

Pregunta 1)

a) Prueba que las siguientes transformaciones son canónicas para cualquier μ .

$$q_1 = x \cos \mu + p_y \sin \mu$$

$$q_2 = y \cos \mu + p_x \sin \mu$$

$$P_1 = p_x \cos \mu - y \sin \mu$$

$$P_2 = p_y \cos \mu - x \sin \mu$$

1- Debemos asegurarnos que los corchetes de Poisson son invariantes.

Ej decir, $\{x_i, p_j\} = \delta_{ij}$ o alternativamente $\{q_i, p_j\} = \delta_{ij}$

Así,

$$\{q_1, P_1\} = \frac{\partial q_1}{\partial x_i} \frac{\partial P_1}{\partial p_i} - \frac{\partial q_1}{\partial p_i} \frac{\partial P_1}{\partial x_i}$$

$$= \frac{\partial q_1}{\partial x} \frac{\partial P_1}{\partial p_x} + \frac{\partial q_1}{\partial y} \frac{\partial P_1}{\partial p_y} - \frac{\partial q_1}{\partial p_x} \frac{\partial P_1}{\partial x} - \frac{\partial q_1}{\partial p_y} \frac{\partial P_1}{\partial y}$$

$$= \cos^2 \mu + \sin^2 \mu = 1$$

$$\{q_1, P_2\} = \frac{\partial q_1}{\partial x} \frac{\partial P_2}{\partial p_x} + \frac{\partial q_1}{\partial y} \frac{\partial P_2}{\partial p_y} - \frac{\partial q_1}{\partial p_x} \frac{\partial P_2}{\partial x} - \frac{\partial q_1}{\partial p_y} \frac{\partial P_2}{\partial y}$$

$$= 0$$

$$\{q_2, P_1\} = \frac{\partial q_2}{\partial x} \frac{\partial P_1}{\partial p_x} + \frac{\partial q_2}{\partial y} \frac{\partial P_1}{\partial p_y} - \frac{\partial q_2}{\partial p_x} \frac{\partial P_1}{\partial x} - \frac{\partial q_2}{\partial p_y} \frac{\partial P_1}{\partial y}$$

$$= 0$$

$$\{q_2, P_2\} = \frac{\partial q_2}{\partial x} \frac{\partial P_2}{\partial p_x} + \frac{\partial q_2}{\partial y} \frac{\partial P_2}{\partial p_y} - \frac{\partial q_2}{\partial p_x} \frac{\partial P_2}{\partial x} - \frac{\partial q_2}{\partial p_y} \frac{\partial P_2}{\partial y}$$

$$= \cos^2 \mu + \sin^2 \mu$$

$$= 1.$$

Por lo tanto, $\{q_i, p_j\} = \delta_{ij}$ //

Además, se debe probar que $\{q_i, q_j\} = 0$ y $\{P_i, P_j\} = 0$

Por propiedades de los corchetes de Poisson: $\{q_1, q_2\} = \{q_2, q_1\} = 0$

$$\{P_1, P_2\} = \{P_2, P_1\} = 0$$

Sólo queda demostrar los restantes:

$$\begin{aligned}\{q_1, q_2\} &= \frac{\partial q_1}{\partial x} \frac{\partial q_2}{\partial p_x} + \frac{\partial q_1}{\partial y} \frac{\partial q_2}{\partial p_y} - \frac{\partial q_1}{\partial p_x} \frac{\partial q_2}{\partial x} - \frac{\partial q_1}{\partial p_y} \frac{\partial q_2}{\partial y} \\ &= \cos \mu \sin \mu - \sin \mu \cos \mu \\ &= 0\end{aligned}$$

$$\begin{aligned}\{P_1, P_2\} &= \frac{\partial P_1}{\partial x} \frac{\partial P_2}{\partial p_x} + \frac{\partial P_1}{\partial y} \frac{\partial P_2}{\partial p_y} - \frac{\partial P_1}{\partial p_x} \frac{\partial P_2}{\partial x} - \frac{\partial P_1}{\partial p_y} \frac{\partial P_2}{\partial y} \\ &= -\sin \mu \cos \mu + \cos \mu \sin \mu \\ &= 0\end{aligned}$$

Por lo tanto, $\{q_i, q_j\} = 0$ y $\{P_i, P_j\} = 0$ //

Así, las transformaciones son canónicas para cualquier μ .

