Transformada de la derivada

❑ Teorema

Si $f, f', .. f^{(n-1)}$ son continuas para $t \geq 0$ y son de orden exponencial y si $f^{(n)}(t)$ es continua por tramos para $t \geq 0$, entonces

$$\boldsymbol{\mathcal{L}(f^{(n)}(t)) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0)}$$

Donde $\boldsymbol{F(s) = \mathcal{L}(f(t))}$

- Caso particular para n=1

$$\mathcal{L}(f'(t)) = sF(s) - f(0)$$

Si: $\boldsymbol{Y_s = L(y(t))}$

$$\boldsymbol{L\left(y'(t)\right) = s.Y_s - y(0)}$$

- Caso particular para n=2

$$\mathcal{L}(f''(t)) = s^2 F(s) - sf(0) - f'(0)$$

$$\boldsymbol{L\left(y''(t)\right) = s^2.Y_s - s.y(0) - y'(0)}$$

$$\boldsymbol{L\left(y'''(t)\right) = s^3.Y_s - s^2.y(0) - s.y'(0) - y''(0)}$$

Transformada de la derivada

Ejemplo:

$$\mathcal{L}(f'(t)) = sF(s) - f(0)$$

$$Y_s = L(y(t))$$

$$L(y'(t)) = s.Y_s - y(0)$$

Resuelva el problema de valores iniciales PVI:

$$y' + y = 1 \ , \ y(0) = 0$$

$$L(y') + L(y) = L(1)$$

$$(sY_s - \underbrace{y(0)}_{0}) + (Y_s) = \frac{1}{s} \Rightarrow Y_s(s+1) = \frac{1}{s}$$

$$Y_s = \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \Rightarrow 1 = A(s+1) + Bs \begin{cases} s=0 \to A=1 \\ s=-1 \to B=-1 \end{cases}$$

$$L(y) = \frac{1}{s} - \frac{1}{s+1} \Rightarrow L\{y\} = L\{1\} - L\{e^{-t}\} \Rightarrow y = 1 - e^{-t}$$

Transformada de la derivada

$$\mathcal{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

$$\boldsymbol{L(y''(t)) = s^2.Y_s - s.y(0) - y'(0)}$$

$$\boldsymbol{L(y'(t)) = s.Y_s - y(0)}$$

$$* \, L(e^{\mp at}) = \frac{1}{s \pm a}$$

Ejemplo: Resuelva el problema de valores iniciales PVI:

$$y'' - 3y' + 2y = e^{-4t}\ ,\ y(0) = 1,\ y'(0) = 5$$

$$L(y'') - 3L(y') + 2L(y) = L(e^{-4t})$$

$$(s^2 Y_s - s.\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{5}) - 3(s.Y_s - \underbrace{y(0)}_{1}) + 2Y_s = \frac{1}{s+4}$$

$$Y_s(s^2 - 3s + 2) = s + 2 + \frac{1}{s+4} = \frac{(s+2)(s+4) + 1}{s+4}$$

$$Y_s = \frac{s^2 + 6s + 9}{(s+4)\underbrace{(s^2 - 3s + 2)}_{(s-2)(s-1)}} = \frac{A}{s+4} + \frac{B}{s-2} + \frac{C}{s-1} \quad \ldots (*)$$

$$\Rightarrow s^2 + 6s + 9 = A(s-2)(s-1) + B(s+4)(s-1) + C(s+4)(s-2)$$

$s = 1 \rightarrow 16 = -5C \rightarrow C = -16/5$

$s = 2 \rightarrow 25 = 6B \rightarrow B = 25/6$

$s = -4 \rightarrow 1 = 30A \rightarrow A = 1/30$

en $(*)$:

$$L(Y_{(t)}) = \frac{1}{30}\cdot\frac{1}{s+4} + \frac{25}{6}\cdot\frac{1}{s-2} - \frac{16}{5}\cdot\frac{1}{s-1}$$

$$L(y_{(t)}) = \frac{1}{30}L(e^{-4t}) + \frac{25}{6}L(e^{2t}) - \frac{16}{5}L(e^{t})$$

$$y_{(t)} = \frac{1}{30}\cdot e^{-4t} + \frac{25}{6}\cdot e^{2t} - \frac{16}{5}\cdot e^{t}$$

