Systems of More Than One Particle

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Systems of particles

- *"An intellect which at a certain moment would know all forces ... and all positions ..."*
- What is it that determines the force on a given particle?
- It is the positions of all the other particles.

Systems of particles

- The fundamental forces are those that act between particles, like gravity and electric forces.
- These depend on a number of things: gravitational forces between particles are proportional to their masses, and electric forces are proportional to their electric charge.
- Masses and charges are considered to be intrinsic properties of a particle, and specifying them is part of specifying the system itself.
- Apart from the intrinsic properties, the forces depend on the location of the particles.
- For example, the distance between objects determines the electric and gravitational force that one exerts on another.

Systems of particles

- The force on particle i is a function of its location as well as the location of all the others.
- We can write it in the form

$$\vec{F}_i = \vec{F}_i(\{\vec{r}\})$$

- The symbol $\{\vec{r}\}$ stands for the collective location of every particle in the system.

Systems of particles

- Newton's equation of motion for particle i is

$$\vec{F}_i(\{\vec{r}\}) = m_i \vec{a}_i$$

- When we express the acceleration as the second derivative of the position, the equation becomes

$$\vec{F}_i(\{\vec{r}\}) = m_i \frac{d^2 \vec{r}_i}{dt^2}$$

- There are three equations for each particle (for each spatial component x_i, y_i, z_i for each $\vec{r}_i$), so if there are N particles there are $3N$ equations.

The space of states of a system of particles

- The formal meaning of the state of a system is, “Everything you need to know (with perfect accuracy) to predict its future, given the dynamical law.”
- In Aristotelian mechanics, assuming that the forces are known, the state is specified by simply knowing the location of the particle.
- In fact, from Aristotle’s law, the force determines the velocity and the velocity tells you where the particle will be at the next moment.

The space of states of a system of particles

- But Newton's law is different from Aristotle's: it tells you the acceleration not the velocity.
- This means that to get started you need to know not only where the particles are but also their velocities.
- Knowing the velocity tells you where the particle will be at the next instant, and knowing the acceleration tells you where the velocity will be.
- Therefore, the state of a system of particle consists of more than just their locations; it also includes their current velocities.
- This obviously means that the space of states is $6N$ dimensional: three position components and three velocity components for of the N particles.

The space of states of a system of particles

- If the state is $6N$ dimensional, why is it that $3N$ components are enough how the system evolves?
- Let's go back to a system of a single particle with the specified forces and write Newton's equation:

$$m\frac{d\vec{v}}{dt} = \vec{F}$$

- Since there is no expression for velocity, let's add another equation expressing that velocity is the rate of change of position:

$$\frac{d\vec{r}}{dt} = \vec{v}$$

The space of states of a system of particles

- Applying the same idea to each individual particle, gives us $6N$ equations governing the motion through the space of states:

$$m_i \frac{d\vec{v}_i}{dt} = \vec{F}_i$$
$$\frac{d\vec{r}_i}{dt} = \vec{v}_i$$

- Wherever you happen to be in the $6N$ dimensional space of states, the above equations tell you where you will be next.
- Therefore, these equations are suitable dynamical laws.

Momentum and phase space

- The momentum of is the product of the velocity and mass

$$\vec{p} = m\vec{v}$$

- We can use momentum and position instead of velocity and position to label the points of the state-space.
- When the state-space is described this way, it has a special name – phase space.
- The term configuration space is used for something else, namely the three-dimensional space of positions.
- The slogan is: *Configuration space plus momentum space equals phase space*.

Momentum and phase space

- The dynamical law in phase space becomes

$$\dot{\vec{p}}_i = \vec{F}_i(\{\vec{r}\})$$
$$\dot{\vec{r}}_i = \frac{\vec{p}_i}{m}$$

Action, reaction, and the conservation of momentum

- The principle of the conservation of momentum is a profound consequence of abstract general principles of classical mechanics that we will study later.
- But it can also be understood at an elementary level from Newton's third law of motion.

  > *For every action there is an equal and opposite reaction.*

Action, reaction, and the conservation of momentum

- The simplest way to think of the third law is to suppose first that particles interact in pairs.
- Each particle j exerts a force on each other particle i, and the total force on any particle is the sum of the forces on it exerted by all the other particles.
- If we denote the force on particle i due to particle j by the symbol $\vec{f}_{ij}$, then the total force acting on particle i is

$$\vec{F}_i = \sum_j \vec{f}_{ij}$$

Action, reaction, and the conservation of momentum

- Newton's law of action and reaction is about the force between pairs of particles $\vec{f}_{ij}$.
- What it says is simple: The force due to one particle j on another particle i is equal and opposite to the force due to particle i acting on particle j

$$\vec{f}_{ij} = -\vec{f}_{ji}$$

Action, reaction, and the conservation of momentum

- Let us rewrite the first equation as

$$\vec{p}_i = \sum_j \vec{f}_{ij}$$

- In words, the rate of change of the momentum of any particle is the sum of the forces due to all the other particles.
- Let's add up all these equations to see how the total momentum changes

$$\sum_i \vec{p}_i = \sum_i \sum_j \vec{f}_{ij} \tag{1}$$

Action, reaction, and the conservation of momentum

- Let's add up all these equations to see how the total momentum changes

$$\sum_i \vec{p}_i = \sum_i \sum_j \vec{f}_{ij} \tag{2}$$

- The lhs is the rate of change of total momentum and the rhs is zero because each $\vec{f}_{ij}$ is offset by $\vec{f}_{ji} = -\vec{f}_{ij}$.

Action, reaction, and the conservation of momentum

- We obtain the equation

$$\frac{d}{dt}\sum_i \vec{p}_i = 0$$

  that is precisely the mathematical expression of the conservation of momentum.
- The total momentum of an isolated system never changes.

Action, reaction, and the conservation of momentum

- Let's consider the $6N$-dimensional space of p's and x's.
- At every point the entire collection of momenta are specified, so it follows that every point in the phase space is (partially) characterized by a value of the total momentum.
- As time evolves, the phase point sweeps out a path in phase space.
- Every point on that path is labelled with the same value of total momentum.
- This is similar to the conservation laws that we studies earlier.