$$L = L(\ddot{q}_i, \dot{q}_i, q, t) \qquad \left.\frac{\partial J}{\partial t}\right|_{t=0} = 0$$

$$J(x) = \int_{x_1}^{x_2} L(\ddot{q}_i, \dot{q}, q, t)\, dt$$

$$\Rightarrow \frac{\partial J}{\partial t} = \frac{\partial}{dt}\int_{t_1}^{t_2} L(q, \dot{q}, \ddot{q}, t)\, dt$$

$$= \int_{t_1}^{t_2} \frac{\partial}{\partial t}\left[L(q, \dot{q}, \ddot{q}, t)\right] dt$$

$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\frac{\partial q}{\partial t} + \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial t} + \frac{\partial L}{\partial \ddot{q}}\frac{\partial \ddot{q}}{\partial t} \right) dt$$

Definimos:

$$\frac{\partial q}{\partial t} = \eta(t) \rightarrow \frac{\partial \dot{q}}{\partial t} = \frac{\partial}{\partial t}\eta(t) \rightarrow \frac{\partial \ddot{q}}{\partial t} = \frac{\partial^2}{\partial t^2}\eta(t)$$

$$\Rightarrow \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q}\eta(t) + \frac{\partial L}{\partial \dot{q}}\frac{\partial}{\partial t}\eta(t) + \frac{\partial L}{\partial \ddot{q}}\frac{\partial^2}{\partial t^2}\eta(t) \right) dt$$

$$= \int_{t_1}^{t_2} \frac{dL}{dq}\eta(t)\, dt \int_{t_1}^{t_2} \frac{dL}{d\dot{q}}\frac{\partial}{\partial t}\eta(t)\, dt + \int_{t_1}^{t_2} \frac{dL}{\partial \ddot{q}}\frac{\partial^2}{\partial t^2}\eta(t)\, dt$$

$$\Rightarrow I_2 \equiv \int_{t_1}^{t_2} \frac{dL}{\partial \dot{q}}\frac{\partial}{\partial t}\eta(t)\, dt$$

Se integra por partes $\Rightarrow u = \eta(t) \qquad dv = \frac{dL}{\partial \dot{q}}\frac{\partial}{\partial t}$

$\qquad du = \frac{d}{dt}\eta(t)\, dt \quad v = \frac{dL}{\partial \dot{q}}$

$$\Rightarrow \int_{t_1}^{t_2} \frac{dL}{\partial \dot{q}}\frac{\partial}{\partial t}\eta(t)\, dt = \left.\frac{dL}{d\dot{q}}\eta(t)\right|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{dL}{d\dot{q}}\frac{d}{dt}\eta(t)\, dt$$

Pero $\eta(t_2) = \eta(t_1) = 0$

Como $\eta(t)$ es una función arbitaria, al momento de hacer la integral esta desaparece.

$$\Rightarrow \int_{t_1}^{t_2} \frac{dL}{d\dot{q}}\frac{d}{\partial t}\eta(t)dt = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

$$I_3 = \int_{t_1}^{2} \frac{dL}{d\ddot{q}} \frac{d^2}{dt^2} \eta(t)\, dt$$

$u = \eta(t)$ $\qquad$ $dv = \frac{dL}{d\ddot{q}} \frac{d^2}{dt^2}$

$du = \frac{d}{dt}\eta(t)\, dt$ $\qquad$ $v = \frac{dL}{d\ddot{q}} \frac{d}{dt}$

$$\Rightarrow \int_{t_1}^{t_2} \frac{dL}{d\ddot{q}} \frac{d^2}{dt^2} \eta(t)\, dt = \frac{dL}{d\ddot{q}} \frac{d}{dt} \eta(t) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{dL}{d\ddot{q}} \frac{d}{dt} \frac{d}{dt} \eta(t)\, dt$$

$$= \underbrace{\int_{t_1}^{t_2} \frac{dL}{d\ddot{q}} \frac{d}{dt} \eta(t)}_{} - \int_{t_1}^{t_2} \frac{dL}{d\ddot{q}} \frac{d\ddot{q}}{dt}\, dt$$