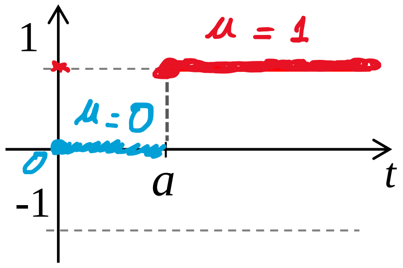
Respuesta: $y(t) = -\frac{16}{5}e^t + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}$

Función escalón unitario y su transformada

- Definición

La **función escalón unitario** o llamada también **función Heaviside** $u(t-a)$ se define como

$$u(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & a \leq t \end{cases}$$



$$\mathcal{L}(\boldsymbol{u}(\boldsymbol{t}-\boldsymbol{a})) = \overbrace{\boxed{\int_0^\infty e^{-st}f(t)dt}}^{\mathcal{L}(f(t))} = \overbrace{\int_0^a e^{-st}(\mathbf{0})dt}^{0} + \int_a^\infty e^{-st}(\mathbf{1})dt = \int_a^\infty e^{-st}dt$$

$$= \frac{1}{-s}e^{-st}\Big|_a^\infty = -\frac{1}{s}\underbrace{e^{-\infty}}_{0} - \left(-\frac{1}{s}e^{-as}\right) = \frac{1}{s}.e^{-as}$$

$$* \int e^{\alpha t}dt = \frac{e^{\alpha t}}{\alpha}$$

$$* e^{-\infty} = 0$$

Entonces:

$$\boxed{\mathcal{L}(\boldsymbol{u}(\boldsymbol{t}-\boldsymbol{a})) = \frac{\boldsymbol{e}^{-\boldsymbol{a}\boldsymbol{s}}}{\boldsymbol{s}}}$$

Función escalón unitario y su transformada

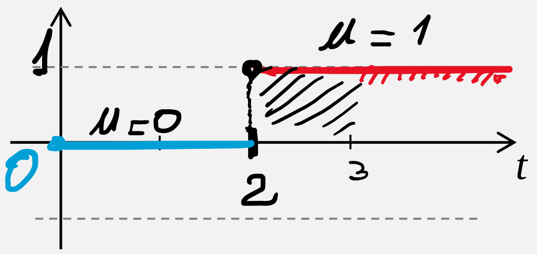
Ejemplo:

Graficar la función f y halle la transformada de Laplace.

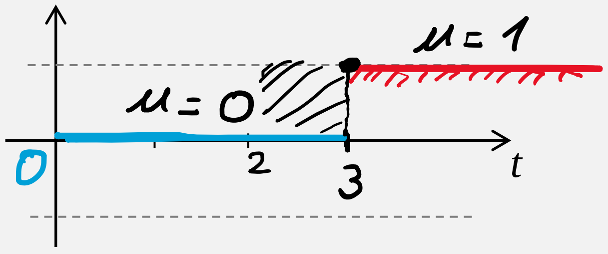
$$f(t) = u(t-2) - u(t-3)$$

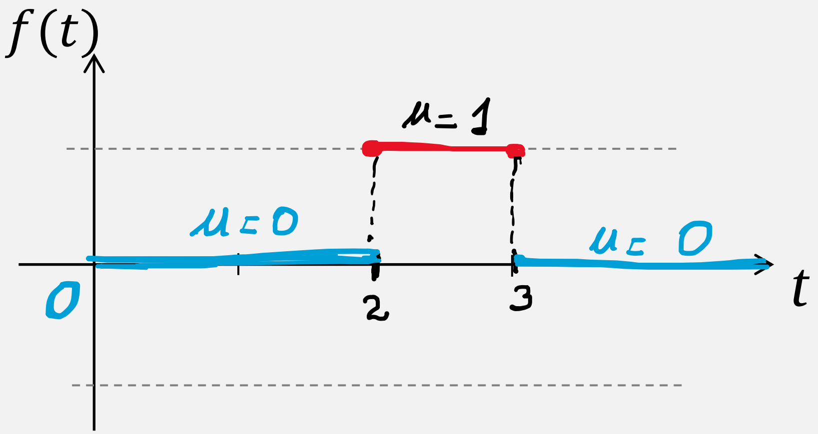
$$\boxed{\mathcal{L}(u(t-a)) = \frac{e^{-as}}{s}}$$

