Edson Oswaldo Ramírez Esqueda 1 de noviembre de 2018

Tarea 3

1.

Sea $L = L(\ddot{q}_i, \dot{q}_i, q_i; t)$

$$J = \int_{t_1}^{t_2} L[\ddot{q}_i, \dot{q}_i, q_i; t]\, dt$$

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha}\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \alpha} + \frac{\partial L}{\partial \ddot{q}_i}\frac{\partial \ddot{q}_i}{\partial \alpha}\right) dt$$

$$\frac{\partial q_i}{\partial \alpha} = \eta(t), \qquad \frac{\partial \dot{q}_i}{\partial \alpha} = \frac{d\eta}{dt}, \qquad \frac{\partial \ddot{q}_i}{\partial \alpha} = \frac{d^2\eta}{dt^2}$$

$$\frac{\partial J}{\partial \alpha} = \frac{\partial \cancel{J}}{\partial \alpha}\left[\int_{t_2}^{t_1} \frac{\partial L}{\partial q_i}\eta(t)\, dt + \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial \dot{q}_i}\frac{d\eta}{dt}\right) dt + \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial \ddot{q}_i}\frac{\partial^2 \eta}{dt^2}\right) dt\right]$$

Integrando por partes el segundo término:

$$\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial \dot{q}_i}\frac{d\eta}{dt}\right) dt = \cancel{\frac{\partial L}{\partial \dot{q}_i}\eta(t)\Big|_{t_1}^{t_2}}^{\,0} - \int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\eta(t)\, dt$$

$$u = \frac{\partial L}{\partial \dot{q}_i}, \quad dv = \frac{d\eta}{dt}\, dt \quad y \quad du = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) dt, \quad v = \eta$$

Integrando el tercer término por Partes:

$$\int_{t_1}^{t_2}\frac{\partial L}{\partial \ddot{q}_i}\frac{d^2\eta}{dt^2}\, dt = \cancel{\frac{\partial L}{\partial \ddot{q}_i}\frac{d\eta}{dt}\Big|_{t_1}^{t_2}}^{\,0} - \int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\frac{d\eta}{dt}\, dt$$

$$= -\int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\frac{d\eta}{dt}\, dt$$

Repitiendo el proceso se tiene que:

$$\frac{\partial J}{\partial \alpha} = \int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}_i}\right]\eta(t)\, dt = 0$$

$$\frac{\partial L}{\partial q_i} - \frac{\partial}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) = 0$$

2.

Recordando las condiciones

i) $g_{ab} = g_{ba}$

ii) g_{ab} es simétrica

$$L(\dot{q}, q; t) = \frac{1}{2} g_{ab}(q^c)\, \dot{q}^a \dot{q}^b$$

$$\begin{cases} \dfrac{\partial L}{\partial q^c} = \dfrac{1}{2} \dfrac{d}{dq^c} g_{ab} \dot{q}^a \dot{q}^b \\ \dfrac{\partial L}{\partial \dot{q}^c} = \dfrac{1}{2} g_{cb} \dot{q}^b + \dfrac{1}{2} g_{ac} \dot{q}^a \end{cases}$$

$$\frac{d}{dt}\left[\frac{1}{2} g_{cb} \dot{q}^b + \frac{1}{2} g_{ac} \dot{q}^a\right] = \underbrace{\frac{1}{2} g_{cb} \ddot{q}^b + \frac{1}{2} g_{ac} \ddot{q}^a}_{\text{Indices mudos}} + \frac{1}{2} \frac{\partial}{\partial q^a}(g_{cb}) \dot{q}^a \dot{q}^b$$

$$+ \frac{1}{2} \frac{\partial}{\partial q^b}(g_{ca}) \dot{q}^a \dot{q}^b$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^c}\right) = g_{ac} \ddot{q}^a + \frac{1}{2} \dot{q}^a \dot{q}^b \left(\frac{\partial}{\partial q^a}(g_{cb}) + \frac{\partial}{\partial q^b}(g_{ac})\right)$$

