1 Question 1

Statement A linear transformation **T** transforms

$$\begin{bmatrix}2\\1\end{bmatrix} \text{ to } \begin{bmatrix}-1\\-5\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}-3\\-2\end{bmatrix} \text{ to } \begin{bmatrix}-15\\-10\end{bmatrix}.$$

We're asked to find where a similar matrix is transformed to, without calculating the linear transformation **A** explicitly.

Givens

$$\text{let} \quad \mathbf{u} = \begin{bmatrix}2\\1\end{bmatrix}, \mathbf{v} = \begin{bmatrix}-3\\-2\end{bmatrix}, \mathbf{w} = \begin{bmatrix}16\\10\end{bmatrix}$$

We know from above that

$$T(\mathbf{u}) = \begin{bmatrix}-1\\-5\end{bmatrix} \quad \text{and} \quad T(\mathbf{v}) = \begin{bmatrix}-15\\-10\end{bmatrix}$$

We also know from previous investigation and work done in class that the linear transformations exhibit the additive property

$$T(c_1\mathbf{u} + c_2\mathbf{v}) = c_1 T(\mathbf{u}) + c_2 T(\mathbf{v})$$

Work So, we can find the coefficients c_1 and c_2 such that

$$\begin{bmatrix}16\\10\end{bmatrix} = c_1 \begin{bmatrix}2\\1\end{bmatrix} + c_2 \begin{bmatrix}-3\\-2\end{bmatrix}$$

Using the usual methods, we get $c_1 = 2$ and $c_2 = -4$. We can then relate this back and see that

$$T(\mathbf{w}) = T(2\mathbf{u} + 4\mathbf{v}) = 2T(\mathbf{u}) - 4T(\mathbf{v}) = 2\begin{bmatrix}-1\\-5\end{bmatrix} - 4\begin{bmatrix}-15\\-10\end{bmatrix}$$

Solution Calculating the above, we get

$$2\begin{bmatrix}-1\\-5\end{bmatrix} - 4\begin{bmatrix}-15\\-10\end{bmatrix} = \begin{bmatrix}58\\30\end{bmatrix}$$

So. Without finding the actual linear transformation, we can use the properties of linear transformations to avoid actually calculating the transformation matrix. Though to have had enough information to do this, we would have had to known the images of two linearly independent vectors.

2 Question 2

2.1 Statement

Ok. This time we want the eigenvalues and eigenvectors of the matrix. We then want to interpret what this means geometrically.

2.2 Givens

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 0 \\ 12 & 2 & 2 \end{bmatrix}$$

2.3 Work

We were given on of the eigenvalues, $\lambda_1 = 4$. Using this we could have used division to find the others, but... We can also just calculate the characteristic polynomial using a calculator and factor back. That's alot of terms to avoid making a mistake with.

$$det(\mathbf{A} - \lambda \mathbf{I}_3) = (5 - x)(x - 4)(x + 2)$$

For $\lambda_1 = 4$

$$\mathbf{A} - \lambda_1 \mathbf{I}_3 = \begin{bmatrix} -3 & 0 & 1 \\ 0 & 0 & 0 \\ 12 & 2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

From this we take

$$x_1 - \frac{1}{3}x_3 = 0$$
$$x_2 + x_3 = 0$$

We let $x_3 = s$ as our free variable, let $s = 3$, and find

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$$

For $\lambda_2 = 5$

$$\mathbf{A} - \lambda_1 \mathbf{I}_3 = \begin{bmatrix} -4 & 0 & 1 \\ 0 & -1 & 0 \\ 12 & 2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From this we take

$$x_1 - \frac{1}{4}x_3 = 0$$
$$x_2 = 0$$

We let $x_3 = s$ as our free variable, let $s = 4$, and find

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

For $\lambda_3 = -2$

$$\mathbf{A} - \lambda_1 \mathbf{I}_3 = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 6 & 0 \\ 12 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From this we take

$$\begin{aligned} x_1 + \frac{1}{3} x_3 &= 0 \\ x_2 &= 0 \end{aligned}$$

We let $x_3 = s$ as our free variable, let $s = 3$, and find

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

2.4 Solution

So, we have the eigenvalues and eigenvectors as follows:

$$\lambda_1 = 4, \mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}; \quad \lambda_2 = 5, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}; \quad \lambda_3 = -2, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