Solución: $u(t-2)$:



$u(t-3)$





$$L(f(t)) = L(u(t-2)) - L(u(t-3))$$

$$L(f(t)) = \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s}$$

$$= \frac{e^{-2s} - e^{-3s}}{s}.$$

Función escalón unitario y su transformada

$$\mathcal{L}(u(t-a)) = \frac{e^{-as}}{s}$$

Ejemplo:

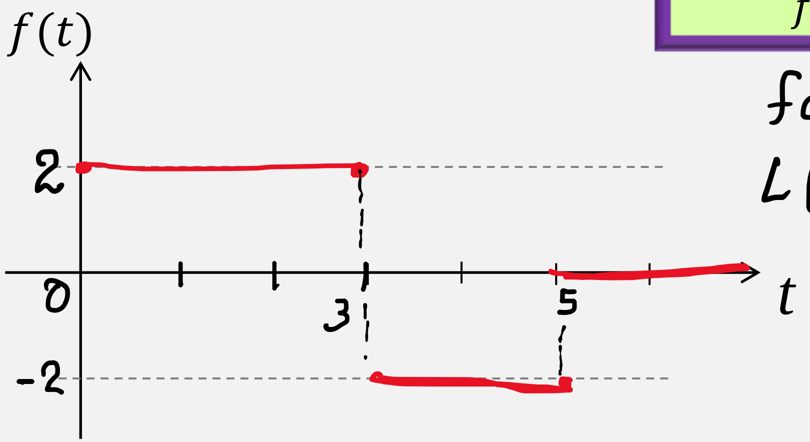
Encuentre la transformada de Laplace de la función f.

$$f(t) = \begin{cases} 2 & , 0 \leq t < 3 \\ -2 & , 3 \leq t < 5 \\ 0 & , t \geq 5 \end{cases}$$

Si se tiene:

$$\boldsymbol{f(t)} = \begin{cases} \boldsymbol{f_1(t)}, & \boldsymbol{0 \leq t < a} \\ \boldsymbol{f_2(t)}, & \boldsymbol{a \leq t < b} \\ \boldsymbol{f_3(t)}, & \boldsymbol{b \leq t} \end{cases}$$

Solución:



Se expresa a f(t) como combinación de escalonadas:

$$f(t) = \boldsymbol{f_1(t)} + \left(\boldsymbol{f_2(t)} - \boldsymbol{f_1(t)}\right).u(t-a) + \left(\boldsymbol{f_3(t)} - \boldsymbol{f_2(t)}\right).u(t-b)$$

$$f(t) = 2 + (-4).\mu(t-3) + (2).\mu(t-5)$$

$$L(f(t)) = L(2) - 4L(t-3) + 2L(t-5)$$

$$L(f(t)) = \frac{2}{s} - 4.\frac{e^{-3s}}{s} + 2.\frac{e^{-5s}}{s}$$

Teorema de traslación

$\mathcal{L}(e^{at}f(t)) = F(s-a)$

$\mathcal{L}(\boldsymbol{f}(\boldsymbol{t}-\boldsymbol{a})u(t-a)) = e^{-as}.\boldsymbol{\mathcal{L}}(\boldsymbol{f}(\boldsymbol{t}))$

$\mathcal{L}(\boldsymbol{g}(\boldsymbol{t})u(t-a)) = e^{-as}.\boldsymbol{\mathcal{L}}(\boldsymbol{g}(\boldsymbol{t}+\boldsymbol{a}))$

