A review of topics, centered around exercises. The writing is sloppy due to my laziness, hence a lot is expected of the reader. Proofs are supplied as main ideas with exercises.
The numberings match the main reference MIRA[Axl20], unless stated otherwise. Note that $0 \notin \mathbb{N}$.

1. Measure Theory

We start the semester with a review of Riemann (or Darboux, according to Ilya) integrals, which are used as motivation for 'a better notion of integral'. This turns out to be the Lebesgue integral, and we spend a preliminary chapter to define the Lebesgue measure.

Definition. Let $f : [a, b] \to \mathbb{R}$. A partition P of $[a, b]$ is a finite sequence $a = x_0 < \cdots < x_n = b$. The **upper** and **lower Riemann integrals** are defined as below, where $\Delta_i = x_i - x_{i-1}$:

$$U(f) = \inf_P \sum_i \sup_{[x_{i-1},x_i]} f(x)\Delta_i$$
$$L(f) = \sup_P \sum_i \inf_{[x_{i-1},x_i]} f(x)\Delta_i$$

f is **Riemann integrable** (RI) if $U(f) = L(f)$, and we denote this value by $\displaystyle\int_a^b f$.

Example (1.14, 1.17)**.** Let $[a, b] = [0, 1]$. If $A \subset [0, 1]$ is finite, χ_A is RI. However, $\chi_{\mathbb{Q}}$ is not RI. Thus, a pointwise limit of RI functions need not be RI. This also shows that both MCT and DCT fail when the Lebesgue integral is replaced with Riemann integral.

We first define the outer measure on all of $\mathbb{R}$ in a relatively intuitive way, then begin restricting to those sets for which the outer measure is additive, i.e. we force it to become a measure.

Definition (Outer measure)**.**

Let $A \subset \mathbb{R}$. An interval cover of A is a countable set $\mathcal{I}$ of *open* intervals that cover A.
The **outer measure** of A is defined by

$$|A| = \inf_{\mathcal{I}} \sum_{(a_i,b_i)\in\mathcal{I}} (b_i - a_i).$$

Proposition. Assume all sets and numbers are in $\mathbb{R}$. Then:

$$(i)\, A \subset B \implies |A| \le |B| \qquad (ii) \left|\bigcup_{i\in\mathbb{N}} A_i\right| \le \sum_{i\in\mathbb{N}} |A_i| \qquad (iii)\, |A + t| = |A|$$

Exercise (2.14)**.** Using Heine-Borel, show that $|[a, b]| = b - a$.

Exercise (2A.13)**.** Exhibit a closed set of arbitrary measure consisting of irrationals.

This exercise is a good lesson that closed (hence, compact) sets can look very strange. I always keep in mind that open sets are expressible as a countable disjoint union of open intervals, but not much can be said about the explicit form of a general closed set.

Exercise (2.18, Vitali construction)**.**

Exhibit a family A_i of disjoint subsets of $\mathbb{R}$ for which $\left|\bigcup_{i\in\mathbb{N}} A_i\right| < \sum_{i\in\mathbb{N}} |A_i|$.

I would say this is the first significant result of the course: the A_i are disjoint transversals of the quotient $\mathbb{R}/\mathbb{Q}$. $|A_i| > 0$, thus by dilation, we have non-measurable sets of arbitrary outer measure. This also implies that there does not exist a measure on $2^{\mathbb{R}}$.

Definition (σ-algebra and measurability)**.**

Let X be a set. A **σ-algebra** on X is a collection $\mathcal{S} \subset 2^X$ satisfying

$$\emptyset \in \mathcal{S} \qquad A \in \mathcal{S} \implies A^{\complement} \in \mathcal{S} \qquad \{A_i\}_{i\in\mathbb{N}} \subset \mathcal{S} \implies \bigcup_{i\in\mathbb{N}} A_i \in \mathcal{S}.$$

The pair $(X, \mathcal{S})$ is a measurable space, and elements of $\mathcal{S}$ are **measurable** (hereafter meas.).

It is weird, even misleading, that the notion of measurability precedes that of measure. Indeed, not every measurable space can be endowed with a measure, as was the case for $(\mathbb{R}, 2^{\mathbb{R}})$.

Definition (Borel σ-algebra)**.**

The **Borel σ-algebra $\mathcal{B}$** is the σ-algebra on $\mathbb{R}$ generated by open intervals.
A function $f : X \to \mathbb{R}$ is ($\mathcal{S}$-)**measurable** (hereon meas.) if $\forall B \in \mathcal{B} : f^{-1}(B) \in \mathcal{S}$.

Exercise (2.39)**.** $f : X \to \mathbb{R}$ meas. $\iff$ $f^{-1}((a, \infty))$ meas.

A tutorial on **σ-induction**: the base case is the generating set, and the induction steps are the complement and countable union. It is not the most commonly used gadget, but I must admit that it can open some locked doors.

The criterion above is useful in its own right, but the corollaries that follow are more powerful.

Exercise. Continuous functions and monotone increasing functions are meas..

Proposition. If $f, g, \{f_n\}_{n\in\mathbb{N}}$ are meas. and $c \in \mathbb{R}$, then the following are also meas.:

$$f+g \qquad cf \qquad f^2 \qquad fg \qquad \lim_{n\to\infty} f_n$$

Proof. The first three are easy, and fg follows from them.
More interesting is the **conversion of quantifiers** technique:

$$\begin{aligned}\lim_{n\to\infty} f_n(x) > a &\iff \exists m \in \mathbb{N}\, \exists N \in \mathbb{N}\, \forall k \geq N : f_k(x) > a + 1/m \\ &\iff x \in \bigcup_{m\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{k\geq N} f_k^{-1}(a + 1/m, \infty)\end{aligned}$$

□

Exercise. Show that fg is meas. directly using (2.39).

Exercise (2B.14)**.** $\{f_n\}_{n\in\mathbb{N}}$ meas. $\implies$ $\{x \in X \mid \lim_{n\to\infty} f_n(x) \in \mathbb{R}\}$ meas.

Definition (measure).

A **measure** on $(X, \mathcal{S})$ is a function $\mu : \mathcal{S} \to [0, \infty]$ satisfying

$$\mu(\emptyset) = 0 \qquad \mu\Big(\dot{\bigcup_{i\in\mathbb{N}}} A_i\Big) = \sum_{i\in\mathbb{N}} \mu(A_i).$$

The triple $(X, \mathcal{S}, \mu)$ is a **measure space**.

X is **finite** if $\mu(X) < \infty$, and **σ-finite** if $X = \bigcup_{i\in\mathbb{N}} A_i$, with $\mu(A_i) < \infty$.

Proposition (continuity of measure)**.** Given $\{A_i\}_{i\in\mathbb{N}}$ meas.,

(continuity from below) $A_i \subset A_{i+1} \implies \mu\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) = \mu\Big(\lim_{i\to\infty} A_i\Big) = \lim_{i\to\infty} \mu(A_i).$

(continuity from above) $\mu(A_1) < \infty,\ A_i \supset A_{i+1} \implies \mu\Big(\bigcap_{i\in\mathbb{N}} A_i\Big) = \mu\Big(\lim_{i\to\infty} A_i\Big) = \lim_{i\to\infty} \mu(A_i).$

Exercise. What happens if we remove the assumption $\mu(A_1) < \infty$ for continuity from above?

Exercise (Amey's Borel-Cantelli)**.** $\sum_{i\in\mathbb{N}} \mu(A_i) < \infty \implies \mu(\limsup_{i\to\infty} A_i) = \mu\Big(\bigcap_{N\in\mathbb{N}} \bigcup_{i\geq N} A_i\Big) = 0.$

Exercise. X finite, $\forall\, B\, \exists\, A : A \subset B,\ 0 < \mu(A) < \mu(B) \implies \exists \{A_i\}_{i\in\mathbb{N}} : \mu(A_i) > 0,\ \lim_{i\to\infty} \mu(A_i) = 0.$

Exercise (2C.3)**.** Let $\mu(\mathcal{S}) = \{\mu(A) \mid A \in \mathcal{S}\}$. Give examples of measure spaces with $\mu(\mathcal{S}) =$

$$(i)\ [0,1] \qquad (ii)\ [0,1] \cup [3,4] \qquad (iii)\ [0,1] \cup [3,\infty]$$

Exercise. $\mu(2^{\mathbb{N}})$ is either finite or uncountable.

Definition (Lebesgue measure)**.**

The **Lebesgue measure** on $\mathbb{R}$ is the outer measure $|\cdot|$ restricted to $\mathcal{B}$.
A set is **Lebesgue** if it differs from a Borel set by a null set.

We thus extend the domain of the Lebesgue measure to include null sets. This procedure is called the **completion** of the measure, and the Lebesgue measure on $\mathbb{R}^n$ is constructed similarly.

Proposition (2.66, strong additivity)**.** $A \subset \mathbb{R}, B \in \mathcal{B}$ disjoint $\implies |A \cup B| = |A| + |B|$

It becomes transparent following this proposition that (i) the Vitali set V of (2.18) is not Borel and (ii) $(\mathbb{R}, \mathcal{B}, |\cdot|)$ is a measure space.

Proposition (useful parts of 2.71)**.** A is Lebesgue iff (TFAE):

$$(i)\ \forall \epsilon > 0\, \exists\, F \overset{\epsilon}{\subset} A \text{ closed} \qquad (ii)\ \forall \epsilon > 0\, \exists\, G \overset{\epsilon}{\supset} A \text{ open} \qquad (iii)\ \exists\, B \overset{0}{\subset} A \overset{0}{\subset} C \text{ Borel sandwich}$$

where we employ the notation $A \overset{\epsilon}{\subset} B : A \subset B$ with $\mu(B \backslash A) < \epsilon$, likewise $A \overset{0}{\subset} B : \mu(B\backslash A) = 0$.

Example (Cantor set)**.**

The **Cantor set** C is constructed by inductively removing the middle 1/3 of every interval, starting from $[0,1]$. Alternatively, C is the collection of ternary expansions with no 1s.
The **Cantor function** $\Lambda : [0,1] \to [0,1]$ is defined by $\sum_i a_i/3^i \mapsto \sum_i (a_i/2)/2^i$ on C, and is extended to $[0,1]$ by defining it to be constant on each removed interval.

C is an uncountable null set, and Λ is constant a.e. but is monotone increasing from 0 to 1. It is thus subject to the 'fundamental inequality of calculus':

$$\int_{[0,1]} \Lambda' < \Lambda(1) - \Lambda(0)$$

Exercise (2D.11)**.** $|B| > 0 \implies \exists\, A \subset B : A$ is not Lebesgue.

Exercise. By considering $x + \Lambda(x) : [0,1] \to [0,2]$, exhibit a Lebesgue set which is not Borel.

Definition (pointwise and uniform convergence of functions)**.** Let $f_n : X \to \mathbb{R}$.

$$f_n \to f : \forall\, x \in X\, \forall\, \epsilon > 0\, \exists\, N \in \mathbb{N}\, \forall\, n \geq N : |f_n(x) - f(x)| < \epsilon$$
$$f_n \rightrightarrows f : \forall\, \epsilon > 0\, \exists\, N \in \mathbb{N}\, \forall\, n \geq N\, \forall\, x \in X : |f_n(x) - f(x)| < \epsilon$$

Exercise (2.84)**.** Does pointwise or uniform limit preserve (uniform) continuity?

Theorem (Egorov)**.**

$\mu(X) < \infty$, $f_n : X \to \mathbb{R}$, $f_n \to f \implies \forall\, \epsilon > 0\, \exists\, A \overset{\epsilon}{\subset} X$ meas.: $f_n \rightrightarrows f$ on A.

Proof. Translating the quantifiers from the definition, we have

$$X = \bigcap_{k\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} (f_n - f)^{-1}((-1/k, 1/k)) := \bigcap_{k\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} A_{n,k}.$$

We want a set $A \overset{\epsilon}{\subset} X$ satisfying $\forall\, k \in \mathbb{N}\, \exists\, N_k \in \mathbb{N} : A \subset \bigcap_{n\geq N_k} A_{n,k}$,

thus it suffices to take $A = \bigcap_{k\in\mathbb{N}} A_k$, where $A_k = \bigcup_{N\leq N_k} \bigcap_{n\geq N} A_{n,k} \overset{2^{-k}\epsilon}{\subset} \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} A_{n,k} = X$. □

I find it illuminating to dissect this proof; it is almost a play on words once all the translation is done. Note that the sacrifice in ϵ is needed since N can no longer quantify both x and $1/k$.