- 2- La segunda forma de probar que la transformación es canónica,
 o probar que el jacobiano es simplectico.

En este caso $J = \begin{pmatrix} \frac{\partial x_i}{\partial q_j} & \frac{\partial x_i}{\partial p_j} \\ \frac{\partial p_i}{\partial q_j} & \frac{\partial p_i}{\partial p_j} \end{pmatrix}$ donde $x_i = (x, y)$
 $p_i = (p_x, p_y)$
 $q_i = (q_1, q_2)$
 $r_i = (P_1, P_2)$

Primero encontramos las expresiones para $x = x(q_i, p_i)$, $y = y(q_i, p_i)$

$$p_x = P_x(q_i, p_i) \quad y \quad p_y = P_y(q_i, p_i)$$

$$\text{De } q_1 = x \cos \mu + y \sin \mu \quad y \quad P_2 = P_y \cos \mu - x \sin \mu \quad \text{despejamos } x$$

$$\Rightarrow x = \frac{q_1 - P_y \sin \mu}{\cos \mu} \quad y \quad x = \frac{P_y \cos \mu - P_2}{\sin \mu}$$

Igualando:

$$\frac{q_1 - P_y \sin \mu}{\cos \mu} = \frac{P_y \cos \mu - P_x}{\sin \mu} \Leftrightarrow q_1 - P_y \sin \mu = \frac{\cos \mu P_y - P_x \cos \mu}{\sin \mu}$$

$$\Leftrightarrow P_y \left(\frac{\cos^2 \mu}{\sin \mu} + \sin \mu \right) = q_1 + \frac{P_x \cos \mu}{\sin \mu} \Rightarrow P_y = q_1 \sin \mu + P_x \cos \mu$$

Sustituyendo en x, $x = \frac{q_1 - q_1 \sin^2 \mu - P_x \cos \mu \sin \mu}{\cos \mu} = q_1 \left(\frac{1 - \sin^2 \mu}{\cos \mu} \right) - P_x \sin \mu$

$$\Rightarrow x = q_1 \cos \mu - P_x \sin \mu$$

Luego, despejamos y de $q_2 = y \cos \mu + P_x \sin \mu$ y $P_1 = P_x \cos \mu - y \sin \mu$

$$\Rightarrow y = \frac{q_2 - P_x \sin \mu}{\cos \mu}, \quad P_1 = \frac{P_x \cos \mu - P_1}{\sin \mu}$$

Igualamos:

$$\frac{P_x \cos \mu - P_1}{\sin \mu} = \frac{q_2 - P_x \sin \mu}{\cos \mu} \Leftrightarrow \frac{P_x \cos \mu - P_1}{\sin \mu} = \frac{q_2 - P_x \sin \mu}{\cos \mu}$$

$$\Rightarrow P_x \cos \mu - P_1 = q_2 \frac{\sin \mu}{\cos \mu} - P_x \frac{\sin^2 \mu}{\cos \mu}$$

$$\Rightarrow P_x \left(\cos \mu + \frac{\sin^2 \mu}{\cos \mu} \right) = q_2 \frac{\sin \mu}{\cos \mu} + P_1 \Rightarrow P_x = q_2 \sin \mu + P_1 \cos \mu$$

Sustituimos P_x en y:

$$y = \frac{q_2 - q_2 \sin^2 \mu - P_1 \cos \mu \sin \mu}{\cos \mu} \Rightarrow y = q_2 \cos \mu - P_1 \sin \mu$$