- **Teorema**

Si $\mathcal{L}(f(t)) = \boldsymbol{F}(\boldsymbol{s})$ se tiene:

Primer teorema: $\mathcal{L}(e^{\pm at}.f(t)) = F(s \mp \boldsymbol{a})$

Segundo teorema: 2 $\mathcal{L}(\boldsymbol{f}(\boldsymbol{t}-\boldsymbol{a})u(t-a)) = e^{-as}.\boldsymbol{\mathcal{L}}(\boldsymbol{f}(\boldsymbol{t}))$

1 $\mathcal{L}(\boldsymbol{g}(\boldsymbol{t})u(\boldsymbol{t}-\boldsymbol{a})) = e^{-\boldsymbol{as}}.\boldsymbol{\mathcal{L}}(\boldsymbol{g}(\boldsymbol{t}+\boldsymbol{a}))$

$\times \; L\{\operatorname{Sen} 2t\} = \dfrac{2}{s^2+4} \Rightarrow L\{e^{+3t}.\operatorname{Sen} 2t\} = \dfrac{2}{(s-3)^2+1}$ ✓

$L\{\cos 3t\} = \dfrac{s}{s^2+9} \Rightarrow L\{e^{-1t}\cos 3t\} = \dfrac{s+1}{(s+1)^2+9}$ ✓

$* \; L\{t^2 u(t-1)\} = ??$

$\underbrace{\phantom{L\{t^2u(t-1)\}}}$

$e^{-1s}.L\{(t+1)^2\}$

$e^{-s}.(L\{t^2+2t+1\})$

$e^{-s}.\left(\dfrac{2!}{s^3} + 2.\dfrac{1!}{s^2} + \dfrac{1}{s}\right)$ ✓

$* \; e^{-2s}.L(t^2) = L((t-2)^2.u(t-2))$ ✓

Teorema de traslación

$\mathcal{L}(e^{at}f(t)) = F(s-a)$

Ejemplo. Encuentre la transformada de Laplace de la función f.

a. $g(t) = e^{2t}t$

$L(g(t)) = L(e^{2t}\cdot t) = ??$

sabemos que: $L(t) = \frac{1}{s^2}$

$\Rightarrow L(e^{2t}\cdot t) = \frac{1}{(s-2)^2}$ ✓

b. $h(t) = e^{3t}\cos 4t$

$L(h(t)) = L(e^{3t}\cdot \cos 4t) = ???$

Sabemos que: $L(\cos 4t) = \frac{s}{s^2+16}$

$\Rightarrow L(e^{3t}\cdot \cos 4t) = \frac{s-3}{(s-3)^2+16}$ ✓

Teorema de traslación

$\mathcal{L}^{-1}F((s-a)) = e^{at}f(t)$

$\mathcal{L}(f(t-a)u(t-a)) = e^{-as}F(s)$

Ejemplo. Encuentre la transformada inversa de:

c. $F(s) = \dfrac{s+5}{s^2+2s+5}$

$$L^{-1}\left(\frac{s+5}{s^2+2s+5}\right) = h(t) = ?? \Rightarrow \frac{s+5}{s^2+2s+5} = L(h(t))$$

$$L(h(t)) = \frac{s+5}{s^2+2s+1+4} = \frac{(s+1)+4}{(s+1)^2+4}$$

$$L(h(t)) = \frac{s+1}{(s+1)^2+4} + 2 \cdot \frac{2}{(s+1)^2+4}$$

$$L\{e^{-t} \cdot \cos 2t\} + 2 \cdot L(e^{-t} \cdot \operatorname{sen} 2t)$$

$$h(t) = e^{-t} \cdot \cos 2t + 2 \cdot e^{-t} \cdot \operatorname{sen} 2t$$

d. $H(s) = e^{-s}\left(\dfrac{1}{s^2}\right) + e^{-2s}\left(\dfrac{4}{s^2+1}\right)$

