1 Question 1

Given We're given a set of points that have different offsets based on two different basis vectors. First, let's grab these points and where they map to in the new basis.

1.1 Part A

$\mathbf{B}_x$	$\mathbf{B}_y$	$\mathbf{B}'_x$	$\mathbf{B}'_y$
1	4	0	1
4	7	1	2
-8	-5	-3	-2
7	1	1	0
$5\frac{1}{2}$	-5	3	$-\frac{1}{2}$
-6	$-1\frac{1}{2}$	$-2\frac{1}{2}$	-1

Where I've denoted $\mathbf{B}$ as the black basis, and $\mathbf{B}'$ as the red. These taken together give us plenty of information to find a change of basis.

1.2 Part B

Alright. I did all this backwards, I'm going to take the L and redo what needs to be. I read these questions backwards, starting with red to black, and undoing my mistakes would be a mess, so, from scratch

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 4 & 7 \end{bmatrix}$$

Where I've combined the first couple points into a system to solve for, to be brief we find $a = 2$, $b = 1$, $c = -1$, and $d = 4$

Checking solution

And again, here we go, let's double check

$$\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Cool, call this $\mathbf{P}$

1.3 Part C

We know from examination in class that we can just take the inverse of the initial matrix to go the other way, so, $\mathbf{Q} = \mathbf{P}^{-1}$

$$\mathbf{Q} = \mathbf{P}^{-1} = \begin{bmatrix} \frac{4}{9} & -\frac{1}{9} \\ \frac{1}{9} & \frac{2}{9} \end{bmatrix}$$

Checking solution

We should expect that $(0, 1)$ should map to $(1, 4)$

$$\mathbf{Q}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{4}{9} & -\frac{1}{9} \\ \frac{1}{9} & \frac{2}{9} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

1.4 Part D

Ok. I'm reading stretch so, eigenvalues, let's give things a name before we get going

Given

First line We are given a stretch factor of 2 corresponding to the line $y = -\frac{1}{2}x$ So that gives us

$$T(\begin{bmatrix} 2 \\ -1 \end{bmatrix}) = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

Where we take $\lambda_1 = 2$ as the eigenvalue

Second line For the second we're given a stretch factor of -1 and a line $y = 4x$. From this

$$T(\begin{bmatrix} 1 \\ 4 \end{bmatrix}) = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$$

Where we take $\lambda_2 = -1$ as the eigenvalue

Forming a system We know that we want a $2x2$ matrix, let's call it $\mathbf{F}$, such that

$$\mathbf{F} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where

$$\mathbf{F}\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}4\\-2\end{bmatrix} \qquad \mathbf{F}\begin{bmatrix}1\\4\end{bmatrix}=\begin{bmatrix}-1\\-4\end{bmatrix}$$

solving for a, b, c, and d we get

$$\begin{aligned}2a-b&=4\\2c-d&=-2\\a+4b&=-1\\c+4d&=-4\end{aligned}$$

which results $a=\frac{5}{3}$, $b=-\frac{2}{3}$, $c=-\frac{4}{3}$, and $d=-\frac{2}{3}$. Giving us a final matrix of

$$\mathbf{F}=\begin{bmatrix}\frac{5}{3}&-\frac{2}{3}\\\frac{4}{3}&-\frac{2}{3}\end{bmatrix}=\frac{1}{3}\begin{bmatrix}5&-2\\4&-2\end{bmatrix}$$

Alright, we have our F that represents the transformation in the standard basis.

i. Convert $[\mathbf{x_r}]$ to e We need to convert first the point we have in the red basis to the standard basis. We're given that $[\mathbf{x_r}]=\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}$

$$\mathbf{P}\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}=\begin{bmatrix}2&1\\-1&4\end{bmatrix}\begin{bmatrix}1\\\frac{1}{2}\end{bmatrix}=\begin{bmatrix}\frac{5}{2}\\1\end{bmatrix}$$