$u = \eta(t)$ $\qquad$ $du = \frac{d}{dt}\eta(t)\, dt$

$dv = \frac{dL}{d\ddot{q}} \frac{d}{dt}$; $\quad v = \frac{dL}{d\ddot{q}}$

$$\Rightarrow \int_{t_1}^{t_2} \frac{dL}{d\ddot{q}} \frac{d^2}{dt^2} \eta(t)\, dt = \cancel{\frac{dL}{d\ddot{q}} \eta(t)}^{0} - \int_{t_1}^{t_2} \frac{dL}{d\ddot{q}} \frac{d}{dt} \cancel{\eta(t)\, dt} - \int_{t_1}^{t_2} \frac{dL}{d\ddot{q}} \frac{d\ddot{q}}{dt}\, dt$$

$$I_3 = -\frac{d}{dt}\left(\frac{dL}{d\ddot{q}}\right) - \cancel{\frac{dL}{d\ddot{q}}\left(\frac{d\ddot{q}}{dt}\right)}^{0}.$$

$$\Rightarrow \frac{dL}{dq} + \frac{d}{dt}\left(\frac{dL}{d\dot{q}}\right) - \frac{d}{dt}\left(\frac{dL}{d\ddot{q}}\right) = 0$$

Encuentra las ecuaciones de Euler-Lagrange del modelo Sigma $L(\dot{q}, q, t) = \frac{1}{2} \underbrace{g_{ab}(q^c)}_{\text{Matriz}} \dot{q}^a \dot{q}^b$

Recordando $\frac{\partial L}{\partial q^\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right)$

Vamos a hacer los cálculos por partes, primero tomamos $\frac{\partial L}{\partial q^\gamma}$ y sustituimos el modelo Lagrangiano

$$\frac{\partial}{\partial q^\gamma}\left(\frac{1}{2} g_{ab}(q^c)\, \dot{q}^a \dot{q}^b\right)$$

Por regla de la cadena

$$\frac{1}{2}\left[\frac{\partial}{\partial q^\gamma} g_{ab}\right] \dot{q}^a \dot{q}^b + \cancel{g_{ab} \frac{\partial}{\partial q^\gamma}(\dot{q}^a \dot{q}^b)} \to 0$$

Pero $\dot{q}^a$ y $\dot{q}^b$ no dependen de q^γ entonces se puede ver como derivar una constante, este término es 0.

$$\Rightarrow \frac{1}{2}\left[\frac{\partial}{\partial q^\gamma} g_{ab}\right] \dot{q}^a \dot{q}^b$$

Utilizando la relación

$$\frac{\partial}{\partial q^\gamma} g_{ab} = \Gamma^\mu_{a\gamma} g_{\mu b} + \Gamma^\mu_{b\gamma} g_{a\mu}$$

Y sustituyendo adecuadamente, llegamos a:

$$\Rightarrow \frac{\partial L}{\partial q^\gamma} = \frac{1}{2}\left[\Gamma^\mu_{a\gamma} g_{\mu b} + \Gamma^\mu_{b\gamma} g_{a\mu}\right] \dot{q}^a \dot{q}^b$$

Ahora hacemos el segundo termino empezando con la derivada respecto a $\dot{q}^{\psi}$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}^{\psi}} = \frac{\partial}{\partial \dot{q}^{\psi}}\left(\frac{1}{2} g_{ab}(q^c)\, \dot{q}^a \dot{q}^b\right)$$

como g_{ab} depende de las coordenadas mas no de la derivada se toma como una constante.

$$= \frac{1}{2} g_{ab}(q^c) \frac{\partial}{\partial \dot{q}^{\psi}}\left(\dot{q}^a \dot{q}^b\right)$$

$$= \frac{1}{2} g_{ab}(q^c)\left[\frac{\partial \dot{q}^a}{\partial \dot{q}^{\psi}} \dot{q}^b + \frac{\partial \dot{q}^b}{\partial \dot{q}^{\psi}} \dot{q}^a\right]$$

$$= \frac{1}{2} g_{ab}\left[\delta^a_{\psi} \dot{q}^b + \delta^b_{\psi} \dot{q}^a\right] = \frac{1}{2} g_{ab}(q^c)\, \delta^a_{\psi} \dot{q}^b + \frac{1}{2} g_{ab}(q^c)\, \delta^b_{\psi} \dot{q}^a$$

Como ψ representa un conjunto de indices libres entonces puede tomar cualquier valor, entonces con este argumento ψ vale a y b respectivamente para que en ambas expresiones la delta sea 1

$$\Rightarrow \frac{\partial L}{\partial \dot{q}^{\psi}} = \frac{1}{2} g_{\underline{a}b}(q^c)\, \dot{q}^{\underline{b}} + \frac{1}{2} g_{a\underline{b}}(q^c)\, \dot{q}^{\underline{a}}$$

