TAREA 4

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Pregunta 1.

a) Prueba que las siguientes transformaciones son canónicas

$$q_1 = x\cos\mu + P_y \operatorname{Sen}\mu \qquad q_2 = y\cos\mu + P_x \operatorname{Sen}\mu$$

$$P_1 = P_x\cos\mu - y\operatorname{Sen}\mu \qquad P_2 = P_y\cos\mu - x\operatorname{Sen}\mu$$

Debemos asegurar o verificar que los corchetes de Poisson son invariantes, $\{q_i, P_j\} = \delta_{ij}$

$$\{q_1, P_1\} = \frac{\partial q_1}{\partial x_i}\frac{\partial P_1}{\partial p_i} - \frac{\partial q_1}{\partial p_i}\frac{\partial P_1}{\partial x_i}$$

$$= \frac{\partial q_1}{\partial x}\frac{\partial P_1}{\partial P_x} + \cancelto{0}{\frac{\partial q_1}{\partial y}}\frac{\partial P_1}{\partial P_y} - \cancelto{0}{\frac{\partial q_1}{\partial P_x}}\frac{\partial P_1}{\partial x} - \frac{\partial q_1}{\partial P_y}\frac{\partial P_1}{\partial y}$$

$$= \cos^2\mu + \operatorname{Sen}^2\mu = 1$$

$$\{q_1, P_2\} = \frac{\partial q_1}{\partial x}\cancelto{0}{\frac{\partial P_2}{\partial P_x}} + \cancelto{0}{\frac{\partial q_1}{\partial y}}\frac{\partial P_2}{\partial P_y} - \cancelto{0}{\frac{\partial q_1}{\partial P_x}}\frac{\partial P_2}{\partial x} - \frac{\partial q_1}{\partial P_y}\cancelto{0}{\frac{\partial P_2}{\partial y}}$$

$$= 0$$

$$\{q_2, P_1\} = \cancelto{0}{\frac{\partial q_2}{\partial x}}\frac{\partial P_1}{\partial P_x} + \cancelto{0}{\frac{\partial q_2}{\partial y}}\frac{\partial P_1}{\partial P_y} - \cancelto{0}{\frac{\partial q_2}{\partial P_x}}\frac{\partial P_1}{\partial x} - \cancelto{0}{\frac{\partial q_2}{\partial P_y}}\frac{\partial P_1}{\partial y}$$

$$= 0$$

$$\{q_2, P_2\} = \frac{\partial q_2}{\partial x}\frac{\partial P_2}{\partial P_x} + \frac{\partial q_2}{\partial y}\frac{\partial P_2}{\partial P_y} - \frac{\partial q_2}{\partial P_x}\frac{\partial P_2}{\partial x} - \frac{\partial q_2}{\partial P_y}\frac{\partial P_2}{\partial y}$$

$$= \cos^2\mu + \operatorname{Sen}^2\mu = 1$$

y se cumple $\{q_i, P_j\} = \delta_{ij}$

b) $H = \frac{1}{2}(q_1^2 + q_2^2 + P_1^2 + P_2^2)$

$= \frac{1}{2}(x^2 \cos^2\mu + \cancel{2x P_y \operatorname{Sen}\mu \cos\mu} + P_y^2 \operatorname{Sen}^2\mu + y^2\cos^2\mu + \cancel{2yP_x \cos\mu \operatorname{Sen}\mu}$

$+ P_x^2 \operatorname{Sen}^2\mu + P_x^2 \cos^2\mu - \cancel{2yP_x \cos\mu\operatorname{Sen}\mu} + y^2 \operatorname{Sen}^2\mu$

$+ P_y^2 \cos^2\mu - \cancel{2xP_y \cos\mu \operatorname{Sen}\mu} + x^2 \operatorname{Sen}^2\mu)$

$= \frac{1}{2}(x^2\cos^2\mu + P_y^2 \operatorname{Sen}^2\mu + y^2\cos^2\mu + P_x^2 \operatorname{Sen}^2\mu + P_x^2\cos^2\mu$

$+ y^2 \operatorname{Sen}^2\mu + P_y^2\cos^2\mu + x^2 \operatorname{Sen}^2\mu)$

$= \frac{1}{2}(x^2 + y^2 + P_x^2 + P_y^2)$

c) Si $y = P_y = 0$, $H = \frac{1}{2}(x^2 + P_x^2)$

Las ecuaciones de Hamilton seran

$\dot{x} = \frac{\partial H}{\partial P_x} = P_x \longrightarrow \dot{x} = P_x$

$\dot{P}_x = \frac{-\partial H}{\partial x} = -x \longrightarrow \dot{P}_x = -x$

Despues derivando $(\dot{x} = P_x)^{\cdot} \Rightarrow \ddot{x} = \dot{P}_x \quad \dot{P}_x = -x$

