DIAGONALIZATION PART I

DEFINITION

Given a square matrix $\boldsymbol{A}$, we wanted to know if it is possible to find an invertible matrix $\boldsymbol{P}$ such that

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{D} \text{ (a diagonal matrix)}$$

A square matrix $\boldsymbol{A}$ is called diagonalizable if there exists an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ is a diagonal matrix.

In here, matrix $\boldsymbol{P}$ is said to diagonalize $\boldsymbol{A}$.

EXAMPLE

$$\boldsymbol{A} = \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix}$$

So $\boldsymbol{A}$ is diagonalizable and $\boldsymbol{P}$ diagonalizes $\boldsymbol{A}$.

Recall that 1 and 0.95 are the eigenvalues of $\boldsymbol{A}$ and

$$E_1 = \text{span}\left\{\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right\}$$

$$E_{0.95} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

Let $\boldsymbol{P} = \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}$.

Then $\boldsymbol{P}$ is invertible (check) and

$$\boldsymbol{P}^{-1}\boldsymbol{AP} = \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0.96 & 0.01 \\ 0.04 & 0.99 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0.95 \end{pmatrix}$$

EXAMPLE

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

So $\boldsymbol{B}$ is diagonalizable and $\boldsymbol{P}$ diagonalizes $\boldsymbol{B}$.

Recall that 3 and 0 are the eigenvalues of $\boldsymbol{B}$ and

$$E_3 = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \quad E_0 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

Let $\boldsymbol{P} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}$.

Then $\boldsymbol{P}$ is invertible (check) and $\boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

EXAMPLE

$$C = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

So C is diagonalizable and P diagonalizes C.

Recall that $1, \sqrt{2}$ and $-\sqrt{2}$ are the eigenvalues of C and

$$E_1 = \text{span}\left\{\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}\right\} \qquad E_{\sqrt{2}} = \text{span}\left\{\begin{pmatrix} -1 \\ \sqrt{2} \\ 1 \end{pmatrix}\right\}$$

Let $P = \begin{pmatrix} -2 & -1 & -1 \\ 2 & \sqrt{2} & -\sqrt{2} \\ 1 & 1 & 1 \end{pmatrix}$.

$$E_{-\sqrt{2}} = \text{span}\left\{\begin{pmatrix} -1 \\ -\sqrt{2} \\ 1 \end{pmatrix}\right\}$$

Then P is invertible (check) and

$$P^{-1}CP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}$$

EXAMPLE

What about $\boldsymbol{M}$e?

$$\boldsymbol{M} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \qquad E_2 = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$

We will now show that $\boldsymbol{M}$ is not diagonalizable.

Suppose $\boldsymbol{M}$ is diagonalizable. Then there exists an invertible matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

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$$\Leftrightarrow \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \Leftrightarrow \begin{cases} \boxed{2a = \lambda a \quad (1)} \\ 2b = \mu b \\ \boxed{a + 2c = \lambda c \quad (2)} \\ b + 2d = \mu d \end{cases}$$

If $a \neq 0$, then $(1) \Rightarrow \lambda = 2$

but now $(2) \Rightarrow a = 0$.

$\boxed{\text{So } a = 0, \lambda = 2.}$

$\boxed{\text{Similarly, } b = 0, \mu = 2.}$

What about $\boldsymbol{M}$e?

EXAMPLE

$$\boldsymbol{M}=\begin{pmatrix}2&0\\1&2\end{pmatrix}\qquad E_2=\text{span}\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}$$

We will now show that $\boldsymbol{M}$ is not diagonalizable.

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}^{-1}\begin{pmatrix}2&0\\1&2\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}\lambda&0\\0&\mu\end{pmatrix}$$

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{pmatrix}0&0\\c&d\end{pmatrix}$$ which is singular, a contradiction.

Is there a more efficient way of showing a matrix is not diagonalizable?

So $\boldsymbol{M}$ is not diagonalizable.

THEOREM

Let $\boldsymbol{A}$ be a square matrix of order n. Then $\boldsymbol{A}$ is diagonalizable if and only if $\boldsymbol{A}$ has n linearly independent eigenvectors.

It is important to emphasize the n eigenvectors have to be linearly independent since

E_λ contains ALL the eigenvectors of $\boldsymbol{A}$ associated with λ.

that is, $\boldsymbol{A}$ already has infinitely eigenvectors associated with a particular eigenvalue λ.

AN ALGORITHM

Purpose: Given a square matrix $\boldsymbol{A}$ of order n, we want to determine whether $\boldsymbol{A}$ is diagonalizable.

If $\boldsymbol{A}$ is diagonalizable, find an invertible matrix $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$ is a diagonal matrix.

Step 1: Solve $\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = 0$ to find all eigenvalues of $\boldsymbol{A}$

$\lambda_1, \lambda_2, \ldots, \lambda_k$ (suppose $\boldsymbol{A}$ has k distinct eigenvalues, $k \le n$)

Step 2: For each λ_i find a basis S_{λ_i} for the eigenspace E_{λ_i}.

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Step 2: For each λ_i find a basis S_{λ_i} for the eigenspace E_{λ_i}.

Step 3: Let $S = S_{\lambda_1} \cup S_{\lambda_2} \cup \ldots \cup S_{\lambda_k}$ (the union of all bases)

(a) If $|S| < n$, then $\boldsymbol{A}$ is not diagonalizable.

$|S| =$ number of vectors in S

(b) If $|S| = n$, say $S = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$, then let

$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_n \end{pmatrix}$ to be the matrix that diagonalizes $\boldsymbol{A}$.

(c) If $|S| > n$, check your working!

SUMMARY

1) Definition of a diagonalizable matrix.

2) A necessary and sufficient condition for a $n \times n$ matrix to be diagonalizable.

3) An algorithm to

(a) determine if a matrix $\boldsymbol{A}$ is diagonalizable

and if it is

(b) find a matrix $\boldsymbol{P}$ that diagonalizes $\boldsymbol{A}$.