W07-06

Slide 01: In this unit, we will introduce a method to find all the eigenvalues of a square matrix.

Slide 02: Recall the definition of eigenvectors and eigenvalues of a matrix $\boldsymbol{A}$. A non zero column vector $\boldsymbol{u}$ is said to be an eigenvector of $\boldsymbol{A}$ if $\boldsymbol{A}\boldsymbol{u} = \lambda\boldsymbol{u}$ for some scalar λ. In this case, we say that λ is an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{u}$ is an eigenvector of $\boldsymbol{A}$ associated with λ.

Slide 03: Consider the matrix $\boldsymbol{A}$, which is precisely the matrix we had in the population movement example discussed in a previous unit. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be the two vectors in $\mathbb{R}^2$ as shown.

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When $\boldsymbol{A}$ is premultiplied to $\boldsymbol{x}$, the result is still the same vector $\boldsymbol{x}$,

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which is 1 times $\boldsymbol{x}$.

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Thus, we see that 1 is an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue 1.

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Premultiplying $\boldsymbol{A}$ to $\boldsymbol{y}$ results in the vector $(0.95, -0.95)$

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which is 0.95 times $\boldsymbol{y}$.

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Thus 0.95 is an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{y}$ is an eigenvector of $\boldsymbol{A}$ associated with the eigenvalue 0.95. Before we move on to the next example, note that at this point, we are not sure if $\boldsymbol{A}$ has other eigenvalues other than 1 and 0.95.

Slide 04: Consider the 3×3 matrix as shown, together with the three vectors from $\mathbb{R}^3$, $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$. Premultiplying $\boldsymbol{B}$ to $\boldsymbol{x}$

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results in $3\boldsymbol{x}$.

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So 3 is an eigenvalue of $\boldsymbol{B}$ and $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{B}$ associated with the eigenvalue 3.

Slide 05: Let's do the same for $\boldsymbol{y}$. Premultiply $\boldsymbol{B}$ to $\boldsymbol{y}$

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gives the zero vector, which can be written as 0 times $\boldsymbol{y}$.

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Thus, 0 is an eigenvalue of $\boldsymbol{B}$ and $\boldsymbol{y}$ is an eigenvector of $\boldsymbol{B}$ associated with the eigenvalue 0.

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Computing $\boldsymbol{B}\boldsymbol{z}$,

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we also have the zero vector as a result

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and thus $\boldsymbol{z}$ is also an eigenvector of $\boldsymbol{B}$ associated with the eigenvalue 0.

Slide 06: We have now found that $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ are all eigenvectors of $\boldsymbol{B}$ where $\boldsymbol{x}$ is associated with 3 and both $\boldsymbol{y}$ and $\boldsymbol{z}$ are associated with 0. Note that we have already seen that if a non zero vector $\boldsymbol{u}$ is an eigenvector of a matrix associated with an eigenvalue λ, then all scalar multiples of $\boldsymbol{u}$ will also be an eigenvector of the matrix associated with the same eigenvalue. However, notice that although both $\boldsymbol{y}$ and $\boldsymbol{z}$ are eigenvectors of $\boldsymbol{B}$ associated with the eigenvalue 0, they are not multiples of each other.
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Now let us compute the product of these three matrices, where the matrix in the middle is in fact our matrix $\boldsymbol{B}$.
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The outcome of the product is actually a diagonal matrix as shown and interestingly,
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you will notice that the second and third columns of the highlighted matrix are the two eigenvectors $\boldsymbol{y}$ and $\boldsymbol{z}$
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while the first column of the matrix is the eigenvector $\boldsymbol{x}$. The position of the 3 eigenvectors in forming the matrix corresponds very nicely with the position of the 3 diagonal entries in the diagonal matrix. These three diagonal entries are precisely the eigenvalues that the eigenvectors are associated with.

Slide 07: We are now ready to derive a method that will allow us to find all the eigenvalues of a matrix. Let $\boldsymbol{A}$ be a square matrix of order n. By definition λ is an eigenvalue of $\boldsymbol{A}$ if and only if there is some non zero column vector $\boldsymbol{u}$ in $\mathbb{R}^n$ such that $\boldsymbol{Au} = \lambda\boldsymbol{u}$.
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Arrange the term as follows,
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this allows us to factorise the left hand side as the square matrix $(\lambda\boldsymbol{I} - \boldsymbol{A})$ premultiplied to $\boldsymbol{u}$ equals to $\boldsymbol{0}$ on the right hand side. The key observation here is that this equation is satisfied for some non zero vector $\boldsymbol{u}$ in $\mathbb{R}^n$.
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Now if we view the equation as a linear system, this would mean that the linear system $(\lambda\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$ has non trivial solutions since $\boldsymbol{u}$ is one such solution.
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By our list of equivalent statements to the invertibility of $\boldsymbol{A}$, we see that this is equivalent to the statement that the determinant of $\lambda\boldsymbol{I} - \boldsymbol{A}$ is zero, or in other words, $\lambda\boldsymbol{I} - \boldsymbol{A}$ is singular.

