1 Question 1

Given We're given a set of points that have different offsets based on two different basis vectors. First, let's grab these points and where they map to in the new basis.

1.1 Part A

$\mathbf{B}_x$	$\mathbf{B}_y$	$\mathbf{B}'_x$	$\mathbf{B}'_y$
1	4	0	1
4	7	1	2
-8	-5	-3	-2
7	1	1	0
$5\frac{1}{2}$	-5	3	$-\frac{1}{2}$
-6	$-1\frac{1}{2}$	$-2\frac{1}{2}$	-1

Where I've denoted $\mathbf{B}$ as the black basis, and $\mathbf{B'}$ as the red. These taken together give us plenty of information to find a change of basis.

1.2 Part B

The points above are of the form $[\mathbf{v}_n]_\mathbf{B} \to [\mathbf{v}_n]_{\mathbf{B'}}\ \forall n$. So we can form a system mapping two columns of the first to that of the second.

The system Let $\mathbf{P}$ be the change of basis we're attempting to find.

$$\mathbf{P}\begin{bmatrix}1 & 4\\ 4 & 7\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 2\end{bmatrix}$$

We need to clear out the left hand side, so take the matrix besides $\mathbf{P}$ as $\mathbf{A}$, and find the inverse. As $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_2$

$$\mathbf{P} = \begin{bmatrix}0 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}1 & 4\\ 4 & 7\end{bmatrix}^{-1} = \begin{bmatrix}0 & 1\\ 1 & 2\end{bmatrix}\frac{1}{9}\begin{bmatrix}7 & -4\\ -4 & 1\end{bmatrix} = \frac{1}{9}\begin{bmatrix}4 & -1\\ 1 & 2\end{bmatrix}$$

Checking solution

Let's check this solution by multiplying $\mathbf{P}$ by another point in the original basis, and see if it maps to the correct point in the new basis.

$$\mathbf{P}\begin{bmatrix}-8\\ -5\end{bmatrix} = \frac{1}{9}\begin{bmatrix}4 & -1\\ 1 & 2\end{bmatrix}\begin{bmatrix}-8\\ -5\end{bmatrix} = \frac{1}{9}\begin{bmatrix}-32+5\\ -8-10\end{bmatrix} = \frac{1}{9}\begin{bmatrix}-27\\ -18\end{bmatrix} = \begin{bmatrix}-3\\ -2\end{bmatrix}$$

This matches the expected output, so our change of basis is correct.

1.3 Part C

Now we're after a way to go in reverse. We could be super diligent and do the same process, but we can also just take the inverse of $\mathbf{P}$.

$$\mathbf{P}^{-1} = \begin{bmatrix} \frac{4}{9} & -\frac{1}{9} \\ \frac{1}{9} & \frac{2}{9} \end{bmatrix}$$

Checking solution

Let's check this solution by multiplying $\mathbf{P}^{-1}$ by another point in the red basis, and see if it maps to the correct point in the black basis.

$$\mathbf{P}^{-1}[\mathbf{v}_2]_{\mathbf{B}'} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix} = [\mathbf{v}_2]_{\mathbf{B}}$$

Again, seems the intuition is correct. For this point atleast, it seems our change of basis is mapping back from black to red.

1.4 Part D

Ok. I'm reading stretch so, eigenvalues, let's give things a name before we get going

Given

First line We are given a stretch factor of 2 corresponding to the line $y = -\frac{1}{2}x$ So that gives us

$$T(\begin{bmatrix} 2 \\ -1 \end{bmatrix}) = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

Where we take $\lambda_1 = 2$ as the eigenvalue

Second line For the second we're given a stretch factor of -1 and a line $y = 4x$. From this

$$T(\begin{bmatrix} 1 \\ 4 \end{bmatrix}) = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