Exercise. What happens if we remove the finiteness assumption?

Definition. A function $f : X \to \mathbb{R}$ is **simple** if it is expressible as a finite sum $f = \sum_{i=1}^{n} c_i \chi_{A_i}$.

Exercise (Standard approximation)**.**

Let $f : X \to \mathbb{R}$ be given. Construct a sequence of simple functions f_n such that $|f_n|$ is monotone increasing and bounded above by $|f|$, $f_n \to f$, and furthermore $f_n \rightrightarrows f$ if f is bounded.

This is a gadget that breaks down a proof into two steps: (i) Prove the statement for simple functions and (ii) pass to the limit. Analogous to induction, (ii) is usually the hard part.
We see the prototype usage in the proof of Luzin's theorem below.

Theorem (Luzin)**.** $g : \mathbb{R} \to \mathbb{R}$ $\mathcal{B}$-meas. $\implies \forall \epsilon > 0 \, \exists F \overset{\epsilon}{\subset} X$ closed: $g|_F$ is continuous.

There is a bit of a scam in this theorem; it does not say that g is continuous on F, it says that the function $g|_F \colon F \to \mathbb{R}$ is continuous. To see the difference, consider $\chi_{\mathbb{Q}}$. We can do (2A.13) to obtain $F \overset{\epsilon}{\subset} \mathbb{R}$ for which $\chi_{\mathbb{Q}}|_F = 0$, but $\chi_{\mathbb{Q}}$ itself is continuous nowhere.

Proof. For simple $g = \sum_{i=1}^{n} c_i \chi_{E_i}$, use (2.71) to approximate by closed and open sets $F_i \overset{\epsilon/2n}{\subset} E_i \overset{\epsilon/2n}{\subset} G_i$, and take $F = \left(\bigcup_{i=1}^{n} F_i \right) \cup \left(\bigcup_{i=1}^{n} G_i \right)^{\complement} \overset{\epsilon}{\subset} \mathbb{R}$.

Exercise. Verify that $g|_F$ is continuous.

It remains to pass to the limit. There are two icebergs: a countable intersection of closed sets is not necessarily closed, and a pointwise limit of continuous functions is not necessarily continuous. The former is not a big deal since we can approximate again, and the latter is solved by Egorov by upgrading the limit to a uniform one. Formally, given the standard approximation $g_n \to g$:

Choose $F_{(n)} \overset{2^{-k}\epsilon}{\subset} \mathbb{R}$ such that $g_n|_{F_{(n)}}$ is continuous, and take $A = \bigcap_{n \in \mathbb{N}} F_{(n)} \overset{\epsilon}{\subset} \mathbb{R}$.

Egorov then gives us $A_m \overset{2^{-k}\epsilon}{\subset} A \cap \pm[m-1, m)$ such that $g_n \rightrightarrows g$ on each A_m.
We conclude by approximating from below: $F \overset{\epsilon}{\subset} B = \bigcup_{m \in \mathbb{N}} A_m \overset{\epsilon}{\subset} A \overset{\epsilon}{\subset} \mathbb{R}$. □

Example (4.29)**.** The 'bad Borel set' B is constructed as $\bigcup_{n \in \mathbb{N}} F_n$, where
$\{I_n\}_{n \in \mathbb{N}}$ enumerates the bounded open intervals with rational endpoints, and
$\{F_n\}_{n \in \mathbb{N}}, \{\hat{F}_n\}_{n \in \mathbb{N}}$ are closed sets consisting of irrationals $F_n \subset I_n \setminus \bigcup_{i=1}^{n-1} \hat{F}_i$, $\hat{F}_n \subset I_n \setminus \bigcup_{i=1}^{n} F_i$,
i.e. they are the 'keep' and 'dump' piles for which each I_n donates a positive measure.

Exercise. Verify that $0 < |B \cap I| < |B|$ for any interval I.

Exercise. Construct a 'fat Cantor set' by removing the middle 1/4 of each interval instead of 1/3. Verify that this set D satisfies $0 < |D \cap I| < |D|$ for any interval $I \subset [0, 1]$.

Exercise (2E.12)**.** Exhibit $g : \mathbb{R} \to \mathbb{R}$ $\mathcal{B}$-meas. s.t. $\nexists B \overset{0}{\subset} X$ Borel: $g|_B$ is continuous.

Before moving on, we take a moment to mention convergence in measure. For $\mu(X) < \infty$, it is the weakest type of convergence, in the sense that it is implied by all other types.

Definition (convergence in measure)**.**

Let $f_n : X \to \mathbb{R}$. $f_n \xrightarrow{\mu} f : \forall \epsilon > 0 : \lim_{n \to \infty} \mu(|f_n - f| > \epsilon) = 0$.

Exercise. Using Egorov's theorem, show that for $\mu(X) < \infty$, $\to$ a.e. implies $\xrightarrow{\mu}$.
Also, exhibit f_n satisfying: $\quad$ *(i)* $\mu(X) = \infty$, $f_n \to f$, $f_n \overset{\mu}{\nrightarrow} f$ $\quad$ *(ii)* $\mu(X) < \infty$, $f_n \nrightarrow f$, $f_n \xrightarrow{\mu} f$

I will close with a secret. Counterexamples for sequences of functions are one of these two: escape to infinity $f_n = \chi_{[n,\infty)}$ or the typewriter $f_{2^n+k} = \chi_{[\frac{k}{2^n}, \frac{k+1}{2^n}]}$ (including thin, fat variants).

2. Integration

We are done with measure theory. Let's do *real* analysis.

Ilya

All functions hereon are measurable. The default range is $[0, \infty]$ until stated otherwise. The main goals are MCT and DCT. As a bonus, we complete our diagram for modes of convergence.

Definition (integral with respect to measure)**.**

Let $(X, \mathcal{S}, \mu)$ be a measure space, and $f : X \to [0, \infty]$. An **$\mathcal{S}$-partition** P of X is a partition $\{A_i\}_{i \le n} \subset \mathcal{S}$ of X. The **integral** of f (w.r.t. μ) is defined as

$$\int_{(X)} f\,(d\mu) = \sup_P \sum_i \inf_{A_i} f(x)\mu(A_i).$$

Note that, unlike Riemann sums, only lower sums are considered. The difficulties in considering upper sums come from dealing with infinite X and unbounded f. In particular, we need to consider infinite partitions. This is possible (according to Ilya), but is harder work for the same results.

Exercise (3.7, 3.9)**.** Show, from the definition: $\displaystyle\int \sum_{i=1}^n c_i \chi_{A_i}\, d\mu = \sum_{i=1}^n c_i \mu(A_i)$.

Use this to show $\displaystyle\int f\, d\mu = \sup_{g \le f} \int g\, d\mu$, where g is simple and bounded.

Theorem (MCT, dubbed non-decreasing convergence theorem)**.**

$f_n : X \to [0, \infty]$, $f_n \nearrow f \implies \displaystyle\int f_n \to \int f$.

Proof. The relevant part is $(\ge)$, and the tool used is the above exercise (3.9).
Fix $g \le f$ simple bounded, and $t \in (0, 1)$. Let $g_{t,n} = g\chi_{\{f_n \ge tg\}}$ s.t. $f_n \ge tg_{t,n}$.
Now, $\displaystyle\lim_{n\to\infty} \int f_n \ge \lim_{n\to\infty} \int tg_{t,n} = t \int g$, whence taking $\displaystyle\sup_g \lim_{t\uparrow 1}$ on the RHS wins the game. $\square$

It is perhaps worthwhile to pause for a moment and scrutinize the second line of the proof. What would make the proof really clean is to just conclude that $f_n \ge g$, or $f_n \ge tg$ for some $n \in \mathbb{N}$. What prevents us from doing so? Try and cook up a counterexample for each.

Exercise. What happens if we consider $f_n \searrow f$ (monotone decreasing)?

Exercise (3.16)**.** By standard approximation, show that $\displaystyle\int$ is linear.

Definition (integral of real functions)**.** Given $f : X \to \mathbb{R}$, define $f_+ = f\chi_{\{f>0\}}$ and $f_- = -f\chi_{\{f<0\}}$. The integral of f is defined as

$$\int f = \int f_+ - \int f_-.$$

We exclude $\pm\infty$ to avoid having to run into $\infty - \infty$; it is possible that we can include one side.

Exercise (3A.11)**.** Given $f_n : X \to \mathbb{R}$, $\displaystyle\sum_{n\in\mathbb{N}} \int |f_n| < \infty \implies f_n \to 0$ a.e.. (cf. Borel-Cantelli)

Exercise (3A.17, Fatou's lemma)**.** $\displaystyle\int \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu.$

Take a moment to think of ways that equality can fail when $\liminf\limits_{n\to\infty} f_n < \limsup\limits_{n\to\infty} f_n$.

Exercise. By considering lim sup, prove MCT assuming Fatou's lemma.

Exercise (3.28, absolute continuity of integral)**.** $\forall\, \epsilon > 0 \,\exists\, \delta > 0 : |A| < \delta \implies \displaystyle\int_A f < \epsilon.$

Another easy cleanup for standard approximation. The naming is a bit premature. Formally, it implies that given $f \in \mathcal{L}^1([a,b])$, the function $F(x) = \displaystyle\int_{[a,x]} f$ is absolutely continuous, and that the measure defined by $\nu(A) = \displaystyle\int_A f \, d\mu$ is absolutely continuous w.r.t. μ.

Theorem (DCT)**.**

$f_n : X \to [-\infty, \infty]$, $f_n \to f$ a.e., $\exists\, h : X \to [0,\infty] : h \in \mathcal{L}^1$, $|f_n| \le h$ a.e. $\implies \displaystyle\int f_n \to \int f.$

Corollary (BCT)**.** $\mu(X) < \infty$, $\exists\, c > 0 : |f_n| \le c$ a.e. $\implies \displaystyle\int f_n \to \int f.$

Exercise. Counterexample for BCT when $\mu(X) = \infty$?

Proof of DCT. Assume $\mu(X) < \infty$. Given $\epsilon > 0$, consider

$$\left| \int f_n - \int f \right| \le \int_{A^\complement} |f_n| + \int_{A^\complement} |f| + \int_A |f_n - f|.$$

Bound each of the three terms on the RHS by $\epsilon/3$ to win the game. We can assign a δ for h to bound the first two by (3.28), and Egorov's theorem conveniently gifts us the magic set $A \overset{\delta}{\subset} X$. Note that the upgrade to $\rightrightarrows$ allows us to bound the third term for all sufficiently large n. $\square$

Exercise (3.29)**.** Let $h \in \mathcal{L}^1$. Then, $\forall\, \epsilon > 0 \,\exists\, B : \mu(B) < \infty$, $\displaystyle\int_{B^\complement} h < \epsilon.$

Exercise. Complete the proof of DCT for $\mu(X) = \infty$ using (3.29).

Exercise (3.34)**.** Let $f : [a,b] \to \mathbb{R}$ be bounded. Then, f is RI $\iff$ f is continuous a.e.. Moreover, $\displaystyle\int_a^b f = \int_{[a,b]} f$, hence the upgrade from Riemann to Lebesgue integral is complete.

Example (3B.12)**.** Compute:

$$(i) \lim_{n\to\infty} \int_{[0,1]} \frac{(1-x)^n \cos(n/x)}{\sqrt{x}} \qquad (ii) \lim_{n\to\infty} \int_{\mathbb{R}} \frac{n \sin(x/n)}{x(1+x^2)} \qquad (iii) \lim_{n\to\infty} \int_{[0,1]} n \log{(1 - \frac{\log x}{n})}$$

Exercise (3B.13)**.** Exhibit a sequence $f_n : [0,1] \to [0,\infty]$ satisfying $\displaystyle\int f_n \to 0$ and $\limsup\limits_{n\to\infty} f_n = \infty$.

