

#### Definition 0.1: Topological Space

A topology  $\mathcal{O} \subseteq \mathcal{P}(M)$  on a set  $M$  forms a *topological space*  $(M, \mathcal{O})$  if it satisfies the following,

- (i)  $\emptyset, M \in \mathcal{O}$ .
- (ii) If  $S \subseteq \mathcal{O}$  then  $\bigcup S \in \mathcal{O}$ .
- (iii) If  $S_1, S_2 \in \mathcal{O}$  then  $S_1 \cap S_2 \in \mathcal{O}$ .

#### Definition 0.2: Fields

A *field*  $(K, +, \cdot)$  is a nonempty set  $K$ , along with two binary operations, addition  $+: K \times K \rightarrow K$  and multiplication  $\cdot: K \times K \rightarrow K$ , satisfying the following,

- (i) For all  $x, y, z \in K$ ,  $(x + y) + z = x + (y + z)$  and  $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ .
- (ii) For all  $x, y \in K$ ,  $x + y = y + x$  and  $x \cdot y = y \cdot x$ .
- (iii) There exists elements  $0, 1 \in K$  such that for all  $x \in K$ ,  $x + 0 = x$  and  $x \cdot 1 = x$ .
- (iv) For all  $x \in K$ , there exists an element  $-x \in K$  such that  $x + (-x) = 0$ , and if  $x \neq 0$ , there exists an element  $x^{-1}$  such that  $x \cdot x^{-1} = 1$ .
- (v) For all  $x, y, z \in K$ ,  $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ .

#### Definition 0.3: Vector Spaces

A *Vector Space*  $(V, +, \cdot)$  over a field  $(K, +, \cdot)$  is a set  $V$  along with two binary operations, vector addition  $+: V \times V \rightarrow V$  and s-multiplication  $\cdot: K \times V \rightarrow V$ , satisfying the following,

- (i) For all vectors  $v, w, u \in V$ ,  $v + (w + u) = (v + w) + u$ .
- (ii) For all vectors  $v, w \in V$ ,  $v + w = w + v$ .
- (iii) There exists a vector  $\mathbf{0} \in V$  such that  $v + \mathbf{0} = v$  for all  $v \in V$ .
- (iv) For every vector  $v \in V$  there exists an element  $-v$  such that  $v + (-v) = \mathbf{0}$ .
- (v) For all scalars  $a, b \in K$  and vector  $v \in V$ ,  $a \cdot_V (b \cdot_V v) = (a \cdot_F b) \cdot_V v$ .
- (vi) For every scalar  $a \in K$  and vectors  $v, w \in V$ ,  $a \cdot (v + w) = a \cdot v + a \cdot w$ .
- (vii) For all scalars  $a, b \in K$  and vector  $v \in V$ ,  $(a + b) \cdot v = a \cdot v + b \cdot v$ .

#### Definition 0.4: The directional derivative

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , for some  $n, m \in \mathbb{N}$ . Then we define the directional derivative of  $f$  along  $v \in \mathbb{R}^n$  as:

$$\partial_v f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m := p \mapsto \lim_{h \rightarrow 0} \frac{f(p + hv) - f(p)}{h}$$

#### Notation 0.1

- (i) Let  $f : \mathbb{R} \rightarrow \mathbb{R}^n$  then define  $f' := \partial f := \partial_1 f$ .
- (ii) Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $e_i$  be a basis for  $\mathbb{R}^n$  then define  $\partial_i f := \partial_{e_i} f$ .
- (iii) Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $e_i$  be a basis for  $\mathbb{R}^n$  then define  $\frac{\partial f}{\partial e_i} := \partial_{e_i} f$ .

#### Example 0.1: Standard topology on the Reals

Let  $\mathbb{R}$  be the set of real numbers, we define the standard topology  $\mathcal{O}_s$  on  $\mathbb{R}$ .

$$\mathcal{O}_s = \{U \in \mathcal{P}(\mathbb{R}) \mid \forall p \in U : \exists a, b \in U : p \in (a, b) \text{ and } (a, b) \subseteq U\}$$

#### Definition 0.5

Let  $(M, \mathcal{O}_M)$  and  $(N, \mathcal{O}_N)$  be topological spaces, define the product topology  $\mathcal{O}_{M \times N}$  for the set  $M \times N$ ,

$$\mathcal{O}_{M \times N} := \{U \in \mathcal{P}(\mathcal{O}_M \times \mathcal{O}_N) \mid \forall p \in U : \exists A \in \mathcal{O}_M : \exists B \in \mathcal{O}_N : p \in A \times B \subseteq U\}$$

#### Definition 0.6: Norm Spaces

A *Norm Space*  $(V, +, \cdot, \|\cdot\|)$  over a field  $(K, +, \cdot)$  is a vector space  $(V, +, \cdot)$  over the field  $(K, +, \cdot)$  along with a *norm*  $\|\cdot\| : V \rightarrow \mathbb{R}$  satisfying the following,

- (i) For every vector  $v \in V$ ,  $\|v\| \geq 0$ .
- (ii)  $\|v\| = 0$  if and only if  $v = \mathbf{0}$ .
- (iii) For every scalar  $\lambda \in K$  and vector  $v \in V$ ,  $\|\lambda v\| = |\lambda| \|v\|$ .

#### Definition 0.7: Sequence

A *Sequence*  $a_n$  is map  $a_n : \mathbb{N} \rightarrow X$  for some target  $X$ . A *Finite Sequence*  $a_n$  with a length  $L \in \mathbb{N}$  is map  $a_n : \{m \in \mathbb{N} \mid m \leq L\} \rightarrow X$  for some target  $X$ .

Definition 0.8: Cauchy Sequence

Let  $(V, +, \cdot, \|\cdot\|)$  be normed space. A Sequence  $a_n : \mathbb{N} \rightarrow V$  is *Cauchy* if:

$$\forall \varepsilon > 0 : \exists N : \forall m, n \geq N \implies \|x_n - x_m\| < \varepsilon.$$

Meaning that elements in the sequence become closer and closer.