UMAT for finite strain viscoelastic strain energy density depending on (I_1, I_2, J) with damage and time-temperature superposition

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Contents

1 Credit and disclaimer

This UMAT was written by Florian Gouhier during his PhD in LMS (Laboratory of Mechanics of Solids), École Polytechnique, under the supervision of Julie Diani. It builds upon the UMAT implementation by Alan Jason Correa (https://github.com/thealanjason/umat_finite_viscoelasticity), which incorporates finite strain viscoelastic constitutive equations based on the work of Reese and Govindjee (Reese and Govindjee, 1998). These equations were formulated for an Ogden strain energy density model with one, two, or three Maxwell branches. Significant modifications have been made to the original UMAT, introducing the following new features:

1. The strain energy density function has been generalized to any expression of the form $\mathcal{W} = \overline{\mathcal{W}}(\bar{I}_1, \bar{I}_2) + \mathcal{U}(J)$,
2. The number of viscoelastic branches has been extended to accommodate n potential relaxation mechanisms,
3. The hydrostatic part of the strain energy density, $\mathcal{U}(J)$, may also be chosen viscoelastic,
4. Deviatoric and hydrostatic strain softening effects have been incorporated through multiplicative damage terms. This feature enables the modeling of phenomena such as Mullins softening in filled rubbers or matrix/filler debonding in propellants,
5. A version compatible with hybrid elements has been implemented for nearly incompressible materials, leveraging the `ABAQUS` option `HYBRID FORMULATION=TOTAL`.

These standard and hybrid UMATs are free to use. Please, cite the appropriate reference (Gouhier and Diani, 2024a or/and Gouhier et al., 2024b) and Github's link (https://github.com/MechMater-project/UMAT_finitestrain_viscoelasticity_withdamage) in presentations, reports, papers...

2 Constitutive equations

This section does not provide a detailed review of the constitutive equations. Readers seeking a comprehensive understanding are referred to (Correa, 2022; Gouhier and Diani, 2024a; Gouhier et al., 2024b; Reese and Govindjee, 1998).
Instead, only useful notations and equations are introduced to help users in making modifications to the model parameters and strain energy density functions.

Viscoelastity In the present formulation, the deformation gradient $\mathbf{F}$ admits the classic volumetric/deviatoric split and elastic/viscoelastic split,

$$\mathbf{F} = J^{1/3}\,\bar{\mathbf{F}} \text{ and } \mathbf{F} = (J_e J_v)^{1/3}\bar{\mathbf{F}}_e\,\bar{\mathbf{F}}_v. \tag{1}$$

Then, we may define the isochoric right Cauchy-Green tensors,

$$\bar{\mathbf{C}} = \bar{\mathbf{F}}^T\,\bar{\mathbf{F}} \text{ and } \bar{\mathbf{C}}_e = \bar{\mathbf{F}}_e^T\,\bar{\mathbf{F}}_e = \bar{\mathbf{F}}_v^{-T}\,\bar{\mathbf{C}}\,\bar{\mathbf{F}}_v^{-1}, \tag{2}$$

and their invariants,

$$\begin{aligned}
\bar{I}_1 &= \mathrm{tr}(\bar{\mathbf{C}}), & \bar{I}_2 &= \frac{1}{2}\left(\mathrm{tr}(\bar{\mathbf{C}}))^2 - \mathrm{tr}(\bar{\mathbf{C}}^2)\right), & J &= \sqrt{\det(\mathbf{C})},\\
\bar{I}_{1e} &= \mathrm{tr}(\bar{\mathbf{C}}_e), & \bar{I}_{2e} &= \frac{1}{2}\left((\mathrm{tr}(\bar{\mathbf{C}}_e))^2 - \mathrm{tr}(\bar{\mathbf{C}}_e^2)\right), & J_e &= \sqrt{\det(\mathbf{C}_e)}.
\end{aligned} \tag{3}$$

Note in passing that $\det(\bar{\mathbf{F}}) = \det(\bar{\mathbf{C}}) = \det(\bar{\mathbf{C}}_e) = 1$.
The strain energy density describing the viscoelastic behavior of a material, represented by a generalized Maxwell rheological scheme with n Maxwell branches and one elastic branch in parallel, including potential damage effects, can be expressed as,

$$\mathcal{W}_D = (1 - D_h)\,\mathcal{U}(J, J_e^1, \ldots, J_e^n) + (1 - D_d)\,\overline{\mathcal{W}}(\bar{\mathbf{C}}, \bar{\mathbf{C}}_e^1, \ldots, \bar{\mathbf{C}}_e^n), \tag{4}$$

with D_h and D_d defining the hydrostatic and deviatoric damages. The hydrostatic and deviatoric parts may write as,

$$\mathcal{U}(J, J_e^1, \ldots, J_e^n) = \mathcal{U}^{eq}(J) + \sum_{k=1}^{n} \mathcal{U}^{neq}(J_e^k) \tag{5}$$

$$\overline{\mathcal{W}}(\bar{\mathbf{C}}, \bar{\mathbf{C}}_e^1, \ldots, \bar{\mathbf{C}}_e^n) = \overline{\mathcal{W}}^{eq}(\bar{\mathbf{C}}) + \sum_{k=1}^{n} \overline{\mathcal{W}}^{neq}(\bar{\mathbf{C}}_e^k) \tag{6}$$

