

Definition 0.1: Topological Space

A topology $\mathcal{O} \subseteq \mathcal{P}(M)$ on a set M forms a *topological space* (M, \mathcal{O}) if it satisfies the following,

- (i) $\emptyset, M \in \mathcal{O}$.
- (ii) If $S \subseteq \mathcal{O}$ then $\bigcup S \in \mathcal{O}$.
- (iii) If $S_1, S_2 \in \mathcal{O}$ then $S_1 \cap S_2 \in \mathcal{O}$.

Terminology 0.1: Open Set

Let (M, \mathcal{O}) be topological space, then a subset $U \subseteq M$ is called open if $U \in \mathcal{O}$.

Terminology 0.2: Open Neighbourhood

Let (M, \mathcal{O}) be topological space, then a subset $U \subseteq M$ is an open neighbourhood around $p \in M$ if it contains p and is open.

$$p \in U \in \mathcal{O}$$

Definition 0.2: Convergence Sequence

Let (M, \mathcal{O}) be a topological space, a sequence $a : \mathbb{N} \rightarrow M$ converges to a point $p \in M$ if:

$$\forall U \in \mathcal{O}(p \in U \implies \exists N \in \mathbb{N} : \forall n \geq N : a_n \in U)$$

This is usually denoted $a_n \rightarrow p$.

Definition 0.3: Hausdorff Space

A topological space (M, \mathcal{O}) is Hausdorff if:

$$\forall p, q \in M(p \neq q \implies \exists U, V \in \mathcal{O} : p \in U, q \in V \text{ and } U \cap V = \emptyset)$$

Proposition 0.1: Unique Convergence in Hausdorff Spaces

Let (M, \mathcal{O}) be a Hausdorff space and assume a sequence $a : \mathbb{N} \rightarrow M$ converges to points p and q in M . Then p and q must be equal.

$$a_n \rightarrow p \text{ and } a_n \rightarrow q \implies p = q$$

Proof. Let (M, \mathcal{O}) be a Hausdorff space and assume $a_n \rightarrow p$ and $a_n \rightarrow q$. Assume for the sake of contradiction that $p \neq q$. By the Hausdorff property there exists open neighbourhoods $U, V \in \mathcal{O}$ such that $p \in U, q \in V$ and $U \cap V = \emptyset$. Since $a_n \rightarrow p$ there exists an $N_1 \in \mathbb{N}$ such that, for all $n \geq N_1, a_n \in U$. Similarly, since $a_n \rightarrow q$ there exists an $N_2 \in \mathbb{N}$ such that, for all $n \geq N_2, a_n \in V$. Combining these statements we can conclude that for all $n \geq N_1$ and $n \geq N_2, a_n \in U \cap V = \emptyset$.

However this is a contradiction, since a_n can not be an element of the emptyset; thus we can conclude that $p = q$. *Quick maths*

Definition 0.4: Continuous Map

Let (M, \mathcal{O}_M) and (N, \mathcal{O}_N) be topological spaces, then a map $f : M \rightarrow N$ is continuous at p if:

$$\forall U \in \mathcal{O}_N (f(p) \in U \implies f^{\text{pre}}(U) \in \mathcal{O}_M)$$

Definition 0.5: Fields

A *field* $(K, +, \cdot)$ is a nonempty set K , along with two binary operations, addition $+: K \times K \rightarrow K$ and multiplication $\cdot: K \times K \rightarrow K$, satisfying the following,

- (i) For all $x, y, z \in K$, $(x + y) + z = x + (y + z)$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
- (ii) For all $x, y \in K$, $x + y = y + x$ and $x \cdot y = y \cdot x$.
- (iii) There exists elements $0, 1 \in K$ such that for all $x \in K$, $x + 0 = x$ and $x \cdot 1 = x$.
- (iv) For all $x \in K$, there exists an element $-x \in K$ such that $x + (-x) = 0$, and if $x \neq 0$, there exists an element x^{-1} such that $x \cdot x^{-1} = 1$.
- (v) For all $x, y, z \in K$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

