MATH 620 HW 10 1) 2)

1) Fit the points $\{(1,2),(2,2),(3,4)\}$

$$\begin{bmatrix}1&1\\1&2\\1&3\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}2\\2\\4\end{bmatrix}\Rightarrow A^TAX=A^Tb$$

$$\text{solve}\left(\begin{bmatrix}3x+6y=8\\6x+14y=18\end{bmatrix},x\right)= \begin{matrix}x=2/3\\y=1\end{matrix}$$

- We were trying to fit a line to minimize $\|e\|^2$ where e is given by

$$e=\begin{bmatrix}\varepsilon_1\\\vdots\\\varepsilon_n\end{bmatrix} \text{ by } \varepsilon_i=y_i-(a+bx_i),\ e=b-Ax$$

- This gives $y=a+bx \Rightarrow y=2/3+(1)x=2/3+x$

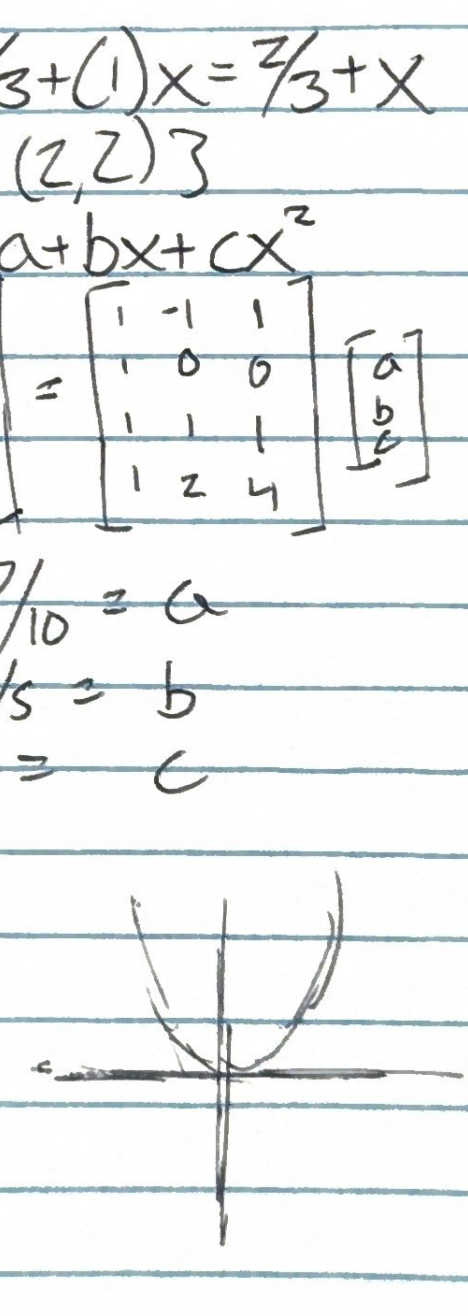
2) Fit the points $\{(-1,1),(0,-1),(1,0),(2,2)\}$ with a parabola given by $y=a+bx+cx^2$

$$\cancel{\begin{bmatrix}1&-1\\1&0\\1&1\\1&2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}1\\-1\\0\\2\end{bmatrix}}\quad \begin{bmatrix}1&x_1&x_1^2\\1&x_2&x_2^2\\1&x_3&x_3^2\\1&x_4&x_4^2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}1\\-1\\0\\2\end{bmatrix}=\begin{bmatrix}1&-1&1\\1&0&0\\1&1&1\\1&2&4\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}4x+2y+6z=2\\2x+6y+8z=3\\6x+8y+18z=9\end{bmatrix}\quad \begin{matrix}x=-7/10=a\\y=-3/5=b\\z=1=c\end{matrix}$$

- So, our fit would be

$$y=a+bx+cx^2=-7/10-3/5x+x^2$$



MATH 626 HW 10 3) a)

3) <u>Definition</u>: if A is a matrix w/ lin. indep. columns then the pseudo-inverse of A, A^+, is given by $A^+ = (A^TA)^{-1}A^T$, thus the solution to the least squares problem $A\mathbf{x} = \mathbf{b}$ w/ a rectangular matrix A is given by $\mathbf{x} = A^+\mathbf{b}$

- Show that if A is a square matrix w/ lin. indep. columns, $A^+ = A^{-1}$
- <u>From</u> the Matrix Inversion Theorem, we know that for a square matrix A w/ lin. indep. columns, A is invertible, as by rank-nullity

$$\dim(col(A)) + \dim(null(A)) = n$$

<u>Proof</u>

Given a square matrix A w/ lin. indep. cols. we know A^{-1} and $(A^T)^{-1}$ exist by rank-nullity. By definition of Pseudo-Inverse we have

$$A^+ = (A^TA)^{-1}A^T, \text{ by the reverse order law}$$

$$A^+ = A^{-1}(A^T)^{-1}A^T \text{ by assoc.}$$

$$A^+ = A^{-1}[(A^T)^{-1}A^T], \text{ by prop of identity}$$

$$A^+ = A^{-1}I_n, \text{ by idempo.}$$

$A^+ = A^{-1}$. as was wanted, we have that, the pseudo-inverse is equivalent

MATH 620 HW 10 3) b)

3) Defn: If A is a matrix w/ lin. indep. cols. Then the pseudo-inverse is given by $A^+ = (A^TA)^{-1}A^T$, thus the solution to the least squares problem $Ax = b$ w/ rect. mat. A is $x = A^+b$.

