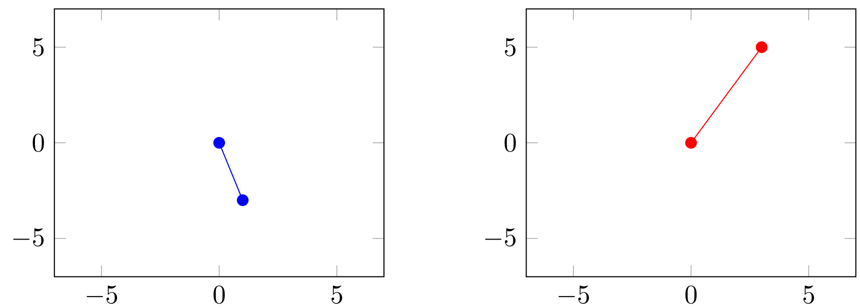
1 Question 1

1.1 Consider

$$4\begin{bmatrix}1\\-3\end{bmatrix} \text{ and } -2\begin{bmatrix}3\\5\end{bmatrix}$$

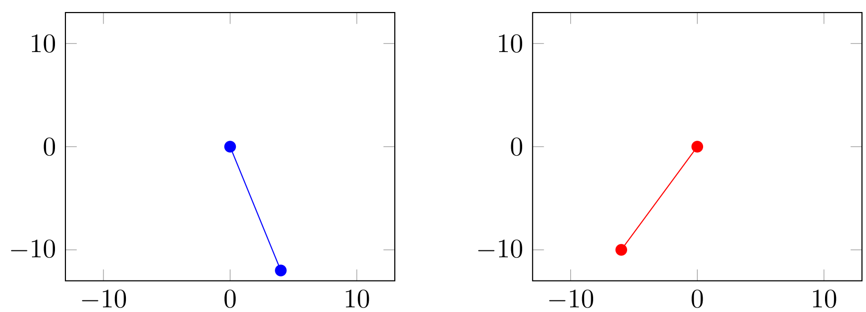
1.2 Graphical representation unscaled

1, -3 3, 5

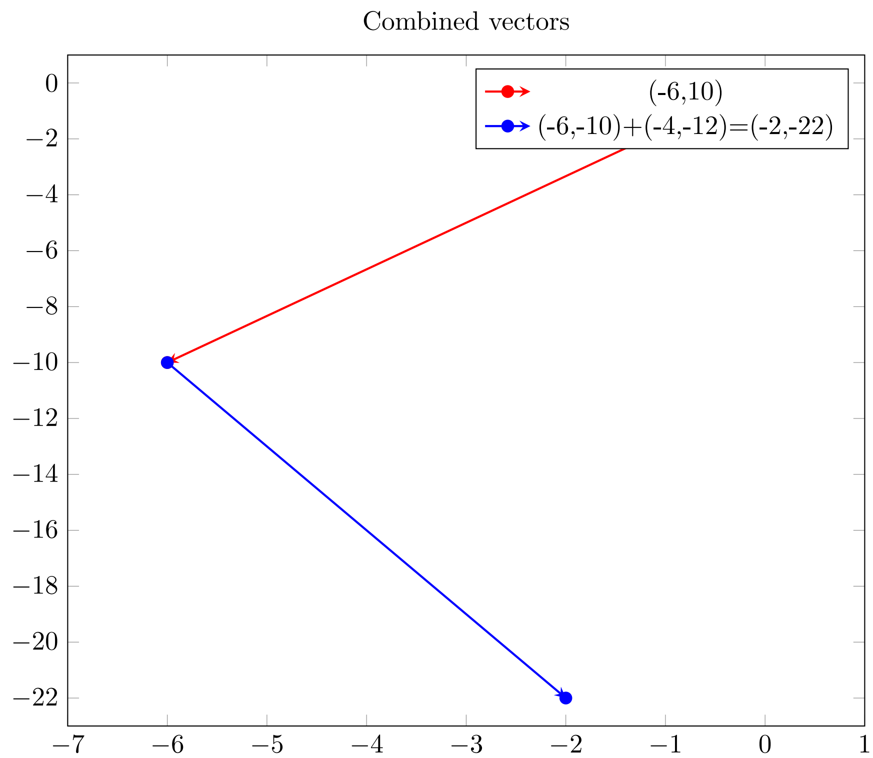


1.3 Graphical representation scaled

4 * (1, -3) -2 * (3, 5)



1.4 Added together



1.5 Changes

As can be seen in the graphs above. The coefficient before each of the column matrices can be seen to scale any offset. For example, were we to take negative one times a matrix, then the offset would extend as far in the negative directions as a matrix extended in the positives. When we scale them by a multiplier and then add (subtract) from one another.