Applying the transformation So, now we have $[\mathbf{x}]_\mathbf{e}=\begin{bmatrix}\frac{5}{2}\\1\end{bmatrix}$ We can apply our transformation matrix **F**

$$\mathbf{F}\begin{bmatrix}\frac{5}{2}\\1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}5&-2\\4&-2\end{bmatrix}\begin{bmatrix}\frac{5}{2}\\1\end{bmatrix}=\frac{1}{3}\begin{bmatrix}\frac{25}{2}-2\\10-2\end{bmatrix}=\frac{1}{3}\begin{bmatrix}\frac{21}{2}\\8\end{bmatrix}=\begin{bmatrix}\frac{7}{2}\\\frac{8}{3}\end{bmatrix}$$

ii. Convert $[\mathbf{x}]_\mathbf{e}$ to r

Going from black to red We're starting in the basis this time, so, we can apply the transformation, then convert. To my understanding, we could also apply this **to** to the matrix **F**, but I'm more confident about this approach. So, given $[\mathbf{x}]_\mathbf{e}=\begin{bmatrix}-3\\3\end{bmatrix}$

Applying the transformation

$$\mathbf{F}\begin{bmatrix}-3\\3\end{bmatrix}=\frac{1}{3}\begin{bmatrix}5&-2\\4&-2\end{bmatrix}\begin{bmatrix}-3\\3\end{bmatrix}=\frac{1}{3}\begin{bmatrix}-15-6\\-12-6\end{bmatrix}=\frac{1}{3}\begin{bmatrix}-21\\-18\end{bmatrix}=\begin{bmatrix}-7\\-6\end{bmatrix}$$

Changing basis Now, we need to convert this back to the red basis.

$$\mathbf{Q}\begin{bmatrix}-7\\-6\end{bmatrix} = \begin{bmatrix}\frac{4}{9} & -\frac{1}{9}\\ \frac{1}{9} & \frac{2}{9}\end{bmatrix}\begin{bmatrix}-7\\-6\end{bmatrix} = \begin{bmatrix}-\frac{22}{9}\\ -\frac{19}{9}\end{bmatrix}$$

Tada! We've transformed and changed basis. Please forgive any formatting for these few, hopefully it should make sense when you read through.

2 Question 2

Our boy Gauss is back, making our lives difficult again. We're keeping the modes of transport we had originally as

$$\beta = \left\{\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}\right\}$$

And, as I would've guessed... changing basis.

2.1 Part A

Given For one old Uncle Cramer, we're given that in the standard basis his house is located at $[\mathbf{w_e}] = \begin{bmatrix}25\\71\end{bmatrix}$

Changing basis I want to clarify my understanding for a moment. Above is simply analogous to saying that

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}25\\71\end{bmatrix} = \begin{bmatrix}25\\71\end{bmatrix}$$

And our desire to change basis forms a system like so

$$\begin{bmatrix}3 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}25\\71\end{bmatrix}$$

Let $\mathbf{P}$ be formed from the columns of our transport basis. It's inverse is then

$$\mathbf{P}^{-1} = \begin{bmatrix}3 & 1\\1 & 2\end{bmatrix}^{-1} = \frac{1}{5}\begin{bmatrix}2 & -1\\-1 & 3\end{bmatrix}$$

Lastly, applying our initial vector we get

$$\mathbf{P}^{-1}[\mathbf{w}]_{\mathbf{e}} = [\mathbf{w}]_\beta = \frac{1}{5}\begin{bmatrix}2 & -1\\-1 & 3\end{bmatrix}\begin{bmatrix}25\\71\end{bmatrix} = \frac{1}{5}\begin{bmatrix}-21\\188\end{bmatrix} = \begin{bmatrix}-\frac{21}{5}\\ \frac{188}{5}\end{bmatrix}$$

So, we'd travel $-\frac{21}{5}$ times by hoverboard, and $\frac{188}{5}$ times by magic carpet. Not exactly satisfying, but, hey.

