Week 09 F2F Example Solutions

1. **Example 8.1**

   (a) Put the vectors as rows of a matrix and then perform Gaussian Elimination to find a row-echelon form.

$$\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 2 & -5 & -3 & -2 & 6 \\ 0 & 5 & 15 & -10 & 0 \\ 2 & 1 & 15 & 8 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & -1 & -3 & -2 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

   So a basis for W is $\{(1,-2,0,0,3),(0,-1,-3,-2),(0,0,0,-2,0)\}$.

   (b) The dimension of W is 3.

   (d) Since a row-echelon form of $\boldsymbol{A}$ has no leading entries in coulmns 3 and 5, we add two vectors $(0,0,1,0,0)$ and $(0,0,0,0,5)$ to extend the basis found in (a) to get a basis for $\mathbb{R}^5$.

2. **Example 8.2**

   (a)

$$\begin{pmatrix} 1 & 0 & -2 & -1 & 1 \\ 1 & 1 & -1 & 2 & 0 \\ 3 & 1 & 0 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

   So a basis for the row space of $\boldsymbol{A}$ is $\{(1,0,0,-1,-1),(0,1,0,3,2),(0,0,1,0,1)\}$.

   (b) A basis for the nullspace of $\boldsymbol{A}$ is $\{(1,-3,0,1,0),(1,-2,-1,0,1)\}$.

   (c) It is easy to check that $\boldsymbol{Ax} = \boldsymbol{b}$.

   (d) A general solution is

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad \text{where } s,t \in \mathbb{R}.$$

3. **Example 8.3**

   (a) Easy to check that $\boldsymbol{u_1} \cdot \boldsymbol{u_2} = \boldsymbol{u_1} \cdot \boldsymbol{u_3} = \boldsymbol{u_2} \cdot \boldsymbol{u_3} = 0$. So S is a linearly independent set of 3 vectors in $\mathbb{R}^3$, thus S is an orthogonal basis for $\mathbb{R}^3$.

   (b) $\boldsymbol{w} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + c_3\boldsymbol{u_3}$ where

$$c_1 = \frac{\boldsymbol{w} \cdot \boldsymbol{u_1}}{\boldsymbol{u_1} \cdot \boldsymbol{u_1}} = -\frac{2}{7}, c_2 = \frac{\boldsymbol{w} \cdot \boldsymbol{u_2}}{\boldsymbol{u_2} \cdot \boldsymbol{u_2}} = \frac{1}{7}, c_3 = \frac{\boldsymbol{w} \cdot \boldsymbol{u_3}}{\boldsymbol{u_3} \cdot \boldsymbol{u_3}} = 0.$$