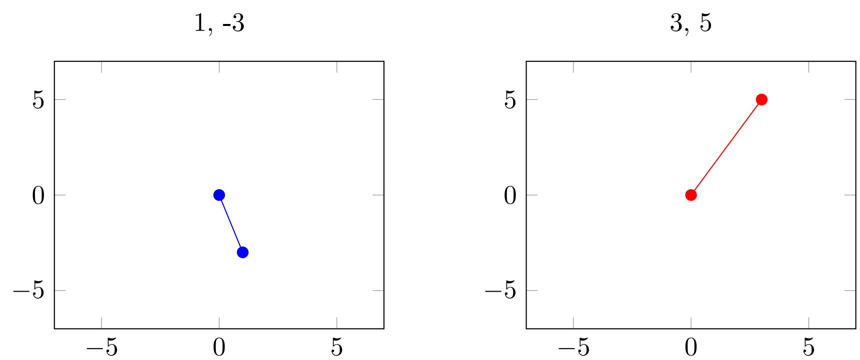
1 Question 1

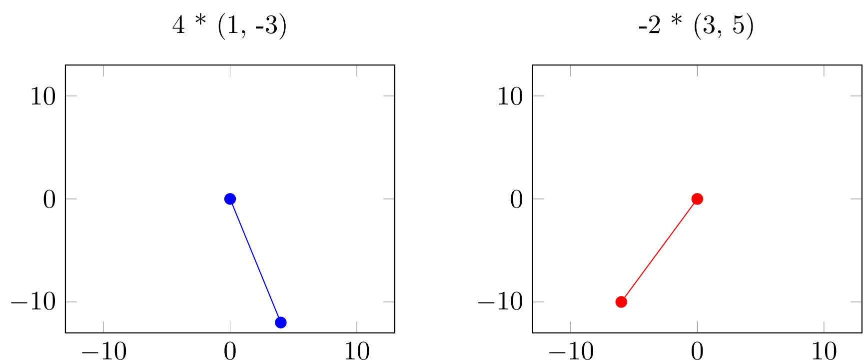
1.1 Consider

$$4 \begin{bmatrix} 1 \\ -3 \end{bmatrix} \text{ and } -2 \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