Where we take $\lambda_2 = -1$ as the eigenvalue

Forming a system We know that we want a $2x2$ matrix, let's call it $\mathbf{F}$, such that

$$\mathbf{F} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where

$$\mathbf{F}\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}4\\-2\end{bmatrix} \qquad \mathbf{F}\begin{bmatrix}1\\4\end{bmatrix}=\begin{bmatrix}-1\\-4\end{bmatrix}$$

solving for a, b, c, and d we get

$$\begin{aligned}2a-b&=4\\2c-d&=-2\\a+4b&=-1\\c+4d&=-4\end{aligned}$$

which results $a=\frac{5}{3}$, $b=-\frac{2}{3}$, $c=-\frac{4}{3}$, and $d=-\frac{2}{3}$. Giving us a final matrix of

$$\mathbf{F}=\begin{bmatrix}\frac{5}{3}&-\frac{2}{3}\\\frac{4}{3}&-\frac{2}{3}\end{bmatrix}=\frac{1}{3}\begin{bmatrix}5&-2\\4&-2\end{bmatrix}$$

Mapping red to black

Given

$$[\mathbf{x}]_{\mathbf{r}}=\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}$$

Work We need to first map this back to the standard basis, then apply our stretch, then map it back to the red basis. We can borrow our **P** from before

$$\mathbf{P}[\mathbf{x}]_{\mathbf{r}}=\begin{bmatrix}\frac{4}{9}&-\frac{1}{9}\\\frac{1}{9}&\frac{2}{9}\end{bmatrix}\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}=\begin{bmatrix}\frac{7}{18}\\\frac{2}{9}\end{bmatrix}=[\mathbf{x}]_{\mathbf{e}}$$

And applying our **F** stretch

$$\mathbf{F}[\mathbf{x}]_{\mathbf{e}}=\frac{1}{3}\begin{bmatrix}5&-2\\4&-2\end{bmatrix}\begin{bmatrix}\frac{7}{18}\\\frac{2}{9}\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\\frac{10}{27}\end{bmatrix}$$

The answer above is in the black basis.

Mapping black to red

Given

$$[\mathbf{x}]_{\mathbf{e}}=\begin{bmatrix}-3\\3\end{bmatrix}$$

We want to apply our transformation, then represent our answer in the red basis.

Work Applying our **F** stretch

$$\mathbf{F}[\mathbf{x}]_{\mathbf{e}} = \frac{1}{3}\begin{bmatrix} 5 & -2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ -6 \end{bmatrix}$$

Now, back to the red basis, we borrow P^{-1} from above

$$\mathbf{P}^{-1}\mathbf{F}[\mathbf{x}]_{\mathbf{e}} = \begin{bmatrix} \frac{4}{9} & -\frac{1}{9} \\ \frac{1}{9} & \frac{2}{9} \end{bmatrix}\begin{bmatrix} -7 \\ -6 \end{bmatrix} = \begin{bmatrix} -\frac{22}{9} \\ -\frac{19}{9} \end{bmatrix}$$

2 Question 2

Our boy Gauss is back, making our lives difficult again. We're keeping the modes of transport we had originally as

$$\beta = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

And, as I would've guessed... changing basis.

2.1 Part A

Given For one old Uncle Cramer, we're given that in the standard basis his house is located at $[\mathbf{w_e}] = \begin{bmatrix} 25 \\ 71 \end{bmatrix}$

Changing basis I want to clarify my understanding for a moment. Above is simply analogous to saying that

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 25 \\ 71 \end{bmatrix} = \begin{bmatrix} 25 \\ 71 \end{bmatrix}$$

And our desire to change basis forms a system like so

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 25 \\ 71 \end{bmatrix}$$

Let **P** be formed from the columns of our transport basis. It's inverse is then

$$\mathbf{P}^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$$

Lastly, applying our initial vector we get

$$\mathbf{P}^{-1}\,[\mathbf{w}]_{\mathbf{e}} = [\mathbf{w}]_{\beta} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 25 \\ 71 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -21 \\ 188 \end{bmatrix} = \begin{bmatrix} -\frac{21}{5} \\ \frac{188}{5} \end{bmatrix}$$

So, we'd travel $-\frac{21}{5}$ times by hoverboard, and $\frac{188}{5}$ times by magic carpet. Not exactly satisfying, but, hey.

2.2 Part B

Given Now in reverse we're given the location of a museum expressed in the transport basis as $[\mathbf{v}]_\beta = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$

Changing basis At the risk of turning one mistake in to two, this should be simply the inverse, ie. the original basis times the vector in the transport basis.

$$\mathbf{P}\,[\mathbf{v}]_\beta = [\mathbf{v}]_\mathbf{e} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 27 \\ 14 \end{bmatrix}$$