Observando bien cada termino nos damos cuenta que en ambos hay un termino libre, es decir, no se repite, para el primer termino este es a y en el segundo es b

$\Rightarrow$ Podemos reescribir como:

$$\frac{\partial L}{\partial \dot{q}^{\psi}} = \frac{1}{2} g_{b\psi}(q^c)\, \dot{q}^b + \frac{1}{2} g_{a\psi}(q^c)\, \dot{q}^a$$

Pero como se tratan de indices libres podemos usar la relación $g_{ab} = g_{ba}$ y haciendo $a = b$

$$\Rightarrow \frac{1}{\cancel{2}}\left[\cancel{2} g_{b\psi}(q^c)\, \dot{q}^b\right]$$

$\Rightarrow$ la derivada del lagrangiano respecto a $\dot{q}^{\psi}$ es

$$\frac{\partial L}{\partial \dot{q}^{\psi}} = g_{b\psi}(q^c)\, \dot{q}^b$$

Ejercicio 2

Esa derivada la tenemos que volver a derivar respecto al tiempo

$$\Rightarrow \frac{\partial}{\partial t}\left[g_{b\psi}(q^c)\,\dot{q}^b\right]$$

la matriz $g_{b\psi}(q^c)$ depende de las coordenadas que dependen del tiempo, es decir, $g_{b\psi}(q^c(t))$

Para hacer esta derivada usamos regla de la cadena y la forma de la derivada de un producto

$$\frac{\partial}{\partial t}\left[g_{b\psi}(q^c)\,\dot{q}^b\right] = g_{b\psi}(q^c)\frac{\partial}{\partial t}\dot{q}^b + \dot{q}^b\left[\frac{\partial g_{b\psi}(q^c)}{\partial q^c}\frac{\partial q^c}{\partial t}\right]$$

$$= \frac{\partial}{\partial t} g_{b\psi}\,\dot{q}^b + g_{b\psi}\,\ddot{q}^b$$

$$= \left[\dot{q}^c\frac{\partial g_{b\psi}}{\partial q^c}\right]\dot{q}^b + \ddot{q}^b g_{b\psi}$$

Recordando

$$\frac{\partial}{\partial q^c} g_{b\psi} = \Gamma^{\mu}_{ac} g_{\mu b} + \Gamma^{\mu}_{bc} g_{a\mu}$$

Sustituyendo

$$\dot{q}^c\left[\Gamma^{\mu}_{ac} g_{\mu b} + \Gamma^{\mu}_{bc} g_{a\mu}\right]\dot{q}^b + \ddot{q}^b g_{b\psi} = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}^\psi}\right)$$

$$\Rightarrow \frac{\partial L}{\partial q^\psi} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}^\psi}\right) = 0$$

$\Rightarrow$ Sustituyendo las expresiones ya obtenidas

$$\frac{1}{2}\left[\Gamma^{\mu}_{a\psi} g_{\mu b} + \Gamma^{\mu}_{b\psi} g_{a\mu}\right]\dot{q}^a\dot{q}^b - \left[\Gamma^{\mu}_{ac} g_{\mu b} + \Gamma^{\mu}_{bc} g_{a\mu}\right]\dot{q}^c\dot{q}^b + \ddot{q}^b g_{b\psi} = 0$$

$$= \frac{1}{2}\Gamma^{\mu}_{a\psi} g_{\mu b}\,\dot{q}^a\dot{q}^b + \frac{1}{2}\Gamma^{\mu}_{b\psi} g_{a\mu}\,\dot{q}^a\dot{q}^b - \Gamma^{\mu}_{ac} g_{\mu b}\,\dot{q}^c\dot{q}^b + \Gamma^{\mu}_{bc} g_{a\mu}\,\dot{q}^c\dot{q}^b + \ddot{q}^b g_{b\psi} = 0$$

