$\sqrt[3]{35}$ Day

Rancho NJHS

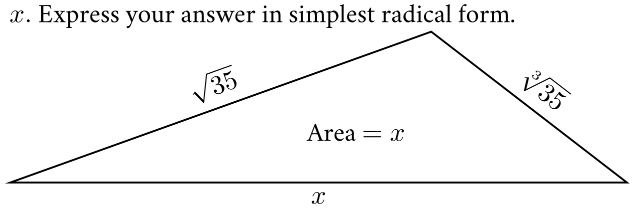
3.28.23

Motivation

$\sqrt[3]{35}$ is the positive real number x such that $x^3 = 35$. In decimal representation, it is approximately 3.27, so rounding up, we celebrate its fascinating day on March 28th. We hope everyone who took a problem enjoyed the experience, and we hope to do more in the future, as part of our RAD-ical celebrations series.

Problems

1. What is the smallest integer value of n such that $\sqrt{n} > \sqrt[3]{35}$?
2. A cube of volume 35 has vertices $ABCDEFGH$, in some order. If segments AB, AC, and AD each have length $\sqrt[3]{35}$, what is the volume of $ABCD$? Express your answer in simplest form.
3. Alawn can mow the top 4 centimeters of grass on his square lawn at a constant rate, 35 square meters per hour (i.e., if Alawn sees a patch of grass with area 35 square meters, each blade of grass will be 4 centimeters shorter, after an hour of uniform mowing). Furthermore, if grass grows at a rate of $\sqrt[3]{35}$ meters per week, what is the minimum side length of his lawn, in meters, such that he can never stop cutting? Express your answer as $\frac{\sqrt[3]{a}}{b}$.
4. In the expression $a\sqrt[b]{c}$, we allow a, b, and c to be each be replaced by either 3 or 5. How many distinct expressions may result?
5. A cube with side length $\sqrt[3]{35}$ is inscribed inside a right square pyramid. If the height of the pyramid is equal to the side length of its base, what is its volume? Express your answer as a common fraction.
6. In the 35th round of a game show, contestants attempt to find the largest possible value that can be attained using each of the operation $\times, \div, +, -$, once in the expression $\sqrt[3]{35} \circ \sqrt[3]{35} \circ \sqrt[3]{35} \circ \sqrt[3]{35} \circ \sqrt[3]{35}$, and valid pairs of parentheses may be inserted. What is the largest possible value attainable?
7. Let S be a subset of the positive integers, possibly with duplicity, such that $\sqrt{P} = 35$, where P is the product of the elements in S. Find the sum of all possible values which may appear in the S (note that, although, for example, $S = \{7, 7, 25\}$ is a possibility, we would still only count 7 once in the sum).
8. Express $\left(\sqrt[3]{35}\right)^6 + \left(\sqrt[3]{35}\right)^{-6}$ as a polynomial with integer coefficients, in terms of t, where $t = \sqrt[3]{35} + \frac{1}{\sqrt[3]{35}}$.
9. Find the minimum sum of a, b, and c, such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{12}{35}$, and a, b, and c are positive integers.
10. Simplify the expression $\sqrt[3]{343 \cdot 5 + 3 \cdot 5 \cdot 7 \cdot x^2 \left(\sqrt[3]{7^2} + \sqrt[3]{5^2}\right) + 125 \cdot 7}$, where $x = \sqrt[3]{35}$.
11. What is the domain of the function $f(x) = \frac{\sqrt{x^3}}{\sqrt[3]{35x^2 - 35}}$? Express your answer in interval notation.
12. A triangle has side lengths of $\sqrt{35}$, $\sqrt[3]{35}$, and x (all in meters). If the area of the triangle is x square meters, find the value of x. Express your answer in simplest radical form.



