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Optimal Tooth Numbers for Compact Standard Spur Gear Sets

The design of a standard gear mesh is treated with the objective of minimizing the gear size for a given ratio, pinion torque, and allowable tooth strength. Scoring, pitting fatigue, bending fatigue, and the kinematic limits of contact ratio and interference are considered. A design space is defined in terms of the number of teeth on the pinion and the diametral pitch. This space is then combined with the objective function of minimum center distance to obtain an optimal design region. This region defines the number of pinion teeth for the most compact design. The number is a function of the gear ratio only. A design example illustrating this procedure is also given.

Introduction

The design of a gear set is a reasonably difficult problem which involves the satisfaction of many design constraints. In recent literature, several approaches to the optimum design of a gear mesh have been presented. Cockerham [1] presents a computer design program for 20 deg pressure angle gearing, which ignores gear-tooth-tip scoring. This program varies the diametral pitch, face width, and gear ratio to obtain an acceptable design. Tucker [2] and Estrin [3] look more closely at the gear mesh parameters, such as addendum ratio and pressure angle, and outline procedures for varying a standard gear mesh to obtain a more favorable gear set. Gay [4] considers gear tip scoring as well and shows how to modify a standard gear set to bring this mode of failure into balance with the pitting fatigue mode. He also adjusts the addendum ratios of the gear and pinion to obtain an optimal design. No general procedure exists, however, to determine the optimal size of a standard gear mesh. The basic approach available is one of checking a given design to verify its acceptability [5-7].

Optimum methods are presented for the design of a gearbox [8-10] with the object of minimizing size and weight. These methods focus on multistage gear reductions and consider the effect of splitting the gear ratios on overall transmission size.

The optimum design of a standard gear set has not been treated in the literature to date. Such a study must be based on a thorough study of the kinematics of the gear mesh, such as those by Buckingham [11] and Andersson [12]. The gear strengths that must be considered include bending fatigue as treated by the AGMA [13], by Gitchel [14], and by Mitchiner and Mabie [15]. Surface pitting of the gear teeth in the full-load region must also be treated [16-18] as must gear scoring at the tip of the gear tooth [19, 20].

The object of the research reported here is to establish an optimal design procedure for standard spur gear pairs. Figure 1 shows a single mesh of the external type while Fig. 2 shows a single mesh of the internal type. The procedure developed in this paper utilizes standard gear geometry and optimizes the design parameters to obtain the most compact standard gear set for a given application of specified speed reduction and input pinion torque. This procedure applies to both internal and external gearing.

Gear Design Problem

A gear set is normally used to transmit an input torque from a shaft turning at one speed to an output torque on a shaft turning at a lower speed. Parameters for this situation are the gear ratio, m_g, the pinion torque, T_p, and the pinion speed, Ω_p.

For an economical solution to this gear design problem, a standard tooth system made from standard tooling is to be sought. The standard American Gear System employs a limited number of diametral pitches, a standard operating pressure angle of 20 or 25 deg, and standard ratios of pinion and gear tooth addenda and dedenda to the reciprocal of diametral pitch. For full depth teeth, these standards are 1.0 for the addendum ratio and 1.25 for the dedendum ratio.

The size of the gear set is another design parameter which affects cost. The gear mesh center distance, C, is a measure of the size of the gear set. Obviously, a smaller gear set will use less material for the gears and for the surrounding housing. If the same life and reliability can be achieved in a small package, a more economical design is achieved.

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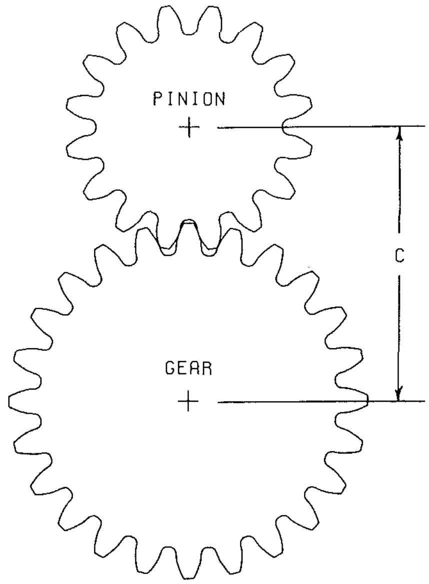

Fig. 1 External gear mesh

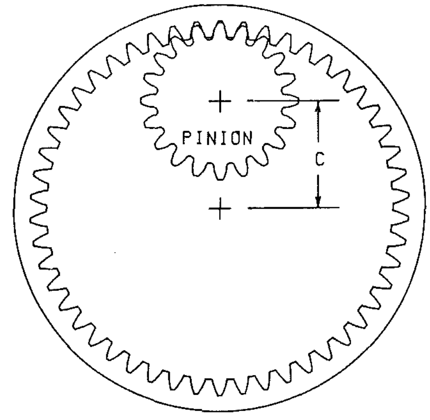

Fig. 2 Internal gear mesh

Finally, the gear materials and their properties are an important facet of the gear design problem. The material cost, its modulus, E, its fatigue strength in bending, σ_B and its surface endurance strength, σ_N, all affect the design.

Parameters and Constraints. One advantage to using an optimization technique is that it enables the designer to consider a spectrum of possible designs. The method employed in this study starts by listing the design parameters available, the equality constraints which must be satisfied, the inequality constraints which define the limits of acceptable designs, and the merit function which is used to compare the relative merit of each possible design on an objective basis.

In Table 1, the primary design parameters and the relevant constraints are listed. In Table 1, the number of pinion teeth, N_1, is not subject to any constraint relation. Parameters of this type will hereinafter be called "free parameters." The gear ratio, m_g, is an input parameter and the number of teeth on the gear, N_2, is related to the number of teeth on the pinion by the gear ratio. For continuous rotation, both N_1 and N_2 must have integer values. The diametral pitch, P_d, is the second free parameter. The pinion pitch radius, R_1, and the gear pitch radius, R_2, define the size of the gears, but are directly related to the tooth numbers by the diametral pitch. The center distance, C, is simply the sum of the pitch radii.