The resulting Cauchy stress $\boldsymbol{\sigma}$ writes as,

$$\begin{aligned}
\boldsymbol{\sigma} &= (1 - D_h)\left(\frac{\partial \mathcal{U}^{eq}(J)}{\partial J} + \sum_{k=1}^{n} \frac{J_e^k}{J}\,\frac{\partial \mathcal{U}_k^{neq}(J_e^k)}{\partial J_e^k}\right)\\
&+ (1 - D_d)\frac{2}{J}\,\mathrm{dev}\left(\bar{\mathbf{F}}\,\frac{\partial \overline{\mathcal{W}}^{eq}(\bar{\mathbf{C}})}{\partial \bar{\mathbf{C}}}\,\bar{\mathbf{F}}^T\right) + \sum_{k=1}^{n}(1 - D_d)\frac{2}{J}\,\mathrm{dev}\left(\bar{\mathbf{F}}_e^k\,\frac{\partial \overline{\mathcal{W}}_k^{neq}(\bar{\mathbf{C}}_e^k)}{\partial \bar{\mathbf{C}}_e^k}\bar{\mathbf{F}}_e^{kT}\right)
\end{aligned} \tag{7}$$

Noting that for isotropic materials,

$$\frac{\partial\overline{\mathcal{W}}(\bar{\mathbf{C}})}{\partial\bar{\mathbf{C}}} = \frac{\partial\overline{\mathcal{W}}(\bar{\mathbf{C}})}{\partial\bar{I}_1}\mathbf{I_d} + \frac{\partial\overline{\mathcal{W}}(\bar{\mathbf{C}})}{\partial\bar{I}_2}\left(\bar{I}_1\mathbf{I_d} - \bar{\mathbf{C}}\right), \tag{8}$$

it is convenient to define the following quantities,

$$\mathcal{A}_{\bar{\mathbf{C}}} = \frac{\partial\overline{\mathcal{W}}^{eq}(\bar{\mathbf{C}})}{\partial\bar{I}_1} + \bar{I}_1\frac{\partial\overline{\mathcal{W}}^{eq}(\bar{\mathbf{C}})}{\partial\bar{I}_2} \qquad \text{et} \qquad \mathcal{B}_{\bar{\mathbf{C}}} = -\frac{\partial\overline{\mathcal{W}}^{eq}(\bar{\mathbf{C}})}{\partial\bar{I}_2}. \tag{9}$$

$$\mathcal{A}_{\bar{\mathbf{C}}_e^k} = \frac{\partial\overline{\mathcal{W}}_k^{neq}(\bar{\mathbf{C}}_e^k)}{\partial\bar{I}_{1e}^k} + \bar{I}_{1e}^k\frac{\partial\overline{\mathcal{W}}_k^{neq}(\bar{\mathbf{C}}_e^k)}{\partial\bar{I}_{2e}^k} \qquad \text{et} \qquad \mathcal{B}_{\bar{\mathbf{C}}_e^k} = -\frac{\partial\overline{\mathcal{W}}_k^{neq}(\bar{\mathbf{C}}_e^k)}{\partial\bar{I}_{2e}^k}. \tag{10}$$

and their derivatives $\dfrac{\partial\mathcal{A}}{\partial\bar{I}_i}$ and $\dfrac{\partial\mathcal{B}}{\partial\bar{I}_i}$ with $i \in \{1,2\}$, for the calculation of the stress and the Jacobian required in the UMAT. As we will demonstrate in section 4, this choice will simplify the transition from the implemented neo-Hookean law,

$$\overline{\mathcal{W}} = \frac{\mu_1}{2}(\bar{I}_1 - 3) \tag{11}$$

to any other strain energy density depending on the invariants $\bar{I}_1$ and $\bar{I}_2$.
The viscoelastic evolution equations for each viscoelastic branch k are defined as follows,

$$\begin{cases} \left(\mathcal{L}_v\,\bar{\mathbf{b}}_e^k\right)\bar{\mathbf{b}}_e^{k\,-1} = -\dfrac{1}{\eta_d^k}\,\mathrm{dev}\left(\boldsymbol{\tau}_k^{neq}\right) = -\dfrac{2}{\eta_d^k}\,\mathrm{dev}\left(\dfrac{\partial\overline{\mathcal{W}}^{neq}(\bar{\mathbf{b}}_e^k)}{\partial\bar{\mathbf{b}}_e^k}\,\bar{\mathbf{b}}_e^k\right) \\ \\ \dot{J}_v^k = \dfrac{J}{\eta_h^k}\dfrac{\partial\mathcal{U}_k^{neq}(J_e^k)}{\partial J_e^k} \end{cases} \tag{12}$$

with $\bar{\mathbf{b}}_e^k = \bar{\mathbf{F}}_e^k\,\bar{\mathbf{F}}_e^{kT}$, $\mathcal{L}_v$ the Lie derivative, and η_d^k and η_h^k the deviatoric and hydrostatic viscosities.

Damage Hydrostatic and deviatoric damages have been defined as,

$$\begin{cases} D_d = 1 - e^{-b_d(\alpha_d^m - \alpha_d^0)^{a_d}} \text{ with } \alpha_d^m = \max(\alpha_d), \text{with multiple choices for } \alpha_d, \\ \\ \alpha_d \in \left\{\sqrt{\frac{\bar{I}_1}{3}} - 1,\, m = \sqrt{\bar{I}_1^2 - 2\bar{I}_2},\, \bar{I}_\gamma = \frac{1}{6}\sqrt{2\bar{I}_1^2 - 6\bar{I}_2},\, \bar{h}_{eq}\right\}, \\ \text{with } \bar{h}_{eq} = \sqrt{\frac{2}{3}(\bar{h}_1^2 + \bar{h}_2^2 + \bar{h}_3^2)} \text{ and } \bar{\mathbf{h}} = \frac{1}{2}\ln\left(\bar{\mathbf{F}}\,\bar{\mathbf{F}}^T\right) \\ \\ \dot{D}_d = \begin{cases} \dfrac{dD_d(\alpha_d)}{d\alpha_d}\dot{\alpha}_d & \text{when} \quad \alpha_d - \alpha_d^m = 0 \quad \text{and} \quad \dot{\alpha}_d > 0, \\ \\ 0 & \text{otherwise,} \end{cases} \\ \\ \dot{\alpha}_d(P) = \dot{\alpha}_d(0)\left(1 - \omega_d\left(1 - e^{-\frac{P}{P_s}}\right)\right) \text{ with } P = \mathrm{tr}(\boldsymbol{\sigma}). \\ \\ D_h = 1 - e^{-b_h(\alpha_h^m - \alpha_h^0)^{a_h}}, \quad \alpha_h \in \{\alpha_d\} \end{cases} \tag{13}$$

