$\sqrt[4]{100\pi}$ Day

Rancho NJHS

4.21.23

Motivation

In a continuation of our series of RAD-ical celebrations, Rancho's NJHS chapter has planned a mathematical outreach event for $\sqrt[4]{100\pi}$ Day, a very close approximation of 4.21. This day, April 21st, NJHS will present a set of engaging math puzzles for Rancho students to solve and experience the joy of experiencing challenging mathematics. If you are interested in applying to NJHS for the 2023–2024 school year, stay tuned for future updates! Below are the problems, an answer key, and detailed solutions.

Instructions

We hope you have fun solving the problems below. When you have the answer to a problem, please check with an NJHS member. The first correct solution to any problem will be awarded a large candy, while subsequent solutions will be awarded smaller prizes. Good luck and happy $\sqrt[4]{100\pi}$ Day!

Problems

1. Plot $\sqrt[4]{e^{100\pi i}}$ on the unit circle (draw the circle too).
2. Alan set 100 pies on a lawn for the lawn bugs to eat. Every second, the lawn bugs eat the smallest integer number of pies above the fourth root of the current number of pies left. How many seconds will the pie last? Assume that the bugs start eating one second after Alan places the pies there, and each second thereafter.
3. 100 pies are stacked on top of each other. Alan randomly chooses four, names them, in some order, A, B, C, and D, and writes the following five statements.
   (a) A is above D.
   (b) B is above C.
   (c) A is below C.
   (d) C is below D.
   (e) A is above B.

   Surprisingly, Alan's statements have an 80% success rate! Which pie is on top?
4. How many of the complex solutions to $z = \sqrt[100]{4\pi}$ have their real part greater than their imaginary part?
5. Evaluate $\sum_{i=4}^{100} \frac{1}{\prod_{j=i}^{i+4} j}$ and given that the value may be expressed as $\frac{1}{a} - d$, where d is some positive real number, find the maximum possible value of a.
6. Four students, A, B, C, and D, each ate some non-negative integer amount of pies. In total, they ate 100 pies. Student A brags to Student B "I ate four times as many pies as you did," while Student X brags to everyone else "I ate four times as many pies as you all did combined." Student X is one of students A, B, C, and D. How many options are there for the distribution of pies eaten?
7. Alan's calculator currently displays the value 0. He can press three buttons, $+$, to add 64 to the current value, $\times$, to multiply the current value by π, or $\sqrt{\circ}$, to take the square root of the current value. Find the minimum number of steps necessary to achieve the value $\sqrt[4]{100\pi}$?

8. A hundred circular pies, each with the same radius, are arranged equidistant from a center, such that adjacent pies are externally tangent. Four pies are chosen and a simple quadrilateral is formed from their centers. For how many quadrilaterals are there perpendicular diagonals?

9. Alan and Nala want more pies. Alan can bake one pie over the course of four consecutive days, while Nala can bake four pies in one day (but not any more or less). How many options are there for baking a total of 100 pies in exactly 25 days, where Alan and Nala can choose which days they bake?

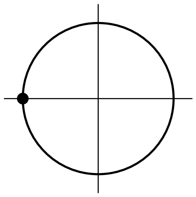
10. A circular lawn with an area of 100π square meters is lit by four matches, each four meters away from the center and equally spaced. Fire spreads radially at a constant rate of 4 centimeters per second. After how long will exactly a fourth of the lawn be burnt? Express your answer as a mixed number of seconds.

11. How many degrees are in $4 \cdot 100\pi$ radians?

12. 100 cuboid pies are stacked together and completely fill a cube. Each pie has integer side lengths and a surface area of 22 units. What is the surface area of the cube?