Interpretation So, geometrically, **if** a point lies on one of the lines defined by the eigenvectors, then when the transformation is applied, the point will be scaled by the corresponding eigenvalue. Also, **if** a point does not lie along a plane, it will be stretched by some combination of the eigenvalues in the directions of the eigenvectors.

Quick example We know that the point

$$3 \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ 9 \end{bmatrix}$$

lies along the plane defined by $\mathbf{v}_1$. Applying the transformation, we get

$$\mathbf{A} \begin{bmatrix} 3 \\ -9 \\ 9 \end{bmatrix} = 3\mathbf{A} \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} = 3 \cdot 4 \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ -36 \\ 36 \end{bmatrix} = \lambda_1 \begin{bmatrix} 3 \\ -9 \\ 9 \end{bmatrix}$$

So, taking a point along that plane, we see that it is scaled by $\lambda_1 = 4$ as expected. We could explore this a little further, but it's going to come up again later, so, I'll save that for then.

3 Question 3

3.1 Statement

Ok, we're back at it, but this time we're given a tranformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that stretches

$$\frac{1}{4} \text{ in } \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \ -3 \text{ in } \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix}, \ -3 \text{ in } \begin{bmatrix} 1 \\ -1 \\ 9 \end{bmatrix}$$

We can take these as eigenvalues and eigenvectors

$$\lambda_1 = \frac{1}{4}, \mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}; \quad \lambda_2 = -3, \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix}; \quad \lambda_3 = -3, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 9 \end{bmatrix}$$

3.2 Question by parts

Part A We want to find why the vector

$$\text{let } \mathbf{A} = \begin{bmatrix} 3 \\ -5 \\ 37 \end{bmatrix}$$

stretches by a factor of -3.

Work We can express $\mathbf{A}$ as a linear combination of the eigenvectors.

$$\begin{bmatrix} 3 \\ -5 \\ 37 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -1 \\ 9 \end{bmatrix}$$

Solving this system, we find that

$$c_1 = 0, c_2 = 2, c_3 = 3$$

So, we can see that

$$\mathbf{A} = 0 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ -1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ -1 \\ 9 \end{bmatrix}$$

Solution Because $\mathbf{A}$ is a linear combination of two eigenvectors that have the same eigenvalue $\lambda_2 = \lambda_3 = -3$, when the transformation is applied, both components will be scaled by -3, resulting in the entire vector being scaled by -3.

Part B Again, without finding the actual transformation, we're going to do this three more times. We follow the same methodology as above, solving the same system of equations, but with the given vector as the augment.

Subpart i

$$\text{given } \mathbf{A} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

So, we can express the given vector as lying on the plane of $\mathbf{v}_1$. Given such, we know that it stretches by $\frac{1}{4}$.

Subpart ii

$$\text{given } \mathbf{A} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{9}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

So, we can express the given vector as lying on the plane of $\mathbf{v}_3$. Given such, we know that it stretches by -3.

Subpart iii

$$\text{given } \mathbf{A} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} \end{bmatrix}$$

This is the tricky one. Because we need to combine the spans of the eigenvectors to reach the point, it scales as a combination of the three. We can represent this like so

$$\mathbf{A} = \frac{2}{3}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 - \frac{1}{3}\mathbf{v}_3$$

Our current thinking is that it scales like this

$$\lambda_1 \frac{2}{3}\mathbf{v}_1 - \lambda_2 \frac{1}{3}\mathbf{v}_2 + \lambda_3 \frac{1}{3}\mathbf{v}_3$$

But as to a perfectly clear picture as to what's going on, I'll leave this until class and hopefully clarify.

