1 Question 1

Given

$$\text{Given } \vec{v_3} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \text{ is } \vec{v_3} \in \text{span}\{\vec{v_1}, \vec{v_2}\}?$$

$$\text{where } \vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \text{ and } \vec{v_2} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

Solution We're given thast we can consider these vectors as

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

We're still asking the same question, so let's see how that breaks down

$$c_1 \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

This gives us the following system of equations

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - c_2 &= 2 \\ c_1 &= 3 \\ c_1 &= 4 \end{aligned}$$

Seen as we can't reconcile the last two equations, we can see that this system is inconsistent.

2 Question 2

Restate Is $\mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

Values

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$

Form $\text{span}\{p(x), q(x), r(x)\}$ is of the form

$$p(x) = 1 - 2x + 0x^2$$
$$q(x) = 0 + 1x - 1x^2$$
$$r(x) = -2 - 3x + 1x^2$$

Investigation

$$1 = 1c_1 + 0c_2 - 2c_3$$
$$1 = -2c_1 + 1c_2 - 3c_3$$
$$1 = 0c_1 - 1c_2 + 1c_3$$

Augment We can express this as an augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ -2 & 1 & -3 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 5/2 \end{array}\right]$$

Analysis Not only consistent, but unique as well! So. Yes. We can use these coefficients to express $\mathbf{v}_4$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

3 Question 3

3.1 Question

Using RREF. How would you determine whether a system of linear equations is consistent, inconsistent, or has infinite solutions?

3.2 Answer

Been looking forwards to this one. Was one of the first things I had to spend a moment clarifying for myself.

Inconsistent This one's the easiest. If the system that I produce in anyway leads to a contradiction, then the system is inconsistent. Too. If there is no solution to a row in the RREF form, then the system is inconsistent.

Example

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

Here we're lead to the conclusion that $0 = 5$, which is clearly a contradiction.

Heuristic If you have a row of all zeros on the left side and a non-zero value on the right side, then the system is inconsistent.

Consistent We can just be lazy and say that consistency is the absense of a contradiction. We can subtype this into two categories, unique solutions and infinite solutions.

Infinite

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Here we can see that there is a free variable, which means that there are infinite solutions. We could express the solution set as

$$\begin{bmatrix} 1+2t \\ -1/2 \\ t \end{bmatrix} \text{ for any } t \in \mathbb{R}$$

Hence, infinite solutions.

Unique

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 5/2 \end{array}\right]$$

Here we can see that there is a unique solution. Way more boring. We can express the solution set as

$$\begin{bmatrix} 1 \\ -1/2 \\ 5/2 \end{bmatrix}$$

Hence, a unique solution.

4 Question 4

4.1 Question

We're going to take a augmented matrix and put it in vector format

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

4.2 Answer

Step 1 We can express this as a system of equations

$$\begin{aligned} x_1 + 2x_3 &= 3 \\ x_2 - x_3 &= 4 \end{aligned}$$

Step 2 Rearrange

$$\begin{aligned} x_1 &= 3 - 2x_3 \\ x_2 &= 4 + x_3 \\ x_3 &= x_3 \end{aligned}$$

Step 3 Then express as a vector as a sum of vectors

$$\vec{x} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

5 Question 5

I knew you'd ask something like this. Alright. Here we go.

5.1 Question

Given

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 0 \\ 1 & 2 & x-1 & -2 \end{array}\right]$$

Find values for x such that

(I) The system has no solutions

(II) The system has a unique solution

(III) The system has infinite solutions

5.2 Answer

Let's reduce as far as we can before trying to make sense first

Step 1 $R_3 \to R_3 - R_1$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 0 \\ 1-1 & 2-1 & (x-1)-1 & -2-2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 0 \\ 0 & 1 & x-2 & -4 \end{array}\right]$$

Step 2 $R_3 \to R_3 - R_2$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 0 \\ 0-0 & 1-1 & (x-2)-(-3) & -4-0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & x+1 & -4 \end{array}\right]$$

Step 3 Alright. We've condensed the relationships dependent on x. Let's see what we can make of this.

$$\begin{aligned} 0x_1 + 0x_2 + (x+1)x_3 &= -4 \\ (x+1)x_3 &= -4 \\ x_3 &= \frac{-4}{x+1} \end{aligned}$$

(I) No solutions when $x + 1 = 0 \Rightarrow x = -1$

(II) Infinite solutions otherwise.

(III) For infinite solutions we'd need to emulate a row of all zeros. This is impossible as $-4 \neq 0$. So. No values of x will yield infinite solutions.