Hacemos las matrices de 2×2 del jacobiano:

$$\frac{\partial x_i}{\partial q_1} = \begin{pmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} \end{pmatrix} = \begin{pmatrix} \cos \mu & 0 \\ 0 & \cos \mu \end{pmatrix}$$

$$\frac{\partial x_i}{\partial P_1} = \begin{pmatrix} \frac{\partial x}{\partial P_1} & \frac{\partial x}{\partial P_2} \\ \frac{\partial y}{\partial P_1} & \frac{\partial y}{\partial P_2} \end{pmatrix} = \begin{pmatrix} 0 & -\sin \mu \\ -\sin \mu & 0 \end{pmatrix}$$

$$\frac{\partial P_i}{\partial q_j} = \begin{pmatrix} \frac{\partial P_i}{\partial q_1} & \frac{\partial P_i}{\partial q_2} \\ \frac{\partial P_i}{\partial q_2} & \frac{\partial P_i}{\partial q_1} \end{pmatrix} = \begin{pmatrix} 0 & \sin \mu \\ \sin \mu & 0 \end{pmatrix}$$

$$\frac{\partial P_i}{\partial p_j} = \begin{pmatrix} \frac{\partial P_i}{\partial p_1} & \frac{\partial P_i}{\partial p_2} \\ \frac{\partial P_i}{\partial p_2} & \frac{\partial P_i}{\partial p_1} \end{pmatrix} = \begin{pmatrix} \cos \mu & 0 \\ 0 & \cos \mu \end{pmatrix}$$

Hacemos JJ^T donde $J^T = \left(\begin{array}{cc} \left(\frac{\partial x_i}{\partial q_j} \right)^T & \left(\frac{\partial P_i}{\partial q_j} \right)^T \\ \left(\frac{\partial x_i}{\partial p_j} \right)^T & \left(\frac{\partial P_i}{\partial p_j} \right)^T \end{array} \right)$

Así,

$$JJ^T = \begin{pmatrix} 0 & II \\ -II & 0 \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \cos \mu & 0 \\ 0 & \cos \mu \end{pmatrix} & \begin{pmatrix} 0 & \sin \mu \\ \sin \mu & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -\sin \mu \\ -\sin \mu & 0 \end{pmatrix} & \begin{pmatrix} \cos \mu & 0 \\ 0 & \cos \mu \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} \begin{pmatrix} 0 & -\sin \mu \\ -\sin \mu & 0 \end{pmatrix} & \begin{pmatrix} \cos \mu & 0 \\ 0 & \cos \mu \end{pmatrix} \\ \begin{pmatrix} -\cos \mu & 0 \\ 0 & -\cos \mu \end{pmatrix} & \begin{pmatrix} 0 & -\sin \mu \\ -\sin \mu & 0 \end{pmatrix} \end{pmatrix}$$

Además, $J = \begin{pmatrix} \begin{pmatrix} \cos \mu & 0 \\ 0 & \cos \mu \end{pmatrix} & \begin{pmatrix} 0 & -\sin \mu \\ -\sin \mu & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \sin \mu \\ \sin \mu & 0 \end{pmatrix} & \begin{pmatrix} \cos \mu & 0 \\ 0 & \cos \mu \end{pmatrix} \end{pmatrix}$

Llamamos $A = \begin{pmatrix} \cos \mu & 0 \\ 0 & \cos \mu \end{pmatrix}$ y $B = \begin{pmatrix} 0 & \sin \mu \\ \sin \mu & 0 \end{pmatrix}$

Así,

$$\mathcal{J}(\mathcal{J} \mathcal{J}^T) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} -B & A \\ -A & -B \end{pmatrix}$$

$$= \begin{pmatrix} -AB + BA & A^2 + B^2 \\ -B^2 - A^2 & BA - AB \end{pmatrix}$$

dónde $AB = \begin{pmatrix} \cos\mu & 0 \\ 0 & \cos\mu \end{pmatrix} \begin{pmatrix} 0 & \sin\mu \\ \sin\mu & 0 \end{pmatrix} = \begin{pmatrix} 0 & \cos\mu \sin\mu \\ \cos\mu \sin\mu & 0 \end{pmatrix}$

$$BA = \begin{pmatrix} 0 & \sin\mu \\ \sin\mu & 0 \end{pmatrix} \begin{pmatrix} \cos\mu & 0 \\ 0 & \cos\mu \end{pmatrix} = \begin{pmatrix} 0 & \cos\mu \sin\mu \\ \cos\mu \sin\mu & 0 \end{pmatrix} = AB$$

$$AA = \begin{pmatrix} \cos\mu & 0 \\ 0 & \cos\mu \end{pmatrix} \begin{pmatrix} \cos\mu & 0 \\ 0 & \cos\mu \end{pmatrix} = \begin{pmatrix} \cos^2\mu & 0 \\ 0 & \cos^2\mu \end{pmatrix}$$

$$BB = \begin{pmatrix} 0 & \sin\mu \\ \sin\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sin\mu \\ \sin\mu & 0 \end{pmatrix} = \begin{pmatrix} \sin^2\mu & 0 \\ 0 & \sin^2\mu \end{pmatrix}$$

