TAREA 4 MECÁNICA ANALÍTICA. Saúl Ali Castillo Rodríguez.

Pregunta 1.

a) Dadas las transformaciones:

$$q_1 = x\cos\mu + P_y \sin\mu \qquad q_2 = y\cos\mu + P_x \sin\mu$$

$$P_1 = P_x\cos\mu - y\sin\mu \qquad P_2 = P_y\cos\mu - x\sin\mu$$

Teorema: El corchete de Poisson es invariante bajo transformaciones canónicas. Inversamente, cualquier transformación que conserve la estructura del corchete de Poisson es canónica. Es decir, dado un conjunto de transformaciones Q_i, P_i, con $Q_i = Q_i(q,p,t)$ y $P_i = P_i(q,p,t)$, donde p y q son las viejas coordenadas y momentos conjugados respectivamente, son canónicas si:

$$\{Q_i, Q_j\} = \{P_i, P_j\} = 0 \quad y \quad \{Q_i, P_j\} = \delta_{ij}$$

Entonces.

- $\{q_1, q_2\} = \frac{\partial q_1}{\partial x}\frac{\partial q_2}{\partial P_x} + \frac{\partial q_1}{\partial y}\frac{\partial q_2}{\partial P_y} - \frac{\partial q_1}{\partial P_x}\frac{\partial q_2}{\partial x} - \frac{\partial q_1}{\partial P_y}\frac{\partial q_2}{\partial y}$

$$= \frac{\partial}{\partial x}(x\cos\mu + P_y\sin\mu)\frac{\partial}{\partial P_x}(y\cos\mu + P_x\sin\mu) + 0 + 0 - \frac{\partial}{\partial P_y}(x\cos\mu + P_y\sin\mu)\frac{\partial}{\partial y}(y\cos\mu + P_x\sin\mu)$$

$$= \cos\mu\sin\mu - \sin\mu\cos\mu = 0$$

- $\{P_1, P_2\} = \frac{\partial P_1}{\partial x}\frac{\partial P_2}{\partial P_x} + \frac{\partial P_1}{\partial y}\frac{\partial P_2}{\partial P_y} - \frac{\partial P_1}{\partial P_x}\frac{\partial P_2}{\partial x} - \frac{\partial P_1}{\partial P_y}\frac{\partial P_2}{\partial y}$

$$= 0 - \sin\mu\cos\mu - \cos\mu(-\sin\mu) + -0 = -\sin\mu\cos\mu + \cos\mu\sin\mu = 0$$

- $\{q_1, P_2\} = \frac{\partial q_1}{\partial x}\frac{\partial P_2}{\partial P_x} + \frac{\partial q_1}{\partial y}\frac{\partial P_2}{\partial P_y} - \frac{\partial q_1}{\partial P_x}\frac{\partial P_2}{\partial x} - \frac{\partial q_1}{\partial P_y}\frac{\partial P_2}{\partial y} = 0 + 0 - 0 - 0 = 0$

$$\circ \{Q_1, P_1\} = \frac{\partial q_1}{\partial x}\frac{\partial P_1}{\partial P_x} + \frac{\partial q_1}{\partial y}\frac{\partial P_1}{\partial P_y} - \frac{\partial q_1}{\partial P_x}\frac{\partial P_1}{\partial x} - \frac{\partial q_1}{\partial P_y}\frac{\partial P_1}{\partial y}$$

$$= \cos\mu \cos\mu + 0 - 0 - \sin\mu(-\sin\mu) = \cos^2\mu + \sin^2\mu = 1$$

$$\circ \{Q_2, P_2\} = \frac{\partial Q_2}{\partial x}\frac{\partial P_2}{\partial P_x} + \frac{\partial Q_2}{\partial y}\frac{\partial Q_2}{\partial P_y} - \frac{\partial Q_2}{\partial P_x}\frac{\partial P_2}{\partial x} - \frac{\partial Q_2}{\partial P_y}\frac{\partial P_2}{\partial y} \qquad (q_1 = Q_1,\ q_2 = Q_2)$$

$$= 0 + \cos\mu\cos\mu - \sin\mu(-\sin\mu) - 0 = \cos^2\mu + \sin^2\mu = 1$$

y mediante las propiedades de los corchetes de Poisson se tiene:

