

# Chapter 1

## The Reals

### Definition 1.0.1: Fields

A *field*  $(\mathbb{F}, +, \cdot)$  is a nonempty set  $\mathbb{F}$ , along with two binary operations, addition  $+: \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$  and multiplication  $\cdot: \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$ , satisfying the following,

1. **Associativity:** For all  $x, y, z \in \mathbb{F}$ ,  $(x + y) + z = x + (y + z)$  and  $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ .
2. **Commutativity:** For all  $x, y \in \mathbb{F}$ ,  $x + y = y + x$  and  $x \cdot y = y \cdot x$ .
3. **Identities:** There exists elements  $0, 1 \in \mathbb{F}$  such that for all  $x \in \mathbb{F}$ ,  $x + 0 = x$  and  $x \cdot 1 = x$ .
4. **Inverses:** For all  $x \in \mathbb{F}$ , there exists an element  $-x \in \mathbb{F}$  such that  $x + (-x) = 0$ , and if  $x \neq 0$ , there exists an element  $x^{-1}$  such that  $x \cdot x^{-1} = 1$ .
5. **Distributive Property:** For all  $x, y, z \in \mathbb{F}$ ,  $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ .

### Definition 1.0.2: Ordered Fields

An *ordered field*  $(\mathbb{F}, P, +, \cdot)$  is a *field*  $(\mathbb{F}, +, \cdot)$  with a subset  $P \subseteq \mathbb{F}$ , satisfying the following,

1. If  $a, b \in P$ , then  $a + b \in P$  and  $a \cdot b \in P$ .
2. If  $a \in \mathbb{F}$  and  $a \neq 0$ , then either  $a \in P$  or  $-a \in P$ , but not both.

Using the subset we define four relations  $>, \geq, <$  and  $\leq$ . Suppose  $a, b \in \mathbb{F}$ .

1.  $a > b$  if  $a - b \in P$ .
2.  $a \geq b$  if  $a > b$  or  $a = b$ .
3.  $a < b$  if  $b > a \in P$ .
4.  $a \leq b$  if  $a < b$  or  $a = b$ .

### Fact 1.0.1

If  $(\mathbb{F}, P, +, \cdot)$  is an ordered field we will assume the following, suppose  $a, b, c \in \mathbb{F}$ .

1. If  $a > b$  then  $a + c > b + c$ .
2. If  $a > b$  and  $b > c$  then  $a > c$ .
3. If  $a > b$  and  $c > 0$  then  $ac > bc$  otherwise if  $c < 0$  then  $ac < bc$ .
4. If  $a \neq 0$  then  $a^2 > 0$ .

Definition 1.0.3: Absolute Value

Suppose  $(F, P, +, \cdot)$  is an ordered field, we define the absolute value function  $|\cdot| : \mathbb{F} \rightarrow \mathbb{F}$  to be,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$