Homework

November 25, 2025

1 Unit 7 - Homework 10

MATH620

Joshua Dunne

1.1 Question 1

1.1.1 Find the singular value decomposition of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

I'm going to borrow substantially from something I wrote for the first question here, we want the SVD, but, we want to show the steps.

[313]:
```
import numpy as np

A = np.array([
    [1,1],
    [1,0],
    [0,1]
])

a_transpose_a = A.T @ A

# Hermitian, so, we can use eigh shortcut
a_transpose_a_eigen_values, a_transpose_a_eigen_vectors = np.linalg.
 ↪eigh(a_transpose_a)
sort_indices = np.argsort(a_transpose_a_eigen_values)[::-1]
# We want that these are ordered, and the corr. eigenvectors are likewise sorted
sorted_eigenvalues = a_transpose_a_eigen_values[sort_indices]
sorted_eigenvectors = a_transpose_a_eigen_vectors[:, sort_indices]
# From those we can compose Sigma
singular_values = np.sqrt(sorted_eigenvalues)
sigma = np.diag(singular_values)
# And compose our left singular
left_singular = (A @ sorted_eigenvectors) / singular_values
```

```python
# And our right singular
right_singular = sorted_eigenvectors.T
# Lastly we check
print(f"left * sigma * right: \n{np.around(left_singular @ sigma @
 ↪right_singular)}")
print(f"left_singular: \n{np.around(left_singular)}")
print(f"sigma: \n{np.around(sigma)}")
print(f"right_singular: \n{np.around(right_singular)}")
```

```
left * sigma * right:
[[ 1.  1.]
 [ 1. -0.]
 [-0.  1.]]
left_singular:
[[ 1.  0.]
 [ 0. -1.]
 [ 0.  1.]]
sigma:
[[2. 0.]
 [0. 1.]]
right_singular:
[[ 1.  1.]
 [-1.  1.]]
```

That should surmise. The SVD is given by above. To summate, we find the eigenvectors of $\mathbf{A}^T\mathbf{A}$ to get $\mathbf{V}$, the eigenvectors of $\mathbf{A}\mathbf{A}^T$ to get $\mathbf{U}$, and the square roots of the eigenvalues of either to get . The only hangup here is getting the sorted order of the singular values, but that is easy enough to do. Lastly, we can just check against the built-in SVD function in numpy to verify our answer.

[314]:
```python
print(f"left_singular: \n{np.around(np.linalg.svd(A)[0])}")
print(f"sigma: \n{np.around(np.diag(np.linalg.svd(A)[1]))}")
print(f"right_singular: \n{np.around(np.linalg.svd(A)[2])}")
```

```
left_singular:
[[-1.  0. -1.]
 [-0. -1.  1.]
 [-0.  1.  1.]]
sigma:
[[2. 0.]
 [0. 1.]]
right_singular:
[[-1. -1.]
 [-1.  1.]]
```

There's some funk going on there, but, I'm happy with the right_singular and sigma matrices. The left_singular matrix seems to have some sign differences, but, that's not a problem since eigenvectors are only defined up to a sign.

1.2 Question 2

1.2.1 Introduction

We're given a few things that are worth listing first here. Given points

$$(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$$

we can find a linear function $y = c_0 + c_1 x$ from

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

via least squares. This is done by solving the normal equations

$$\mathbf{A}^T \mathbf{A} \mathbf{c} = \mathbf{A}^T \mathbf{y}$$

where $\mathbf{A}$ is the matrix on the left hand side above, $\mathbf{c}$ is the vector of coefficients, and $\mathbf{y}$ is the vector of y values. If we substitue in the terms we can see the relationship more clearly:

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

Using calculus we can do this another way.

$$||r(c)||^2 = ||y - Ac||^2 = [y_1 - (c_0 + c_1 x_1)]^2 + \dots + [y_m - (c_0 + c_1 x_m)]^2 = f(c_0, c_1)$$

1.3 Question 2

1.3.1 Part a

We want to show that if we set the partial derivatives equal to 0 we get the same normal equations. We have two variables here we're actually interested in, c_0 and c_1. So we compute the partial derivatives with respect to each variable. I think we can just get by with throwing this in sympy, so, let's see.