Teorema de traslación

sea $h(t) = \begin{cases} h_1 & 0 \le t < a \\ h_2 & a \le t < b \\ h_3 & b \le t < \infty \end{cases}$

en escalonadas

$h(t) = h_1 + (h_2 - h_1)U(t-a) + (h_3 - h_2)U(t-b)$

$\mathcal{L}(e^{at}f(t)) = F(s-a)$

$\mathcal{L}(\boldsymbol{f(t-a)}u(t-a)) = e^{-as}.\mathcal{L}(\boldsymbol{f(t)})$

$\mathcal{L}(\boldsymbol{g(t)}u(t-a)) = e^{-as}.\mathcal{L}(\boldsymbol{g(t+a)})$

Ejemplo: Resuelva el PVI:

$$y''-y = \begin{cases} \boldsymbol{1} & \boldsymbol{0 \le t < 1} \\ \boldsymbol{0} & \boldsymbol{1 \le t} \end{cases}, \quad y(0) = y'(0) = 0$$

Solución

$$y'' - y = 1 + (0-1)u(t-1) \Rightarrow \mathcal{L}(y'') - \mathcal{L}(y) = \mathcal{L}(1) - \mathcal{L}\{u(t-1)\}$$

$$[\boldsymbol{s^2 Y_s} - \boldsymbol{s\underbrace{y(0)}_{0}} - \boldsymbol{\underbrace{y'(0)}_{0}}] - Y_s = \frac{1}{s} - \frac{e^{-1s}}{s} \qquad (s^2-1)Y_s = \frac{1}{s} - \frac{e^{-s}}{s} \qquad \Rightarrow \quad (s-1)(s+1)Y_s = \frac{1}{s} - e^{-s}.\frac{1}{s}$$

$$\Rightarrow \quad Y_s = \frac{1}{s(s-1)(s+1)} - e^{-s}\frac{1}{s(s-1)(s+1)}$$

$$* \quad \underbrace{\frac{1}{s(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+1}}$$

$1 = A(s-1)(s+1) + Bs(s+1) + Cs(s-1)$

$s = -1, \quad s = 1, \quad s = 0$

$$\Rightarrow \quad Y = \frac{-1}{s} + \frac{1/2}{s-1} + \frac{1/2}{s+1} - e^{-s}\left(-\frac{1}{s} + \frac{1/2}{s-1} + \frac{1/2}{s+1}\right)$$

$$y(t) = -1 + \frac{e^t}{2} + \frac{e^{-t}}{2} - u(t-1)\left(-1 + \frac{e^{t-1}}{2} + \frac{e^{-t+1}}{2}\right)$$

$$\Rightarrow \ Y = \frac{-1}{s} + \frac{1/2}{s-1} + \frac{1/2}{s+1} - e^{-s}\left(\frac{-1}{s} + \frac{1/2}{s-1} + \frac{1/2}{s+1}\right)$$

$$\mathcal{L}(y) = -\mathcal{L}(1) + \frac{1}{2}\cdot\mathcal{L}(e^{t}) + \frac{1}{2}\mathcal{L}(e^{-t}) - e^{-s}\left(\mathcal{L}(1) + \frac{1}{2}\cdot\mathcal{L}(e^{t}) + \frac{1}{2}\mathcal{L}(e^{-t})\right)$$

$$\mathcal{L}(y) = \mathcal{L}\left(-1 + \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}\right) - \underbrace{e^{-s}\cdot\mathcal{L}\left(-1 + \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}\right)}_{\mathcal{L}\left(u(t-1)\cdot\left(-1+\frac{1}{2}e^{t-1}+\frac{1}{2}e^{-(t-1)}\right)\right)}$$

$$y = -1 + \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} - u(t-1)\left(-1 + \frac{1}{2}e^{t-1} + \frac{1}{2}e^{-(t-1)}\right)$$

$$\mathcal{L}(f(t-a)u(t-a)) = e^{-as}.\mathcal{L}(f(t))$$

$$y(t) = -1 + \frac{e^{t}}{2} + \frac{e^{-t}}{2} - u(t-1)\left(-1 + \frac{e^{t-1}}{2} + \frac{e^{-t+1}}{2}\right)$$