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MATH 620 HW 9 1) a)

(1) Let A be square w/ λ and v.

a) prove that $\forall n \in \mathbb{N}$, λ^n is an eigenvalue w/ v_1, call the eigenvector x

Proof

By definition $Ax = \lambda x$. From this the case $n=1$ gives $A^1 x = \lambda^1 x$, which is trivially true. By inductive hypothesis we can state that $\forall k \in \mathbb{N}$ $A^k x = \lambda^k x$. Now by induction, let $n = k+1$, and, we're hoping to show that $A^{k+1}x = \lambda^{k+1}x$.

$$
\begin{aligned}
A^{k+1}x &= \lambda^{k+1}x \\
A(A^k)x &= \\
A(A^k x) &= && \text{by associativity} \\
A(\lambda^k x) &= && \text{by hypothesis} \\
\lambda^k (Ax) &= && \text{by scalar mult.} \\
\lambda^k (\lambda x) &= && \text{by definition} \\
\checkmark\ (\lambda^k \lambda)x &= && \text{by associativity} \\
\lambda^{k+1}x &= \lambda^{k+1}x && \text{by prop. of expons.}
\end{aligned}
$$

As was to be shown. By induction we know that $A^n x = \lambda^n x$ $\forall n \in \mathbb{N}$ where

MATH 620 HW 9 1) b)

1) Let A be square w/ λ & x as corresponding eigenvalue/eigenvector

b) We want to prove that if $\exists A^{-1}$ For A, then λ^{-1} is an eigenvalue

Proof

Let's deal w/ $\lambda = 0$ First, we'd expect to encounter a contradiction. We know

$Ax = \lambda x$; suppose $\lambda = 0$

$A^{-1}Ax = A^{-1}(0)x$

$x = 0$

Which it can't. So, constraining ourselves to the case where $\lambda \neq 0$, we can start as above

$$Ax = \lambda x$$

$A^{-1}(Ax) = A^{-1}\lambda x$	by left mult.
$(A^{-1}A)x =$	by assoc.
$I_n x =$	by prop. of inverses.
$x =$	by idempotence
$= \lambda(A^{-1}x)$	by scalar mult.
$1/\lambda\, x = A^{-1}x$	by division

As was to be shown. To conclude, we know that were $\lambda \neq 0$ and $\exists A, A^{-1}$, $1/\lambda$ is an eigenvalue of A^{-1}

MATH 620 HW 9 1) c)

1) Let A be square w/ corresponding eigens λ & x

c) If A is invertible then $\forall n \in \mathbb{Z}$, λ^n is an eigenvalue of A^n with x

Proof

For $n \in \mathbb{N}$ the proof is as above, we want to show that this holds for $n = 0$ & $n < 0$. For the simpler case where $n = 0$ we have

$$\underbrace{A^0 x = \lambda^0 x}_{\text{by subst.}} \Rightarrow \underbrace{Ix = (1)x}_{\text{by prop of iden.}} \Rightarrow \underbrace{x = x}_{\text{by iden.}}$$

Next we want to show it holds for $n < 0$. We can save some effort and use the above proof again. Let's state as given

& $A^{-n} = (A^{-1})^n$

& $\lambda^{-n} = (1/\lambda)^n$, if we take $Ax = \lambda x$

$$(A^{-1})^{-n} x = (1/\lambda)^{-n} x, \text{ let } m = -n$$

$$(A^{-1})^m x = (1/\lambda)^m x, \cancel{\text{let } B = A^{-1} \text{ \& } \lambda_2 = (1/\lambda)}$$

$$\cancel{B^{-m} x = \lambda_2^{-m} x}$$

$$A^{-m} x = \lambda^{-n} x$$

$$A^n x = \lambda^n x$$

As was to be shown. We now know that A^n & λ^n correspond for $n \in \mathbb{Z}$.

MATH 620 HW 9 2)

2) • Let A be 3×3 w/

$$\left\{\lambda_1 = \frac{-1}{3} : \begin{bmatrix}1\\0\\0\end{bmatrix}\ \lambda_2 = \frac{1}{3} : \begin{bmatrix}1\\1\\0\end{bmatrix}\ \lambda_3 = 1 : \begin{bmatrix}1\\1\\1\end{bmatrix},\ x = \begin{bmatrix}2\\1\\2\end{bmatrix}\right\}$$

We want to find $A^{20}x$. We can suspect that this will go the same way as given in class, that the dominant eigenvalue λ_3 and its corresponding eigenvector will become dominant. We need proportions first

$$\begin{bmatrix}2\\1\\2\end{bmatrix} = c_1\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2\begin{bmatrix}1\\1\\0\end{bmatrix} + c_3\begin{bmatrix}1\\1\\1\end{bmatrix},\ c_1=1,\ c_2=-1,\ c_3=2$$

Grabbing from the class work this gives

$$A^{20}x = c_1(\lambda_1^{20}v_1) + c_2(\lambda_2^{20}v_2) + c_3(\lambda_3^{20}v_3)$$

$$= 1\left(\frac{-1}{3}\right)^{20}v_1 - 1\left(\frac{1}{3}\right)^{20}v_2 + 2(1)^{20}v_3$$

We can see where this is going

$$= \begin{bmatrix}2\\2\\2\end{bmatrix} + \begin{bmatrix}0\\(-1/3)^{20}\\0\end{bmatrix} = \begin{bmatrix}2\\\text{almost } 2\\2\end{bmatrix}$$

As $k\to\infty$, $A^kx \to \begin{bmatrix}2\\2\\2\end{bmatrix} = 2\begin{bmatrix}1\\1\\1\end{bmatrix} = 2\bar{\lambda}_3$

MATH 620 HW 3)

3) Given an idempotent matrix A show that $\lambda = 0$ & $\lambda = 1$ are the only possibilities.