Como se tratan de indices libres podemos reescribir

$$\frac{1}{2}\Gamma^{\mu}_{\gamma\psi}\, g_{\mu b}\, \dot{q}^{\gamma}\dot{q}^{b} + \frac{1}{2}\Gamma^{\mu}_{\gamma\psi}\, g_{\gamma\mu}\, \dot{q}^{\psi}\dot{q}^{\gamma} - \Gamma^{\mu}_{ac}\, g_{\mu b}\, \dot{q}^{b}\dot{q}^{c}$$

$$- \Gamma^{\mu}_{bc}\, g_{a\mu}\, \dot{q}^{b}\dot{q}^{c} - \ddot{q}^{b} g_{b\psi} = 0$$

$$= \underbrace{\cancel{\Gamma^{\mu}_{\gamma\psi}\, g_{\mu b}\, \dot{q}^{\gamma}\dot{q}^{b} - \Gamma^{\mu}_{ca}\, g_{\mu b}\, \dot{q}^{c}\dot{q}^{b}}} - \Gamma^{\mu}_{bc}\, g_{a\mu}\, \dot{q}^{b}\dot{q}^{c} - \ddot{q}^{b} g_{b\psi} = 0$$

Quedan ψ y a como indices libres por lo que se eliminan

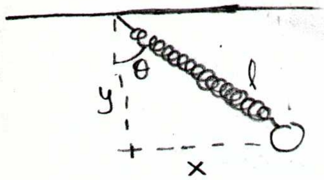
$$-\Gamma^{\mu}_{bc}\, g_{a\mu}\, \dot{q}^{b}\dot{q}^{c} - \ddot{q}^{b} g_{b\psi} = 0$$

Volviendo a reescribir tomando en cuenta que los indices son libres y pueden tomar cualquier valor

$$\Rightarrow -g_{\mu a}\left(\Gamma^{\mu}_{bc}\, \dot{q}^{b}\dot{q}^{c} + \ddot{q}^{\mu}\right) = 0$$

Pero $g_{\mu a} \neq 0 \Rightarrow \underbrace{\Gamma^{\mu}_{bc}\, \dot{q}^{b}\dot{q}^{c} + \ddot{q}^{\mu} = 0}_{\text{Ecuación Geodésicas}}$

Ejercicio 2



$r = \sqrt{x^2+y^2}$

$\tan\theta = \frac{x}{y} \Rightarrow \theta = \arctan\left(\frac{x}{y}\right)$

$x = (l+z(t))\sin\theta \;;\; \dot{x} = \frac{d}{dt}(l+z(t))\sin\theta = \dot{z}\sin\theta + (l+z)\cos\theta\,\dot{\theta}$

$y = (l+z(t))\cos\theta \;;\; \dot{y} = \frac{d}{dt}(l+z(t))\cos\theta = \dot{z}\cos\theta - (l+z)\sin\theta\,\dot{\theta}$

Para construir el lagrangiano

$L = T - U$

Para este sistema $T = \frac{1}{2}m\dot{\underline{r}}^2 \;;\; U = U_{pendulo} + U_{resorte}$

$U = mgy + \frac{k}{2}z^2$

$\Rightarrow L = \frac{1}{2}m\dot{\underline{r}}^2 - \left(mgy + \frac{k}{2}z^2\right)$

Reescribiendo:

$L = \frac{1}{2}m\left[(\dot{z}S_\theta + (l+z)C_\theta\dot{\theta})^2 + (\dot{z}C_\theta - (l+z)S_\theta\dot{\theta})^2\right] - mg(l+z)C_\theta - \frac{k}{2}z^2$

$\Rightarrow L(z,\dot{z},\theta,\dot{\theta})$ No hay dependencia explicita del tiempo

$\Rightarrow$ La energía se conserva

Recordando: $\frac{\partial L}{\partial q} - \frac{\partial}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$

$\Rightarrow q$ puede tomar valores de $z, \theta \rightarrow \dot{q} = \dot{z}, \dot{\theta}$

Para $q = z$ $\quad \frac{\partial L}{\partial z} - \frac{\partial}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right)$

$\frac{\partial}{\partial z}\left[\frac{1}{2}m\left[(\dot{z}S_\theta + (l+z)C_\theta\dot{\theta})^2 + (\dot{z}C_\theta - (l+z)S_\theta\dot{\theta})^2\right] - mg(l+z)C_\theta - \frac{k}{2}z^2\right]$

$= \frac{1}{\cancel{2}}m\left[\cancel{2}(\dot{z}S_\theta + (l+z)\dot{\theta}C_\theta)(C_\theta\dot{\theta}) + \cancel{2}(\dot{z}C_\theta - (l+z)S_\theta\dot{\theta})(S_\theta\dot{\theta})\right] - mgC_\theta - kz$

