
Constraint Satisfaction Problems

AIMA: Chapter 6

Big idea

- Represent the **constraints** that solutions must satisfy in a uniform **declarative** language
- Find solutions by **GENERAL PURPOSE** search algorithms with no changes from problem to problem
 - No hand built transition functions
 - No hand built heuristics
- Just specify the problem in a formal declarative language, and a general purpose algorithm does everything else!

Constraint Satisfaction Problems

A CSP consists of:

- *Finite set of variables* X_1, X_2, \dots, X_n
- *Nonempty domain of possible values* for each variable D_1, D_2, \dots, D_n *where* $D_i = \{v_1, \dots, v_k\}$
- *Finite set of constraints* C_1, C_2, \dots, C_m
 - Each *constraint* C_i limits the values that variables can take, e.g., $X_1 \neq X_2$ A *state* is defined as an *assignment* of values to some or all variables.
- A *consistent* assignment does not violate the constraints.
- Example problem: Sudoku

Constraint satisfaction problems

- An assignment is **complete** when every variable is assigned a value.
- A **solution** to a CSP is a **complete, consistent** assignment.
- Solutions to CSPs can be found by a completely **general purpose** algorithm, given only the formal specification of the CSP.
- Beyond our scope: CSPs that require a solution that maximizes an **objective function**.

Applications

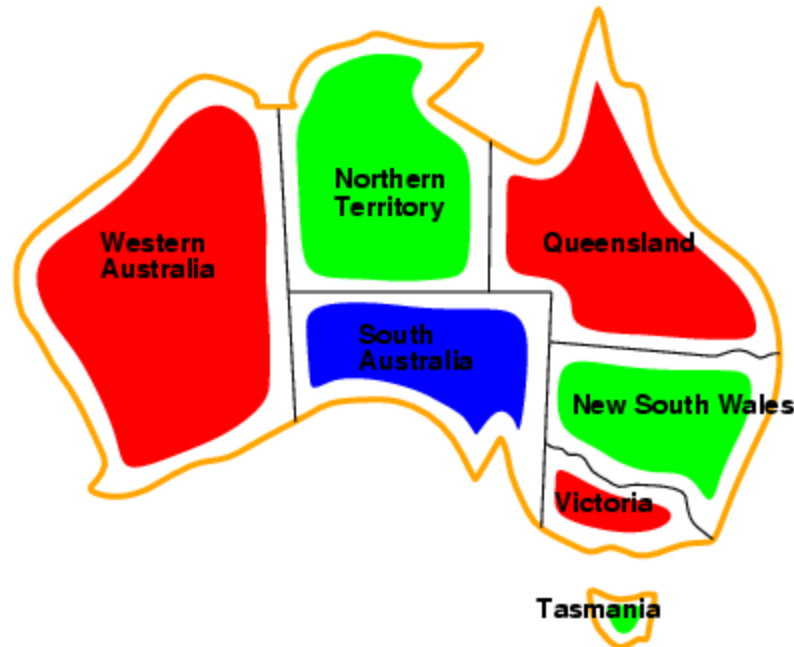
- **Map coloring**
- **Scheduling problems**
 - Job shop scheduling
 - Scheduling the Hubble Space Telescope
- **Floor planning for VLSI**
- **Sudoku**
- ...

Example: Map-coloring



- **Variables:** *WA, NT, Q, NSW, V, SA, T*
- **Domains:** $D_i = \{\text{red, green, blue}\}$
- **Constraints:** adjacent regions must have different colors
 - e.g., $WA \neq NT$
 - So (WA, NT) must be in $\{(\text{red, green}), (\text{red, blue}), (\text{green, red}), \dots\}$

Example: Map-coloring



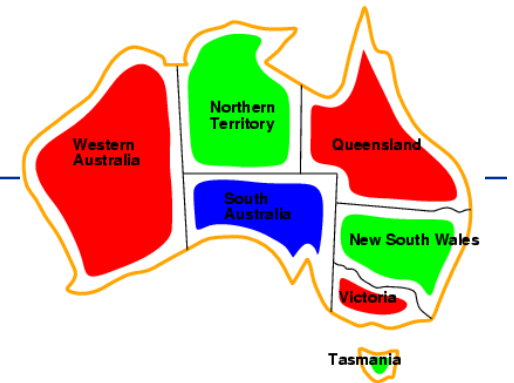
Solutions: **complete** and **consistent** assignments

- e.g., WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green

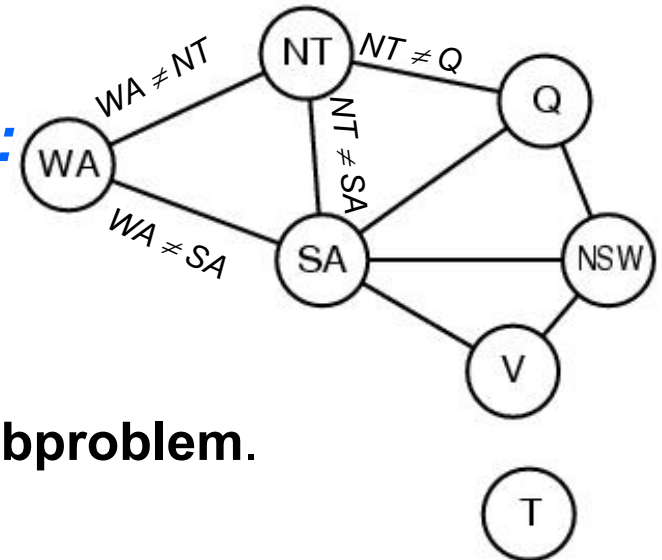
Benefits of CSP

- **Clean specification of many problems, generic goal, successor function & heuristics**
 - Just represent problem as a CSP & solve with general package
- **CSP “knows” which variables violate a constraint**
 - And hence where to focus the search
- **CSPs: Automatically prune off all branches that violate constraints**
 - (State space search could do this only by *hand-building constraints into the successor function*)

CSP Representations



- **Constraint graph:**
 - *nodes* are variables
 - *arcs* are (binary) constraints
- **Standard representation pattern:**
 - variables with values
- **Constraint graph** simplifies search.
 - e.g. Tasmania is an independent subproblem.
- **This problem: A binary CSP:**
 - each constraint relates two variables



Varieties of CSPs

- **Discrete variables**

- finite domains:
 - n variables, domain size $d \rightarrow O(d^n)$ complete assignments
 - e.g., Boolean CSPs, includes Boolean satisfiability (NP-complete)
- infinite domains:
 - integers, strings, etc.
 - e.g., job scheduling, variables are start/end days for each job
 - need a constraint language, e.g., $StartJob_1 + 5 \leq StartJob_3$

- **Continuous variables**

- e.g., start/end times for Hubble Space Telescope observations
- linear constraints solvable in polynomial time by linear programming

Varieties of constraints

- **Unary** constraints involve a single variable,
 - e.g., $SA \neq \text{green}$
- **Binary** constraints involve pairs of variables,
 - e.g., $SA \neq WA$
- **Higher-order** constraints involve 3 or more variables
 - e.g., crypt-arithmetic column constraints
- **Preference** (soft constraints) e.g. *red is better than green* can be represented by a cost for each variable assignment
 - Constrained optimization problems.