The pressure dependence has been introduced to capture experimental observations in propellants (Gouhier et al., 2024b), which indicate a delay in damage progression when an external pressure is applied. The quantity P_s acts as a saturation pressure, above which further increases in pressure do not affect the damage evolution.
Therefore, the deviatoric damage is characterized by four parameters, a_d, b_d, ω_d and P_s, while the hydrostatic damage is defined by two parameters, a_h and b_h. As formulated, the hydrostatic damage also depends on the pressure P.

Time-temperature superposition The UMAT was originally developed for propellants, which consist of a polymer binder that is highly filled with energetic particles. The viscoelastic binder can undergo large strains because its glass transition temperature is well below room temperature. Furthermore, it satisfies the time-temperature superposition principle, which has been incorporated into the model through the WLF equation,

$$\log a_{T_{ref}}(T) = \frac{-C_1\,(T - T_{ref})}{C_2 + (T - T_{ref})}, \tag{14}$$

To account for this property, three parameters T_{ref}, C_1 and C_2 are defined. Then, the Maxwell branch k viscosities, η_d^k and η_h^k, are initially defined at T_{ref}. For any other temperature, the deviatoric and hydrostatic viscosities are given by $\eta^k(T) = 10^{a_{T_{ref}}(T)}.\,\eta^k(T_{ref})$.

For a neo-Hookean material, the list of parameters is as follows,

Parameters input	
Elastic branch	μ^e, K^e
Each viscoelastic branch	μ^k, K^k, η_d^k, η_h^k
Damage	a_d, b_d, a_h, b_h, ω_d and P_s
Time-temperature superposition	T_{ref}, C_1, C_2

3 Validation

Several uniaxial tension and simple shear tests were performed to validate the UMAT. The numerical results were compared with analytical expressions computed using a Python routine. These tests are illustrated in the following for information purposes. Moduli are expressed in MPa, and viscosities in MPa.s.

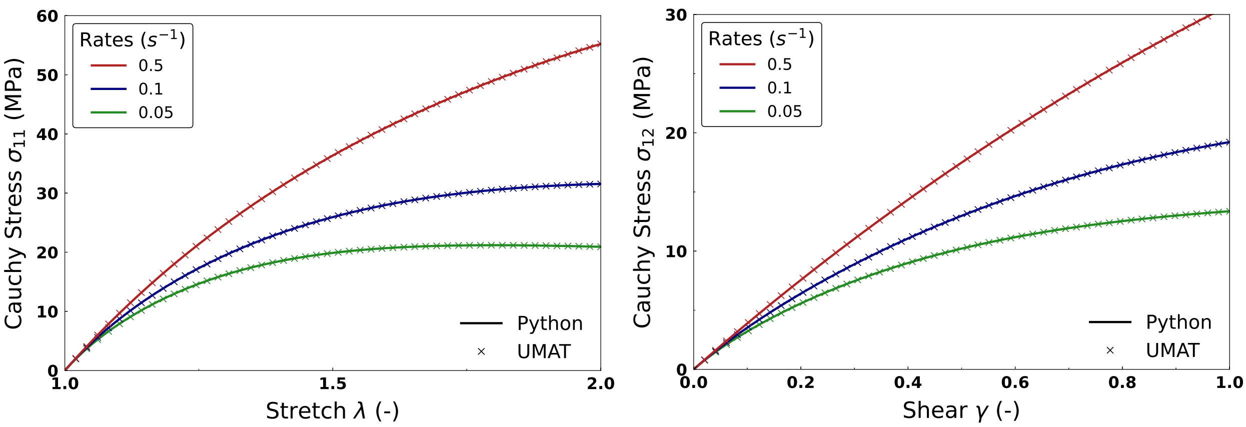
Unless stated otherwise, the deviatoric part of the strain energy density is defined by Eq. (11) and the hydrostatic part is given by $\mathcal{U}(J) = \frac{K}{4}\left(J^2 - 2\ln(J) - 2\right)$, with $(\mu, K) = (\mu^e, K^e)$ for the elastic branch, and $(\mu, K) = (\mu^k, K^k)$ for the viscoelastic branches.

Example

Strain rate: Uniaxial tension and simple shear at constant displacements of 0.05/L s^{-1}, 0.1/L s^{-1} and 0.5/L s^{-1}, with L the cube length.