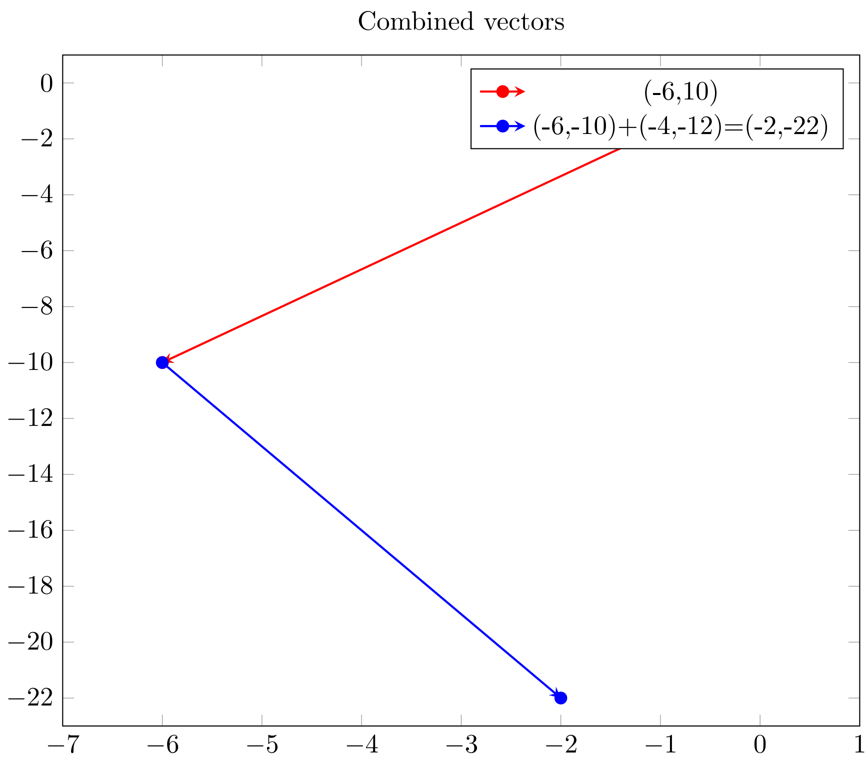
1.2 Graphical representation unscaled



1.3 Graphical representation scaled



1.4 Added together



1.5 Changes

As can be seen in the graphs above. The coefficient before each of the column matrices can be seen to scale any offset. For example, were we to take negative one times a matrix, then the offset would extend as far in the negative directions as a matrix extended in the positives. When we scale them by a multiplier and then add (subtract) from one another.

2 Question 2

2.1 Quick reiteration

Having found that we were unable to reach old man Gauss's house using only one form of transport, justify this conclusion with two approaches.

2.2 First approach

Induction For this to be true we'd, much like before, need a coefficient with which we'd be able to multiply by our column vector to get to $\begin{bmatrix}107\\64\end{bmatrix}$

$$c_1 \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}107\\64\end{bmatrix}$$

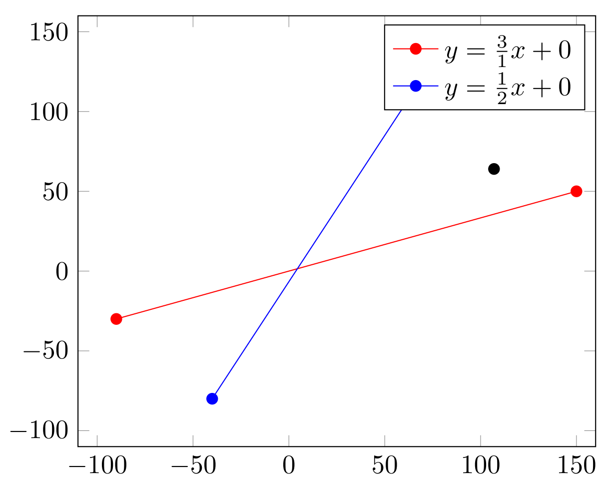
$$c_2 \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}107\\64\end{bmatrix}$$

However, there simply are not c_1 or $c_2 \in \mathbb{R}$ that satisfy either equation. It is therefor impossible to reach his home.

2.3 Second approach

Visual We can illustrate this as a form of argument. As $\begin{bmatrix}3\\1\end{bmatrix}$ forms a line $y = \frac{3}{1}x + 0$ That is, by scaling, we can point the end of this vector to any point on the line defined. However $(107, 64)$ is not on that line. No possible scaling could result in a point that is not on the line.

3, 1 as a line



Analysis As can be seen as depicted, either vector, when scaled, is insufficient to reach the point. It is only when we combine the two that we can reach both old man Gauss's house and the entirety of $\mathbb{R}^2$

3 Question 3

3.1 Quick reiteration

Introduction We *want* to show that given vectors $\vec{b_1}$ and $\vec{b_2}$ where $\vec{b_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{b_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ can be taken in a linear combination to form $\mathbb{R}^2$. *If* this were the case then we could expect that given any combination r_1, $r_2 \in \mathbb{R}$ would have solutions
$c_1, c_2 \in \mathbb{R}$ such that $c_1\vec{b_1} + c_2\vec{b_2} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$

Method To show this we can take a matrix approach. We can form an augmented matrix and then row reduce to find the solutions for c_1 and c_2 in terms of r_1 and r_2.