The next four items listed in Table 1 are the pinion and gear addendum and dedendum ratios. For a standard gear set, these ratios are 1 and 1.25 as noted. However, these ratios are among the first parameters changed when nonstandard gearing is utilized in order to improve properties available from standard gear sets [2-4]. The involute profile is also modified at its tip and root by a process called tip relief to improve the dynamic loading in the gear teeth at high speeds [21]. Dynamic loading and tip relief will not be treated in this study since we are searching for an optimal standard geometry which can be modified.

The gear mesh face width, f, is the next item listed. As long as the tooth load remains uniform, this width is directly proportional to the resulting strength of the gear set. For the tooth load to be uniform, a common criterion is to limit the length to diameter ratio, λ, of the line of contact [7]. This is the ratio of the gear face width to the pitch diameter of the smallest gear in the mesh, λ, and is related to the face width by the number of teeth on the pinion and the diametral pitch as shown. The mesh face width is thus determined for a given design by this limit.

The pitch line pressure angle is another parameter that has been standardized to limit the required tooling inventory. Current values of common usage today are 20 and 25 deg.

The last series of items in Table 1 are the relevant material properties, which become fixed once materials and their heat treatments are specified for the pinion and the gear. Table 1 includes four free parameter groups: addendum and dedendum ratios, face width, pressure angle, and material constants, which are tied down by standard practice but which

Nomenclature

a = addendum ratio
C = center distance (m)
d = dedendum ratio
E = elastic modulus (Pa)
F = tangential tooth load (N)
f = gear face width (m)
k = slope of equal size design line in design space
m_g = gear ratio
m_p = contact ratio
N = tooth number
n = number of million stress cycles
p_b = base pitch (m)
P_d = diametral pitch
R = pitch radius (m)
R_a = addendum radius (m)
R_b = base circle radius (m)
S = probability of survival
T_p = pinion torque (Nm)
V = stress volume (m^3)
Y = Lewis tooth form factor
Z = length of action (m)
z_o = depth to maximum shearing stress (m)
θ_β = pinion roll angle to lowest point of single tooth contact (rad)
θ_C = pinion roll angle to lowest point of tooth contact (rad)
λ = length to diameter ratio
υ = Poisson's ratio
ρ = radius of curvature (m)
σ_B = bending fatigue stress (Pa)
σ_{BT} = bending fatigue stress for load at the pinion tooth tip (Pa)
σ_N = surface pressure (Pa)
σ_{NT} = surface pressure at the gear tooth tip (Pa)
τ = shearing stress (Pa)
ϕ = pressure angle (deg)
Ω_p = pinion speed (rpm)

Subscripts

1 = pinion
2 = gear

Table 1 Gear mesh parameter constraints

Parameter	Description	Equality constraint
N_1	pinion tooth number	
m_g	gear ratio	$m_g = (m_g)$ design
N_2	gear tooth number	$N_2 = m_g N_1$
P_d	diametral pitch	
R_1	pinion pitch radius	$R_1 = N_1/2P_d$
R_2	gear pitch radius	$R_2 = N_2/2P_d$
C	center distance	$C = R_2 \pm R_1$
a_1	pinion addendum ratio	1 for standard tooth form
a_2	gear addendum ratio	1 for standard tooth form
d_1	pinion dedendum ratio	1.25 for standard tooth form
d_2	gear dedendum ratio	1.25 for standard tooth form
f	mesh face width	$f = \lambda N_1/P_d$
ϕ	pitch line pressure angle	$\phi = \phi_{\mathrm{STD}}$
E_1	pinion modulus	
υ_1	pinion Poisson's ratio	pinion
σ_{B1}	pinion bending design stress	material properties
σ_{N1}	pinion surface design stress	
E_2	gear modulus	
υ_2	gear Poisson's ratio	gear
σ_{B2}	gear bending design stress	material properties
σ_{N2}	gear surface design stress	

could be varied in the design of nonstandard gearing. Several other parameters are determined exactly in terms of the input specifications and the free parameters. One parameter is a design input quantity, and only two parameters are free to be varied over an arbitrary range of values. These two free parameters in this formulation of the gear mesh design problem are the number of teeth on the pinion and the diametral pitch. This leaves the designer with a two-dimensional design space for standard gears and a six-dimensional design space for nonstandard gears.

The design spaces are limited by the constraints of the problem. These constraints could generate a null design space by placing conflicting requirements on the free parameters. In a well-posed design problem, they will enclose a bounded area of acceptable designs.

To assure a quiet mesh, the contact ratio of the mesh should be greater than some minimum value. The value 1.4 is commonly stated [6, 7], but higher values will make the mesh even quieter. For reasonable manufacturing of the gear teeth, both the tooth tip and the tool tip should be wider than a specified minimum [3]. These widths permit proper surface hardening of the tooth and prevent excessive tool tip wear in manufacture. For standard gearing, these three inequalities are automatically satisfied by the standard.

For proper tooth engagement and disengagement, involute interference (contact below the base circle) and secondary interference between the pinion tooth tip and the internal gear tooth tip for internal gearing must be avoided.

For proper gear mesh strength and life, the possibility of failure by three different mechanisms must be avoided. These mechanisms are pinion tooth bending fatigue, surface fatigue or spalling in the region of single tooth load, and gear tip scoring. To treat these three modes of failure on a common basis, a nominal stress approach is used. All three modes of failure are affected by more than the nominal design stress used herein. The bending fatigue is dependent on the surface finish of the tooth, among other factors [6, 7]. The surface fatigue of the tooth is influenced by the stress volume and does not have an infinite life endurance limit [17]. The gear tip scoring failure is highly temperature dependent [22]. However, this temperature is a direct result of the Hertz (contact) stress and sliding velocity at the gear tip. The Hertz stress is thus a meaningful parameter to predict the severity of both surface pitting and tip scoring. The nominal tooth bending stress will also be used as a measure of the bending fatigue severity.