Definition 0.6: Vecotor Spaces

A *Vector Space* $(V, +, \cdot)$ over a field $(K, +, \cdot)$ is a set V along with two binary operations, vector addition $+: V \times V \rightarrow V$ and s-multiplication $\cdot: K \times V \rightarrow V$, satisfying the following,

- (i) For all vectors $v, w, u \in V$, $v + (w + u) = (v + w) + u$.
- (ii) For all vectors $v, w \in V$, $v + w = w + v$.
- (iii) There exists a vector $\mathbf{0} \in V$ such that $v + \mathbf{0} = v$ for all $v \in V$.
- (iv) For every vector $v \in V$ there exists an element $-v$ such that $v + (-v) = \mathbf{0}$.
- (v) For all scalars $a, b \in K$ and vector $v \in V$, $a \cdot_V (b \cdot_V v) = (a \cdot_F b) \cdot_V v$.
- (vi) For every scalar $a \in K$ and vectors $v, w \in V$, $a \cdot (v + w) = a \cdot v + a \cdot w$.
- (vii) For all scalars $a, b \in K$ and vector $v \in V$, $(a + b) \cdot v = a \cdot v + b \cdot v$.

Definition 0.7: The directional derivative

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, for some $n, m \in \mathbb{N}$. Then we define the directional derivative of f along $v \in \mathbb{R}^n$ as:

$$\partial_v f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m := p \mapsto \lim_{h \rightarrow 0} \frac{f(p + hv) - f(p)}{h}$$

Notation 0.1

- (i) Let $f : \mathbb{R} \rightarrow \mathbb{R}^n$ then define $f' := \partial f := \partial_1 f$.
- (ii) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and e_i be a basis for \mathbb{R}^n then define $\partial_i f := \partial_{e_i} f$.
- (iii) Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and e_i be a basis for \mathbb{R}^n then define $\frac{\partial f}{\partial e_i} := \partial_{e_i} f$.

Example 0.1: Standard topology on the Reals

Let \mathbb{R} be the set of real numbers, we define the standard topology \mathcal{O}_s on \mathbb{R} .

$$\mathcal{O}_s = \{U \in \mathcal{P}(\mathbb{R}) \mid \forall p \in U : \exists a, b \in U : p \in (a, b) \text{ and } (a, b) \subseteq U\}$$

Definition 0.8

Let (M, \mathcal{O}_M) and (N, \mathcal{O}_N) be topological spaces, define the product topology $\mathcal{O}_{M \times N}$ for the set $M \times N$,

$$\mathcal{O}_{M \times N} := \{U \in \mathcal{P}(\mathcal{O}_M \times \mathcal{O}_N) \mid \forall p \in U : \exists A \in \mathcal{O}_M : \exists B \in \mathcal{O}_N : p \in A \times B \subseteq U\}$$

Definition 0.9: Norm Spaces

A *Norm Space* $(V, +, \cdot, \|\cdot\|)$ over a field $(K, +, \cdot)$ is a vector space $(V, +, \cdot)$ over the field $(K, +, \cdot)$ along with a *norm* $\|\cdot\| : V \rightarrow \mathbb{R}$ satisfying the following,

- (i) For every vector $v \in V$, $\|v\| \geq 0$.
- (ii) $\|v\| = 0$ if and only if $v = \mathbf{0}$.
- (iii) For every scalar $\lambda \in K$ and vector $v \in V$, $\|\lambda v\| = |\lambda| \|v\|$.

Definition 0.10: Sequence

A *Sequence* a_n is map $a_n : \mathbb{N} \rightarrow X$ for some target X . A *Finite Sequence* a_n with a length $L \in \mathbb{N}$ is map $a_n : \{m \in \mathbb{N} \mid m \leq L\} \rightarrow X$ for some target X .

Definition 0.11: Cauchy Sequence

Let $(V, +, \cdot, \|\cdot\|)$ be normed space. A Sequence $a_n : \mathbb{N} \rightarrow V$ is *Cauchy* if:

$$\forall \varepsilon > 0 : \exists N : \forall m, n \geq N \implies \|x_n - x_m\| < \varepsilon.$$

Meaning that elements in the sequence become closer and closer.