Group Theory in Cryptography

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Basics

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 - 1.3 There is an identity element 0 in the group.
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- 2. Group of rational number excluding 0 under multiplication $(\mathbb{Q} \setminus \{0\}, \times)$
 - 2.1 Multiplying 2 rational number will result in a rational number.
 - 2.2 Multiplication in \mathbb{Q} is associative.
 - 2.3 There is an identity element 1 in the group.
 - 2.4 Every rational number other than 0 has an inverse.



Applications

- ▶ Group is one of the algebraic structure in the studies of abstract algebra.
- ► There are many applications of group theory and mathematicians have used them to solve hard math problems.

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Fermat's last theorem

No three positive integers a, b, and c satisfy the equation $a^n+b^n=c^n$ for any integer value of n greater than 2

Abel-Ruffini theorem

There is no solution in radicals to general polynomial equations of degree five or higher with arbitrary coefficients.



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 - 1.2 $\forall a, b \in G, a \diamond b \in G$
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- 3. Identity
 - 3.1 There exists an unique identity element e such that for all element a in G, $a \diamond e = a$ and $e \diamond a = a$.
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 - 3.2 $\exists ! e \in G, \forall a \in G, (a \diamond e = a) \land (e \diamond a = a)$
- 4. Inverse
 - 4.1 For each element a in G, there exists an unique element a^{-1} in G such that $a \diamond a^{-1} = e$ and $a^{-1} \diamond a = e$.
 - $4.2 \ \forall a \in G, \exists! a^{-1} \in G, (a \diamond a^{-1} = e) \land (a^{-1} \diamond a = e)$



Abelian Group

Abelian/Commutative group are groups that satisfy one more axiom.

- Commutativity
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Finite Group

Finite group are groups that has finite amount of elements in the sets.

- ▶ One example is the integer group under addition modulo n $(\mathbb{Z}/n\mathbb{Z},+)$
- Most of the groups commonly used in cryptography are finite group.

Prove that Integer under addition $(\mathbb{Z},+)$ is an abelian group

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- 2. Associativity
 - 2.1 Take any 3 arbitrary integers a, b, c, (a + b) + c = a + (b + c)
- 3. Identity
 - 3.1 Take any integer a, a + 0 = a and 0 + a = a
- 4. Inverse
 - 4.1 Take any integer a, a + (-a) = 0 and (-a) + a = 0
- 5. Commutativity
 - 5.1 Take any integer a,b, a + b = b + a

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- 1. Closure
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 - 2.1 Take any 3 arbitrary rational numbers a, b, c, $(a \times b) \times c = a \times (b \times c)$
- 3. Identity
 - 3.1 Take any rational number a, $a \times 1 = a$ and $1 \times a = a$
- 4. Inverse
 - 4.1 Take any rational number a, $a \times (\frac{1}{a}) = 1$ and $(\frac{1}{a}) \times a = 1$
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 - 5.1 Take any integer a,b, $a \times b = b \times a$

Which of the following are groups?

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 - Yes this is indeed a group and it is called the general linear group
 - ▶ It is denoted by $GL_n(\mathbb{R})$ or $GL(n,\mathbb{R})$ where n is the dimension of the matrix
 - ightharpoonup Special linear group $SL_n(\mathbb{R})$ is general linear group but with determinant 1

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$$0+1\equiv 1\ (\mathsf{mod}\ 3)$$

$$0+2\equiv 2\ (\mathsf{mod}\ 3)$$

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+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Table: Cayley Table

2. $(\mathbb{Z}/5\mathbb{Z},+)$

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+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

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2	2	3	4	0	1
3	3	4	0	1	2
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Multiplicative Finite Group

Group of integer under multiplication modulo n $(\mathbb{Z}/n\mathbb{Z}\setminus\{0\},\times)$

- n must be a prime number
- ▶ The inverse of 2 modulo 4 does not exists

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×	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Definition

let (G, \diamond) be a group. let H be a subset of G. (H, \diamond) is a subgroup if H also forms a group under \diamond This is denoted by $H \leq G$

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A proper subgroup of a group G is a subgroup H which is a proper subset of G (that is, $H \neq G$). This is usually represented notationally by H < G