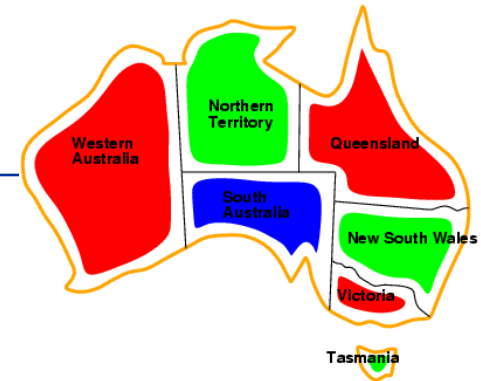
Idea 1: CSP as a search problem

- **A CSP can easily be expressed as a search problem**
 - *Initial State*: the empty assignment {}.
 - *Successor function*: Assign value to any unassigned variable *provided that there is not a constraint conflict*.
 - *Goal test*: the current assignment is complete.
 - *Path cost*: a constant cost for every step.
- **Solution is always found at depth n , for n variables**
 - Hence Depth First Search can be used

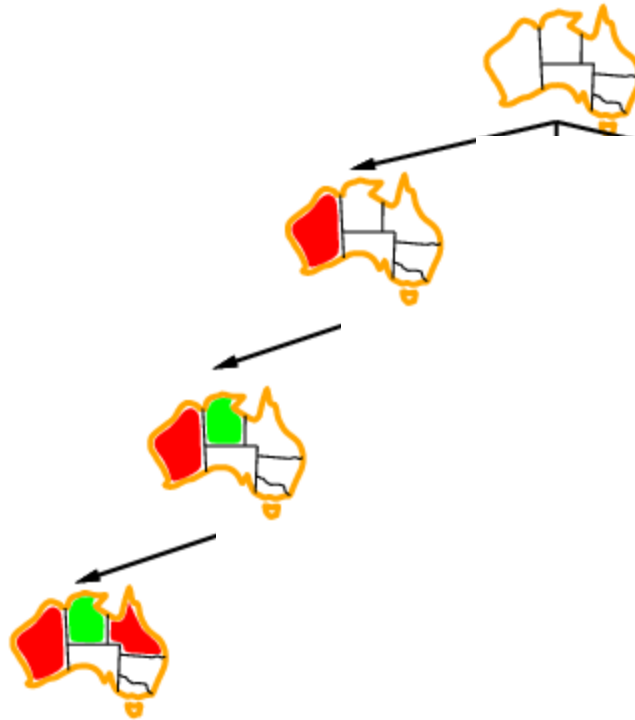
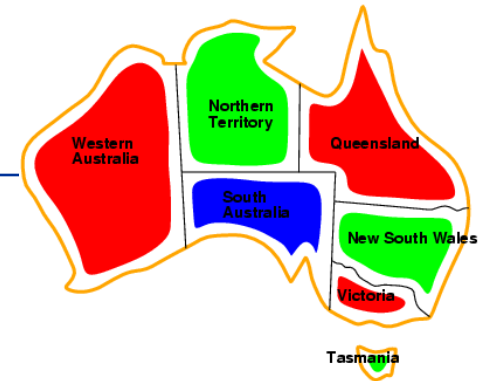
Backtracking search

- Note that variable assignments are *commutative*
 - Eg [step 1: **WA = red**; step 2: **NT = green**]
equivalent to [step 1: **NT = green**; step 2: **WA = red**]
 - Therefore, a *tree search*, not a *graph search*
- Only need to consider assignments to a single variable at each node
 - $b = d$ and there are d^n leaves (n variables, domain size d)
- Depth-first search for CSPs with single-variable assignments is called *backtracking* search
- Backtracking search is the basic *uninformed* algorithm for CSPs
- Can solve n -queens for $n \approx 25$

Backtracking example



Backtracking example



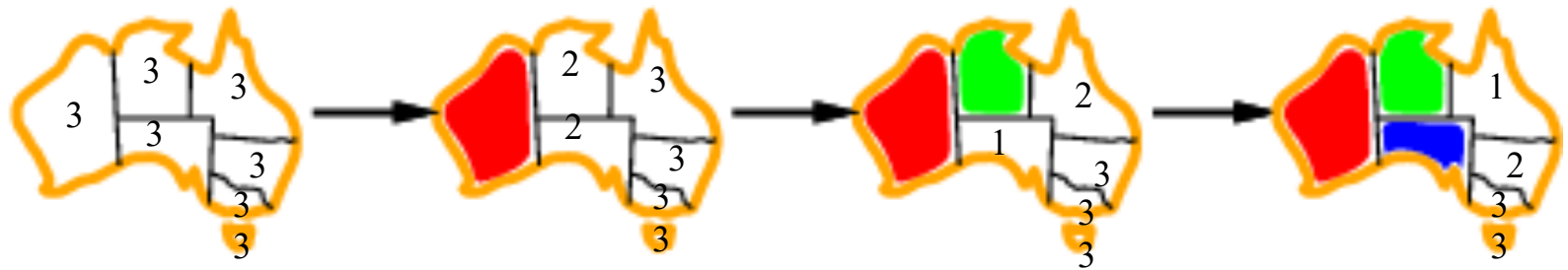
And so on....

Idea 2: Improving backtracking efficiency

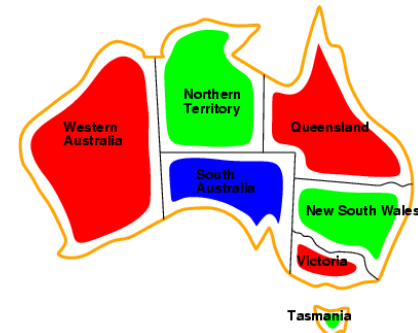
- **General-purpose** methods & **general-purpose** heuristics can give huge gains in speed, **on average**
- **Heuristics:**
 - Q: Which variable should be assigned next?
 1. **Most constrained** variable
 2. (if ties:) **Most constraining** variable
 - Q: In what order should that variable's values be tried?
 3. **Least constraining** value
 - Q: Can we detect inevitable failure early?
 4. **Forward checking**

Heuristic 1: Most constrained variable

- Choose a variable with the *fewest legal values*



- a.k.a. *minimum remaining values (MRV)* heuristic

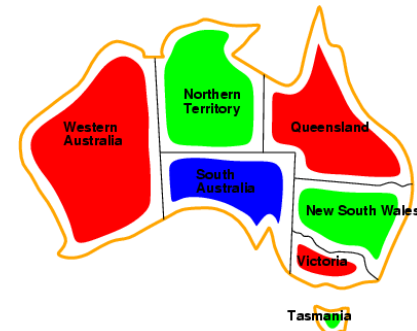


Heuristic 2: Most constrain^{ing} variable

- Tie-breaker among most constrained variables
- Choose the variable with the *most constraints on remaining variables*



These two heuristics together lead to immediate solution of our example problem

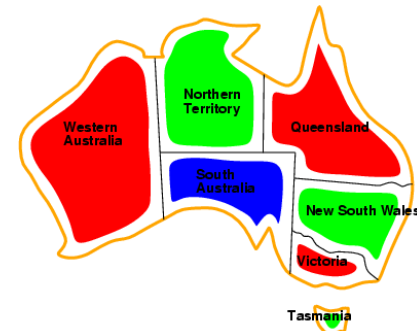


Heuristic 3: Least constraining *value*

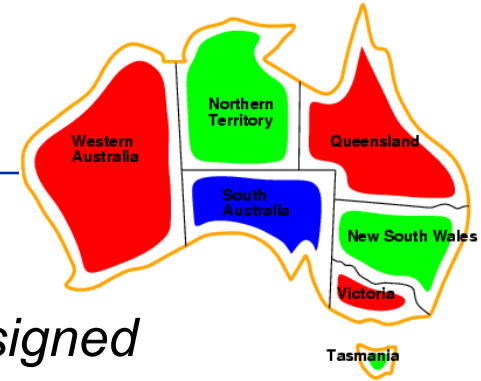
- Given a variable, *choose the least constraining value*:
 - the one that rules out the fewest values in the remaining variables



Note: demonstrated here independent of the other heuristics



Heuristic 4: Forward checking



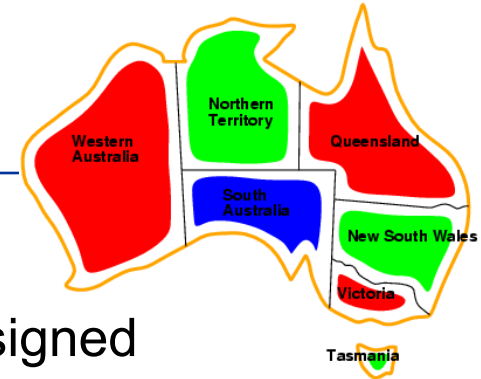
- **Idea:**
 - Keep track of *remaining* legal values for *unassigned* variables
 - Terminate search when any unassigned variable has no remaining legal values