Exercise (3B.15)**.** Exhibit a bounded open set $A \subset \mathbb{R}$ for which χ_A is not RI.

On this page, we consider functions $X \to \mathbb{R}$.

Definition (convergence in $\mathcal{L}^1$)**.**

Let $f_n : X \to \mathbb{R}$. $f_n \to f$ in $\mathcal{L}^1$: $\displaystyle\int |f_n - f| \to 0$, i.e. $||f_n - f||_1 \to 0$.

In particular, MCT and DCT give sufficient conditions for $\mathcal{L}^1$ convergence.

Exercise (Markov's inequality)**.**

$h \in \mathcal{L}^1 \implies \mu(\{|h| \geq c\}) \leq \dfrac{1}{c} \displaystyle\int |h|.$

Probably the most cost-efficient result in the course.

Exercise (Chebyshev)**.** $\mu(X) = 1$, $h \in \mathcal{L}^1 \implies \mu\Big(\Big\{\Big|h - \displaystyle\int h\Big| \geq c\Big\}\Big) \leq \dfrac{1}{c^2}\Big(\displaystyle\int h^2 - \Big(\int h\Big)^2\Big).$

Exercise. Use Markov's inequality to show that $\to$ in $\mathcal{L}^1 \implies \xrightarrow{\mu}$.
Also, exhibit f_n in $\mu(X) < \infty$ satisfying: (i) $f_n \xrightarrow{\mu} f$, $f_n \not\to f$ in $\mathcal{L}^1$ $\quad (ii)$ $f_n \not\to f$, $f_n \to f$ in $\mathcal{L}^1$

Call the type of convergence implied by Egorov's theorem 'almost uniform', and denote 'a. $\rightrightarrows$'.

Exercise. Show that a. $\rightrightarrows \implies \to$ a.e..

As a culmination of previous results, we have the following diagram:

$$\begin{tikzcd} \to \text{a.e.} \arrow[r, dotted, shift left, "\text{Egorov}"] & \text{a. } \rightrightarrows \arrow[l, shift left, "\text{easy}"] \arrow[d, "\text{easy}"] \\ \text{in } \mathcal{L}^1 \arrow[r, "\text{Markov}"] & \xrightarrow{\mu} \end{tikzcd}$$

The dotted line indicates that the implication holds only for $\mu(X) < \infty$. The exhibitions thus far hopefully convinces you that non-arrows are non-implications. Some extra snacks:

Exercise. Let $\mu(X) < \infty$. Then, $f_n \xrightarrow{\mu} f \implies f_n^2 \xrightarrow{\mu} f^2$.

Exercise. Let $\mu(X) < \infty$, $f_n : X \to [0, \infty)$. Then, $f_n \xrightarrow{\mu} 0 \iff \dfrac{f_n}{1 + f_n} \to 0$ in $\mathcal{L}^1$.

Proposition. $f_n \xrightarrow{\mu} 0 \implies \exists$ subsequence $f_{n_k} \to 0$ a.e..

Proof. We have $\forall\, k \in \mathbb{N}\, \exists\, n_k \in \mathbb{N} : \mu(\{f_{n_k} \geq 1/k\}) < 2^{-k}$.

Exercise. Apply Borel-Cantelli to the sets $A_k = \{f_{n_k} \geq 1/k\}$ to conclude the proof.

$\square$

3. Differentiation

Unlike in calculus, analysis of derivatives is harder than integrals. Intuitively, this is because the former needs good local behavior to be defined, while the latter is automatically well-defined.
The proofs of HLMI and LDT rely on some things that were omitted; they are left as exercises, but I do not think much is lost by skipping them.

Definition (Hardy-Littlewood maximal function)**.**

Let $h : \mathbb{R} \to \mathbb{R}$. Then, $h^*(x) := \sup\limits_{t>0} \dfrac{1}{2t} \displaystyle\int_{x-t}^{x+t} |h|$.

Example (4.7)**.** Compute $h^*(x)$ for $h(x) = \quad (i)\, \chi_{[0,1]} \quad (ii)\, x\chi_{[0,1]} \quad (iii)\, \chi_{[0,1]\cup[3,4]}$

Exercise (baby Vitali)**.** Given a finite set of intervals $\{I_k\}_{k\leq n}$, there exists a subset consisting of disjoint intervals $\{I_{k_i}\}_{i\leq m}$ such that $\displaystyle\bigcup_{i\leq m} 3 * I_{k_i} \supset \bigcup_{k\leq n} I_k$.

Theorem (HLMI)**.**

$h \in \mathcal{L}^1(\mathbb{R}) \implies |\{h^* > c\}| \leq \dfrac{3}{c} \displaystyle\int |h|$.

Proof. We want to be in a situation where we can apply baby Vitali. The key is:

Exercise (2D.24)**.** $|A| = \sup\limits_{F\subset A} |F|$, taken over F compact.

from which we can apply Heine-Borel to interval covers. Set $A = \{h^* > c\}$, and fix F.
Our interval cover $\mathcal{I}$ of F consists of $I_x = (x - t_x, x + t_x)$ with $x \in F$, $\dfrac{1}{|I_x|} \displaystyle\int_{I_x} |h| > c$.
With the tools in hand, we now have a finite disjoint set of intervals $I_i = (x_i - t_i, x_i + t_i)$:

$$|F| \leq \sum_i 3|I_i| < \sum_i \frac{3}{c} \int_{I_i} |h| \leq \frac{3}{c} \int |h|.$$

□

Theorem (LDT)**.**

$f \in \mathcal{L}^1(\mathbb{R}) \implies \lim\limits_{t\downarrow 0} \dfrac{1}{2t} \displaystyle\int_{x-t}^{x+t} |f - f(x)| = 0$ $(*)$ a.e..

It is noted that we only need $f \in \mathcal{L}^1$ 'locally', i.e. it suffices that $\forall\, a, b \in \mathbb{R} : \displaystyle\int_a^b |f| < \infty$.

Proof. We make a slight upgrade to the standard approximation to make f_n continuous:

Exercise (3.48)**.** Construct a sequence of continuous functions f_n with bounded support satisfying all the properties given by standard approximation.

The main advantage of this is the reduction of the base case:

Exercise. Show that $(*)$ holds everywhere for f continuous.

thus the induction step given by

$$\frac{1}{2t}\int_{x-t}^{x+t}|f-f(x)| \leq (f-f_n)^*(x) + \frac{1}{2t}\int_{x-t}^{x+t}|f_n - f_n(x)| + |f-f_n|(x)$$

is reduced to bounding the first and third terms in the RHS, which we can do in the dumb way, by defining $A_n = \{(f-f_n)^* > 1/n\} \cup \{|f-f_n| > 1/n\}$ and applying Markov and HLMI:

$$|A_n| \leq |\{(f-f_n)^* > 1/n\}| + |\{|f-f_n| > 1/n\}| \leq 4n\int|f-f_n|$$

Given $\epsilon > 0$, replace $\{f_n\}_{n\in\mathbb{N}}$ with a subsequence such that $\displaystyle\int|f-f_n| < 2^{-n}\epsilon/4n$. □

Exercise. Let $A = \displaystyle\bigcap_{n\in\mathbb{N}} A_n$. Show that $|A| < \epsilon$ and $(*)$ holds outside A. Conclude the theorem.

It is healthy to pause for a moment and think about why we are able to say

$$\forall\,\epsilon > 0\,\exists\, A : A^{\complement} \subset \{(*)\},\ |A| < \epsilon \implies (*) \text{ a.e..}$$

Compare to Egorov's theorem; a. $\rightrightarrows$ does not imply $\rightrightarrows$ a.e..

The difference is that $\rightrightarrows$ is a global property, while $(*)$ is local. More explicitly, we may characterize the set of points such that $(*)$ does not hold, but the same cannot be said for $\rightrightarrows$.

Corollary (4.21)**.** $f \in \mathcal{L}^1(\mathbb{R}) \implies f(x) = \displaystyle\lim_{t\downarrow 0}\frac{1}{2t}\int_{x-t}^{x+t} f$ a.e..

Exercise (7A.24)**.** Use (4.21) to show, for $p > 0 : f \in \mathcal{L}^p(\mathbb{R}) \implies \displaystyle\lim_{t\downarrow 0}\frac{1}{2t}\int_{x-t}^{x+t}|f-f(x)|^p = 0$ a.e..

Exercise (4B.2)**.** $f \in \mathcal{L}^1(\mathbb{R}) \implies f(x) = \displaystyle\lim_{t\downarrow 0}\sup_{|I|=t,\, b\in I}\frac{1}{t}\int_I\left|f - \frac{1}{t}\int_I f\right|$ a.e..

> **Exercise** (FTC I)**.**
> $f \in \mathcal{L}^1(\mathbb{R}) \implies f(x) = \dfrac{d}{dx}\displaystyle\int_{(-\infty,x]} f$ a.e..

Exercise (4B.5)**.** Let $f : \mathbb{R} \to \mathbb{R}$. Then, $|f| \leq f^*$ a.e..

Definition (density)**.** Let $A \subset \mathbb{R}$. The **density** of A at $x \in \mathbb{R}$ is $\rho_A(x) := \displaystyle\lim_{t\downarrow 0}\frac{1}{2t}|A \cap (x-t, x+t)|$.

Exercise (density theorem)**.** $\rho_A = \chi_A$ a.e..

Exercise (4B.9)**.** Exhibit sets whose density at 0 is: $(i)\ 1/2 \quad (ii)\ t \in (0,1) \quad (iii)$ not defined

It is nice to have an interpretation for density as how much of the interval the set takes up as we zoom in near the point: an undefined density means that two different ways of zooming in gives two different interpretations of how much the set takes up: Consider $A = \displaystyle\bigcup_{n\in\mathbb{N}} 2^{-n}[1/2, 3/4]$.

(To be precise, different ways of zooming in correspond to distinct converging subsequences, whose existence is guaranteed by Bolzano-Weierstrass.)

Ilya takes a detour from MIRA here to talk a bit more about monotone functions, functions of bounded variation, and absolute continuity. The main motivation to study these is the 'FIC (fundamental *inequality* of calculus)' and characterize functions for which equality holds.

The reference is I.P. Natanson's 'Theory of Functions of a Real Variable' [Nat64], chapters 8, 9. We employ the notation $f \nearrow$ to denote that f is monotone increasing, and $f(x\pm) = \lim_{t \to x\pm} f(t)$.

For the rest of this section, we consider functions $f : [a, b] \to \mathbb{R}$.

Exercise (8.1.1)**.** Let $f \nearrow$. Then, the set of discontinuities of f is at most countable.

This allows us to define the **jump** (or *saltus*) **function** of $f \nearrow$:

$$j(x) = (f(a+) - f(a)) + \sum_{t \in (a,x)} (f(t+) - f(t-)) + (f(x) - f(x-)).$$

Exercise (8.1.2)**.** Let $f \nearrow$. Then, $(f - j) : [a, b] \nearrow \mathbb{R}$ is continuous.

We now work towards understanding the derivative of monotone functions:

The set of **derivative numbers** of f at x is defined by

$$Df(x) = \left\{ \lim_{n \to \infty} \frac{f(x + a_n) - f(x)}{a_n} \,\middle|\, a_n \to 0 \right\} \subset [-\infty, \infty].$$

Bolzano-Weierstrass tells us that for $x \in [a, b]$, $Df(x) \neq \emptyset$, and if $f \nearrow$, then $Df(x) \subset [0, \infty]$. Note that $f'(x) = s \iff Df(x) = \{s\}$.

We need the following to proceed:

Theorem (8.2.4)**.** Let $f \nearrow$. Then, f' is defined a.e., and furthermore $f' < \infty$ a.e..

The proof is long and grueling. It is perhaps better to defer it and first do what we are here for. To define f' on $[a, b]$, we simply replace $f'(x) = 0$ where $f'(x)$ is undefined.

Theorem (FIC)**.**

$f : [a, b] \nearrow \mathbb{R} \implies \int_{[a,b]} f' \leq f(b) - f(a)$, i.e. $f' \in \mathcal{L}([a, b])$.