[315]:
```
from sympy import Matrix, symbols, diff, Sum, IndexedBase, Idx

# We have four things we need
c_0, c_1, x, y = symbols('c0 c1 x y')
# From a function we can define as such
error_term_squared = (y - (c_0 + c_1*x))**2
# If we take the derivative of each
d_c_0 = diff(error_term_squared, c_0)
d_c_1 = diff(error_term_squared, c_1)
print(f"c_0 derivative: {d_c_0}")
print(f"c_1 derivative: {d_c_1}")
```

```
c_0 derivative: 2*c0 + 2*c1*x - 2*y
c_1 derivative: -2*x*(-c0 - c1*x + y)
```

We can restate the results a little more cleanly here. This was given in terms of a sum of all the possible points given, so, the derivative is better written for c_0 as

$$\frac{\partial f}{\partial c_0} = -2\sum_{i=1}^{m}[y_i - (c_0 + c_1 x_i)]$$

and for c_1 as

$$\frac{\partial f}{\partial c_1} = -2\sum_{i=1}^{m}[y_i - (c_0 + c_1 x_i)]x_i$$

We can afford a quick aside and do this symbolically. We want to set the partials equal to zero, so, let's use sympy and see where we get.

[316]:
```
import sympy as sp

critical_points_d_c_0 = sp.solve([d_c_0], [c_0])
critical_points_d_c_1 = sp.solve([d_c_1], [c_1])
print(f"c_0 partials against 0: {sp.latex(critical_points_d_c_0)}")
print(f"c_1 partials against 0: {sp.latex(critical_points_d_c_1)}")
```

```
c_0 partials against 0: \left\{ c_{0} : - c_{1} x + y\right\}
c_1 partials against 0: \left\{ c_{1} : \frac{- c_{0} + y}{x}\right\}
```

$$\{c_0 : -c_1 x + y\}$$

$$\left\{c_1 : \frac{-c_0 + y}{x}\right\}$$

We can write than in terms of c_0 and c_1 as

$$\sum_{i=1}^{m} y_i = \sum_{i=1}^{m}(c_0 + c_1 x_i)$$

and

$$\sum_{i=1}^{m} x_i y_i = \sum_{i=1}^{m}(c_0 + c_1 x_i)x_i$$

Rearranging these gives us

$$mc_0 + c_1\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i$$

and

$$c_0\sum_{i=1}^{m} x_i + c_1\sum_{i=1}^{m} x_i^2 = \sum_{i=1}^{m} x_i y_i$$

This is exactly the same as the normal equations we had before, so, we've shown what we wanted to show.

1.4 Question 2

1.4.1 Part b

We want to use a quadratic to fit $\{(-1,1),(0,-1),(1,0),(2,2)\}$, so, our system is going to look like

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

Substituting in our points gives us

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

Where we are trying to find coefficients c_0, c_1, c_2 to fit the quadratic $y = c_0 + c_1 x + c_2 x^2$. We can use sympy for the lulz.

[317]:

```python
import sympy as sp

from sympy import symbols, Matrix, latex

x1, x2, x3, x4 = symbols('x_1 x_2 x_3 x_4')
y1, y2, y3, y4 = symbols('y_1 y_2 y_3 y_4')

A = Matrix([
    [1, x1, x1**2],
    [1, x2, x2**2],
    [1, x3, x3**2],
    [1, x4, x4**2]
])



y_vec = Matrix([y1, y2, y3, y4])

ATA = A.T * A
ATy = A.T * y_vec
substitutions = {
    x1: -1, x2: 0, x3: 1, x4: 2,
    y1: 1, y2: -1, y3: 0, y4: 2
}
print("--- A Transpose A (Left side of Normal Eq) ---")
print(f"ATA: {latex(ATA.subs(substitutions))}")
print(f"ATy: {latex(ATy.subs(substitutions))}")
```

```
--- A Transpose A (Left side of Normal Eq) ---
ATA: \left[\begin{matrix}4 & 2 & 6\\2 & 6 & 8\\6 & 8 & 18\end{matrix}\right]
ATy: \left[\begin{matrix}2\\3\\9\end{matrix}\right]
```

