1 Question 1

Determine Examine the transformations below and Determine whether they are linear. Justify this by the definition as given in class, showing that it does indeed hold, or showing where on which condition things break down.

1.1 Part a

$T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(\mathbf{x}) = a\mathbf{x} + \mathbf{b} \quad \forall b \in \mathbf{R}^{n \times n}$

Checking homogeneity

Left hand side $T(c\mathbf{x}) = \mathbf{a}(c\mathbf{x}) + \mathbf{b} = c(\mathbf{ax}) + \mathbf{b}$

Right hand side $cT(\mathbf{x}) = c(\mathbf{ax} + \mathbf{b}) = c(\mathbf{ax}) + c\mathbf{b}$

Conclusion Homogeneity breaks as the two sides are not equivalent. The relationship is not a linear transormation.

1.2 Part B

$T : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ by $T(\mathbf{x}) = \mathbf{Ax} - \mathbf{xA}$

Checking homogeneity

Left hand side $cT(\mathbf{x}) = c(\mathbf{Ax} - \mathbf{xA}) = c\mathbf{Ax} - c\mathbf{xA}$

Right hand side $T(c\mathbf{x}) = \mathbf{A}(c\mathbf{x}) - c\mathbf{xA}$

Conclusion Homogeneity checks out. We can bubble the c outwards and equate the two sides.

Checking additivity

Left hand side $T(\mathbf{u}+\mathbf{v}) = \mathbf{A}(\mathbf{u}+\mathbf{v}) - (\mathbf{u}+\mathbf{v})\mathbf{A} = \mathbf{Au} + \mathbf{Av} - \mathbf{uA} - \mathbf{vA}$

Right hand side

$$(\mathbf{u}) + T(\mathbf{v}) = (\mathbf{Au} - \mathbf{uA}) + (\mathbf{Av} - \mathbf{vA}) = \mathbf{Au} + \mathbf{Av} - \mathbf{uA} - \mathbf{vA}$$

Conclusion Unlike above, additivity checks out. I've seen some other places have more requirements, such as $\langle 0, 0, \dots \rangle \to \langle 0, 0, \dots \rangle$ but all of those follow from homogeneity and additivity. So, it is a linear transformation.

1.3 Part C

$\phi : \mathbb{C} \to \mathbb{C}$ by $\phi(a + bi) = b + ai$ and $c \in \mathbb{C}$ where $c = d + ei$

Checking homogeneity

Left hand side $\phi(c[a + bi]) = \phi(ca + bi) = (d + ei)(a + bi) = ad - be$

Right hand side $c\phi(a + bi) = c[a + bi] = (d + ei)(a + bi) = ca + bi$

Conclusion Homogeneity breaks. Having done this incorrectly before though, with a non-complex scalar it will work. The relationship given is not a linear transformation. I want to check additivity just out of curiosity.

Examining additivity too Let $c = d + ei$ and $f = g + hi$ as $c,\ f \in \mathbb{C}$.

Left hand side $\phi(c + f) = \phi([d + ei] + [g + hi]) = \phi([d + g] + [e + h]i) = d + g - e - h$

Right hand side $\phi(c) + \phi(f) = \phi(d + ei) + \phi(g + hi) = e + di + h + gi$

Conclusion Guess it doesn't work here either. As $\phi(c + f) \neq \phi(c) + \phi(f)$... well, atleast it isn't always true, the transformation isn't linear.

2 Question 2

Considering a matrix and finding $col(\mathbf{A})$, $row(\mathbf{A})$, $rank(\mathbf{A})$, $nul(\mathbf{A})$ We're to develop a basis for each, and I'm going to recite definitions as we go so they're a little fresher.

