Definition 0.1: Topological Space

A *topology* $\mathcal{O} \subseteq \mathcal{P}(M)$ on a set M forms a *topological space* $(M, \mathcal{O})$ if it satisfies the following,

(i) $\emptyset, M \in \mathcal{O}$.

(ii) If $S \subseteq \mathcal{O}$ then $\bigcup S \in \mathcal{O}$.

(iii) If $S_1, S_2 \in \mathcal{O}$ then $S_1 \cap S_2 \in \mathcal{O}$.

Terminology 0.1: Open Set

Let $(M, \mathcal{O})$ be topological space, then a subset $U \subseteq M$ is called open if $U \in \mathcal{O}$.

Terminology 0.2: Open Neighbourhood

Let $(M, \mathcal{O})$ be topological space, then a subset $U \subseteq M$ is an open neighbourhood around $p \in M$ if it contains p and is open.

$$p \in U \in \mathcal{O}$$

Definition 0.2: Connected Space

Let $(M, \mathcal{O})$ be a topological space. The space is connected if:

$$\nexists U, V \in \mathcal{O} : U \cap V = \emptyset, U \cup V = M \text{ and } U, V \neq M, \emptyset$$

Proposition 0.1

Let $(M, \mathcal{O})$ be topological space and $U, V \in \mathcal{O}$ such that:

$$U \cup V = M \qquad U \cap V = \emptyset \qquad V, U \neq M, \emptyset$$

Then, $V = U^c$.

Proof. Let $(M, \mathcal{O})$ be a topological space. Assume that U, V are open subsets satisfying the following conditions:

$$U \cup V = M \qquad U \cap V = \emptyset \qquad V, U \neq M, \emptyset$$

Assume for the sake of contradiction that $U^c \neq V$. From this we can conclude that either, there exists an element $x \in U^c$ such that $x \notin V$, or there exists an element $y \in V$ such that $x \notin U^c$. First consider the case where $x \in U^c$ and $x \notin V$. We can further conclude that $x \notin U$, and so $x \notin U \cup V$. Since there exists an element in M namely x that isn't in $U \cup V$ we can conclude that $M \neq U \cup V$, a contradiction. Now, consider the second case where $y \in V$ and $y \notin U^c$. Since $y \notin U^c$, then $y \in U$. Further we conclude that $y \in U \cap V$, showing that $U \cap V \neq \emptyset$, a contradiction. Since either option leads to a contradiction we can conclude that our initial assumption is wrong, and that $U^c = V$. *Quick maths*

Definition 0.3: Convergence Sequence

Let $(M, \mathcal{O})$ be a topological space, a sequence $a : \mathbb{N} \to M$ converges to a point $p \in M$ if:

$$\forall U \in \mathcal{O}(p \in U \implies \exists N \in \mathbb{N} : \forall n \geq N : a_n \in U)$$

This is usually denoted $a_n \to p$.

Definition 0.4: Hausdorff Space

A topological space $(M, \mathcal{O})$ is Hausdorff if:

$$\forall p, q \in M(p \neq q \implies \exists U, V \in \mathcal{O} : p \in U, q \in V \text{ and } U \cap V = \emptyset)$$

Proposition 0.2: Unique Convergence in Hausdorff Spaces

Let $(M, \mathcal{O})$ be a Hausdorff space and assume a sequence $a : \mathbb{N} \to M$ converges to points p and q in M. Then p and q must be equal.

$$a_n \to p \text{ and } a_n \to q \implies p = q$$

Proof. Let $(M, \mathcal{O})$ be a Hausdorff space and assume $a_n \to p$ and $a_n \to q$. Assume for the sake of contradiction that $p \neq q$. By the Hausdorff property there exists open neighbourhoods $U, V \in \mathcal{O}$ such that $p \in U, q \in V$ and $U \cap V = \emptyset$. Since $a_n \to p$ there exists an $N_1 \in \mathbb{N}$ such that, for all $n \geq N_1, a_n \in U$. Similarly, since $a_n \to q$ there exists an $N_2 \in \mathbb{N}$ such that, for all $n \geq N_2, a_n \in V$. Combining these statements we can conclude that for all $n \geq N_1$ and $n \geq N_2, a_n \in U \cap V = \emptyset$. However this is a contradiction, since a_n can not be an element of the emptyset; thus we can conclude that $p = q$. *Quick maths*

Definition 0.5: Continuous Map

Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be topological spaces, then a map $f : M \to N$ is continuous at p if:

$$\forall U \in \mathcal{O}_N \left(f(p) \in U \implies f^{\text{pre}}(U) \in \mathcal{O}_M\right)$$

Definition 0.6: Fields

A *field* $(K, +, \cdot)$ is a nonempty set K, along with two binary operations, addition $+ : K \times K \to K$ and multiplication $\cdot : K \times K \to K$, satisfying the following,

(i) For all $x, y, z \in K$, $(x + y) + z = x + (y + z)$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(ii) For all $x, y \in K$, $x + y = y + x$ and $x \cdot y = y \cdot x$.

