Edson Oswaldo Ramírez Esqueda 5 de diciembre de 2018

Tarea 4

Pregunta 1

a) Prueba que las siguientes transformaciones son canónicas para cualquier μ.

$$q_1 = x\cos\mu + P_y \operatorname{sen}\mu \qquad q_2 = y\cos\mu + P_x \operatorname{sen}\mu$$

$$P_1 = P_x\cos\mu - y\operatorname{sen}\mu \qquad P_2 = P_y\cos\mu - x\operatorname{sen}\mu$$

$$\{q_1, q_2\} = \frac{\partial q_1}{\partial x}\frac{\partial q_2}{\partial P_x} - \frac{\partial q_2}{\partial x}\frac{\partial q_1}{\partial P_x} + \frac{\partial q_1}{\partial y}\frac{\partial q_2}{\partial P_y} - \frac{\partial q_1}{\partial P_y}\frac{\partial q_2}{\partial y}$$

$$= (\cos\mu)(\operatorname{sen}\mu) - (0)(0) + (0)(0) - (\operatorname{sen}\mu)(\cos\mu)$$

$$= \cancel{\cos\mu\operatorname{sen}\mu} - \cancel{\cos\mu\operatorname{sen}\mu}$$

$$\{q_1, q_2\} = 0$$

$$\{P_1, q_1\} = \frac{\partial P_1}{\partial x}\frac{\partial q_1}{\partial P_x} - \frac{\partial q_1}{\partial x}\frac{\partial P_1}{\partial P_x} + \frac{\partial P_1}{\partial y}\frac{\partial q_1}{\partial P_y} - \frac{\partial q_1}{\partial y}\frac{\partial P_1}{\partial P_y}$$

$$= (0)(0) - (\cos\mu)(\cos\mu) + (\operatorname{sen}\mu)(\operatorname{sen}\mu) - (0)(0)$$

$$= -\cos^2\mu - \operatorname{sen}^2\mu$$

$$\{P_1, q_1\} = -1$$

$$\{q_1, P_1\} = \frac{\partial q_1}{\partial x}\frac{\partial P_1}{\partial P_x} - \frac{\partial P_1}{\partial x}\frac{\partial q_1}{\partial P_x} + \frac{\partial q_1}{\partial y}\frac{\partial P_1}{\partial P_y} - \frac{\partial P_1}{\partial y}\frac{\partial q_1}{\partial P_y}$$

$$= (\cos\mu)(\cos\mu) - (0)(0) + (0)(0) - (-\operatorname{sen}\mu)(\operatorname{sen}\mu)$$

$$= \cos^2\mu + \operatorname{sen}^2\mu$$

$$\{q_1, P_1\} = 1$$

Se muestra que $\{q_1, P_1\} = -\{P_1, q_1\}$

$$\{q_2, P_2\} = \frac{\partial q_2}{\partial x}\frac{\partial P_2}{\partial P_x} - \frac{\partial P_2}{\partial x}\frac{\partial q_2}{\partial P_x} + \frac{\partial q_2}{\partial y}\frac{\partial P_2}{\partial P_y} - \frac{\partial P_2}{\partial y}\frac{\partial q_2}{\partial P_y}$$

$$= (0)(0) - (-\text{sen}\,\mu)(\text{sen}\,\mu) + (\cos\mu)(\cos\mu) - (0)(0)$$

$$= \text{sen}^2\mu + \cos^2\mu$$

$$\underline{\{q_2, P_2\} = 1} \;//$$

$$\{P_2, q_2\} = \frac{\partial P_2}{\partial x}\frac{\partial q_2}{\partial P_x} - \frac{\partial q_2}{\partial x}\frac{\partial P_2}{\partial P_x} + \frac{\partial P_2}{\partial y}\frac{\partial q_2}{\partial P_y} - \frac{\partial q_2}{\partial y}\frac{\partial P_2}{\partial P_y}$$

$$= (-\text{sen}\,\mu)(\text{sen}\,\mu) - (0)(0) + (0)(0) - (\cos\mu)(\cos\mu)$$

$$= -\text{sen}^2\mu - \cos^2\mu$$

$$\underline{\{P_2, q_2\} = -1} \;//$$

Probando la antisimetría:

$$\{q_2, P_2\} = -\{P_2, q_2\}$$

$$\{P_1, P_2\} = \frac{\partial P_1}{\partial x}\frac{\partial P_2}{\partial P_x} - \frac{\partial P_2}{\partial x}\frac{\partial P_1}{\partial P_x} + \frac{\partial P_1}{\partial y}\frac{\partial P_2}{\partial P_y} - \frac{\partial P_2}{\partial y}\frac{\partial P_1}{\partial P_y}$$

$$= (0)(0) - (-\text{sen}\,\mu)(\cos\mu) + (-\text{sen}\,\mu)(\cos\mu) - (0)(0)$$

$$= \cos\mu\,\text{sen}\,\mu - \cos\mu\,\text{sen}\,\mu$$

$$\underline{\{P_1, P_2\} = 0} \;//$$

$$\{P_1, q_2\} = \frac{\partial P_1}{\partial x}\frac{\partial q_2}{\partial P_y} - \frac{\partial P_1}{\partial P_y}\frac{\partial q_2}{\partial x} + \frac{\partial P_1}{\partial y}\frac{\partial q_2}{\partial P_y} - \frac{\partial P_1}{\partial P_y}\frac{\partial q_2}{\partial y}$$