Given First, let's RREF this bad boi, knock it's numbers around a little

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 4 & 4 & 0 \\ 1 & -1 & 3 & 9 & 11 \\ 2 & 5 & -1 & -3 & -13 \\ 0 & 6 & -6 & 4 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2.1 col(A)

Definition The column space is the set of all possible linear combinations of a matrices column vectors.

$$\mathrm{col}(\mathbf{a}) = \mathrm{span}(\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n)$$

Basis I feel that we could be cheaty for all these, as, simply giving each column a coefficient would still generate a the same space, however, if we look at the terms of the reduced matrix we see that only three rows actually start with a one. So, disreguarding the 3^{rd} and 5^{th} columns we can more efficiently generate the space with

$$\text{col}(\mathbf{A}) = c_1 \begin{bmatrix} 3 \\ 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \\ 5 \\ 6 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 9 \\ -3 \\ 4 \end{bmatrix}$$

2.2 row(A)

Definition The row space is the set of all possible linear combinations of a matrices row vectors.

$$\text{row}(\mathbf{A}) = \text{span}(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n)$$

Basis Likewise as before, we can ditch those rows that have no leading ones.

$$\text{row}(\mathbf{A}) = c_1 \begin{bmatrix} 3 \\ 2 \\ 4 \\ 4 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 3 \\ 9 \\ 11 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 5 \\ -1 \\ -3 \\ -13 \end{bmatrix}$$

I think I should be writing this a little differently, like, $basis_{\text{row}(\mathbf{A})} = \ldots$ but that's a niggling issue for another day.

2.3 column(A)

Definition The column space is the set of all possible linear combinations of a matrices column vectors.

$$\text{row}(\mathbf{A}) = \text{span}(\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n)$$

Ran in to a little uncertainty here that I plan to clarify sometime I get the time. The prevailing wisdom is that I should take only those original column vectors that have corresponding pivots in the reduced matrix. We had pivots in the 1^{st}, 2^{nd}, and 4^{th} columns, so if we retain only those we can create a basis like so.

Basis

$$\text{col}(\mathbf{A}) = c_1 \begin{bmatrix} 3 \\ 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \\ 5 \\ 6 \end{bmatrix} + c_3 \begin{bmatrix} 4 \\ 9 \\ -3 \\ 4 \end{bmatrix}$$

We could have just as easily enumerated all the possible column vectors, but I'm reasonably confident in the way I'm interepreting the reduced form. Do need to double check and better understand why though.

2.4 null(A)

Definition The null space is the of all possible linear combinations that, when multiplied by the original matrix, result in the zero vector

$$\text{null}(\mathbf{A}) = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{0}\}$$

Starting with our reduced form, we can augment said matrix with the zero vector and solve

Solution We imagine a column of zero's along the right side of the previous. We have two free, so, choose $x_3 = 0$ and $x_5 = 1$ so that we don't end up with a trivial solution. Doing this we get... $x_1 = 0$, $x_2 = 2$, $x_4 = -1$ We still need one more though, with two free variables. So, for the other, let $x_3 = 1$ and $x_5 = 0$. We get $x_4 = 0$, $x_1 = -2$, $x_2 = 1$.

Basis

$$\text{null}(\mathbf{A}) = c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

2.5 rank(A)

We've already done all the heavy lifting here. We know that $\text{rank}(\mathbf{A}) = 3$ from way back. It was the number of pivots in our reduced form.

3 Question 3

We want to find all possible values of a matrix $\mathbf{A}$ such that $\text{rank}(\mathbf{A}) = 3$ and that $\text{rank}(\mathbf{A}) = 2$.

Given

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & \alpha \\ 4 & 4 & 8 \\ \alpha & 1 & 4 \end{bmatrix}$$

Finding where rank is not 3 Let's jump in and take the determinant to start, this should give us a relationship we can work with.

$$\det(\mathbf{A}) = \alpha(16 - 4\alpha) - 1(8 - 4\alpha) + 4(4 - 8) = -4\alpha^2 + 20\alpha - 24$$

When the determinant is 0, we can't be spanning all of $\mathbb{R}^3$ so

$$-4\alpha^2 + 20\alpha - 24 = 0 \to \alpha^2 - 5\alpha + 6 = 0$$

So, we have solutions where $\alpha = 2$ and $\alpha = 3$. If either of these is true, we do not span R^3