4 Question 4

4.1 Statement

Theorem A $n \times n$ matrix $\mathbf{A}$ is diagonizable if and only if $\mathbf{A}$ has n linearly independent eigenvectors. We're looking for a 2×2 matrix that is invertible, but not diagonizable, and a 2×2 **non-diagonal** matrix that is diagonizable, but not invertible.

Simplification I distilled this down a bit to make checking a little easier. For invertibility I was looking for $det(\mathbf{A}) = ad - bc \neq 0$ For diagonizable I was looking for $\mathbf{v}_1 \neq \mathbf{v}_2$ More specifically, I was looking for $\mathbf{v}_1 \neq c\mathbf{v}_2$ where $c \in \mathbb{R}$. As this is simply a restatement of the theorem above.

Subpart A Ok, here we're looking for invertible not diagonizable, doesn't have to be a diagonal. We started more generally, but a simple example is

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Invertibility is easy, it's trivial to see that $det(\mathbf{A} = 1 \neq 0)$ It's an upper triangular, so, the eigenvalues are just those in the diagonal. If we use those to get the eigenvectors, we get... obviously,

$$\mathbf{v}_1 = \mathbf{v}_2 = c\mathbf{v}_2$$

Which is a no go, per the theorem we need them to be nonscalar non... multiples?

Subpart B Alright, the other way. We want a **non-diagonal** matrix that **is** diagonizable, but that is **not** invertible. We (Jack and I) settled on...

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Alright, so, the determinant is $det(\mathbf{A}) = 1 - - - 1 = 1 - 1 = 0$. So this puppy definitely isn't invertible. If we plug and chug for the eigens

$$\begin{aligned} \text{for } \lambda_1 =& 2 : \mathbf{v}_{1,1} = -\mathbf{v}_{1,2} \\ \text{for } \lambda_2 =& 0 : \mathbf{v}_{2,1} = \mathbf{v}_{2,2} \\ \mathbf{v}_1 =& \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ \mathbf{v}_2 =& \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

From this we can double check that it actually diagonalizes

$$\mathbf{A} \stackrel{?}{=} \mathbf{PDP}^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \mathbf{A}$$

QED et cetera.

5 Question 5

5.1 Statement

We're back with the Markov chains! Alright, let's make our matrix and run through

5.2 Subpart A

We're attempting to find the transition matrix, the data is given in the same order. We're filling slots, keeping in mind that our columns should always add up to 1

Transition matrix

$$\mathbf{A} = \begin{bmatrix} .6 & .1 & .2 \\ .2 & .7 & .4 \\ .2 & .2 & .4 \end{bmatrix}$$

5.3 Subpart B

Alright, now we want the probability that a short person will have tall grandchilden. I almost you-knowed the goose here and took $\mathbf{A}^3$, but, alas, this should just be the corresponding entry in $\mathbf{A}^2$

$$\mathbf{A}^2 = \begin{bmatrix} .42 & .17 & .24 \\ .34 & .59 & .40 \\ .24 & .24 & .28 \end{bmatrix}$$

where the entry in the top right, $\mathbf{A}^2_{1,3} = .24$ should correspond to the probability of... a short person having a tall grandchild.

5.4 Subpart C

Alright, we're given some numbers here, that I really don't care about... we could do the diagonalization, but there's no need, steady state don't given two hoots about initial conditions with Markov Chains.

$$(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 - 2x_3 &= 0 \end{aligned}$$

From this we get a vector of form $\langle 1, 2, 1 \rangle$. If we go ahead and normalize it, we get $\langle \frac{1}{4}, \frac{1}{2}, \frac{1}{4} \rangle$. So, given a million billion generations we'd expect to see twice as many medium height people as there are short or tall people... that passes a quick intuition check.