Elastic branch	$\mu^e = 1,\ K^e = 20$
2 viscoelastic branches	$\mu^1 = 9,\ \ K^1 = 20,\ \eta_d^1 = 9,\ \ \ \eta_h^1 = 20$
	$\mu^2 = 30,\ K^2 = 40,\ \eta_d^2 = 300,\ \eta_h^2 = 400$
Damage	*null*

Result: Uniaxial tension (left) - Simple shear (right)

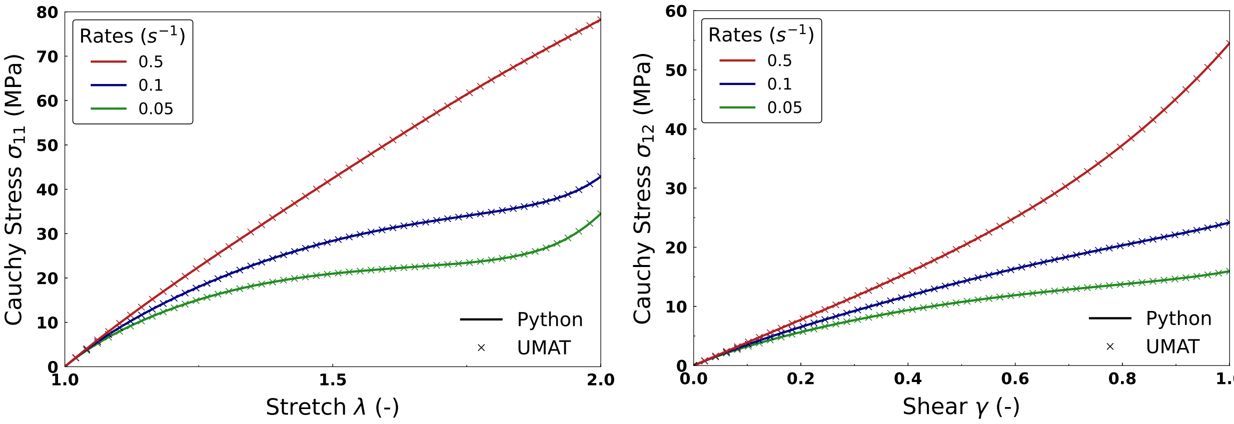


Example

Strain energy density: Uniaxial tension and simple shear at a constant displacements of 0.05/L s^{-1}, 0.1/L s^{-1} and 0.5/L s^{-1}, with L the cube length, for a Gent material defined by $\overline{W}(\bar{\mathbf{C}}) = -\frac{\mu}{2} J_m \ln\left(1 - \frac{\bar{I}_1 - 3}{J_m}\right)$.

Elastic branch	$\mu^e = 1,\ K^e = 20,\ J_m^e = 1.5$
2 viscoelastic branches	$\mu^1 = 9,\ \ K^1 = 20,\ \eta_d^1 = 9,\ \ \ \eta_h^1 = 20,\ \ J_m^1 = 1.5$
	$\mu^2 = 30,\ K^2 = 40,\ \eta_d^2 = 300,\ \eta_h^2 = 400,\ J_m^2 = 1.5$
Damage	null

Result: Uniaxial tension (left) - Simple shear (right)

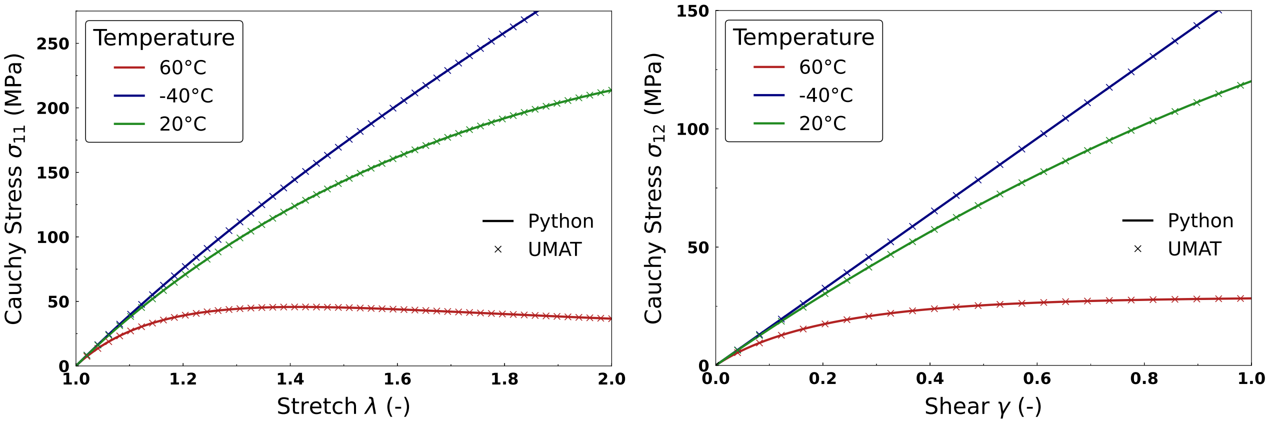


Example

Temperature dependence: Uniaxial tension and simple shear at a constant displacement of 0.1/L s^{-1}, with L the cube length, for several temperatures.

Elastic branch	$\mu^e = 1,\ K^e = 20$
4 viscoelastic branches	$\mu^1 = 9,\ K^1 = 20,\ \eta_d^1 = 9,\ \eta_h^1 = 20$
	$\mu^2 = 30,\ K^2 = 40,\ \eta_d^2 = 300,\ \eta_h^2 = 400$
	$\mu^3 = 40,\ K^3 = 80,\ \eta_d^3 = 1000,\ \eta_h^3 = 2000$
	$\mu^4 = 80,\ K^4 = 160,\ \eta_d^4 = 4000,\ \eta_h^4 = 8000$
Damage	*null*
Time-temperature superposition	$T_{ref} = 20\ ^{\circ}\mathrm{C},\ C_1 = 6.5,\ C_2 = 166.2\ ^{\circ}\mathrm{C}$

Result: Uniaxial tension (left) - Simple shear (right)