New data structure

(A first step towards Arc Consistency & AC-3)

Forward checking

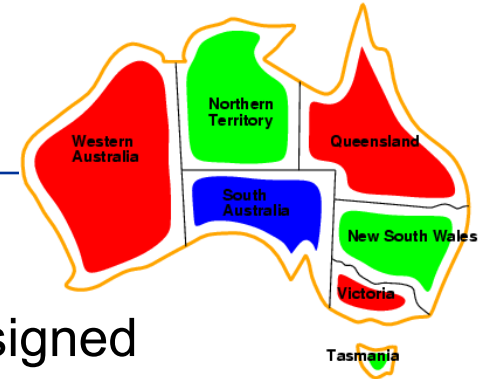


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WA	NT	Q	NSW	V	SA	T
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Forward checking

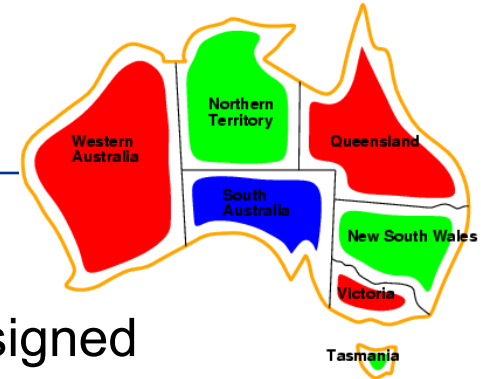


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Forward checking



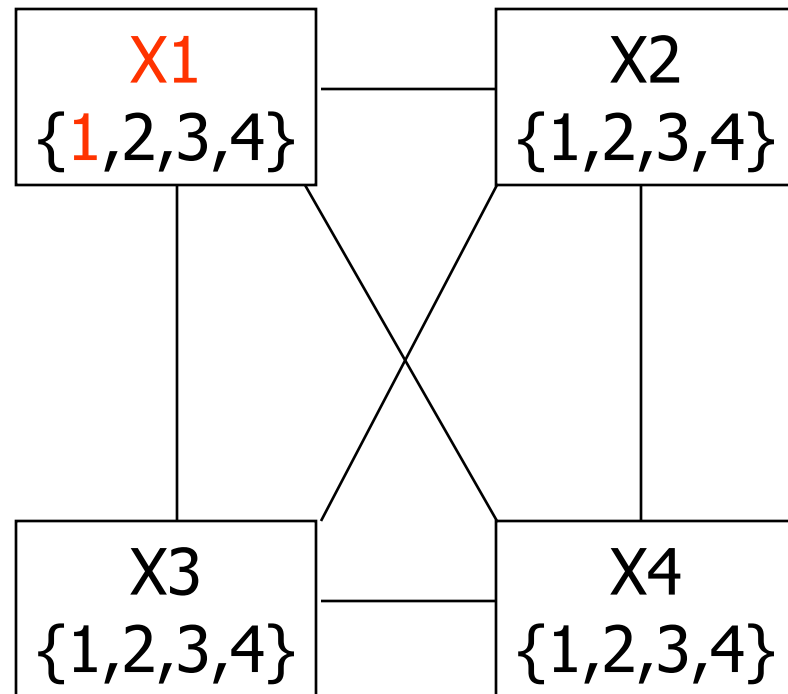
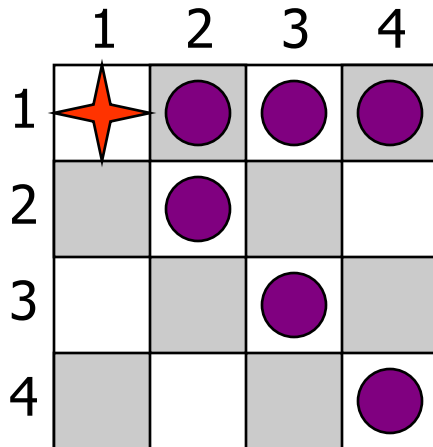
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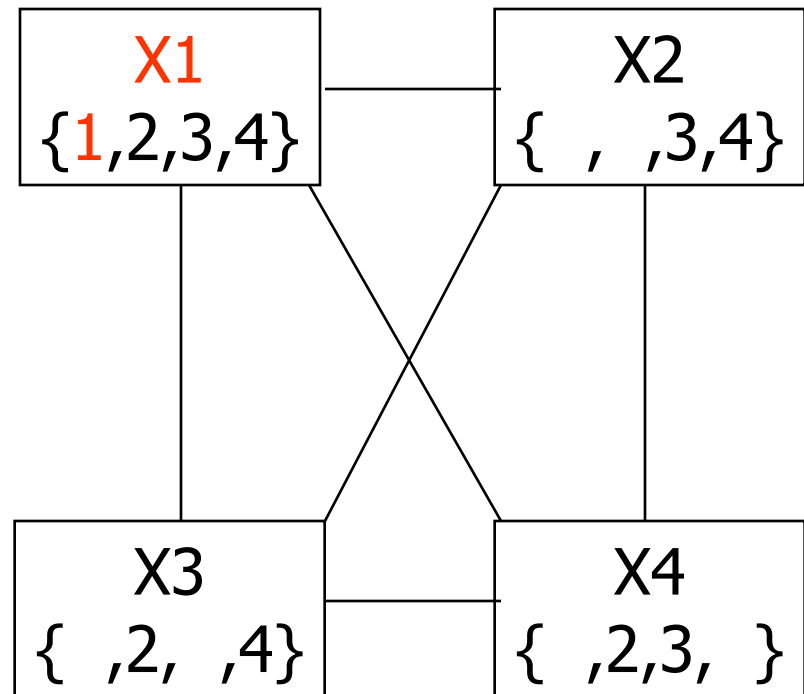
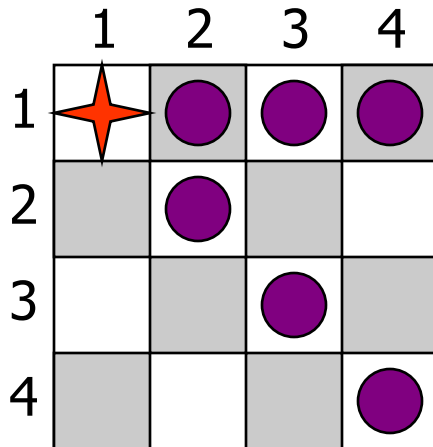
Terminate! No possible value for SA

Example: 4-Queens Problem





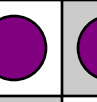
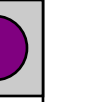
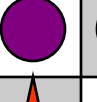

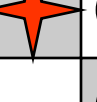
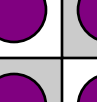



Assign value to
unassigned variable

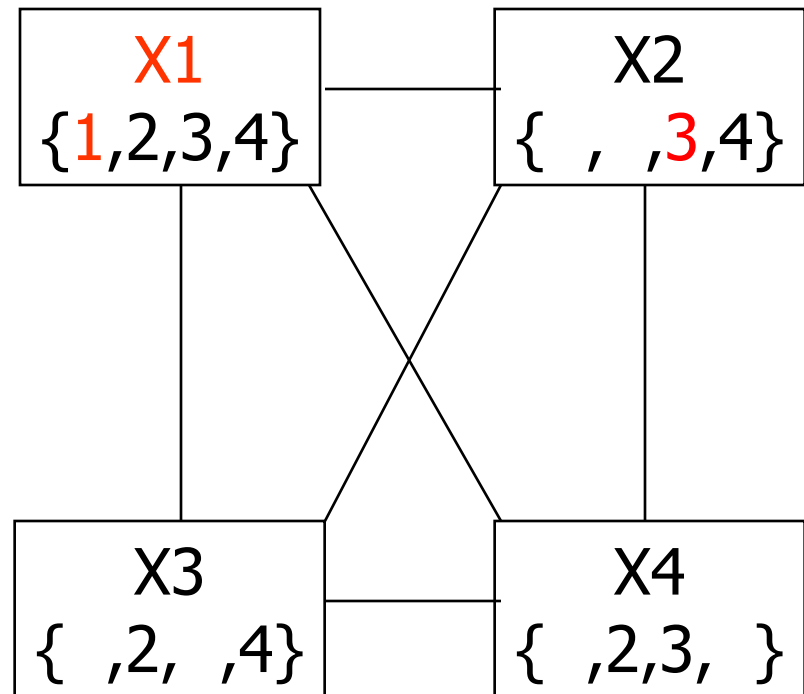
Example: 4-Queens Problem



Forward check!












