El lagrangiano en terminos de Q y $\dot{Q}$

$$L = \frac{1}{2} m\left(\dot{Q}^2 - b\dot{Q}Q + \frac{b^2}{4}Q^2\right) - \frac{1}{2}k^2 Q^2$$

Como el Lagrangiano que tenemos no aparece explisitamente el tiempo podemos sospechar cual sea la simetria

osea tenemos

$$\frac{dL}{dt} = 0$$

$$\frac{dL}{dt} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} \qquad \frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$$

$$\frac{dL}{dt} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}\right)$$

$$\frac{d}{dt}\left(L - \frac{\partial L}{\partial \dot{q}}\dot{q}\right) = 0$$

$\downarrow$

cantidad que se conserva

$$\frac{\partial L}{\partial \dot{Q}} = m\dot{Q} - \frac{bmQ}{2} \qquad \frac{\partial L}{\partial \dot{Q}}\dot{Q} = m\dot{Q}^2 - \frac{bm}{2}Q\dot{Q}$$

$$L - \frac{\partial L}{\partial \dot{Q}}\dot{Q} = \frac{1}{2}m\dot{Q}^2 - \cancel{\frac{bm}{2}\dot{Q}Q} + \frac{b^2 m Q^2}{8} - \frac{1}{2}k^2Q^2 - m\dot{Q}^2 + \cancel{\frac{bm}{2}Q\dot{Q}}$$

$$= -\frac{1}{2}\dot{Q}^2 + Q^2\left(\frac{b^2 m}{8} - \frac{1}{2}k^2\right)$$

Pregunta 4

Considere el siguiente Lagrangiano con un grado de libertad

$L = e^{bt}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}k^2q^2\right)$ k, b, m constantes positivas

a) Encontrar las ecuaciones de Euler-Lagrange

$\frac{\partial L}{\partial q} = -e^{bt}k^2 q \qquad \frac{\partial L}{\partial \dot{q}} = e^{bt} m\dot{q}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = e^{bt} m\ddot{q} + b e^{bt} m\dot{q}$

$e^{bt} m\ddot{q} + b e^{bt} m\dot{q} + e^{bt}k^2 q = 0$

$m\ddot{q} + bm\dot{q} + k^2 q \longrightarrow \ddot{q} + b\dot{q} + \omega_0^2 k q = 0$

$\rightarrow$ El sistema al que se parece es a un pendulo amortiguado.

b) Realiza un cambio de variable $Q = e^{bt/2} q$ y construye el Lagrangiano. Encuentra la simetria continua y deduce la cantidad conservada asociada a ella usando Teorema de Noether. Re-escribe la cantidad en termino de q

Podemos sustituir al nivel del Lagrangiano

$q = Q e^{-bt/2}$

$\dot{q} = \dot{Q} e^{-bt/2} - \frac{b}{2} e^{-bt/2} Q$

$L = e^{bt}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}k^2q^2\right)$

$q^2 = Q^2 e^{-bt}$

$\dot{q}^2 = e^{-bt}\left(\dot{Q} - \frac{b}{2}Q\right)^2 = e^{-bt}\left(\dot{Q}^2 - b\dot{Q}Q + \frac{b^2}{4}Q^2\right)$

$L = \frac{1}{2}m\left(\dot{Q}^2 - b\dot{Q}Q + \frac{b^2}{4}Q^2\right) - \frac{1}{2}k^2 Q^2$

$\theta = 0$ $\qquad$ $\theta = \pi$

$-g + \frac{k}{m} r = 0$ $\qquad$ $g + \frac{k}{m} r = 0$ $\qquad$ Puntos de equilibrio

$\theta_1 = 0 \qquad \theta_2 = \pi$

$r_1 = \frac{gm}{k}$ $\qquad$ $r_2 = -\frac{gm}{k}$ $\qquad$ $r_1 = \frac{gm}{k} \quad r_2 = -\frac{gm}{k}$

c) Haz una exposición alrededor de los puntos de equilibrio y resuelve el sistema.

