1. Let a denote a positive constant. If y is the solution of

$$\frac{dy}{dx} = \frac{252}{\left((252)^2 + x^2\right) e^y}$$

with $y(0) = 0$ and $y(a) = 0.8778$, find the value of a. Give your answer correct to the nearest integer.

$$\int_0^a \frac{252}{(252)^2 + x^2}\, dx = \int_0^{0.8778} e^y\, dy$$

$$\tan^{-1}\left(\frac{x}{252}\right)\Big|_0^a = e^y\Big|_0^{0.8778}$$

$$\tan^{-1}\frac{a}{252} = e^{0.8778} - 1$$

$$a = 252 \tan\left(e^{0.8778} - 1\right)$$

$$= 1511.57\ldots$$

$$\approx \underline{\underline{1512}}$$

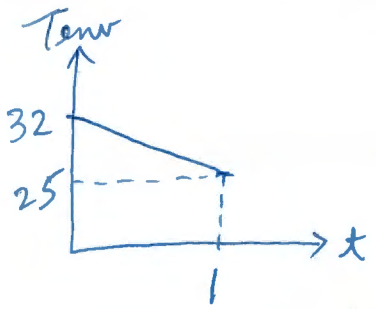
2. Billionaire engineer Tan Ah Lian attributes her success to her ability in making mathematical observations. Once at the beginning of a semester when she was an engineering undergraduate at NUS her chemistry Prof assigned her a project. She was given an unknown amount of an unknown radioactive substance at time $t = 0$ and was told to take measurements of the amount in milligram of that radioactive substance at time t_1, t_2 and t_3 where time is measured in days. Here $0 < t_1 < t_2 < t_3$. She did her project and recorded the amounts as a, b, c respectively in her report. At the end of the semester the Prof asked her to hand in her answer for the value of b. To her horror, when she went home after class that day, her report was nowhere to be found. (She claimed that her mother in law accidentally threw the report out while cleaning her study desk.) Undeterred, she thought hard and was able to recall two mathematical observations that she made while writing up her report. The first was that t_1, t_2 and t_3 formed an arithmetic progression and the second was that the product $ac = 1512$. Based on these two recalled observations she was able to figure out the answer for the value of b which she submitted to her Prof and got an A+ for her project. What was her answer? Give your answer correct to two decimal places.

$$y = Ae^{-kt} \Rightarrow a = Ae^{-kt_1}$$
$$b = Ae^{-kt_2}$$
$$c = Ae^{-kt_3}$$

$$\therefore \frac{b}{a} = e^{-k(t_2-t_1)}, \quad \frac{c}{b} = e^{-k(t_3-t_2)}$$

$$t_1, t_2, t_3 \text{ A.P.} \Rightarrow t_2 - t_1 = t_3 - t_2$$
$$\Rightarrow \frac{b}{a} = \frac{c}{b}$$
$$\therefore b = \sqrt{ac} = \sqrt{1512} = 38.884\ldots$$
$$\approx \underline{\underline{38.88}}$$

3. At time $t = 0$ you brought a cup of hot coffee at 80^oC into a room and turned on the airconditioning at the same time. It is known that the temperature of the coffee T (measured in ^{o}C) at any subsequent time t (measured in hour) satisfies the differential equation $\frac{dT}{dt} = -(T - T_{\text{env}})$ where T_{env} denotes the room temperature at that time. The room temperature was 32^oC at time $t = 0$ and the airconditioning caused it to decrease uniformly to 25^oC at time $t = 1$. What is the temperature of your coffee at time $t = 1$? Give your answer in ^{o}C and correct to two decimal places.



$$T_{env} = 32 - t(32-25)$$
$$= 32 - 7t$$

$$\frac{dT}{dt} + T = 32 - 7t$$

$$R = e^{\int dt} = e^t$$

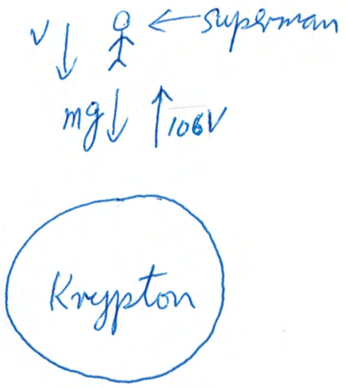
$$T = e^{-t} \int e^t (32 - 7t)\, dt$$
$$= e^{-t} \{ 39e^t - 7te^t + c \}$$
$$= 39 - 7t + ce^{-t}$$

$$80 = 39 + C \Rightarrow C = 41$$

$$T = 39 - 7t + 41e^{-t}$$

$$t = 1 \Rightarrow T = 39 - 7 + 41e^{-1} = 47.083\ldots$$
$$\approx \underline{\underline{47.08}}$$

4. Superman lives on the planet Krypton where the acceleration due to gravity is not constant but varies with time. One day while flying high over Krypton he stops flying suddenly and starts a free fall towards Krypton under the gravitational pull of Krypton. It is known that if we start the clock with $t = 0$ at the moment when he stops flying suddenly then the acceleration due to gravity on Krypton at time t measured in second is given by $g = 10\,(1 + \sin t)$ metre per second square. It is also known that the air resistance on Krypton at a time t seconds after Superman starts his free fall is given by $106v$ Newtons, where v is his velocity measured in metre per second at that time. If the mass of Superman is 106 kg, find the value of v at time $t = \frac{\pi}{4}$ second. Give your answer correct to two decimal places.



$$m\frac{dv}{dt} = mg - 106v$$

$$\frac{dv}{dt} + v = 10(1+\sin t)$$

$$R = e^{\int dt} = e^{t}$$

$$v = e^{-t}\int e^{t}\,10(1+\sin t)\,dt$$

$$= e^{-t}\left\{10e^{t} - 5e^{t}\cos t + 5e^{t}\sin t + c\right\}$$

$$= 10 - 5\cos t + 5\sin t + ce^{-t}$$

$$v(0) = 0 \Rightarrow 0 = 10 - 5 + c \Rightarrow c = -5$$

$$v = 10 - 5\cos t + 5\sin t - 5e^{-t}$$

$$v(\tfrac{\pi}{4}) = 7.720\ldots$$

$$\approx \underline{\underline{7.72}}$$