# $\sqrt{5}$ Day

Rancho NJHS

2.23.23

**Motivation**

$\sqrt{5}$ is the positive real number $x$ such that $x^2 = 5$. In decimal representation, it is approximately $2.236$, so today, February 23rd, is cause for celebration. The square root of 5 is a mathematical concept, and so today is an opportunity to solve engaging math problems.

**Instructions**

All problems below have answers in the set $\{\, n \in \mathbb{N} \cup \{0\} \mid 0 \leq n \leq 50 \,\}$, and are arranged in roughly increasing order of difficulty. Units are not required for any problem. When you have the answer to a problem, please check with an NJHS member. Correct solutions will be awarded prizes. Good luck and happy $\sqrt{5}$ Day!

**Problems**

1. A circle contains 2 intersecting chords, $AC$ and $BD$, where $AC \cap BD = P$. The length of chord $AC$ is $2\sqrt{5}$ inches. Both $AP$ and $PC$ have a length of $\sqrt{5}$. The length of $BP$ is 5. What is the length of $DP$?

2. An equilateral triangle has a side length of $\sqrt{5}$. If the inradius of the triangle can be expressed as $\frac{\sqrt{a}}{b}$, where $a$ and $b$ are integers and $m$ is square-free, find $a + b$.
   *Note: The original answer was* $\boxed{19}$*, but that turned out to be incorrect. The correct answer is* $\boxed{21}$.

3. A sequence of positive real numbers begins with $a_1 = \sqrt{5}$. For all $n \in \mathbb{N} > 1$, the recurrence $a_n = \sum_{i=1}^{n-1} \sqrt{a_i}$ is defined. For example, the first few terms are $a_2 = \sqrt[4]{5}$, $a_3 = \sqrt[4]{5} + \sqrt[8]{5}$, etc. Let $n$ be the first term such that $a_n \geq 2023$. What is $\lfloor a_n - a_{n-1} \rfloor$?

4. What is the floor of $\sqrt{5}$? Note that the floor of $x$ is the greatest integer below $x$.

5. What is the 1st digit behind the decimal point of $\sqrt{5}$?

6. In every $\sqrt{5}$ minute interval, Alan solves a math problem. What is the minimum number of problems that can be solved in 60 minutes?

7. Alan has a giant cubic box with a side length of $\sqrt{5}$ meters. What is the surface area of the box in square centimeters? If your answer is expressed in scientific notation as $a \times 10^b$, find $a + b$.

8. The first 10 characters of $\sqrt{5}$, where the decimal point is considered a character, are $2.23606797$. Alan chooses a random subset of 6 of these characters (possibly including the decimal point) without replacement, and arranges them in the order he chose them. What is the probability that the characters form a number equivalent to $2.236$, where trailing or leading zeroes do not affect number equivalence. Given that the answer is $\frac{a}{b}$, find the sum of the digits of $a$ plus the sum of the digits of $b$.

9. Alan lives in a huge square-shaped town. Interestingly, the perimeter of the town in kilometers is $\sqrt{5}$ times the area in square kilometers. Given that the perimeter can be expressed as $\frac{a\sqrt{b}}{c}$, where $a$, $b$, and $c$ are integers and $b$ is square-free while $a$ and $c$ have a greatest common factor of $1$, what is $a+b+c$?

10. The quadratic equation $x^2+bx+c=0$ has two roots, $x=2023\pm\sqrt{5}$. What is the value of $b^2-4c \pmod{50}$?

11. Let $f_1(x)=\sqrt{5}x$, $f_2(x)=\sqrt{5}-x$, $f_3(x)=\sqrt{5}+x$, and $f_4(x)=\frac{\sqrt{5}}{x}$. We also denote $f_5^1(x)=f_1(f_2(f_3(f_4(x))))$, and $f_5^n(x)=f_5^{n-1}(f_5^n(x))$ for $n\geq 2$. If $|f_5^{2023}(x)|$ is expressed as $a\sqrt{b}$, where $a$ and $b$ are integers and $b$ is square-free, what is $a+b$?

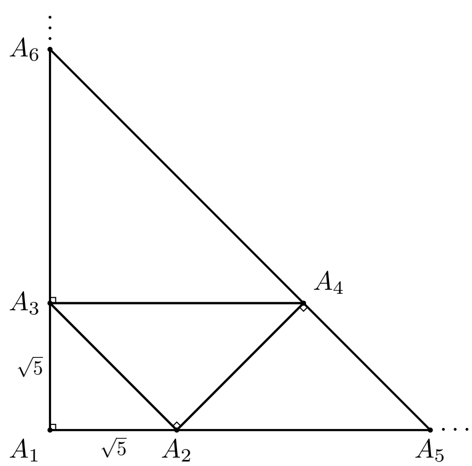
12. A number is defined as a "ten-value" if it has 2 and 5 as its only prime factors. Alan finds the product of the first 10 ten-values, and denotes the value by $P$. If $\sqrt{P}=n\sqrt{5}$, find the sum of the digits of $n$.
*Note: This problem was edited to contain $\sqrt{P}$ instead of $P$, and to ask for the value of $n$, instead.*

13. The square root of 2232023 (today's date written without punctuation), is $1493.99564926$. What is the value of the next perfect square after 2232023 $\pmod{50}$? Do not use a calculator, please.