Once all these limits have been applied to the design space, the designer is in a position to survey the acceptable designs and select the optimum. The criterion of this selection, called the merit function, is established as the center distance of the gears. By minimizing the center distance at a specified load, one produces the most economical gear set for the stated conditions, since it would use the least material for the gears and permit the gearbox to assume a minimum size. This criterion could be inverted very easily if a fixed size were available for the gear set. The merit function would then become the maximum transmitted load for the given size or the maximum reliability for a given size and load.

Kinematic Considerations. The contact ratio for a given spur gear mesh is the ratio of the length of contact along the line of action between the two addendum circles to the base pitch [7]:

$$m_p = \frac{\overline{BD}}{p_b} \tag{1}$$

The base pitch, p_b, is defined as the distance from one tooth to the next tooth measured along the line of action.

$$p_b = \frac{\pi}{P_d} \cos \phi \tag{2}$$

And the distance $\overline{BD}$, called the length of contact Z, is shown in Figs. 3 and 4 for contact with external and internal gears.

In terms of the pinion addendum ratio, a_1, and the gear addendum ratio, a_2, the expression for the contact ratio becomes

$$m_p = \frac{N_1}{2\pi \cos \phi} \left\{ \sqrt{\left(1 + \frac{2a_1}{N_1}\right)^2 - \cos^2 \phi} - \sqrt{\left(m_g \pm \frac{2a_2}{N_1}\right)^2 - m_g^2 \cos^2 \phi} - (1 \pm m_g) \sin \phi \right\} \tag{3}$$

which is independent of the physical size of the gears. In this equation, the two positive signs hold for external gear contact, while the two negative signs hold for internal gear contact.

This contact ratio should be greater than 1.4 and is normally less than 2.0 for standard spur gears. For contact with an internal gear at low ratios and for nonstandard addenda and dedenda, it can exceed 2.0. For the strength modeling of this design study, it will be assumed to be less than 2.0.

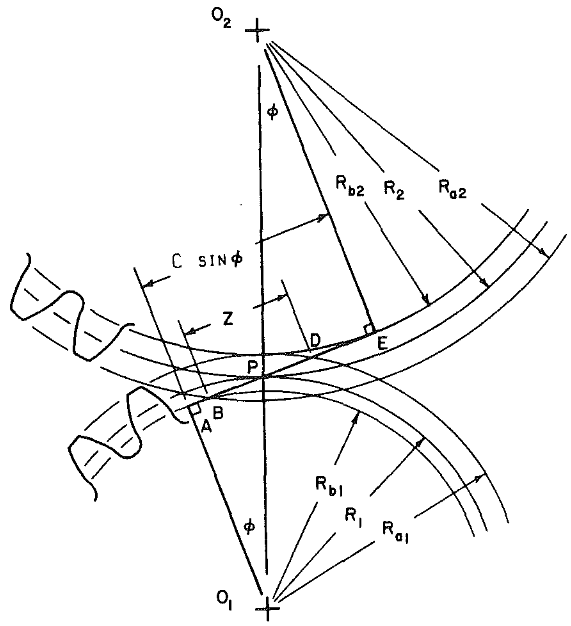

Fig. 3 Pinion-external gear mesh geometry

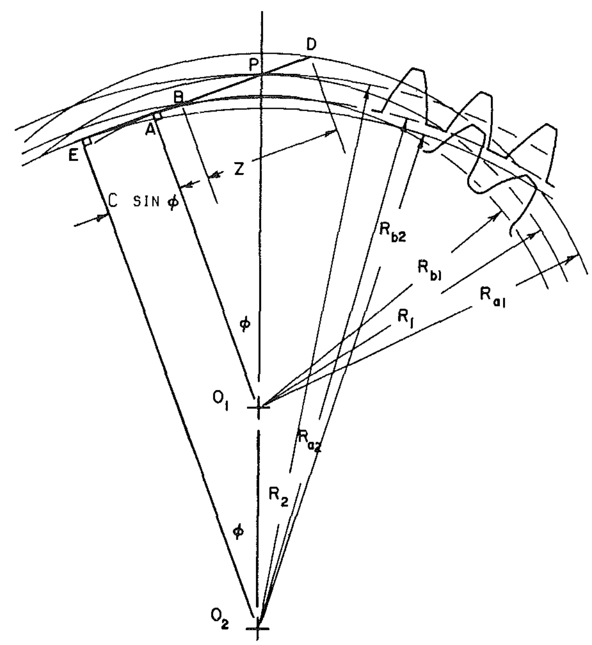

Fig. 4 Pinion-internal gear mesh geometry

Involute interference occurs when the addendum circle of one gear crosses the line of action past its point of tangency with the base circle on its mating gear. For standard tooth systems with equal addenda, the pinion will always be the gear on which contact occurs below the base circle. This is shown in Fig. 3 for contact with an external gear and in Fig. 4 for contact with an internal gear. For contact with an external gear, involute interference is avoided if

$$C \sin \phi > \overline{BE} \tag{4}$$

For contact with an internal gear, involute interference is avoided if

$$C \sin \phi < \overline{BE} \tag{5}$$

These conditions can be manipulated into a single lower bound on the number of teeth in the pinion for no involute interference,

$$N_1 > \left| \frac{2a_2/m_g}{1-\sqrt{\cos^2\phi + \left(\dfrac{1}{m_g} \pm 1\right)^2 \sin^2\phi}} \right| \tag{6}$$

where the positive sign holds for contact with an external gear and the negative sign holds for contact with an internal gear.

A second type of kinematic interference is possible for contact with an internal gear, in which the tip of the internal gear tooth contacts the tip of the pinion tooth as they come into engagement [11, 12]. This interference is called fouling. For standard gearing geometry, this condition is not present for internal gear ratios greater than 2.5:1 [23].

