Chapter 1

The Reals

Definition 1.0.1: Fields

A *field* $(\mathbb{F}, +, \cdot)$ is a nonempty set $\mathbb{F}$, along with two binary operations, addition $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ and multiplication $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$, satisfying the following,

I. **Closure:** For all $x, y \in \mathbb{F}$, $x + y \in \mathbb{F}$ and $x \cdot y \in \mathbb{F}$.

II. **Associativity:** For all $x, y, z \in \mathbb{F}, (x + y) + z = x + (y + z)$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

III. **Commutativity:** For all $x, y \in \mathbb{F}$, $x + y = y + x$ and $x \cdot y = y \cdot x$.

IV. **Identities:** There exists elements $0, 1 \in \mathbb{F}$ such that for all $x \in \mathbb{F}, x + 0 = x$ and $x \cdot 1 = x$.

V. **Inverses:** For all $x \in \mathbb{F}$, there exists an element $-x \in \mathbb{F}$ such that $x + (-x) = 0$, and if $x \neq 0$, there exists an element x^{-1} such that $x \cdot x^{-1} = 1$.

VI. **Distributive Property:** For all $x, y, z \in \mathbb{F}$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

Notation 1.0.1

Multiplication will usually be written like xy or $(x)(y)$ instead of $x \cdot y$. For example the distributive property would be written as $x(y + z) = xy + xz$.

Notation 1.0.2

Addition of inverse elements will usually be written like $x - y$ instead of $x + (-y)$.

Lemma 1.0.1: unique additive identity

For any field $(\mathbb{F}, +, \cdot)$ there exists only one additive identity.

Proof. Assume $(\mathbb{F}, +, \cdot)$ forms a field, as well assume that 0 and $0'$ are additive identities in $\mathbb{F}$. Recall that this means $x = x + 0 = x + 0'$ for all $x \in \mathbb{F}$. Using these identities as well as commutativity we obtain,

$$0 = 0 + 0' = 0' + 0 = 0'.$$

This shows that any two additive identities must be the equal, completing the proof. *Quick maths*

Lemma 1.0.2: unique multiplicative identity

For any field $(\mathbb{F}, +, \cdot)$ there exists only one multiplicative identity.

Proof. Assume $(\mathbb{F}, +, \cdot)$ forms a field, as well assume that 1 and $1'$ are additive identities in $\mathbb{F}$. Recall that this means $x = 1x = 1'x$ for all $x \in \mathbb{F}$. Using these identities as well as commutativity we obtain,

$$1 = 1 \cdot 1' = 1' \cdot 1 = 1'.$$

This shows that any two multiplicative identities must be equal, completing the proof. *Quick maths*

Lemma 1.0.3: $0x = 0$

For any field $(\mathbb{F}, +, \cdot)$, $0x = 0$ for all $x \in \mathbb{F}$.

Proof. Assume $(\mathbb{F}, +, \cdot)$ forms a field and $x \in \mathbb{F}$. Using the additive identity, distributive property and additive inverses we obtain,

$$0 = x - x = 1x - x = (1 + 0)x - x = 1x + 0x - x = x - x + 0x = 0x.$$

This shows that $0 = 0x$ for all $x \in \mathbb{F}$, completing the proof. *Quick maths*

Lemma 1.0.4: $-x = (-1)x$

For any field $(\mathbb{F}, +, \cdot)$, $-x = (-1)x$ for any $x \in \mathbb{F}$.

Proof. Assume $(\mathbb{F}, +, \cdot)$ forms a field and $x \in \mathbb{F}$. Using properties of a field and the previous lemma we obtain,

$$-x = -x + 0 = -x + 0x = -x + x(1 - 1) = -x + 1x + (-1)x = -x + x + (-1)x = 0 + (-1)x = (-1)x.$$

This shows that $-x = (-1)x$ for all $x \in \mathbb{F}$, completing the proof. *Quick maths*

Lemma 1.0.5: $x = -(-x)$

For any field $(\mathbb{F}, +, \cdot)$, $-(-x) = x$ for any $x \in \mathbb{F}$.

