Chapter 1

The Reals

Definition 1.0.1: Fields

A *field* $(\mathbb{F}, +, \cdot)$ is a nonempty set $\mathbb{F}$, along with two binary operations, addition $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ and multiplication $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$, satisfying the following,

1. **Associativity:** For all $x, y, z \in \mathbb{F}$, $(x + y) + z = x + (y + z)$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
2. **Commutativity:** For all $x, y \in \mathbb{F}$, $x + y = y + x$ and $x \cdot y = y \cdot x$.
3. **Identities:** There exists elements $0, 1 \in \mathbb{F}$ such that for all $x \in \mathbb{F}, x + 0 = x$ and $x \cdot 1 = x$.
4. **Inverses:** For all $x \in \mathbb{F}$, there exists an element $-x \in \mathbb{F}$ such that $x + (-x) = 0$, and if $x \neq 0$, there exists an element x^{-1} such that $x \cdot x^{-1} = 1$.
5. **Distributive Property:** For all $x, y, z \in \mathbb{F}$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.

Definition 1.0.2: Ordered Fields

An *ordered field* $(\mathbb{F}, P, +, \cdot)$ is a *field* $(\mathbb{F}, +, \cdot)$ with a subset $P \subseteq \mathbb{F}$, satisfying the following,

1. If $a, b \in P$, then $a + b \in P$ and $a \cdot b \in P$.
2. If $a \in \mathbb{F}$ and $a \neq 0$, then either $a \in P$ or $-a \in P$, but not both.

Using the subset we define four relations $>, \geq, <$ and $\leq$. Suppose $a, b \in \mathbb{F}$.

1. $a > b$ if $a - b \in P$.
2. $a \geq b$ if $a > b$ or $a = b$.
3. $a < b$ if $b > a \in P$.
4. $a \leq b$ if $a < b$ or $a = b$.