Gear Tooth Strengths. Spur gear teeth have three primary modes of failure. A tooth may fail in bending fatigue at its root; it may fail by pitting fatigue on its surface; or it may fail by scoring.

The basic model for bending failure of a gear tooth was developed by Wilfred Lewis in 1893 [15]. The knowledge of this mechanism of failure has increased significantly since then and compensating factors now exist which make the Lewis model of bending failure reasonably accurate. As a result, it is still in use today.

Lewis has written this exprssion as

$$\sigma_L = \frac{FP_d}{fY} \tag{7}$$

where F is the tangential load on the tooth and Y is the dimensionless Lewis form factor which can be determined from the tooth shape by an iterative technique [15, 23]. For standard teeth, this factor is a function of the number of pinion teeth, N_1, the gear ratio, m_g, the location of the load, and the pressure angle, ϕ.

It has been shown that the surface endurance of gear teeth behaves much like that of rolling element bearings [17]. The life of the contact is represented by

$$\ln \frac{1}{S} \sim \frac{\tau^c V n^e}{z_o^h} \tag{8}$$

where S is the probability of survival: τ is the critical shear stress; n is the number of million stress cycles; z_o is the depth to the critical stress; e is the Weibull exponent; and c and h are material constants. The three design parameters in this expression are the shear stress, its depth, and the volume of material subjected to the stress. Since the depth of the critical shear stress is nearly fixed and the stress is raised to a power in excess of 3 relative to the volume, it is assumed in this study that the magnitude of Hertz stress is a reasonable measure of the tendency of the surface to pit. It is thus used as a criterion for design comparison.

The Hertz stress which produces pitting can be modeled by

$$\sigma_N = \sqrt{\frac{F}{\pi f \cos\phi} \left[\frac{1/\rho_1 + 1/\rho_2}{\dfrac{1-\nu_1^2}{E_1} + \dfrac{1-\nu_2^2}{E_2}} \right]} \tag{9}$$

where ρ_1 and ρ_2 are the radii of curvature at the point of contact.

The scoring mode of failure is more difficult to predict since it is influenced by a combination of factors. Scoring is caused by an instability in oil film thickness at high speed and high load with inadequate cooling. In addition to surface pressure, this mode of failure is affected by the relatively large sliding velocity present at the gear tip and by the temperature of the teeth. Because of the sliding velocity present, some elastohydrodynamic effects are present in the contact which alter the stress distribution from the simple Hertzian model of equation (9). Oil cooling has a major effect in preventing scoring.

A second factor exists which increases the tip stress above that of the simple model. That factor is dynamic loading, which is largest at the initial point of contact. When the pinion drives the gear making the mesh a speed reduction, this point of initial contact is the gear tooth tip and the base of the pinion tooth. Because of the complexities involved in dynamic load estimation, it is felt that it is reasonable to model the contact pressure at the gear tip by equation (9).

Although contact pressure is only one factor in this mode of failure, it is a significant one. If this pressure is extreme, the design is not in balance and excessive measures must be taken in the other factors to compensate. It is felt then that the contact pressure at the gear tip should be kept to the same level as that in the single tooth load region.

Design Space. As shown in Table 1 and described in the section on parameters and constraints, the standard gear design problem for gears made of a chosen material can be reduced to a two-dimensional design problem where the two free parameters are the number of teeth on the pinion and the diametral pitch.

The inequality constraints which bound this design space are the minimum number of teeth required to prevent interference as given by equation (6) and the three strength limits. These strength limits can be converted to expressions for the maximum allowable diametral pitch for the given problem as functions of the number of teeth on the pinion. As stated in Table 1, the face width can be expressed in terms of the length to diameter ratio of the gear tooth contact

$$f = \lambda \frac{N_1}{P_d} \tag{10}$$

The load can also be expressed in terms of the diametral pitch

$$F = \frac{T_p}{R_1} = \frac{2T_pP_d}{N_1} \tag{11}$$

For the bending limit, where Y is the tooth form factor for the highest point of tooth contact for the given ratio and the number of teeth on the pinion, an upper bound on the diametral pitch can be determined from bending fatigue

$$P_d \leq \sqrt[3]{\frac{\sigma_B Y \lambda N_1^2}{T_p}} \tag{12}$$

This limit is more severe than that due to the full load taken at the highest point of single tooth contact.

Based on the assumption that Hertz stress is a measure of the tendency to pit, the diametral pitch limit based on the contact stress at the lowest point of single tooth contact is

$$P_d \leq \sqrt[3]{\frac{(1-\upsilon^2)\,\pi\lambda N_1^3 \sigma_N^2 \cos^2\phi\, \theta_B}{2\,T_p \mathrm{E} \sin\phi}\left[\sin\phi - \frac{\theta_B \cos\phi}{(1 \pm m_g)}\right]} \tag{13}$$

where θ_B is the roll angle of the pinion to the lowest point of single tooth contact [23].

Finally, a similar equation can be found for the limit on P_d based on gear tooth tip wear in terms of θ_c, the roll angle on the pinion to the initiaiton of contact.

$$P_d \leq \sqrt[3]{\frac{(1-\upsilon^2)\,\pi\lambda N_1^3 \sigma_N^2 \cos^2\phi\, \theta_c}{2\,T_P \mathrm{E} \sin\phi}\left[\sin\phi - \frac{\theta_c \cos\phi}{(1 \pm m_g)}\right]} \tag{14}$$

These limits are plotted in Fig. 5 for the case $\phi = 20$ deg, $m_g = 1.0$, $\lambda = 0.25$, $T_p = 113$ Nm (1000 lb in.), E = 205 GPa (30×10^6psi), $\upsilon = 0.25$, $\sigma_N = 1.38$ GPa (200,000 psi) and $\sigma_B = 414$ MPa (60,000 psi).

The gear for this design problem is external. The acceptable design space is the upper left hand corner of the plot.