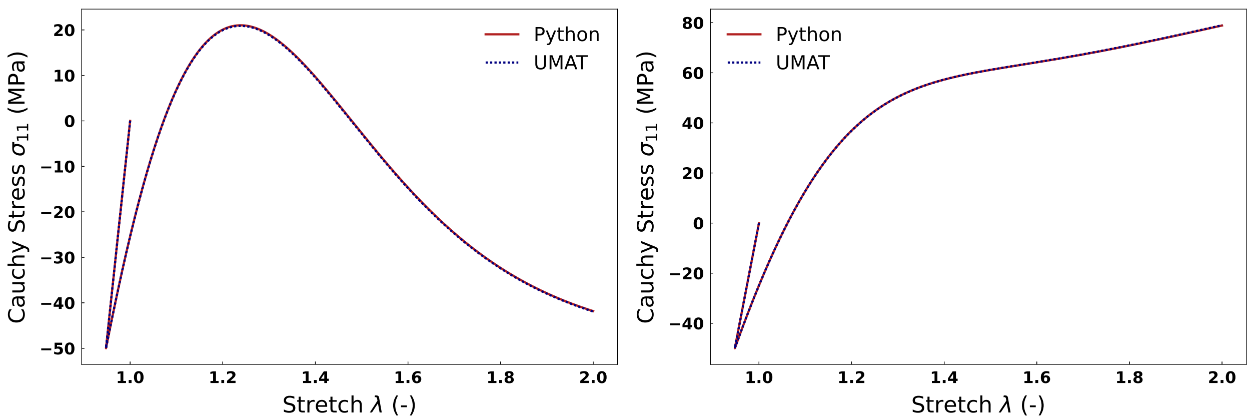


Example

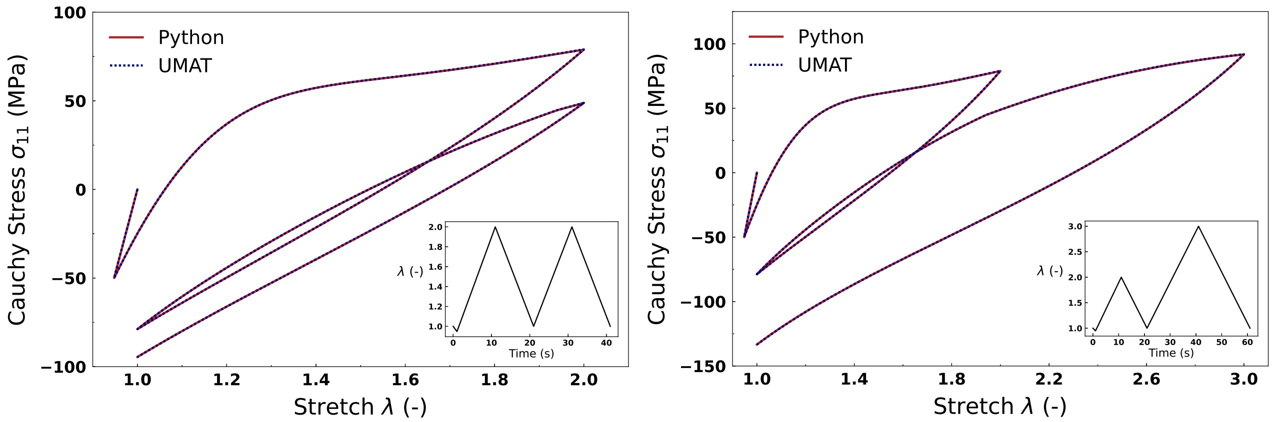
Deviatoric damage: Uniaxial tension at a constant displacement of 0.1/L s^{-1}, with L the cube length, monotonic and cyclic loadings.

Elastic branch	$\mu^e = 1,\ K^e = 20$
4 viscoelastic branches	$\mu^1 = 9,\ K^1 = 20,\ \eta_d^1 = 9,\ \eta_h^1 = 20$
	$\mu^2 = 30,\ K^2 = 40,\ \eta_d^2 = 300,\ \eta_h^2 = 400$
	$\mu^3 = 40,\ K^3 = 80,\ \eta_d^3 = 1000,\ \eta_h^3 = 2000$
	$\mu^4 = 80,\ K^4 = 160,\ \eta_d^4 = 4000,\ \eta_h^4 = 8000$
Damage variable	$\bar{h}_{eq}$
Damage parameters	$a_d = 1.65,\ b_d = 6.5,\ w_d = 0.95,\ P_s = 50\,\text{MPa}$

Result: no hydrostatic pressure pressure (left) - with pressure dependence (right)



