NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

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| **Module:** | MA1508E Linear Algebra for Engineering |
| **Year/Semester:** | 2018-2019 (Semester 2) |
| **Practice Problem Set:** | 4 |

**Name:**

**Matriculation Number:**

**Tutorial Group:**

Write down the solutions to the problems below, showing all working involved (imagine it is an examination question). Hand in your answers **together with this question paper** before you leave the classroom. You may refer to any materials while answering the questions. You may also discuss with your friends but do not **copy blindly**.

1. Let $\boldsymbol{v_1}, \boldsymbol{v_2}, \cdots, \boldsymbol{v_k}$ be points in $\mathbb{R}^3$ and suppose that for $j = 1, 2, \cdots, k$, an object with mass $m_j$ is located at point $\boldsymbol{v_j}$. Physicists call such objects *point masses.* The total mass of the system of point masses is

$$m = m_1 + m_2 + \cdots + m_k.$$

The *center of gravity* (or *center of mass*) of the system is

$$\boldsymbol{v} = \frac{1}{m}(m_1\boldsymbol{v_1} + m_2\boldsymbol{v_2} + \cdots + m_k\boldsymbol{v_k}).$$

(a) Compute the center of gravity of the system consisting of the following point masses as given in the table.

| Point | Mass |
|---|---|
| $\boldsymbol{v_1} = (5, -4, 3)$ | 2g |
| $\boldsymbol{v_2} = (4, 3, -2)$ | 5g |
| $\boldsymbol{v_3} = (-4, -3, 1)$ | 2g |
| $\boldsymbol{v_4} = (-9, 8, 6)$ | 1g |

(b) Does $\boldsymbol{v}$ belong to $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$?

**Solution:**

(a) As $m = 2+5+2+1 = 10$, so the center of gravity is at

$$\begin{aligned}
\boldsymbol{v} &= \frac{1}{5}\boldsymbol{v_1} + \frac{1}{2}\boldsymbol{v_2} + \frac{1}{5}\boldsymbol{v_3} + \frac{1}{10}\boldsymbol{v_4} \\
&= \frac{1}{5}\begin{pmatrix} 5 \\ -4 \\ 3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} -4 \\ -3 \\ 1 \end{pmatrix} + \frac{1}{10}\begin{pmatrix} -9 \\ 8 \\ 6 \end{pmatrix} \\
&= \begin{pmatrix} \frac{13}{10} \\ \frac{9}{10} \\ \frac{2}{5} \end{pmatrix}
\end{aligned}$$

(b) Yes, since $\boldsymbol{v}$ is a linear combination of $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}$, it belongs to $\text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}\}$.

2. Let $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}, \boldsymbol{v_4}$ be 4 points in $\mathbb{R}^3$.

(a) Suppose for each $j = 1, 2, 3, 4$, an object with mass $m_j$ is located at point $\boldsymbol{v_j}$ as given in the table below. Show that the center of gravity (or *center of mass*) of the system is located at $\boldsymbol{v} = (9/5, 1, 3/5)$.

| Point | Mass |
|---|---|
| $\boldsymbol{v_1} = (2, 1, 0)$ | 2g |
| $\boldsymbol{v_2} = (1, 1, -1)$ | 1g |
| $\boldsymbol{v_3} = (3, 1, 1)$ | 1g |
| $\boldsymbol{v_4} = (1, 1, 3)$ | 1g |

(b) Suppose we are allowed to change the values of $m_j$, $j = 1, 2, 3, 4$ to any **positive integer** values such that $m_1 + m_2 + m_3 + m_4 \leq 15$. Find **all** $(m_1, m_2, m_3, m_4)$ such that the location of the center of gravity remains unchanged.

**Solution:**

(a) The center of gravity is at

$$\begin{aligned}
\boldsymbol{v} &= \frac{1}{m}(m_1\boldsymbol{v_1} + m_2\boldsymbol{v_2} + m_3\boldsymbol{v_3} + m_4\boldsymbol{v_4}) \\
&= \frac{1}{5}(2\boldsymbol{v_1} + \boldsymbol{v_2} + \boldsymbol{v_3} + \boldsymbol{v_4}) \\
&= \frac{1}{5}\left[2\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}\right] \\
&= \frac{1}{5}\begin{pmatrix} 9 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{9}{5} \\ 1 \\ \frac{3}{5} \end{pmatrix}.
\end{aligned}$$