The merit function for this problem is the center distance, C,

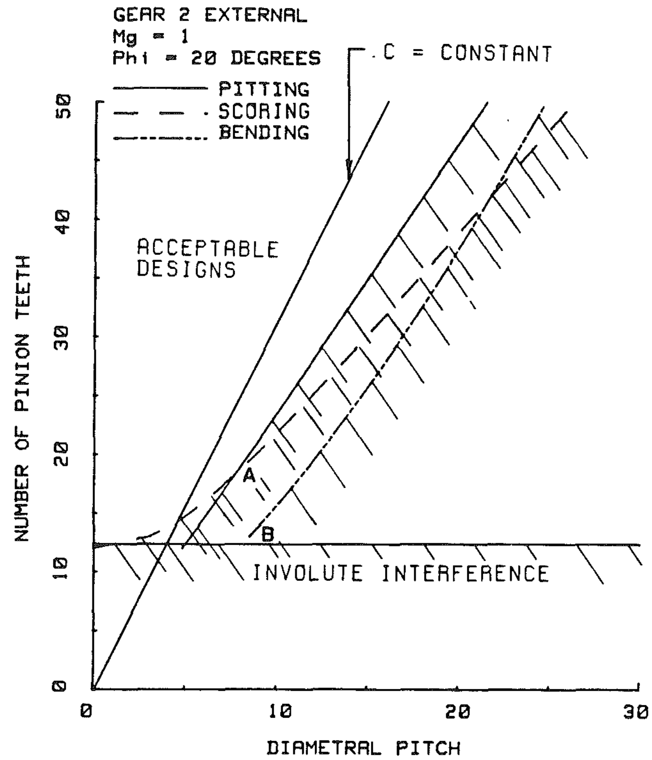

Fig. 5 External gear mesh design space with $\lambda = 0.25$, $T_p = 113$ Nm (1000 lb in.), $\sigma_N = 1.38$ GPa (200 ksi), $\sigma_B = 414$ MPa (60 ksi), E = 205 GPa (30×10^6 psi), and $\upsilon = 0.25$

$$C = \frac{N_1}{2P_d}(m_g \pm 1) \tag{15}$$

where the plus sign is valid for an external gear while the negative sign holds for an internal gear in equations (14-16). For a given gear ratio, the locus of equally optimum designs can be found by considering C to be constant. This produces the relation

$$N_1 = \left[\frac{2C}{(m_g \pm 1)}\right] P_d \tag{16}$$

which states that constant center distance designs lie on a straight line through the origin. Since the smaller C is, the better the design is, the best design corresponds to the line of least slope (smallest center distance) drawn through the origin which lies within the design space. For this design, the line of smallest center distance within the design space crosses the two surface pressure limits at their intersection, point A, in Fig. 5.

Optimal Designs. In this paper, arbitrarily chosen values are used for the tooth width to pinion pitch diameter ratio, λ, and the pinion torque T_p. Values of 0.25 and 113 Nm (1000 lb in.) were chosen to reflect reasonable geometry and a nominal load. The design spaces do not change in character as these quantities are varied, so no loss in generality is incurred due to their use. The values of elastic modulus, Poisson's ratio, allowed surface stress, and bending fatigue strength are for hardened steel. If another material is used for the gears, then the design space would be altered. However, the only real difference would be a shift in the relative importance of the allowed surface stress and the bending strength.

For high-speed, high-cycle operations, point A in Fig. 5 is an important design point since it identifies the minimum number of pinion teeth required to keep the gear tooth tip

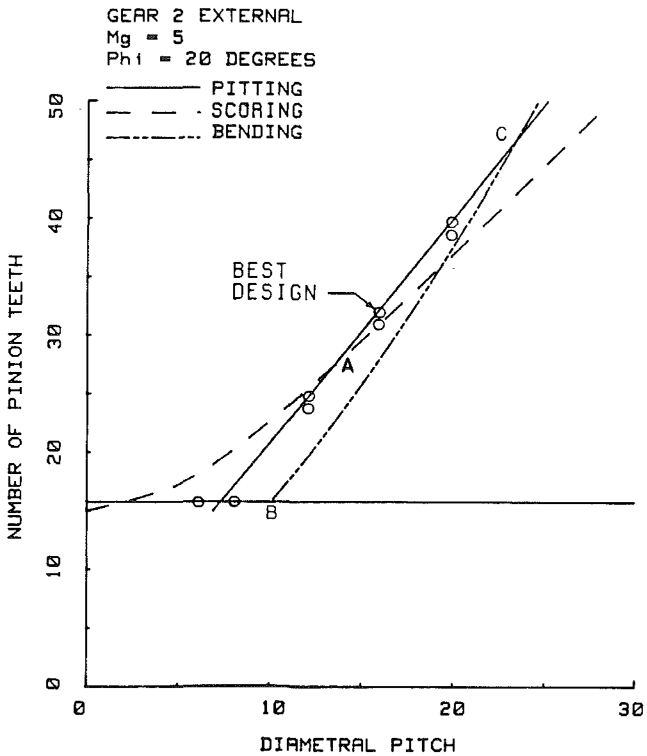

Fig. 6 External gear mesh design space with $\lambda = 0.25$, $T_p = 113$ Nm (1000 lb in.), $\sigma_N = 1.38$ GPa (200 ksi), $\sigma_B = 414$ MPa (60 ksi), $E = 205$ GPa (30×10^6 psi), and $\upsilon = 0.25$