Proof. Assume $(\mathbb{F}, +, \cdot)$ forms a field and $x \in \mathbb{F}$. By the definition of a field there exists an element $-x \in \mathbb{F}$ such that $x - x = 0$. Furthermore, there exists an element $-(-x) \in \mathbb{F}$ such that $-(-x) - x = 0$. Using the properties of a field we obtain,

$$x = x + 0 = x - x - (-x) = -(-x).$$

This shows that $x = -(-x)$ for any $x \in \mathbb{F}$, completing the proof. *Quick maths*

Lemma 1.0.6: $0 = -0$

For any field $(\mathbb{F}, +, \cdot)$, $0 = -0$.

Proof. Assume $(\mathbb{F}, +, \cdot)$ forms a field. Using properties of field as well as previously proven statements we obtain,

$$0 = 0 \cdot 0 = 0(1 - 1) = (1)0 + (-1)0 = 0 - 0 = -0.$$

Quick maths

Definition 1.0.2

An *ordered set* $(S, >)$ is a set S with a relation $>$ called an *ordering* such that,

I. For all $x, y \in S$ either $x > y$, $y > x$ or $x = y$.

II. If $x > y$ and $y > z$ then $x > z$.

We also define relations $\geq, <$ and $\leq$,

I. $x \geq y$ if $x > y$ or $x = y$.

II. $x < y$ if $y > x$.

III. $x \leq y$ if $x < y$ or $x = y$.

Definition 1.0.3: Ordered Fields

A field $(\mathbb{F}, +, \cdot)$ is an *ordered field* $(\mathbb{F}, +, \cdot, >)$ if $(\mathbb{F}, >)$ forms an ordered set satisfying the following,

I. For all $x, y, z \in \mathbb{F}$, if $x > y$ then $x + z > y + z$.

II. For all $x, y \in \mathbb{F}$, if $x > 0$ and $y > 0$ then $xy > 0$.

Notation 1.0.3

We say a number is *positive* if $x > 0$ and *negative* if $x < 0$.

Lemma 1.0.7: $x > y \implies -y > -x$

If $(\mathbb{F}, +, \cdot, >)$ is an ordered field, $x, y \in \mathbb{F}$ and $x > y$ then $-y > -x$.

Proof. Assume that $(\mathbb{F}, +, \cdot, >)$ is an ordered field, $x, y \in \mathbb{F}$ and $x > y$. By the second property of an ordered field $0 > y - x$ furthermore $-y > -x$. This completes the proof. *Quick maths*

Definition 1.0.4: Absolute Value

Suppose $(\mathbb{F}, P, +, \cdot)$ is an ordered field, we define the *absolute value* function $|\cdot| : \mathbb{F} \to \mathbb{F}$ to be,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Proposition 1.0.1: $|x| > 0$

If $(\mathbb{F}, +, \cdot, >)$ is an ordered field, then $|x| > 0$ for all $x \in \mathbb{F}$ where $x \neq 0$. If $x = 0$ then $|x| = 0$, which is clear from the definition.

Proof. Assume $(\mathbb{F}, +, \cdot, >)$ forms an ordered field, $x \in \mathbb{F}$ and $x \neq 0$. By the definition of an ordered field $x > 0$ or $x < 0$.

First consider the case when $x > 0$. Notice, by the definition of an ordered set $x \geq 0$ as well. By the definition of the absolute value $|x| = x$. Thus, $|x| > 0$.

Now, consider the case when $x < 0$. By the definition of the absolute value $|x| = -x$. By lemma 1.0.7 $-x > 0$, as well as $|x| > 0$.

In either case $|x| > 0$ completing the proof. *Quick maths*

Proposition 1.0.2: $|x| = |-x|$

If $(\mathbb{F}, +, \cdot, >)$ is an ordered field, then $|x| = |-x|$ for all $x \in \mathbb{F}$.

Proof. Assume $(\mathbb{F}, +, \cdot, >)$ is an ordered field. Notice that $0 \geq 0$ and thus $|0| = |-0| = 0$. There are two other cases $x > 0$ or $x < 0$. The argument is the same for either case, so we will just consider the case when $x > 0$. By the definition of the absolute value $|x| = x$. Using lemma 1.0.7, we know that $-x < 0$. By the definition of the absolute value $|-x| = -(-x) = x$. This shows that $|x| = |-x|$ for all $x \in \mathbb{F}$, completing the proof. *Quick maths*

Proposition 1.0.3: $|x| \geq x \geq -|x|$

If $(\mathbb{F}, +, \cdot, >)$ is an ordered field, then $|x| \geq x \geq -|x|$ for all $x \in \mathbb{F}$.