Example: 4-Queens Problem

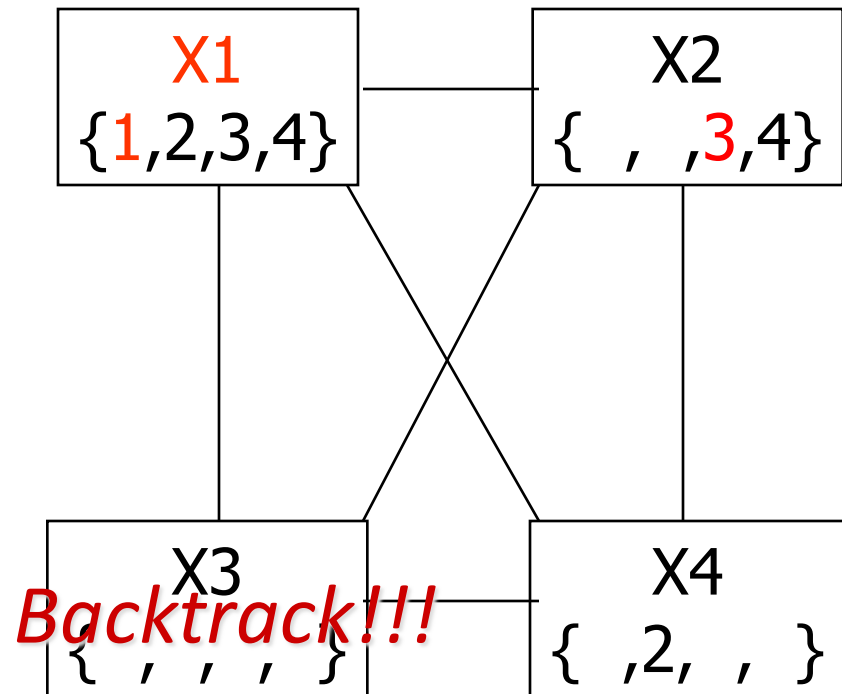
	1	2	3	4
1				
2				
3				
4				



Assign value to
unassigned variable

Example: 4-Queens Problem

	1	2	3	4
1				
2				
3				
4				

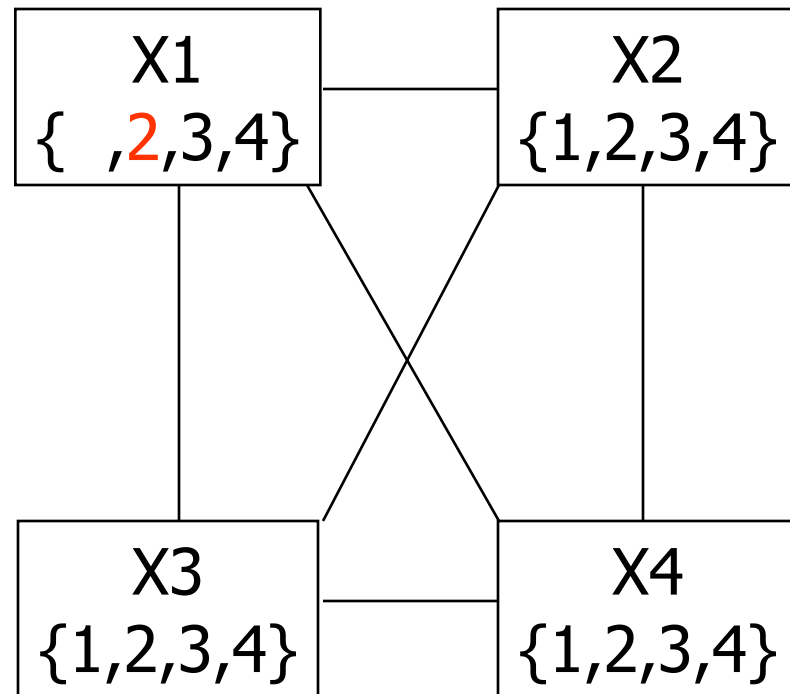


Forward check!

Example: 4-Queens Problem

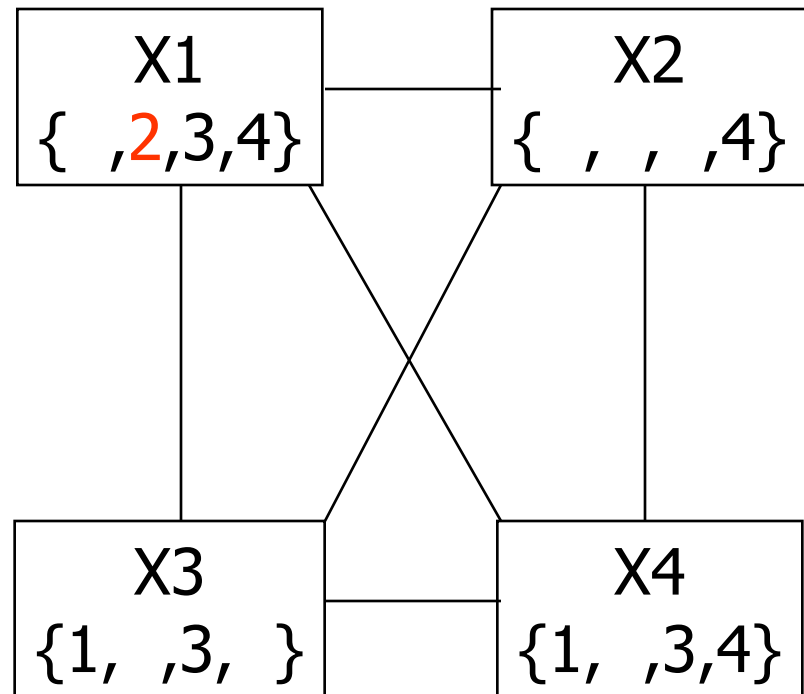
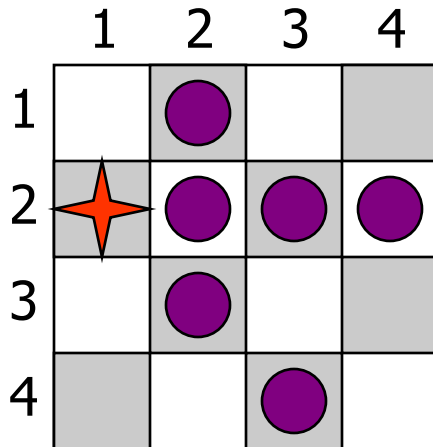
Picking up a little later after two steps of backtracking....