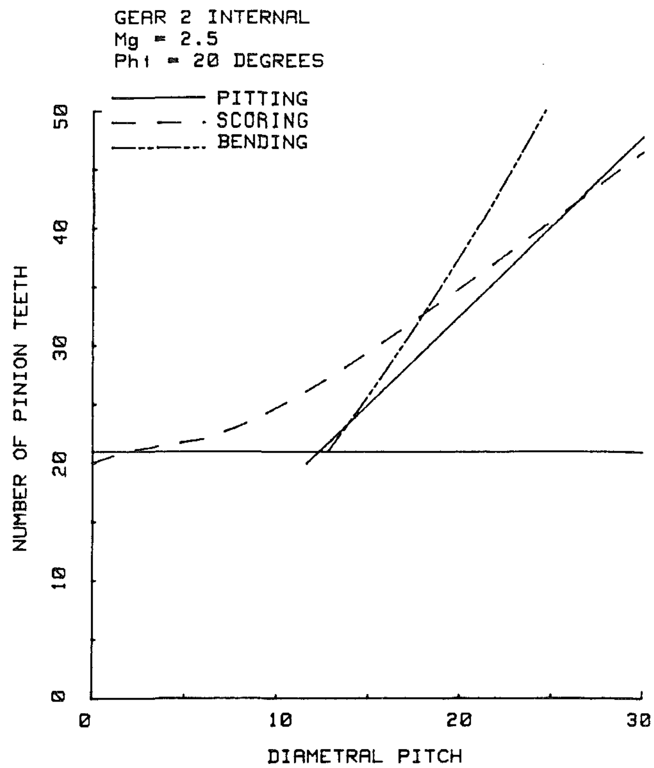

Fig. 7 Internal gear mesh design space with $\lambda = 0.25$, $T_p = 113$ Nm (1000 lb in.), $\sigma_N = 1.38$ GPa (200 ksi), $\sigma_B = 414$ MPa (60 ksi), $E = 205$ GPa (30×10^6 psi), and $\upsilon = 0.25$

contact stress at or below the maximum Hertz stress in the full load region of the tooth surface. By having at least this number of teeth in the pinion, a well-balanced compact design can be achieved by sizing the gears based on the full-load contact surface pressure.

A common design approach [7] in the design of gear sets has been to use the interference limit and the tooth bending strength limit to define the "best" design. This gives point B in the design space of Fig. 5. This point will produce a smaller gear set since $\overline{OB}$ has smaller slope than $\overline{OA}$, but it ignores the pitting problems in the full-load region and totally ignores the situation at the tip of the gear tooth. It thus will produce gears which are not balanced in their design and which may have pitting and scoring problems in service.

The design space of Fig. 5 can be used to study the effects of varying the parameters N_1 and P_d on a design with a gear ratio of 1 and a pressure angle of 20 deg. Similar design spaces can be drawn for different gear ratios and different pressure angles. Figures 6 and 7 are two more design spaces for a pressure angle of 20 deg. In Fig. 6, the gear is external and the ratio is 5:1, while in Fig. 7, the gear is internal and the ratio is 2.5:1.

A continuous change in characteristics occurs as the external gear ratio increases to infinity (contact with a rack) and the internal gear ratio reduces from infinity. Three distinct shifts occur in the design space as the gear ratio increases. First, the involute interference limit increases to a higher number of pinion teeth. Second, point A, the condition of equal surface contact pressure at the gear tip and at the lowest point of single tooth contact, shifts to a higher N_1 and a higher P_d. Third, the bending fatigue strength limit becomes more critical as the gear ratio increases.

The design tradeoff between pitting fatigue and bending fatigue is shown in the design space curves at the point where these two constraint curves cross (point C in Fig. 6). If one were just concerned with bending fatigue failure, moving from point C along the bending constraint line to get a reduction in the number of teeth would produce a better design. Conversely, if one were just concerned with pitting fatigue, then moving from point C along the pitting constraint line to get an increase in the number of teeth would produce a better design.

In [23], it is shown that increasing the pressure angle to 25 deg has the effect of reducing both the number of pinion teeth, N_1, and the diametral pitch, P_d, for the optimal design point A. Thus, increasing the pressure angle does not significantly improve the design from a pitting fatigue standpoint, although it favorably affects the bending fatigue of the teeth.

These design spaces indicate that the most compact designs will have a minimal number of teeth. However, this number is not based on involute interference as implied by point B, but is based on maximum allowable Hertz stress at the base of the pinion and on the region of single tooth contact. Figures 8 and 9 are plots of the numbers of teeth which produce this equality of Hertz stress assuming equal load sharing when two tooth sets are in contact. The upper branch of each curve is for contact with an internal gear, while the lower branch is for contact with an external gear. Figure 8 is for 20 deg pressure angle while Fig. 9 is for 25 deg pressure angle standard tooth systems. A truly optimal design might use a slightly higher number of teeth to obtain a combination of gears with a standard diametral pitch and whole teeth.

Design Example. To illustrate the use of this method in design, consider a gear ratio of 5 and a pressure angle of 20 deg. Consider the pinion torque to be 113 Nm (1000 lb in.) and the other design values to be identical to those of Figs. 5-7 for convenience. For contact with an external gear, Fig. 8 indicated that for an optimal design using standard teeth, the pinion should have about 27 teeth. The design space for this particular example is shown in Fig. 6. One could check the