Fining where the rank is 2 So, we know we don't span $\mathbb{R}^3$ is $\alpha = 3$ or $\alpha = 2$. Now we need to figure out which of these might limit our span even down to $\mathbb{R}$. I'm going to borrow a trick from Jack and make this simple. Because we have a 2x2 submatrix whose determinant is nonzero,

$$det(\begin{bmatrix} 1 & 2 \\ 4 & 4 \end{bmatrix}) \neq 0$$

For both values $\alpha = 2$ and $\alpha = 3$, the span$(\mathbf{A}) = \mathbb{R}^2$

Finiding where the rank is 3 Likewise with before, we've already done all the work there. There are only 2 values of α that will lead to nonzero determinants, therefor any value for which $\alpha \neq 2$ and $\alpha \neq 3$ the span$(\mathbf{A}) = \mathbb{R}^3$.

4 Question 4

We want the null$(\mathbf{A})$ and null$(\mathbf{B})$

Given

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 3 & 6 & 3 & 6 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

For A This looks a little awkward, but, we're still after the same thing

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It should be easy to see that any values x_1, x_2, x_3, x_4 in $\mathbb{R}$ will satisfy this equation. So, we can write the basis for the null space as

$$\text{null}(\mathbf{A}) = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{I}_4$$

For B Ew. This looks nasty, but, we can row reduce it and make it a little easier.

$$\begin{bmatrix} 3 & 6 & 3 & 6 \\ 1 & 2 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Associating this with the corresponding vector we get. . .

$$x_1 + 2x_2 + x_3 + 2x_4 = 0$$

We could totally just leave things here, we're saying that you can pick anything you want for the last three, and that only the first has to be determined. Anything that satisfies above will solve $\mathbf{Bx} = \mathbf{0}$. Continuing on though, we can do a little rearranging. We have three free variables, and this kind of flies in the face of intuition, but we can follow the methodology presented in class. If we specify the relationships for each x_n in terms of the free variables, taking x_2, x_3, and x_4 as free, we get...

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_3 - 2x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

How's that for nasty? We have so many free variables we can write the null space in terms of the other variables. This is both our nullspace **and** the basis for our nullspace...

5 Question 5

Starting with a 4×8, and adding as many ones as possible so that x_2, x_4, x_5, and x_6 are free.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix}$$

Restrictions We know that we can't have pivots in columns for those entires. However, we do have to have pivots for those variables that aren't free. Let's do the latter first, then we can pad right. Also, because we're assuming RREF'ed, nix anything in the lower diagonal.

With pivots

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

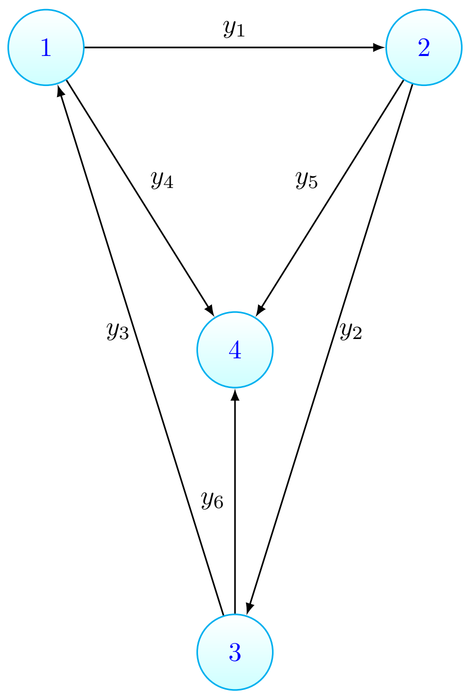
Padding right From there, we have considerable freedom, **if** we have a pivot in that column, then we **don't** have a 1 anywhere else.

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

That should satisfy. I can't think where else I could place a one and not break anything else.

6 Question 6

Depiction



This looks familiar, let's take a column of zero's as our augment, then we can equate

Analysis

$$0 = -y_1 + y_3 + y_4$$
$$0 = y_1 - y_2 - y_5$$
$$0 = y_2 - y_3 - y_6$$
$$0 = y_4 + y_5 + y_6$$

$$\begin{bmatrix} -1 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We're after the null space of this matrix, so, let's reduce.

