César David Aguirre Gutiérrez

TAREA 3

Pregunta 1.

$$L = L(\ddot{q}, \dot{q}, q, t)$$

Deduciremos las ecuaciones de Euler-Lagrange para dichos parámetros.

Por el principio de minima acción tenemos

$$\delta S[X^A] = \delta \int_t L(\ddot{q}, \dot{q}, q, t)\, \delta t$$

$$\Rightarrow \int_t \delta L(\ddot{q}, \dot{q}, q, t)\, \delta t = \int_t \Big[\frac{\partial L}{\partial \ddot{X}^A}(\delta \ddot{X}^A) + \frac{\partial L}{\partial \dot{X}^A}(\delta \dot{X}^A) - \frac{\partial L}{\partial X^A}(\delta X^A) \Big] \delta t$$

$$= \int_t \Big[\frac{\partial L}{\partial \ddot{X}^A}\Big(\frac{\partial}{\partial t} \delta \dot{X}^A\Big) + \frac{\partial L}{\partial \dot{X}^A}\Big(\frac{\partial}{\partial t} \delta X^A\Big) + \frac{\partial L}{\partial X^A}(\delta X^A) \Big] \delta t$$

De la integral tomamos: $U = \frac{\partial L}{\partial \ddot{X}^A}$ $\quad \delta U = \frac{\partial}{\partial t}\Big(\frac{\partial L}{\partial \ddot{X}^A}\Big)$

$\delta V = \frac{\partial}{\partial t}(\delta \dot{X}^A)$ $\quad V = \delta \dot{X}^A$

$S = \frac{\partial L}{\partial X^A}$ $\quad \delta S = \frac{\partial}{\partial t}\Big(\frac{\partial L}{\partial \dot{X}^A}\Big)$ $\quad \delta t = \frac{\partial}{\partial t} \delta X^A$ $\quad t = \delta X^A$

Sustituyendo en la integral

$$\int_t \Big[-\frac{\partial}{\partial t}\Big(\frac{\partial L}{\partial \dot{X}^A}\Big)(\delta X^A) - \frac{\partial}{\partial t}\Big(\frac{\partial L}{\partial \dot{X}^A}\Big)(\delta X^A) + \frac{\partial L}{\partial X^A}(\delta X^A) \Big] dt = 0$$

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Pregunta 2.

Encontrar las ecs. de E-L del modelo sigma

$(\dot{q}, q, t)\, L = \frac{1}{2} g_{ab}(q^c)\, \dot{q}^a \dot{q}^b$

Ec. de Euler-Lagrange para el modelo

$$\boxed{\frac{\partial L}{\partial q^\gamma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\gamma}\right) = 0}$$

*Primera derivada

$$\frac{\partial L}{\partial q^\gamma} = \frac{\partial}{\partial q^\gamma}\left(\frac{1}{2} g_{ab}(q^c)\,\dot{q}^a \dot{q}^b\right) = \frac{\dot{q}^a \dot{q}^b}{2} \frac{\partial}{\partial q^\gamma}\left(g_{ab}(\ddot{q}^c)\right)$$

Haremos la derivada covariante, ya que se trata de un tensor de 2do Orden

$$g_{ab;\gamma} = \frac{\partial}{\partial q^\gamma} g_{ab} - \Gamma^\mu_{a\gamma}\, g_{\mu b} - \Gamma^\mu_{b\gamma}\, g_{a\mu} \qquad \gamma$$

Tenemos por una identidad:

$g_{ab;\gamma} = 0$

$\Rightarrow$ Despejando: $\frac{\partial}{\partial q^\gamma} g_{ab} = \Gamma^\mu_{a\gamma}\, g_{\mu b} + \Gamma^\mu_{b\gamma}\, g_{a\mu}$

Sustituyendo en $\frac{\partial L}{\partial q^\gamma}$

$$\frac{\partial L}{\partial q^\gamma}=\frac{\dot{q}^a\dot{q}^b}{2}\left[\Gamma^\mu_{a\gamma}g_{\mu b}+\Gamma^\mu_{b\gamma}g_{a\mu}\right]$$

$$=\frac{1}{2}\dot{q}^a\dot{q}^b\Gamma^\mu_{a\gamma}g_{\mu b}+\frac{1}{2}\dot{q}^a\dot{q}^b\Gamma^\mu_{b\gamma}g_{a\mu}$$

$$=\frac{\dot{q}^a\dot{q}^b}{2}\left[2\Gamma^\mu_{a\gamma}g_{\mu b}\right]=\underline{\dot{q}^a\dot{q}^b\Gamma^\mu_{a\gamma}g_{\mu b}}$$