Para $\theta = 0$ el punto de equilibrio será

$$P_e = \ell + \frac{gm}{k}$$

Para $\theta = \pi$ punto de equilibrio será

$$P_e' = \ell - \frac{gm}{k}$$

las ecuaciones de movimiento que obtuvimos fueron

$$\ddot{r} - (\ell + r)\dot{\theta}^2 - g + \frac{k}{m} r = 0$$

$$(\ell + r)\ddot{\theta} + 2\dot{\theta}\dot{r} + g \operatorname{sen} \theta = 0$$

Linealizamos $\cos\theta \simeq 1$ y $\operatorname{sen}\theta \simeq \theta$

$$\ddot{r} - (\ell + r)\dot{\theta}^2 - g + \frac{k}{m} r = 0$$

$$(\ell + r)\ddot{\theta} + 2\dot{\theta}\dot{r} + g\theta = 0$$

Tomamos punto de equilibrio

$$\left(\ell + \frac{gm}{k}\right)\ddot{\theta} + 2\dot{\theta}\left(\ell + \frac{gm}{k}\right)^{\cdot} + g\theta = 0$$

$$\left(\ell + \frac{gm}{k}\right)\ddot{\theta} + g\theta = 0 \longrightarrow \text{Ecuación de de un péndulo}$$

$$\ddot{\theta} = -\omega^2\theta \qquad \theta = A\cos(\omega\theta - \delta) \quad \text{donde } \omega = \sqrt{\frac{gk}{k\ell + gm}}$$

obtenemos las ecuaciones de movimiento

$$\frac{\partial \mathcal{L}}{\partial r} = m(e+r)\dot{\theta}^2 + mg\cos\theta - kr$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r} \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = m\ddot{r}$$

$$m\ddot{r} - m(e+r)\dot{\theta}^2 + mg\cos\theta - kr = 0$$

$$\ddot{r} - (e+r)\dot{\theta}^2 - g\cos\theta + \omega_0^2 r = 0$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = mg(e+r)\operatorname{sen}\theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m(e+r)^2\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = m(e+r)^2\ddot{\theta} + 2m(e+r)\dot{\theta}\dot{r}$$

$$m(e+r)^2\ddot{\theta} + 2m(e+r)\dot{\theta}\dot{r} + mg(e+r)\operatorname{sen}\theta = 0$$

$$(e+r)\ddot{\theta} + 2\dot{\theta}\dot{r} + g\operatorname{sen}\theta = 0$$

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los ecuaciones de Movimiento son

$$\ddot{r} - (e+r)\dot{\theta}^2 - g\cos\theta + \omega_0^2 r = 0$$

$$(e+r)\ddot{\theta} + 2\dot{\theta}\dot{r} + g\operatorname{sen}\theta = 0$$

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b) Encuentre los puntos de equilibrio y describe su estabilidad

para encontrar los puntos de equilibrio

$$\ddot{r} = \dot{r} = \ddot{\theta} = \dot{\theta} = 0$$

$$-g\cos\theta + \frac{k}{m}r = 0$$

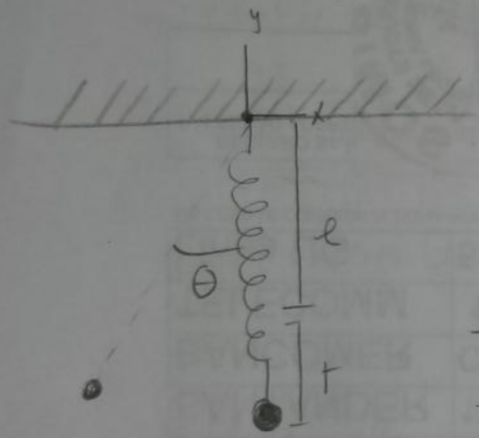
$$g\operatorname{sen}\theta = 0 \Longrightarrow \theta = n\pi \quad n = 0, 1 \ldots \qquad \theta = 0 \text{ y } \theta = \pi$$

sustituimos para obtener los puntos en r

Pregunta 3

A un pendulo simple se remplaza por un resorte de longitud en reposo l y constante k

a) Construye el Lagrangiano del sistema y deducir las ecuaciones de Euler - Lagrange.



$$\mathcal{L} = T - V$$

donde $T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$

$\longrightarrow$ 2 grados de libertad

$\longrightarrow$ hacemos cambio de variable

$X = R \operatorname{sen} \theta$ donde $R = (e + r)$

$y = -R \cos \theta$

$x = (e+r) \operatorname{sen} \theta \qquad \dot{x} = (e+r)\cos\theta\, \dot{\theta} + \dot{r} \operatorname{sen}\theta$

$y = -(e+r) \cos \theta \qquad \dot{y} = (e+r)\operatorname{sen}\theta\, \dot{\theta} - \dot{r}\cos\theta$

$\dot{x}^2 = (e+r)^2 \cos^2\theta\, \dot{\theta}^2 + \cancel{2(e+r)\cos\theta\, \dot{\theta}\, \dot{r} \operatorname{sen}\theta} + \dot{r}^2 \operatorname{sen}^2\theta$