	1	2	3	4
1		●		
2	★	●	●	●
3		●		
4			●	



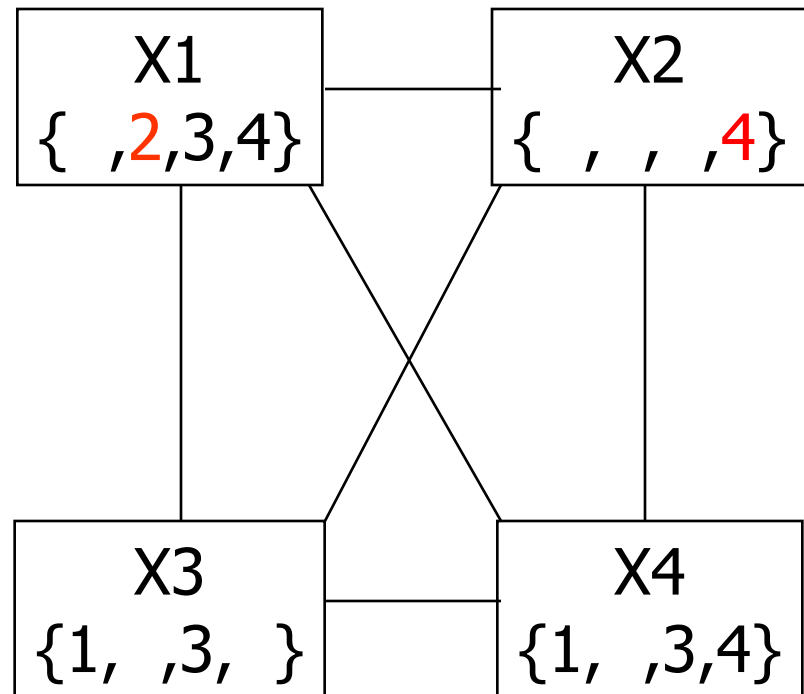
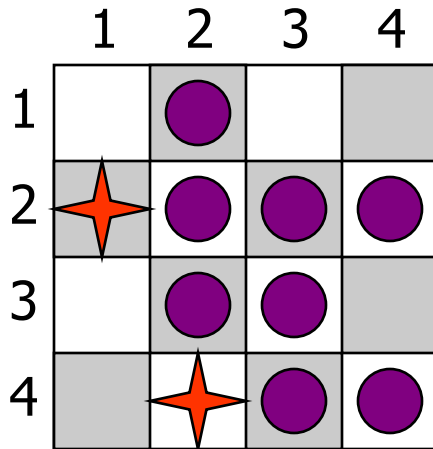
Assign value to
unassigned variable

Example: 4-Queens Problem



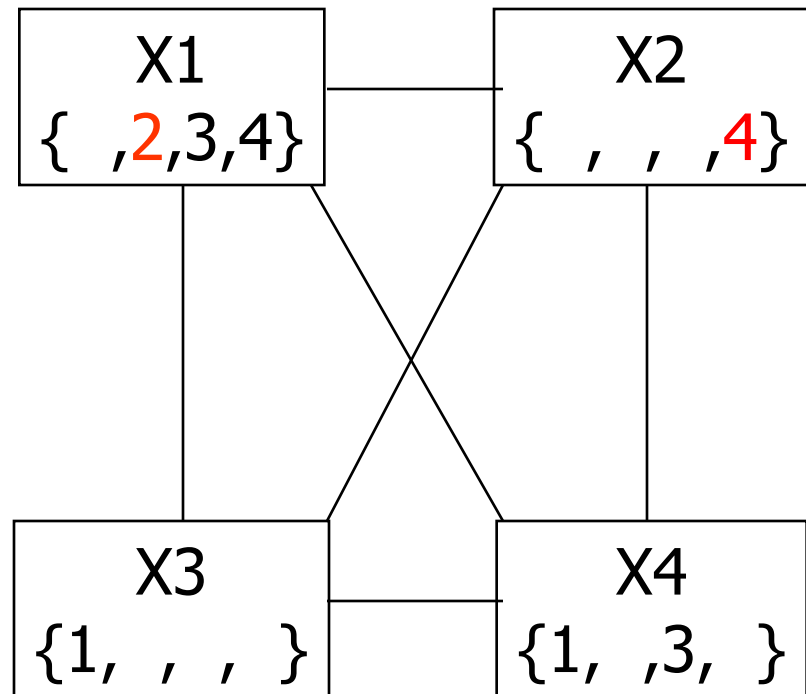
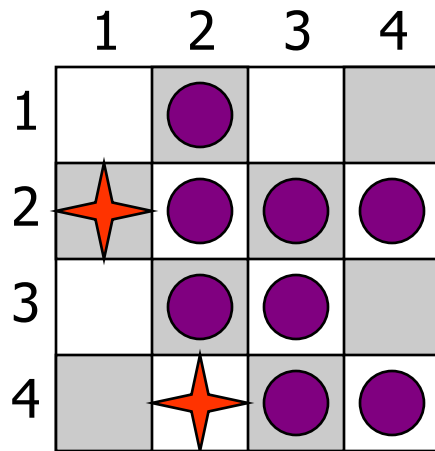
Forward check!

Example: 4-Queens Problem



Assign value to
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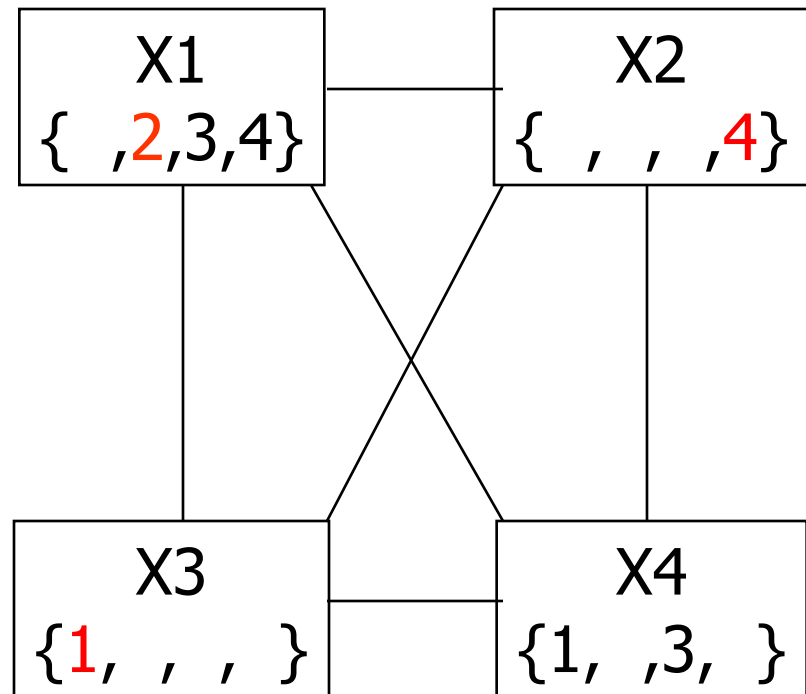
Example: 4-Queens Problem



Forward check!

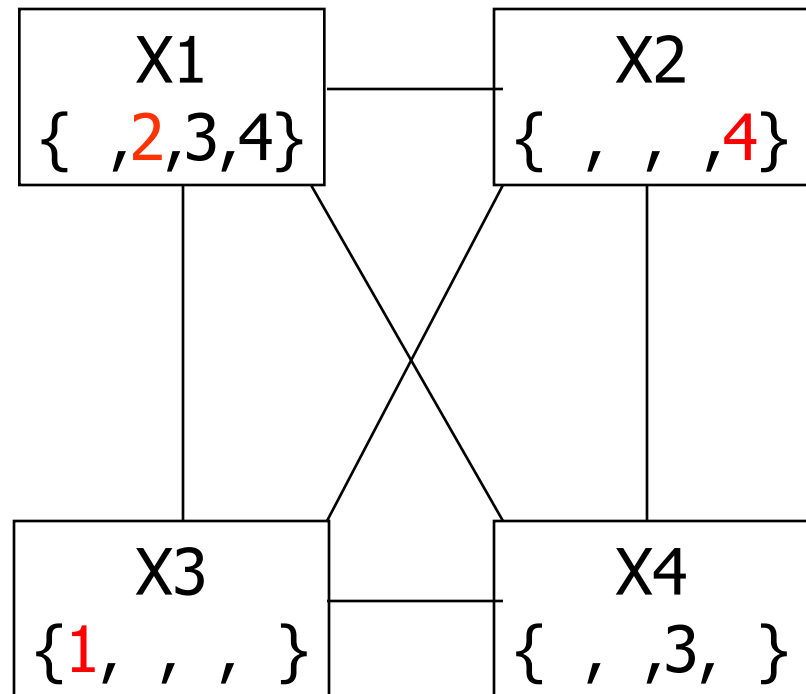
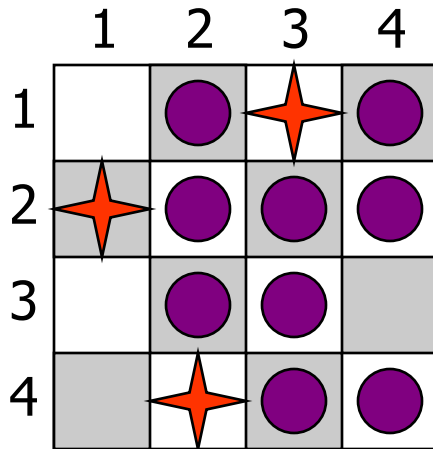
Example: 4-Queens Problem

