1 Question 1

Determine Examine the transformations below and Determine whether they are linear. Justify this by the definition as given in class, showing that it does indeed hold, or showing where on which condition things break down.

1.1 Part a

$T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(\mathbf{x}) = a\mathbf{x} + \mathbf{b} \quad \forall b \in \mathbf{R}^{n\times n}$

Checking homogeneity

Left hand side $T(c\mathbf{x}) = \mathbf{a}(c\mathbf{x}) + \mathbf{b} = c(\mathbf{ax}) + \mathbf{b}$

Right hand side $cT(\mathbf{x}) = c(\mathbf{ax} + \mathbf{b}) = c(\mathbf{ax}) + c\mathbf{b}$

Conclusion Homogeneity breaks as the two sides are not equivalent. The relationship is not a linear transormation.

1.2 Part B

$T : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ by $T(\mathbf{x}) = \mathbf{Ax} - \mathbf{xA}$

Checking homogeneity

Left hand side $cT(\mathbf{x}) = c(\mathbf{Ax} - \mathbf{xA}) = c\mathbf{Ax} - c\mathbf{xA}$

Right hand side $T(c\mathbf{x}) = \mathbf{A}(c\mathbf{x}) - c\mathbf{xA}$

Conclusion Homogeneity checks out. We can bubble the c outwards and equate the two sides.

Checking additivity

Left hand side $T(\mathbf{u}+\mathbf{v}) = \mathbf{A}(\mathbf{u}+\mathbf{v}) - (\mathbf{u}+\mathbf{v})\mathbf{A} = \mathbf{Au} + \mathbf{Av} - \mathbf{uA} - \mathbf{vA}$

Right hand side

$$(\mathbf{u}) + T(\mathbf{v}) = (\mathbf{Au} - \mathbf{uA}) + (\mathbf{Av} - \mathbf{vA}) = \mathbf{Au} + \mathbf{Av} - \mathbf{uA} - \mathbf{vA}$$

Conclusion Unlike above, additivity checks out. I've seen some other places have more requirements, such as $\langle 0, 0, \dots \rangle \to \langle 0, 0, \dots \rangle$ but all of those follow from homogeneity and additivity. So, it is a linear transformation.

1.3 Part C

$\phi : \mathbb{C} \to \mathbb{C}$ by $\phi(a + bi) = b + ai$ and $c \in \mathbb{C}$ where $c = d + ei$

Checking homogeneity

Left hand side $\phi(c[a + bi]) = \phi(ca + bi) = (d + ei)(a + bi) = ad - be$

Right hand side $c\phi(a + bi) = c[a + bi] = (d + ei)(a + bi) = ca + bi$

Conclusion Homogeneity breaks. Having done this incorrectly before though, with a non-complex scalar it will work. The relationship given is not a linear transformation. I want to check additivity just out of curiosity.

Examining additivity too Let $c = d + ei$ and $f = g + hi$ as $c,\ f \in \mathbb{C}$.

Left hand side $\phi(c + f) = \phi([d + ei] + [g + hi]) = \phi([d + g] + [e + h]i) = d + g - e - h$

Right hand side $\phi(c) + \phi(f) = \phi(d + ei) + \phi(g + hi) = e + di + h + gi$

Conclusion Guess it doesn't work here either. As $\phi(c + f) \neq \phi(c) + \phi(f)$... well, atleast it isn't always true, the transformation isn't linear.

2 Question 2

Considering a matrix and finding $col(\mathbf{A})$, $row(\mathbf{A})$, $rank(\mathbf{A})$, $nul(\mathbf{A})$ We're to develop a basis for each, and I'm going to recite definitions as we go so they're a little fresher.

Given First, let's RREF this bad boi, knock it's numbers around a little

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 4 & 4 & 0 \\ 1 & -1 & 3 & 9 & 11 \\ 2 & 5 & -1 & -3 & -13 \\ 0 & 6 & -6 & 4 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2.1 col(A)

Definition The column space is the set of all possible linear combinations of a matrices column vectors.

$$\mathrm{col}(\mathbf{a}) = \mathrm{span}(\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n)$$

Basis I feel that we could be cheaty for all these, as, simply giving each column a coefficient would still generate a the same space, however, if we look at the terms of the reduced matrix we see that only three rows actually start with a one. So, disreguarding the 3rd and 5th columns we can more efficiently generate the space with

$$\text{col}(\mathbf{A}) = c_1 \begin{bmatrix} 3 \\ 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \\ 5 \\ 6 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 9 \\ -3 \\ 4 \end{bmatrix}$$

2.2 row(A)

Definition The row space is the set of all possible linear combinations of a matrices row vectors.

$$\text{row}(\mathbf{A}) = \text{span}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n)$$

Basis Likewise as before, we can ditch those rows that have no leading ones.

$$\text{row}(\mathbf{A}) = c_1 \begin{bmatrix} 3 \\ 2 \\ 4 \\ 4 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 3 \\ 9 \\ 11 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 5 \\ -1 \\ -3 \\ -13 \end{bmatrix}$$

I think I should be writing this a little differently, like, $basis_{\text{row}(\mathbf{A})} = \ldots$ but that's a niggling issue for another day.

2.3 column(A)

Definition The column space is the set of all possible linear combinations of a matrices column vectors.

$$\text{row}(\mathbf{A}) = \text{span}(\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n)$$

Ran in to a little uncertainty here that I plan to clarify sometime I get the time. The prevailing wisdom is that I should take only those original column vectors that have corresponding pivots in the reduced matrix. We had pivots in the 1st, 2nd, and 4th columns, so if we retain only those we can create a basis like so.

Basis

$$\text{col}(\mathbf{A}) = c_1 \begin{bmatrix} 3 \\ 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \\ 5 \\ 6 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 9 \\ -3 \\ 4 \end{bmatrix}$$

We could have just as easily enumerated all the possible column vectors, but I'm reasonably confident in the way I'm interepreting the reduced form. Do need to double check and better understand why though.

2.4 null(A)

Definition The null space is the of all possible linear combinations that, when multiplied by the original matrix, result in the zero vector

$$\text{null}(\mathbf{A}) = \{\mathbf{x} \mid \mathbf{Ax} = \mathbf{0}\}$$

Starting with our reduced form, we can augment said matrix with the zero vector and solve

Solution We imagine a column of zero's along the right side of the previous. We have two free, so, choose $x_3 = 0$ and $x_5 = 1$ so that we don't end up with a trivial solution. Doing this we get… $x_1 = 0$, $x_2 = 2$, $x_4 = -1$ We still need one more though, with two free variables. So, for the other, let $x_3 = 1$ and $x_5 = 0$. We get $x_4 = 0$, $x_1 = -2$, $x_2 = 1$.

Basis

$$\text{null}(\mathbf{A}) = c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

2.5 rank(A)

We've already done all the heavy lifting here. We know that $\text{rank}(\mathbf{A}) = 3$ from way back. It was the number of pivots in our reduced form.