14. Alan has a defunct calculator, with only two operations, $\times\sqrt{5}$, and $+1$, which are performed on the current value. If Alan starts with a value of $\sqrt{5}$, what is the fewest number of operations he can make to get to 2023?

15. A lawn can be mowed by Alan at a rate of $\sqrt{5}$ square meters per $\sqrt{5}$ seconds. The lawn then grows back, and needs to be remowed once a week. Given that the lawn is a square with one side of $100\sqrt{5}$ meters, what fraction of Alan's time is spent mowing his lawn? As the answer may be expressed as $\frac{1}{b}$, find the closest integer to $b$.

16. In the diagram below (drawn to scale), all interior angles of triangles are either $45^\circ$ or $90^\circ$, and are distinguished by right-angle marks. Note that the diagram is simply a construction of triangles around the initial $\triangle A_1A_2A_3$. Therefore, as $\triangle A_1A_5A_6\sim\triangle A_1A_2A_3$, we can indefinitely repeat this construction, but around larger and larger triangles instead, assigning indices of $A_n$ in the same way we did for the previous triangles. If $\overline{A_1A_2}$ has length $\sqrt{5}$, the length of $\overline{A_{20}A_{23}}$ can be expressed as $a\sqrt{5}$. What is the sum of the digits of $a$?




**Bonus Problems**

17 A *cool* expression of the $n^{\text{th}}$ order contains the values $t\sqrt{5}$, for $t\in\{1,2,...,n\}$, with the operations $+$, $-$, $\div$, or $\times$, as well as parentheses between them (only one operation between two terms). For example, the general cool expression of order 3 is $1\sqrt{5}\circ 2\sqrt{5}\circ 3\sqrt{5}$, where the circles represent any aforementioned operation. If a cool expression of the $23^{\text{rd}}$ order is equal to 2023, determine the operations and parentheses used.

## Answers

1. 1
2. 21
3. 44
4. 2
5. 2
6. 26
7. 8
8. 19
9. 26
10. 20
11. 6
12. 11
13. 36
14. 25
15. 12
16. 18
17. $\left(1\sqrt{5}+2\sqrt{5}+3\sqrt{5}+4\sqrt{5}\right)\cdot 5\sqrt{5}+\frac{6\sqrt{5}+7\sqrt{5}+8\sqrt{5}+9\sqrt{5}}{10\sqrt{5}}+\left(11\sqrt{5}+12\sqrt{5}+13\sqrt{5}\right)\cdot 14\sqrt{5}-15\sqrt{5}\left(-16\sqrt{5}-17\sqrt{5}-18\sqrt{5}+19\sqrt{5}+20\sqrt{5}+21\sqrt{5}-22\sqrt{5}+23\sqrt{5}\right).$

# Solutions

1. Power of a point.
2. The inradius of any triangle is $\frac{K}{s}$, where $K$ is the area and $s$ the semi-perimeter. Plugging in, we have $\frac{\frac{\sqrt{3}}{4}(\sqrt{5})^2}{\frac{3}{2}\sqrt{5}} = \frac{\sqrt{15}}{6}$. Our answer is then $15 + 6 = 21$.
3. We see that the recurrence $a_i = a_{i-1} + \sqrt{a_{i-1}}$ is defined. Now, we want to find the difference between the least term greater than 2023, and the greatest term below. Clearly, the greatest term below is between $2023 - \sqrt{2023}$ and 2023, and the square root of this has a floor of 44. Therefore, the difference is also 44, by definition.
4. We proceed by computation.
5. We proceed by computation.
6. $\lfloor \frac{60}{\sqrt{5}} \rfloor = 26$.
7. The surface area of a cube is given by $6s^2$, where $s$ is the side length of the box. Clearly, this equals 30 square meters. However, in square centimeters, we multiply by the conversion factor of $10^4$, so our final answer is $3 \times 10^5$, which has a sum of $a$ and $b$ equal to 8.
8. There are two numbers that could be equivalent to 2.236, those being 02.236 and 2.2360. Both require at one point choosing a zero out of the one zero in the group, a two out of the two twos, a decimal point out of the decimal point, etc. Thus, the probability of the first being correct is $\frac{1}{10} \cdot \frac{2}{9} \cdot \frac{1}{8} \cdot \frac{1}{7} \cdot \frac{1}{6} \cdot \frac{2}{5}$, which simplifies to $\frac{1}{37800}$. However, we have two of these, so the final probability is $\frac{1}{18900}$. The sum of 1, 1, 8 and 9 is 19.
9. Let the side length of the town be $s$ kilometers. Then, the area of the town is $A = s^2$ in square kilometers, while the perimeter is $p = 4s$ in meters. If $\sqrt{5}A = p$, then $\sqrt{5}s^2 = 4s$. Clearly, the side length $s$ is $\frac{4\sqrt{5}}{5}$, so the perimeter is $\frac{16\sqrt{5}}{5}$, so $a + b + c = 26$.
10. From the quadratic equation, we have $\frac{\sqrt{b^2-4c}}{2} = \sqrt{5}$, so $b^2 - 4c$ is clearly equal to 20.
11. After applying the rules, $f_5^1(x) = -\frac{5}{x}$. Therefore, $f_5^2(x) = x$. In addition, $f_5^{2022}(x) = x$, so $f_5^{2023}(\sqrt{5}) = f_5^1(\sqrt{5}) = -\sqrt{5}$. The magnitude is $\sqrt{5}$, so $a = 1, b = 5$, and their sum is 6.
12. We need to find the square root of the product of the first ten ten-value numbers, which are $10, 20, 40, 50, 80, 100, 160, 200, 250, 320$. The prime factorizations of these numbers are as follows:

