Proofs problems

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Chapter 1

Intuitive Proofs

Fact 1.0.1: The pigeonhole principle

Simple form: If $n + 1$ objects are placed into n boxes, then at least one box has at least two objects in it.
General form: If $kn+1$ objects are placed into n boxes, then at least one box has at least $k+1$ objects in it.

Proposition

If one chooses $n+1$ numbers from $\{1, 2, 3, \ldots, 2n\}$, it is guaranteed that two of the numbers they chose are consecutive.

Proof. TODO *Quick maths*

Proposition

If one selects any $n + 1$ numbers from the set $\{1, 2, \ldots, 2n\}$, then two of the selected numbers will sum to $2n + 1$.

Proof. TODO *Quick maths*

Proposition

If one chooses 31 numbers from the set $\{1, 2, 3, \ldots, 60\}$, then two of the numbers must be relatively prime.

Proof. TODO *Quick maths*

Problem

Determine whether or not the pigeonhole principle guarantees that two students at your school have the same 3-letter initals.

TODO

Chapter 2

Direct proofs

Fact 2.0.1

The sum of integers in an integer, the difference of integers is an integer, and the product of integers is an integer.

Definition 2.0.1: Even and odd integers

- An integer n is even if $n = 2k$ for some integer k;
- An integer n is odd if $n = 2k + 1$ for some integer k.

Fact: Any integer is either even or odd.

Proposition

The sum of an even integer and an odd integer is odd.

Proof. Assume that n is an even integer and that m is an odd integer. By the definition of even and odd numbers $n = 2a$ and $m = 2b + 1$ for some integers a and b. Then,

$$n + m = (2a) + (2b + 1) = 2a + 2b + 1 = 2(a + b) + 1.$$

And since $a+b$ is an integer by Fact 2.0.1, we have shown that $n+m = 2k+1$ where $k = a+b$. Therefore by the definition of an odd integer this means that $a + b$ is odd. *Quick maths*

Proposition

The product of two even integers is even.

Proof. Assume that n and m are even integers. By the definition of an even integer $n = 2a$ and $m = 2b$ for some integers a and b. Then,

$$nm = (2a)(2b) = 4ab = 2(2ab).$$

And since $2ab$ is an integer by Fact 2.0.1, we have shown that $nm = 2k$ where $k = 2ab$. Therefore by the definition of an even integer this means that nm is even. *Quick maths*

Proposition

The product of two odd integers is odd.

Proof. Assume that n and m are odd integers. By the definition of an odd integer this means that $n = 2a + 1$ and $m = 2b + 1$ for some integers a and b. Then,

$$nm = (2a + 1)(2b + 1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1.$$

And since $2ab + a + b$ is an integer by Fact 2.0.1, we have shown that $nm = 2k + 1$ where $k = 2ab + a + b$. Therefore by the definition of an odd integer this means that nm is odd. *Quick maths*

Proposition

The product of an even integer and an odd integer is even.

Proof. Assume that n is an even integer and m is an odd integer. By the definition of an even and odd integer this means that $n = 2a$ and $m = 2b + 1$ for some integers a and b. Then,

$$nm = (2a)(2b + 1) = 4ab + 2a = 2(2ab + a).$$

Since $2ab + a$ is an integer by Fact 2.0.1, we have shown that $nm = 2k$ where $k = 2ab + a$. Therefore by the definition of an even integer this means that nm is even. *Quick maths*

Proposition

An even integer squared is an even integer.

Proof. Assume that n is an even integer. By the definition of an even integer $n = 2a$ for some integer a. Then,

$$n^2 = (2a)^2 = 4a^2 = 2(2a^2).$$

Since $2a^2$ is an integer by Fact 2.0.1, we have shown that $n^2 = 2k$ where $k = 2a^2$. Therefore by the definition of an even integer this means that n^2 is even. *Quick maths*

Definition 2.0.2

A nonzero integer a is said to *divide* an integer b if $b = ak$ for some integer k. When a does divide b, we write "$a \mid b$" and when a does not divide b we write "$a \nmid b$."