So, we've taken $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}^T\mathbf{y}$. From this we get

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix}$$

and

$$\mathbf{A}^T\mathbf{y} = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix}$$

[318]:
```
# We can then solve for c_0, c_1, c_2
ata_substituted = ATA.subs(substitutions)
aty_substitued = ATy.subs(substitutions)
print(latex(ata_substituted.solve(aty_substitued)))
```

```
\left[\begin{matrix}- \frac{7}{10}\\- \frac{3}{5}\\1\end{matrix}\right]
```

Which gives us

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{7}{10} \\ -\frac{3}{5} \\ 1 \end{bmatrix}$$

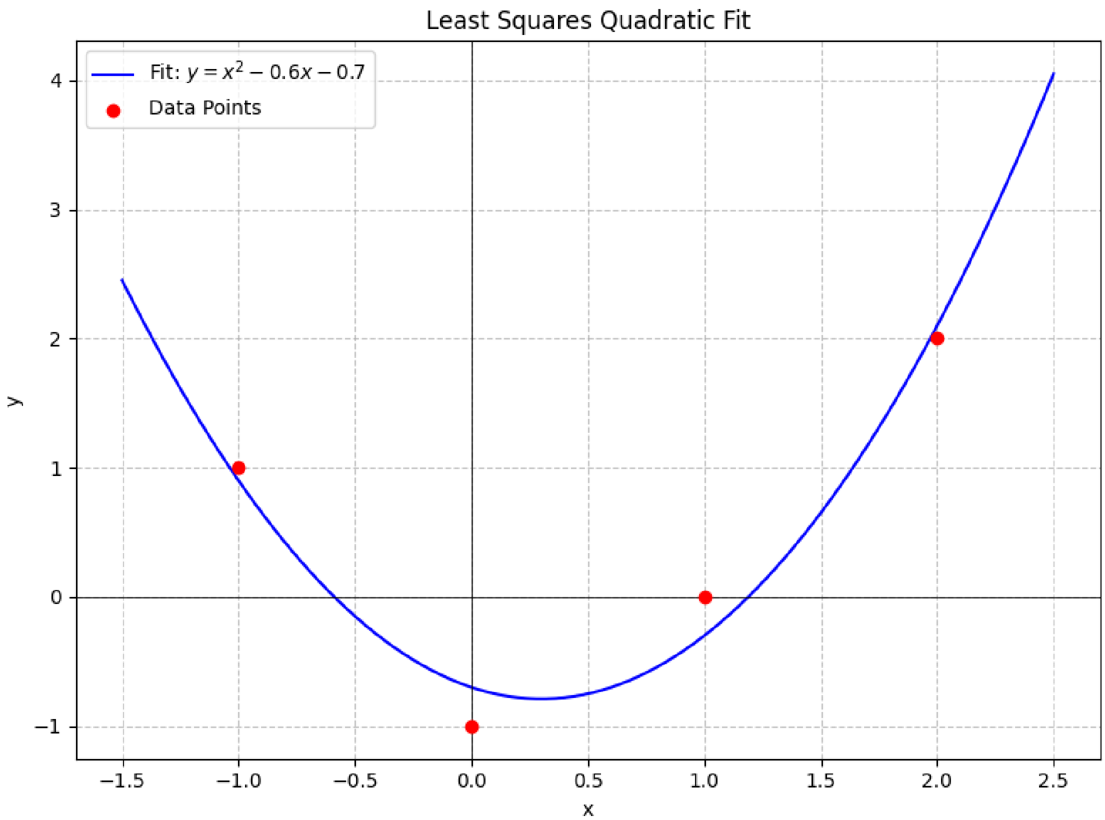
And putting that back in to our original equation gives us the quadratic fit of

