section_4

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1 Section 4. Existence and uniqueness of roots

1.1 Definintions

1.1.1 Intermediate value theorem

Let f be a continuous function defined on $[a, b]$, and let s be a number such that $f(a) < s < f(b)$. Then there exists some x between a and b such that $f(x)=s > https://en.wikipedia.org/wiki/Intermediate_value_theorem

1.1.2 Rolle's theorem

Suppose $f \in C[a, b]$ and f is differentiable on (a, b). If $f(a) = f(b)$ then $\exists c \in (a, b)$ such that $f'(c) = 0$ > https://drive.google.com/drive/folders/1UU0Re7ZDqMQLyX4_QSW5pb9W_3gnIHsp

1.1.3 Solving $expr_1(x) = expr_2(x)$

Rearrange so that $f(x) = expr_1(x) - expr_2(x)$ >https://drive.google.com/drive/folders/1UU0Re7ZDqMQLyX4_Q

2 Work

(a) Consider the polynomial $f(x) = x^5 + x - 1$. Use IVT to show f has atleast one real root on $(0, 1)$ $f(0) = 0^5 + 0 - 1 = -1 < 0$

$f(1) = 1^5 + 1 - 1 = 1 > 0$

$f(0) * f(1) = (-1) * (1) = -1 < 0$

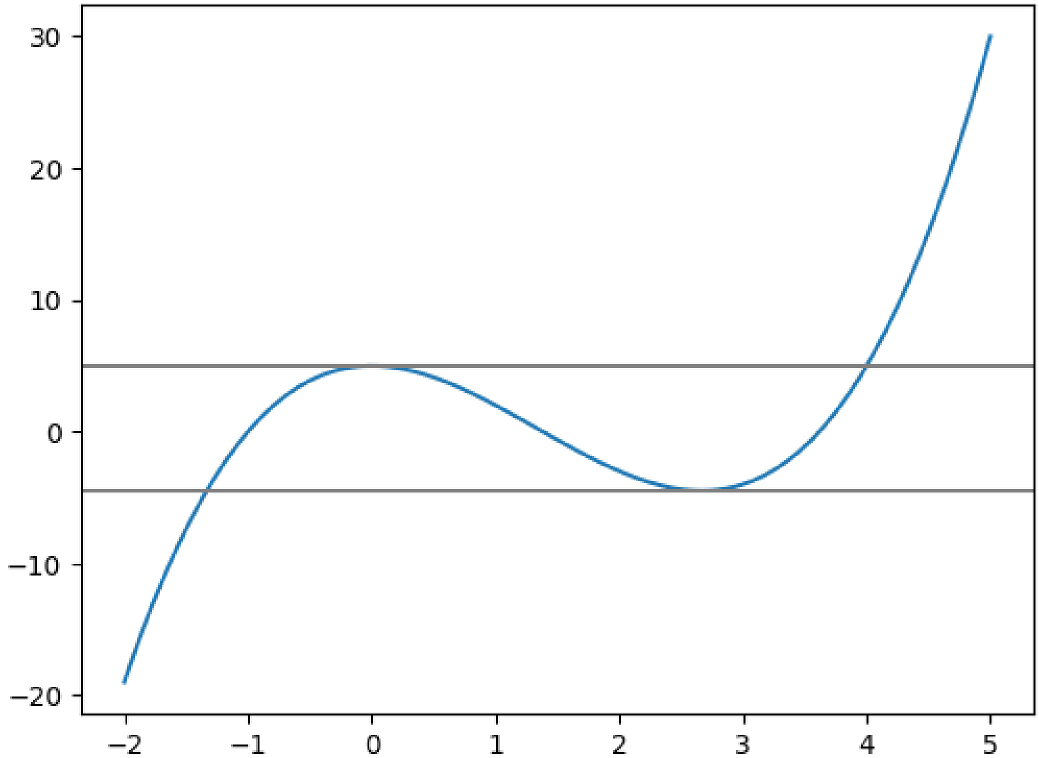
We can be sure that f crosses the axis because of the change in sign of its values

(b) Explain how you would use Rolle's Theorem to show f has at most one real root on all of $\mathbb{R}$

[16]:
```
import matplotlib.pyplot as plt
from numpy import linspace

f = lambda x : x**3 -4*(x**2) + 5
plt.plot(linspace(-2, 5, 100), f(linspace(-2, 5, 100)))
plt.axhline(y=5, color="gray")
plt.axhline(y=-4.48, color="gray")
```

[16]: <matplotlib.lines.Line2D at 0x1238ce930>



Let's take a look at the plot. Rolle's theorem here works much in the same way as the Intermediate Value Theorem, but with the derivative. For the graph to have passed the x-axis multiple times, the derivative of the function must be 0 at some point, because if $f(x)$ were only strictly increasing or decreasing, then we'd only go past the x-axis at one point. Put another way

For there to exist points $a, b \in \mathbb{R}$ such that $a \neq b$ and $f(a) = f(b) = 0$, $f'(x)$ must be 0 at some point for the graph to return to cross the x-axis again

(c) Develop a general theory for how the number of solutions to $f'(x) = 0$ can be used to determine an upper bound for the number of roots of f

Examining Firstly, let's just disreguard the case where we have an infinite number of roots along a interval, if $[a, b] \in \mathbb{R}$. If $f'(x) = 0$ and $f(x) = 0$ for $x \in [a, b]$ where $a \neq b$. Then because of the properties of real numbers, we have an infinite number of roots in the interval.

We can do nothing with Rolle's to guarantee that there is a root, but, if $f'(x) \neq 0$ for $x \in [a, b]$ we can't return to cross the x-axis in the other direction, so, let's state this succinctly

Stating There are at most $n + 1$ roots for $x \in [a, b]$ for the n times $f'(x) = 0$