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Ok, we're missing pivots in... the 3$^{\text{rd}}$, 5$^{\text{th}}$, and 6$^{\text{th}}$ columns. So, we can write the relationships for those in terms of the other variables.

$$\begin{aligned} y_1 &= y_3 - y_4 \\ y_2 &= y_3 + y_6 \\ y_3 &= y_3 \\ y_4 &= y_4 \\ y_5 &= -y_6 - y_4 \\ y_6 &= y_6 \end{aligned}$$

And this should form the basis for our null space, we can take a linear combination of the free variables to produce...

$$\text{null}(\mathbf{A}) = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Meaning the basis for the null space.

7 Question 7

We can create rank one matrices by multiplying two column vectors like so $\mathbf{u}\mathbf{v}^T$.

Given

$$\mathbf{A} = \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix}$$

7.1 Part A

We want to find the two vectors such that we can generate **A**. Let's set that up.

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix}$$

$$\begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix}$$

Keeping everything as a matrix to save from the massive system of equations that arrises. Figuring by looking we can devise that $u_1 = 3$, $u_2 = 1$, $u3 = 4$ and $v_1 = 1$, $v_2 = 2$, $v_3 = 2$.

$$\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix}$$

7.2 Part B

We want to show that the product of two rank one matrices also has rank one. Let's borrow the expanded matrix from above as a clean place to start. We know from above we'd create two rank one matrices like so...

$$\begin{bmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{bmatrix} \begin{bmatrix} x_1y_1 & x_1y_2 & x_1y_3 \\ x_2y_1 & x_2y_2 & x_2y_3 \\ x_3y_1 & x_3y_2 & x_3y_3 \end{bmatrix}$$

Expanding these on paper, then bringing back the factored form

$$\begin{bmatrix} u_1(v_1x_1 + v_2x_2 + v_3x_3)y_1 & u_2(v_1x_1 + v_2x_2 + v_3x_3)y_1 & u_3(v_1x_1 + v_2x_2 + v_3x_3)y_1 \\ u_1(v_1x_1 + v_2x_2 + v_3x_3)y_2 & u_2(v_1x_1 + v_2x_2 + v_3x_3)y_2 & u_3(v_1x_1 + v_2x_2 + v_3x_3)y_2 \\ u_1(v_1x_1 + v_2x_2 + v_3x_3)y_3 & u_2(v_1x_1 + v_2x_2 + v_3x_3)y_3 & u_3(v_1x_1 + v_2x_2 + v_3x_3)y_3 \end{bmatrix}$$

$$= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} (v_1x_1 + v_2x_2 + v_3x_3) \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}$$

Where rank is one We can see that this is still a product of two vectors, and therefore has rank one. The scalar in the middle doesn't change anything, it just scales the resulting matrix.

Where rank is zero The only way this could fail to be rank one is if the scalar in the middle is zero. This would happen if $v_1x_1 + v_2x_2 + v_3x_3 = 0$. In this case, the resulting matrix would be a zero matrix, which has rank zero.

7.3 Part C

We want to find $\mathbf{A}_1$ and $\mathbf{A}_2$ such that $\operatorname{rank}(\mathbf{A}_1\mathbf{B}) = 1$ and $\operatorname{rank}(\mathbf{A}_2\mathbf{B}) = 0$.

Given

$$\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Finding $\mathbf{A}_1$ Let's borrow from above, we know that if we multiply two rank one matrices we get another rank one matrix. So, let's just pick something simple.

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{A}_1\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\operatorname{rank}(\mathbf{A}_1\mathbf{B}) = 1$$

Finding $\mathbf{A}_2$ We know from above that if the scalar in the middle is zero, then the resulting matrix will be a zero matrix, which has rank zero. So, let's pick something that will make that happen.

$$\mathbf{A}_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{A}_2\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\operatorname{rank}(\mathbf{A}_2\mathbf{B}) = 0$$

Very much rushing through this last part, but the intuition is there.