2.2 Part B

Given Now in reverse we're given the location of a museum expressed in the transport basis as $[\mathbf{v}]_\beta = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$

Changing basis At the risk of turning one mistake in to two, this should be simply the inverse, ie. the original basis times the vector in the transport basis.

$$\mathbf{P}\,[\mathbf{v}]_\beta = [\mathbf{v}]_\mathbf{e} = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 27 \\ 14 \end{bmatrix}$$

3 Question 3

Given

$$\mathbf{A} = \begin{bmatrix} -3 & 4 \\ -2 & 6 \end{bmatrix}$$

Running the characteristic Gonna just do the work here, then answer the questions as they come along we're gonna need the characteristic polynomial for the first two questions anyways

3.1 Characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}_2) = 0 = \begin{bmatrix} -3 & 4 \\ -2 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -3-\lambda & 4 \\ -2 & 6-\lambda \end{bmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}_2) = 0 = (-3-\lambda)(6-\lambda) + 8 = -\lambda^2 - 3\lambda - 10$$

So, take $\lambda_1 = 5$ and $\lambda_2 = -2$... and back we go again for the eigenvectors.

3.2 Finding eigenvectors

For $\lambda_1 = 5$

$$\mathbf{A} - 5\mathbf{I}_2 = \begin{bmatrix} -3 & 4 \\ -2 & 6 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -3-5 & 4 \\ -2 & 6-5 \end{bmatrix} = \begin{bmatrix} -8 & 4 \\ -2 & 1 \end{bmatrix}$$

Reducing this matrix gives us

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

So we can take $\mathbf{v}_{1,1} = 1$ and $\mathbf{v}_{1,2} = 2$ giving us $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

For $\lambda_2 = -2$ Let's imagine, all that work, but with only slight changes. Borrowing from my friend pen and paper, we get $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

3.3 Questions

Part A The stretch factors are simply the eigenvalues we found above. So, $\lambda_1 = 5$ and $\lambda_2 = -2$. The stretch directions are the eigenvectors we found above. To relate this back to how we were presented this before, the stretch directions can be thought of as forming the axis along the lines $y = 2x$ and $y = \frac{1}{4}x$ respectively.

Part B The eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -2$. The eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Part C Alright, now for the tricky wicket. We want a similarity transformation that is similar to a diagonal matrix. That was given in the handout as

Definition Two matrices $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ are similar if $\exists\, \mathbf{C} \in \mathbb{R}^{n\times n}$ where $\exists\, \mathbf{C}^{-1}$ such that

$$\mathbf{B} = \mathbf{C}^{-1}\mathbf{A}\mathbf{C}$$

Work We can take $\mathbf{C} = [\mathbf{v}_1\, \mathbf{v}_2]$ as the matrix, then find our inverse

$$\mathbf{C}^{-1} = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{-7}\begin{bmatrix} 1 & -4 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} & \frac{4}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{bmatrix}$$

then, plug and chug in to $\mathbf{B} = \mathbf{C}^{-1}\mathbf{A}\mathbf{C}$

$$\mathbf{B} = \begin{bmatrix} -\frac{1}{7} & \frac{4}{7} \\ \frac{2}{7} & -\frac{1}{7} \end{bmatrix} \begin{bmatrix} -3 & 4 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$$

Which is indeed a diagonal matrix.

4 Question 4

Ok. I went down a few rabbit holes before I got a grasp here. The question is to explain why, that if the rows add up to a scalar k, then k is an eigenvalue. This made so much more sense once I put together that k is my stretch factor, and the vector of all ones is my eigenvector. Problems I would have avoided by actually reading the entire question.

$\mathbf{A}\mathbf{v}_1$

$$\mathbf{A}\mathbf{v}_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix}$$

$\lambda\mathbf{v}_1$

$$\lambda\mathbf{v}_1 = k \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} k \\ k \end{bmatrix}$$

Conclusion As we can see above, $\mathbf{A}\mathbf{v}_1 = \lambda\mathbf{v}_1$ where $\lambda = k$, so by definition k is an eigenvalue of $\mathbf{A}$. This is the case as $\mathbf{A}\mathbf{v}_1$ should result in a stretch of $\mathbf{v}_1$ by the factor of k.