$$y = -\frac{7}{10} - \frac{3}{5}x + x^2$$

We can take a moment and plot this

[319]:
```
import numpy as np
import matplotlib.pyplot as plt

# Points: (-1, 1), (0, -1), (1, 0), (2, 2)
x_points = np.array([-1, 0, 1, 2])
y_points = np.array([1, -1, 0, 2])

# Coefficients: c0 = -0.7, c1 = -0.6, c2 = 1
def quadratic_model(x):
    return -0.7 - 0.6 * x + 1.0 * x**2

x_line = np.linspace(-1.5, 2.5, 100)
y_line = quadratic_model(x_line)

plt.figure(figsize=(8, 6))
plt.plot(x_line, y_line, label=r'Fit: $y = x^2 - 0.6x - 0.7$', color='blue')
plt.scatter(x_points, y_points, color='red', zorder=5, label='Data Points')

plt.title('Least Squares Quadratic Fit')
plt.xlabel('x')
plt.ylabel('y')
```

```python
plt.axhline(0, color='black', linewidth=0.5) # x-axis
plt.axvline(0, color='black', linewidth=0.5) # y-axis
plt.grid(True, linestyle='--', alpha=0.7)
plt.legend()

plt.tight_layout()
plt.show()
```



Pure gravey. I'm liking the fit.

1.5 Question 3

1.5.1 Part A

Exponential growth by

$$p = 2.56e^{kt} \rightarrow ln(p) = ln(2.56) + ln(e^{kt}) \rightarrow lp(p) = ln(2.56) + kt$$

Year	Population	k	t	ln(p)
1950	2.56 billion	$k = 0$	0	$ln(2.56) = 0.939$

Year	Population	k	t	ln(p)
1960	3.04 billion	$k =$ 0.172	$3.04 = 2.56e^{1k}$	$ln(\frac{3.04}{2.56}) = k, k =$ 0.17185
1970	3.71 billion	$2k =$ 0.317	$3.71 = 2.56e^{2k}$	$ln(\frac{3.71}{2.56}) = 2k, k =$ 0.185512
1980	4.46 billion	$3k =$ 0.555	$4.46 = 2.56e^{3k}$	$ln(\frac{4.46}{2.56}) = 3k, k =$ 0.185047
1990	5.28 billion	$4k =$ 0.724	$5.28 = 2.56e^{4k}$	$ln(\frac{5.28}{2.56}) = 4k, k =$ 0.18098
2000	6.08 billion	$5k =$ 0.865	$6.08 = 2.56e^{5k}$	$ln(\frac{6.08}{2.56}) = 5k, k =$ 0.173

If we write this as a matrix equation we'd be looking for

$$\begin{bmatrix} 1k \\ 2k \\ 3k \\ 4k \\ 5k \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} k = \begin{bmatrix} \ln\left(\frac{3.04}{2.56}\right) \\ \ln\left(\frac{3.71}{2.56}\right) \\ \ln\left(\frac{4.46}{2.56}\right) \\ \ln\left(\frac{5.28}{2.56}\right) \\ \ln\left(\frac{6.08}{2.56}\right) \end{bmatrix}$$

[320]:
```
from sympy import Matrix, log, symbols, exp, N

A = Matrix([1, 2, 3, 4, 5])

b = Matrix([
    log(3.04 / 2.56),
    log(3.71 / 2.56),
    log(4.46 / 2.56),
    log(5.28 / 2.56),
    log(6.08 / 2.56)
])

ATA = A.T * A
ATb = A.T * b
print(f"ATA: {ATA}")
print(f"ATb: {ATb.evalf(4)}")

k_matrix = ATA.solve(ATb)
k_value = k_matrix[0]

t = symbols('t')
model = 2.56 * exp(k_value * t)
```