Proof. Take $f_n = \dfrac{f(x + 1/n) - f(x)}{1/n} \to f'$ a.e. by (8.2.4). Then, $f' : [a, b] \to [0, \infty]$ is meas., and

$$\int_{[a,b]} f' \leq \liminf_{n \to \infty} \int_{[a,b]} f_n = \liminf_{n \to \infty} n\Big(\int_{[b,b+1/n]} f - \int_{[a,a+1/n]} f \Big) \leq f(b) - f(a).$$

The first inequality is Fatou's lemma, and we extend $f(x) = f(b)$ for $x \geq b$ wlog. □

Alright, now to the proof of (8.2.4): Skip if you're busy. The key is this set of twin lemmas:

Lemma (8.2.2)**.** Let $f \nearrow$ be strictly increasing, and $A \subset [a, b]$, $t \in (0, \infty)$ satisfy $\forall\, x \in A\, \exists\, s \in Df(x) : s \leq t$ (resp. $s \geq t$). Then, $|f(A)| \leq t|A|$ (resp. $|f(A)| \geq t|A|$).

Proof. Given $\epsilon > 0, t_0 > t$, take $G \overset{\epsilon}{\supset} A$ open, $a_n \to 0$ s.t. $[x, x + a_n] \subset G$, $\dfrac{f(x + a_n) - f(x)}{a_n} < t_0$. Note we do not assume $a_n > 0$. Nevertheless, $[x, x + a_n]$ denotes the interval between x and $x + a_n$. Write $f([x, x + a_n]) \subset [f(x), f(x + a_n)] = I_n(x)$, $|I_n| < t_0|a_n|$. Now, a sublemma, if you will.

Lemma (Vitali II)**.** An interval cover $\mathcal{I}$ of $A \subset \mathbb{R}$ is said to be of **Vitali-type** (hereon V-type) if $\forall\, x \in A\, \forall\, \epsilon > 0\, \exists\, I \in \mathcal{I} : x \in I,\ |I| < \epsilon$. Let $\mathcal{I}$ be an V-type interval cover of $A \subset \mathbb{R}$.
Then, there exists a countable subcover of disjoint intervals $\{I_n\}_{n\in\mathbb{N}}$ such that $\left|A \setminus \dot{\bigcup_{n\in\mathbb{N}}} I_n\right| = 0$.

Proof. Assume A is bounded. At the n-th step, choose I_n with $|I_n| > \dfrac{1}{2} \sup_{I\in\mathcal{I}} \left\{ |I| \,\middle|\, I \subset \Big(\bigcup_{k<n} I_k\Big)^{\complement} \right\}$.

Exercise. Show that $\forall\, n \in \mathbb{N} : x \in \Big(\bigcup_{k\le n} I_k\Big)^{\complement} \implies x \in \bigcup_{k>n} (5 * I_k)$. Complete the proof.

$\square$

Exercise. $\mathcal{I} = \{I_n(x)\}_{x\in A,\, n\in\mathbb{N}}$ is a V-type cover of $f(A)$.

Vitali II yields a subcover $\{I_{n_i}(x_i)\}_{i\in\mathbb{N}}$, whence $|f(A)| \le \sum_{i\in\mathbb{N}} |I_{n_i}(x_i)| < \sum_{i\in\mathbb{N}} t_0 |a_{n_i}| \le t_0 |G|$.
Taking the limits $t_0 \downarrow t$ and $\epsilon \to 0$ wins us the first game.

As much as I would like to skip the proof of $(\ge)$, running the same thing gives the wrong inequality at the end, so we need to be a bit smarter. The idea is to cover A instead of $f(A)$.
Given $\epsilon > 0, t_0 < t$, take $G \overset{\epsilon}{\supset} f(A)$ this time, and likewise $a_n \to 0$ with $\dfrac{f(x+a_n) - f(x)}{a_n} < t_0$.
By (8.1.1), $B = \{f \text{ continuous}\} \overset{0}{\subset} A$, and we may assume $I_n(x) \subset G$ for all $x \in B$.

Exercise. $\mathcal{I} = \{[x, x+a_n]\}_{x\in B,\, n\in\mathbb{N}}$ is a V-type cover of B. Conclude the proof of $(\ge)$.

$\square$

Proof of 8.2.4. We satisfy the hypotheses of (8.2.2) by replacing f with $f(x) + x$ wlog.

Exercise. $|\{\infty \in Df(x)\}| = 0$.

For each rational pair $0 < q_1 < q_2 < \infty$, let $A_{q_1,q_2} = \{\exists\, s_1, s_2 \in Df(x),\ s_1 < q_1 < q_2 < s_2\}$.
(8.2.2) then makes magic happen: $q_2|A_{q_1,q_2}| \le |f(A_{q_1,q_2})| \le q_1 |A_{q_1,q_2}|$, hence $|A_{q_1,q_2}| = 0$.
$\{\#Df(x) > 1\} \subset A_{q_1,q_2}$ for some pair $q_1, q_2 \in \mathbb{Q}$. Combine with the previous exercise to win. $\square$

Definition (bounded variation)**.**
Let $f : [a,b] \to \mathbb{R}$. The **total variation** of f is

$$V_a^b f = \sup_P \Big\{ \sum_i |f(x_i) - f(x_{i-1})| \Big\},$$

taken over all partitions P of $[a,b]$. f is of **bounded** (or *finite*) **variation** if $V_a^b f < \infty$.

Observe that $f \nearrow \implies V_a^b f = f(b) - f(a)$. In particular, $V_a^b f < \infty$.
Conversely, $f(x) \le f(a) + V_a^x f \le f(a) + V_a^b f$, thus $V_a^b(f) < \infty \implies f$ bounded.

Exercise. $V_a^b f, V_a^b g < \infty,\ c \in \mathbb{R} \implies V_a^b(cf), V_a^b(f+g), V_a^b(fg) < \infty$.

Theorem (8.3.6)**.** $V_a^b f < \infty \iff \exists\, g, h : [a,b] \nearrow \mathbb{R} : f = g - h$.

Proof. $\implies$: Take $g(x) = V_a^x f$, $h = g - f$. Then, $h(y) - h(x) = V_x^y - (f(y) - f(x))$. $\square$

Now, we come to the central definition in the study of derivatives: The main result is that absolute continuity characterizes functions for which FIC is equality.

Definition (absolute continuity)**.**

$f : [a,b] \to \mathbb{R}$ is **absolutely continuous** (hereon a.c.) if

$$\forall\, \epsilon > 0 \,\exists\, \delta > 0 : \sum_{i \le n} (b_i - a_i) < \delta \implies \sum_{i \le n} |f(b_i) - f(a_i)| < \epsilon,$$

where $\{[a_i, b_i] \subset [a,b]\}_{i \le n}$ are disjoint.

Another definition, one that probably should be common knowledge but I bother to state in a colored box due to my ignorance:

Definition (Lipschitz)**.**

$f : [a,b] \to \mathbb{R}$ is **Lipschitz** if

$$\forall\, x, y \in [a,b] \,\exists\, K \in [0, \infty) : |f(y) - f(x)| < K\,|x - y|.$$

Exercise. Lipschitz $\implies$ absolutely cont. $\implies$ uniformly cont. $\implies$ pointwise cont..

Exercise. Absolute continuity without the disjoint condition is equivalent to Lipschitz.

Exercise (9.1.1)**.** f, g a.c., $c \in \mathbb{R} \implies cf, f+g, fg$ a.c..

Exercise (9.2.1)**.** f a.c. $\implies V_a^b f < \infty$.

Note that (8.3.6) combined with (8.2.4) implies that $f' < \infty$ a.e., and FIC tells us $f' \in \mathcal{L}^1([a,b])$. We keep the convention that $f' = 0$ where it is not defined by the usual definition.

Exercise. f Lipschitz $\iff$ f' bounded.

Example. On $[0,1]$, $\sqrt{x}$ is absolutely continuous but not Lipschitz, and the Cantor function Λ is uniformly continuous but not absolutely continuous.

In concrete terms, it is convenient to think of Lipschitz as equivalent to having bounded derivative, and absolute continuity as a check for whether FTC holds. Recall that FIC is not equality for Λ.

Exercise. On $[0,1]$, $x\,|\sin(1/x)|$ is not a.c., but $x^2 \sin^2(1/x)$ is.

Writing $x\,|\sin(1/x)| = \sqrt{x^2 \sin^2(1/x)}$, we see that composition of a.c. functions need not be a.c.. However, the following criteria are useful to keep in mind. Maybe even a colored box.

Exercise (9.1.2, 9.1.3)**.**

Let $f : [a,b] \to [c,d], g : [c,d] \to \mathbb{R}$ be a.c.. Then,

$$(i)\, g \text{ Lipschitz} \implies g \circ f \text{ a.c.} \qquad (ii)\, f \nearrow \text{ strictly} \implies g \circ f \text{ a.c.}$$

Note that the example prior shows that (i) and (ii) both fail if the roles of f and g are reversed.

Exercise. $f : [a,b] \to (0, \infty)$ a.c. $\implies 1/f$ a.c.. If $f \nearrow$ strictly, then $\sqrt{f}$ is also a.c..

Okay, enough foreshadowing. Let's prove the fundamental theorem of (Lebesgue) calculus.

Theorem (FTC II)**.**

$f : [a,b] \to \mathbb{R}$ a.c. $\implies \displaystyle\int_{[a,b]} f' = f(b) - f(a)$.

Proof. By (8.3.6), this exercise, and (9.1.1), we may assume $f \nearrow$ wlog.

Exercise. f a.c. $\implies g(x) = V_a^x f$ a.c..

Let $A = \left\{ \lim_{t\downarrow 0} \frac{1}{t} \int_{[x,x+t]} f' = f' \right\}$. Note that this condition implies that of (4.21), thus $A \overset{0}{\subset} [a,b]$. Given $\epsilon > 0$, $x \in A$, take $a_n \in (0, 1/n) : \left| \frac{f(x+a_n) - f(x)}{a_n} - f'(x) \right|, \left| \frac{1}{a_n} \int_{[x,x+a_n]} f' - f'(x) \right| < \epsilon$.

Similarly as in the proof of (8.2.2), $\{[x, x+a_n]\}_{x\in A,\, n\in\mathbb{N}}$ is a V-type cover of A, thus Vitali II extracts a countable subset of disjoint intervals $\{I_i = [x_i, x_i + a_{n_i}(x_i)]\}_{i\in\mathbb{N}}$ with $\left| A \setminus \dot{\bigcup_{i\le N}} I_i \right| = 0$.

Since $A \overset{0}{\subset} [a,b]$, we have $\forall\, \delta > 0\, \exists\, N \in \mathbb{N} : \left| [a,b] \setminus \dot{\bigcup_{i\le N}} I_i \right| < \delta$, and $f \nearrow$ gives us

$$f(b) - f(a) = \sum_{i\le N} |f(I_i)| + \sum_{i\le N+1} |f(J_i)|, \qquad \int_{[a,b]} f' = \sum_{i\le N} \int_{I_i} f' + \int_{[a,b]\setminus \bigcup_{i\le N} I_i} f'$$

where J_i denotes the intervals of $[a,b] \setminus \dot{\bigcup_{i\le N}} I_i$.

Let's put the pieces together. A good enough choice of δ will bound the 2nd term in the RHS for both equalities by ϵ, using that f and $\int$ are both a.c. (see (3.28)). On the other hand, by our initial construction, we have $\big|\, |f(I_i)| - f'(x_i)|I_i| \,\big|, \left| \int_{I_i} (f' - f'(x_i)) \right| < |I_i|\epsilon$. Now, win the game by observing that both $f(b) - f(a)$ and $\int_{[a,b]} f'$ differ from $\sum_{i\le N} f'(x_i)|I_i|$ by at most $(b - a + 1)\epsilon$. $\square$

This proof Ilya gives differs a bit from Natanson's (9.4.3), where it is first shown that for f, g a.c., $f' = g'$ implies $f - g$ constant, whence we deduce $f(x) = f(a) + \int_{[a,x]} f'$ by LDT.

