Generalized inverse

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In mathematics, a generalized inverse of a matrix A is a matrix that has some properties of the inverse matrix of A but not necessarily all of them. Formally, given a matrix $A \in \mathbb{R}^{n \times m}$ and a matrix $A^{\mathrm{g}} \in \mathbb{R}^{m \times n}$, A^{g} is a generalized inverse of A if it satisfies the condition $AA^{\mathrm{g}}A = A$.

The purpose of constructing a generalized inverse is to obtain a matrix that can serve as the inverse in some sense for a wider class of matrices than invertible ones. A generalized inverse exists for an arbitrary matrix, and when a matrix has an inverse, then this inverse is its unique generalized inverse. Some generalized inverses can be defined in any mathematical structure that involves associative multiplication, that is, in a semigroup.

Contents

Motivation for the generalized inverse

Consider the linear system

$$Ax = y$$

where A is an $n \times m$ matrix and $y \in \mathcal{R}(A)$, the range space of A. If the matrix A is nonsingular then $x = A^{-1}y$ will be the solution of the system. Note that, if a matrix A is nonsingular

$$AA^{-1}A = A.$$

Suppose the matrix A is singular or $n \neq m$ then we need a right candidate G of order $m \times n$ such that

$$AGy = y.$$

That is Gy is a solution of the linear system $Ax = y$. Equivalently, G of order $m \times n$ such that

$$AGA = A.$$

Hence we can define the generalized inverse as follows: Given a $n \times m$ matrix A, a $m \times n$ matrix G is said to be generalized inverse of A if $AGA = A$.

Construction of generalized inverse

[1]

The following characterizations are easy to verify.

1. If $A = BC$ is a rank factorization, then $G = C_r^- B_l^-$ is a g-inverse of A where C_r^- is a right inverse of C and B_l^- is left inverse of B.
2. If $A = P\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}Q$ for any non-singular matrices P and Q, then $G = Q^{-1}\begin{bmatrix} I_r & U \\ W & V \end{bmatrix}P^{-1}$ is a generalized inverse of A for arbitrary U, V and W.
3. Let A be of rank r. Without loss of generality, let

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}.$$

where $B_{r\times r}$ is the non-singular submatrix of A.
Then,

$$G = \begin{bmatrix} B^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$ is a g-inverse of A.

Types of generalized inverses

The Penrose conditions are used to define different generalized inverses: for $A \in \mathbb{R}^{n\times m}$ and $A^{\mathrm{g}} \in \mathbb{R}^{m\times n}$,

1.) $AA^{\mathrm{g}}A = A$
2.) $A^{\mathrm{g}}AA^{\mathrm{g}} = A^{\mathrm{g}}$
3.) $(AA^{\mathrm{g}})^{\mathrm{T}} = AA^{\mathrm{g}}$
4.) $(A^{\mathrm{g}}A)^{\mathrm{T}} = A^{\mathrm{g}}A$.

If A^{g} satisfies condition (1.), it is a generalized inverse of A, if it satisfies conditions (1.) and (2.) then it is a generalized reflexive inverse of A, and if it satisfies all 4 conditions, then it is a Moore–Penrose pseudoinverse of A.

Other various kinds of generalized inverses include

- One-sided inverse (left inverse or right inverse) If the matrix A has dimensions $n \times m$ and is full rank then use the left inverse if $n > m$ and the right inverse if $n < m$
  - Left inverse is given by $A_{\text{left}}^{-1} = \left(A^{\mathrm{T}}A\right)^{-1}A^{\mathrm{T}}$, i.e. $A_{\text{left}}^{-1}A = I_m$ where I_m is the $m \times m$ identity matrix.
  - Right inverse is given by $A_{\text{right}}^{-1} = A^{\mathrm{T}}\left(AA^{\mathrm{T}}\right)^{-1}$, i.e. $AA_{\text{right}}^{-1} = I_n$ where I_n is the $n \times n$ identity matrix.
- Drazin inverse
- Bott–Duffin inverse
- Moore–Penrose pseudoinverse

Uses

Any generalized inverse can be used to determine if a system of linear equations has any solutions, and if so to give all of them.[2] If any solutions exist for the n × m linear system

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

with vector $\boldsymbol{x}$ of unknowns and vector b of constants, all solutions are given by

$$\boldsymbol{x} = \boldsymbol{A}^{\mathrm{g}}\boldsymbol{b} + [\boldsymbol{I} - \boldsymbol{A}^{\mathrm{g}}\boldsymbol{A}]\boldsymbol{w}$$

parametric on the arbitrary vector w, where $\boldsymbol{A}^{\mathrm{g}}$ is any generalized inverse of $\boldsymbol{A}$. Solutions exist if and only if $\boldsymbol{A}^{\mathrm{g}}\boldsymbol{b}$ is a solution - that is, if and only if $\boldsymbol{A}\boldsymbol{A}^{\mathrm{g}}\boldsymbol{b} = \boldsymbol{b}$.

See also

- Inverse element

References

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