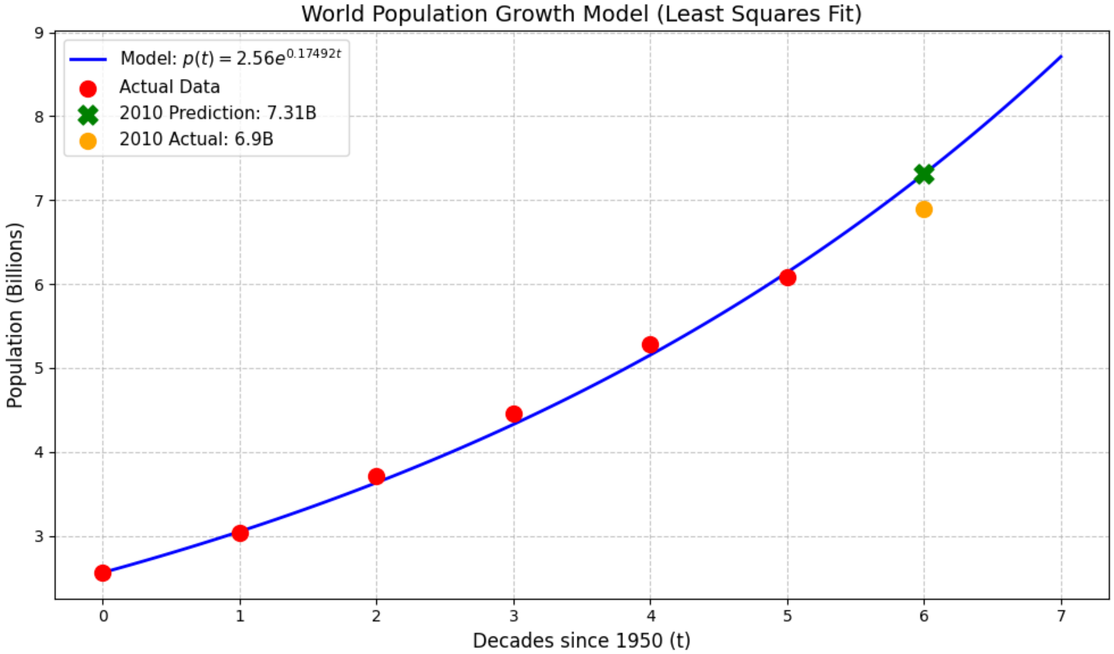
```
p_2010 = model.subs(t, 6)
print(f"model: {latex(model)}")
print(f"\nprediction for 2010 (t=6): {p_2010.evalf(3)} billion")
```

```
ATA: Matrix([[55]])
ATb: Matrix([[9.800]])
model: 2.56 e^{0.178181573820795 t}

prediction for 2010 (t=6): 7.46 billion
```

[321]:

```
import numpy as np
import matplotlib.pyplot as plt

t_data = np.array([0, 1, 2, 3, 4, 5])
p_data = np.array([2.56, 3.04, 3.71, 4.46, 5.28, 6.08])

k_optimal = 0.17492
p0 = 2.56

def model_func(t):
    return p0 * np.exp(k_optimal * t)

t_smooth = np.linspace(0, 7, 100)
p_smooth = model_func(t_smooth)

t_2010 = 6
p_2010_pred = model_func(t_2010)
p_2010_actual = 6.9

plt.figure(figsize=(10, 6))

plt.plot(t_smooth, p_smooth, label=f'Model: $p(t) = 2.56e^{{{k_optimal}t}}$',␣
↪color='blue', linewidth=2)

plt.scatter(t_data, p_data, color='red', s=100, zorder=5, label='Actual Data')

plt.scatter(t_2010, p_2010_pred, color='green', marker='X', s=150, zorder=5,␣
↪label=f'2010 Prediction: {p_2010_pred:.2f}B')
plt.scatter(t_2010, p_2010_actual, color='orange', marker='o', s=100, zorder=5,␣
↪label=f'2010 Actual: {p_2010_actual}B')

plt.title('World Population Growth Model (Least Squares Fit)', fontsize=14)
plt.xlabel('Decades since 1950 (t)', fontsize=12)
plt.ylabel('Population (Billions)', fontsize=12)
plt.grid(True, linestyle='--', alpha=0.7)
plt.legend(fontsize=11)
```

```
plt.tight_layout()
plt.show()
```