	1	2	3	4
1		●	★	●
2	★	●	●	●
3		●	●	
4		★	●	●



Assign value to
unassigned variable

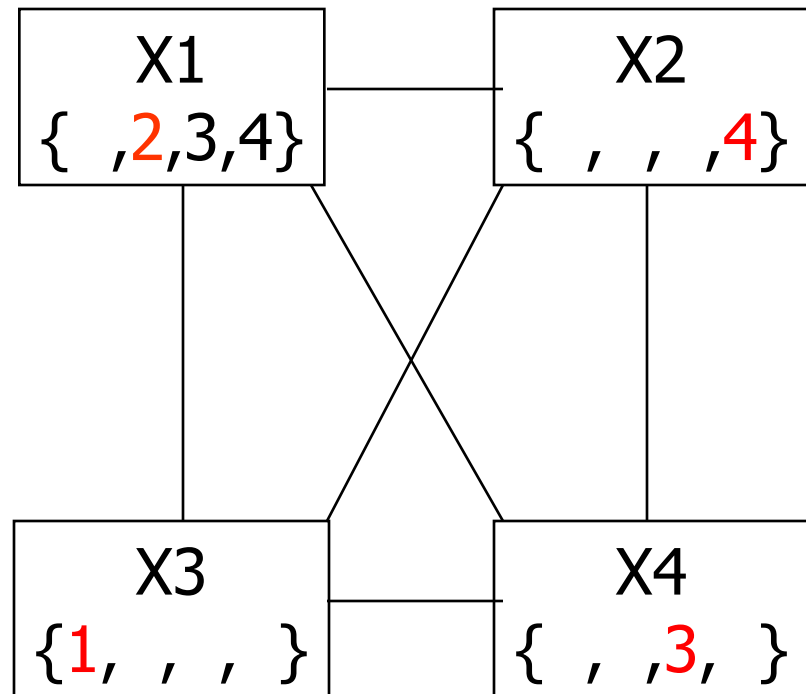
Example: 4-Queens Problem



Forward check!

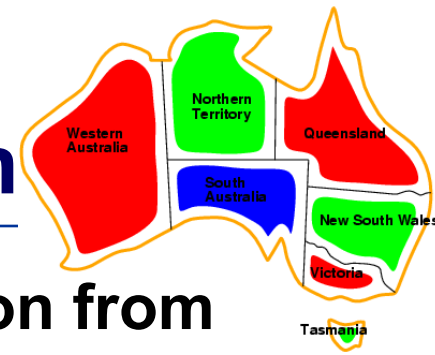
Example: 4-Queens Problem

	1	2	3	4
1		●	★	●
2	★	●	●	●
3		●	●	★
4	●	★	●	●

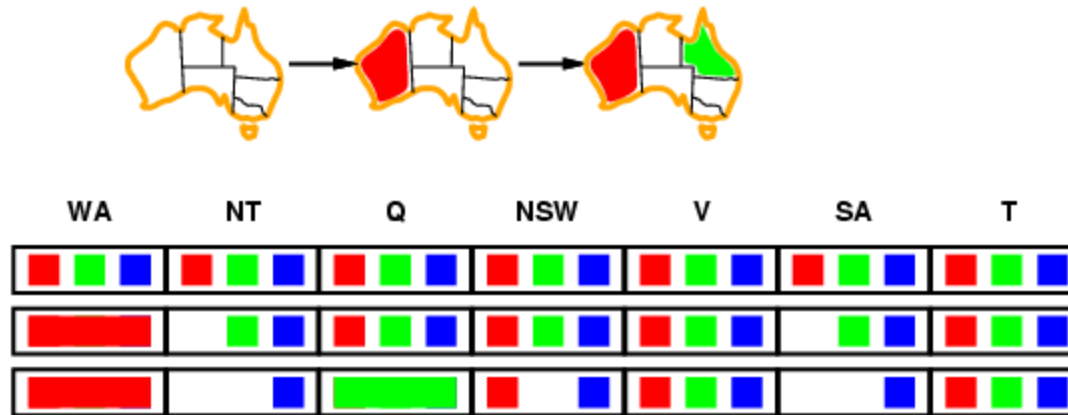


Assign value to
unassigned variable

Towards Constraint propagation



- Forward checking propagates information from *assigned* to *unassigned* variables, but doesn't provide early detection for all failures:



- NT and SA cannot both be blue!
- **Constraint propagation** goes beyond forward checking & repeatedly enforces constraints locally

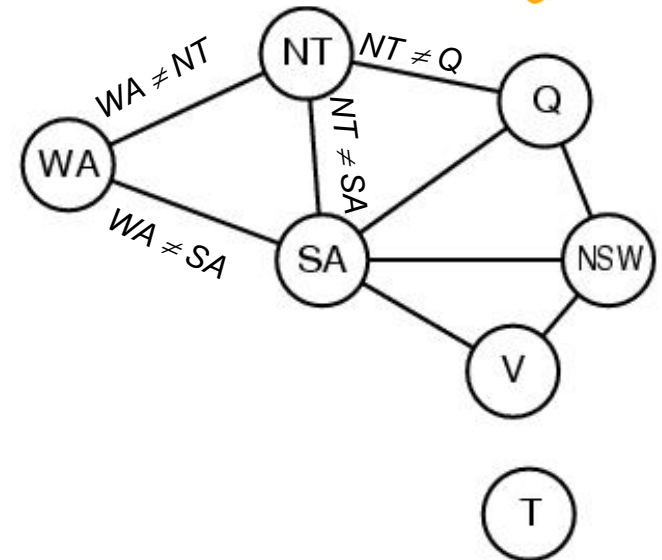
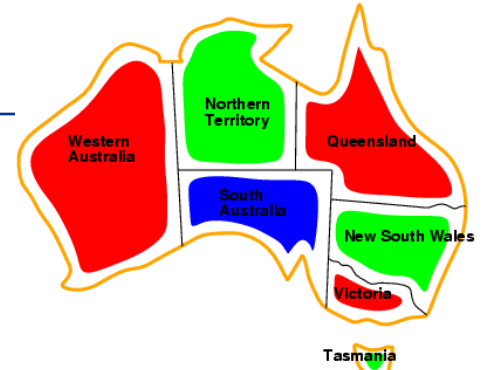
Arc Consistency, Constraint Propagation & AC-3

Idea 3 (*big idea*): *Inference* in CSPs

- **CSP solvers combine search and inference**
 - Search
 - assigning a value to a variable
 - *Constraint propagation (inference)*
 - Eliminates possible values for a variable if the value would violate *local consistency*
 - *Can do inference first, or intertwine it with search*
 - You'll investigate this in the Sudoku homework
- **Local consistency**
 - *Node consistency*: satisfies unary constraints
 - This is trivial!
 - *Arc consistency*: satisfies binary constraints
 - (X_i is arc-consistent w.r.t. X_j if for every value v in D_i , there is some value w in D_j that satisfies the binary constraint on the arc between X_i and X_j)

