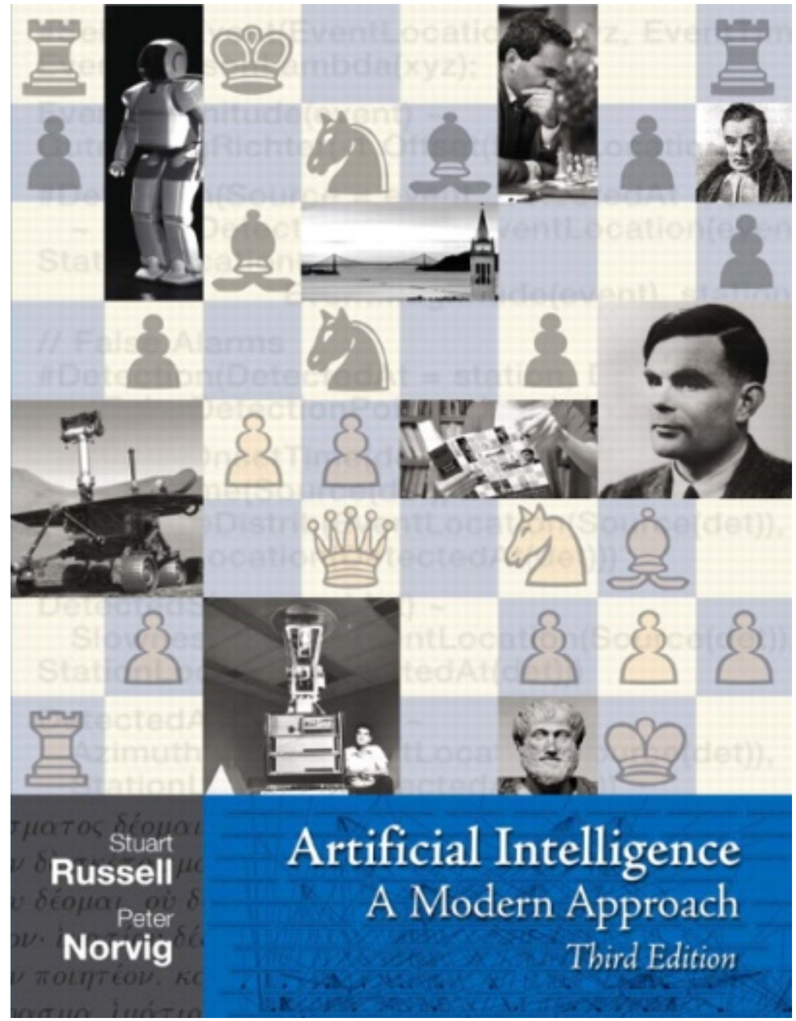
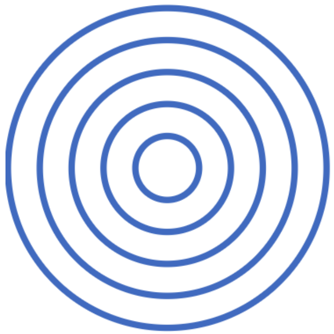
A* Wrap Up

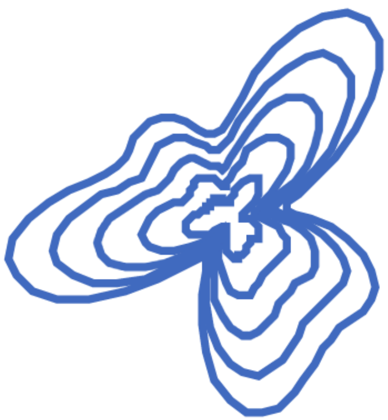
Read AIMA 3.1-3.6.
Some materials will not be covered in lecture, but will be on the exam.



Review: Shape of Search



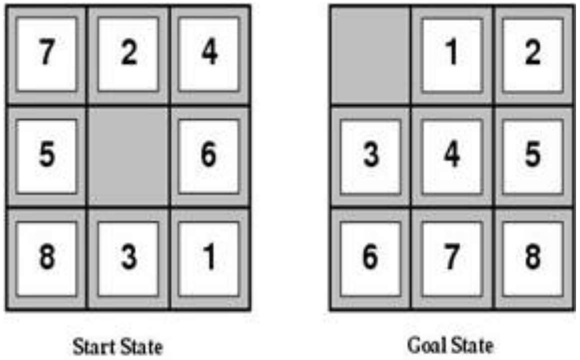
- **Breadth First Search** explores equally in all directions. It's frontier is implemented as a FIFO queue. It is applicable to unweighted graphs.



- **Uniform Cost Search** lets us prioritize which paths to explore. Instead of exploring all possible paths equally, it favors lower cost paths. It's frontier is a priority queue.