*Para la segunda ecuación

$$\frac{\partial L}{\partial \dot{q}^\gamma}=\frac{\partial}{\partial \dot{q}^\gamma}\left(\frac{1}{2}g_{ab}\dot{q}^a\dot{q}^b\right)=\frac{1}{2}g_{ab}\frac{\partial}{\partial \dot{q}^\gamma}\left(\dot{q}^a\dot{q}^b\right)$$

$$=\frac{1}{2}g_{ab}\left[\frac{\partial \dot{q}^a}{\partial \dot{q}^\gamma}\dot{q}^b+\frac{\partial \dot{q}^b}{\partial \dot{q}^\gamma}\dot{q}^a\right]=\frac{1}{2}g_{ab}\left[\delta^a_\gamma\dot{q}^b+\delta^b_\gamma\dot{q}^a\right]$$

$$=\frac{1}{2}g_{ab}\delta^a_\gamma\dot{q}^b+\frac{1}{2}g_{ab}\delta^b_\gamma\dot{q}^a=\frac{1}{2}g_{\gamma b}\dot{q}^b+\frac{1}{2}g_{a\gamma}\dot{q}^a$$

$$=\frac{1}{2}\left[2g_{a\gamma}\dot{q}^a\right]=\underline{g_{a\gamma}\dot{q}^a}$$

Despues, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\gamma}\right)=\frac{d}{dt}\left(g_{a\gamma}\dot{q}^a\right)=\frac{dg_{a\gamma}}{dt}\dot{q}^a+\ddot{q}^a g_{a\gamma}$

$$\frac{dg_{a\gamma}}{dt}=\frac{\partial g_{a\gamma}}{\partial q^c}\frac{\partial q^c}{\partial t}=\dot{q}^c\frac{\partial g_{ab}}{\partial q^c}=\dot{q}^c\left[\Gamma^\mu_{ac}g_{\mu b}+\Gamma^\mu_{bc}g_{a\mu}\right]$$

$$=\Gamma^\mu_{ac}g_{\mu b}\dot{q}^c+\Gamma^\mu_{bc}g_{a\mu}\dot{q}^c \longrightarrow$$ Haciendo uso de la simetria de los simbolos de Christoffel

$$=2\Gamma^\mu_{ac}g_{\mu b}\dot{q}^c$$

Asi,

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}^{\gamma}}\right)=\left[\left(\frac{\partial g_{a\gamma}}{\partial t}\right)\dot{q}^{a}+\ddot{q}^{a}g_{a\gamma}\right]$$

$$=\left[2\Gamma^{\mu}_{ac}g_{\mu b}\dot{q}^{c}\right]\dot{q}^{a}+\ddot{q}^{a}g_{a\gamma}$$

$$=2\Gamma^{\mu}_{ac}g_{\mu b}\dot{q}^{c}\dot{q}^{a}+\ddot{q}^{a}g_{a\gamma}$$

Ahora substituimos en la ec. E-L,

$$\frac{\partial L}{\partial q^{\gamma}}-\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}^{\gamma}}\right)=0$$

$$\dot{q}^{a}\dot{q}^{b}\Gamma^{\mu}_{a\gamma}g_{\mu b}-2\Gamma^{\mu}_{ac}g_{\mu b}\dot{q}^{c}\dot{q}^{a}-\ddot{q}^{a}g_{a\gamma}=0$$

$$\Rightarrow -\Gamma^{\mu}_{ac}g_{\mu b}\dot{q}^{c}\dot{q}^{a}-\ddot{q}^{a}g_{a\gamma}=0$$

Si hacemos el cambio de $b=\gamma$ en el primer termino

$$-\Gamma^{\mu}_{ac}g_{\mu\gamma}\dot{q}^{c}\dot{q}^{a}-\ddot{q}^{a}g_{a\gamma}=0$$

