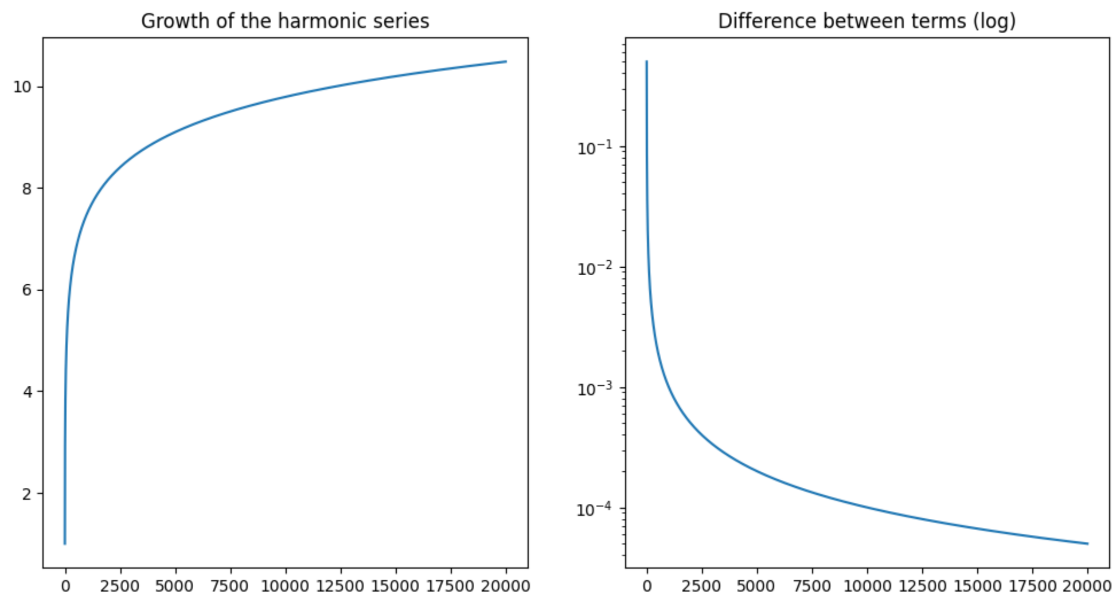
section_5

February 7, 2025

(i) Let p_n be the sequence defined by $p_n = \sum_{k=1}^{n} \frac{1}{k}$.

Show that p_n diverges even though $\lim_{n\to\infty}(p_n - p_{n-1}) = 0$

[7]:
```
import numpy as np
import matplotlib.pyplot as plt

def harmonic_series(n):
    return np.sum(1 / np.arange(1, n + 1))

n_vals = range(1, 20000)
p_n = [harmonic_series(x) for x in range(1, 20000)]
p_n_diffs = np.diff(p_n)

plt.figure(figsize=(12, 6))
plt.subplot(1, 2, 1)
plt.plot(n_vals, p_n)
plt.title("Growth of the harmonic series")

plt.subplot(1, 2, 2)
plt.plot(n_vals[1:], p_n_diffs)
plt.title("Difference between terms (log)")
plt.yscale('log')
```



Well, numerically this doesn't seem to be converging, though the difference between terms is decreasing, given 20,000 iterations, we'd expect to see something more

Proof by contradiction, the harmonic series diverges *Let* $H = \sum_{k=1}^{n} \frac{1}{k}$