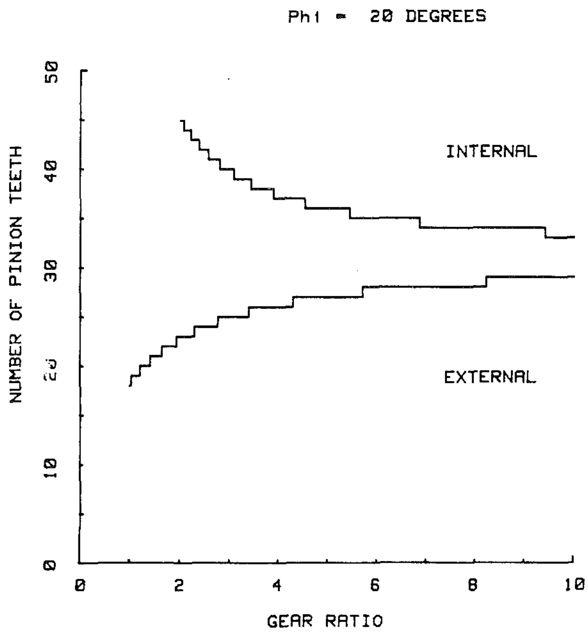

Fig. 8 Optimal number of pinion teeth for a 20 deg pressure angle

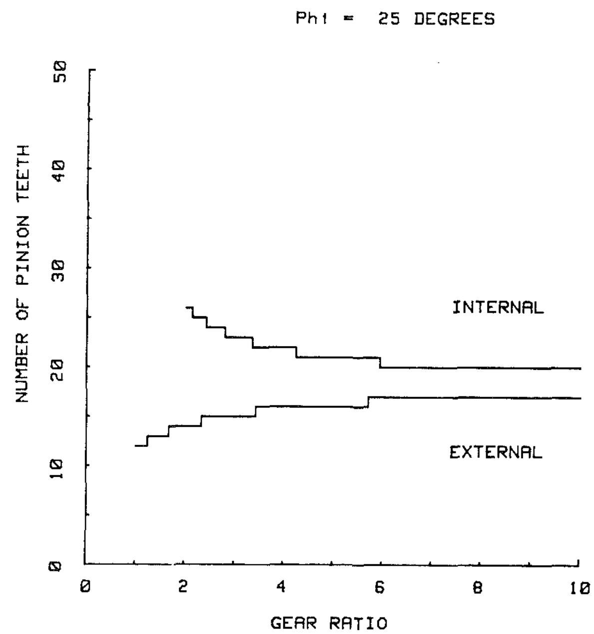

Fig.9 Optimal number of pinion teeth for a 25 deg pressure angle

design space of Fig. 6 to identify point A for which the diametral pitch is about 13.5.

Since this pitch would require nonstandard tooling, the best design will be shifted away from point A. However, it will lie near the pitting constraint line. This pitting constraint line can be closely approximated by a straight line from the origin through point A, with the slope given by

$$k = N_1/P_d \tag{17}$$

In this case

$$K = 27/13.5 = 2.0$$

The standard pitches near 13.5 can be used to find the numbers of teeth for near optimal designs with equation (17). Pitches of 12, 16, and 20 could be used to obtain minimal numbers of teeth of 24, 32, and 40, respectively. The best design will be in the set of trial designs with pinion teeth numbers near these limits.

Table 2 lists the possible optimal designs determined from equation (17) with diametral pitches of 12, 16, or 20. The best design is that with a P_d of 16 and 32 pinion teeth. This design has a center distance of 0.152 m (6.0 in.), a maximum contact pressure of 1.35 GPa (196 ksi) and a maximum fillet bending stress of 303 MPa (44 ksi). The design is shown in Fig. 10. Figure 11 is a plot of the maximum surface compression on the pinion tooth as a function of pinion roll angle.

It can be noted from Table 2 that an equally compact design exists with a diametral pitch of 20 and 40 pinion teeth. This design also has a center distance of 0.152 m (6.0 in.) and does not exceed any of the design stress limits. However, it has 25 percent higher bending stresses with no improvement in compactness, so the lower pitch design is to be preferred.

A popular approach to the design of a gear mesh [7] suggests that the strongest gear design results when the pinion has the smallest number of teeth possible. This makes the teeth large. Based on the minimum number of teeth required to eliminate interference, N_1 would be equal to 16. The last two designs in Table 2 show this design with a diametral pitch of 8 and 6.

As can be seen in Table 2, even decreasing the diametral pitch to 6 to further increase the tooth size does not reduce the scoring Hertz stress σ_{NT}, to the levels present in the optimum design. Figure 12 shows the minimum tooth design with a

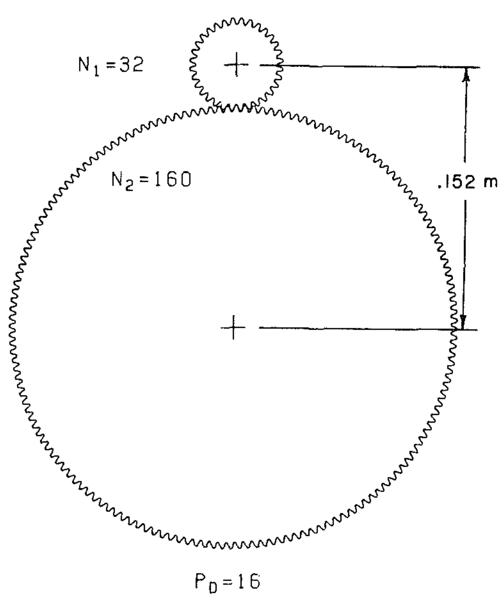

Fig. 10 Optimum design for $m_g = 5$, = 20 deg, and $T_p = 113$ Nm (1000 lb in.) for mesh with and external gear

diametral pitch of 8 and a center distance of 0.152 m (6.0 in.), which is the same as that of the optimal design. This design is stronger than the best design of Fig. 10 in bending fatigue and slightly weaker in pitting fatigue, but is extremly overloaded with contact pressure at the gear tip where scoring may occur. Figure 13 is a plot of the maximum surface pressure on the pinion tooth of this design as a function of pinion roll angle.

Design Procedure. The design space developed in this paper can be used to obtain optimal designs for gear meshes using standard spur gears. A procedure using this design space was followed in the design example of the previous section. Required input to the procedure is the gear ratio, the pinion torque, the pitch line pressure angle, the maximum allowable length to diameter ratio for the pinion pitch cylinder, and the material properties of the gears. Equation (6) can then be used to determine the kinematic interference limit for the mesh, and equations (12-14) can be used to determine the strength

Table 2 External gear designs; [$m_g = 5$, $\phi = 20$ deg]

P_d	N_1	N_2	f,	C,	σ_{NT},	σ_N	σ_B,	σ_{BT},
–	–	–	m	m	GPa	GPa	MPa	MPa
12	24	120	0.0127	0.152	1.51	1.39	228	248
12	25	125	0.0132	0.159	1.37	1.30	207	228
16	31	155	0.0123	0.148	1.34	1.43	297	331
16	32	160	0.0127	0.152	1.26	1.35	276	303
20	39	195	0.0124	0.149	1.21	1.39	338	379
20	40	200	0.0127	1.152	1.16	1.34	324	359
8	16	80	0.0127	0.152	6.87	1.49	193	200
6	16	80	0.0169	0.203	4.45	0.97	83	83