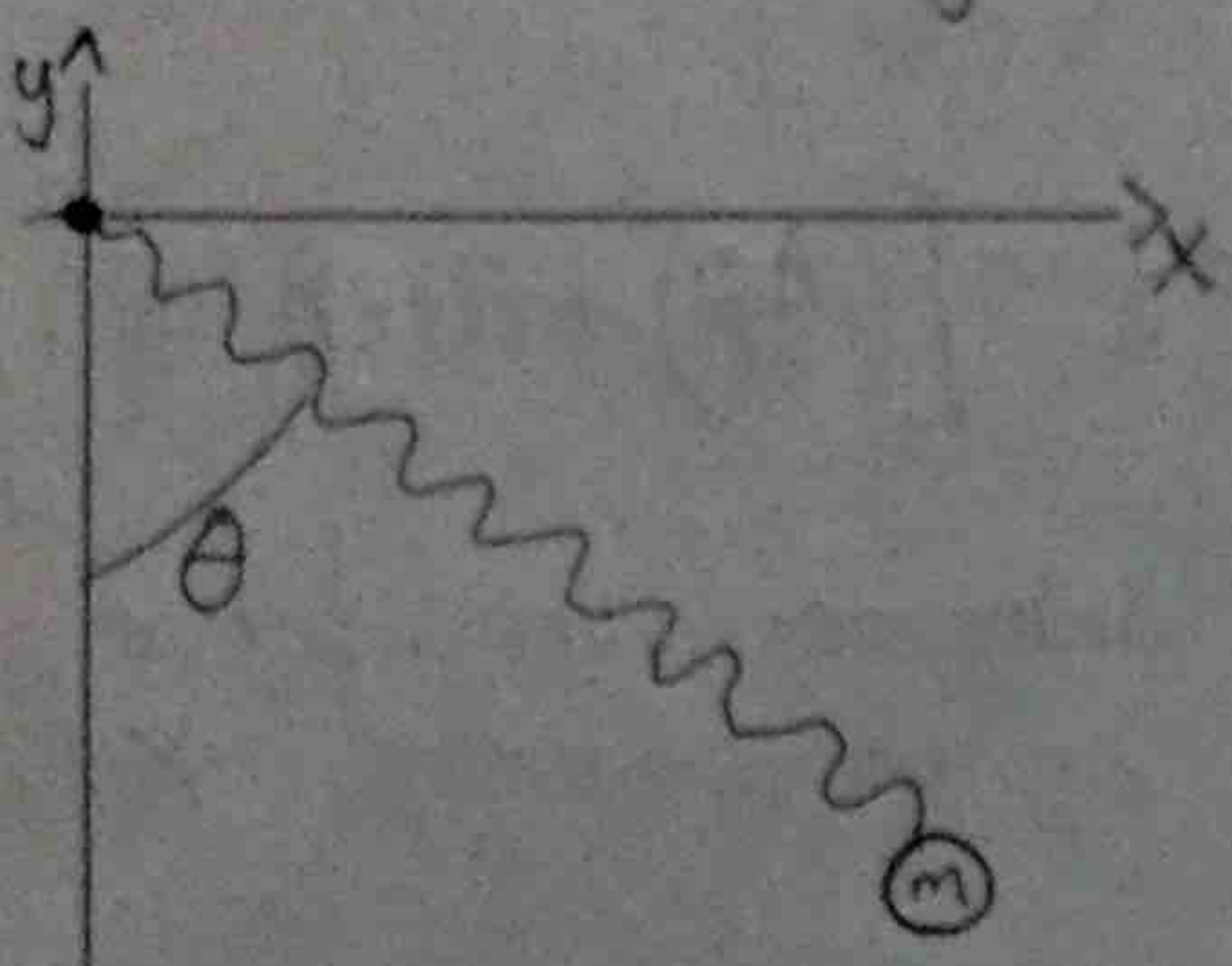
Factorizando

$$-g_{\mu\gamma}\left[\Gamma^{\mu}_{ac}\dot{q}^{c}\dot{q}^{a}+\ddot{q}^{a}\right]=0 \qquad g_{\mu\gamma}\neq 0$$

$$\therefore\ \Gamma^{\mu}_{ac}\dot{q}^{c}\dot{q}^{a}+\ddot{q}^{a}=0 \longrightarrow \boxed{\Gamma^{\mu}_{ac}\dot{q}^{a}\dot{q}^{c}+\ddot{q}^{\mu}=0}$$

Pregunta 3.

