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Tarea #3 Mecánica Analítica.

1.- Definimos el lagrangiano $L = L(\ddot{x}^A, \dot{x}^A, x^A)$

Definimos la acción como:

$$S[x^A] = \int_{t_0}^{t_f} L(\ddot{x}^A, \dot{x}^A, x^A)\, dt$$

El cambio en la acción es:

$$\delta S[x^A] = \int_{t_0}^{t_f} \delta L(\ddot{x}^A, \dot{x}^A, x^A)\, dt$$

Como se vio en clases es posible aproximar $\delta f(x)$:

$$\delta f(x) = f'(x)\,\delta x + O(\delta x^2)$$

Entonces

$$(1)\quad \delta S[x^A] = \int_{t_0}^{t} \left(\frac{\partial L}{\partial \ddot{x}^A} \delta \ddot{x} + \frac{\partial L}{\partial \dot{x}^A} \delta \dot{x}^A + \frac{\partial L}{\partial x} \delta x \right) dt$$

Para el segundo termino de la integral 1:

$$\frac{\partial L}{\partial \dot{x}^A} \delta \dot{x}^A = \frac{\partial L}{\partial \dot{x}^A} \frac{d}{dt} \delta x$$

$$\int_{t_0}^{t_f} \left(\frac{\partial L}{\partial \dot{x}^A} \frac{d}{dt} \delta x \right) dt \qquad \text{integrando por partes}$$

$$U = \frac{\partial L}{\partial \dot{x}^A} \qquad \int \frac{dv}{dt} = \int \frac{d\,\delta x^A}{dt} dt$$

$$\frac{dU}{dt} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^A} dt \qquad v = \delta x^A$$

$$\frac{\partial L}{\partial \dot{x}^A} \delta x^A \Big|_{t_0}^{t_f} - \int_{t_0}^{t_f} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^A} \right) \delta x^A\, dt \qquad \text{pero } \delta x^A \Big|_{t_0}^{t_f} = 0$$

Entonces solo queda $-\int_{t_0}^{t_f} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^A} \right) \delta x^A dt$

Para el primer termino de la integral 1

$$\frac{\partial L}{\partial \ddot{x}^A}\,\delta \ddot{x}^A = \frac{\partial L}{\partial \ddot{x}^A}\frac{d}{dt}\,\delta \dot{x}^A$$

Entonces :

$$\int_{t_0}^{t_f} \frac{\partial L}{\partial \ddot{x}^A}\frac{d}{dt}\,\delta \dot{x}^A\,dt$$ integrando por partes:

$$u = \frac{\partial L}{\partial \ddot{x}^A} \qquad \frac{dv}{dt} = \frac{d}{dt}(\delta \dot{x}^A)$$

$$\frac{du}{dt} = \frac{d}{dt}\frac{\partial L}{\partial \ddot{x}^A}\,dt \qquad v = \delta \dot{x}^A$$

$$\frac{\partial L}{\partial \ddot{x}^A}\,\delta \dot{x}^A \Big|_{t_0}^{t} - \int_{t_0}^{t_f} \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}^A}\right)\delta \dot{x}^A\,dt$$ integrando por partes otra vez

con $\delta \dot{x}^A = \frac{d}{dt}(\delta x^A)$ $\qquad u = \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}^A}\right) \qquad \frac{dv}{dt} = \frac{d}{dt}(\delta x^A)$

$$\frac{du}{dt} = \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}^A}\right)dt \qquad v = \delta x^A$$

$$\cancel{\frac{\partial L}{\partial \ddot{x}^A}\,\delta \dot{x}^A \Big|_{t_0}^{t_f}}^{\,0} - \cancel{\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}^A}\right)\delta x^A \Big|_{t_0}^{t_f}}^{\,0} + \int_{t_0}^{t_f} \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}^A}\right)\delta x^A\,dt$$

