1 Systems of Equations

The goal is to create systems of linear equations that have no solution, one unique solution, and infinitely many solutions.

- **No solution (Inconsistent System)**

  We can create an inconsistent system by having two parallel lines. For example:

$$\begin{aligned} x + 2y &= 3 \\ x + 2y &= 5 \end{aligned}$$

  If we subtract the first equation from the second, we get $0 = 2$, which is a contradiction. This means there is no pair of (x, y) that can satisfy both equations simultaneously.

- **One unique solution (Consistent and Independent)**

  A system with one solution consists of two lines that intersect at a single point. The vectors representing the equations must be linearly independent.

$$\begin{aligned} 2x + 3y &= 7 \\ x - y &= 1 \end{aligned}$$

  We can solve this using substitution or elimination. From the second equation, $x = y + 1$. Substituting into the first:

$$\begin{aligned} 2(y + 1) + 3y &= 7 \\ 2y + 2 + 3y &= 7 \\ 5y &= 5 \\ y &= 1 \end{aligned}$$

  Then $x = 1 + 1 = 2$. The unique solution is $(2, 1)$.

- **Infinitely many solutions (Consistent and Dependent)**

  This occurs when the equations in the system are multiples of each other, essentially representing the same line.

$$\begin{aligned} x + 2y &= 4 \\ 2x + 4y &= 8 \end{aligned}$$

  The second equation is just the first equation multiplied by 2. Any point (x, y) that lies on the line $x + 2y = 4$ is a solution. We can express the solutions in terms of a parameter, say t. Let $y = t$, where $t \in \mathbb{R}$. Then $x = 4 - 2t$. The solution set is all points of the form $(4 - 2t, t)$.

2 To check whether using rref

- No solution
$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & 2 & 5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 2 \end{array}\right]$$
The second row translates to $0 = 2$, which is a contradiction, confirming no solution. If we ever run in to a contradiction like this, we know there is no solution.

- One solution
$$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 3 \end{array}\right]$$
Here we have no codependencies. That is, we do not have to presume a value for one variable to find the other. This means there is one unique solution, which is $(-1, 3)$.

- Infinitely many solutions
$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 4 & 8 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & 0 \end{array}\right]$$
So. One equation. Two variables. We can let $y = t$, where $t \in \mathbb{R}$ Then $x = 4 - 2t$. The solution set is all points of the form $(4 - 2t, t)$.