

1 Question 1

Given We're given a set of points that have different offsets based on two different basis vectors. First, let's grab these points and where they map to in the new basis.

1.1 Part A

\mathbf{B}_x	\mathbf{B}_y	\mathbf{B}'_x	\mathbf{B}'_y
1	4	0	1
4	7	1	2
-8	-5	-3	-2
7	1	1	0
$5\frac{1}{2}$	-5	3	$-\frac{1}{2}$
-6	$-1\frac{1}{2}$	$-2\frac{1}{2}$	-1

Where I've denoted \mathbf{B} as the black basis, and \mathbf{B}' as the red. These taken together give us plenty of information to find a change of basis.

1.2 Part B

The points above are of the form $[\mathbf{v}_n]_{\mathbf{B}} \rightarrow [\mathbf{v}_n]_{\mathbf{B}'} \forall n$. So we can form a system mapping two columns of the first to that of the second.

The system Let \mathbf{P} be the change of basis we're attempting to find.

$$\mathbf{P} \begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

We need to clear out the left hand side, so take the matrix besides \mathbf{P} as \mathbf{A} , and find the inverse. As $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}_2$

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \frac{1}{9} \begin{bmatrix} 7 & -4 \\ -4 & 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$$

Checking solution

Let's check this solution by multiplying \mathbf{P} by another point in the original basis, and see if it maps to the correct point in the new basis.

$$\mathbf{P} \begin{bmatrix} -8 \\ -5 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -8 \\ -5 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} -32 + 5 \\ -8 - 10 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} -27 \\ -18 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$$

This matches the expected output, so our change of basis is correct.

1.3 Part C

Now we're after a way to go in reverse. We could be super diligent and do the same process, but we can also just take the inverse of \mathbf{P} .

$$\mathbf{P}^{-1} = \begin{bmatrix} \frac{4}{9} & -\frac{1}{9} \\ \frac{1}{9} & \frac{2}{9} \end{bmatrix}$$

Checking solution

Let's check this solution by multiplying \mathbf{P}^{-1} by another point in the red basis, and see if it maps to the correct point in the black basis.

$$\mathbf{P}^{-1}[\mathbf{v}_2]_{\mathbf{B}'} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix} = [\mathbf{v}_2]_{\mathbf{B}}$$

Again, seems the intuition is correct. For this point atleast, it seems our change of basis is mapping back from black to red.

1.4 Part D

Ok. I'm reading stretch so, eigenvalues, let's give things a name before we get going

Given

First line We are given a stretch factor of 2 corresponding to the line $y = -\frac{1}{2}x$ So that gives us

$$T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

Where we take $\lambda_1 = 2$ as the eigenvalue

Second line For the second we're given a stretch factor of -1 and a line $y = 4x$. From this

$$T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

Where we take $\lambda_2 = -1$ as the eigenvalue

Forming a system We know that we want a 2×2 matrix, let's call it \mathbf{F} , such that

$$\mathbf{F} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where

$$\mathbf{F} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$\mathbf{F} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

solving for a , b , c , and d we get

$$2a - b = 4$$

$$2c - d = -2$$

$$a + 4b = -1$$

$$c + 4d = -4$$

which results $a = \frac{5}{3}$, $b = -\frac{2}{3}$, $c = -\frac{4}{3}$, and $d = -\frac{2}{3}$. Giving us a final matrix of

$$\mathbf{F} = \begin{bmatrix} \frac{5}{3} & -\frac{2}{3} \\ \frac{4}{3} & -\frac{2}{3} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 & -2 \\ 4 & -2 \end{bmatrix}$$

Mapping red to black

Given

$$[\mathbf{x}]_{\mathbf{r}} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

Work We need to first map this back to the standard basis, then apply our stretch, then map it back to the red basis. We can borrow our \mathbf{P} from before

$$\mathbf{P}[\mathbf{x}]_{\mathbf{r}} = \begin{bmatrix} \frac{4}{9} & -\frac{1}{9} \\ \frac{1}{9} & \frac{2}{9} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{7}{18} \\ \frac{2}{9} \end{bmatrix} = [\mathbf{x}]_{\mathbf{e}}$$

And applying our \mathbf{F} stretch

$$\mathbf{F}[\mathbf{x}]_{\mathbf{e}} = \frac{1}{3} \begin{bmatrix} 5 & -2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \frac{7}{18} \\ \frac{2}{9} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{10}{27} \end{bmatrix}$$

The answer above is in the black basis.

Mapping black to red

Given

$$[\mathbf{x}]_{\mathbf{e}} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

We want to apply our transformation, then represent our answer in the red basis.

Work Applying our \mathbf{F} stretch

$$\mathbf{F}[\mathbf{x}]_{\mathbf{e}} = \frac{1}{3} \begin{bmatrix} 5 & -2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} -7 \\ -6 \end{bmatrix}$$

Now, back to the red basis, we borrow P^{-1} from above

$$\mathbf{P}^{-1}\mathbf{F}[\mathbf{x}]_{\mathbf{e}} = \begin{bmatrix} \frac{4}{9} & -\frac{1}{9} \\ \frac{1}{9} & \frac{2}{9} \end{bmatrix} \begin{bmatrix} -7 \\ -6 \end{bmatrix} = \begin{bmatrix} -\frac{22}{9} \\ -\frac{19}{9} \end{bmatrix}$$