13. Express $\frac{4}{100\pi}$ in simplest form with a denominator composed of rational values.

Answers

1. 
2. 34
3. A
4. 50
5. 3360
6. 11
7. 30
8. 60 025
9. 1425
10. $62\frac{1}{2}$
11. 72 000
12. 384
13. *There are multiple answers to this question, depending on which identity of π is used, but one possible answer is*

$$\frac{\pi}{\frac{4}{1}-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\cdots}.$$

Solutions

1. Equivalent to $e^{25i\pi}$ which is equivalent to $e^{i\pi}$ or $\boxed{-1}$.

2. Each second, $\lceil n^{\frac{1}{4}} \rceil$ are eaten. When $n \neq k^4$ for any integer k, we have $\lceil n^{\frac{1}{4}} \rceil$ be constant. Before any bugs are eaten, there are more than 81 pies remaining and the bugs eat 4 pies per second. So, the bugs will take $\lceil \frac{100-81}{4} \rceil = 5$ seconds to get to 80 pies. From there, they can only eat 3 pies per second. Repeating the above process, we find that the pies will last 22 seconds before it reaches 14 pies remaining, and the bugs can eat only 2 pies per second. The pies will then last a further $\frac{14}{2} = 7$ seconds. Adding the times together, we find the the 100 pies last is $5 + 22 + 7 = \boxed{34 \text{ seconds.}}$

3. Note that if all five statements were true, then we would have a contradiction. For example, statements $1, 3,$ and 4, and statements $1, 2,$ and 5, can not simultaneously be true, because that would imply a loop. The only way to resolve both loops is to falsify statement 1. Now, we have either the arrangement A, B, D, C or A, D, B, C, and in either case, $\boxed{\text{A}}$ is on top.

4. We first of all note that we are equivalently finding the 100 roots of unity, because we can simply scale by any value to make the the roots be of 4π. Now, if the real part is equal to the imaginary part, then the angle would be either $45°$ or $135°$, but we can easily check that neither 4500 or 13500 is a multiple of 360. Thus, by symmetry, there are the same number of complex solutions with real part greater than with real part less than their imaginary part. Clearly, $\frac{100}{2} = \boxed{50}$.

5. We note that $\frac{1}{\prod_{j=i}^{i+4} j}$ is equivalent to $\frac{1}{4}\left(\frac{1}{i(i+1)(i+2)(i+3)} - \frac{1}{(i+1)(i+2)(i+3)(i+4)}\right)$. By telescoping, we calculate a final value of $\frac{1}{4}\left(\frac{1}{840} - \frac{1}{101 \cdot 102 \cdot 103 \cdot 104}\right)$. Clearly, $a = 4 \cdot 840 = \boxed{3360}$, as the second term is negligible (and of course, $\frac{1}{3361}$ is greater than the sum).

6. From the first statement, $A = 4B$. From the second statement, we know that student X ate 80 pies and everyone else at a sum of 20 pies. We could have $A = 4B + 4C + 4D$, so $C = D = 0$, which is our first possibility, or some other student be student X. Clearly, student B could not be student X, because student B ate at most as much as student A. If student C were student X, then we would have $C = 4A + 4B + 4D = 20B + 4D = 80$, or $5B + D = 20$. If $B \in \{0, 1, 2, 3, 4\}$, we have new solutions, and so this is 5 new solutions. Furthermore, if student D were student X, we would have essentially the same 5 new solutions. In total, there are $1 + 5 + 5 = \boxed{11}$ solutions.

7. We first note that $\boxed{30}$ is possible (add 64 to 256, square root to 16, add 64 to 1296, square root to 36, add 64, multiply by π, and then square root twice). Now, let us prove that 30 is minimal (we will do so by working backward from 100). First of all, note that at one point, we will reach 100. We can get here either by adding 64 from 36, or taking the square root from 10 000. Furthermore, the only way we could have gotten to 36 is by taking the square root of 1296. Now, $1296 \equiv 10\,000 \equiv 16 \pmod{64}$, so if we can get to 10 000, we must have gotten to 1296 first. Furthermore, to get to 1296, our two options are $1296 - 64$ or 1296^2. We can eliminate 1296^2 as an option, because it takes more than 30 steps just to get there. The same applies to all $n \equiv 16 \pmod{64}$ except 16 itself. Now, we want to find the quickest way to get to 16. The number before 16 must have been 256, which, being a multiple of 64, is fastest to get to just by adding 64 to 0 four times. This is exactly our method, so it must be optimal.