Exercise (9.2.2)**.** Mimic Ilya's proof of FTC II above to show: f a.c., $f' = 0 \implies f$ constant.

Exercise. Combine FTC I and FTC II to show, for arbitrary $f : [a,b] \to [c,d]$ and a.c. $g : [c,d] \to \mathbb{R}$ such that $(f \circ g) \cdot g' \in \mathcal{L}^1[a,b] : \int_{[a,b]} (f \circ g) \cdot g' = \int_{[g(a),\, g(b)]} f$.

4. Product Measures

Back to measure theory for a bit. The goal of this brief section is to prove Tonelli and Fubini, and do some interesting stuff with layered cake. We start with a bunch of definitions.
The groundwork is quite boring, so this page may be mostly skipped if you are busy.

Definition. Let $\mathcal{S}, \mathcal{T}$ be σ-algebras. We denote by $\mathcal{S} \otimes \mathcal{T}$ the σ-algebra generated by $\mathcal{S} \times \mathcal{T}$.
Let X, Y be sets. For $A \subset X \times Y$ and $(x, y) \in X \times Y$, we define the **cross sections**

$$[A]_x = \{(x, -) \in A\} \subset Y, \quad [A]^y = \{(y, -) \in A\} \subset X.$$

Similarly, given $f : X \times Y \to \mathbb{R}$, we have $[f]_x = f(x, -) : Y \to \mathbb{R}$, $[f]^y = f(-, y) : X \to \mathbb{R}$.

For the next four exercises, let $(X, \mathcal{S}), (Y, \mathcal{T})$ be measurable spaces. By the above definition, we may also view $(X \times Y, \mathcal{S} \otimes \mathcal{T})$ as a measurable space.

Exercise (5.6)**.** $A \subset X \times Y$ meas. $\implies \forall\, (x, y) \in X \times Y : [A]_x, [A]^y$ meas..

Exercise (5A.2)**.** $A \subset X \times X$ meas. $\implies \{x \in X \,|\, (x, x) \in A\}$ meas..

The above are exercises in σ-induction.

Exercise (5.9)**.** $f : X \times Y \to \mathbb{R}$ meas. $\implies [f]_x, [f]^y$ meas..

Exercise (5A.4)**.** $g : X \to \mathbb{R}$, $h : Y \to \mathbb{R}$ meas. $\implies f(x, y) = g(x)h(y) : X \times Y \to \mathbb{R}$ meas..

Definition. An **algebra** $\mathcal{A}$ on X is a collection of sets closed under complements and finite unions.
A **monotone class** $\mathcal{E}$ on X is a collection closed under increasing and decreasing sequences.

In the above language, a σ-algebra is an algebra that is also a monotone class.

Theorem (monotone class theorem)**.** Let $\mathcal{A}$ be an algebra on X, and $\mathcal{E}$ the minimal monotone class on X containing $\mathcal{A}$. Then, $\mathcal{E}$ is a σ-algebra.

Proof. It suffices to show that $\mathcal{E}$ is an algebra. Let $\mathcal{D} = \{D \in \mathcal{E} \,|\, \forall\, E \in \mathcal{E} : D \cup E \in \mathcal{E}\}$.

Exercise. Verify that $\mathcal{D}$ is a monotone class containing $\mathcal{A}$. Conclude $\mathcal{E}$ is closed under finite unions.

Exercise. By a similar construction, show that $\mathcal{E}$ is closed under complements.

$\square$

This is a gadget that allows us to upgrade σ-induction by treating $\mathcal{A}$ as a base case, then reducing most of the work to showing that given property of a set is preserved under monotone limits.
We now work towards defining the product measure. First, we need the following:

Theorem (5.20)**.** Let $(X, \mathcal{S}, \mu), (Y, \mathcal{T}, \nu)$ be σ-finite. Then, $\forall\, A \in \mathcal{S} \otimes \mathcal{T}\ \forall\, (x, y) \in X \times Y$:
the functions $x \mapsto \nu([A]_x) : X \to \mathbb{R}$ and $y \mapsto \mu([A]^y) : Y \to \mathbb{R}$ are meas..

Proof. Let $\mathcal{A}$ be the algebra generated by $\mathcal{S} \times \mathcal{T}$ and $\mathcal{E} = \{E \in \mathcal{S} \otimes \mathcal{T} \,|\, x \mapsto \nu([A]_x) \text{ meas.}\}$.

Exercise. Show that $\mathcal{E}$ contains $\mathcal{A}$.

Assume Y is finite. Then, continuity of measure shows that $\nu([\lim_{i \to \infty} A_i]_x) = \lim_{i \to \infty} \nu[A_i]_x$, for $\{A_i\}_{i \in \mathbb{N}}$ monotone, thus $\mathcal{E}$ is a monotone class since limits preserve measurability of functions.
Conclude by the monotone class theorem that $\mathcal{E} = \mathcal{S} \otimes \mathcal{T}$.
For σ-finite Y, express Y as an increasing limit of finite Y_i, and repeat the argument.

Exercise. Copy the argument to show that $y \mapsto \mu([A]^y)$ is meas..

$\square$

Definition (iterated integrals, product measure)**.**

Let $(X, \mathcal{S}, \mu), (Y, \mathcal{T}, \nu)$ be measure spaces, and $f : X \times Y \to [0, \infty]$. We define

$$\int_{(X)} \int_{(Y)} f \, d\nu d\mu = \int_X \Big(\int_Y [f]_x \, d\nu \Big) d\mu.$$

The measure $\mu \times \nu$ on $X \times Y$ is then defined by

$$(\mu \times \nu)(A) = \int_X \int_Y \chi_A \, d\nu d\mu = \int_X \Big(\int_Y [\chi_A]_x \, d\nu \Big) d\mu = \int_X \nu([A]_x) d\mu.$$

Note that by the above definition, the iterated integral for f need not be defined, but (5.20) shows that the product measure $\mu \times \nu$ is necessarily well-defined. The theorems of Tonelli and Fubini give sufficient conditions for when we are allowed to interchange the order of iterated integrals. In particular, it is not clear yet that $\mu \times \nu$ is the same measure as $\nu \times \mu$.

Theorem (Tonelli)**.**

Let $(X, \mathcal{S}, \mu), (Y, \mathcal{T}, \nu)$ be σ-finite, and $f : X \times Y \to [0, \infty]$ meas.. Then,

$$\int_{X \times Y} f(x, y) \, d(\mu \times \nu) = \int_X \int_Y f \, d\nu d\mu = \int_Y \int_X f \, d\mu d\nu.$$

Proof. Our strategy is to use standard approximation and MCT to reduce to the simple case, for which the proof is similar to that of (5.20), with $\mathcal{E} = \Big\{ E \in \mathcal{S} \otimes \mathcal{T} \,\Big|\, \iint \chi_E \, d\nu d\mu = \iint \chi_E \, d\mu d\nu \Big\}$.

Exercise. Likewise as in (5.20), show that $\mathcal{E}$ is a monotone class assuming X, Y are finite.

As before, we pass to the σ-finite case, whence MCT concludes the proof. $\square$

Corollary (Fubini)**.**

Let $(X, \mathcal{S}, \mu), (Y, \mathcal{T}, \nu)$ be σ-finite, and $f \in \mathcal{L}^1(X \times Y)$. Then,

$$\int_{X \times Y} f(x, y) \, d(\mu \times \nu) = \int_X \int_Y f \, d\nu d\mu = \int_Y \int_X f \, d\mu d\nu.$$

In particular, $[f]_x \in \mathcal{L}^1(X)$ X-a.e., and $[f]^y \in \mathcal{L}^1(Y)$ Y-a.e..

Proof. Write $f = f_+ + f_-$, and apply Tonelli to f_+ and f_-. $\square$

Some obligatory examples to convince you that the hypotheses are optimized:

Exercise (5.30)**.** Tonelli fails for $f(x, y) = \chi_{\{x=y\}}$ on $X \times Y = [0, 1] \times [0, 1]$, where we use the counting measure on Y. Note that $([0, 1], \mathcal{B})$ is not σ-finite with the counting measure.

Exercise (5B.1)**.** Tonelli and Fubini both fail for $f(x, y) = \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$ on $X \times Y = [0, 1] \times [0, 1]$.

We omit the hypotheses of σ-finite measure spaces hereon. The most useful application arises in computing the area under the graph of a function, or the 'layered cake representation':

Exercise (layered cake)**.**

Let $f : X \to [0, \infty]$. Then, denoting the Lebesgue measure as λ, we have:

$$\int f \, d\mu = \int_{[0,\infty)} \mu(\{f \geq t\}) \, d\lambda(t).$$

The intuition is that we are measuring the set $\{(x,t) \in X \times [0,\infty) \mid f(x) \geq t\}$ in two different ways. The nontriviality of Tonelli becomes transparent when we omit the function we start with and state only the middle steps $\displaystyle\int_X f \, d\mu$ and $\displaystyle\int_Y f \, d\nu$.

For the following exercises, we assume $f : X \to [0, \infty]$ and that all integrals are finite.

Exercise. $\displaystyle\frac{1}{p}\int_X f^p \, d\mu = \int_{[0,\infty)} \lambda(\{f \geq t\}) \cdot t^{p-1} \, d\lambda(t)$ for $p > 0$.

Exercise. $\displaystyle\mu(X) < \infty \implies \int_X e^f \, d\mu = \mu(X) + \int_{[0,\infty)} \mu(\{f \geq t\}) \cdot e^t \, d\lambda(t)$.

It is useful to generalize the previous two exercises to the following:

Exercise. $\displaystyle\int_X (h \circ f) \, d\mu = \int_{h^{-1}([0,\infty))} \mu(\{f \geq t\}) \cdot h'(t) \, d\lambda(t)$, where $h : [0,\infty) \nearrow [0,\infty)$ strictly.

5. Real Measures

We extend our notion of measure to measures taking values in $\mathbb{R}$. This section starts with a Jun sub, and Ilya starts speedrunning to Radon-Nikodym and L^p duality. I admit that I have not done much to appreciate the usefulness of these concepts, so this section might feel very dry.
Likewise as the last section, not much is lost by skipping this first page.
Throughout, $(X, \mathcal{S})$ is a measurable space and all sets are assumed to be measurable.

Definition. A **real** (or *signed*) **measure** on X is a countably additive function $\nu : \mathcal{S} \to \mathbb{R}$.
The **total variation measure** of ν is defined by $|\nu|\colon A \mapsto \sup_P \sum_{i \le n} |\nu(A_i)|$, where P ranges over all finite disjoint collections (or wlog, partitions of A) $\{A_i \subset A\}_{i \le n}$.

Note that we do not allow $\pm\infty$, thus by splitting into the signed parts, we observe that $\sum_{i \in \mathbb{N}} \nu(A_i)$ converges absolutely for all disjoint collections $\{A_i\}_{i \in \mathbb{N}}$. The same argument also shows that it suffices to consider $n = 2$ in the definition of $|\nu|$.

Exercise (9.11)**.** Show that $|\nu|$ is finitely additive, i.e. $|\nu|\Big(\dot{\bigcup_{i \le n}} A_i\Big) = \sum_{i \le n} |\nu|(A_i)$.
Deduce that $|\nu|$ is in fact countably additive, thus is a measure on X.

Proposition (9.17)**.** If ν is a real measure on X, then $|\nu|(X) < \infty$.

Proof. Suppose not. The contradiction is derived by exhibiting $A = \bigcap_{n \in \mathbb{N}} A_n$ with $\nu(A) \notin \mathbb{R}$.
Set $A_0 = X$, and construct A_n inductively with $|\nu|(A_n) = \infty$, $\nu(A_n) \ge n$ as follows:
$\exists\, B_n \subset A_n : |\nu(B_n)| \ge (n+1) + |\nu(A_n)|$, whence $|\nu(B_n \backslash A_n)| \ge |\nu(B_n)| - |\nu(A_n)| \ge n+1$,
and it suffices to set A_{n+1} to be either B_n or $A_n \backslash B_n$ to satisfy $|\nu|(A_{n+1}) = \infty$. □

As is the case with most proofs by contradiction, it is not the most illuminating. Fortunately, the Jordan decomposition theorem will tell us that we can in fact do the naive thing as before and simply split X into two (purely) signed parts.