Answer Key

1. 11
2. $\frac{35}{6}$
3. $\frac{\sqrt[3]{35}}{5}$
4. 8
5. $\frac{280}{3}$
6. $2\sqrt[3]{1225}-1$
7. 1767
8. $t^6-6t^4+9t^2-2$
9. 41
10. $7\sqrt[3]{5}+5\sqrt[3]{7}$
11. $[0,35)\cup(35,\infty)$
12. $\sqrt{31}+\sqrt{\sqrt[3]{1225}-4}$

Solutions

1. We raise both sides of the inequality to the sixth power (which will not change the sign, as both sides are positive). Thus, $n^3 > 1225$. The smallest such integer value of n is 11.

2. By the given, A is a vertex adjacent to B, C, and D. Thus, $ABCD$ is a pyramid, and has a volume equal to a sixth of the cube's volume. In other words, the volume of $ABCD$ is equal to $\frac{35}{6}$.

3. Approximating grass to have a volume, we need the condition to be that the volume of the grass cut per time is equivalent to the volume of the grass grown per time. The volume cut per time is clearly $0.04 \cdot 35$ cubic meters per hour, while the grass grown is $\sqrt[3]{35} \cdot s^2$, where s is the side length in square meters. Setting these equal, we have $s^2 = 0.04\sqrt[3]{35^2}$, so the positive value of s is $\frac{\sqrt[3]{35}}{5}$.

4. Note that there are 2 possible values each for a, b, and c, as they do not intersect. As such, 8 distinct expressions can be formed.

5. Let's say the height of the pyramid is h. Then the side length of the base is s. We can compute s by similar triangles. From depth 0 to $h - \sqrt[3]{35}$, the side length of the pyramid increases from 0 to $\sqrt[3]{35}$. From $h - \sqrt[3]{35}$ to h, the side length increases from $\sqrt[3]{35}$ to s. In other words, $\frac{s-\sqrt[3]{35}}{\sqrt[3]{35}} = \frac{\sqrt[3]{35}}{h-\sqrt[3]{35}}$. When $h = s$, we have a single solution $h = 2\sqrt[3]{35}$. For this combination, the volume is given by $\frac{1}{3}(2\sqrt[3]{35})^3 = \frac{280}{3}$.

6. The largest value attainable can be expressed in the form of $x \times (x + x) - \frac{x}{x}$ where $x = \sqrt[3]{35}$, and evaluates to $2\sqrt[3]{35^2} - 1 = 2\sqrt[3]{1225} - 1$.

7. We will prove that all and only factors of 1225 may be elements of S. First of all, the equation simplifies to $P = 1225$. If a number that was not a factor of 1225 were an element, then as we can only multiply by integers greater than one, we can not remove that factor (a number which is not a factor of 1225 is either greater than 1225 or has a prime other than 5 or 7). However, any factor of 1225 can be a member, as by definition, a factor can be multiplied by some other numbers to end with the whole. Thus, we find the sum of the factors of 1225, which is 1767.

8. It works.

9. We already see that $\frac{1}{7} + \frac{1}{5} = \frac{12}{35}$, and that there is no other combination that makes $\frac{12}{35}$ without them, so the best thing to do here is to split one of them into two common fractions that must be unique. $\frac{1}{5} = \frac{1}{6} + \frac{1}{30}$. These are the smallest possible values that work (we can also try the case by splitting 7, but that leads to a larger sum), so they sum to sum to $5 + 6 + 30 = 41$.

10. We recognize this as the expression $(\sqrt[3]{(7\sqrt[3]{5} + 5\sqrt[3]{7})})^3$.

11. First of all, the function is undefined for all negative values. Furthermore, when the denominator is 0, the function will also be undefined, which occurs at $x = \pm 35$. Thus, our final interval is $[0, 35) \cup (35, \infty)$.

12. We find the altitude to x, which is 2. The altitude makes two right angles, splitting the base of x into two lengths, a and b. Using the Pythagorean Theorem, we have (assuming that $a < b$) $a^2 + 2^2 = (\sqrt{35})^2$, so $a = \sqrt{31}$. Furthermore, $b^2 + 2^2 = (\sqrt[3]{35})^2$, so $b = \sqrt{\sqrt[3]{1225} - 4}$. Therefore, $x = a + b = \sqrt{31} + \sqrt{\sqrt[3]{1225} - 4}$.