1.6 Question 3

1.6.1 Part B

We're given another table for the growth of the US population from 1950 to 2000. Just doing the same here. | Year | Population | k (approx) | Exponential Model | Log Calculation | | :— | :— | :— | :— | :— | | 1950 | 150 million | $k = 0$ | 0 | $\ln(1) = 0$ | | 1960 | 179 million | $k = 0.172$ | $179 = 150e^{1k}$ | $\ln(\frac{179}{150}) = k, \ k \approx 0.177$ | | 1970 | 203 million | $2k = 0.303$ | $203 = 150e^{2k}$ | $\ln(\frac{203}{150}) = 2k, \ k \approx 0.151$ | | 1980 | 227 million | $3k = 0.414$ | $227 = 150e^{3k}$ | $\ln(\frac{227}{150}) = 3k, \ k \approx 0.138$ | | 1990 | 250 million | $4k = 0.511$ | $250 = 150e^{4k}$ | $\ln(\frac{250}{150}) = 4k, \ k \approx 0.128$ | | 2000 | 281 million | $5k = 0.628$ | $281 = 150e^{5k}$ | $\ln(\frac{281}{150}) = 5k, \ k \approx 0.126$ |

[322]:
```
from sympy import Matrix, log, symbols, exp, N

A = Matrix([1, 2, 3, 4, 5])

b = Matrix([
    log(179/150),
    log(203/150),
    log(227/150),
    log(250/150),
    log(281/150)
])
```

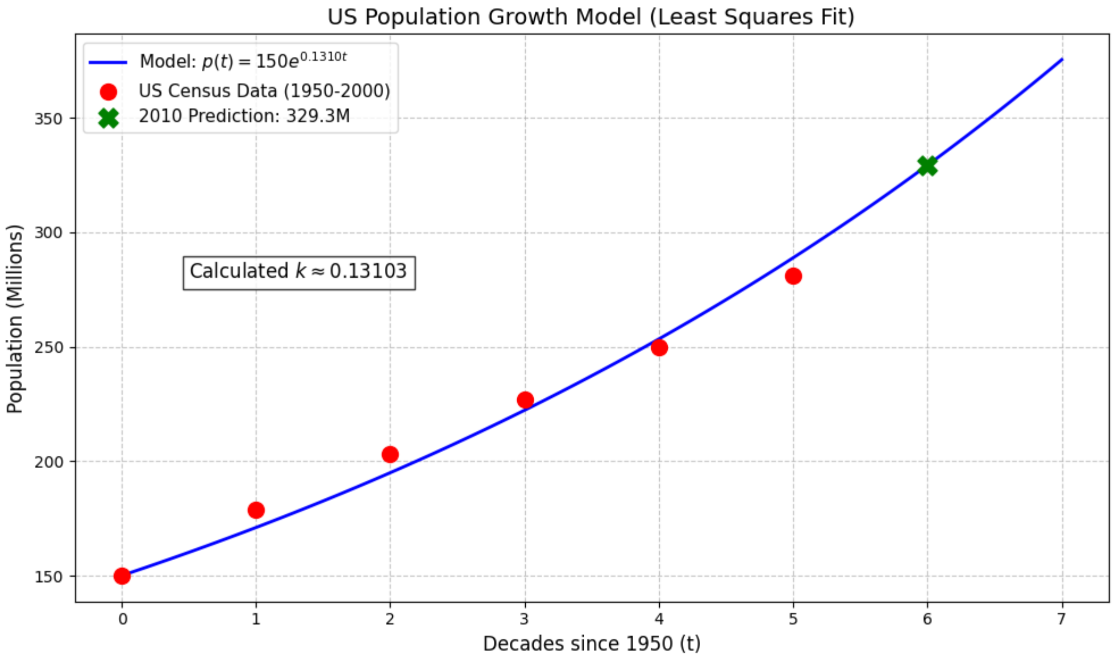
```
ATA = A.T * A
ATb = A.T * b
print(f"ATA: {ATA}")
print(f"ATb: {ATb.evalf(4)}")

k_matrix = ATA.solve(ATb)
k_value = k_matrix[0]

t = symbols('t')
model = 150 * exp(k_value * t)

p_2010 = model.subs(t, 6)
print(f"model: {latex(model)}")
print(f"\nprediction for 2010 (t=6): {p_2010.evalf(3)} million")
```

```
ATA: Matrix([[55]])
ATb: Matrix([[7.207]])
model: 150 e^{0.131031553146226 t}

prediction for 2010 (t=6): 329 million
```