Por lo tanto,

$$\mathcal{J} \mathcal{J} \mathcal{J}^T = \begin{pmatrix} 0 & 0 & (\cos^2\mu \ 0) (\sin^2\mu \ 0) \\ -(\cos^2\mu \ 0) - (\sin^2\mu \ 0) & 0 & 0 \end{pmatrix}$$

$$\mathcal{J} \mathcal{J} \mathcal{J}^T = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$$

El Jacobiano es simplectico
 \therefore las transformaciones son canónicas

b) Si el Hamiltoniano original es $H = \frac{1}{2}(q_1^2 + q_2^2 + p_1^2 + p_2^2)$ se encuentra en un nuevo Hamiltoniano como función de x y y y sus momentos conjugados.

Sustituimos q_1, q_2, p_1 y p_2 .

$$\begin{aligned}
 H &= \frac{1}{2} (q_1^2 + q_2^2 + p_1^2 + p_2^2) \\
 &= \frac{1}{2} (x^2 \cos^2 \mu + 2x p_y \sin \mu \cos \mu + p_y^2 \sin^2 \mu + y^2 \cos^2 \mu + 2y p_x \sin \mu \cos \mu + \\
 &\quad + p_x^2 \sin^2 \mu + p_x^2 \cos^2 \mu - 2y p_x \sin \mu \cos \mu + y^2 \sin^2 \mu + p_y^2 \cos^2 \mu - 2x p_y \sin \mu \cos \mu + \\
 &\quad + x^2 \sin^2 \mu) \\
 &= \frac{1}{2} (x^2 \cos^2 \mu + x^2 \sin^2 \mu + p_y^2 \sin^2 \mu + p_y^2 \cos^2 \mu + p_x^2 \sin^2 \mu + p_x^2 \cos^2 \mu + \\
 &\quad + y^2 \cos^2 \mu + y^2 \sin^2 \mu) \\
 &= \frac{1}{2} (x^2 + y^2 + p_x^2 + p_y^2)
 \end{aligned}$$

c) Usa el nuevo Hamiltoniano para resolver la dinámica con la restricción $y = p_y = 0$

$$\text{Si } y = p_y = 0 \Rightarrow H = \frac{1}{2}(x^2 + p_x^2)$$

Así, las ecuaciones de Hamilton son

$$\dot{x} = \frac{\partial H}{\partial p_x} = p_x \Rightarrow \dot{x} = p_x \quad (1)$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -x \Rightarrow \dot{p}_x = -x \quad (2)$$

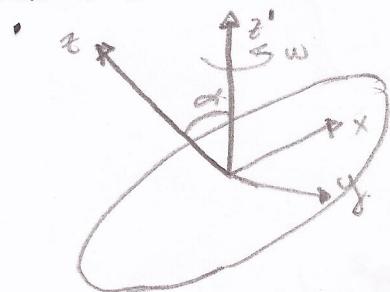
desenvolvemos (1): $(\dot{x} = p_x) \Rightarrow \ddot{x} = \dot{p}_x$, y sustituimos $\dot{p}_x = -x$.

$$\therefore \ddot{x} = -x \Rightarrow x(t) = A \cos(t + \delta) \quad \text{dónde } A, \delta \in \mathbb{R}$$

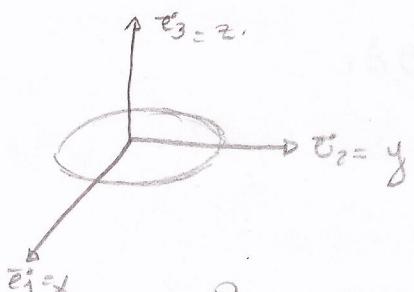
$$\text{Y de (2), } p_x(t) = -A \sin(t + \delta)$$

Pregunta 2.