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Simple Group

Simple group are groups that only has 2 subgroup (Trivial Subgroup and itself)



```
1. \mathbb{Z}/8\mathbb{Z} or (\{0,1,2,3,4,5,6,7\},+)

1.1 (\{0,1,2,3,4,5,6,7\},+)

1.2 (\{0,2,4,6\},+)

1.3 (\{0,4\},+)

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2. \mathbb{Z}/10\mathbb{Z} or (\{0,1,2,3,4,5,6,7,8,9\},+)
    2.1 (\{0,1,2,3,4,5,6,7,8,9\},+)
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     2.2 (\{0, 2, 4, 6, 8\}, +)
     2.3 (\{0,5\},+)
     2.4 (\{0\}, +)
3. (\mathbb{Z}/7\mathbb{Z})^{\times} or (\{1,2,3,4,5,6\},\times)
     3.1 (\{1,2,3,4,5,6\},\times)
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```

Cosets

let (G, \diamond) be a group. let H be a subgroup of G.

The left cosets of H in G are the sets $g \diamond H = \{g \diamond h \mid h \in H\}$ for each $g \in G$

Similarly for right cosets, $H \diamond g = \{h \diamond g \mid h \in H\}$ for each $g \in G$

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Properties

- Cosets of H are equal size and disjoint.
- left cosets is identical with right cosets in abelian group.

Examples

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let $H = \{0, 2, 4, 6\}$ The cosets of H are

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Examples

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Generating set of a group

If S is a subset of a group G, then $\langle S \rangle$, the subgroup generated by S, is the smallest subgroup of G containing every element of S

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$$S = \{0,3\}, G = \mathbb{Z}/9\mathbb{Z}$$
 , $\langle S \rangle = \{0,3,6\}$

Cyclic Group

A cyclic group or monogenous group is a group that is generated by a single element

 $(\mathbb{Z}/5\mathbb{Z})^{\times}$ is a cyclic group because 2 can be the generator

 $\mathbb{Z}/n\mathbb{Z}$ is also a cyclic group because 1 can be the generator



Group Order

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Order of an element

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The order of every element of a finite group is finite. The order of the element also equal to the order of the subgroup generated by the element.

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Examples

1. $(\mathbb{Z}/5\mathbb{Z})^{\times}$ 1.1 $1^4 = 1 \equiv 1 \pmod{5}$ 1.2 $2^4 = 16 \equiv 1 \pmod{5}$ 1.3 $3^4 = 81 \equiv 1 \pmod{5}$ 1.4 $4^4 = 256 \equiv 1 \pmod{5}$

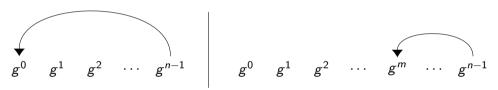


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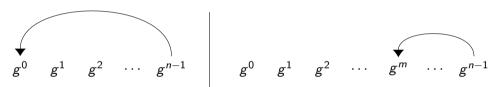
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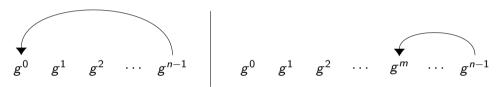


Prove

1. Let $g \in G$ and consider the set $S = \{g^0 = e, g^1, g^2, \dots\}$

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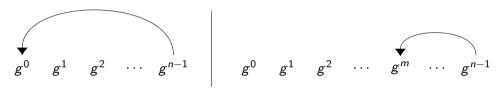
The order of the element is equal to the order of the subgroup generated by the element



- 1. Let $g \in G$ and consider the set $S = \{g^0 = e, g^1, g^2, \dots\}$
- 2. Since G is a finite group, S must also be a finite group. $S = \{e, g^1, g^2, \dots, g^{n-1}\}$

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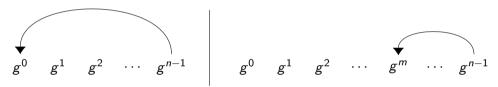
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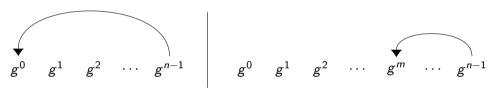


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- 2. Since G is a finite group, S must also be a finite group. $S = \{e, g^1, g^2, \dots, g^{n-1}\}$
- 3. Let n be the number of elements in S and $g^n = g^m$, where n > m
- 4. $g^{n-m} = g^0 = e$. Since n > m, Therefore n m > 0



The order of every element of a finite group is finite.

The order of the element is equal to the order of the subgroup generated by the element



- 1. Let $g \in G$ and consider the set $S = \{g^0 = e, g^1, g^2, \dots\}$
- 2. Since G is a finite group, S must also be a finite group. $S = \{e, g^1, g^2, \dots, g^{n-1}\}$
- 3. Let n be the number of elements in S and $g^n = g^m$, where n > m
- 4. $g^{n-m} = g^0 = e$. Since n > m, Therefore n m > 0
- 5. Hence, there exists a positive integer k such that $g^k \equiv e$



Lagrange's Theorem

For any finite group G, the order (number of elements) of every subgroup of G divides the order of G.

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Fermat's Little Theorem

 $a^{p-1} \equiv 1 \pmod{p}$, where p is a prime number

Euler's Theorem