[323]:

```
import numpy as np
import matplotlib.pyplot as plt

t_data = np.array([0, 1, 2, 3, 4, 5])
p_data = np.array([150, 179, 203, 227, 250, 281])
p0 = 150

A = np.array([1, 2, 3, 4, 5])
b = np.log(p_data[1:] / p0)

ATA = np.dot(A, A)
ATb = np.dot(A, b)

k_optimal = ATb / ATA

def model_func(t):
    return p0 * np.exp(k_optimal * t)


t_smooth = np.linspace(0, 7, 100)
p_smooth = model_func(t_smooth)

t_2010 = 6
p_2010_pred = model_func(t_2010)
```

```
plt.figure(figsize=(10, 6))

plt.plot(t_smooth, p_smooth, label=f'Model: $p(t) = 150e^{{{k_optimal:.
  ↪4f}t}}$', color='blue', linewidth=2)
plt.scatter(t_data, p_data, color='red', s=100, zorder=5, label='US Census Data␣
  ↪(1950-2000)')
plt.scatter(t_2010, p_2010_pred, color='green', marker='X', s=150, zorder=5,␣
  ↪label=f'2010 Prediction: {p_2010_pred:.1f}M')

plt.title('US Population Growth Model (Least Squares Fit)', fontsize=14)
plt.xlabel('Decades since 1950 (t)', fontsize=12)
plt.ylabel('Population (Millions)', fontsize=12)
plt.grid(True, linestyle='--', alpha=0.7)
plt.legend(fontsize=11)
plt.text(0.5, 280, f'Calculated $k \\approx {k_optimal:.5f}$', fontsize=12,␣
  ↪bbox=dict(facecolor='white', alpha=0.8))

plt.tight_layout()
plt.show()
```



1.7 Question 4

1.7.1 Statement

We're being given a matrix

$$\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}$$

and we want the singular values from this.

[324]:
```python
import numpy as np

A = np.array([
    [3, -1],
    [2, 4]
])

ATA = A.T @ A
eigenvalues, eigenvectors = np.linalg.eigh(ATA)
ordering = np.argsort(eigenvalues)[::-1]
eigenvalues = eigenvalues[ordering]
eigenvectors = eigenvectors[:, ordering]
print(latex(np.sqrt(eigenvalues)))
```

```
\mathtt{\text{[4.51499333 3.10077977]}}
```

$$_n = [4.51499333\ 3.10077977]$$

1.8 Question 4

1.8.1 Part A

This would mean that we could calculate the Frobenius norm as

$$||\mathbf{A}||_F = \sqrt{(4.51499333)^2 + (3.10077977)^2} = 5.464985704219043$$

Similarly, we can calculate the Frobenius norm directly from the matrix as

$$||\mathbf{A}||_F = \sqrt{3^2 + (-1)^2 + 2^2 + 4^2} = \sqrt{9 + 1 + 4 + 16} = \sqrt{30} = 5.477225575051661$$

1.9 Question 4

1.9.1 Part B

We know want the l-2 norm, $||\mathbf{A}||_2 = \max(\sigma)$.

$$||\mathbf{A}||_2 = \max(\ ) = \sigma_1 = 4.51499333$$