(b) Let the masses be $m_1, m_2, m_3, m_4$ and let $m = m_1+m_2+m_3+m_4$. We would like to determine the values for $m_1, m_2, m_3, m_4$ such that

$$\frac{1}{m}\left[m_1\begin{pmatrix}2\\1\\0\end{pmatrix}+m_2\begin{pmatrix}1\\1\\-1\end{pmatrix}+m_3\begin{pmatrix}3\\1\\1\end{pmatrix}+m_4\begin{pmatrix}1\\1\\3\end{pmatrix}\right]=\begin{pmatrix}\frac{9}{5}\\1\\\frac{3}{5}\end{pmatrix}.$$

$$\Leftrightarrow \begin{pmatrix}2&1&3&1\\1&1&1&1\\0&-1&1&3\end{pmatrix}\begin{pmatrix}m_1\\m_2\\m_3\\m_4\end{pmatrix}=\begin{pmatrix}\frac{9m}{5}\\m\\\frac{3m}{5}\end{pmatrix}.$$

Solving the above linear system,

$$\left(\begin{array}{cccc|c}2&1&3&1&\frac{9m}{5}\\1&1&1&1&m\\0&-1&1&3&\frac{3m}{5}\end{array}\right)\xrightarrow{R_2-\frac{1}{2}R_1}\left(\begin{array}{cccc|c}2&1&3&1&\frac{9m}{5}\\0&\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}&\frac{m}{10}\\0&-1&1&3&\frac{3m}{5}\end{array}\right)\xrightarrow{R_3+2R_2}$$

$$\left(\begin{array}{cccc|c}2&1&3&1&\frac{9m}{5}\\0&\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}&\frac{m}{10}\\0&0&0&4&\frac{4m}{5}\end{array}\right)\xrightarrow[2R_2,\ \frac{1}{4}R_3]{\frac{1}{2}R_1}\left(\begin{array}{cccc|c}1&\frac{1}{2}&\frac{3}{2}&\frac{1}{2}&\frac{9m}{10}\\0&1&-1&1&\frac{m}{5}\\0&0&0&1&\frac{m}{5}\end{array}\right)\xrightarrow[R_2-R_3]{R_1-\frac{1}{2}R_3}$$

$$\left(\begin{array}{cccc|c}1&\frac{1}{2}&\frac{3}{2}&0&\frac{4m}{5}\\0&1&-1&0&0\\0&0&0&1&\frac{m}{5}\end{array}\right)\xrightarrow{R_1-\frac{1}{2}R_2}\left(\begin{array}{cccc|c}1&0&2&0&\frac{4m}{5}\\0&1&-1&0&0\\0&0&0&1&\frac{m}{5}\end{array}\right)$$

So a general solution to the system is

$$\begin{cases}m_1&=&\frac{4m}{5}-2s\\m_2&=&s\\m_3&=&s\\m_4&=&\frac{m}{5}\end{cases}$$

Since $m_1+m_2+m_3+m_4 \leq 15$ and $m_4 = \frac{m}{5}$ is an integer, the possible values of $m$ are $m = 5$, $m = 10$ or $m = 15$.

**Case 1:** $m = 5$ In this case there is only one possible $m_1, m_2, m_3, m_4$, namely $(m_1, m_2, m_3, m_4) = (2, 1, 1, 1)$.

**Case 2:** $m = 10$ In this case the possible values for $m_1, m_2, m_3, m_4$ are

$$\begin{cases}m_1&=&8-2s\\m_2&=&s\\m_3&=&s\\m_4&=&2\end{cases}$$

The possible values for $(m_1, m_2, m_3, m_4)$, are $(6, 1, 1, 2)$, $(4, 2, 2, 2)$ and $(2, 3, 3, 2)$.

**Case 3:** $m = 15$ In this case the possible values for $m_1, m_2, m_3, m_4$ are

$$\begin{cases}m_1&=&12-2s\\m_2&=&s\\m_3&=&s\\m_4&=&3\end{cases}$$

The possible values of $(m_1, m_2, m_3, m_4)$ subjected to are $(10, 1, 1, 3)$, $(8, 2, 2, 3)$, $(6, 3, 3, 3)$, $(4, 4, 4, 3)$ and $(2, 5, 5, 3)$.

There are a total of 9 possible answers.