2 Question 2

2.1 Quick reiteration

Having found that we were unable to reach old man Gauss's house using only one form of transport, justify this conclusion with two approaches.

2.2 First approach

Induction For this to be true we'd, much like before, need a coefficient with which we'd be able to multiply by our column vector to get to $\begin{bmatrix} 107 \\ 64 \end{bmatrix}$

$$c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 107 \\ 64 \end{bmatrix}$$

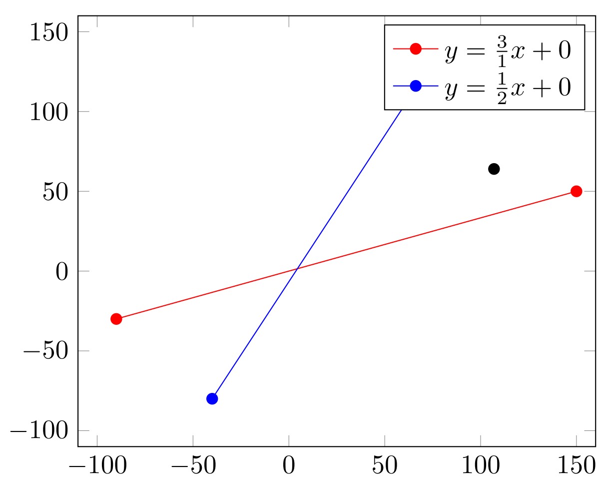
$$c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 107 \\ 64 \end{bmatrix}$$

However, there simply are not c_1 or $c_2 \in \mathbb{R}$ that satisfy either equation. It is therefor impossible to reach his home.

2.3 Second approach

Visual We can illustrate this as a form of argument. As $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ forms a line $y = \frac{3}{1}x + 0$ That is, by scaling, we can point the end of this vector to any point on the line defined. However $(107, 64)$ is not on that line. No possible scaling could result in a point that is not on the line.

3, 1 as a line



Analysis As can be seen as depicted, either vector, when scaled, is insufficient to reach the point. It is only when we combine the two that we can reach both old man Gauss's house and the entirety of $\mathbb{R}^2$

3 Question 3

3.1 Quick reiteration

Introduction We *want* to show that given vectors $\vec{b_1}$ and $\vec{b_2}$ where $\vec{b_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\vec{b_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ can be taken in a linear combination to form $\mathbb{R}^2$. *If* this were the case then we could expect that given any combination r_1, $r_2 \in \mathbb{R}$ would have solutions
$c_1, c_2 \in \mathbb{R}$ such that $c_1\vec{b_1} + c_2\vec{b_2} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}$

Method To show this we can take a matrix approach. We can form an augmented matrix and then row reduce to find the solutions for c_1 and c_2 in terms of r_1 and r_2.

3.2 Matrix approach

$$\left[\begin{array}{cc|c} 3 & 1 & r_1 \\ 1 & 2 & r_2 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2}$$

$$\left[\begin{array}{cc|c} 1 & 2 & r_2 \\ 3 & 1 & r_1 \end{array}\right]$$

$$\xrightarrow{R_2 - 3R_1 \to R_2}$$

$$\left[\begin{array}{cc|c} 1 & 2 & r_2 \\ 0 & -5 & r_1 - 3r_2 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2 \to R_2}$$

$$\left[\begin{array}{cc|c} 1 & 2 & r_2 \\ 0 & 1 & \frac{3r_2 - r_1}{5} \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_2 \to R_1}$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{2r_1 + r_2}{5} \\ 0 & 1 & \frac{3r_2 - r_1}{5} \end{array}\right]$$

3.3 Conclusion

As can be seen, we have found c_1 and c_2 in terms of r_1 and r_2.

$$c_1 = \frac{2r_1 + r_2}{5}$$

$$c_2 = \frac{3r_2 - r_1}{5}$$

4 Question 4

4.1 Quick reiteration

Relationship between terms Now we're supposing in the more general case again. Let's suppose we want to find what restrictions we'd have to place on the terms in $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ such that

$$c_1 \begin{bmatrix} a \\ b \end{bmatrix} + c_2 \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

where $e_1, e_2 \in \mathbb{R}$.