Un disco delgado uniforme de masa M y radio A , rota sin fricción con una velocidad angular uniforme ω sobre un eje vertical fijo que pasa sobre su centro y tiene un ángulo alfa con el eje de simetría del disco.



a) Determina los momentos de inercia y los ejes principales.



$$\text{uniforme} \Rightarrow I = \frac{M}{\pi A^2} \text{ radio } A \text{ masa } M$$

Para cuerpos continuos:

$$I = \int d^3r f(r) \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{pmatrix}$$

Y como no hay variación en z y hay simetría en x y y , los únicos valores diferentes de cero en el tensor de inercia son:

$$I_{uu} = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}$$

Así, calculamos I_{11} .

coordenadas polares

$$I_{11} = \iint r^2 y^2 dx dy = \iint r^2 r^2 \sin^2 \theta r d\theta dr = \int_0^{2\pi} \int_0^A r^3 \sin^2 \theta dr d\theta$$

$$\begin{aligned}
 I_{11} &= \rho \int_0^{2\pi} \int_0^A \sin^2 \theta d\theta \left(\frac{1}{4} r^4 \right) = \frac{1}{4} A^4 \left(\frac{M}{\pi A^2} \right) \int_0^{2\pi} \left(\frac{1 - \cos 2\theta}{2} \right) d\theta \\
 &= \frac{1}{4} \frac{A^2 M}{\pi} \left(\frac{1}{2} \theta \Big|_0^{2\pi} - \frac{1}{4} \sin 2\theta \Big|_0^{2\pi} \right) \\
 &= \frac{1}{4} \frac{A^2 M}{\pi} \frac{1}{2} (2\pi) \\
 &= \frac{1}{4} M A^2
 \end{aligned}$$

Y por simetria en x e y y, $I_{22} = I_{11} = \frac{1}{4} M A^2$

Luego

$$\begin{aligned}
 I_{33} &= \int_A^{\infty} \int_0^A \int_0^{2\pi} (\rho^2 + y^2) dxdy d\theta = \rho \int_0^A \int_0^{2\pi} r^2 r dr d\theta = \rho \int_0^A \int_0^{2\pi} r^3 dr d\theta \\
 &= \frac{M}{\pi A^2} (2\pi) \frac{1}{4} r^4 \Big|_0^A \\
 &= \frac{1}{4} \frac{M}{\pi A^2} (2\pi) A^4 \\
 &= \frac{1}{2} M A^2
 \end{aligned}$$

Por lo tanto, el tensor de inercia es:

$$I_{in} = \begin{pmatrix} \frac{1}{4} M A^2 & 0 & 0 \\ 0 & \frac{1}{4} M A^2 & 0 \\ 0 & 0 & \frac{1}{2} M A^2 \end{pmatrix}$$

Los ejes principales son los (autovectores) del tensor de inercia.

Primero, obtenemos la ecuación característica:

$$\begin{vmatrix} \frac{1}{4}MA^2 - \lambda & 0 & 0 \\ 0 & \frac{1}{4}MA^2 - \lambda & 0 \\ 0 & 0 & \frac{1}{2}MA^2 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda_1 = \frac{1}{4}MA^2, \quad \lambda_2 = \frac{1}{4}MA^2 \quad y \quad \lambda_3 = \frac{1}{2}MA^2$$

Así, los autovectores correspondientes a cada λ son:

- Para $\lambda_1 = \lambda_2 = \frac{1}{4}MA^2$

$$\begin{pmatrix} \frac{1}{4}MA^2 - \lambda_1 & 0 & 0 \\ 0 & \frac{1}{4}MA^2 - \lambda_2 & 0 \\ 0 & 0 & \frac{1}{2}MA^2 - \lambda_2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}MA^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A_1 = \text{libre}$$

$$A_2 = \text{libre}$$

$$A_3 = 0$$

$$\therefore \vec{A} = \begin{pmatrix} s \\ t \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix}}_{\text{dos ejes principales}} + \underbrace{\begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}}_{\text{dado que } s, t \in \mathbb{R}}$$