Theorem: Let A be a matrix w/ lin. indep. cols., then the pseudo-inverse satisfies the following properties, Penrose conditions,

$\underbrace{AA^+A = A}_{\#1}$ $\underbrace{A^+AA^+ = A^+}_{\#2}$ $\underbrace{AA^+ \text{ \& } A^+A \text{ are symm.}}_{\#3}$

Proof: Let A be a matrix w/ lin. indep. cols. w/ the pseudo-inverse given by $A^+ = (A^TA)^{-1}A^T$ and $(AB)^T = B^TA^T$ also as given.

#1 #2

$AA^+A = A$ $A^+AA^+ = A^+$

by sub. $A[(A^TA)^{-1}A^T]A = A$, $[(A^TA)^{-1}A^T]A[(A^TA)^{-1}A^T] = A^+$, by sub.

by assoc. $A[(A^TA)^{-1}][A^TA] = A$, $[(A^TA)^{-1}(A^TA)]I[(A^TA)^{-1}A^T] = A^+$, by assoc

by iden. $AI = A$ $I[(A^TA)^{-1}A^T] = A^+$, by identity

by idempo. $A = A$ $A^+ =$, by defn.

#3

$AA^+ = A[(A^TA)^{-1}A^T]$ $(AA^+)^T = [A(A^TA)^{-1}A^T]^T$ by sub.

so, $AA^+ = (AA^+)^T$ is symmetric by definition. $= (A^T)^T[(A^TA)^{-1}]^TA^T$ by transp.

$= A[(A^TA^{-1})]^TA^+$ by prop exp.

$A^+A = (A^+A)^T$ $= A(A^TA)^{-1}A^T$ by transp.

$\hookrightarrow = (A^TA)^{-1}A^TA$ $= AA^+$ by defn.

by sub. $= (A^TA)^{-1}(A^TA)$, and, as $A^+A = (A^+A)^T$, it

by assoc $= I = I^T = (A^+A)^T$ is symmetric by definition

by identity too.

MATH 620 HW 10 4)

4) Find the least squares solutions for

a) $$\begin{bmatrix} 1 & -2 \\ 0 & -3 \\ 2 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -2 \\ 4 \end{bmatrix} \Rightarrow A^TAx = A^Tb$$

- gives $a = 4/3$, $b = -5/6$

$$y = a + bx \Rightarrow y = 4/3 - 5/6x$$

b) $$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & -1 & 1 & 0 \end{bmatrix} \Rightarrow A^TAx = A^Tb,\ b = \begin{bmatrix} 1 \\ -3 \\ 2 \\ 4 \end{bmatrix}$$

c is free, so let x be of the form

$$x = \begin{bmatrix} (-c+3)/2 \\ (c-5)/2 \\ c \\ (-c-5)/2 \end{bmatrix},\quad w/\ c=1 \Rightarrow \begin{bmatrix} (-1+3)/2 \\ (1-5)/2 \\ 1 \\ (-1-5)/2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ -3 \end{bmatrix}$$

so we'd have a family of equivalently fittable quadratics, but w/ $c=1$ we'd get

$$y = 1 - 2x + x^2 - 3x^3$$

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(5)

given
$$\begin{aligned} x+y-z&=2 \\ -y+2z&=6 \\ 3x+2y-z&=11 \\ -x\quad +z&=0 \end{aligned}$$
, we're trying to minimize $\|\mathbf{b}-A\mathbf{x}\|$, so back to using $A^TA\mathbf{x}=A^T\mathbf{b}$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 3 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 11 \\ 0 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 3 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 2 \\ 6 \\ 11 \\ 0 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$A^TA = \begin{bmatrix} 11 & 7 & -5 \\ 7 & 6 & -5 \\ -5 & -5 & 7 \end{bmatrix}, \quad A^T\mathbf{b} = \begin{bmatrix} 35 \\ 18 \\ -1 \end{bmatrix}$$

$$x = 42/11, \ y = 19/11, \ z = 42/11$$

so the best approximation to the solution would be

$$\mathbf{x} = \begin{bmatrix} 42/11 \\ 19/11 \\ 42/11 \end{bmatrix}$$

MATH 620 HW 10 6)

6) Let $(x_1, y_1) \ldots (x_n, y_n)$ be given Show that if they do not ~~all~~ lie on the same vertical line, then they have a unique squares approx. line.

Proof

Let's borrow from what we were given initially and go from there

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

. We know that we need lin. indep columns, so, let's represent A as V_1 & V_2.

Let $A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$. We'd need $k \in \mathbb{R}$ such that $V_2 = kV_1$. Implying $V_{2,n} = kV_{1,n}$ $\forall n \in \mathbb{N}$.

We can do this a little differently than the obvious... we know $\dim(\text{null}(A))$ must be $\emptyset$

$$a \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + b \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow a + bx_n = 0 \quad \forall n \in \mathbb{N}$$

if $b = 0$ then $a = 0$, so we'd only have the trivial solution as expected. However if $b \neq 0$ then $x_n = -a/b$, so x_n would be a constant. So, in order to have $\text{null}(A) = \emptyset$, all x_n must be the same, and by contrapositive, x_n must be different for a unique solution (not on a vertical line).