. How can we compute this efficiently?
- Determinant of some special groups of matrices.

# Review of Week 04 (Units 018-022) content

- Since the determinant of a triangular matrix is easy to compute, can we use ERO to help us?
- What does the determinant of a matrix tell us about the invertibility?
- What properties does determinants have?
- Establishing an "if and only if" statement for a 2x2 matrix to be invertible.
- Definition of the determinant of a square matrix (cofactor expansion).
- Special formula to remember the determinant of a 3x3 matrix.
- Computing determinant via cofactor expansion along any row or column.
- Determinant of special matrices
  - Triangular matrices
  - A square matrix and its transpose
  - Square matrices with identical rows or columns

# Week 05 (units 023-028) overview

## 023 Determinants and elementary row operations

- How the 3 types of elementary row operations change the determinant of a square matrix
- Computing the determinant of a matrix using elementary row operations

## 024 Equivalent statements Part II

- $\boldsymbol{A}$ is invertible if and only if $\det(\boldsymbol{A})$ is not zero

## 025 Properties of determinants

- $\det(c\boldsymbol{A})$; $\det(\boldsymbol{AB})$; $\det(\boldsymbol{A}^{-1})$

## 026 Euclidean vectors

- A vector and the components of a vector
- Addition and subtraction of vectors; and multiplying a scalar to a vector
- Identifying a vector with a matrix. Some vector operation laws
- Definition of the Euclidean $n$-space, $\boldsymbol{R}^n$; subsets of $\boldsymbol{R}^n$

# Week 05 (units 023-028) overview

## 027 More on Euclidean vectors

- Definition of norm of a vector; distance between two vectors; angle between two vectors; dot product between two vectors
- Dot product and matrix product

## 028 Orthogonality

- Some results on length and dot product for Euclidean vectors
- When are two vectors orthogonal?
- When is a set of vectors orthogonal?
- Orthonormal sets; normalising an orthogonal set

# Cramer’s Rule

Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ is a linear system where $\boldsymbol{A}$ is an invertible square matrix of order $n$. For $i = 1, 2, \ldots, n$, let $\boldsymbol{A}_i$ be the square matrix of order $n$ where the $i$th column of $\boldsymbol{A}$ is replaced by $\boldsymbol{b}$.

Then the unique solution is $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ where

Cramer's Rule

$$x_i = \frac{\det(\boldsymbol{A}_i)}{\det(\boldsymbol{A})}, \; i = 1, 2, \ldots, n.$$

# Example

Solve the following linear system by Cramer's Rule.

$$\begin{cases} x + y + z = \boxed{-1} \\ 2x - y - z = 4 \\ x + 2y - 3z = 7 \end{cases}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -3 \end{pmatrix} \quad \boldsymbol{A}_1 = \begin{pmatrix} -1 & 1 & 1 \\ 4 & -1 & -1 \\ 7 & 2 & -3 \end{pmatrix} \quad \boldsymbol{A}_2 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 4 & -1 \\ 1 & 7 & -3 \end{pmatrix} \quad \boldsymbol{A}_3 = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 4 \\ 1 & 2 & 7 \end{pmatrix}$$

# Example

$$\det(\boldsymbol{A}) = 15 \quad \det(\boldsymbol{A}_1) = 15 \quad \det(\boldsymbol{A}_2) = 0 \quad \det(\boldsymbol{A}_3) = -30$$

$$x = \frac{15}{15} = 1 \qquad y = \frac{0}{15} = 0 \qquad z = \frac{-30}{15} = -2$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 2 & -3 \end{pmatrix} \quad \boldsymbol{A}_1 = \begin{pmatrix} -1 & 1 & 1 \\ 4 & -1 & -1 \\ 7 & 2 & -3 \end{pmatrix} \quad \boldsymbol{A}_2 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 4 & -1 \\ 1 & 7 & -3 \end{pmatrix} \quad \boldsymbol{A}_3 = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 4 \\ 1 & 2 & 7 \end{pmatrix}$$

# Example 4.1

Let

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & -1 \\ -2 & 1 & 0 & -2 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} -1 & 3 & 4 & -2 \\ 0 & 10 & 1 & 1 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

(a) Find $\det(\boldsymbol{C})$.

(b) Without computing the matrix $\boldsymbol{AC}$, explain why the homogeneous linear system $\boldsymbol{ACx} = \boldsymbol{0}$ has infinitely many solutions.

# Example 4.2

Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$

# Example 4.3

Let $A$ be a $4 \times 4$ matrix such that $\det(A) = 9$. Find

(a) $\det(3A)$ (b) $\det(A^{-1})$ (c) $\det(3A^{-1})$ (d) $\det((3A)^{-1})$

# Example 4.4

Let $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$ be matrices such that both $\boldsymbol{A}$ and $\boldsymbol{B}$ can be obtained from $\boldsymbol{C}$ by elementary row operations:

$$\boldsymbol{C} \xrightarrow[3R_2]{} \xrightarrow[R_3 + 2R_1]{} \boldsymbol{A} \qquad \boldsymbol{C} \xrightarrow[R_1 + R_2]{} \xrightarrow[R_2 \leftrightarrow R_3]{} \xrightarrow[R_4 - R_2]{} \boldsymbol{B}$$

(a) Describe how $\boldsymbol{A}$ can be obtained from $\boldsymbol{B}$ by elementary row operations.

(b) Let $\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 7 & \frac{1}{11} \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. Find $\det(\boldsymbol{B})$.

# Finally…

THE END