$$a^{\varphi(n)} \equiv 1 \pmod{n}$$

Where $\varphi(n)$ is the Euler's totient function i.e the number of positive integers up to a given integer n that are relatively prime to n.



Prove

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- 5. Since the order of H equal to the order of coset, Lagrange's theorem is true.

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Types

- ► Monomorphism (Injective)
- Epimorphism (Surjective)
- Isomorphism (Bijective)
- ► Endomorphism (Same domain and codomian)
- Automorphism (Endomorphism and Bijective)



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2.2 Cayley Table

$$\begin{array}{c|cccc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ \hline 1 & 1 & 0 \\ \end{array}$$

Table:
$$\mathbb{Z}/2\mathbb{Z}$$

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 $2.3 ext{ } f$ is a Group Isomorphism

Properties

- ▶ By homomorphism, you can transfer group from one to other. Sometimes a problem can be easier to solve in other group
- lacktriangle All the groups that has prime order are isomorphic with the additive group $\mathbb{Z}/p\mathbb{Z}$
- ▶ All the groups that has order less than 3 are isomorphic with each other
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Kernel Group

Let $f: G \rightarrow H$ be a group homomorphism.

The kernel group is the set of elements from G which maps to the identity element in H i.e. $f(g) = e_H$



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1. R is an abelian group under addition

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- 4. Distributivity
 - 4.1 For all element a, b, c in R, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
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- 3. $\mathbb{Z}[x]$
- **4**. \mathbb{R}
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 - 2.1 Let additive identity be e_a , multiplicative identity be e_m
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- $2. \mathbb{Z}/10\mathbb{Z}$
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Domain

A domain is a ring which has no zero divisor.

Integral Domain

A commutative domain is also called an integral domain.



Definition

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Other Interpretation

- 1. Fields are Rings with multiplicative inverses and no zero divisors.
- 2. Field are combination of groups which relates through distributivity.



- 1. Every field must have order of p^m where p is a prime number. p is also the characteristic of the field
- 2. All field with the same order are isomorphic with each other
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+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

\times	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
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- 5. Therefore, Cauchy's Theorem is True

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- 6. Since every element of G has the same order, the order of every element must be a prime and $|G| = p^m$

Prime Field

Field with prime order is isomorphic with $\mathbb{Z}/p\mathbb{Z}$, where p is a prime

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Field with order p^m where m > 1 and p is prime

▶ Is $\mathbb{Z}/p^m\mathbb{Z}$ a field?

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- ightharpoonup How do we construct a field with order p^m ?
 - 1. Since we need $p \cdot p \cdot ... \cdot p$ elements, the simplest way is to use polynomial.
 - 2. $GF(p^m) = a_{m-1} \cdot x^{m-1} + a_{m-2} \cdot x^{m-2} + \dots + a_1 \cdot x + a_0$, where $a_i \in GF(p)$
 - 3. Instead of number arithmetic, we now use polynomial arithmetic.

- Addition
 - 1. Let $g, h \in GF(p^m)$
 - 2. $g + h = (g_0 + h_0) \mod p + ((g_1 + h_1) \mod p) \cdot x + \dots + ((g_0 + h_0) \mod p) \cdot x^{m-1}$

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 - 3. $g^{-1} \cdot g \equiv 1 \mod (P)$



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Quiz

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 - 2.1 Yes, this is an irreducible polynomial
- 3. $x^2 + 1$ in GF(4)
 - 3.1 No, this is not an irreducible polynomial
 - 3.2 $(x+1)^2 = x^2 + 2x + 1 \equiv x^2 + 1$

Examples

1. $GF(2^2)$ with $p = 1 + x + x^2$

+	0	1	×	$\times + 1$
0	0	1	×	$\times + 1$
1	1	0	$\times + 1$	×
×	X	$\times + 1$	0	1
x + 1	x + 1	×	1	0

+	0	1	2	3	
0	0	1	2	3	
1	1	0	3	2	
2	2	3	0	1	
3	3	2	1	0	

×	0	1	x	x + 1
0	0	0	0	0
1	0	1	Х	x + 1