8. My computer says the answer is $\boxed{60\,025}$ (see appendix). But regardless, let us split into cases by the number of pairs of antipodal points. If there are two pairs of antipodal points, we have a square, and there are 25 such squares. If there is only one pair, we have a kite. Let us first choose the pair of antipodal points (50 options). There are then 48 choices for the other pair, for a total of 2400 options (51 choices on one side, but 2 of them are already chosen and one of them would lead to a second pair of antipodal points). Now, there could also be no pairs of antipodal points. However, for any of these cases, we simply have a modified kite, moving the antipodal pair off center. There are 48 positions to move it to, but any of these quadrilaterals could have come from 2 kites, so effectively we have an extra 24 quadrilaterals for every kite (a division by two once we distinguish between the movement off center in either direction). Thus, we have in total $50 \cdot 48 \cdot (24 + 1) + 25 = \boxed{60\,025}$.

9. Nala can earn four pies all 25 days, without Alan doing anything, which is one option. Nala can also earn four pies over the span of 16 days, and choose one day to stop earning four pies, leaving it to Alan — but if she skipped two days, it would be impossible to attain the hundred. There are 20 options of when to stop, but now we need to count how many ways there are to distribute the four groups of four consecutive days. We can simply pretend that those groups are only one day each, so we have eight total elements with four to choose to be the earning days, which is $\binom{8}{4} = 70$. In total, we have $25 + 20 \cdot 70 = \boxed{1425}$.

10. The area that must be burnt is 25π square meters. Thus, let us first assume that each match burns exactly a circle. Then, each match burns $\frac{25}{4}\pi$ square meters, equivalent to a radius of 2.5 meters. Now, the distance between two matches is $4\sqrt{2}$ meters, which is greater than double the radius, which means that the matches do indeed each burn a circle in area. Now, to burn 2.5 meters, it would take $\frac{250}{4} = 62.5$ seconds. As a mixed number, this is $\boxed{62\frac{1}{2}.}$

11. 180° are equivalent to π radians, so 400π radians are equivalent to $\boxed{72\,000}$ degrees.

12. The surface area of a cube is given by twice the sum of the areas of each type of face. Let a possible set of perpendicular side lengths of the pie be a, b, and c. Then, $ab + ac + bc = 11$. Thus, $(a, b, c) = (1, 1, 5)$ or $(1, 2, 3)$, in some order. The two possible volumes are 5 or 6. This means that the volume of the cube is between 500 and 600, so it must be 512, which is 8^3. The surface area is then $6 \cdot 8^2 = \boxed{384}$.

13. We know that $\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$. So, multiplying the numerator by $\frac{\pi}{4}$ and the denominator by $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$, we get $\boxed{\dfrac{\pi}{\frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots}}$.

Appendix

This is Python code used to compute the answer to problem 8.

```python
from itertools import combinations

count = 0

for choice in combinations(range(100), 4): # choice[0] < choice[1] < choice[2] < choice[3]
  # points at an angle 2 * PI * PT / 100, where PT is stored value of the point, to the positive x-axis, if we consider the unit circle

  diff = choice[2] - choice[0] # useful to align their centers

  if diff % 2 == 0: # there is an odd number of points STRICTLY between the two points, so we make the chord from choice[0] to choice[2] vertical
    offset = choice[0] + int(diff / 2)
    choice = [pt - offset for pt in choice]

    if (choice[1] + choice[3]) % 100 == 50: # needs to be horizontal
      count += 1
  else: # there is an even number of points STRICTLY between the two points, so we make the chord from choice[0] to choice[2] diagonal (the two points would be symmetric about the line y=x)
    offset = choice[0] + int((diff - 25) / 2)
    choice = [pt - offset for pt in choice]

    if (choice[1] + choice[3]) % 100 == 75: # the other two points have to be diagonal (slope +1)
      count += 1

print(count)
```