$\dot{y}^2 = (e+r)^2 \operatorname{sen}^2\theta\, \dot{\theta}^2 - \cancel{2(e+r)\cos\theta\, \dot{\theta}\, \dot{r} \operatorname{sen}\theta} + \dot{r}^2 \cos^2\theta$

$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m\left((e+r)^2 \dot{\theta}^2 + \dot{r}^2\right)$

Para el potencial sera debido al potencial gravitacional + el del resorte

$V = mgy + \frac{1}{2} k r^2$

$V(\theta, r) = -mg(e+r)\cos\theta + \frac{1}{2} k r^2$

$\mathcal{L}(r,\theta) = \frac{1}{2} m\left((e+r)^2 \dot{\theta}^2 + \dot{r}^2\right) + mg(e+r)\cos\theta - \frac{1}{2} k r^2$

Jugamos, con indices contraidos

$$\delta S = \frac{1}{2}\int \left(g_{pa} \left\{ {p \atop bc} \right\} + g_{pa} \left\{ {p \atop bc} \right\} \right) \dot{q}^a \dot{q}^b \delta q^c + 2 g_{ab} \delta \dot{q}^a \dot{q}^b \, dt$$

$$\delta S = \int g_{pa} \left\{ {p \atop bc} \right\} \dot{q}^a \dot{q}^b \delta q^c + g_{ab} \delta \dot{q}^a \dot{q}^b \, dt$$

$$\delta S = \int g_{ab} \left\{ {b \atop pc} \right\} \dot{q}^a \dot{q}^p \delta q^c + g_{ab} \delta \dot{q}^a \dot{q}^b \, dt$$

$$\delta S = \int g_{ap} \left\{ {p \atop bc} \right\} \dot{q}^a \dot{q}^b \delta q^c + g_{ap} \delta \ddot{q}^a \dot{q}^p \, dt$$

$$\delta S = \int g_{ap} \left\{ {a \atop bc} \right\} \dot{q}^p \dot{q}^b \delta q^c + g_{ap} \delta \dot{q}^a \dot{q}^p \, dt = 0$$

Como vemos podemos sacar factor comun, la metrica y la variacion de q^p

$$\left\{ {a \atop bc} \right\} \dot{q}^b \dot{q}^c + \ddot{q}^a = 0$$

Con un pequeño analisis tensorial.

Una conexión metrica de 1° especie es la siguente

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

Tomamos otra conexión con indices cambiados

$$[ik,j] = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}\right)$$

si sumamos ambas

$$[ij,k] + [ik,j] = \frac{1}{2}\left(\cancel{\frac{\partial g_{ik}}{\partial x^j}} + \frac{\partial g_{jk}}{\partial x^i} - \cancel{\frac{\partial g_{ij}}{\partial x^k}} + \cancel{\frac{\partial g_{ij}}{\partial x^k}} + \frac{\partial g_{kj}}{\partial x^i} - \cancel{\frac{\partial g_{ik}}{\partial x^j}}\right)$$

$$[ij,k] + [ik,j] = \frac{\partial g_{kj}}{\partial x^i}$$

En terminos de conexiones de 2° especie

$$\left\{ {e \atop ij} \right\} = g^{ek}[ij,k] \longrightarrow g_{em}\left\{ {e \atop ij} \right\} = [ij,m]$$

$$[ij,k] + [ik,j] = \frac{\partial g_{kj}}{\partial x^i}$$

$$\frac{\partial g_{kj}}{\partial x^i} = g_{ej}\left\{ {e \atop ki} \right\} + g_{ek}\left\{ {e \atop ij} \right\}$$

Retomamos

$$\delta S = \int \left(\frac{\partial g_{ab}}{\partial q^u} \frac{\partial q^u}{\partial q^c} \delta q^c \dot{q}^a \dot{q}^b + 2 g_{ab} \delta \dot{q}^a \dot{q}^b \right) dt$$

donde

$$\frac{\partial g_{ab}}{\partial q^u} = g_{pa}\left\{ {p \atop bu} \right\} + g_{pb}\left\{ {p \atop au} \right\}$$

$$\delta S = \int \left(\left(g_{pa}\left\{ {p \atop bu} \right\} + g_{pb}\left\{ {p \atop au} \right\} \right) \delta^u_c \dot{q}^a \dot{q}^b \delta q^c + 2 g_{ab} \delta \dot{q}^a \dot{q}^b \right) dt$$

$$\delta S = \int_{t_0}^{t} \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\delta q_i \, dt + \int_{t_0}^{t} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \, dt + \int_{t_0}^{t} \frac{\partial L}{\partial q_i}\delta q_i \, dt$$

