1 Topology

Definition 1.1: Topological Space

A *topology* $\mathcal{O} \subseteq \mathcal{P}(X)$ on a set X forms a *topological space* $(X, \mathcal{O})$ if it satisfies the following,

(i) $\emptyset, X \in \mathcal{O}$.

(ii) If $S \subseteq \mathcal{O}$ then $\bigcup S \in \mathcal{O}$.

(iii) If $S_1, S_2 \in \mathcal{O}$ then $S_1 \cap S_2 \in \mathcal{O}$.

2 Hilbert Spaces

Definition 2.1: Fields

A *field* $(\mathbb{F}, +, \cdot)$ is a nonempty set $\mathbb{F}$, along with two binary operations, addition $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ and multiplication $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$, satisfying the following,

(i) For all $x, y, z \in \mathbb{F}, (x + y) + z = x + (y + z)$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(ii) For all $x, y \in \mathbb{F}$, $x + y = y + x$ and $x \cdot y = y \cdot x$.

(iii) There exists elements $0, 1 \in \mathbb{F}$ such that for all $x \in \mathbb{F}, x + 0 = x$ and $x \cdot 1 = x$.

(iv) For all $x \in \mathbb{F}$, there exists an element $-x \in \mathbb{F}$ such that $x + (-x) = 0$, and if $x \neq 0$, there exists an element x^{-1} such that $x \cdot x^{-1} = 1$.

(v) For all $x, y, z \in \mathbb{F}$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

Definition 2.2: Vecotor Spaces

A *Vector Space* $(V, +, \cdot)$ over a field $(\mathbb{F}, +, \cdot)$ is a set V along with two binary operations, vector addition $+ : V \times V \to V$ and s-multiplication $\cdot : \mathbb{F} \times V \to V$, satisfying the following,

(i) For all vectors $v, w, u \in V$, $v + (w + u) = (v + w) + u$.

(ii) For all vectors $v, w \in V$, $v + w = w + v$.

(iii) There exists a vector $\mathbf{0} \in V$ such that $v + \mathbf{0} = v$ for all $v \in V$.

(iv) For every vector $v \in V$ there exists an element $-v$ such that $v + (-v) = \mathbf{0}$.

(v) For all scalars $a, b \in \mathbb{F}$ and vector $v \in V$, $a \cdot_V (b \cdot_V v) = (a \cdot_F b) \cdot_V v$.

(vi) For every scalar $a \in \mathbb{F}$ and vectors $v, w \in V$, $a \cdot (v + w) = a \cdot v + a \cdot w$.

(vii) For all scalars $a, b \in \mathbb{F}$ and vector $v \in V$, $(a + b) \cdot v = a \cdot v + b \cdot v$.

Definition 2.3: Norm Spaces

A *Norm Space* $(V, +, \cdot, \|\cdot\|)$ over a field $(\mathbb{F}, +, \cdot)$ is a vector space $(V, +, \cdot)$ over the field $(\mathbb{F}, +, \cdot)$ along with a *norm* $\|\cdot\| : V \to \mathbb{R}$ satisfying the following,

(i) For every vector $v \in V$, $\|v\| \geq 0$.

(ii) $\|v\| = 0$ if and only if $v = \mathbf{0}$.

(iii) For every scalar $\lambda \in \mathbb{F}$ and vector $v \in V$, $\|\lambda v\| = |\lambda|\|v\|$.

Definition 2.4: Sequence

A *Sequence* a_n is map $a_n : \mathbb{N} \to X$ for some target X. A *Finite Sequence* a_n with a length $L \in \mathbb{N}$ is map $a_n : \{m \in \mathbb{N} \mid m \leq L\} \to X$ for some target X.

Definition 2.5: Cauchy Sequence

Let $(V, +, \cdot, \|\cdot\|)$ be normed space. A Sequence $a_n : \mathbb{N} \to V$ is *Cauchy* if:

$$\forall \varepsilon > 0 : \exists N : \forall m, n \geq N \implies \|x_n - x_m\| < \varepsilon.$$

Meaning that elements in the sequence become closer and closer.