Suppose p_n converges to a value, then

$H = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + ...$

$H \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} + ...$

Now grouping terms together

$H \geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

And substituting from above

$H \geq \frac{1}{2} + H$

We arrive at a contradiction, therefor, H diverges

https://web.williams.edu/Mathematics/lg5/harmonic.pdf

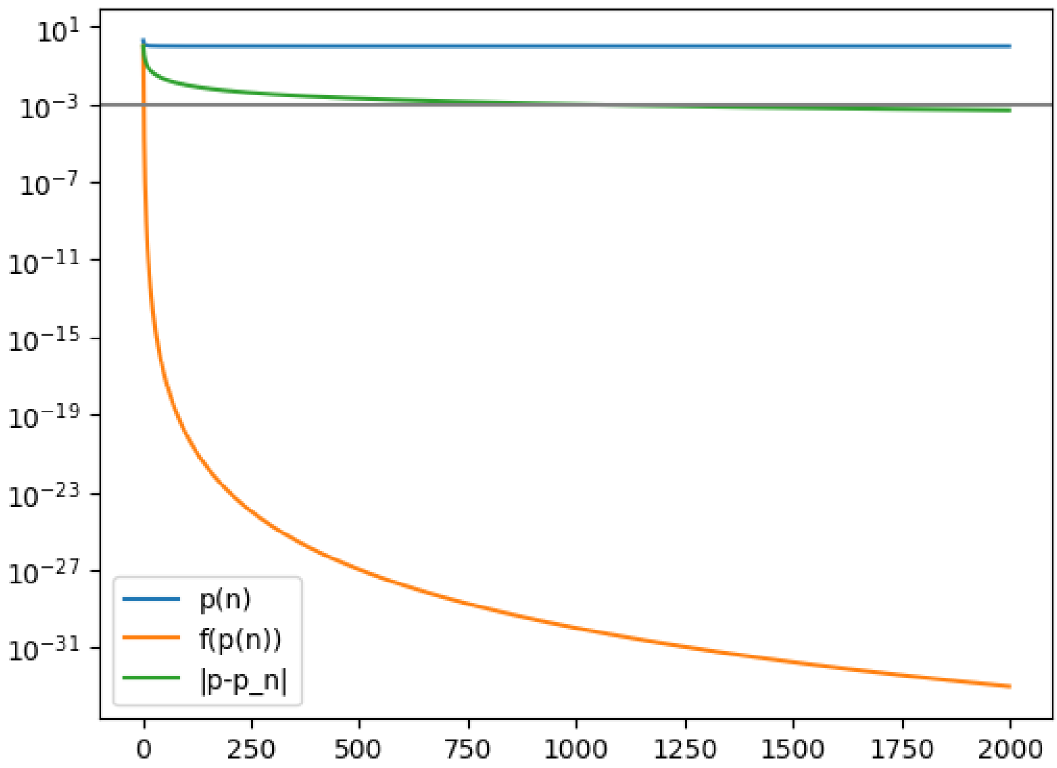
(ii) Let $f(x) = (x-1)^{10}$, $p = 1$**, and** $p_n = 1 + \frac{1}{n}$

Show that $|f(p_n)| < 10^{-3}$ **whenever** $n > 1$ **but that** $|p - p_n| < 10^{-3}$ **requires that** $n > 1000$

[8]:
```
import numpy as np
import matplotlib.pyplot as plt

def exp_func(x):
    return (x - 1) ** 10
```

```
p = 1
n_values = np.arange(1, 2000)
p_n_values = 1 + (1 / n_values)

f_p_n = exp_func(p_n_values)
abs_p_minus_p_n = abs(p - p_n_values)

plt.plot(n_values, p_n_values, label='p(n)')
plt.plot(n_values, f_p_n, label='f(p(n))')
plt.plot(n_values, abs_p_minus_p_n, label='|p-p_n|')
plt.axhline(1e-3, color='grey')
plt.yscale('log')
plt.legend()
```

[8]: <matplotlib.legend.Legend at 0x211c7efbe90>



Alright, we've shown this numerically as above. Let's get down to moving some numbers around.

Substituting in to $|p - p_n| < 10^{-3}$ $\quad |1 - (1 + (\frac{1}{n}))| < 10^{-3} \Rightarrow |-\frac{1}{n}| < 10^{-3} \Rightarrow \frac{1}{n} < 10^{-3}$

$\Rightarrow n > 10^3$

Restating ■ $|p - p_n| > 10^{-3} \Leftrightarrow n > 1000$