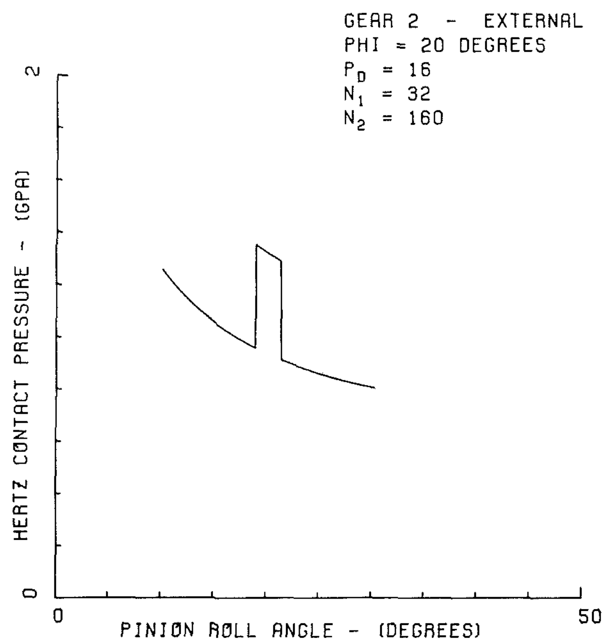

Fig. 11 Optimal design – tooth surface pressure

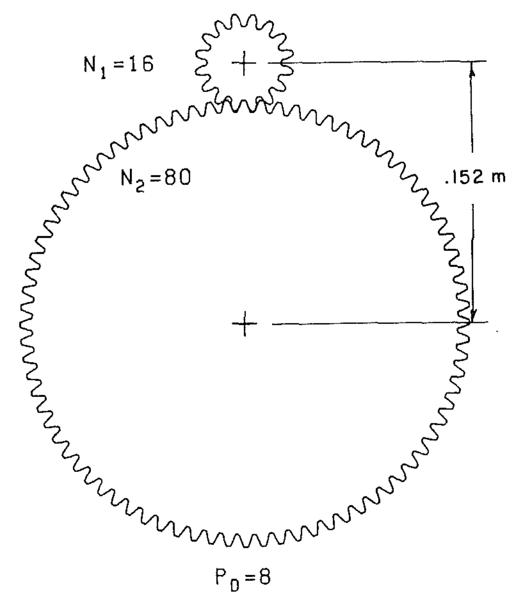

Fig. 12 Minimum tooth number design for $M_g = 5$, $\varphi = 20$ deg, and $T_p = 113$ Nm (1000 lb in.) for mesh with an external gear

constraints on the design space. By plotting these curves on a graph of pinion tooth number versus diametral pitch, the region of acceptable designs can be established as that region of high pinion tooth numbers and low diametral pitches bounded by these curves. The most compact designs lie on the line of least slope inside this region. For designs in which surface pressure dominates, the graphs of Figs. 8 and 9 show the optimal number of pinion teeth as a function of the gear ratio and pressure angle. By using the number of pinion teeth indicated by these graphs and a straight line through the origin, a set of standard pitches and corresponding minimum numbers of pinion teeth can be found near this optimal position. A small set of standard pitch practical designs can now be obtained by considering designs with these pitches and valid numbers of pinion teeth greater than the corresponding minimum numbers. By analyzing these designs and comparing their properties, a practical optimum design can be selected.

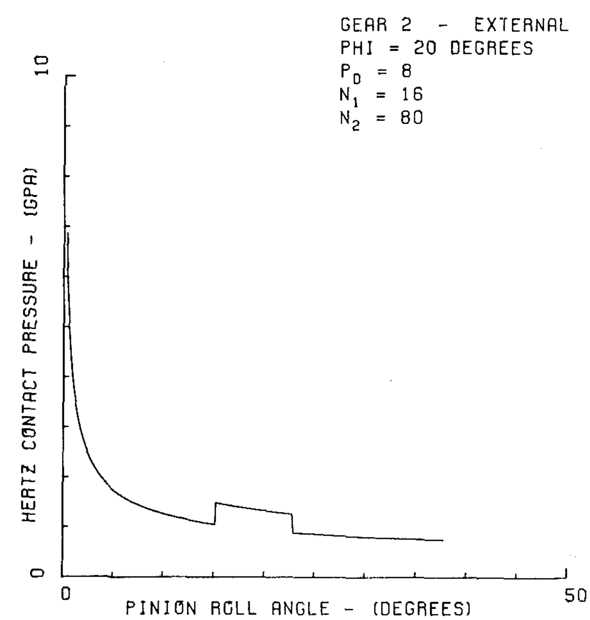

Fig. 13 Minimum tooth number design – tooth surface pressure

Summary

A design methodology for sizing standard involute spur gears was developed in this paper and applied to configure an optimal design. From the methodology presented, a design space may be formulated for either external gear contact or for internal gear contact. The design space includes kinematic considerations of involute interference, tip fouling, and contact ratio. Also included are design constraints based on bending fatigue in the pinion fillet and Hertzian (contact) stress in the full load region and at the gear tip where scoring is possible. This design space is two dimensional, giving the gear-mesh center distance as a function of diametral pitch and the number of pinion teeth.

The following results were obtained

1 The constraint equations were identified for kinematic interference (equation (6)), fillet bending fatigue (equation (12)), pitting fatigue (equation (13)), and scoring pressure

(equation (14)), which define the optimal design space for a given gear design.

2 The locus of equally sized optimum designs was identified as the straight line through the origin which has the least slope in the design region.

3 For designs in which bending fatigue is not dominant, the optimal design condition was identified as the point in the design space where the tooth tip contact pressure equals the maximum full load contact stress.

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