

1 Question 1

Determine Examine the transformations below and Determine whether they are linear. Justify this by the definition as given in class, showing that it does indeed hold, or showing where on which condition things break down.

1.1 Part a

$$T : \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ by } T(\mathbf{x}) = \mathbf{a}\mathbf{x} + \mathbf{b} \quad \forall \mathbf{b} \in \mathbb{R}^{n \times n}$$

Checking homogeneity

$$\text{Left hand side} \quad T(c\mathbf{x}) = \mathbf{a}(c\mathbf{x}) + \mathbf{b} = c(\mathbf{a}\mathbf{x}) + \mathbf{b}$$

$$\text{Right hand side} \quad cT(\mathbf{x}) = c(\mathbf{a}\mathbf{x} + \mathbf{b}) = c(\mathbf{a}\mathbf{x}) + c\mathbf{b}$$

Conclusion Homogeneity breaks as the two sides are not equivalent. The relationship is not a linear transformation.

1.2 Part B

$$T : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n} \text{ by } T(\mathbf{x}) = \mathbf{A}\mathbf{x} - \mathbf{x}\mathbf{A}$$

Checking homogeneity

$$\text{Left hand side} \quad cT(\mathbf{x}) = c(\mathbf{A}\mathbf{x} - \mathbf{x}\mathbf{A}) = c\mathbf{A}\mathbf{x} - c\mathbf{x}\mathbf{A}$$

$$\text{Right hand side} \quad T(c\mathbf{x}) = \mathbf{A}(c\mathbf{x}) - c\mathbf{x}\mathbf{A}$$

Conclusion Homogeneity checks out. We can bubble the c outwards and equate the two sides.

Checking additivity

$$\text{Left hand side} \quad T(\mathbf{u} + \mathbf{v}) = \mathbf{A}(\mathbf{u} + \mathbf{v}) - (\mathbf{u} + \mathbf{v})\mathbf{A} = \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v} - \mathbf{u}\mathbf{A} - \mathbf{v}\mathbf{A}$$

Right hand side

$$(\mathbf{u}) + T(\mathbf{v}) = (\mathbf{A}\mathbf{u} - \mathbf{u}\mathbf{A}) + (\mathbf{A}\mathbf{v} - \mathbf{v}\mathbf{A}) = \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v} - \mathbf{u}\mathbf{A} - \mathbf{v}\mathbf{A}$$

Conclusion Unlike above, additivity checks out. I've seen some other places have more requirements, such as $\langle 0, 0, \dots \rangle \rightarrow \langle 0, 0, \dots \rangle$ but all of those follow from homogeneity and additivity. So, it is a linear transformation.

1.3 Part C

$\phi : \mathbb{C} \rightarrow \mathbb{C}$ by $\phi(a + bi) = b + ai$ and $c \in \mathbb{C}$ where $c = d + ei$

Checking homogeneity

Left hand side $\phi(c[a + bi]) = \phi(ca + bi) = (d + ei)(a + bi) = ad - be$

Right hand side $c\phi(a + bi) = c[a + bi] = (d + ei)(a + bi) = ca + bi$

Conclusion Homogeneity breaks. Having done this incorrectly before though, with a non-complex scalar it will work. The relationship given is not a linear transformation. I want to check additivity just out of curiosity.

Examining additivity too Let $c = d + ei$ and $f = g + hi$ as $c, f \in \mathbb{C}$.

Left hand side $\phi(c + f) = \phi([d + ei] + [g + hi]) = \phi([d + g] + [e + h]i) = d + g - e - h$

Right hand side $\phi(c) + \phi(f) = \phi(d + ei) + \phi(g + hi) = e + di + h + gi$

Conclusion Guess it doesn't work here either. As $\phi(c + f) \neq \phi(c) + \phi(f)$... well, atleast it isn't always true, the transformation isn't linear.