$$\begin{aligned} 10 &= 2 \cdot 5 \\ 20 &= 2^2 \cdot 5 \\ 40 &= 2^3 \cdot 5 \\ 50 &= 2 \cdot 5^2 \\ 80 &= 2^4 \cdot 5 \\ 100 &= 2^2 \cdot 5^2 \\ 160 &= 2^5 \cdot 5 \\ 200 &= 2^3 \cdot 5^2 \\ 250 &= 2 \cdot 5^3 \\ 320 &= 2^6 \cdot 5 \end{aligned}$$

Multiplying these values together, we get

$$10 \cdot 20 \cdot 40 \cdot 50 \cdot 80 \cdot 100 \cdot 160 \cdot 200 \cdot 250 \cdot 320 = 2^28 \cdot 5^15.$$

Thus, the square root of the product of the first ten ten-value numbers is $\sqrt{2^{28} \cdot 5^{15}} = 2^{14} \cdot 5^7\sqrt{5} = 1,280,000,000\sqrt{5}$. $1 + 2 + 8 = 11$, so our answer is $\boxed{11}$.

13. Clearly, 2232023 is pretty close to the next perfect square. Indeed, the next one would be $1494^2$. Note that $1494 \equiv 0 \pmod{9}$, so $1494^2 \equiv 0 \pmod{81}$. The nearest number to 2232023 that is $0 \pmod{81}$ is 2232036. Then, $2232036 \pmod{50}$ is 36.

14. Clearly, as $2023$ is rational, there is never any reason to multiply by $\sqrt{5}$ an odd number of times, before adding $1$. Therefore, we will always multiply by $\sqrt{5}$ an even number of times, so we can count just the number of times we multiply by $5$, and then double that, to get the equivalent number of operations by $\sqrt{5}$. Now, our first step is obviously to multiply by $\sqrt{5}$, following our above reasoning. Now, let us write $2023_{10}$ in base $5$. It happens to be $31043_5$, although our current number begins with $10_5$. Note that when we add $1$, we increment the last digit, but when we multiply by $5$, we shift the digits one place to the left. Clearly, then, only by adding $1$ can we change the actual values of the digits. So, since we start with $10_5$, but our objective starts with $31_5$, we need to add $31_5 - 10_5 = 21_5 = 11_{10}$ ones, before anything else. Now, we have $31$, and to get to $310$, it is obviously easiest just to multiply by $5$. We then multiply by $5$ again, so that we can increment the last digit to $4$ (the alternative would be to add a very big number of 1s, as once we multiply by $5$, that would be the only way). This requires $4$ operations. We then multiply by $5$ and add $3$ again, giving us a total of $1 + 11 + 2 + 2 + 4 + 2 + 3 = 25$ operations.

15. We first find the area, and then divide by the rate of mowing, to find the time to mow. Then, we determine how often the lawn needs to be mowed, and divide.

16. By construction, we see that after each step, all the lengths triple. In addition, by taking all indices modulo $3$, which is how the indices should be defined, we see that $A_{20}$ and $A_{23}$ are both equivalent to $A_2$, modulo $3$, so they are points along the bottom ray. We now define the function $d(N) =$ the distance between $A_N$ and $A_{N+3}$, for all $n \equiv 2 \pmod 3$. Now, we see that $d(2) = 2\sqrt{5}$, by isosceles right triangles, so $d(5)$ is three times this, and so on. Continuing the pattern, $d(N) = 2\sqrt{5} \cdot 3^{\frac{N-2}{3}}$, so $d(20) = 1458$. The sum of the digits is $18$.