Result: Cyclic loadings with pressure dependence

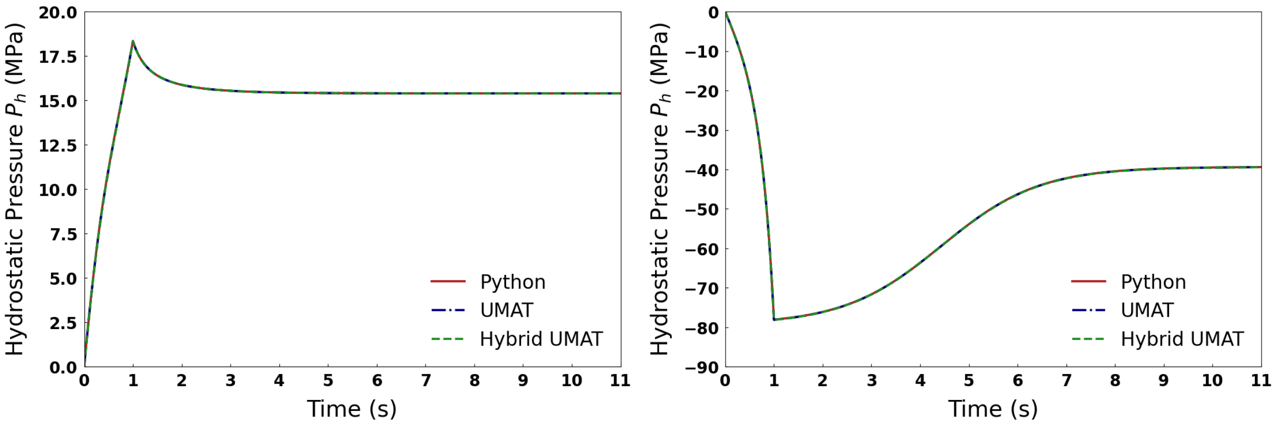


Example

Hydrostatic loading: A last test corresponding to hydrostatic displacements of 0.5/L $^{-1}$ with L the length of the cube, in compression or in tension and followed by a relaxation step.

Elastic branch	$\mu^e = 0.5,\ K^e = 10$
1 viscoelastic branch	$\mu^1 = 0.5, \quad K^1 = 10, \quad \eta_d^1 = 10^5, \qquad \eta_h^1 = 1$
No damage	

Result: Hydrostatic tension (left) - Hydrostatic compression (right)



4 How to use

Using the UMAT as distributed The UMAT may be tested using single-element input files, `Shear.inp`, `UTension.inp`, and Hydrostatic.inp which correspond to simple shear, uniaxial tension and hydrostatic compression tests, respectively. The distributed UMAT defines a material with a strain energy density given by:

$$\begin{aligned} \mathcal{W}(\mathbf{C}, \mathbf{C}_e^1, \ldots, \mathbf{C}_e^n) &= (1 - D_d)\left(\tfrac{\mu_e}{2}(\bar{I}_1 - 3) + \textstyle\sum_k \tfrac{\mu^k}{2}(\bar{I}_{1e}^k - 3)\right) \\ &+ (1 - D_h)\left(\tfrac{K_e}{4}(J^2 - 2\ln(J) - 1) + \textstyle\sum_k \tfrac{K^k}{4}\left({J_e^k}^2 - 2\ln(J_e^k) - 1\right)\right) \end{aligned} \tag{15}$$

Considering n viscoelastic branches, $12 + n \times 4$ parameters must be provided, in the following order:

- Parameters for temperature dependence T_{ref}, C_1 and C_2
- The choice for the deviatoric damage variable $n \in \{0, 1, 2, 3, 4\}$
  `0:` no damage, `1:` $\sqrt{\frac{\bar{I}_1}{3}} - 1$, `2:` m, `3:` $\bar{I}_\gamma$, `4:` $\bar{h}_{eq}$

- Damage parameters a_d, b_d, a_h, b_h, ω_d and P_s
- The elastic parameters μ_e and K_e
- The viscoelastic parameters μ^1, K^1, η_d^1, η_h^1, $\ldots$, μ^n, K^n, η_d^n, η_h^n

Note: Following Abaqus conventions, parameters are provided by lines of eight values. Consequently, the parameters for a neo-Hookean material with two viscoelastic branches will appear as follows:

Input changes

```
*User Material, constants=20
20, -6.5, 166., 1, 0.0, 0.0, 0.0, 0.0,
0.0, 1.0, 1.0, 20.0, 9.0, 20.0, 9.0, 20.0,
30.0, 40.0, 300.0, 400.0,
```

The `*Depvar` parameter specifies the number of internal variables that can be tracked. In this implementation, the variables are defined in the following order: 6 variables for each viscoelastic branch representing the components of the symmetric tensor $\bar{\mathbf{b}}_e^k$,($\bar{\mathbf{b}}_{e\,11}^k$, $\bar{\mathbf{b}}_{e\,22}^k$, $\bar{\mathbf{b}}_{e\,33}^k$, $\bar{\mathbf{b}}_{e\,12}^k$, $\bar{\mathbf{b}}_{e\,13}^k$, $\bar{\mathbf{b}}_{e\,23}^k$), followed by the damage variables α_d and α_h, and the damage functions D_d and D_h. Thus, for two viscoelastic branches:

Input changes

```
*Depvar
16,
```

Final recommendations:

Parameters input

- If temperature dependence is not required, keep the temperature fixed at $T = T_{ref}$ using the `*INITIAL CONDITIONS, TYPE=TEMP` option.
- If damage modeling is not needed, set the damage choice parameter to `0`.
- For hydrostatic damage, the pressure dependence is directly implemented in the UMAT as `DDAMVARHDT = ( ( DAMVARD1 - DAMVARD0 ) / DTIME ) * ( ONE - (NINE/TEN) * ( ONE - EXP(-PRESSURE/P_SAT) ) )`, implying the existence of a parameter ω_h exists, which is set here to $1 - \frac{9}{10} = 0.9$.
- For any branch k where one of the viscosities is null, avoid setting η_d or η_h to zero directly. Instead, set K^k or μ^k to zero accordingly.
- If K becomes significantly larger than μ, consider using the hybrid version of the UMAT.