$= m\left[\dot{z}\dot{\theta}S_\theta C_\theta + (l+z)\dot{\theta}^2C_\theta^2 + \dot{z}\dot{\theta}S_\theta C_\theta - (l+z)\dot{\theta}^2S_\theta^2\right] - mgC_\theta - kz$

$= m\left[2\dot{z}\dot{\theta}S_\theta C_\theta + (l+z)\dot{\theta}^2\left[C_\theta^2 - S_\theta^2\right]\right] - mgC_\theta - kz$

$= m\left[\dot{z}\dot{\theta}S_{2\theta} + (l+z)\dot{\theta}^2\left[C_\theta^2 - S_\theta^2\right]\right] - mgC_\theta - kz$

$$\frac{\partial L}{\partial \dot{q}} = \frac{1}{\cancel{2}} m\left[\cancel{2}(\dot{z} S_\theta + (l+z) C_\theta \dot{\theta})(S_\theta) + \cancel{2}(\dot{z} C_\theta - (l+z) S_\theta \dot{\theta}) C_\theta\right]$$

$$= m\left[\dot{z} S_\theta^2 + \cancel{(l+z)\dot{\theta} C_\theta S_\theta} + \dot{z} C_\theta^2 - \cancel{(l+z)\dot{\theta} C_\theta S_\theta}\right]$$

$$= m\dot{z}$$

$$\frac{\partial}{\partial t}(m\dot{z}) = m\ddot{z} \quad \Rightarrow \frac{\partial}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = m\ddot{z}$$

$$= m\left[\dot{z}\dot{\theta} S_{2\theta} + (l+z)\dot{\theta}^2 \left[C_\theta^2 - S_\theta^2\right] - mg C_\theta - \frac{k}{m} z\right] - m\ddot{z} = 0$$

Para θ

$$\frac{\partial L}{\partial \theta} = \frac{1}{\cancel{2}} m\left[\cancel{2}(\dot{z} S_\theta + (l+z)\dot{\theta} C_\theta)(\dot{z} C_\theta - (l+z)\dot{\theta} S_\theta) + \cancel{2}[(\dot{z} C_\theta - (l+z) S_\theta \dot{\theta}) \ldots\right.$$

$$\left.\ldots(-\dot{z} S_\theta - (l+\dot{z})\dot{\theta} C_\theta)\right] + mg(l+z) S_\theta$$

$$= m\left[\dot{z}^2 S_\theta C_\theta - \dot{z}(l+z)\dot{\theta} S_\theta^2 + \dot{z}\dot{\theta}(l+z) C_\theta^2 - (l+z)^2 \dot{\theta}^2 S_\theta C_\theta - \dot{z}^2 C_\theta S_\theta - \dot{z}(l+z)\dot{\theta} C_\theta^2\right.$$

$$\left. + (l+z)\dot{\theta} S_\theta^2 \dot{z} + (l+z)^2 \dot{\theta}^2 S_\theta C_\theta\right] + mg(l+z) S_\theta$$

$$= mg(l+z) S_\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{\cancel{2}} m\left(\cancel{2}[\dot{z} S_\theta + (l+z)\dot{\theta} C_\theta](l+z) C_\theta\right) + \cancel{2}[\dot{z} C_\theta - (l+z) S_\theta \dot{\theta}][-(l+z) S_\theta]$$

$$= m\left[(l+z)^2 \dot{\theta} C_\theta^2 + (l+z)^2 \dot{\theta} S_\theta^2\right] = m\dot{\theta}(l+z)^2$$

$$\frac{\partial}{dt}\left[m\dot{\theta}(l+z)^2\right] = m\left[(l+z)^2 \ddot{\theta} + \dot{\theta}[2(l+z)(\dot{z})]\right]$$

$$= m\left[\ddot{\theta}(l+z)^2 + 2\dot{\theta}\dot{z}(l+z)\right]$$

$$\Rightarrow \frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{d\dot{\theta}}\right) = mg(l+z) S_\theta + m\left[\ddot{\theta}(l+z)^2 + 2\dot{\theta}\dot{z}(l+z)\right]$$

$$= mg(l+z) S_\theta + \ddot{\theta} m(l+z)^2 + 2\dot{\theta}\dot{z}(l+z)$$

b) Puntos de equilibrio

A partir de las expresiones obtenidas por Euler-Lagrange

Para z

$$= m\left[\dot{z}\dot{\theta}S_{2\theta} + (l+z)\dot{\theta}^2(C_\theta^2 - S_\theta^2) - mg\,C_\theta - \frac{k}{m}z\right] - m\ddot{z}$$