Proof. Assume $(\mathbb{F}, +, \cdot, >)$ is ordered field. Notice that this proposition can be split into two, firstly that $|x| \geq x$ and secondly that $x \geq -|x|$. We will first prove that $|x| \geq x$ for all x then we will prove that $x \geq -|x|$ for all x.

Assume that $x \in \mathbb{F}$. By the definition of an ordered field $x \geq 0$ or $x < 0$. First consider the case when $x \geq 0$. By the definition of the absolute value $|x| = x$. Thus, $|x| \geq x$. Now consider the case when $x < 0$. Notice that this is equivalent as saying $0 > x$. By the definition of the absolute value $|x| = -x$. Furthermore, by proposition 1.0.1 $|x| > 0$. Applying the transitive property of an ordered field we get that $|x| \geq x$. Together both cases show that $|x| \geq 0$ for all $x \in \mathbb{F}$, completing the first half of the proof.

Just as before, assume that $x \in \mathbb{F}$. By the definition of an ordered field $x \geq 0$ or $x < 0$. First consider the case when $x \geq 0$. By the definition of the absolute value $|x| = x$. Thus, $|x| \geq 0$. Applying lemmas 1.0.7 and 1.0.6 we obtain that $0 \geq -|x|$. Furthermore, applying the transitive property of an ordered field we get that $x \geq -|x|$. Now consider the case when $x < 0$. By the definition of the absolute value $|x| = -x$. Multiplying both sides by -1 and applying lemmas 1.0.4 and 1.0.5 we get that $-|x| = x$. This shows that $x \geq -|x|$. Together both cases show that $x \geq -|x|$ for all $x \in \mathbb{F}$, completing the second half of the proof. *Quick maths*

Proposition 1.0.4: $|xy| = |x||y|$

If $(\mathbb{F}, +, \cdot, >)$ is an ordered field, then $|xy| = |x||y|$ for all $x, y \in \mathbb{F}$.

Proof. I will do this later, right now I am lazy *Quick maths*

Fact 1.0.1: $|a| \leq b \iff -b \leq a \leq b$

If $(\mathbb{F}, +, \cdot, >)$ is an ordered field, then $|a| \leq b$ if and only if $-b \leq a \leq b$ for all $a, b \in \mathbb{F}$.

Fact 1.0.2: The triangle inequality

If $(\mathbb{F}, +, \cdot, >)$ is an ordered field, then $|a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{F}$.

Fact 1.0.3: The reverse triangle inequality

If $(\mathbb{F}, +, \cdot, >)$ is an ordered field, then $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{F}$.

Definition 1.0.5: Metric on ordered fields

If $(\mathbb{F}, +, \cdot, >)$ is an ordered field, then we define a metric $d : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ on $(\mathbb{F}, +, \cdot, >)$.

$$d(x, y) := |x - y|$$

Lemma 1.0.8: Distance function commutes

If $(\mathbb{F}, +, \cdot, >)$ is an ordered field, then $d(x, y) = d(y, x)$ for all $x, y \in \mathbb{F}$.

Proof. Assume $(\mathbb{F}, +, \cdot, >)$ is an ordered field and that $x, y \in \mathbb{F}$. Then, by proposition 1.0.2,

$$d(x, y) = |x - y| = |y - x| = d(y, x).$$

Quick maths

Definition 1.0.6: Upper and Lower bounds on sets

If X is an ordered set and $Y \subseteq X$, then an *upper bound* b on Y is an element $b \in X$ such that $b > y$ for all $y \in Y$. Suppose b is an upper bound on Y, if for every upper bound b' on Y $b \leq b'$ then b is the *least upper bound* on Y denoted $\sup(Y)$. You can similarly define *lower bounds* and the *greatest lower bound*, greatest lower bounds are denoted with $\inf(X)$.

Definition 1.0.7: Bounded sets

A subset $Y \subseteq X$ of an ordered set X is bounded above if there exists an upper bound on Y and is bounded below if there exists a lower bound on Y, if the set is both bounded above and below we say the set is bounded.

Definition 1.0.8: Least upper bound property

An ordered field $(\mathbb{F}, +, \cdot, >)$ has the least upper bound property if for every subset $E \subseteq \mathbb{F}$ that is bounded above there exists a least upper bound on E in $\mathbb{F}$.

Fact 1.0.4: The existence of $\mathbb{R}$

There exists a unique ordered field $(\mathbb{R}, +, \cdot, >)$ with the least upper bound property.