$$= (0)(0) - (0)(0) + (-\text{sen}\,\mu)(0) - (0)(\cos\mu)$$

Scanned by CamScanner

$$\{P_1, q_2\} = 0$$

$$\{P_2, q_1\} = \frac{\partial P_2}{\partial x}\frac{\partial q_1}{\partial P_x} - \frac{\partial P_2}{\partial P_x}\frac{\partial q_1}{\partial x} + \frac{\partial P_2}{\partial y}\frac{\partial q_2}{\partial P_y} - \frac{\partial P_2}{\partial P_y}\frac{\partial q_2}{\partial y}$$

$$= (-\operatorname{sen}\mu)(0) + -(0)(\cos\mu) + (0)(0) - (\cos\mu)(0)$$

$$\{P_2, q_1\} = 0$$

b) Si el Hamiltoniano original es $H = (q_1^2 + q_2^2 + P_1^2 + P_2^2)/2$ encuentra un nuevo Hamiltoniano como función de x, y y sus momentos conjugados

$q_1^2 = x^2\cos^2\mu + 2xP_y \operatorname{sen}\mu\cos\mu + P_y^2\operatorname{sen}^2\mu$; $P_1^2 = P_x^2\cos^2\mu - 2yP_x\operatorname{sen}\mu\cos\mu + y^2\operatorname{sen}^2\mu$

$q_2^2 = y^2\cos^2\mu + 2yP_x\operatorname{sen}\mu\cos\mu + P_x^2\operatorname{sen}^2\mu$; $P_2^2 = P_y^2\cos^2\mu - 2xP_y\operatorname{sen}\mu\cos\mu + x^2\operatorname{sen}^2\mu$

$$H = \frac{1}{2}\Big(x^2\cos^2\mu + \cancel{2xP_y\operatorname{sen}\mu\cos\mu} + P_y^2\operatorname{sen}^2\mu + y^2\cos^2\mu + \cancel{2yP_x\operatorname{sen}\mu\cos\mu} + P_x^2\operatorname{sen}^2\mu$$
$$+ P_x^2\cos^2\mu - \cancel{2yP_x\operatorname{sen}\mu\cos\mu} + y^2\operatorname{sen}^2\mu + P_y^2\cos^2\mu - \cancel{2xP_y\operatorname{sen}\mu\cos\mu} + x^2\operatorname{sen}^2\mu\Big)$$

$$H = \frac{1}{2}\left(x^2 + y^2 + P_x^2 + P_y^2\right)$$

c) Usa el nuevo Hamiltoniano para resolver la dinámica con la ~~restr~~ restricción:
Si $y = P_y = 0$ $\qquad H = \frac{1}{2}(x^2 + P_x^2)$

Las ecuaciones de Hamilton son:

$$\dot{q}_i = \frac{\partial H}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial H}{\partial q}$$

donde $P_1 = P_x$, $P_2 = P_y$, $q_1 = x$, $q_2 = y$

$$\begin{cases} \dot{q}_1 = \dfrac{\partial H}{\partial P_1} = \dfrac{\partial H}{\partial P_x} = P_x & (1) \\ \dot{q}_2 = \dfrac{\partial H}{\partial P_2} = \dfrac{\partial H}{\partial P_y} = 0 & (2) \end{cases}$$

$$\begin{cases} \dot{P}_x = -\dfrac{\partial H}{\partial x} = -x & (3) \\ \dot{P}_y = -\dfrac{\partial H}{\partial y} = 0 & (4) \end{cases}$$

de las ecuaciones (1) y (3) se tiene que

$$\dot{q}_1 = P_x \,, \qquad \dot{P}_x = -x$$

$$\dot{q}_1 - P_x = 0$$

$$\ddot{q}_1 = -x = \dot{P}_x = m\ddot{x}$$

$$m\ddot{x} = -x$$

$$m\ddot{x} + x = 0$$

Pregunta 2

Un disco delgado uniforme de masa M y radio A rota sin fricción con una velocidad angular uniforme ω sobre un eje vertical fijo que pasa sobre su centro θ y tiende un ángulo alfa con el eje de simetría del disco.

a) Determina los momentos de inercia y los ejes principales.

Dada la simetría del disco se tiene que $I_1 = I_2$ e $I_1 + I_2 = I_3$

$$I_1 = \int \rho x^2 d^2x \qquad \text{con} \qquad \rho = \frac{M}{\pi A^2}$$

$$I_2 = \int \rho y^2 d^2x$$

$$I_3 = \int \rho (x^2+y^2) d^2x$$

$$\therefore I_3 = I_1 + I_2 = 2\pi\rho \int_0^A r'^3 dr' = \frac{1}{2}MA^2$$

$$\boxed{I_1 = \frac{1}{4}MA^2} \qquad \boxed{I_2 = \frac{1}{4}MA^2} \qquad \boxed{I_3 = \frac{1}{2}MA^2}$$

Los ejes principales son: en el centro del disco, dada la rotación que tiene el disco.

es decir, $I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$

b) Encuentra el vector de momento angular

$$L = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} (0, 0, \omega) = (0, 0, I_3\omega)$$

c) ¿Cuál es la magnitud y ~~rotación~~ dirección de la torca relativa al sistema de referencia del cuerpo (x, y, z)?

$$\tau = \frac{d\vec{L}}{dt} = \frac{d(\omega I_3)}{dt}\hat{k} = \omega I_3 \frac{d}{dt}\hat{k} = 0$$