section_6

February 7, 2025

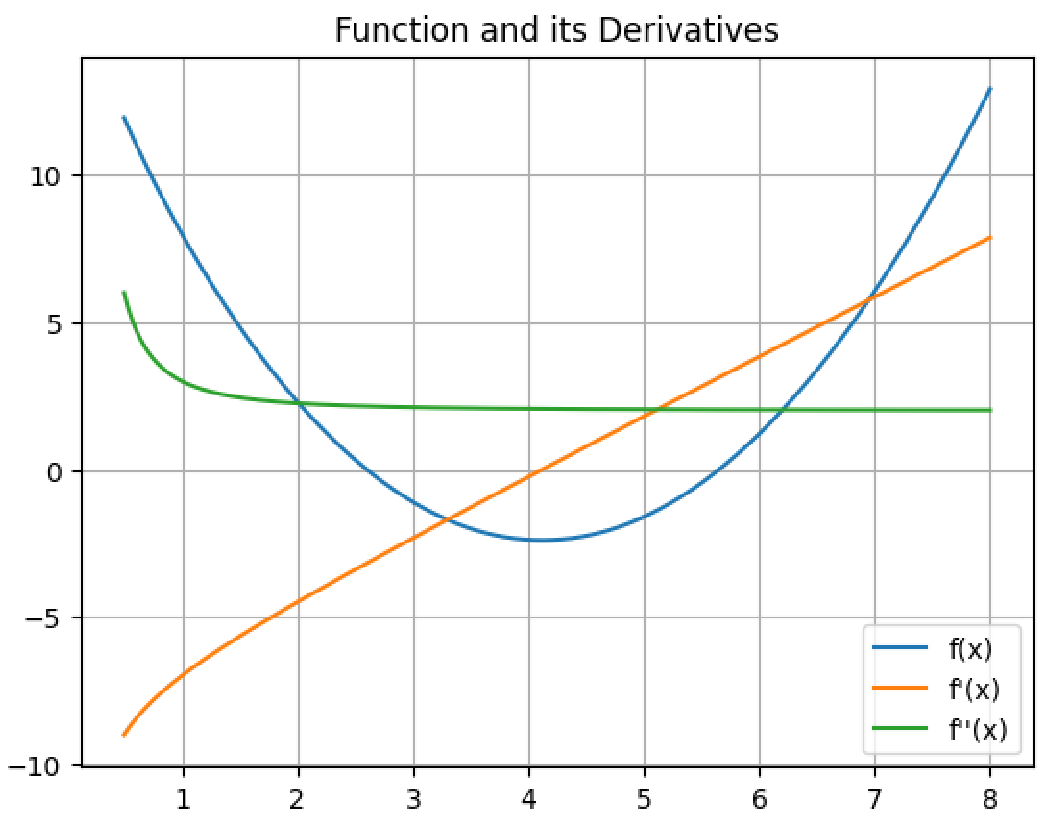
1 Section 6. Newton's method

(a) Consider the equation $ln(x) = (x - 4)^2 - 1$

[50]:
```python
import numpy as np
import matplotlib.pyplot as plt
import sympy

x = sympy.symbols('x')

f_of_x = (-sympy.log(x)) + ((x - 4) ** 2) - 1
f_prime_of_x = sympy.diff(f_of_x, x)
f_prime_prime_of_x = sympy.diff(f_prime_of_x, x)

# Lambdify the symbolic expressions to numerical functions
f_of_x_func = sympy.lambdify(x, f_of_x, 'numpy')
f_prime_of_x_func = sympy.lambdify(x, f_prime_of_x, 'numpy')
f_prime_prime_of_x_func = sympy.lambdify(x, f_prime_prime_of_x, 'numpy')


x_values = np.linspace(0.5, 8.0, 1000)

y_values = f_of_x_func(x_values)
y_prime_values = f_prime_of_x_func(x_values)
y_prime_prime_values = f_prime_prime_of_x_func(x_values)

f_prime_of_x_roots = sympy.solve(f_prime_of_x, x)
print(f"Roots of f': {f_prime_of_x_roots}")
for index, item in enumerate(f_prime_of_x_roots, 1):
    print(f'Root number {index} stated numerically: {item.evalf()}')
print(f"Root of f' stated numerically: {f_prime_of_x_roots[1].evalf()}")

plt.plot(x_values, y_values, label="f(x)")
plt.plot(x_values, y_prime_values, label="f'(x)")
plt.plot(x_values, y_prime_prime_values, label="f''(x)")
plt.legend()
```

```
plt.title("Function and its Derivatives")
plt.grid(True)
plt.show()
```

```
Roots of f': [2 - 3*sqrt(2)/2, 2 + 3*sqrt(2)/2]
Root number 1 stated numerically: -0.121320343559643
Root number 2 stated numerically: 4.12132034355964
Root of f' stated numerically: 4.12132034355964
```