$\{q_2, q_1\} = -\{q_1, q_2\} = -0 = 0$

$\{P_2, P_1\} = -\{P_1, P_2\} = -0 = 0$ y dado que $\delta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ se tiene entonces que las transformaciones q_1, q_2, P_1, P_2 son canónicas ya que preservan la estructura del corchete de Poisson.

b) $H = (q_1^2 + q_2^2 + P_1^2 + P_2^2)/2$ (Hamiltoniano original)

Dadas las transformaciones canónicas en a) obtenemos la matriz:

$$\begin{vmatrix} \cos\mu & 0 & 0 & \sin\mu \\ 0 & \cos\mu & \sin\mu & 0 \\ 0 & -\sin\mu & \cos\mu & 0 \\ -\sin\mu & 0 & 0 & \cos\mu \end{vmatrix} = A$$

cuyo determinante está dado por:

$$\det(A) = \cos\mu \begin{vmatrix} \cos\mu & \sin\mu & 0 \\ -\sin\mu & \cos\mu & 0 \\ 0 & 0 & \cos\mu \end{vmatrix} - \sin\mu \begin{vmatrix} 0 & \cos\mu & \sin\mu \\ 0 & -\sin\mu & \cos\mu \\ -\sin\mu & 0 & 0 \end{vmatrix}$$

$$\det(A) = \cos\mu\left(\cos\mu\begin{vmatrix}\cos\mu & 0\\ 0 & \cos\mu\end{vmatrix} - \sin\mu\begin{vmatrix}-\sin\mu & 0\\ 0 & \cos\mu\end{vmatrix}\right) - \sin\mu\left(-\cos\mu\begin{vmatrix}0 & \cos\mu\\ -\sin\mu & 0\end{vmatrix} + \sin\mu\begin{vmatrix}0 & -\sin\mu\\ -\sin\mu & 0\end{vmatrix}\right) = \cos\mu\left(\cos\mu\cos^2\mu - \sin\mu(-\sin\mu\cos\mu)\right) - \sin\mu\left(-\cos\mu(\cos\mu\sin\mu) + \sin\mu(-\sin^2\mu)\right)$$

$$= \cos\mu(\cos^3\mu + \sin^2\mu\cos\mu) - \sin\mu(-\cos^2\mu\sin\mu - \sin^3\mu)$$

$$= \cos^4\mu + \sin^2\mu\cos^2\mu + \sin^2\mu\cos^2\mu + \sin^4\mu = \cos^2\mu(\cos^2\mu + \sin^2\mu) + \sin^2\mu(\cos^2\mu + \sin^2\mu)$$

$= \cos^2\mu + \sin^2\mu = 1$ $\therefore$ A es invertible y mediante el calculo de la matriz adjunta se obtuvo (mediante saftware) la inversa de A: A^{-1}:

$$A^{-1} = \begin{pmatrix}\cos\mu & 0 & 0 & -\sin\mu\\ 0 & \cos\mu & -\sin\mu & 0\\ 0 & \sin\mu & \cos\mu & 0\\ \sin\mu & 0 & 0 & \cos\mu\end{pmatrix}$$

, por lo tanto las transformaciones inversas son:

$$x = q_1\cos\mu - P_2\sin\mu \qquad y = q_2\cos\mu - P_1\sin\mu$$

$$P_x = q_2\sin\mu + P_1\cos\mu \qquad P_y = q_1\sin\mu + P_2\cos\mu$$

Para encontrar la función generadora F tal que:

$$K = H + \frac{\partial F_1}{\partial t}$$

, donde K es el nuevo Hamiltoniano.

