W05-08

Slide 01: In this unit, we continue our discussion on the dimension of a vector space.

Slide 02: The following is a theorem which will be stated without proof. Let V be a vector space of dimension k and suppose S is a subset of V. The following three statements are logically equivalent, meaning that they are either all true or all false. The first statement is S is a basis for V.

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The second statement states that S is a linearly independent set which contains exactly k vectors.

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The third statement states that S spans the vector space V and it contains exactly k vectors.

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What is the significance of this theorem and how can we use it?

Slide 03: Basically, the theorem allows us to make certain conclusions when we know that the dimension of the vector space is k.

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Firstly, any subset S of V with the correct number of vectors, which is k, and are linearly independent will be a basis for V. Note that we can make this conclusion even though we have not verified that S spans V. Having the correct number of vectors, together with the linear independence property would be good enough to conclude that S is a basis for V.

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Secondly, any subset S of V with the correct number of vectors, which is k, and can be shown to span V will be a basis for V. Note that we can make this conclusion even though we have not verified that S is a linearly independent set. Once again, having the correct number of vectors, together with the knowledge that S spans V would be good enough to conclude that S is a basis for V. In a way, this theorem tells us that if we have knowledge of the dimension of the vector space, verifying whether a set, with the correct number of vectors, is a basis for the vector space or not, can be done by checking either linear independence or linear span without having to do both.

Slide 04: Consider the following example. We would like to show that $\boldsymbol{u_1}, \boldsymbol{u_2}$ and $\boldsymbol{u_3}$ forms a basis for $\mathbb{R}^3$.

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As mentioned in the previous slide, if we go strictly by definition, we would need to show that the three vectors are linearly independent, and they span $\mathbb{R}^3$.

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However, since we know that the dimension of $\mathbb{R}^3$ is 3, meaning that we have the correct number of vectors in $\boldsymbol{u_1}, \boldsymbol{u_2}$ and $\boldsymbol{u_3}$,

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it suffices for us to show either the 3 vectors are linearly independent or they span $\mathbb{R}^3$.

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We will proceed to show that the three vectors are linearly independent.

Slide 05: To do so, we go through the usual procedure of first setting up a vector equation as shown here.

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Writing down the corresponding homogeneous linear system,

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and the augmented matrix,

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we proceed to obtain the reduced row-echelon form of the augmented matrix as shown.

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The conclusion that $\boldsymbol{u_1}, \boldsymbol{u_2}$ and $\boldsymbol{u_3}$ are linearly independent is thus obtained.

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Following what we have discussed earlier, since the dimension of $\mathbb{R}^3$ is 3,

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and 3 linearly independent vectors in $\mathbb{R}^3$ will always form a basis for $\mathbb{R}^3$.

Slide 06: The next theorem relates the dimension of subspaces to the dimension of the vector space where the subspace belongs to. Let U be a subspace of a vector space V. Then the dimension of U does not exceed the dimension of V. Furthermore, if U is not equal to V, then the dimension of U will be strictly smaller than the dimension of V.

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Essentially, given a vector space V

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and a subspace U found inside V,

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the relationship between the dimension of the two spaces is immediate.

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In addition, if U is not equal to V, meaning that there are vectors in V that are not in U, then

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the dimension of U will be strictly smaller than the dimension of V. In other words, the only subspace of V that can have the same dimension as V is V itself.

Slide 07: Consider the following example where V is a plane in $\mathbb{R}^3$ containing the origin. We can treat V as a vector space and the dimension of V is 2.

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Suppose U is a subspace of V and U is not equal to V. Then by the preceding theorem, the dimension of U is strictly less than 2, meaning that it can either be 0 or 1.

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If the dimension of U is 0, then U is the zero subspace of $\mathbb{R}^3$, which is simply the origin, a single point.

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If the dimension of U is 1, then U will be a straight line that passes through the origin in $\mathbb{R}^3$.

Slide 08: To summarise this unit,

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We saw that knowing the dimension of a vector space V helps in determining whether a set S is a basis for V.

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We also saw that the dimension of all subspaces of a vector space V cannot exceed the dimension of V itself.

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In fact, the only subspace of V that has the same dimension as V is V itself.