Review: CSP Representations

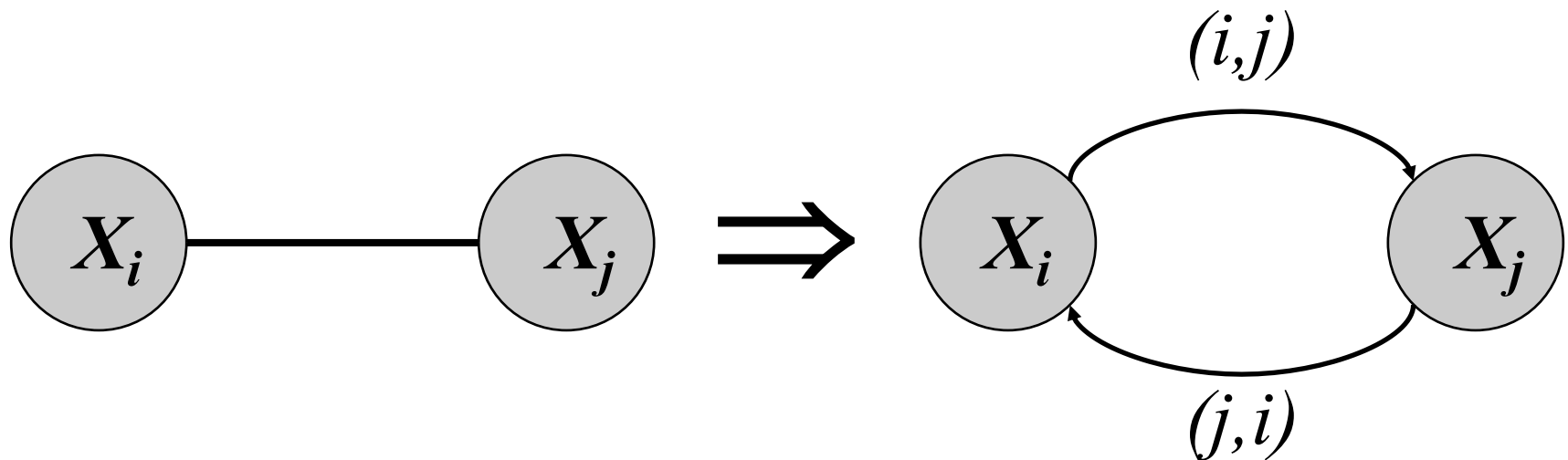
- **Constraint graph:**
 - *nodes* are variables
 - *edges are constraints*



Edges to Arcs:

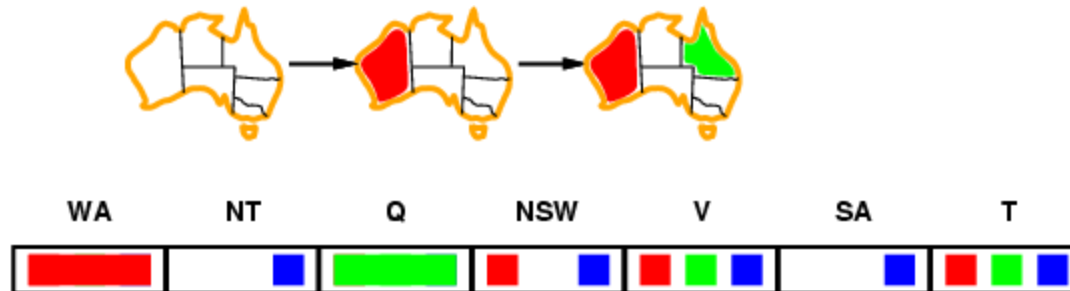
From Constraint Graph to DAG

- Given a pair of nodes X_i and X_j connected by a constraint **edge**, we represent this not by a single undirected edge, but a **pair of directed arcs**.
 - For a connected pair of nodes X_i and X_j , there are **two** arcs that connect them: (i,j) and (j,i) .



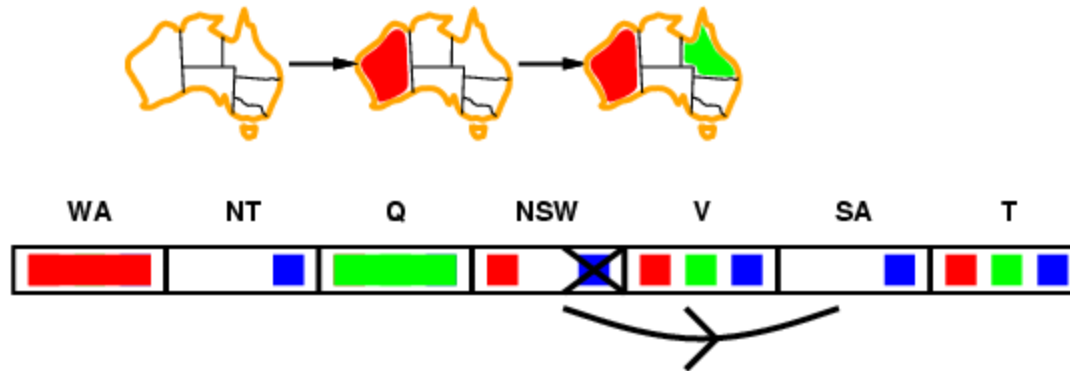
Arc consistency

- Simplest form of propagation makes each arc **consistent**
- $X \rightarrow Y$ is consistent iff
for **every** value x of X there is **some** allowed y



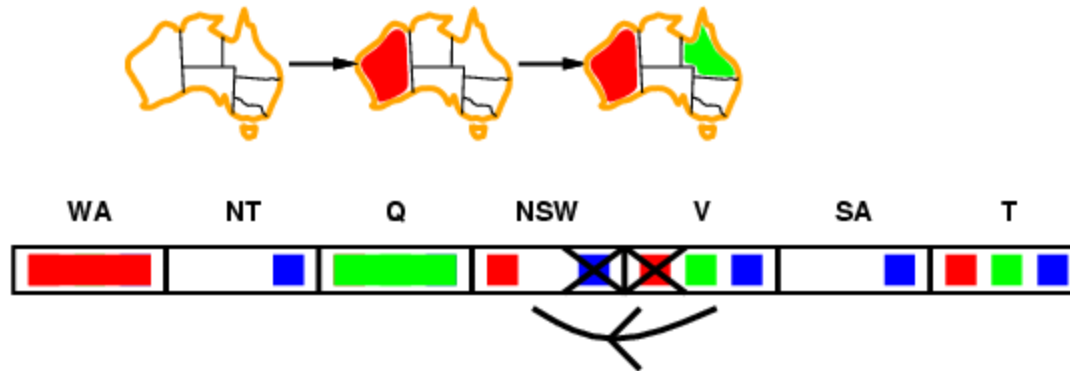
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Arc consistency

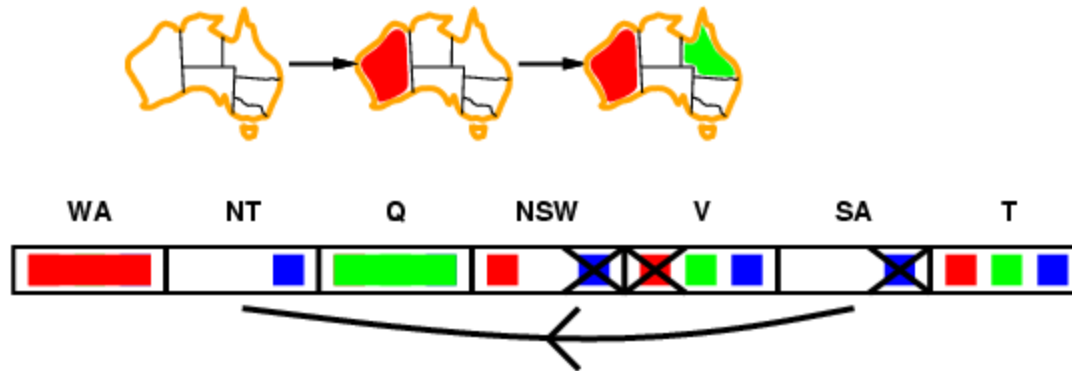
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- If X loses a value, recheck neighbors of X

Arc consistency

- Simplest form of propagation makes each arc **consistent**
- $X \rightarrow Y$ is consistent iff
for **every** value x of X there is **some** allowed y



- If X loses a value, we need to recheck neighbors of X
- Detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment

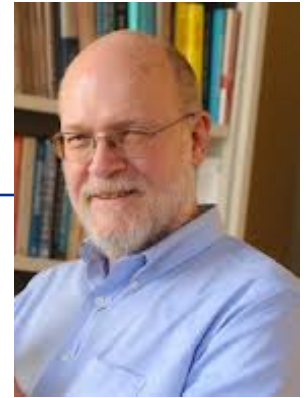
Arc Consistency

An arc (i,j) is **arc consistent** if and only if every value v on X_i is consistent with some label on X_j .

