1 Question 1

Given We are given three different vectors to try to map through a given function. We are then to plot everything.

1. $\mathbf{A} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$
2. $T : \mathbb{R}^2 \to \mathbb{R}^2$
3. $T(\mathbf{x}) = \mathbf{Ax}$

Find

1. The image of u under T where $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$
2. The image of c under T where $\mathbf{v} = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}$
3. The image of $\mathbf{u} + \mathbf{v}$

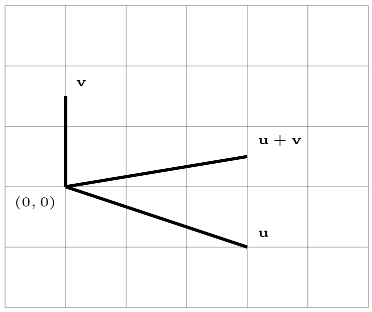
1.1 Work

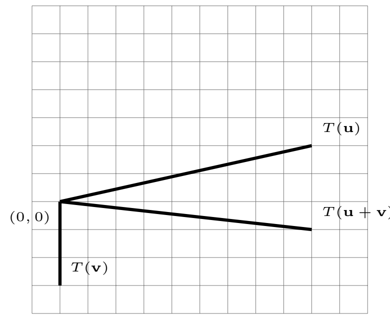
Images The image under is just going to be the result of having the function applied. We can do as such like so, for each of the given terms

1. $T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \end{bmatrix}$
2. $T(\mathbf{v}) = A\mathbf{v} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$
3. $T(\mathbf{u} + \mathbf{v}) = A(\mathbf{u} + \mathbf{v}) = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 9 \\ -1 \end{bmatrix}$

1.2 Illustration

Scaling Throughout we should expect to see a few things. The origin should remain constant, and relationships about perpendicularity and being parallel should contiue to hold.





2 Question 2

Given the matrix A:

$$A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$$

Define the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$.

2.1 Prompts

(a) Find the image under T of $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ So. We have a 2x3 and we're multiplying by a 3x1. The result is going to be a 2x1

Checking solution

$$\begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

(b) Find a vector $\mathbf{x}$ whose image under T is $\mathbf{b} = \begin{bmatrix} -12 \\ 12 \end{bmatrix}$

Thinking We can se up the system of equations so that we're solving for this. As long as the solution $y \in \text{span}\{\mathbf{u}, \mathbf{v}\}$

Solution

$$\begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -12 \\ 12 \end{bmatrix}$$

$$\begin{aligned} x_1 - 5x_2 - 7x_3 &= -12 \\ -3x_1 + 7x_2 + 5x_3 &= 12 \end{aligned}$$

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ x_2 + 2x_3 &= 3 \end{aligned}$$

Particular solution Let $x_3 = 0$ then $x_1 = 3$ and $x_2 = 3$.

Checking solution

$$\begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -12 \\ 12 \end{bmatrix}$$

3 Question 3

3.1 Restate

We're out to show that something is not a linear transformation

Given

$$T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} x_1 + 5 \\ x_2 \end{bmatrix}$$

Clarification As was discussed in classes and previously in other classes. The two things we need to consider something a linear transformation are

- **Homogeneity** We need that $T(c\mathbf{v}) = cT(\mathbf{v}), \forall c \in \mathbb{R}$
- **Additivity** We need that $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

3.2 Breaking things

Alright. We have two requirements. Let's see which is going to be the one to break.

Additivity Let's define two vectors, $\mathbf{u}$ and $\mathbf{v}$, to test this property.

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

First, we'll find the sum of the vectors and then apply the transformation T.

$$T(\mathbf{u}+\mathbf{v}) = T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}\right) = \begin{bmatrix} (u_1+v_1)+5 \\ u_2+v_2 \end{bmatrix}$$

Next, we'll apply the transformation to each vector individually and then add the results.

$$T(\mathbf{u}) = \begin{bmatrix} u_1+5 \\ u_2 \end{bmatrix}, \quad T(\mathbf{v}) = \begin{bmatrix} v_1+5 \\ v_2 \end{bmatrix}$$

$$T(\mathbf{u}) + T(\mathbf{v}) = \begin{bmatrix} u_1+5 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1+5 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1+v_1+10 \\ u_2+v_2 \end{bmatrix}$$

By comparing the two outcomes, we can see that $T(\mathbf{u}+\mathbf{v}) \neq T(\mathbf{u}) + T(\mathbf{v})$ because:

$$\begin{bmatrix} (u_1+v_1)+5 \\ u_2+v_2 \end{bmatrix} \neq \begin{bmatrix} u_1+v_1+10 \\ u_2+v_2 \end{bmatrix}$$

Since the additivity property does not hold, T is not a linear transformation. We could go through the trouble to double check the other condition, but, there's no need.

4 Question 4

4.1 Restated

Given AB is defined with B having linearly dependent columns. Are the columns of AB linearly independent? If not, counterexample.

4.2 Slow

Let's take our time with this one. It doesn't give us as far as much to work with as have some of the others previously.

Given So. The columns of B are linearly dependent. We know a few things because of this that are worth listing here before further investigation

1. We know that we will have a free variable
2. That one of the columns can be expressed as a linear combination of the others
3. $B\mathbf{x} = \mathbf{0}$ will have a nontrivial solution

From this we can state that there exists a non-zero vector $\mathbf{x}$ such that $B\mathbf{x} = \mathbf{0}$. This means that at least one column of B can be written as a linear combination of the other columns.

Deduction

$$\begin{gathered} B\mathbf{x} = \mathbf{0} \text{ for some } \mathbf{x} \\ AB\mathbf{x} = (AB)\mathbf{x} = A(B\mathbf{x}) \\ A(B\mathbf{x}) = A\mathbf{0} = \mathbf{0} \end{gathered}$$

So. There exists an $\mathbf{x}$ such that when $\mathbf{x}$ is applied to B we get the zero vector. Then, when we apply A to that zero vector, we still get the zero vector. This means that $(AB)\mathbf{x} = \mathbf{0}$ has a non-trivial solution (the same $\mathbf{x}$ that makes $B\mathbf{x} = \mathbf{0}$). Therefore, the columns of AB are linearly dependent.

Arr. Thar be a counterexample We could do this backwards as was done above. Find an $\mathbf{x}$ such that $B\mathbf{x} = \mathbf{0}$. Then we'd easily find a nontrivial solution for $AB\mathbf{x} = \mathbf{0}$

Sounds like alot of work That sounds like it might take more than a single step, so... let's just guess and hope for the best.

A guess Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. The columns of B are linearly dependent because the second column is twice the first column. Specifically, if $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, then $B\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2-2 \\ 4-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Now, let's compute AB:

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

The columns of AB are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$. These columns are linearly dependent, as the second column is twice the first. And, as expected, $(AB)\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which is a non-trivial solution.