Exercise. Let μ be a measure on X, and $f \in \mathcal{L}^1(X)$. Then, $\nu : A \mapsto \int_A f \, d\mu$ defines a real measure, and the associated total variation measure is $|\nu|\colon A \mapsto \int_A |f| \, d\mu$. Write $\nu = f \, d\mu$, $|\nu| = |f| \, d\mu$.

Theorem (Jordan decomposition)**.** Let ν be a real measure on X. Then,

$$\exists\, E \in \mathcal{S}\, \forall\, A \in \mathcal{S} : \nu_+(A) := \nu(E \cap A) \ge 0,\ -\nu_-(A) := \nu(E^{\complement} \cap A) \le 0.$$

In other words, $\nu = \nu_+ - \nu_-$, where ν_+, ν_- are measures on X.

Proof. Let $a = \sup_{A \in \mathcal{S}} \nu(A)$. By (9.17), we have $a \in [0, \infty)$, and we may assume $a > 0$ wlog.
$\forall\, n \in \mathbb{N}\, \exists\, E_n \in \mathcal{S} : \nu(E_n) \ge a - 2^{-n}$. Let $E = \bigcap_{N \in \mathbb{N}} \bigcup_{n \ge N} E_n$.

Exercise. Show that $\nu(E) = a$. Complete the proof.

□

It is convenient to call E and $E^{\complement}$ the positive and negative parts of X, respectively.

Definition (absolute continuity of measures)**.**

Let μ be a measure on X, and ν a real measure on X.
ν is **absolutely continuous** w.r.t. μ, written $\nu \ll \mu$, if $\forall\, A \in \mathcal{S} : \mu(A) = 0 \implies \nu(A) = 0$.
ν, μ are **mutually singular**, written $\nu \perp \mu$, if $\exists\, E \in \mathcal{S} : \mu(E) = 0,\, \nu(E^{\complement}) = 0$.

Exercise (9B.14)**.** $\nu \ll \mu \iff \forall\, \epsilon > 0\, \exists\, \delta > 0\, \forall\, A \in \mathcal{S} : \mu(A) < \delta \implies |\nu(A)| < \epsilon$.

Note that ν_+, ν_- are mutually singular in the Jordan decomposition.

Theorem (Lebesgue decomposition)**.** Let μ be a measure on X, and ν a real measure on X. Then, there exist unique real measures ν_1, ν_2 on X such that $\nu = \nu_1 + \nu_2$, $\nu_1 \ll \mu$, $\nu_2 \perp \mu$.

Proof. Similarly to Jordan decomposition, let $a = \sup\limits_{A \in \mathcal{S},\, \mu(A)=0} \nu(A)$, and assume $a > 0$ wlog.
$\forall\, n \in \mathbb{N}\, \exists\, E_n \in \mathcal{S} : \nu(E_n) \geq a - 1/n$. Let $E = \bigcup\limits_{n \in \mathbb{N}} E_n$.

Exercise. Show that $\nu_2(A) := \nu(E \cap A)$, $\nu_1(A) := \nu(E^{\complement} \cap A)$ yields the desired decomposition.

Exercise (9.34)**.** $\nu \ll \mu,\, \nu \perp \mu \implies \nu = 0$. Deduce uniqueness of the decomposition.

$\square$

Theorem (Radon-Nikodym)**.**

Let (X, μ) be σ-finite, and ν a real measure on X. Then,

$$\nu \ll \mu \iff \exists\, f \in \mathcal{L}^1(X, \mu) : \nu = f\, d\mu.$$

f is called the **Radon-Nikodym derivative** of ν w.r.t. μ. Write $f = \dfrac{d\nu}{d\mu}$.

Proof. The relevant direction is $\implies$. Start with the case that μ, ν are both finite measures on X.
Define $\mathcal{F} = \{f \in \mathcal{L}^1(X) \,|\, \forall\, A \in \mathcal{S} : \nu(A) \geq \int_A f\, d\mu\}$, and assume $a = \sup\limits_{f \in \mathcal{F}} \int_X f\, d\mu > 0$ wlog.
$\forall\, n \in \mathbb{N}\, \exists\, f_n \in \mathcal{F} : \int_X f_n\, d\mu \geq a - 1/n$. Let $f = \limsup\limits_{n \to \infty} f_n$.

Exercise. Using MCT, show that $f \in \mathcal{F}$.

The claim is that $\nu = f\, d\mu$. If not, choose $\epsilon > 0$ such that $\nu(X) - \int_X f\, d\mu > \epsilon \mu(X)$.
Applying Jordan decomposition to the real measure $\nu - f\, d\mu - \epsilon\mu$, let E be the positive part of X.

Exercise. Show that $f + \chi_E \in \mathcal{F}$, and $\int_X (f + \chi_E)\, d\mu > a$. Conclude the proof for the finite case.

Exercise. Extend to the σ-finite case, and use Jordan decomposition on ν to conclude the proof.

$\square$

The proof differs from Axler, mainly due to lack of coverage of earlier chapters. Ilya credits this proof to Wikipedia, which seems to be similar to be that in Folland[Fol99]. The obligatory counterexample to Radon-Nikodym for when (X, μ) is not σ-finite is the point set $X = \{x\}$ assigned with $\mu(X) = \infty$, for which the only real measure on X of the form $f\, d\mu$ is 0.

6. L^p Spaces

The last theme of the course is L^p spaces. The most useful thing here is Hölder's inequality. Throughout this section, let (X, μ) be a measure space. All functions are assumed measurable. Recall that $\mathcal{L}^1(X)$ is the space of measurable functions $f : X \to \mathbb{R}$ such that $||f||_1 = \int_X |f|\, d\mu < \infty$. We extend the definition to $p \in (0, \infty]$:

Definition (L^p norm)**.**

For $p \in (0, \infty)$, the **p-norm** of $f : X \to \mathbb{R}$ is $||f||_p := \left(\int |f|^p \, d\mu \right)^{1/p}$.
The **∞-norm** of $f : X \to \mathbb{R}$ is $||f||_\infty := \operatorname{ess\,sup} |f| = \inf\{t \in [0, \infty] \mid |f| \le t \text{ a.e.}\}$.
$\mathcal{L}^p(X)$ is the space of measurable functions with $||f||_p < \infty$.
$L^p(X)$ is the quotient space of $\mathcal{L}^p(X)$ by the equivalence relation $f \sim g$ if $f = g$ a.e..

Our first job is to show that $||\cdot||_p$ is actually a norm on L^p: The equivalence relation imposes positive definiteness, homogeneity follows from definition, and it remains to show triangle inequality. For $p \in [1, \infty]$, we define the **dual exponent** p' of p to be the number satisfying $\frac{1}{p} + \frac{1}{p'} = 1$.

Exercise (Young's inequality)**.** $a, b \in [0, \infty)$, $p \in (1, \infty) \implies ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}$.

Theorem (Hölder's inequality)**.**

$p \in [1, \infty] \implies \int |fg| \le ||f||_p \, ||g||_{p'}$.

Proof. Assume $||f||_p, ||g||_{p'} \in (0, \infty)$ wlog. The case $p = 1, p' = \infty$ is also easy.
By rescaling, we assume $||f||_p, ||g||_{p'} = 1$, from which the result follows from Young's inequality:

$$\int |fg| \le \int \left(\frac{|f|^p}{p} + \frac{|g|^{p'}}{p'} \right) = \frac{1}{p} + \frac{1}{p'} = 1.$$

□

The following generalization is useful to keep in mind:

Exercise (7A.7, 7A.8)**.** $p, q, r \in (0, \infty]$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \implies ||fg||_r \le ||f||_p \, ||g||_q$.
Generalize to: $p_1, \cdots, p_n, r \in (0, \infty]$, $\frac{1}{r} = \sum_{i \le n} \frac{1}{p_i} \implies \Big|\Big| \prod_{i \le n} f_i \Big|\Big|_r \le \prod_{i \le n} ||f_i||_{p_i}$.

Independent of Hölder, we also justify the definition of the ∞-norm:

Proposition (7A.16)**.** Let $f \in L^\infty$. Then, $\exists\, q \in (0, \infty) : f \in L^q \implies \lim_{p \to \infty} ||f||_p = ||f||_\infty$.

Proof. Assume $||f||_\infty = 1$ wlog. Given $\epsilon \in (0, 1)$, let $A_\epsilon = \{f > 1 - \epsilon\}$. Then, we have $||f||_p \ge \left(\int_{A_\epsilon} (1 - \epsilon)^p \right)^{1/p} = \mu(A_\epsilon)^{1/p}(1 - \epsilon)$ and $||f||_p \le ||f||_q^{q/p}$, the latter assuming $p > q$.
The former for $p = q$ implies $\mu(A_\epsilon) < \infty$. Take the limits $\epsilon \to 0$ and $p \to \infty$ to win. □

We take some time to explore containment of L^p spaces and various examples of L^p functions. The next five exercises assume $p, q \in (0, \infty], p < q$. Warmup: $1/x \in L^2([1, \infty))$, and $1/x \notin L^1([0, 1])$. In fact, $1/x \in L^p([1, \infty)) \iff p \in (1, \infty]$, and $1/x \in L^p([0, 1]) \iff p \in (0, 1)$.

Exercise (7.10)**.** $\mu(X) < \infty \implies L^q \subset L^p$. Exhibit $f \in L^p \backslash L^q$.

Exercise (7A.14)**.** Exhibit $f \in L^q \backslash L^p$. In general, there is no containment between L^p and L^q.

Exercise (7A.10)**.** $\ell^p := L^p(\mathbb{N})$, where $\mathbb{N}$ has the counting measure. Given $f : \mathbb{N} \to \mathbb{R}$, show that $||f||_p \geq ||f||_q$, hence $\ell^p \subset \ell^q$. Exhibit $f \in \Big(\bigcap\limits_{q>p} \ell^q\Big) \backslash \ell^p$.

The first part is a bit tricky, but it becomes almost trivial once it is observed that we may assume $||f||_p = 1$ wlog, as in the proofs of Hölder and $(7A.16)$. It is perhaps an opportunity to realize that the fractional exponents are extremely cumbersome to deal with without this assumption.

It is also noted that we may view ℓ^p as a subspace of $L^p([0, \infty))$ (or $L^p(\mathbb{R})$) by identifying $f \in \ell^p$ with $\sum\limits_{n \in \mathbb{N}} f(n)\chi_{[n-1,n)}$. In particular, $(7A.10)$ also gives a function $f \in \Big(\bigcap\limits_{q>p} L^q(\mathbb{R})\Big) \backslash L^p(\mathbb{R})$. As a variation, examples for $L^p([0, 1])$ may also be constructed by considering $\sum\limits_{n \in \mathbb{N}} f(n)\chi_{(2^{-n},\, 2^{1-n}]}$.

Exercise (7A.12, 7A.13)**.** Let $X = [0, 1]$. Exhibit $f : \mathbb{N} \to [0, \infty)$ such that $\sum\limits_{n \in \mathbb{N}} f(n)\chi_{(2^{-n},\, 2^{1-n}]} \in$

$$(i) \Big(\bigcap_{p<\infty} L^p\Big) \backslash L^\infty \qquad (ii) \Big(\bigcap_{p<q} L^p\Big) \backslash L^q \qquad (iii)\, L^p \backslash \Big(\bigcup_{q>p} L^q\Big)$$

The last trick is to convert vertical sticks into horizontal ones. To be precise, write $f(n) = 2^n g(n)$, then: $\displaystyle\int \sum_{n \in \mathbb{N}} f(n)\chi_{(2^{-n},\, 2^{1-n}]} = \sum_{n \in \mathbb{N}} g(n) = \int \sum_{n \in \mathbb{N}} 2^{-n} g(n)\chi_{[2^n,\, 2^{n+1})}$.

In particular, $\displaystyle\sum_{n \in \mathbb{N}} 2^n g(n)\chi_{(2^{-n},\, 2^{1-n}]} \in L^1([0, 1]) \iff \int \sum_{n \in \mathbb{N}} 2^{-n} g(n)\chi_{[2^n,\, 2^{n+1})} \in L^1([2, \infty))$.