Si elaboramos un grafico del sistema, se veria de la siguiente Forma,



- Representa un pendulo, en donde se le reemplazo el cable por un resorte de longitud en reposo ℓ y constante del resorte "K"

a) El pendulo presentará deformación en el eje $z\hat{k}$, donde cuando se estire, le añadiremos una elongación "W" al resorte. Y su longitud sera $\ell + w$

La posición de la particula (masa), estara en dos ejes que serán x, y. Se considerará su longitud Fija

$$x = (\ell + w) \, Sen\,\theta$$
$$y = (\ell + w) \, Cos\,\theta$$

Para obtener la velocidad se deriva la posición respecto al tiempo, de x y y.

$$\frac{dx}{dt} = \frac{dw}{dt} Sen\,\theta + (\ell + w) Cos\,\theta \frac{d\theta}{dt}$$

$$\frac{dy}{dt} = -\frac{dw}{dt} Cos\,\theta + (\ell + w) Sen\,\theta \frac{d\theta}{dt}$$

Construyamos ahora el Lagrangiano $L = T - V$

-Para la energia cinetica tenemos,

$$T = \frac{1}{2}mv^2 \qquad v^2 = \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]$$

$$\Rightarrow T = \frac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] = \frac{1}{2}m\left[\left(\frac{dw}{dt}\right)^2 + (l+w)\left(\frac{d\theta}{dt}\right)^2\right]$$

-Para la energia potencial debemos considerar la energia potencial normal de movimiento y aparte la del resorte,

$$V = mgh + \frac{1}{2}Kw^2 \qquad \text{donde } h \text{ sera: } (l+w)\cos\theta$$

$$V = mg(l+w)\cos\theta + \frac{1}{2}Kw^2$$

Asi,

$$L = \frac{1}{2}m\left[\left(\frac{dw}{dt}\right)^2 + (l+w)\left(\frac{d\theta}{dt}\right)^2\right] - mg(l+w)\cos\theta + \frac{1}{2}Kw^2$$

Reescribiendo el Lagrangiano, en otra notación similar

$$L = \frac{1}{2}m(\dot{w}^2 + (l+w)\dot{\theta}^2 - mg(l+w)\cos\theta + \frac{1}{2}Kw^2$$

Aplicando Euler-Lagrange a nuestro Lagrangiano

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

Primero lo haremos para la coordenada θ

$$(l+w)\frac{d^2\theta}{dt^2}+2\frac{d\theta}{dt}\frac{dz}{dt}+g\,\mathrm{Sen}\,\theta=0$$

Ahora para z,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right)-\frac{\partial L}{\partial z}=0$$

$$\frac{d^2 z}{dt^2}+\frac{k}{m}z-(l+w)\left(\frac{d\theta}{dt}\right)^2-g\cos\theta=0$$

Y obtuvimos un sistema de dos ecuaciones diferenciales

Pregunta 4

$$L = e^{bt}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}k^2q^2\right)$$

k, b y m
ctes. positivas

a) $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$

$$\frac{\partial L}{\partial q} = e^{bt}\left(-\frac{1}{2}k^2(2q)\right) = -e^{bt}k^2q$$

$$\frac{\partial L}{\partial \dot{q}} = e^{bt}\left(\frac{1}{2}m(2\dot{q})\right) = e^{bt}m\dot{q}$$

$$\frac{d}{dt}(e^{bt}m\dot{q}) = m(e^{bt}\ddot{q} + \dot{q}be^{bt})$$

Substituyendo

$$m\ddot{q}e^{bt} + m\dot{q}be^{bt} - (-e^{bt}k^2q) = 0$$

$$m\ddot{q}e^{bt} + m\dot{q}be^{bt} + (e^{bt}k^2q) = 0$$

$$e^{bt}(m\ddot{q} + m\dot{q}b + k^2q) = 0$$

Si $e^{bt} \neq 0$, entonces

$$m\ddot{q} + m\dot{q}b + k^2q = 0$$

$$= \ddot{q} + \dot{q}b + \frac{k^2q}{m} = 0$$

Y podemos ver que se parece al oscilador armonico amortiguado

$$\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0 x = 0}$$