Proof

We could go by cases and end in cofusion here... or, use our old friend

$Ax = \lambda x$

$AAx = A\lambda x$ by left mul.

$A^2 x =$ by prop of exp

$Ax =$ by givens

$= \lambda A x$ by scalar mult.

$= \lambda \lambda x$ by definition

$= \lambda^2 x$ by prop of exp.

$\lambda x =$ by definition

$\lambda x - \lambda^2 x = 0$ by subtraction

$\lambda^2 x - \lambda x = 0$ by scalar mult

$\lambda(\lambda x - x) = 0$ by factoring

$\lambda = 0$ OR $(\lambda x - x) = 0$

$\lambda x = x$

$\lambda = 1$

As was to be shown. An idempotent matrix can only have eigenvalues of either 0, or 1.

MATH 620 HW 9 4)

4) $L = \begin{bmatrix} 1 & 5 & 3 \\ 1/3 & 0 & 0 \\ 0 & 2/3 & 0 \end{bmatrix}$

$\lambda_1 = 2$ by calculator ✓

characteristic: $3\lambda^3 - 3\lambda^2 - 5\lambda - 2$

$$\begin{array}{r|rrrr} 2 & 3 & -3 & -5 & -2 \\ & & 6 & 6 & 2 \\ \hline & 3 & 3 & 1 & 0 \end{array} \Rightarrow 3\lambda^2 + 3\lambda + 1$$

$x = \frac{-3 \pm \sqrt{9-12}}{6} = \frac{-3 \pm \sqrt{-3}}{6} = \frac{-3 \pm i\sqrt{3}}{6}$

$= \frac{-1}{2} \pm i\frac{\sqrt{3}}{6}$, let $\lambda_2 = \frac{-1}{2} + i\frac{\sqrt{3}}{6}$, $\lambda_3 = \frac{-1}{2} - i\frac{\sqrt{3}}{6}$

$|\lambda_2| = |\lambda_3| = \sqrt{\frac{1}{4} + \frac{3}{36}} = \frac{1}{3}$ take λ_1 as dominant

for $\lambda_1 = 2 \Rightarrow v_{1,1} - 18v_{1,3} = 0$ & $v_{1,2} - 3v_{1,3}$

$v_1 = (18v_{1,3}, 3v_{1,3}, v_{1,3})^T = (18, 3, 1)^T$ ✓

$v_{1,1} + v_{1,2} + v_{1,3} = 22$

$\text{norm}\left(\begin{bmatrix} 18 \\ 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 18/22 \\ 3/22 \\ 1/22 \end{bmatrix} = \begin{bmatrix} 9/11 \\ 3/22 \\ 1/22 \end{bmatrix} = \text{norm}(v_1)$

We'd expect the population distribution to converge to this over time.

MATH 620 HW 9 5)

5) If a Leslie matrix has a unique positive eigenvalue λ_1, what is the significance for the population if $\lambda_1 > 1, \lambda_1 = 1, \lambda_1 < 1$.

- Quick aside about uniqueness. Apparently we can guarantee that we will get a unique and positive eigenvalue, as any matrix with only positive entries, like a Leslie matrix, will have a positive, dominant, unique eigenvalue. So, the extra work on the last page was a bit moot.

The rest is straightforward.

if $\lambda_1 < 1$
the population is in exponential decline

✓ if $\lambda_1 = 1$
the population is stable

if $\lambda_1 > 1$
the population is in exponential increase

We'll keep the ratio's, the stretch direction, this is for the total number of initial populations

MATH 620 HW 9 6) a) b)

6) The Fib. recurrance $f_n = f_{n-1} + f_{n-2}$ has the assoc. matrix eq. $X_n = AX_{n-1}$ where

$$X_n = \begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ w/ } f_0 = 0 \text{ \& } f_1 = 1$$

a) <u>Proof</u>

- Base case, $n=1$, $A^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$

$$A^1 = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} f_2 & f_1 \\ f_1 & f_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \checkmark$$

- Inductive hypothesis, assume for $k \geq 1$

$$A^k = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k = \begin{bmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{bmatrix}$$

- Induction, For $n = k+1$, so we want to show

✓ $$A^{k+1} = AA^k = \begin{bmatrix} f_{k+1+1} & f_{k+1} \\ f_{k+1} & f_k \end{bmatrix} = \begin{bmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{bmatrix} \text{ from above}$$

- Stepping through.

$$\begin{bmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} f_{k+1} + f_k & f_{k+1} \\ f_k + f_{k-1} & f_k \end{bmatrix}, \quad f_{k+2} = f_{k+1} + f_k, \quad f_{k+1} = f_k + f_{k-1}$$

$$= \begin{bmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{bmatrix}$$

b) I could... but I recognice this and want to save paper

By prop of deters, $\det(A^k) = (\det(A))^k$ so

$$\det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) = -1 \text{ \& } \det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k\right) = (-1)^k$$