- Para $\lambda_3 = \frac{1}{2}MA^2$

$$\begin{pmatrix} \frac{1}{4}MA^2 - \frac{1}{2}MA^2 & 0 & 0 \\ 0 & \frac{1}{4}MA^2 - \frac{1}{2}MA^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{1}{2}MA^2 & 0 & 0 \\ 0 & \frac{1}{2}MA^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow B_1 = 0$$

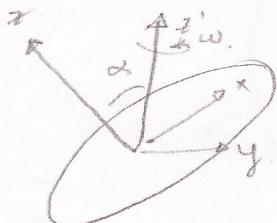
$$B_2 = 0$$

$B_3 = \text{libre}$ → eje principal

$$\therefore \vec{B} = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \text{ donde } u \in \mathbb{R}.$$

Por lo tanto, los ejes principales son $(s, 0, 0)$, $(0, t, 0)$ y $(0, 0, u)$ donde $s, t, u \in \mathbb{R}$, los cuales son x, y y z .

b) Encuentra el vector de momento angular (magnitud y dirección)



Si proyectamos \vec{w} en los ejes principales:

$$w_1 = w \cos(90 - \alpha) = w \sin \alpha.$$

$$w_2 = 0$$

$$w_3 = w \cos \alpha.$$

Así, de $L_a = I_{ab}w_b$, las componentes del momento angular son

$$L_1 = I_{11}w_1 = \frac{1}{4}MA^2w \sin \alpha.$$

$$L_2 = I_{22}w_2 = 0$$

$$L_3 = I_{33}w_3 = \frac{1}{2}MA^2w \cos \alpha$$

Por lo tanto, $\vec{L} = \frac{1}{2}MA^2w (\frac{1}{2} \sin \alpha, 0, \cos \alpha)$

Y su magnitud es

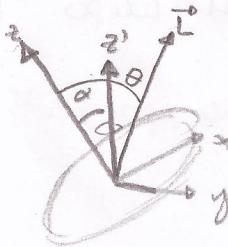
$$\|\vec{L}\| = \sqrt{\frac{1}{4}MA^4w^2 \left(\frac{1}{4} \sin^2 \alpha + 0 + \cos^2 \alpha \right)}$$

$$= \frac{1}{2}MA^2w \sqrt{\frac{1}{4} \sin^2 \alpha + \cos^2 \alpha}$$

$$= \frac{1}{2}MA^2w \sqrt{1 - \frac{3}{4} \sin^2 \alpha}$$

La dirección de \vec{L} está dada por el ángulo θ que hace con respecto a \vec{z} , donde

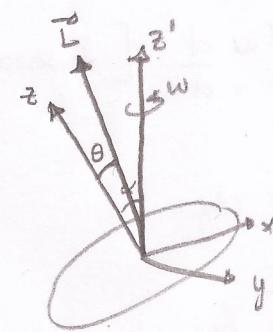
$$\theta = \tan^{-1} \left(\frac{L_1}{L_3} \right)$$



$$= \tan^{-1} \left(\frac{\frac{1}{4}MA^2w\sin\alpha}{\frac{1}{2}MA^2w\cos\alpha} \right)$$

$$= \tan^{-1} \left(\frac{1}{2} \tan\alpha \right)$$

Por lo tanto, $\theta = \tan^{-1} \left(\frac{1}{2} \tan\alpha \right)$



D
en
continuar
la
siguiente
sección.

c) ¿Cuál es la magnitud y dirección de la torca relativa al sistema de referencia del cuerpo (x, y, z) ?

El torque está dado por $\vec{\tau} = \frac{d\vec{L}}{dt}$

Así,

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d}{dt} \left[\frac{1}{2} M A^2 w \left(\frac{1}{2} \sin \alpha, 0, \cos \alpha \right) \right]$$

$$\vec{\tau} = \frac{1}{2} M A^2 w \frac{d}{dt} \left[\frac{1}{2} \sin \alpha \hat{i} + \cos \alpha \hat{k} \right]$$

Pero $\alpha = \alpha t$.

$$\therefore \vec{\tau} = 0 \quad //$$