×	0	Х	$\times + 1$	1
$\overline{x+1}$	0	x + 1	1	×

×	0	1	2	2
0	0	0	0	0
1	0	1	2	3
2	0	2	3	1
3	0	3	1	2

Application in Cryptography

Definition

Let (G, \diamond) be a group.

Let
$$g, b \in G$$
 such that $b = g^k = g \diamond g \diamond \cdots \diamond g$

Find
$$k = \log_g b$$

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- 2. In group $\mathbb{Z}/13\mathbb{Z}$. Find k, where $5 \equiv k \cdot 2 \pmod{13}$
 - 2.1 $k \equiv 5 \cdot 2^{-1} \pmod{13}$
 - 2.2 $k \equiv 5 \cdot 7 \equiv 35 \equiv 9 \pmod{13}$

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- 2. Applicable to many groups
 - 2.1 Discrete Logarithm is hard to compute generally.
 - 2.2 Take any group and you can invent a new cryptographic scheme based on DLP

Why discrete logarithm is easy for some groups?

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Field extension

Given some group (G, +), if there exist some field F which is extended from G, then the DLP in G is just the multiplication in F.

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Isomorphism

If we can find a group isomorphism between two groups G, H and given that DLP in H is easy, we can easily solve DLP in G too.

Prime order

Since all group with prime order are isomorphic with $\mathbb{Z}/n\mathbb{Z}$. If a group G has prime order, then there exist a group isomorphism between G and $\mathbb{Z}/n\mathbb{Z}$.

Anomalous elliptic curves

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Smooth order

If G is an abelian group and the order of G is smooth, then the DLP in G is easy to compute using Pohlig-Hellman algorithm.

A number is said to be smooth if the highest prime factor of the number is small.

Diffie-Hellman Key Exchange

Two parties Alice, Bob wish to exchange a shared key through an public channel.

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Preparation

Alice, Bob agree on a group (G,\diamond) a generator $g\in G$ and function $f:G\to k$ that maps the group element to the set of keys

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- 3. Alice get the shared key by computing $f(B^a)$

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Aftermath

After exchanging keys, two parties Alice and Bob can now communicate securely by encrypting/decrypting their messages using a symmetric encryption system such as AES through an unsecure channel

Example

- 1. $(\mathbb{Z}/13\mathbb{Z})^{\times}$, g=21.1 Alice : a = 5, $A \equiv 2^a \equiv 6 \pmod{13}$ 1.2 Bob : b = 7, $B \equiv 2^b \equiv 11 \pmod{13}$ 1.3 Alice send A to Bob, Bob send B to Alice
 - 1.4 Alice : $S = B^a \equiv 7 \pmod{13}$ 1.5 Bob : $S = A^b \equiv 7 \pmod{13}$

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- 2. $(\mathbb{Z}/23\mathbb{Z})^{\times}$, g = 22.1 Alice: a = 14, $A \equiv 2^{a} \equiv 8 \pmod{23}$ 2.2 Bob: b = 5, $B \equiv 2^{b} \equiv 9 \pmod{23}$ 2.3 Alice send A to Bob, Bob send B to Alice 2.4 Alice: $S = B^{a} \equiv 16 \pmod{23}$ 2.5 Bob: $S = A^{b} \equiv 16 \pmod{23}$

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Alice, Bob agree on a group (G, \diamond) a generator $g \in G$ and a bijective function $f: M \to G$ that maps the message to a group element

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Encryption

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Decryption

1. Bob compute $S = A^b$ and $g_m = k \diamond S^{-1}$



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- 2. Alice generate a random secret a and computes $A = g^a$, $S = B^a$.
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- 4. Alice computes $k = g_m \diamond S$ and sends A, k to Bob

Decryption

- 1. Bob compute $S = A^b$ and $g_m = k \diamond S^{-1}$
- 2. Bob decrypt the message by mapping g_m back to m using $f^{-1}(g_m) = m$



Elliptic Curve Cryptography

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Properties

- 1. Diffie-Hellman Key Exchange, Elgamal Encryption are both applicable with Elliptic Curve Group.
- 2. It offers a better bit security compare to multiplicative group modulo n.
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Reading Material

- 1. The Arithmetic of Elliptic Curves by Joseph H. Silverman
- 2. Elliptic Tales: Curves, Counting, and Number Theory by Avner Ash, Robert Gross



Challenges

PolyRSA

RSA with polynomial... Is it even possible?

Diagonal

Solving Discrete Logarithm is hard, it should be hard in this matrix group too?