Jose Salvador Negrete Serrato Tarea 4

1.-

a) Probar que son canónicas para cualquier μ

$$q_1 = x\cos\mu + P_y \operatorname{sen}\mu \qquad q_2 = y\cos\mu + P_x \operatorname{sen}\mu$$

$$P_1 = P_x\cos\mu - y\operatorname{sen}\mu \qquad P_2 = P_y\cos\mu - x\operatorname{sen}\mu$$

Las ecuaciones de transformación canónica cumplen con las siguientes propiedades de los corchetes de poisson:

$\{q_i, q_j\} = 0$; $\{P_i, P_j\} = 0$ y $\{q_i, P_j\} = \delta_{ij}$ si $i=j$

en donde los corchetes de Poisson se definen como:

$$\{f, g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$$

con $f(q_i, p_i)$ y $g(q_i, p_i)$.

Entonces:

$$\{q_1, q_2\} = \frac{\partial q_1}{\partial q_i}\frac{\partial q_2}{\partial p_i} - \frac{\partial q_1}{\partial p_i}\frac{\partial q_2}{\partial q_i}$$

$$= \frac{\partial q_1}{\partial x}\frac{\partial q_2}{\partial P_x} + \frac{\partial q_1}{\partial y}\frac{\partial q_2}{\partial P_y} - \frac{\partial q_1}{\partial P_x}\frac{\partial q_2}{\partial x} - \frac{\partial q_1}{\partial P_y}\frac{\partial q_2}{\partial y}$$

$$= (\cos\mu)(\operatorname{sen}\mu) + (0)(0) - (0)(0) - (\operatorname{sen}\mu)(\cos\mu)$$

$$= \underline{\underline{0}}\,//$$

$$\{P_1, P_2\} = \frac{\partial P_1}{\partial x}\frac{\partial P_2}{\partial P_x} + \frac{\partial P_1}{\partial y}\frac{\partial P_2}{\partial P_y} - \frac{\partial P_1}{\partial P_x}\frac{\partial P_2}{\partial x} - \frac{\partial P_1}{\partial P_y}\frac{\partial P_2}{\partial y}$$

$$= (0)(0) + (-\operatorname{sen}\mu)(\cos\mu) - (\cos\mu)(-\operatorname{sen}\mu)$$

$$-(0)(0) = \underline{\underline{0}}\,//$$

$$\{q_1, P_2\} = \frac{\partial q_1}{\partial x}\frac{\partial P_2}{\partial P_x} + \frac{\partial q_1}{\partial y}\frac{\partial P_2}{\partial P_y} - \frac{\partial q_1}{\partial P_x}\frac{\partial P_2}{\partial x} - \frac{\partial q_1}{\partial P_y}\frac{\partial P_2}{\partial y}$$

$$= (\cos \mu)(0) + (0)(\cos \mu) - (0)(-\operatorname{sen}\mu) - (\operatorname{sen}\mu)(0)$$

$$= \underline{\underline{0}}$$

$$\{q_2, P_1\} = \frac{\partial q_2}{\partial x}\frac{\partial P_1}{\partial P_x} + \frac{\partial q_2}{\partial y}\frac{\partial P_1}{\partial P_y} - \frac{\partial q_2}{\partial P_x}\frac{\partial P_1}{\partial x} - \frac{\partial q_2}{\partial P_y}\frac{\partial P_1}{\partial y}$$

$$= (0)(\cos \mu) + (\cos \mu)(0) - (\operatorname{sen}\mu)(0) - (0)(-\operatorname{sen}\mu)$$

$$= \underline{\underline{0}}$$

$$\{q_1, P_1\} = \frac{\partial q_1}{\partial x}\frac{\partial P_1}{\partial P_x} + \frac{\partial q_1}{\partial y}\frac{\partial P_1}{\partial P_y} - \frac{\partial q_1}{\partial P_x}\frac{\partial P_1}{\partial x} - \frac{\partial q_1}{\partial P_y}\frac{\partial P_1}{\partial y}$$

$$= (\cos \mu)(\cos \mu) + (0)(0) - (0)(0) - (\operatorname{sen}\mu)(-\operatorname{sen}\mu)$$

$$= \cos^2 \mu + \operatorname{sen}^2 \mu = \underline{\underline{1}}$$

$$\{q_2, P_2\} = \frac{\partial q_2}{\partial x}\frac{\partial P_2}{\partial P_x} + \frac{\partial q_2}{\partial y}\frac{\partial P_2}{\partial P_y} - \frac{\partial q_2}{\partial P_x}\frac{\partial P_2}{\partial x} - \frac{\partial q_2}{\partial P_y}\frac{\partial P_2}{\partial y}$$