Para encontrar puntos de equilibrio, el sistema tiene que estar en reposo, es decir

$$\dot{z} = \ddot{z} = 0 \; ; \; \dot{\theta} = 0 = \ddot{\theta}$$

$$\Rightarrow \; -mg\cos\theta - \frac{k}{m}z = 0$$

Para θ

$$mg(l+z)S_\theta + \ddot{\theta}m(l+z)^2 + 2\dot{\theta}\dot{z}(l+z)$$

$$\dot{z} = 0 = \ddot{z} \; ; \; \dot{\theta} = 0 = \ddot{\theta}$$

$$\Rightarrow mg(l+z)\sin\theta = 0$$

Ahora tenemos un sistema de ecuaciones

① $-m^2 g\cos\theta - kz = 0$

② $mg(l+z)\sin\theta = 0$

Resolvemos el sistema de ecuaciones

$$\cos\left(-\frac{kz}{m^2 g}\right) = \theta$$

$$mg(l+z)\sin\Bigl(\underbrace{\cos\left(-\frac{kz}{m^2 g}\right)}_{\varphi}\Bigr) \Rightarrow mgl\sin\varphi + mgz\sin\varphi = 0$$

$$mgz\sin\varphi = -mgl\sin\varphi$$

$$z = -l$$

$$\Rightarrow \cos\left(-\frac{kl}{m^2 g}\right) = \theta$$

c)

$\cos\left(-\frac{kl}{m^2 g}\right)$ Siendo l la unica variable que cambia

$\Rightarrow$ Haciendo la expansión en serie de McLauren

$$-\frac{k}{m^2 g}\cos(l) = -\frac{k}{m^2 g}\sum_{k=0}^{\infty}(-1)^k \frac{l^{2k}}{(2k)!}$$

$$= -\frac{k}{m^2 g}\left[1 - \frac{l^2}{2!} + \frac{l^4}{4!} - \frac{l^6}{6!} + \frac{l^8}{8!} + \cdots\right]$$

(a) $L = e^{bt}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}k^2 q^2\right)$

$$\frac{\partial}{\partial q}L - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial q}\left[e^{bt}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}k^2 q^2\right)\right]$$

$$\frac{\partial L}{\partial q} = -e^{bt}k^2 q$$

$$\frac{\partial}{\partial \dot{q}}L = e^{bt} m\dot{q}(t) \Rightarrow \frac{d}{dt}\left(e^{bt}m\dot{q}\right) = e^{bt}(m\ddot{q}) + m\dot{q}(b e^{bt})$$

$\Rightarrow$ Euler langrange $: -e^{bt}\left(k^2 q - m\ddot{q} - bm\dot{q}\right) = 0$

La función exponencial siempre es diferente de cero

$\therefore \; k^2 q - bm\dot{q} - m\ddot{q} = 0$

Esto tiene la forma de un oscilador amortiguado

b) $Q(t) = e^{\frac{bt}{2}} q \Rightarrow q = \frac{Q}{e^{\frac{bt}{2}}}$; $q^2 = \frac{Q^2}{e^{bt}}$

$$\dot{q} = \frac{d}{dt}\left(\frac{Q}{e^{\frac{bt}{2}}}\right) \Rightarrow \dot{q} = \frac{\dot{Q}\left(e^{\frac{bt}{2}}\right) - Q\left(\frac{b}{2}e^{\frac{bt}{2}}\right)}{e^{bt}}$$

$$\dot{q} = \frac{\left(\dot{Q} - \frac{b}{2}Q\right)}{e^{\frac{bt}{2}}} \Rightarrow \dot{q}^2 = \frac{\left(\dot{Q} - \frac{b}{2}Q\right)^2}{e^{bt}}$$

$$L = \cancel{e^{bt}}\left(\frac{1}{2}m\left(\frac{\left(\dot{Q} - \frac{b}{2}Q\right)^2}{\cancel{e^{bt}}}\right) - \frac{1}{2}k^2\left(\frac{Q^2}{\cancel{e^{bt}}}\right)\right)$$

$$L = \frac{1}{2}m\left(\dot{Q} - \frac{b}{2}Q\right)^2 - \frac{1}{2}k^2 Q^2$$