NATIONAL UNIVERSITY OF SINGAPORE
Department of Mathematics

Module: MA1508E Linear Algebra for Engineering
Year/Semester: 2018-2019 (Semester 2)
Tutorial: 7

1. Let

$$\boldsymbol{u_1} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \quad \boldsymbol{u_2} = \begin{pmatrix} -1 \\ 1 \\ 3 \\ 1 \end{pmatrix}, \quad \boldsymbol{u_3} = \begin{pmatrix} 4 \\ 5 \\ -6 \\ -1 \end{pmatrix}.$$

   (a) Show that $\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$ is a linearly independent set. Is $\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$ a basis for $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$? What is the dimension of $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$? Write down a basis for $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\}$.

   (b) Show that $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is a linearly dependent set. What is the dimension of $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$? Write down a basis for $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$.

   (c) Find a vector $\boldsymbol{u_4}$ such that the dimension of $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_4}\}$ is 3.

   (d) Find a basis for $\mathbb{R}^4$ that contains $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$.

2. Let V and W be subspaces of $\mathbb{R}^n$. Suppose S_1 and S_2 are two sets such that $\text{span}(S_1) = V$ and $\text{span}(S_2) = W$. Define the set $V + W$ as

$$V + W = \{\boldsymbol{v} + \boldsymbol{w} \mid \boldsymbol{v} \in V, \boldsymbol{w} \in W\}.$$

   (a) Show that $S_1 \cup S_2$ spans $V + W$, that is, $V + W = \text{span}(S_1 \cup S_2)$. This would establish the result that $V + W$ is always a subspace.

   (b) For each of the following,

      (i) Find S_1 and S_2 that spans V and W respectively. Check if S_1 and S_2 are bases for V and W respectively. What is the dimension of V and W?

      (ii) Write $V + W$ as a linear span. Find a basis for $V + W$ and state its dimension.

      (iii) Is $V \cap W$ a subspace of $\mathbb{R}^n$? Explain your answer. If $V \cap W$ is a subspace, find a basis for $V \cap W$ and state its dimension.

      (1) $V = \{(s, 0) \mid s \in \mathbb{R}\}$, $W = \{(0, t) \mid t \in \mathbb{R}\}$.

      (2) $V = \{(x, y, z) \mid 2x - y + 3z = 0\}$, $W = \{(a, a, a) \mid a \in \mathbb{R}\}$.

      (3) $V = \{(a, b, c, d) \mid a - 2b + c - d = 0 \text{ and } 2a + c + 2d = 0\}$,
      $W = \{(r, 2r, r, -r) \mid r \in \mathbb{R}\}$.

3. For each of the following cases, write down a matrix $\boldsymbol{A}$ with the required property or explain why no such matrix exists.

(a) The column space of $\boldsymbol{A}$ contains vectors $(1,0,0)^T$, $(0,0,1)^T$ and the row space of $\boldsymbol{A}$ contains vectors $(1,1)$, $(1,2)$.

(b) The column space $= \mathbb{R}^4$ and the row space $= \mathbb{R}^3$.

(c) The column space of $2\boldsymbol{A}$ = the row space of $-\boldsymbol{A} = \text{span}\{(1,2,3)\}$.

(d) $\boldsymbol{A}$ is a square matrix of order 2 where the column space of $\boldsymbol{A}$ is the solution space of the homogeneous linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$.

4. In $\mathbb{R}^4$, let X be the subspace of all vectors of the form $(x_1, x_2, 0, 0)$ and let Y be the subspace of all vectors of the form $(0, y_1, y_2, 0)$. What are the dimensions of X, Y, $X \cap Y$, $X + Y$? Find a basis for each of these four subspaces.

5. Is it possible to find two subspaces V and W of $\mathbb{R}^3$ such that $V \cap W = \{\boldsymbol{0}\}$ (meaning that these two subspaces have only the zero vector in common)? Explain your answer.