Modifying the UMAT The constitutive equations for the strain energy density are implemented directly within the UMAT. To modify the deviatoric part of the strain energy density, the following lines need to be adjusted,

Input changes

For the elastic part $\overline{\mathcal{W}}^{eq}$:	For the viscoelastic part $\overline{\mathcal{W}}^{neq}$:
ALPHAEQ = (MU1/TWO)	ALPHANEQ(N_B) = MUVIS(N_B)/TWO
DALPHAEQDI1 = ZERO	DALPHANEQDI1(N_B) = ZERO
DALPHAEQDI2 = ZERO	DALPHANEQDI2(N_B) = ZERO
BETAEQ = ZERO	BETANEQ(N_B) = ZERO
DBETAEQDI1 = ZERO	DBETANEQDI1(N_B) = ZERO
DBETAEQDI2 = ZERO	DBETANEQDI2(N_B) = ZERO

with the following definition from Eq. (10),

$$
\begin{cases}
\mathtt{ALPHAEQ} = \dfrac{\partial \overline{\mathcal{W}}^{\mathtt{eq}}(\bar{\mathbf{C}})}{\partial \bar{\mathtt{I}}_1} + \bar{\mathtt{I}}_1 \dfrac{\partial \overline{\mathcal{W}}^{\mathtt{eq}}(\bar{\mathbf{C}})}{\partial \bar{\mathtt{I}}_2} \\
\mathtt{DALPHAEQDI1} = \dfrac{\partial \mathtt{ALPHAEQ}}{\partial \bar{\mathtt{I}}_1} \\
\mathtt{DALPHAEQDI2} = \dfrac{\partial \mathtt{ALPHAEQ}}{\partial \bar{\mathtt{I}}_2} \\
\mathtt{BETAEQ} = -\dfrac{\partial \overline{\mathcal{W}}^{\mathtt{eq}}(\bar{\mathbf{C}})}{\partial \bar{\mathtt{I}}_2} \\
\mathtt{DBETAEQDI1} = \dfrac{\partial \mathtt{BETAEQ}}{\partial \bar{\mathtt{I}}_1} \\
\mathtt{DBETAEQDI2} = \dfrac{\partial \mathtt{BETAEQ}}{\partial \bar{\mathtt{I}}_2}
\end{cases}
\quad
\begin{cases}
\mathtt{ALPHANEQ} = \dfrac{\partial \overline{\mathcal{W}}^{\mathtt{neq}}(\bar{\mathbf{C}}_{\mathtt{e}})}{\partial \bar{\mathtt{I}}_{1\mathtt{e}}} + \bar{\mathtt{I}}_{1\mathtt{e}} \dfrac{\partial \overline{\mathcal{W}}^{\mathtt{neq}}(\bar{\mathbf{C}}_{\mathtt{e}})}{\partial \bar{\mathtt{I}}_{2\mathtt{e}}} \\
\mathtt{DALPHANEQDI1} = \dfrac{\partial \mathtt{ALPHANEQ}}{\partial \bar{\mathtt{I}}_{1\mathtt{e}}} \\
\mathtt{DALPHANEQDI2} = \dfrac{\partial \mathtt{ALPHANEQ}}{\partial \bar{\mathtt{I}}_{2\mathtt{e}}} \\
\mathtt{BETANEQ} = -\dfrac{\partial \overline{\mathcal{W}}^{\mathtt{neq}}(\bar{\mathbf{C}}_{\mathtt{e}})}{\partial \bar{\mathtt{I}}_{2\mathtt{e}}} \\
\mathtt{DBETANEQDI1} = \dfrac{\partial \mathtt{BETANEQ}}{\partial \bar{\mathtt{I}}_{1\mathtt{e}}} \\
\mathtt{DBETAEQDI2} = \dfrac{\partial \mathtt{BETANEQ}}{\partial \bar{\mathtt{I}}_{2\mathtt{e}}}
\end{cases}
$$

$$
\begin{aligned}
\text{with}\quad \bar{I}_1 &= \mathtt{PVBAR(1)} + \mathtt{PVBAR(2)} + \mathtt{PVBAR(3)}, \\
\bar{I}_2 &= \mathtt{PVBAR(1)} * \mathtt{PVBAR(2)} + \mathtt{PVBAR(2)} * \mathtt{PVBAR(3)} + \mathtt{PVBAR(3)} * \mathtt{PVBAR(1)}, \\
\bar{I}_{1e} &= \mathtt{PVBeBAR(N\_B,1)} + \mathtt{PVBeBAR(N\_B,2)} + \mathtt{PVBeBAR(N\_B,3)}, \\
\bar{I}_{2e} &= \mathtt{PVBeBAR(N\_B,1)} * \mathtt{PVBeBAR(N\_B,2)} + \mathtt{PVBeBAR(N\_B,2)} * \mathtt{PVBeBAR(N\_B,3)} \\
&\quad + \mathtt{PVBeBAR(N\_B,3)} * \mathtt{PVBeBAR(N\_B,1)}.
\end{aligned}
$$

For the hydrostatic part of the strain energy density,

Input changes

For the elastic part $\mathcal{U}^{eq}$:

```
PVTAUEQH = KELAS/TWO*(DET*DET - ONE)
CABHYD = KELAS*DET*DET
```

For the viscoelastic part $\mathcal{U}^{neq}$:

```
PVTAUNEQH(N_B) = KVIS(N_B)/TWO*(Je(N_B)*Je(N_B) - ONE)
DPVHYDTAUDEPSe(N_B) = KVIS(N_B)*Je(N_B)*Je(N_B)
```

with

$$\begin{cases} \texttt{PVTAUEQH} = J\,\dfrac{\partial \mathcal{U}^{eq}}{\partial J} \\[2ex] \texttt{CABHYD} = J\,\dfrac{\partial\left(J\frac{\partial \mathcal{U}^{eq}}{\partial J}\right)}{\partial J} \\[2ex] \texttt{PVTAUNEQH(N\_B)} = J_e\,\dfrac{\partial \mathcal{U}^{neq}}{\partial J_e} \\[2ex] \texttt{DPVHYDTAUDEPSe(N\_B)} = J_e\,\dfrac{\partial\left(J_e\frac{\partial \mathcal{U}^{neq}}{\partial J_e}\right)}{\partial J_e} \end{cases}$$

It is recommended to define the material parameters directly in the input file. Consequently, one may need to adjust the order and number of input parameters. Furthermore, the input variables are passed through the variable `PROPS`.