To make an arc (i,j) arc consistent,
for each value v on X_i ,
if there is no label on X_j consistent with v
then remove v from X_i

- Given d values, checking arc (i,j) takes $O(d^2)$ time worst case

Replacing Search: Constraint Propagation Invented...



Dave Waltz's insight:

- By **iterating** over the graph, the arc-consistency **constraints** can be **propagated** along arcs of the graph.
- **Search**: Use constraints to **add** labels to find **one solution**
- **Constraint Propagation**: Use constraints to **eliminate labels** to simultaneously find **all solutions**

The Waltz/Mackworth Constraint Propagation Algorithm

1. Assign every node in the constraint graph a set of *all* possible values
2. Repeat until there is no change in the set of values associated with any node:
 3. For each node i :
 4. For each neighboring node j in the picture:
 5. Remove any value from i which is not arc consistent with j .

Inefficiencies: Towards AC-3

1. At each iteration, we only need to examine those X_i *where at least one neighbor of X_i has lost a value in the previous iteration.*
2. If X_i loses a value only because of arc inconsistencies with X_j , we *don't need to check X_j on the next iteration.*
3. Removing a value on X_i can only make X_j arc-inconsistent with respect to X_i itself. Thus, we only need to check that *(j,i)* is still arc-consistent.

These insights lead a much better algorithm...

AC-3

function **AC-3**(*csp*) return the CSP, possibly with reduced domains

inputs: *csp*, a binary csp with variables $\{X_1, X_2, \dots, X_n\}$

local variables: *queue*, a queue of arcs initially the arcs in *csp*

while *queue* is not empty do

$(X_i, X_j) \leftarrow \text{REMOVE-FIRST}(\text{queue})$

 if **REMOVE-INCONSISTENT-VALUES**(X_i, X_j) then

 for each X_k in **NEIGHBORS**[X_i] – $\{X_j\}$ do

 add (X_k, X_i) to queue

function **REMOVE-INCONSISTENT-VALUES**(X_i, X_j) return *true* iff we remove a value

removed \leftarrow *false*

 for each x in **DOMAIN**[X_i] do

 if no value y in **DOMAIN**[X_j] allows (x, y) to satisfy
 the constraints between X_i and X_j

 then delete x from **DOMAIN**[X_i]; *removed* \leftarrow *true*

 return *removed*

AC-3: Worst Case Complexity Analysis

- All nodes can be connected to *every* other node,
 - so each of n nodes must be compared against $n-1$ other nodes,
 - so total # of arcs is $2*n*(n-1)$, i.e. $O(n^2)$
- If there are d values, checking arc (i,j) takes $O(d^2)$ time
- Each arc (i,j) can only be inserted into the queue d times
- Worst case complexity: $O(n^2d^3)$

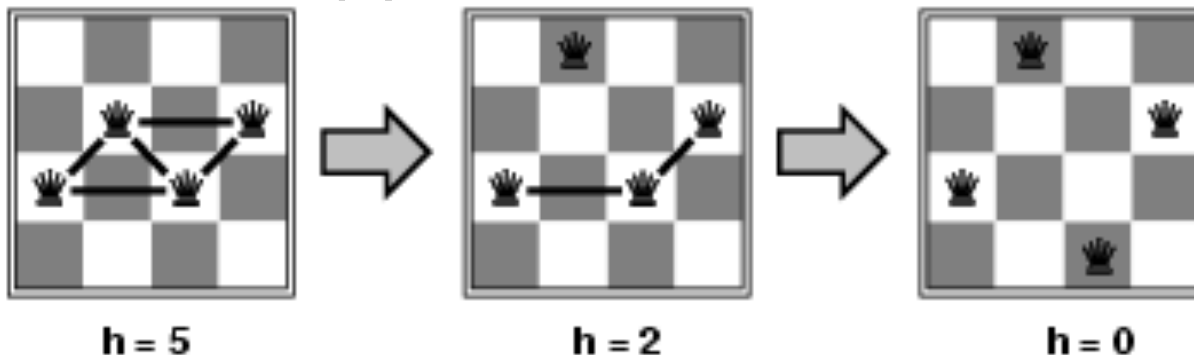
(For *planar* constraint graphs, the number of arcs can only be *linear in N* and the time complexity is only $O(nd^3)$)

Local search for CSPs

- Hill-climbing, simulated annealing typically work with "complete" states, i.e., all variables assigned
- To apply to CSPs:
 - allow states with unsatisfied constraints
 - operators **reassign** variable values
- Variable selection: randomly select any conflicted variable
- Value selection by **min-conflicts** heuristic:
 - choose value that violates the fewest constraints
 - i.e., hill-climb with $h(n)$ = total number of violated constraints

Example: n-queens

- **States:** 4 queens in 4 columns ($4^4 = 256$ states)
- **Actions:** move queen in column
- **Goal test:** no attacks
- **Evaluation:** $h(n)$ = number of attacks



- Given random initial state, local min-conflicts can solve n -queens in almost constant time for arbitrary n with high probability (e.g., $n = 10,000,000$)

Beyond binary constraints:

Path consistency

- Generalizes arc-consistency from individual binary constraints to multiple constraints
- A pair of variables X_i, X_j is path-consistent w.r.t. X_m if for every assignment $X_i=a, X_j=b$ consistent with the constraints on X_i, X_j there is an assignment to X_m that satisfied the constraints on X_i, X_m and X_j, X_m
- **Global constraints**
 - Can apply to any number of variables
 - E.g., in Sudoku, all numbers in a row must be different
 - E.g., in cryptarithmic, each letter must be a different digit
 - Example algorithm:
 - If any variable has a single possible value, delete that variable from the domains of all other constrained variables
 - If no values are left for any variable, you found a contradiction

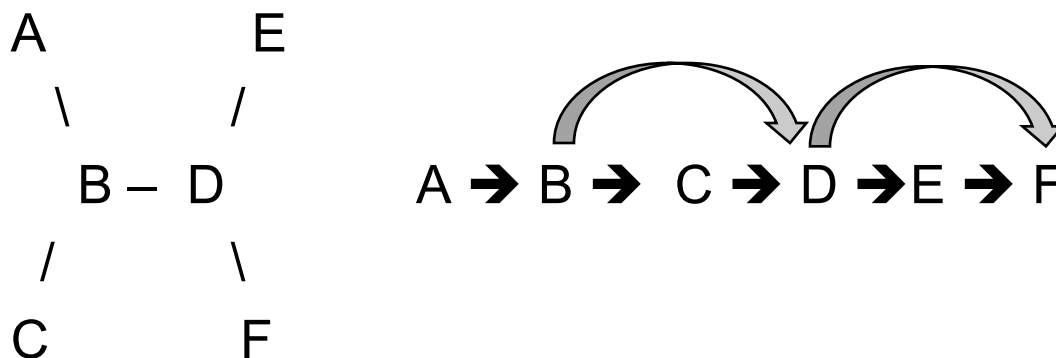
Simple CSPs can be solved quickly

1. Completely independent subproblems

- e.g. Australia & Tasmania
- Easiest

2. Constraint graph is a tree

- Any two variables are connected by only a single path
- Permits solution in time linear in number of variables
- Do a topological sort and just march down the list



Simplifying hard CSPs: Cycle Cutsets

- **Constraint graph can be decomposed into a tree**
 - Collapse or remove nodes
 - *Cycle cutset* S of a graph G : any subset of vertices of G that, if removed, leaves G a tree
- **Cycle cutset algorithm**
 - Choose some cutset S
 - For each possible assignment to the variables in S that satisfies all constraints on S
 - Remove any values for the domains of the remaining variables that are not consistent with S
 - If the remaining CSP has a solution, then you have are done
 - For graph size n , domain size d
 - Time complexity for cycle cutset of size c :
 $O(d^c * d^2(n-c)) = O(d^{c+2}(n-c))$