Week 10 IVLE Quiz

1. Which of the following statements below is/are definitely correct?

   (I) If $\boldsymbol{u}$ is a least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$, then $\boldsymbol{u}$ is an exact solution to $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$.

   (II) If $\boldsymbol{u}$ is an exact solution to $\boldsymbol{Ax} = \boldsymbol{b}$, then $\boldsymbol{u}$ is an exact solution to $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$.

   (A) (I) only.

   (B) (II) only.

   (C) Both (I) and (II).

   (D) Inconclusive, more information is required.

   Answer: (C). Recall the Theorem in Unit 54 (slide 8). $]bbu$ is a least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$ if and only if $\boldsymbol{u}$ is a solution to $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$. So statement (I) is true. Statement (II) is also true since an exact solution to $\boldsymbol{Ax} = \boldsymbol{b}$ is the **best** least squares solution there is.

2. Suppose $\boldsymbol{b} \in \mathbb{R}^n$ is a vector that is not in the column space of a $n \times m$ matrix $\boldsymbol{A}$. How many statements below is/are definitely correct?

   (I) $\boldsymbol{Ax} = \boldsymbol{b}$ is inconsistent.

   (II) The projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$ is a least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$.

   (III) If $\boldsymbol{u}$ is a solution to $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$, then $\boldsymbol{Au}$ is the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$.

   (A) None.

   (B) Exactly one.

   (C) Exactly two.

   (D) All three.

   Answer: (C). Statement (I) is true since $\boldsymbol{b}$ is not in the column space of $\boldsymbol{A}$. Statement (II) is incorrect since the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$, say $\boldsymbol{p}$, is **NOT** a least squares solution. Instead, solving $\boldsymbol{Ax} = \boldsymbol{p}$ yields a least squares solution. Statement (III) is correct since $\boldsymbol{u}$ is a solution to the normal equation means that $\boldsymbol{u}$ is a least squares solution to $\boldsymbol{Ax} = \boldsymbol{b}$. This implies that $\boldsymbol{u}$ is a solution to $\boldsymbol{Ax} = \boldsymbol{p}$.

3. Let V be a subspace of $\mathbb{R}^n$. If $\boldsymbol{b} \in \mathbb{R}^n$ does not belong to V and $\boldsymbol{p}$ is the orthogonal projection of $\boldsymbol{b}$ onto V, which of the following statements is/are correct?

   (I) $\boldsymbol{p} \neq \boldsymbol{b}$.

   (II) $\boldsymbol{p} - \boldsymbol{b}$ is orthogonal to $\boldsymbol{p}$.

   (III) $\boldsymbol{p} - \boldsymbol{b}$ is orthogonal to $\boldsymbol{b}$.

   (A) (II) and (III) only

   (B) (I) and (II) only

   (C) (I) and (III) only

   (D) None of the given combinations is correct

   Answer: (B). Statement (I) is correct since $\boldsymbol{b}$ is not in V, then the vectors $\boldsymbol{b}$ and its projection onto V ($\boldsymbol{p}$) would be different. Statement (II) is correct since the difference $\boldsymbol{b} - \boldsymbol{p}$ (and thus $\boldsymbol{p} - \boldsymbol{b}$ as well) is a normal vector to V, meaning that $\boldsymbol{p} - \boldsymbol{b}$ is orthogonal to all vectors in V, including $\boldsymbol{p}$. Statement (III) is incorrect as $\boldsymbol{p} - \boldsymbol{b}$ is orthogonal to $\boldsymbol{p}$, not to $\boldsymbol{b}$.

4. Suppose I am applying Gram-Schmidt Process on $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\}$ (a basis) to convert S into an orthogonal set $T = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}\}$. How many statements below is/are definitely true?

   (I) $\boldsymbol{v_i} \cdot \boldsymbol{v_j} = 0$ for all $i \neq j$.

   (II) $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \cdots, \boldsymbol{u_k}\} = \text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}\}$.

   (III) It is possible that T contains the zero vector.

   (A) Exactly one.

   (B) Exactly two.

   (C) All three.

   (D) None.

   Answer: (B). Note that S is a basis, so S is a linearly independent set. Statement (I) is correct since the vectors in T are pairwise orthogonal. Statement (II) is correct since both S and T are bases for the same subspace. Statement (III) is incorrect since T is a basis, it cannot contain the zero vector (else it would be linearly dependent).

5. Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ is an inconsistent linear system. Which of the statements below is definitely correct?

   (A) The least squares solution for $\boldsymbol{Ax} = \boldsymbol{b}$ is unique.

   (B) The projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$ is unique.

   (C) $\boldsymbol{b}$ belongs to the column space of $\boldsymbol{A}$.

(D) $\boldsymbol{A}$ is not of full rank.

Answer: (B). Statement (A) is not necessarily correct since a linear system can be infinitely many least squares solution (imagine a consistent linear system with inifinitely many exact solutions, all of which are least squares solutions). Statement (C) is incorrect since $\boldsymbol{Ax} = \boldsymbol{b}$ is inconsistent. Statement (D) is not necessarily correct since $\boldsymbol{A}$ can be full rank and still $\boldsymbol{Ax} = \boldsymbol{b}$ is inconsistent. Statement (B) is correct since the projection is always unique.