Input changes

To add or modify the internal variables of interest, the `STATEV` array, which defines the internal variables, must be updated. Additionally, the `DEPVAR` parameter in the input file needs to be adjusted accordingly.

Using the hybrid version (`UMAT_hybrid_FG.for`) When the bulk modulus K becomes significantly larger than the shear modulus μ, a hybrid formulation is necessary to ensure convergence. In Abaqus/Standard, hybrid elements (H) and a specific hybrid formulation must be used. As recommended in the ABAQUS User's Manual, the total formulation based on the Lagrange multiplier has been applied, using the option `HYBRID FORMULATION=TOTAL`. For a concise and clear explanation of this formulation, we recommend reading (Lefèvre et al., 2024). In this respect, ABAQUS requires the following quantities,

$$\begin{cases} \hat{K} = J\dfrac{\partial^2 \mathcal{U}(\hat{J})}{\partial \hat{J}^2} \\[2ex] \dfrac{\partial \hat{K}}{\partial \hat{J}} = J\dfrac{\partial^3 \mathcal{U}(\hat{J})}{\partial \hat{J}^3} \end{cases} \tag{16}$$

with $\hat{J} = \texttt{STRESS(NTENS + 1)}$, a read-only variable.
The modifications from the standard formulation involve the Cauchy stress and the Jacobian, which are now expressed as,

$$\begin{cases} \sigma = \sigma^D + \dfrac{\partial \mathcal{U}(\hat{J})}{\partial \hat{J}}\mathbf{I_d} \\[2ex] \mathcal{L}_{ijkl} = \mathcal{L}^D_{ijkl} + \hat{K}\,\delta_{ij}\delta_{kl} \end{cases} \tag{17}$$

where $(\bullet)^D$ denotes the deviatoric part of the quantity $(\bullet)$.

The hybrid formulation has been implemented for the strain energy density as follows:

$$\mathcal{W} = \overline{\mathcal{W}}^{eq}(\bar{I}_1, \bar{I}_2) + \sum_{k=1}^{n} \overline{\mathcal{W}}_k^{neq}(\bar{I}_{1\,e}^k, \bar{I}_{2\,e}^k) + \frac{K_\infty}{4}\left(J^2 - 2\ln J - 1\right) + \frac{K_v}{4}\left(J_e^2 - 2\ln Je - 1\right). \tag{18}$$

To run simulations with the hybrid version of the UMAT, modify the input file following the requirements below:

Input changes

Change the element type `C3D8` $\rightarrow$ `C3D8H`

```
*Depvar
7*N_B+4, with N_B the number of viscoelastic branches

*User Material, constants=16 →
*User Material, constants=16, HYBRID FORMULATION=TOTAL
```

Now for each branch J_v is an internal variable that is updated by solving equation $\dot{J}_v^k = \frac{J}{\eta_h^k}\frac{\partial \mathcal{U}_k^{neq}(J_e^k)}{\partial J_e^k}$ with $J = \hat{J}$, indepently with a Newton-Raphson scheme.
Examples are provided in the files `cube_shearH.inp`, `cubeUH.inp` and `cube_pressureH.inp`.

Note: There is no strict threshold for $\frac{\mu}{K}$ below which the regular formulation will fail to converge or produce erroneous results. This likely depends on the material parameters and the loading conditions. However, when $\frac{\mu}{K}$ increases, both the hybrid and standard versions provide similar results.

Changing the deviatoric part of the strain energy is identical to the procedure for the standard `UMAT`. For the hydrostatic part, one will need to modify the following lines,

Input changes

```
DUEDJHAT = KELAS/TWO*(AJHAT - ONE/AJHAT)
D2UEDJHAT2 = KELAS/TWO*(ONE + ONE/(AJHAT*AJHAT))
D3UEDJHAT3 = -KELAS/(AJHAT*AJHAT*AJHAT)

DUEDJHATNB = (KVIS(N_B)/TWO)*(Je(N_B)**2-ONE)/AJHAT
D2UEDJHAT2NB = KVIS(N_B)/TWO*(Je(N_B)**2+ONE)/(AJHAT*AJHAT)
D3UEDJHAT3NB = -KVIS(N_B)/AJHAT**3
```

with `DUEDJHAT` $= \left.\frac{\partial \mathcal{U}^{eq}(J)}{\partial J}\right|_{\hat{J}}$, `D2UEDJHAT2` $= \left.\frac{\partial^2 \mathcal{U}^{eq}(J)}{\partial J^2}\right|_{\hat{J}}$, `D3UEDJHAT3` $= \left.\frac{\partial^3 \mathcal{U}^{eq}(J)}{\partial J^3}\right|_{\hat{J}}$, and with `DUEDJHATNB` $= \left.\frac{\partial \mathcal{U}^{neq}(J)}{\partial J}\right|_{\hat{J}}$, `D2UEDJHAT2NB` $= \left.\frac{\partial^2 \mathcal{U}^{neq}(J)}{\partial J^2}\right|_{\hat{J}}$, `D2UEDJHAT2NB` $= \left.\frac{\partial^3 \mathcal{U}^{neq}(J)}{\partial J^3}\right|_{\hat{J}}$.
Finally, note that in order to validate the Jacobian estimates, both UMAT were tested on

structures with several elements as the Pokerchip test for the hybrid case that is provided here as `PokerUH.inp`.

References

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