Hacemos uso de las ecuaciones diferenciales: acopladas:

$$P_x = \frac{\partial F_1}{\partial x}, \quad P_y = \frac{\partial F_1}{\partial y}, \quad P_1 = -\frac{\partial F_1}{\partial q_1}, \quad P_2 = -\frac{\partial F}{\partial q_2}$$

* Nota Erré, esto no era necesario.

$$H = \frac{q_1^2+q_2^2+P_1^2+P_2^2}{2} \Rightarrow$$

$$q_1^2 = (x\cos\mu + P_y \sin\mu)^2$$
$$= x^2\cancel{\cos^2\mu} + \cancel{2x\cos\mu P_y \sin\mu} + P_y^2\cancel{\sin^2\mu}$$

$$q_2^2 = (y\cos\mu + P_x\sin\mu)^2 = y^2\cancel{\cos^2\mu} + \cancel{2y\cos\mu P_x\sin\mu} + P_x^2\cancel{\sin^2\mu}$$

$$P_1^2 = (P_x\cos\mu - y\sin\mu)^2 = P_x^2\cancel{\cos^2\mu} - \cancel{2P_x\cos\mu y\sin\mu} + y^2\cancel{\sin^2\mu}$$

$$P_2^2 = (P_y\cos\mu - x\sin\mu)^2 = P_y^2\cancel{\cos^2\mu} - \cancel{2P_y\cos\mu x\sin\mu} + x^2\cancel{\sin^2\mu}$$

$$\Rightarrow q_1^2+q_2^2+P_1^2+P_2^2 = x^2+y^2+P_x^2+P_y^2$$

$$\boxed{\therefore H = \frac{x^2+y^2+P_x^2+P_y^2}{2}}$$

c) De las ecuaciones: $\dot{Q}_1 = \frac{\partial H}{\partial P_1}$, $\dot{P}_1 = -\frac{\partial H}{\partial Q_1}$ con la restricción $y = P_y = 0$

Se tiene $H = \frac{x^2+P_x^2}{2}$, por lo tanto: se tienen las siguientes ecuaciones de movimiento.

$$\dot{p}_1 = \frac{\partial H}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{x^2+P_x^2}{2}\right) = -\left(x + \frac{P_x^2}{2}\right), \quad \dot{p}_2 = 0$$

$$\dot{q}_1 = \frac{\partial H}{\partial P_x} = \frac{\partial}{\partial P_x}\left(\frac{x^2+P_x^2}{2}\right) = \frac{x^2}{2} + P_x$$

$$\therefore \dot{p}_1 = -x - \frac{P_x^2}{2} \quad y \quad \dot{q}_1 = \frac{x^2}{2} + P_x$$

Pregunta 2.

a) Momentos de inercia y ejes principales:

Considerando la definición de tensor de momento de inercia:

$$I_{ij} = \int_V \rho(\bar{r}')\left(\delta_{ij}\sum_k x_k^2 - x_i x_j\right)dv \qquad \text{con } dv = dx_1 dx_2 dx_3$$
$$dv = dx\,dy\,dz$$

Calculamos los momentos de inercia del disco.

$$I_{11} = \int_V \rho\left(x_1^2 + x_2^2 + x_3^3 - x_1^2\right)dv = \rho\int_V (y^2+z^2)\,dx\,dy\,dz,$$ dado que el disco el delgado podemos despreciar la contribución de la integral en el eje z de modo que:

- $I_{11} = \rho\int_V y^2\,dx\,dy$, en coordenadas polares: $I_{11} = \rho\int_V r^2\sin^2\theta\, r\,dr\,d\theta$

$$I_{11} = \rho\int_0^A\int_0^{2\pi} r^3\sin^2\theta\,d\theta\,dr = \rho\int_0^A r^3\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right)\Big]_0^{2\pi} dr = \rho\int_0^A r^3\left(\frac{1}{2}2\pi - \frac{1}{4}\sin 4\pi + \frac{1}{4}\sin 0\right)dr$$

$$= \rho\int_0^A r^3\pi\,dr = \rho\pi\left(\frac{1}{4}r^4\right)\Big]_0^A = \rho\pi\frac{1}{4}A^4 = \frac{M}{\pi A^2}\frac{1}{4}A^4 = \frac{MA^2}{4}$$