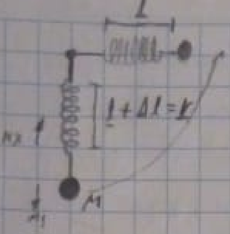
Entonces reescribiendo la integral 1:

$$\delta S[x^A] = \int_{t_0}^{t_f}\left[\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}^A} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^A}\right) + \frac{\partial L}{\partial x^A}\right]\delta x\,dt$$

como $\frac{\delta S}{\delta x^A} = 0$, el integrando debe de ser cero.

$$\therefore \boxed{\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}^A} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^A} + \frac{\partial L}{\partial x^A} = 0}$$ Ecuaciones de Euler Lagrange.

3.-



sistema con 2 grados de libertad, en coor. polares.

dirección radial ; r

dirección angular ; θ

$r = l + \Delta l$

$V = -mgr\cos\theta + \frac{1}{2}K(r-l)^2$

$T = \frac{1}{2}m^2(\dot{r} + r^2\dot{\theta}^2)$

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}K(r-l)^2 + mgr\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^A}\right) - \frac{\partial L}{\partial x^A} = 0$$

para r : $m\ddot{r} + mr\dot{\theta}^2 - \frac{1}{2}K(r-l) + mg\cos\theta = 0$

para θ : $mr\dot{r}\dot{\theta}^2 + mr^2\ddot{\theta} - mgr\,\mathrm{sen}\,\theta = 0$

puntos de equilibrio:

1) $\theta = 0$ $\quad m\ddot{r} = \frac{1}{2}K(r-l) - mg$ $\quad \frac{d}{dt}\cdot\frac{d}{dt}(r)$

$$\iint \frac{2m\,dr}{K(r-l) - mg} = \iint dt^2$$

$$2m\int \frac{dr}{Kr - Kl - mg} = \iint dt^2$$

2) $\dot{\theta} = 0$

$mg\cos\theta = 0$ $\quad \cos\theta = 0$ en $n\pi$ $\quad$ para $n = 0, 1, 2, \ldots$

$m\ddot{r} - \frac{1}{2}K(r-l) + mg \quad = 0$

4.- $L = e^{bt}\left(\frac{1}{2} m \dot{q}^2 - \frac{1}{2} k^2 q^2\right)$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}$

Entonces $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{d}{dt}\left(e^{bt} m \dot{q}\right) = e^{bt} m \ddot{q} + m b e^{bt} \dot{q}$

$\frac{\partial L}{\partial q} = -e^{bt} k^2 q$

entonces el resultado es:

$m e^{bt}\left(\ddot{q} + b\dot{q} + \frac{k^2}{m} q\right) = 0$

$\ddot{q} + b\dot{q} - \frac{k^2}{m} q = 0$ es la ec. de oscilador armonico amortiguado.

con $Q = e^{bt/2} q$

$q = Q e^{-bt/2}$

$q^2 = Q^2 e^{-bt}$

$\frac{d}{dt} q = \dot{q} = \dot{Q} e^{-bt/2} - \frac{1}{2} Q b e^{-bt/2}$

$\dot{q}^2 = \dot{Q}^2 e^{-bt} - \dot{Q} Q b e^{-bt} + \frac{1}{4} Q^2 e^{-bt}$

$L = \frac{1}{2} m e^{bt} \dot{q}^2 - \frac{1}{2} k^2 e^{bt} q^2$

Sustituyendo

$L = \frac{1}{2} m e^{bt}\left(\dot{Q}^2 e^{-bt} - \dot{Q} Q b e^{-bt} + \frac{1}{4} Q^2 e^{-bt}\right) - \frac{1}{2} e^{bt} k^2 Q^2 e^{-bt}$

$L = \frac{1}{2} m\left(\dot{Q}^2 - b\dot{Q}Q\right) + Q^2\left(\frac{m}{8} b^2 - \frac{1}{2} k^2\right)$

No se conserva nada, puesto que $L = L(q, \dot{q}, t)$