Method

Determinant We can use the determinant of the matrix formed by the two column vectors to determine if the two vectors are linearly independent. If they are, then we can reach any point in $\mathbb{R}^2$. If they are not, then we can only reach points on the line formed by the two vectors.

$$\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc \neq 0$$

If we restrict ourselves to the case where $ad - bc = 0$ then we know that either one row is a multiple of another, or one row is a linear combination of the other. In either case, we can only reach points on the line formed by the two vectors.

Matrix approach We can also use a matrix approach to determine the restrictions on e_1 and e_2.

$$\left[\begin{array}{cc|c} a & c & e_1 \\ b & d & e_2 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2}$$

$$\left[\begin{array}{cc|c} b & d & e_2 \\ a & c & e_1 \end{array}\right]$$

$$\xrightarrow{R_2 - \frac{a}{b} R_1 \rightarrow R_2}$$

$$\left[\begin{array}{cc|c} b & d & e_2 \\ 0 & c - \frac{ad}{b} & e_1 - \frac{ae_2}{b} \end{array}\right]$$

$$\xrightarrow{b(c - \frac{ad}{b}) = 0}$$

$$\left[\begin{array}{cc|c} b & d & e_2 \\ 0 & 0 & e_1 - \frac{ae_2}{b} \end{array}\right]$$

For this to be consistent, we need $e_1 - \frac{ae_2}{b} = 0$.

$$e_1 = \frac{ae_2}{b}$$

$$be_1 = ae_2$$

$$ae_2 - be_1 = 0$$

4.2 Conclusion

We can see that for the two vectors to be linearly independent, we need $ad - bc \neq 0$. We can also see that for the system to be consistent (*ie. one solution*), we need $ae_2 - be_1 = 0$. So. If we get linearly independent without consistency, we have no solutions. If we get consistency without linear independence, we have infinite solutions. If we get both, we have one solution. If we get neither, we have no solutions. [1]

5 Question 5

5.1 Restating

Given two vectors $\vec{v_1}$ and $\vec{v_2}$ in $\mathbb{R}^2$, Is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in $span\{\vec{v_1}, \vec{v_2}\}$?

5.2 Answer

But of course! The zero vector is in the span of any set of vectors in $\mathbb{R}^2$. This is because we can always find coefficients c_1 and c_2 such that

$$c_1\vec{v_1} + c_2\vec{v_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Specifically, we can take $c_1 = 0$ and $c_2 = 0$.

$$0\vec{v_1} + 0\vec{v_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is true regardless of what $\vec{v_1}$ and $\vec{v_2}$ are, as long as they are in $\mathbb{R}^2$.

6 Question 6

6.1 Restating

We want to find one vector $\in \mathbb{R}^3$, that is in the span of $span\{\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ -6 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\}$ and one that isn't.

Noticed The second vector is simply the first vector scaled by -2. This means that the span of these vectors is simply the span of the first and third vectors.

$$span\{\begin{bmatrix}2\\3\\1\end{bmatrix}, \begin{bmatrix}-4\\-6\\-2\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\} = span\{\begin{bmatrix}2\\3\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\}$$

In the span So, in the span, we can generate using a combination of two scalars

$$c_1 \begin{bmatrix}2\\3\\1\end{bmatrix} + c_2 \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}2c_1\\3c_1 + c_2\\c_1\end{bmatrix}$$

Let $c_1 = 2$ and $c_2 = 3$.

$$2 \begin{bmatrix}2\\3\\1\end{bmatrix} + 3 \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}4\\12\\2\end{bmatrix}$$

Outside of the span Outside of the span we can be a little devious and just maniupulate what we found before. Examine $\begin{bmatrix}4\\12\\3\end{bmatrix} + \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}5\\12\\3\end{bmatrix}$ This is not in the span. To show this, we can set up a quick system of equations.

$$\begin{gathered}2 * c_1 + 0 * c_2 = 5\\3 * c_1 + 1 * c_2 = 12\\1 * c_1 + 0 * c_2 = 3\end{gathered}$$

Taking from this, we can see an obvious problem. From the first and third equations, we can see that c_1 cannot be both 2.5 and 3. This means that there are no c_1 and c_2 that satisfy this equation. Therefore, $\begin{bmatrix}5\\12\\3\end{bmatrix}$ is not in the span.

7 Question 7

7.1 Trying to find

We're trying to find vectors for each set of conditions. If we think that they might not exist, we can try to explain.