- $I_{22} = \rho\int_V x^2\,dx\,dy = \rho\int_V r^2\cos^2\theta\,d_r\,d\theta = \rho\int_V r^3\cos^2\theta\,d_r\,d\theta = \rho\int_0\int_0 r^3\cos^2\theta\,d\theta\,dr$

$$= \rho\int_0^A r^3\left(\frac{1}{2}\theta + \frac{1}{4}\sin(4/2\theta)\right]_0^{2\pi}\Big)dr = \rho\int_0^A r^3\left[\frac{1}{2}2\pi + \frac{1}{4}\sin(4\pi) - 0 - \frac{1}{4}\sin(0)\right]dr$$

$$= \rho\int_0^A r^3\pi = \rho\pi\frac{1}{4}A^4 = \frac{MA^2}{4}$$

- $I_{33} = \rho\int_V (x^2+y^2)\,dx\,dy = \rho\int_V (r^2\cos^2\theta + \sin^2\theta\, r^2)\,r\,dr\,d\theta = \rho\int_V r^3\,dr\,d\theta = \rho\int_0^A\int_0^{2\pi} r^3\,d\theta\,dr$

$$I_{33} = \rho \int_0^A r^3\theta \Big]_0^{2\pi} dr = \rho \int_0^A r^3 2\pi\, dr = \rho \frac{1}{4} r^4 2\pi \Big]_0^A = \rho \frac{1}{4} A^4 2\pi = \rho \frac{1}{2} A^4 \pi = \frac{M}{\pi A^2} \frac{1}{2} A^4 \cancel{\pi}$$

$\therefore$ 1. $I_{33} = \frac{1}{2} M A^2$

Por lo tanto los momentos de inercia son:

$$I_{11} = I_{22} = \frac{1}{4} M A^2 \quad y \quad I_{33} = \frac{1}{2} M A^2$$

Calculando los componentes del tensor de momento de inercia I_{ik} restantes son:

- $I_{12} = \rho \int_V - x_1 x_2\, dv = -\rho \int_V xy\, dx\, dy = -\rho \int_V r\cos\theta\, r\sin\theta\, r\, dr\, d\theta = -\rho \int_0^A \underbrace{\int_0^{2\pi} r^3 \cos\theta \sin\theta\, d\theta}_{=a} dr$

$$a = \int_0^{2\pi} r^3 \cos\theta \sin\theta\, d\theta = r^3 \int_0^{2\pi} \cos\theta\, u \frac{du}{\cos\theta} = r^3 \int_0^{2\pi} u\, du = r^3 \frac{1}{2} u^2 = r^3 \frac{1}{2} u^2 = r^3 \frac{1}{2} \sin^2\theta \Big]_0^{2\pi} = 0$$

$u = \sin\theta \Rightarrow du = \cos\theta\, d\theta$

$\therefore I_{21} = I_{12} = 0$

- $I_{13} = \rho \int_V -xz\, dv = -\rho \int_V r\cos\theta\, r\, dr\, d\theta = -\rho \int_0^A \int_0^{2\pi} r^2 \cos\theta\, dr\, d\theta = 0 \quad \therefore I_{13} = I_{31} = 0$

- $I_{32} = -\rho \int_V zy\, dx\, dy\, dz = -\rho \int_V y\, dx\, dy = -\rho \int_V r\sin\theta\, r\, dr\, d\theta = -\rho \int_0^A \int_0^{2\pi} r^2 \sin\theta\, d\theta\, dr$

$$= -\rho \int_0^A r^2 \cos\theta \Big]_0^{2\pi} dr = 0 \quad \therefore I_{32} = I_{23} = 0$$

Por lo tanto el tensor de inercia es:

$$I_{ij} = \begin{pmatrix} \frac{1}{4}MA^2 & 0 & 0 \\ 0 & \frac{1}{4}MA^2 & 0 \\ 0 & 0 & \frac{1}{2}MA^2 \end{pmatrix} = \frac{1}{4} M A^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

y los eigenvalores de I_{ij}:

$$\det\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} = (1-\lambda)(1-\lambda)(2-\lambda) \Rightarrow \lambda_1 = 1,\ \lambda_2 = 1 \ \text{y} \ \lambda_3 = 2$$