Slide 08: We have thus arrived at the equivalence between the statement that λ is an eigenvalue of $\boldsymbol{A}$ and that the determinant of $\lambda\boldsymbol{I} - \boldsymbol{A}$ is zero.
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Thus, in order to find all the eigenvalues of a square matrix $\boldsymbol{A}$, we need to find out all the numbers λ that will make the matrix $(\lambda\boldsymbol{I} - \boldsymbol{A})$ singular.

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As $\boldsymbol{A}$ is a square matrix of order n, if we compute, by cofactor expansion, the determinant of $\lambda\boldsymbol{I} - \boldsymbol{A}$, we will obtain a polynomial in λ of degree n. We can represent this polynomail as shown. Here c_0, c_1 and so on till c_n are just some real numbers.

Slide 09: This polynomial is known as the characteristic polynomial of the matrix $\boldsymbol{A}$.

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If we set the polynomial to equal to zero, we will have an equation known as the characteristic equation of the matrix $\boldsymbol{A}$. We now know that the values of λ that satisfies this characteristic equation will be the eigenvalues of $\boldsymbol{A}$. In other words, the eigenvalues of $\boldsymbol{A}$ will be the roots of the characteristic equation.

Slide 10: Returning to our population movement example matrix $\boldsymbol{A}$. We asked previously whether $\boldsymbol{A}$ had other eigenvalues other than 1 and 0.95. To answer the question, we have to find all the roots of the characteristic equation. First, write down the matrix $\lambda\boldsymbol{I} - \boldsymbol{A}$.

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The determinant of $\lambda\boldsymbol{I} - \boldsymbol{A}$ can be found easily.

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This experession involving λ can be simplified

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and further factorised into the product of two factors, $(\lambda - 1)$ and $(\lambda - 0.95)$.

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So now we see that the characteristic polynomial is zero if and only if λ is 1 or 0.95.

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Therefore, we can conclude that 1 and 0.95 are the only two eigenvalues of $\boldsymbol{A}$.

Slide 11: What about our second matrix $\boldsymbol{B}$? We saw earlier that 0 and 3 are both eigenvalues of $\boldsymbol{B}$. Are there others? To answer this question, we start off by writing down the matrix $\lambda\boldsymbol{I} - \boldsymbol{B}$.

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You would need to do some hard work here as to find the determinant of $\lambda\boldsymbol{I} - \boldsymbol{B}$, in terms of λ requires some careful cofactor expansion. You may wish to verify that the characteristic polynomial is $\lambda^3 - 3\lambda^2$.

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The characteristic polynomial can be factorised into λ^2 times $(\lambda - 3)$.

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Thus the characteristic polynomial is zero if and only if λ is 0 or 3.

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Once again, we can now conclude that 0 and 3 are the only two eigenvalues of $\boldsymbol{B}$, both of which we have seen in the earlier example.

Sllde 12: Let's look at another matrix $\boldsymbol{C}$. To find all the eigenvalues of $\boldsymbol{C}$, we first find the characteristic polynomial by writing down the matrix $\lambda\boldsymbol{I} - \boldsymbol{C}$.

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Again, some careful cofactor expansion is required before we obtain the following polynomial $\lambda^3 - \lambda^2 - 2\lambda + 2$.

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At this point, we need to find the roots of the characteristic equation and you may wonder what are the roots in this case?

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One suggestion is to try a few simple values of λ, namely $-2, -1, 0, 1$ or 2 and see whether when these values are substituted into the characteristic polynomial we will obtain the value of 0.

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For example when we substitute $\lambda = 1$ into the polynomial,

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we do obtain 0.

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So this tells us that $\lambda = 1$ is a root of the characteristic equation.

Slide 13: After obtaining this first root, we can now factorise the characteristic polynomial into $(\lambda - 1)$ and $(\lambda^2 - 2)$.

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The term $(\lambda^2 - 2)$ is obtained after dividing the characteristic polynomial by the factor we have found, namely $(\lambda - 1)$.

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The term $(\lambda^2 - 2)$ can now be factorised further into $(\lambda - \sqrt{2})$ and $(\lambda + \sqrt{2})$.

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This means the chacteristic polynomial is zero if and only if λ is equal to 1, $\sqrt{2}$ or $-\sqrt{2}$.

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We have thus found all the 3 eigenvalues of $\boldsymbol{C}$.

Slide 14: Let us summarise the main points in this unit.

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We defined the characteristic polynomial and characteristic equation of a square matrix.

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We then derived a method to find all the eigenvalues of a square matrix. This is done by solving for the roots of the characteristic equation.