$$= (0)(0) + (\cos \mu)(\cos \mu) - (\operatorname{sen}\mu)(-\operatorname{sen}\mu)$$

$$+ (0)(0) = \cos^2 \mu + \operatorname{sen}^2 \mu = \underline{\underline{1}}$$

Los unos resultantes de $\{q_1, P_1\}$ y $\{q_2, P_2\}$ son indicativo de que $\{q_i, P_j\} = \begin{cases} 0 & \text{si } i \neq j \\ 1 & \text{si } i = j \end{cases} \Rightarrow \delta_{ij}$

Esto muestra que son ecuaciones de transformación canónicas.

b) $H = (q_1^2 + q_2^2 + P_1^2 + P_2^2)/2$

Encontrar el nuevo hamiltoniano como función de X, Y y sus momentos conjugados:

Tenemos: $q_1 = X\cos\mu + P_y \operatorname{sen}\mu$ $\quad q_2 = Y\cos\mu + P_x \operatorname{sen}\mu$

$P_1 = P_x\cos\mu - Y\operatorname{sen}\mu \quad P_2 = P_y\cos\mu - X\operatorname{sen}\mu$

$$H = \left[(X\cos\mu + P_y\operatorname{sen}\mu)^2 + (Y\cos\mu + P_x\operatorname{sen}\mu)^2 + (P_x\cos\mu - Y\operatorname{sen}\mu)^2 + (P_y\cos\mu - X\operatorname{sen}\mu)^2\right]/2$$

$$= \left[X^2\cos^2\mu + \cancel{2XP_y\cos\mu\operatorname{sen}\mu} + P_y^2\operatorname{sen}^2\mu + Y^2\cos^2\mu + \cancel{2YP_x\cos\mu\operatorname{sen}\mu} + P_x^2\operatorname{sen}^2\mu + P_x^2\cos^2\mu - \cancel{2YP_x\cos\mu\operatorname{sen}\mu} + Y^2\operatorname{sen}^2\mu + P_y^2\cos^2\mu - \cancel{2XP_y\cos\mu\operatorname{sen}\mu} + X^2\operatorname{sen}^2\mu\right]/2$$

$$= \left[X^2(\cos^2\mu + \operatorname{sen}^2\mu) + Y^2(\cos^2\mu + \operatorname{sen}^2\mu) + P_x^2(\cos^2\mu + \operatorname{sen}^2\mu) + P_y^2(\cos^2\mu + \operatorname{sen}^2\mu)\right]/2$$

$$= (X^2 + Y^2 + P_x^2 + P_y^2)/2 \;//$$

c) $Y = P_y = 0$

Entonces:

$$H = (X^2 + P_x^2)/2$$

y las ecuaciones de movimiento para hamiltonianos son:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \quad y \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i$$

Entonces

$$\frac{\partial H}{\partial p_i} = \frac{\partial H}{\partial p_x} = \frac{\partial}{\partial p_x}\left[(x^2 + p_x^2)/2\right]$$

$$= \frac{2p_x}{2} = p_x$$

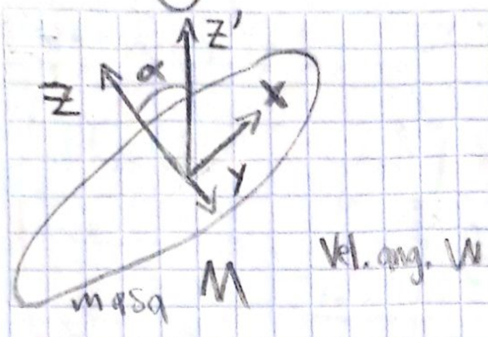
$$\frac{\partial H}{\partial q_i} = \frac{\partial H}{\partial x} = \frac{\partial}{\partial x}\left[(x^2 + p_x^2)/2\right]$$

$$= \frac{2x}{2} = x$$

$$\boxed{\begin{aligned} \dot{q}_x &= p_x \\ \dot{p}_x &= -x \end{aligned}} \Rightarrow \dot{p}_x + x = 0 \Rightarrow \boxed{m\ddot{x} + x = 0}$$

oscilador armónico

2.- αy



radio A

a)