$$\int_{t_0}^{t} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \, dt \qquad u = \frac{\partial L}{\partial \dot{q}_i} \qquad dv = \delta \dot{q}_i \, dt$$

$$du = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)dt \qquad v = \delta q_i$$

$$\int_{t_0}^{t} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \, dt = \underbrace{\frac{\partial L}{\partial \dot{q}_i}\cancel{\delta q_i}\Big|_{t_0}^{t}}_{0} - \int_{t_0}^{t} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i \, dt$$

$$\delta S = \int_{t_0}^{t} \left\{ \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\delta q_i - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i + \frac{\partial L}{\partial q_i}\delta q_i \right\} dt$$

$$\delta S = 0$$

$$\int_{t_0}^{t} \left\{ \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial L}{\partial q_i} \right\} \delta q_i \, dt = 0$$

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial L}{\partial q_i} = 0$$

Pregunta 2

Encuentra del modelo sigma las Ecuaciones de Euler-Lagrange

$$L(\dot{q}, q, t) = \frac{1}{2} g_{ab}(q^c)\, \dot{q}^a \dot{q}^b$$

$$S = \int L(\dot{q}, q, t)\, dt \qquad \delta S = \int \delta L(\dot{q}, q, t)\, dt = 0$$

$$\delta S = \frac{1}{2}\int \left(\delta g_{ab}(q^c)\, \dot{q}^a \dot{q}^b + g_{ab}(q^c)\, \delta \dot{q}^a \dot{q}^b + g_{ab}(q^c)\, \delta \dot{q}^b \dot{q}^a\right) dt$$

$$\delta S = \frac{1}{2}\int \left(\frac{\partial g_{ab}}{\partial q^\mu}\delta q^\mu \dot{q}^a \dot{q}^b + 2\, g_{ab}(q^c)\, \delta \dot{q}^a \dot{q}^b\right) dt$$

Tarea # 3 Juan Manuel López Vega

Pregunta 1

Deduce paso a paso las Ecuaciones de Euler – Lagrange para un lagrangiano que dependa de la aceleración.

$$L = L(\ddot{q}_i, \dot{q}_i, q_i; t)$$

Partiendo de obtener la acción y extremisarla

$$S = \int_{t_0}^{t} L(\ddot{q}_i, \dot{q}_i, q_i; t)\, dt$$ por el principio de mínima acción $$\delta S[q_i] = 0$$

$$\delta S = \int_{t_0}^{t} \delta L(\ddot{q}_i, \dot{q}_i, q_i; t)\, dt$$

$$\delta S = \int_{t_0}^{t} \left[\frac{\partial L}{\partial \ddot{q}_i}\delta \ddot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \frac{\partial L}{\partial q_i}\delta q_i \right] dt$$

$$\delta S = \int_{t_0}^{t} \frac{\partial L}{\partial \ddot{q}_i}\delta \ddot{q}_i\, dt + \int_{t_0}^{t} \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\, dt + \int_{t_0}^{t} \frac{\partial L}{\partial q_i}\delta q_i\, dt$$

integrando por partes los 2 primeros miembros

$$\int_{t_0}^{t} \frac{\partial L}{\partial \ddot{q}_i}\delta \ddot{q}_i\, dt \qquad u = \frac{\partial L}{\partial \ddot{q}_i} \qquad dv = \delta \ddot{q}_i\, dt$$

$$du = \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) \qquad v = \delta \dot{q}_i$$

$$\int_{t_0}^{t} \frac{\partial L}{\partial \ddot{q}_i}\delta \ddot{q}_i\, dt = \frac{\partial L}{\partial \ddot{q}_i}\delta \dot{q}_i \Big|_{t_0}^{t} - \int_{t_0}^{t} \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\delta \dot{q}_i\, dt$$

$$\delta \dot{q}_i = \frac{d}{dt}\cancel{\delta q_i}^{\,0}\Big|_{t_0}^{t}$$ como q_i está fijo en los extremos

$$\int_{t_0}^{t} \frac{\partial L}{\partial \ddot{q}_i}\delta \ddot{q}_i\, dt = -\int_{t_0}^{t} \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\delta \dot{q}_i\, dt \qquad u = \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) \qquad dv = \delta \dot{q}_i\, dt$$

$$du = \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) dt \qquad v = \delta q_i$$

$$\int_{t_0}^{t} \frac{\partial L}{\partial \ddot{q}_i}\delta \ddot{q}_i\, dt = -\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\cancel{\delta q_i}^{\,0}\Big|_{t_0}^{t} + \int_{t_0}^{t} \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\delta q_i\, dt$$