Evaluando la matriz en $\lambda_1 = \lambda_2 = 1$.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow z=0 \Rightarrow \bar{x} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$, evaluando en $\lambda_3 = 2$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} x=0 \\ y=0 \end{matrix} \Rightarrow \bar{x} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\therefore$ Los eigenvalores de I_{ij} son: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ y $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, los cuales determinan los ejes principales, es decir, los ejes principales son los ejes del sistema coordenado del disco.

b) Vector de momento angular.

El vector de velocidad angular desde el sistema coordenado del disco está dado por:

$$\vec{\omega} = \begin{pmatrix} \omega\cos(\pi/2-\alpha) \\ \omega\cos(\pi/2) \\ \omega\cos\alpha \end{pmatrix} = \begin{pmatrix} \omega\cos(\pi/2-\alpha) \\ 0 \\ \omega\cos(\alpha) \end{pmatrix}$$, donde $\|\vec{\omega}\| = \omega$, $\vec{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$

Dado que $\vec{L} = I\vec{\omega} \Rightarrow L_a = I_{ab}\omega_b = \sum_b I_{ab}\omega_b$, por lo tanto:

$$L_1 = I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 = I_{11}\omega_1 = \frac{1}{4}MA^2\omega\cos(\pi/2-\alpha)$$

$$L_2 = \cancelto{0}{I_{21}}\omega_1 + I_{22}\omega_2 + \cancelto{0}{I_{23}\omega_3} = I_{22}\omega_2 = \frac{1}{4}MA^2\omega\cos(\pi/2) = 0$$

$$L_3 = I_{33}\omega_3 = \frac{1}{2}MA^2\omega\cos(\alpha)$$

Por lo tanto, el vector de momento angular es:

$$\vec{L} = \begin{pmatrix} \frac{1}{4}MA^2 w\cos(\pi/2-\alpha) \\ 0 \\ \frac{1}{2}MA^2 w\cos(\alpha) \end{pmatrix} = \frac{1}{4}MA^2 \begin{pmatrix} w\cos(\pi/2-\alpha) \\ 0 \\ 2w\cos(\alpha) \end{pmatrix} = \frac{1}{4}MA^2 w \begin{pmatrix} \sin(\alpha) \\ 0 \\ 2\cos(\alpha) \end{pmatrix}$$

Cuya magnitud es:

$$\|\vec{L}\| = \sqrt{\left[\frac{1}{4}MA^2 w\cos(\pi/2-\alpha)\right]^2 + \left(\frac{1}{2}MA^2 w\cos\alpha\right)^2}$$

$$= \sqrt{\left(\frac{1}{4}MA^2 w\right)^2 \sin^2(\alpha) + \left(\frac{1}{4}MA^2 w\right)^2 (2\cos(\alpha))^2}$$

$$= \frac{1}{4}MA^2 w\sqrt{\sin^2(\alpha) + 2^2\cos^2(\alpha)} = \frac{1}{4}MA^2 w\sqrt{\sin^2(\alpha) + 4\cos^2(\alpha)} = \|\vec{L}\|$$

c) Magnitud y dirección de la torca relativa al sistema de referencia del disco

$$\text{Si } \frac{d\vec{L}}{dt} = \vec{\tau} \Rightarrow \vec{\tau} = \frac{d\vec{L}}{dt} = \frac{d}{dt}\begin{pmatrix} \frac{1}{4}MA^2 w\sin\alpha \\ 0 \\ \frac{1}{2}MA^2 w\cos(\alpha) \end{pmatrix} = \frac{1}{4}MA^2 \dot{w} \begin{pmatrix} \sin(\alpha) \\ 0 \\ 2\cos(\alpha) \end{pmatrix}$$

$$\text{y } \|\vec{\tau}\| = \frac{1}{4}MA^2 \dot{w}\sqrt{\sin^2(\alpha) + 4\cos^2(\alpha)}$$