Let $f(x) = (x-4)^2 - 1 - ln(x) \Rightarrow f'(x) = 2x^2 - 8x - 1 \Rightarrow f''(x) = x^{-2} + 2$

We can see the derivative crossing the x-axis only once, if we zoom out further, we'd see that $f'(x)$ is increasing from $x \approx 4.12$, we can also see that $f(x)$ is crossing the x-axis twice, for $f'(x)$ there's two roots, but, $ln(x)$ is *undefined* for $x \leq 0$

We could prove that there's no other roots to $f'(x)$ but, sympy has already told us otherwise. To borrow from work I've done previously, and tossing out the errogenous root, we know that as $f'(x)$ only crosses the x-axis once in the interval, we know we have at most two roots

■ There are at most two roots in the interval

(*ii*) Let α be the smaller of the two solutions. Use Newton's Method to find an approximation of α with an absolute error of less than 10^{-3}

[51]:
```
from typing import Callable, Optional

def newtons_method(
        fOfX : Callable[[float], float],
        dfOfX : Callable[[float], float],
        guess : float,
        maxIterations : int = 100,
        maxError : float = 1e-6) -> Optional[float]:

    def formula() -> float:
        return guess - (fOfX(guess) / dfOfX(guess))

    if maxIterations == 0 and fOfX(guess) > maxError:
        return None
    elif fOfX(guess) < maxError:
        return guess
    else:
        return newtons_method(fOfX, dfOfX, formula(), maxIterations - 1,␣
 ↪maxError)

f_of_x = lambda x : (-np.log(x)) + ((x - 4) ** 2) - 1
f_prime_of_x = lambda x : 2 * (x ** 2) - (8 * x) - 1
guess = newtons_method(
    fOfX = f_of_x,
    dfOfX = f_prime_of_x,
    guess = 1
)

print(f'Root at ({guess}, {f_of_x(guess)})')
```

```
Root at (2.601409918180815, 6.426907868117837e-07)
```

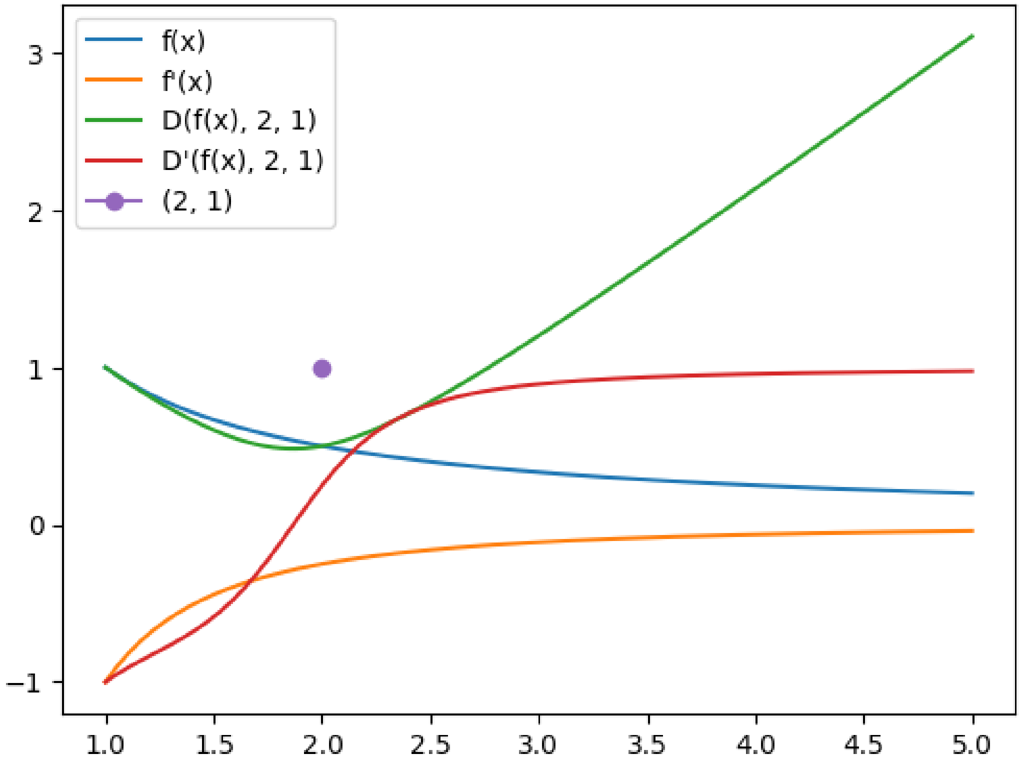
We can look at the graph before for a quick double check, this seems totally reasonable, the root is at $x \approx 2.6$