Calculamos los momentos de inercia $\quad \rho = \frac{M}{V} = \frac{M}{\pi A^2}$

$$I_{11} = \frac{M}{\pi A^2} \int x^2 \, dx\, dy$$

$$I_{22} = \frac{M}{\pi A^2} \int y^2 \, dx\, dy$$

$r^2 = x^2 + y^2$

$$I_{33} = I_{11} + I_{22} = \frac{M}{\pi A^2} \int (x^2 + y^2)\, dx\, dy$$

$$= \frac{M}{\pi A^2} \int r^2 (r\, dr\, d\theta)$$

$$= \frac{M}{\pi A^2} \int_0^A \int_0^{2\pi} r^3 \, dr\, d\theta$$

$$= \frac{M}{\pi A^2} 2\pi \int_0^A r^3 \, dr$$

$$= \frac{2M}{A^2} \left. \frac{r^4}{4} \right|_0^A = \frac{MA^2}{2}$$

Pero $I_{11} = I_{22}$, entonces:

$$I_{11}=I_{22}=\frac{MA^2}{4}$$

de modo que:

$$I_{ij}=\begin{pmatrix} \frac{MA^2}{4} & 0 & 0 \\ 0 & \frac{MA^2}{4} & 0 \\ 0 & 0 & \frac{MA^2}{2} \end{pmatrix}$$

Ahora, para los ejes principales:

$$\det(I_{ij}-\lambda I)=\begin{pmatrix} \frac{MA^2}{4}-\lambda & 0 & 0 \\ 0 & \frac{MA^2}{4}-\lambda & 0 \\ 0 & 0 & \frac{MA^2}{2}-\lambda \end{pmatrix}$$

$$=\left(\frac{MA^2}{4}-\lambda\right)\left(\frac{MA^2}{4}-\lambda\right)\left(\frac{MA^2}{2}-\lambda\right)=0$$

$$\Rightarrow \lambda_1=\lambda_2=\frac{MA^2}{4}$$

$$\lambda_3=\frac{MA^2}{2}$$

Para λ_1 y λ_2:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{MA^2}{4} \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$(0)(x^1) = 0 \qquad x^1 = t$

$(0)(x^2) = 0 \Rightarrow x^2 = s$

$\frac{MA^2}{4}(x^3) = 0 \qquad x^3 = 0$

de modo que uno de los ejes es:

$\vec{e}_1 = (t, s, 0)$

$$\Rightarrow \begin{pmatrix} t \\ s \\ 0 \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Para λ_3:

$$\begin{pmatrix} -\frac{MA^2}{4} & 0 & 0 \\ 0 & -\frac{MA^2}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(-\frac{MA^2}{4}\right)(x^1)=0 \qquad x^1=0$$

$$\left(-\frac{MA^2}{4}\right)(x^2)=0 \Rightarrow x^2=0$$

$$(0)(x^3)=0 \qquad x^3=r$$

de modo que:

$$\vec{e}_2=\begin{pmatrix}0\\0\\r\end{pmatrix} \Rightarrow \vec{e}_2=r\begin{pmatrix}0\\0\\1\end{pmatrix}$$

b) Encontrar $\underline{L}$

Tenemos que: $\underline{L}=\underline{I}\,\omega \Rightarrow$ Para este problema

$$\begin{pmatrix}L_1\\L_2\\L_3\end{pmatrix}=\begin{pmatrix}\frac{MA^2}{4} & 0 & 0\\ 0 & \frac{MA^2}{4} & 0\\ 0 & 0 & \frac{MA^2}{2}\end{pmatrix}\begin{pmatrix}0\\ \omega\cos\alpha\\ \omega\operatorname{sen}\alpha\end{pmatrix}$$

$$\boxed{\begin{aligned} L_1&=0\\ L_2&=\frac{MA^2\omega}{4}\cos\alpha\\ L_3&=\frac{MA^2\omega}{2}\operatorname{sen}\alpha\end{aligned}} \Rightarrow \underline{L}=\left(0,\frac{MA^2\omega}{4}\cos\alpha,\frac{MA^2\omega}{2}\operatorname{sen}\alpha\right)$$

$$\|\underline{L}\| = \sqrt{\left(\frac{MA^2 w}{4}\cos\alpha\right)^2 + \left(\frac{MA^2 w}{2}\operatorname{sen}\alpha\right)^2}$$

$$= \sqrt{\frac{M^2A^4w^2}{16}\cos^2\alpha + \frac{M^2A^4w^2}{4}\operatorname{sen}^2\alpha}$$

c) $\underline{\tau} = \frac{d\underline{L}}{dt} = \dot{\underline{L}}$

Pero $\underline{L}$ no cambia en el tiempo, entonces:

$$\boxed{\underline{\tau} = 0}$$