(iii) There exists elements $0, 1 \in K$ such that for all $x \in K, x + 0 = x$ and $x \cdot 1 = x$.

(iv) For all $x \in K$, there exists an element $-x \in K$ such that $x + (-x) = 0$, and if $x \neq 0$, there exists an element x^{-1} such that $x \cdot x^{-1} = 1$.

(v) For all $x, y, z \in K$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

Definition 0.7: Vecotor Spaces

A *Vector Space* $(V, +, \cdot)$ over a field $(K, +, \cdot)$ is a set V along with two binary operations, vector addition $+ : V \times V \to V$ and s-multiplication $\cdot : K \times V \to V$, satisfying the following,

(i) For all vectors $v, w, u \in V$, $v + (w + u) = (v + w) + u$.

(ii) For all vectors $v, w \in V$, $v + w = w + v$.

(iii) There exists a vector $\mathbf{0} \in V$ such that $v + \mathbf{0} = v$ for all $v \in V$.

(iv) For every vector $v \in V$ there exists an element $-v$ such that $v + (-v) = \mathbf{0}$.

(v) For all scalars $a, b \in K$ and vector $v \in V$, $a \cdot_V (b \cdot_V v) = (a \cdot_F b) \cdot_V v$.

(vi) For every scalar $a \in K$ and vectors $v, w \in V$, $a \cdot (v + w) = a \cdot v + a \cdot w$.

(vii) For all scalars $a, b \in K$ and vector $v \in V$, $(a + b) \cdot v = a \cdot v + b \cdot v$.

Definition 0.8: The directional derivative

Let $f : \mathbb{R}^n \to \mathbb{R}^m$, for some $n, m \in \mathbb{N}$. Then we define the directional derivative of f along $v \in \mathbb{R}^n$ as:

$$\partial_v f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m := p \mapsto \lim_{h \to 0} \frac{f(p + hv) - f(p)}{h}$$

Notation 0.1

(i) Let $f : \mathbb{R} \to \mathbb{R}^n$ then define $f' := \partial f := \partial_1 f$.

(ii) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and e_i be a basis for $\mathbb{R}^n$ then define $\partial_i f := \partial_{e_i} f$.

(iii) Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and e_i be a basis for $\mathbb{R}^n$ then define $\frac{\partial f}{\partial e_i} := \partial_{e_i} f$.

Example 0.1: Standerd topology on the Reals

Let $\mathbb{R}$ be the set of real numbers, we define the standerd topology $\mathcal{O}_s$ on $\mathbb{R}$.

$$\mathcal{O}_s = \{U \in \mathcal{P}(\mathbb{R}) \mid \forall p \in U : \exists a, b \in U : p \in (a, b) \text{ and } (a, b) \subseteq U\}$$

Definition 0.9

Let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be topological spaces, define the product topology $\mathcal{O}_{M\times N}$ for the set $M \times N$,

$$\mathcal{O}_{M\times N} := \{U \in \mathcal{P}(\mathcal{O}_M \times \mathcal{O}_N) \mid \forall p \in U : \exists A \in \mathcal{O}_M : \exists B \in \mathcal{O}_N : p \in A \times B \subseteq U\}$$

Definition 0.10: Norm Spaces

A *Norm Space* $(V, +, \cdot, \|\cdot\|)$ over a field $(K, +, \cdot)$ is a vector space $(V, +, \cdot)$ over the field $(K, +, \cdot)$ along with a *norm* $\|\cdot\| : V \to \mathbb{R}$ satisfying the following,

(i) For every vector $v \in V$, $\|v\| \geq 0$.

(ii) $\|v\| = 0$ if and only if $v = \mathbf{0}$.

(iii) For every scalar $\lambda \in K$ and vector $v \in V$, $\|\lambda v\| = |\lambda|\|v\|$.

Definition 0.11: Sequence

A *Sequence* a_n is map $a_n : \mathbb{N} \to X$ for some target X. A *Finite Sequence* a_n with a length $L \in \mathbb{N}$ is map $a_n : \{m \in \mathbb{N} \mid m \leq L\} \to X$ for some target X.

Definition 0.12: Cauchy Sequence

Let $(V, +, \cdot, \|\cdot\|)$ be normed space. A Sequence $a_n : \mathbb{N} \to V$ is *Cauchy* if:

$$\forall \varepsilon > 0 : \exists N : \forall m, n \geq N \implies \|x_n - x_m\| < \varepsilon.$$

Meaning that elements in the sequence become closer and closer.