(*b*) Use Newton's method to approximate to within 10^{-4}, the value of x that produces the point on the graph of $f(x) = \frac{1}{x}$ that is closest to $(2, 1)$.

[52]:
```
f_of_x_sympy = 1/x

distance_of_f_of_x_sympy = sympy.sqrt((x - 2)**2 + (f_of_x_sympy - 1)**2)
distance_prime_of_f_of_x_sympy = sympy.diff(distance_of_f_of_x_sympy, x)
distance_prime_prime_of_f_of_x_sympy = sympy.
 ↪diff(distance_prime_of_f_of_x_sympy, x)
```

```python
f_of_x = sympy.lambdify(x, f_of_x_sympy, 'numpy')
f_prime_of_x = sympy.lambdify(x, sympy.diff(f_of_x_sympy, x), 'numpy')

distance_of_f_of_x = sympy.lambdify(x, distance_of_f_of_x_sympy, 'numpy')
distance_prime_of_f_of_x = sympy.lambdify(x, distance_prime_of_f_of_x_sympy,␣
↪'numpy')
distance_prime_prime_of_f_of_x = sympy.lambdify(x,␣
↪distance_prime_prime_of_f_of_x_sympy, 'numpy')

x_values = np.linspace(1, 5, 1000)

plt.plot(x_values, f_of_x(x_values), label = "f(x)")
plt.plot(x_values, f_prime_of_x(x_values), label = "f'(x)")
plt.plot(x_values, distance_of_f_of_x(x_values), label = "D(f(x), 2, 1)")
plt.plot(x_values, distance_prime_of_f_of_x(x_values), label = "D'(f(x), 2, 1)")
plt.plot(2, 1, label = "(2, 1)", marker = 'o')
plt.legend()
guess = 2

guess = newtons_method(
    fOfX = distance_prime_of_f_of_x,
    dfOfX = distance_prime_prime_of_f_of_x,
    guess = 1
)

print(f'Root at ({guess}, {f_of_x(guess)})')
```

```
Root at (1, 1.0)
```



Alright, this is going to take a... little explaining - We write the distance equation $D(f(x), 2, 1) = \sqrt{(x-2)^2 + (\frac{1}{x} - 1)^2}$ - We could just minimize the distance equation, but, in order to use the Newton's method, we're gonna use the first and second derivative - Where the first derivative has a root, the distance equation is minimized - The root we found is at $f(x \approx 1.87) = 0.54$

(c) **Use Newton's method to approximation for λ, accurate to within 10^{-4}, for the population growth equation**

[64]:
```
f_of_x_lambda = lambda x: 1000000*sympy.exp(x) + (435000 / x) * (sympy.exp(x) -
↪1) - 1564000
f_of_x_sympy = sympy.sympify(f_of_x_lambda(x))
f_prime_of_x_sympy = sympy.diff(f_of_x_sympy, x)

f_of_x = sympy.lambdify(x, f_of_x_sympy)
f_prime_of_x = sympy.lambdify(x, f_prime_of_x_sympy)

root = newtons_method(
    f_of_x,
    f_prime_of_x,
    guess = 2,
    maxError = 10 ** (-4)
)
```

```
print(f'Root found at ({root}, {f_of_x_lambda(root)}')
```

```
Root found at (0.10099792968651021, 0.000001018401235342O3
```