1 Question 1

Statement A linear transformation **T** transforms

$$\begin{bmatrix}2\\1\end{bmatrix} \text{ to } \begin{bmatrix}-1\\-5\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}-3\\-2\end{bmatrix} \text{ to } \begin{bmatrix}-15\\-10\end{bmatrix}.$$

We're asked to find where a similar matrix is transformed to, without calculating the linear transformation **A** explicitly.

Givens

$$\text{let} \quad \mathbf{u} = \begin{bmatrix}2\\1\end{bmatrix}, \mathbf{v} = \begin{bmatrix}-3\\-2\end{bmatrix}, \mathbf{w} = \begin{bmatrix}16\\10\end{bmatrix}$$

We know from above that

$$T(\mathbf{u}) = \begin{bmatrix}-1\\-5\end{bmatrix} \quad \text{and} \quad T(\mathbf{v}) = \begin{bmatrix}-15\\-10\end{bmatrix}$$

We also know from previous investigation and work done in class that the linear transformations exhibit the additive property

$$T(c_1\mathbf{u} + c_2\mathbf{v}) = c_1T(\mathbf{u}) + c_2T(\mathbf{v})$$

Work So, we can find the coefficients c_1 and c_2 such that

$$\begin{bmatrix}16\\10\end{bmatrix} = c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}-3\\-2\end{bmatrix}$$

Using the usual methods, we get $c_1 = 2$ and $c_2 = -4$. We can then relate this back and see that

$$T(\mathbf{w}) = T(2\mathbf{u} + 4\mathbf{v}) = 2T(\mathbf{u}) - 4T(\mathbf{v}) = 2\begin{bmatrix}-1\\-5\end{bmatrix} - 4\begin{bmatrix}-15\\-10\end{bmatrix}$$

Solution Calculating the above, we get

$$2\begin{bmatrix}-1\\-5\end{bmatrix} - 4\begin{bmatrix}-15\\-10\end{bmatrix} = \begin{bmatrix}58\\30\end{bmatrix}$$

So. Without finding the actual linear transformation, we can use the properties of linear transformations to avoid actually calculating the transformation matrix. Though to have had enough information to do this, we would have had to known the images of two linearly independent vectors.

2 Question 2

2.1 Statement

Ok. This time we want the eigenvalues and eigenvectors of the matrix. We then want to interpret what this means geometrically.

2.2 Givens

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 0 \\ 12 & 2 & 2 \end{bmatrix}$$

2.3 Work

We were given on of the eigenvalues, $\lambda_1 = 4$. Using this we could have used division to find the others, but... We can also just calculate the characteristic polynomial using a calculator and factor back. That's alot of terms to avoid making a mistake with.

$$det(\mathbf{A} - \lambda \mathbf{I}_3) = (5 - x)(x - 4)(x + 2)$$

For $\lambda_1 = 4$

$$\mathbf{A} - \lambda_1 \mathbf{I}_3 = \begin{bmatrix} -3 & 0 & 1 \\ 0 & 0 & 0 \\ 12 & 2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

From this we take

$$x_1 - \frac{1}{3} x_3 = 0$$
$$x_2 + x_3 = 0$$

We let $x_3 = s$ as our free variable, let $s = 3$, and find

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$$

For $\lambda_2 = 5$

$$\mathbf{A} - \lambda_1 \mathbf{I}_3 = \begin{bmatrix} -4 & 0 & 1 \\ 0 & -1 & 0 \\ 12 & 2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From this we take

$$x_1 - \frac{1}{4} x_3 = 0$$
$$x_2 = 0$$

We let $x_3 = s$ as our free variable, let $s = 4$, and find

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

For $\lambda_3 = -2$

$$\mathbf{A} - \lambda_1 \mathbf{I}_3 = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 6 & 0 \\ 12 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

From this we take

$$\begin{aligned} x_1 + \frac{1}{3}x_3 &= 0 \\ x_2 &= 0 \end{aligned}$$

We let $x_3 = s$ as our free variable, let $s = 3$, and find

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

2.4 Solution

So, we have the eigenvalues and eigenvectors as follows:

$$\lambda_1 = 4, \mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}; \quad \lambda_2 = 5, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}; \quad \lambda_3 = -2, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

Interpretation So, geometrically, **if** a point lies on one of the lines defined by the eigenvectors, then when the transformation is applied, the point will be scaled by the corresponding eigenvalue. Also, **if** a point does not lie along a line, it will be stretched by some combination of the eigenvalues in the directions of the eigenvectors.

Quick example We know that the point

$$3 \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ 9 \end{bmatrix}$$

lies along the line defined by $\mathbf{v}_1$. Applying the transformation, we get

$$\mathbf{A} \begin{bmatrix} 3 \\ -9 \\ 9 \end{bmatrix} = 3\mathbf{A} \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} = 3 \cdot 4 \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ -36 \\ 36 \end{bmatrix} = \lambda_1 \begin{bmatrix} 3 \\ -9 \\ 9 \end{bmatrix}$$

So, taking a point along that line, we see that it is scaled by $\lambda_1 = 4$ as expected.