1 Question 1

Determine Examine the transformations below and Determine whether they are linear. Justify this by the definition as given in class, showing that it does indeed hold, or showing where on which condition things break down.

1.1 Part a

$T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(\mathbf{x}) = a\mathbf{x} + \mathbf{b}$ $\quad \forall b \in \mathbf{R}^{n \times n}$

Checking homogeneity

    Left hand side $\quad T(c\mathbf{x}) = \mathbf{a}(c\mathbf{x}) + \mathbf{b} = c(\mathbf{ax}) + \mathbf{b}$

    Right hand side $\quad cT(\mathbf{x}) = c(\mathbf{ax} + \mathbf{b}) = c(\mathbf{ax}) + c\mathbf{b}$

    Conclusion Homogeneity breaks as the two sides are not equivalent.

1.2 Part B

$T : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ by $T(\mathbf{x}) = \mathbf{Ax} - \mathbf{xA}$

Checking homogeneity

    Left hand side $\quad cT(\mathbf{x}) = c(\mathbf{Ax} - \mathbf{xA}) = c\mathbf{Ax} - c\mathbf{xA}$

    Right hand side $\quad T(c\mathbf{x}) = \mathbf{A}(c\mathbf{x}) - c\mathbf{xA}$

    Conclusion Homogeneity checks out. We can bubble the c outwards and equate the two sides.

Checking additivity

    Left hand side $\quad T(\mathbf{u} + \mathbf{v}) = \mathbf{A}(\mathbf{u} + \mathbf{v}) - (\mathbf{u} + \mathbf{v})\mathbf{A} = \mathbf{Au} + \mathbf{Av} - \mathbf{uA} - \mathbf{vA}$

    Right hand side $\quad T(\mathbf{u}) + T(\mathbf{v}) = (\mathbf{Au} - \mathbf{uA}) + (\mathbf{Av} - \mathbf{vA}) = \mathbf{Au} + \mathbf{Av} - \mathbf{uA} - \mathbf{vA}$

    Conclusion Unlike above, additivity checks out. I've seen some other places have more requirements, such as $\langle 0, 0, \dots \rangle \to \langle 0, 0, \dots \rangle$ but all of those follow from homogeneity and additivity.