$\therefore \ddot{x} = -x$

Por lo tanto $\boxed{x(t) = A\cos(t + B)}$

y $\boxed{P_x(t) = -A\operatorname{Sen}(t+B)}$

Pregunta 2.

a) ρ: uniforme $\quad \rho = \dfrac{M}{\pi A^2}$

Tenemos en el caso continuo,

$$I = \int d^3r\, \rho(r) \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix}$$

Existe simetria en x y y, y en z no hay variación y los unicos valores que seran diferentes de cero en el tensor seran

$$I_{ij} = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}$$

$$I_{11} = \int \rho y^2\, dx\, dy = \int \rho r^2 \operatorname{sen}^2\theta\, r\, d\theta\, dr = \int_0^A \int_0^{2\pi} \rho r^3 \operatorname{sen}^2\theta\, d\theta\, dr$$

$$= \rho \int_0^{2\pi} \operatorname{sen}^2\theta\, d\theta \left(\frac{1}{4}r^4\right)\Big|_0^A = \frac{A^4}{4}\left(\frac{M}{\pi A^2}\right)\int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{4}\frac{A^2 M}{\pi}\left(\frac{1}{2}\theta\Big|_0^{2\pi} - \frac{1}{4}\operatorname{sen} 2\theta\Big|_0^{2\pi}\right)$$

$$= \frac{1}{4}\frac{A^2M}{\pi}\frac{1}{2}2\pi = \underline{\frac{1}{4}A^2M}$$

Por simetria en x y y, $I_{22} = I_{11} = \underline{\frac{1}{4}MA^2}$

$$I_{33} = \int \rho (x^2+y^2)\, dx\, dy = \rho \int r^2 r\, dr\, d\theta = \rho \int_0^A \int_0^{2\pi} r^3\, d\theta\, dr$$

$$= \frac{M}{\pi A^2}\, 2\pi\, \frac{1}{4} r^4 \Big|_0^A$$

$$= \frac{1}{4} \frac{M}{\pi A^2}\, 2\pi A^4 = \frac{1}{2} M A^2$$

El tensor de inercia es

$$I_{ij} = \begin{pmatrix} \frac{1}{4}MA^2 & 0 & 0 \\ 0 & \frac{1}{4}MA^2 & 0 \\ 0 & 0 & \frac{1}{2}MA^2 \end{pmatrix}$$

b) $\omega_1 = \omega \cos(90-\gamma) = \omega \operatorname{Sen} \gamma$

$\omega_2 = 0$

$\omega_3 = \omega \cos \gamma$

$L_a = I_{ab}\, \omega_b$, los componentes del momento angular seran

$$L_1 = I_{11}\, \omega_1 = \frac{1}{4} M A^2 \omega \operatorname{Sen} \gamma$$

$$L_2 = I_{22}\, \omega_2 = 0$$

$$L_3 = I_{33}\, \omega_3 = \frac{1}{2} M A^2 \omega \cos \gamma$$

$$\vec{L} = \frac{1}{2} M A^2 \omega \left(\frac{1}{2} \operatorname{Sen}\gamma, 0, \cos\gamma\right)$$

$$\|\vec{L}\| = \sqrt{\frac{1}{4} M^2 A^4 \omega^2 \left(\frac{1}{4}\operatorname{Sen}^2\gamma + \cos^2\gamma\right)} = \frac{1}{2} M A^2 \omega \sqrt{\frac{1}{4}\operatorname{Sen}^2\gamma + \cos^2\gamma}$$

$$= \frac{1}{2} M A^2 \omega \sqrt{1 - \frac{3}{4}\operatorname{Sen}^2\gamma}$$

Para la dirección tendremos el angulo que hace con respecto a z

$$\theta = \tan^{-1}\left(\frac{L_1}{L_3}\right) = \tan^{-1}\left(\frac{\frac{1}{4}MA^2 w \operatorname{Sen}\gamma}{\frac{1}{2}MA^2 w \operatorname{Cos}\gamma}\right)$$

$$= \underline{\tan^{-1}\left(\frac{1}{2}\tan\gamma\right)}$$

c) Por definición, la torca es $\vec{\tau} = \frac{d\vec{L}}{dt}$

$$\vec{\tau} = \frac{d}{dt}\left(\frac{1}{2}MA^2 w\left(\frac{1}{2}\operatorname{Sen}\gamma, 0, \cos\gamma\right)\right)$$

$$= \frac{1}{2}MA^2 w \frac{d}{dt}\left(\frac{1}{2}\operatorname{Sen}\gamma\,\hat{i} + \operatorname{Cos}\gamma\,\hat{k}\right)$$

Pero como el argumento es γ y no t,

$\gamma = cte.$

$$\underline{\underline{\vec{\tau} = 0}}$$