1 Field Axioms

Definition 1. A *field* $\mathbb{F}$ is a nonempty set $\mathbb{F}$, along with two binary operations, addition $+$ and multiplication $\cdot$, satisfying the following axioms.

1. **Additive Closure:** For all $a, b \in \mathbb{F}, a + b \in \mathbb{F}$.
2. **Additive Associativity:** For all $a, b, c \in \mathbb{F}, (a + b) + c = a + (b + c)$.
3. **Additive Identity:** There exists an element $0 \in \mathbb{F}$ such that for all $a \in \mathbb{F}, a + 0 = a$.
4. **Additive Inverse:** For every $a \in \mathbb{F}$, there exists an element $-a \in \mathbb{F}$ such that $a + (-a) = 0$.
5. **Additive Commutativity:** For all $a, b \in \mathbb{F}$, $a + b = b + a$.
6. **Multiplicative Closure:** For all $a, b \in \mathbb{F}$, $a \cdot b \in \mathbb{F}$.
7. **Multiplicative Associativity:** For all $a, b, c \in \mathbb{F}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
8. **Multiplicative Identity:** There exists an element $1 \in \mathbb{F}$ such that for all $a \in \mathbb{F}$, $a \cdot 1 = a$.
9. **Multiplicative Inverse:** For every non-zero element $a \in \mathbb{F}$, there exists an element $a^{-1} \in \mathbb{F}$ such that $a \cdot a^{-1} = 1$.
10. **Multiplicative Commutativity:** For all $a, b \in \mathbb{F}$, $a \cdot b = b \cdot a$.
11. **Distributive Property:** For all $a, b, c \in \mathbb{F}$, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

2 Ordered Field Axioms

Definition 2. A field $\mathbb{F}$ is an *oredered field*, if there exsits a subet $P \subseteq \mathbb{F}$ such that:

1. **Additive Closure Positives:** For all $a, b \in P, a + b \in P$.
2. **Multiplicative Closure Positives:** For all $a, b \in P, a \cdot b \in P$.
3. **Positive or Negative:** For all non-zero $a \in \mathbb{F}$, either $a \in P$ or $-a \in P$ but never both.

Definition 3. If $\mathbb{F}$ is an oredered field, and $a, b \in \mathbb{F}$, then

1. **Greater than:** $a > b$ if $a - b \in P$.
2. **Less than:** $a < b$ if $b - a \in P$.
3. **Greater than or equal:** $a \geq b$ if $a > b$ or $a = b$.
4. **Less than or equal:** $a \leq b$ if $a < b$ or $a = b$.

Definition 4. If $\mathbb{F}$ is an oredered field, the *absolute value* function $|x| : \mathbb{F} \to \mathbb{F}$ to be

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Field Proofs

Lemma 1. Given a field $\mathbb{F}$ and an element $a \in \mathbb{F}$. $a = -(-a)$.

Proof.

1	$a \in \mathbb{F}$	(**Premise**)
2	$a - a = 0$	(**Additive Inverse : 1**)
3	$a - a - (-a) = 0 - (-a)$	(**Add to both sides : 2**)
4	$a - a - (-a) = -(-a)$	(**Additive Identity : 3**)
5	$a = -(-a)$	(**Additive Inverse : 4**)

$\square$