3.2 Matrix approach

$$\left[\begin{array}{cc|c} 3 & 1 & r_1 \\ 1 & 2 & r_2 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2}$$

$$\left[\begin{array}{cc|c} 1 & 2 & r_2 \\ 3 & 1 & r_1 \end{array}\right]$$

$$\xrightarrow{R_2 - 3R_1 \to R_2}$$

$$\left[\begin{array}{cc|c} 1 & 2 & r_2 \\ 0 & -5 & r_1 - 3r_2 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2 \to R_2}$$

$$\left[\begin{array}{cc|c} 1 & 2 & r_2 \\ 0 & 1 & \frac{3r_2 - r_1}{5} \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_2 \to R_1}$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{2r_1 + r_2}{5} \\ 0 & 1 & \frac{3r_2 - r_1}{5} \end{array}\right]$$

3.3 Conclusion

As can be seen, we have found c_1 and c_2 in terms of r_1 and r_2.

$$c_1 = \frac{2r_1 + r_2}{5}$$

$$c_2 = \frac{3r_2 - r_1}{5}$$

4 Question 4

4.1 Quick reiteration

Relationship between terms Now we're supposing in the more general case again. Let's suppose we want to find what restrictions we'd have to place on the terms in $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ such that

$$c_1 \begin{bmatrix} a \\ b \end{bmatrix} + c_2 \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

where $e_1, e_2 \in \mathbb{R}$.

Method

Determinant We can use the determinant of the matrix formed by the two column vectors to determine if the two vectors are linearly independent. If they are, then we can reach any point in $\mathbb{R}^2$. If they are not, then we can only reach points on the line formed by the two vectors.

$$\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc \neq 0$$

If we restrict ourselves to the case where $ad - bc = 0$ then we know that either one row is a multiple of another, or one row is a linear combination of the other. In either case, we can only reach points on the line formed by the two vectors.

Matrix approach We can also use a matrix approach to determine the restrictions on e_1 and e_2.

$$\left[\begin{array}{cc|c} a & c & e_1 \\ b & d & e_2 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2}$$

$$\left[\begin{array}{cc|c} b & d & e_2 \\ a & c & e_1 \end{array}\right]$$

$$\xrightarrow{R_2 - \frac{a}{b} R_1 \rightarrow R_2}$$

$$\left[\begin{array}{cc|c} b & d & e_2 \\ 0 & c - \frac{ad}{b} & e_1 - \frac{ae_2}{b} \end{array}\right]$$

$$\xrightarrow{b(c - \frac{ad}{b}) = 0}$$

$$\left[\begin{array}{cc|c} b & d & e_2 \\ 0 & 0 & e_1 - \frac{ae_2}{b} \end{array}\right]$$

For this to be consistent, we need $e_1 - \frac{ae_2}{b} = 0$.

$$e_1 = \frac{ae_2}{b}$$

$$be_1 = ae_2$$

$$ae_2 - be_1 = 0$$

4.2 Conclusion

We can see that for the two vectors to be linearly independent, we need $ad - bc \neq 0$. We can also see that for the system to be consistent (*ie. one solution*), we need $ae_2 - be_1 = 0$. So. If we get linearly independent without consistency, we have no solutions. If we get consistency without linear independence, we have infinite solutions. If we get both, we have one solution. If we get neither, we have no solutions. [1]

5 Question 5

5.1 Restating

Given two vectors $\vec{v_1}$ and $\vec{v_2}$ in $\mathbb{R}^2$, Is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in $span\{\vec{v_1}, \vec{v_2}\}$?

5.2 Answer

But of course! The zero vector is in the span of any set of vectors in $\mathbb{R}^2$. This is because we can always find coefficients c_1 and c_2 such that

$$c_1\vec{v_1} + c_2\vec{v_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Specifically, we can take $c_1 = 0$ and $c_2 = 0$.

$$0\vec{v_1} + 0\vec{v_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is true regardless of what $\vec{v_1}$ and $\vec{v_2}$ are, as long as they are in $\mathbb{R}^2$.

References

[1] Dependent Consistent Independent and Inconsistent YouTube. Consistent independent, dependent and inconsistent, 2012. Accessed: September 8, 2025.