Exercise (7A.15)**.** Use the above construction to exhibit $f \in L^q \backslash \Big(\bigcup\limits_{p<q} L^p\Big)$.

Combine the previous results to construct a function $f \in L^p \backslash \Big(\bigcup\limits_{r \neq p} L^r\Big)$.

Alright, back on track. Reminder that we need to show that $||\cdot||_p$ satisfies the triangle inequality.

Exercise (7.12)**.** Use Hölder to show, for $p \in [1, \infty)$: $||f||_p = \sup\limits_{||h||_{p'}=1} \Big|\displaystyle\int fh\Big|$.

Generalize to: $p, q, r \in (0, \infty)$, $\dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q} \implies ||f||_p = \sup\limits_{||h||_q=1} ||fh||_r$.

Considering $X = \{x\}$, $\mu(X) = \infty$ shows that (7.12) does not hold for $p = \infty$. However:

Exercise (7A.9)**.** If X is σ-finite, then for $p \in (0, \infty) : ||f||_\infty = \sup\limits_{||h||_p=1} ||fh||_p$.

For $(\leq)$, do proof by contradiction, with $\sup\limits_{||h||_p=1} ||fh||_p < c < ||f||_\infty$ and considering $A = \{|f| \geq c\}$.

Exercise (Minkowski's inequality)**.** Use (7.12) to show, for $p \in [1, \infty]$: $||f+g||_p \leq ||f||_p + ||g||_p$.

Theorem (7.20)**.** For $p \in [1, \infty)$, L^p is a **Banach space**, i.e. a complete normed space.

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be Cauchy in L^p, and assume $\sum_{n\in\mathbb{N}} ||f_n - f_{n-1}||_p < \infty$ wlog. Take $g_m = \sum_{n\leq m} |f_n - f_{n-1}|$, and $g = \sum_{n\in\mathbb{N}} |f_n - f_{n-1}|$. Then, $\int g^p = \lim_{m\to\infty} \int (g_m)^p \leq \Big(\sum_{n\in\mathbb{N}} ||f_n - f_{n-1}||_p\Big)^p$ by MCT and Minkowski, respectively, hence $g \in L^p$, i.e. $g < \infty$ a.e.. This allows us to write $f_m = \sum_{n\leq m} (f_n - f_{n-1}) \to f < \infty$ a.e., and $|f_m - f|^p \leq 2^p(|f_m|^p + |f|^p) \leq 2^p((g_m)^p + g^p) \leq 2^{p+1} g^p$.
DCT now applies to conclude $\lim_{m\to\infty} \int |f_m - f|^p = \int \Big(\lim_{m\to\infty} |f_m - f|\Big)^p = 0$, i.e. $f_m \to f$ in L^p. □

Exercise. Mimic the proof for $p \in [1, \infty)$ to show that L^∞ is also a Banach space.

Okay, now towards L^p duality, the final result of the course. Some definitions are needed.

Definition. Let V be a normed vector space. A **linear functional** $\phi : V \to \mathbb{R}$ is a linear map. The **norm** of a linear functional is defined by $||\phi|| = \sup_{v\in V,\, ||v||=1} |\phi(v)|$.
The **dual space** V^* of V is the space of bounded linear functionals $V \to \mathbb{R}$.

Exercise. For any normed vector space V, V^* is a Banach space.

> **Theorem** (L^p duality)**.**
>
> For $p \in (1, \infty)$, the map $\Phi : L^{p'} \to (L^p)^*$ defined by $\Phi(h) := \phi_h : f \mapsto \int fh$ is an isometric isomophism of Banach spaces.

Proof. The assertion that Φ is an isometry is equivalent to (7.12), from which injectivity follows. For surjectivity, start with the case $\mu(X) < \infty$. Given $\phi \in (L^p)^*$, let $\nu_\phi(A) := \phi(\chi_A)$.

Exercise. Show that ν_ϕ is a real measure, satisfying $\nu_\phi \ll \mu$.

By Radon-Nikodym, we may write $\nu_\phi = h\, d\mu$ for some $h \in L^1$. By construction, $\phi = \phi_h$ holds for simple functions, and standard approximation with DCT applies to $f \in L^p$.
It follows that $||h||_{p'} = ||\phi|| < \infty$, i.e. $h \in L^{p'}$.

Exercise. First, extend to the σ-finite case. For the general case, let $a = \sup_{A\subset X\ \sigma\text{-finite}} ||h_A||_{p'}$, $\{A_n\}_{n\in\mathbb{N}}$ a sequence such that $||h_{A_n}||_{p'} \to a$, and $h_{A_n} \to h$. Conclude the proof.

□

The last lecture ended at the finite case. The argument of the exercise follows Folland (6.15). Note that by (7.12), Φ is an injective isometry for $p = \infty$. However, considering $X = \{x\}$, $\mu(X) = \infty$ shows that duality fails for both $p = 1$ and $p = \infty$.

Exercise (7B.4)**.** Let $p \in (1, \infty)$. If f satisfies $\forall\, h \in L^p : fh \in L^1$, then $f \in L^{p'}$.

Given $h \in L^p$, Hölder tells us that if $f \in L^{p'}$, then $fh \in L^1$. In a sense, this is a converse to that. Do proof by contradiction by taking $\{h_n\}_{n\in\mathbb{N}} \subset L^p$ with $||h_n||_p = 1$, $\int fh_n \geq n^2$, and $h = \sum_{n\in\mathbb{N}} \frac{h_n}{n^2}$.

Appendix

1. List of notations
   - $A^{\complement}$: set complement of A in X, or $X\backslash A$.
   - 2^X : power set of a set X
   - $A \overset{\epsilon}{\subset} B : A \subset B$ and $\mu(B\backslash A) < \epsilon$.
   - $A \overset{0}{\subset} B : A \subset B$ and $\mu(B\backslash A) = 0$.
   - χ_A : characteristic function of a set A.
   - $f \in \mathcal{L}^1(X) : f$ is measurable, and $\displaystyle\int_X f < \infty$.
   - $\{P\}$: abuse of notation for $\{x \in X \mid P(x) \text{ holds}\}$, e.g. $\{f \leq g\} = \{x \in X \mid f(x) \leq g(x)\}$.
   - $f_n \nearrow f : f_n$ is a monotone increasing sequence of functions converging to f.
   - $f : [a, b] \nearrow \mathbb{R}$ or $f \nearrow$: f is a monotone increasing function $f : [a, b] \to \mathbb{R}$.
   - $n * I$: the dilation of I at its center by a factor of n.
2. List of abbreviations
   - TFAE = the following are equivalent
   - wlog = without loss of generalization
   - meas. = measurable
   - a.e. = almost everywhere
   - a.c. = absolutely continuous
3. Omitted topics

There is a selection of things that were covered but I chose to omit, the main reasons being my lack of understanding and laziness. A better excuse would be that we did not do anything substantial with them to motivate their study.

The two major ones are decomposition of monotone functions and Lebesgue integration on $\mathbb{R}^n$. It is also noted that the concept of real measure is also easily extendable to complex measures, and the proofs for real measures extend to complex ones by treating real and imaginary components separately. I did not feel that this is significant enough to warrant cluttering up the text.

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Solutions

For up to August 2024, consult Hong's notes. These are for the quals from January 2025 onwards. The problem statements may differ here from the original, for the sake of brevity and consistency of notation. Please visit the website for clarification.

January 2025

1. $f_n : [0,1] \to [0,\infty)$, $f_n \to 0$ in $\mathcal{L}^1 \implies \exists$ subsequence $f_{n_k} \to 0$ a.e.. Exhibit $f_n \not\to 0$ a.e..

Solution. By Markov, we have $\forall\, k \in \mathbb{N}\, \exists\, n_k \in \mathbb{N} : \mu(\{f_{n_k} \geq 1/k\}) \leq k \displaystyle\int f_{n_k} < 2^{-k}$.
Let $A_k = \{f_{n_k} \geq 1/k\}$, $A = \displaystyle\bigcap_{N\in\mathbb{N}} \bigcup_{k\geq N} A_k$. Then, $\mu(A) = \displaystyle\lim_{N\to\infty} \mu\Big(\bigcup_{k\geq N} A_k\Big) \leq \lim_{N\to\infty} \sum_{k\geq N} \mu(A_k) = 0$.
Now, $x \notin A \iff \exists\, N \in \mathbb{N}\, \forall\, k \geq N : f_{n_k}(x) < 1/k$, thus $\displaystyle\lim_{k\to\infty} f_{n_k}(x) = 0$ a.e..
For an example where $f_n \not\to 0$, consider the typewriter $f_{2^m+k} = \chi_{[\frac{k}{2^m}, \frac{k+1}{2^m}]}$, where $0 \leq k < 2^m$.
$\displaystyle\int f_{2^m+k} = 2^{-m} \to 0$ as $n = 2^m + k \to \infty$, but $\forall\, x \in [0,1]\, \forall\, m \in \mathbb{N}\, \exists\, k < 2^m : f_{2^m+k}(x) = 1$,
thus f_n does not converge anywhere on $[0,1]$.

2. Let $\mu(X) < \infty$, $p, q \in [1,\infty]$, and $p \leq q$. Then, $L^q \subset L^p$.

Solution. Assume $p < q$ wlog. For $q = \infty$, note that $||f||_\infty = 0 \implies ||f||_p = 0$, and if $||f||_\infty \in (0,\infty)$, by rescaling we may assume $||f||_\infty = 1$, whence $||f||_p \leq ||1||_p = \mu(X)^{1/p} < \infty$.
For $q < \infty$ and $f \in L^q$, we have $||f||_p^p = ||f^p||_1 \leq ||f^p||_{q/p} ||1||_{(q/p)'} = ||f||_q^p\, \mu(X)^{1-p/q} < \infty$ by Hölder.

3. $f, g : [0,1] \to \mathbb{R}$ differentiable $\implies C = \{(f(t), g(t)) \,|\, t \in [0,1]\}$ has measure 0 in $\mathbb{R}^2$.

Solution. Layered cake without cake. Apply Tonelli to χ_C:

$$\begin{aligned}\lambda^2(C) &= \int_{\mathbb{R}^2} \chi_C\, d\lambda^2 = \int_{\mathbb{R}^2} \chi_{\{(x,y)\,|\, x=f(t), y=g(t)\}}\, d\lambda^2 = \int_{[0,1]^2} \chi_{\{(f(t),g(s))\,|\, t=s\}}\, f'(t) d\lambda(t)\, f'(s) d\lambda(s) \\ &= \int_{[0,1]} \int_{\{(f(s),g(s))\}} f'(t) d\lambda(t)\, f'(s) d\lambda(s) = \int_{[0,1]} 0 \cdot f'(s) d\lambda(s) = 0.\end{aligned}$$

4. $\mu(X) = 1$, $h \in L^1$, $c > 0 \implies \mu\Big(\Big\{\Big|h - \displaystyle\int h\Big| \geq c\Big\}\Big) \leq \dfrac{1}{c^2}\Big(\displaystyle\int h^2 - \Big(\int h\Big)^2\Big)$.

Solution. By Markov, we have:

$$\mu\Big(\Big\{\Big|h - \int h\Big| \geq c\Big\}\Big) = \mu\Big(\Big\{\Big(h - \int h\Big)^2 \geq c^2\Big\}\Big) \leq \frac{1}{c^2} \int \Big(h - \int h\Big)^2$$
$$= \frac{1}{c^2} \int \Big(h^2 - 2h \int h + \Big(\int h\Big)^2\Big) = \frac{1}{c^2}\Big(\int h^2 - (2 - \mu(X))\Big(\int h\Big)^2\Big) = \frac{1}{c^2}\Big(\int h^2 - \Big(\int h\Big)^2\Big).$$

5. $\mu(X, \mathcal{S}) < \infty, \forall\, A \in \mathcal{S}\, \exists\, B \in \mathcal{S} : B \subset A, 0 < \mu(B) < \mu(A) \implies \forall\, \epsilon > 0\, \exists\, A \in \mathcal{S} : 0 < \mu(A) < \epsilon$.