Por la condición i) $\cancel{\forall}$ $g_{ab} = g_{ba}$

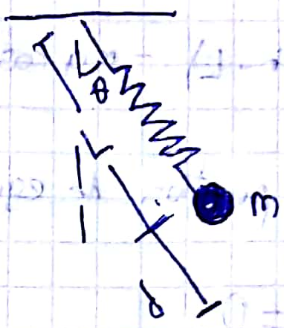
$$\frac{dL}{\partial q^c} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^c}\right) = g_{ac} \ddot{q}^a + \frac{1}{2} \dot{q}^a \dot{q}^b \left(\frac{\partial}{\partial q^a}(g_{cb}) + \frac{\partial}{\partial q^b}(g_{ac})\right) - \frac{1}{2} g_{ab} \dot{q}^a \dot{q}^b$$

$$= g_{ac} \ddot{q}^a + \frac{1}{2}\left(\frac{\partial}{\partial q^a} g_{cb} + \frac{\partial}{\partial q^b} g_{ac} - \frac{\partial}{\partial q^c} g_{ab}\right) \dot{q}^a \dot{q}^b$$

$$\Gamma^a_{bc} = \frac{1}{2}\left(\frac{\partial g_{cb}}{\partial q^a} + \frac{\partial g_{ac}}{\partial q^b} - \frac{\partial g_{ab}}{\partial q^c}\right)$$

$$\frac{\partial L}{\partial q^c} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^c}\right) = g_{ac} \ddot{q}^a + \Gamma^a_{bc} \dot{q}^a \dot{q}^b$$

3.



$$L = T - V$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$$

$$V = \frac{1}{2} k (r - \ell)^2 - mgr\cos\theta \qquad r = L + d$$

$$L = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2} k (r - L)^2 - mgr\cos\theta$$

Luego,

$$\frac{\partial L}{\partial \theta} = mgr\,\mathrm{sen}\,\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(r^2 \dot{\theta}\right) = 2r\dot{\theta} + r^2\ddot{\theta} \qquad \text{(Para } \theta\text{)}$$

Y las ecuaciones de Euler-Lagrange se escriben:

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mgr\,\mathrm{sen}\,\theta - 2r\dot{\theta} - r^2\ddot{\theta} = 0 \qquad \text{(Para } \theta\text{)}$$

$$\frac{\partial L}{\partial r} = k(r - L) - mg\cos\theta$$

$$\left(\frac{\partial L}{\partial \dot{r}}\right) = m\dot{r} \quad \text{y} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r} \qquad \text{(Para } r\text{)}$$

$$\frac{\partial L}{\partial r} = k(r - L) - mg\cos\theta \quad \text{y} \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$$

Y las ecuaciones de E-L

$$k(r-L) - mg\cos\theta - m\ddot{r} = 0$$

Los puntos de equilibrio:

$$\ddot{r} = 0$$
$$r = 0$$

$$\begin{cases} mg\,\text{sen}\,\theta - 2\theta - r\ddot{\theta} = 0 \\ k(r-L) - mg\cos\theta - m\ddot{r} = 0 \end{cases}$$

4.

a) $L = e^{bt}\left(\frac{1}{2} m \dot{q}^2 - \frac{1}{2} k^2 q^2\right)$

$\frac{\partial L}{\partial q} = -e^{bt} k^2 q^2$ $\qquad \frac{\partial L}{\partial \dot{q}} = e^{bt} m \dot{q} \ddot{q}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = bme^{bt} \dot{q} \ddot{q} + e^{bt}\left[(\ddot{q})^2 + \dot{q} \dddot{q}\right]$

$e^{bt} m \dot{q} \ddot{q} - bme^{bt} \dot{q} \ddot{q} + e^{bt} (\ddot{q})^2 + e^{bt} \dot{q} \dddot{q} = 0$ //

b) $Q = e^{bt/2} q$, $\quad q = Q e^{-bt/2} \quad \dot{q} = \dot{Q} e^{-bt/2} + \left(-\frac{b}{2}\right) Q e^{-\frac{bt}{2}}$

$L = \left(e^{bt}\left(\frac{1}{2} m e^{-bt}\left(\dot{Q} - \frac{b}{2} Q\right)^2\right) - \frac{1}{2} e^{bt} k^2 Q^2 e^{-bt}$

$L = \frac{1}{2}\left[m\left(\dot{Q} - \frac{b}{2} Q\right)^2 - k^2 Q^2\right]$ //