7.2 Questions

a A set of three vectors in $\mathbb{R}^2$ that doesn't span $\mathbb{R}^2$

Answer: $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}3\\0\end{bmatrix}\right\}$

Explanation: No change no any coefficient will make any subsequent change in the second component.

b A set of three vectors in $\mathbb{R}^2$ that spans $\mathbb{R}^2$

Answer: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

Explanation: If we reduce these against an arbitrary vector $\begin{bmatrix} a \\ b \end{bmatrix}$, we can see that

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & 1 & 2 & b \end{array}\right] \xrightarrow{R_1 - R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & -1 & a-b \\ 0 & 1 & 2 & b \end{array}\right]$$

As we have a pivot in each row we can device that this will span $\mathbb{R}^2$

c A set of three vectors in $\mathbb{R}^2$ that spans $\mathbb{R}^2$ from which you can remove one vector and still span $\mathbb{R}^2$ with the remaining two.

Answer:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Explanation: With one removed we can form any of three combinations of the two remaining vectors. These being

$$\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}, \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\}, \{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$$

All of these pairs will span $\mathbb{R}^2$.

d A set of three vectors in $\mathbb{R}^2$ that spans $\mathbb{R}^2$ from which you can remove two vectors and still span $\mathbb{R}^2$ with the remaining one.

Answer: No such set of vectors exists.

Explanation: If we were to take a vector, any vector, alone, the most we can form is a line through the origin. This is because we can only scale the vector by a coefficient. $\nexists\, b_1,\, b_2 \in \mathbb{R}$ such that $c_1 b_1 = b_2\ \forall\ c_1 \in \mathbb{R}$.

8 Question 8

8.1 Restating

Take $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subset \mathbb{R}^3$ such that $span\{S\} = \mathbb{R}^3$.
What can be said about $c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} = \mathbf{0}$?

8.2 Deduction

We can take as given that no entire vector is entirely empty as it would be linearly dependent. We can also take as given that no two vectors are multiples of one another as they would also be linearly dependent.

8.3 Answer

$$\begin{aligned}
c_1\mathbf{v}_{1,1} + c_2\mathbf{v}_{2,1} + c_3\mathbf{v}_{3,1} &= 0\\
c_1\mathbf{v}_{1,2} + c_2\mathbf{v}_{2,2} + c_3\mathbf{v}_{3,2} &= 0\\
c_1\mathbf{v}_{1,3} + c_2\mathbf{v}_{2,3} + c_3\mathbf{v}_{3,3} &= 0
\end{aligned}$$

The only way we can asure that this is true is if $c_1 = c_2 = c_3 = 0$. This is because if any of the coefficients were non-zero, then we could rearrange the equation to show that one vector is a linear combination of the other two. This would contradict our initial assumption that the vectors spanned $\mathbb{R}^3$.

9 Question 9

9.1 Restating

Given east as the positive x and the north as the positive y. We're given three modes of transport

$$\mathbf{broomstick} = \begin{bmatrix}-1\\2\end{bmatrix}, \mathbf{magic\_carpet} = \begin{bmatrix}3\\4\end{bmatrix}, \mathbf{jet\_pack} = \begin{bmatrix}5\\0\end{bmatrix}$$

9.2 Questions

a Describe traveling to $\begin{bmatrix}3\\-6\end{bmatrix}$.

Answer: To reach $\begin{bmatrix}3\\-6\end{bmatrix}$

$$\left[\begin{array}{ccc|c} -1 & 3 & 5 & 3\\ 2 & 4 & 0 & -6 \end{array}\right]$$

Row reducing this matrix gives:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6\\ 0 & 1 & 0 & -3 \end{array}\right]$$

So, to make it to our destination, we'd take the broomstick... 6 times, and the magic carpet -3 times.

$$6\begin{bmatrix}-1\\2\end{bmatrix} + -3\begin{bmatrix}3\\4\end{bmatrix} = \begin{bmatrix}3\\-6\end{bmatrix}$$

b Write the system of equations that has the same soltion as above.

Answer:

$$\begin{aligned}-1c_1 + 3c_2 &= 3\\ 2c_1 + 4c_2 &= -6\end{aligned}$$

c Can $\begin{bmatrix} 3 \\ -6 \end{bmatrix}$, be reached in *i.* one way, *ii.* many ways, *ii.* no ways?

Answer: *ii.* many ways

Explanation:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -3 \end{array}\right]$$

If we just let $c_1 = -3$ and $c_2 = 2$ then for any c_3 we can reach our destination. This might make some sense, as, the jetpack was the only method without a y component.

References

[1] Dependent Consistent Independent and Inconsistent YouTube. Consistent independent, dependent and inconsistent, 2012. Accessed: September 8, 2025.