Solution. Let $X = A_0$. Construct A_n from A_{n-1} inductively, as follows: $\exists\, B \subset A : 0 < \mu(B) < \mu(A)$, choose $A_n = B$ or $A_{n-1} \backslash B$ such that $0 < \mu(A_n) \leq \mu(A_{n-1})/2$. Then, $0 < \mu(A_n) \leq 2^{-n}\mu(X) < \infty$, thus given $\epsilon > 0$, we may choose $n \in \mathbb{N}$ satisfying $\mu(A_n) \leq 2^{-n}\mu(X) < \epsilon$.

August 2025

1. Let $f : \mathbb{R} \to \mathbb{R}$ be arbitrary. Then, $A = \{x \in \mathbb{R} \,|\, f \text{ is continuous at } x\}$ is Borel.

Solution. Let $A_n = \{x \in \mathbb{R} \,|\, \exists\, \delta > 0 : y, z \in (x-\delta, x+\delta) \implies f(y) - f(z) \in (-1/n, 1/n)\}$.
A_n is open since $x \in A_n \implies (x-\delta, x+\delta) \subset A_n$, and $A = \bigcap\limits_{n\in\mathbb{N}} A_n$ since

$$\begin{aligned} x \in A &\iff \forall\, \epsilon > 0\, \exists\, \delta > 0 : (y \in (x-\delta, x+\delta) \implies f(y) \in (f(x)-\epsilon, f(x)+\epsilon) \\ &\iff \forall\, n \in \mathbb{N}\, \exists\, \delta > 0 : (y, z \in (x-\delta, x+\delta) \implies f(y) - f(z) \in (-1/n, 1/n)) \\ &\iff \forall\, n \in \mathbb{N} : x \in A_n. \end{aligned}$$

For the second $\iff$, take $\epsilon < 1/2n$ in the forward direction, and $z = x$, $n > 1/\epsilon$ in the reverse.

2. For each pair of spaces below, show an inclusion or disprove that either inclusion exists:
$(i)\, L^2([0,1]), L^3([0,1]) \qquad (ii)\, L^2(\mathbb{R}), L^3(\mathbb{R}) \qquad (iii)\, \ell^2(\mathbb{Z}), \ell^3(\mathbb{Z})$

Solution. (i) $\supset$: $f \in L^3([0,1]) \implies ||f||_2^2 = ||f^2||_1 \le ||f^2||_{3/2} ||1||_3 = ||f||_3^2 < \infty$ by Hölder.
(ii) No inclusion. $x^{-1/2}\chi_{[1,\infty)} \in L^3(\mathbb{R}) \backslash L^2(\mathbb{R})$, and $x^{-1/3}\chi_{(0,1]} \in L^2(\mathbb{R}) \backslash L^3(\mathbb{R})$.
(iii) $\subset$: Let $f \in \ell^2(\mathbb{Z})$. $||f||_2 = 0 \implies ||f||_3 = 0$. Otherwise, by rescaling, assume $||f||_2 = 1$ wlog.
Then, $\forall\, n \in \mathbb{Z} : f(n) \le 1$, $f(n)^3 \le f(n)^2$, thus $||f||_3 \le ||f||_2 = 1$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be meas.. The graph of f, $\Gamma_f = \{(x, y) \in \mathbb{R}^2 \,|\, y = f(x)\}$ has measure 0 in $\mathbb{R}^2$.

Solution. $\lambda^2(\Gamma_f) = \displaystyle\int_{\mathbb{R}^2} \chi_{\Gamma_f}\, d\lambda^2 = \int_{\mathbb{R}} \mu(\{y \in \mathbb{R} \,|\, y = f(x)\})\, d\lambda(x) = \int_{\mathbb{R}} 0\, d\lambda(x) = 0$ by Tonelli.

4. Let $\mu(X) < \infty$, $f_n : X \to [0, \infty)$ meas.. Then, $f_n \xrightarrow{\mu} 0 \iff \dfrac{f_n^2}{1+f_n^2} \to 0$ in $\mathcal{L}^1$.

Solution. $\implies$: Given $\epsilon > 0$, we have $\displaystyle\int_X \frac{f_n^2}{1+f_n^2} \le \int_{\{f \ge \epsilon\}} 1 + \int_{\{f < \epsilon\}} \frac{\epsilon^2}{1+\epsilon^2} \le \mu(\{f_n \ge \epsilon\}) + \epsilon^2 \mu(X)$.
It follows that $\displaystyle\lim_{n\to\infty} \int_X \frac{f_n^2}{1+f_n^2} \le \lim_{n\to\infty} \mu(\{f_n \ge \epsilon\}) + \lim_{\epsilon\to 0} \epsilon^2 \mu(X) = 0$.
$\impliedby$: Given $\epsilon > 0$, we have $\mu(\{f_n \ge \epsilon\}) = \mu\left(\left\{\dfrac{f_n^2}{1+f_n^2} \ge \dfrac{\epsilon^2}{1+\epsilon^2}\right\}\right) \le \dfrac{1+\epsilon^2}{\epsilon^2} \displaystyle\int \frac{f_n^2}{1+f_n^2}$ by Markov.
It follows that $\displaystyle\lim_{n\to\infty} \mu(\{f_n \ge \epsilon\}) \le \frac{1+\epsilon^2}{\epsilon^2} \lim_{n\to\infty} \int \frac{f_n^2}{1+f_n^2} = 0$.

5. Let $\mu(X) < \infty$, $f : X \to (0, \infty)$. Then, $\forall\, \epsilon > 0\, \exists\, \delta > 0\, \forall\, A \subset X : \mu(A) \ge \epsilon \implies \displaystyle\int_A f \ge \delta$.

Solution. Note that $\displaystyle\lim_{n\to\infty} \mu(\{f < 1/n\}) = \mu\Big(\bigcap_{n\in\mathbb{N}} \{f < 1/n\}\Big) = 0$, thus given $\epsilon > 0$, we may choose $n \in \mathbb{N}$ satisfying $\mu(\{f < 1/n\}) < \epsilon/2$. Then, we have $\displaystyle\int_A f < \delta = \epsilon/2n \implies \mu(A) < \epsilon$, since $\mu(A \cap \{f \ge 1/n\}) \le \displaystyle\int_A nf < \epsilon/2$ implies that $\mu(A) = \mu(A \cap \{f \ge 1/n\}) + \mu(A \cap \{f < 1/n\}) < \epsilon$.

January 2026

1. $B = \{x \in R \,|\, \text{the decimal expansion of } x \text{ has infinitely many 5s}\}$ is Borel.

Solution. Let $A = \{x \in [0,1) \,|\, \text{the decimal expansion of } x \text{ has finitely many 5s}\}$.
Noting that $B^{\complement} = \bigcup_{m\in\mathbb{Z}} (m + A)$, it suffices to show that A is Borel, which follows since

$A = \bigcup_{n\in\mathbb{N}} \Big(\bigcup_{|a|=n-1} [0.a5, 0.a6) \Big\backslash \bigcup_{m>n} \bigcup_{|b|=m-1} [0.b5, 0.b6) \Big)$, where a, b are strings consisting of $\{0, \cdots, 9\}$,

and $|\cdot|$ denotes the length, i.e. $\bigcup_{|a|=n-1} [0.a5, 0.a6)$ is the set of decimals having 5 as the n-th digit.

2. $\mu(X) < \infty, f_n : X \to \mathbb{R}, \forall\, x \in X : f_n(x) \to \infty \implies \forall\, \epsilon > 0 \,\exists\, A \overset{\epsilon}{\subset} X : f_n \rightrightarrows \infty$ on A,
where $f_n \rightrightarrows \infty$ on A means $\forall\, M \in \mathbb{N}\,\exists\, N_M \in \mathbb{N}\,\forall\, n > N_M\,\forall\, x \in A : f_n(x) > M$.

Solution. Copy the proof of Egorov: $f_n \to \infty \iff \forall\, M \in \mathbb{N} : X = \bigcup_{N\in\mathbb{N}} \bigcap_{n>N} f_n^{-1}((M, \infty))$.

$\forall\, M \in \mathbb{N}\,\exists\, N_M \in \mathbb{N} : \bigcap_{n>N_M} f_n^{-1}((M, \infty)) = \bigcup_{N\le N_M} \bigcap_{n>N} f_n^{-1}((M, \infty)) \overset{2^{-M}\epsilon}{\subset} X$.

$A = \bigcap_{M\in\mathbb{N}} \bigcap_{n>N_M} f_n^{-1}((M, \infty)) \iff \forall\, M \in \mathbb{N}\,\forall\, n > N_M\,\forall\, x \in A : f_n(x) > M$.

$\mu(A^{\complement}) = \mu\Big(\bigcup_{M\in\mathbb{N}} \Big(\bigcap_{n\ge N_M} f_n^{-1}((M, \infty)) \Big)^{\complement} \Big) < \sum_{M\in\mathbb{N}} 2^{-M}\epsilon = \epsilon.$

3. $f : \mathbb{R} \to \mathbb{R}$ a.c.. $\forall\, \epsilon > 0\,\exists\, \delta > 0\,\forall\, A \in \mathcal{B} : \mu(A) < \delta \implies \mu(f(A)) < \epsilon$.

Solution. Given $\epsilon > 0$, choose $\delta > 0$ for $\epsilon/2$ in the definition of a.c.. Given $A \in \mathcal{B}$ with $\mu(A) < \delta$, take $G \overset{\delta-\mu(A)}{\supset} A$ open such that $\mu(G) < \delta$. Write $G = \dot{\bigcup}_{n\in\mathbb{N}} I_n$, where I_n are disjoint intervals, then:

$$\mu(f(A)) \le \mu(f(G)) = \mu\Big(f\Big(\dot{\bigcup_{n\in\mathbb{N}}} I_n \Big) \Big) = \mu\Big(\bigcup_{N\in\mathbb{N}} f\Big(\dot{\bigcup_{n\le N}} I_n \Big) \Big) = \lim_{N\to\infty} \mu\Big(f\Big(\dot{\bigcup_{n\le N}} I_n \Big) \Big) \le \epsilon/2 < \epsilon.$$

4. (X, μ) σ-finite, $f : X \to [0, \infty) \implies \int_X f\, d\mu = 2 \int_{[0,\infty)} s\mu(\{\sqrt{f} > s)\, d\lambda(s)$.

Solution. Layered cake. Apply Tonelli to $\chi_{\{(x,t)\in X\times[0,\infty)|f(x)>t\}}$:

$$\begin{aligned} \int_X f\, d\mu &= \int_X \mu(\{t \in [0, \infty) \,|\, f \ge t\})\, d\mu = \int_{[0,\infty)} \mu(\{x \in X \,|\, f(x) \ge t\})\, d\lambda(t) \\ &= \int_{[0,\infty)} \mu(\{x \in X \,|\, f(x) \ge s^2\}) \cdot 2s\, d\lambda(s) = 2 \int_{[0,\infty)} s\mu(\{\sqrt{f} > s\})\, d\lambda(s). \end{aligned}$$

5. $\mu(X) = 1$. $f \in L^\infty \iff \forall\, p \in [1, \infty) : f \in L^p$ and $\sup_{p\in[1,\infty)} ||f||_p < \infty$.

Solution. $\implies$: $||f||_p \le \Big(\int ||f||_\infty^p \Big)^{1/p} = \mu(X)^{1/p} ||f||_\infty = ||f||_\infty < \infty$, hence $\sup_{p\in[1,\infty)} ||f||_p < \infty$.
$\impliedby$: Let $A_n = \{x \in X \,|\, |f| \ge n\}$. If $||f||_\infty = \infty$, then $\forall\, k \in \mathbb{N}\,\exists\, n > k : \mu(A_n) > 0$, thus
$|f| \ge n\chi_{A_n}$, $||f||_p \ge n\mu(A_n)^{1/p}$, i.e. $\sup_{p\in[1,\infty)} ||f||_p \ge \lim_{p\to\infty} n\mu(A_n)^{1/p} = n > k$, and $\sup_{p\in[1,\infty)} ||f||_p = \infty$.