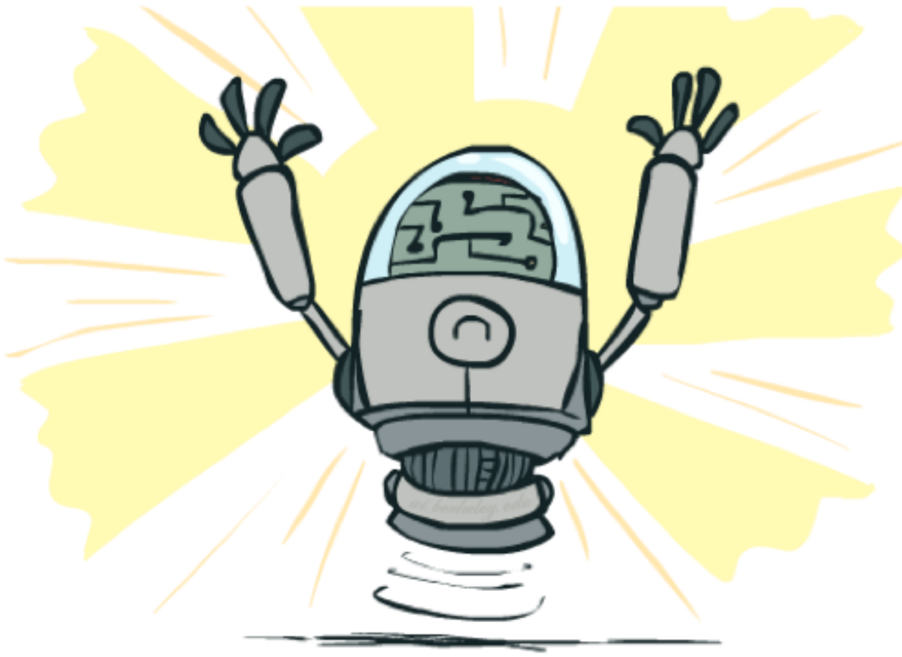
- **A*** prioritizes paths that seem to be leading towards the goal. It uses a priority queue sorted based on a combination of path cost plus a heuristic cost to the goal.

Review: Heuristic functions



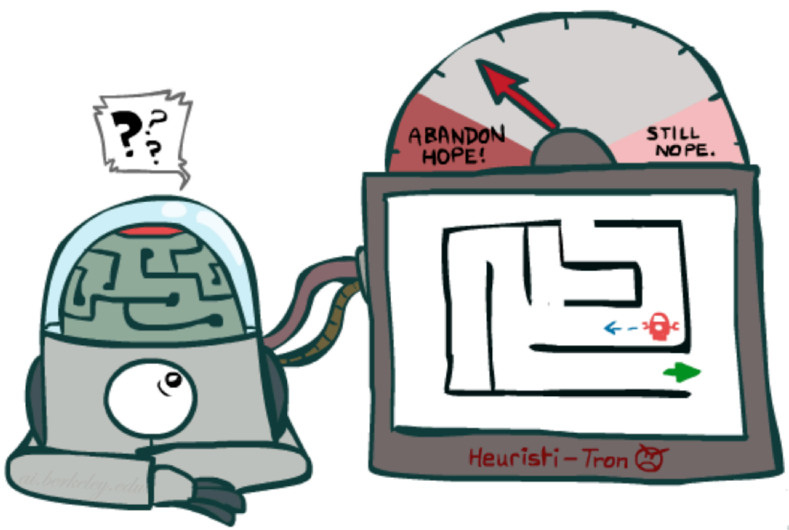
- For the 8-puzzle
  - Avg. solution cost is about 22 steps
    - (branching factor ≤ 3)
  - Exhaustive search to depth 22: ***3.1 x 10^{10} states***
  - A good heuristic function can reduce the search process

Optimality of A* Tree Search

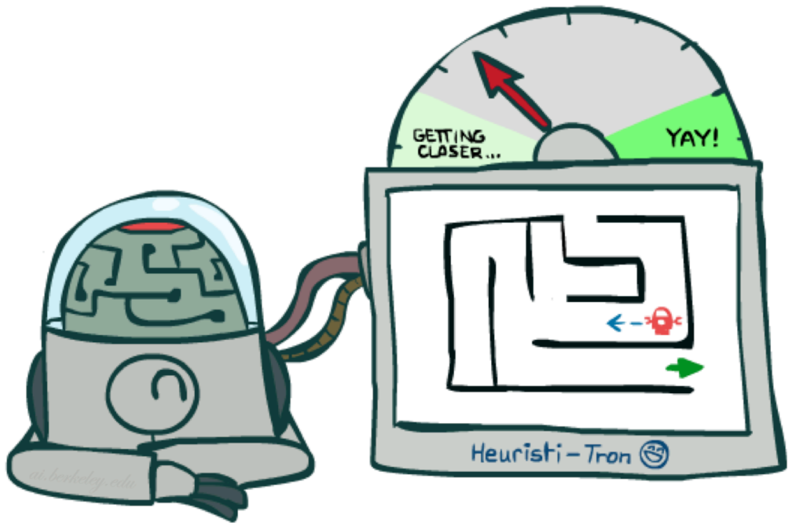


Slide credit: Dan Klein and Pieter Abbeel
http://ai.berkeley.edu

Admissibility



Inadmissible (pessimistic) heuristics break optimality by trapping good plans on the fringe



Admissible (optimistic) heuristics slow down bad plans but never outweigh true costs

Slide credit: Dan Klein and Pieter Abbeel
http://ai.berkeley.edu

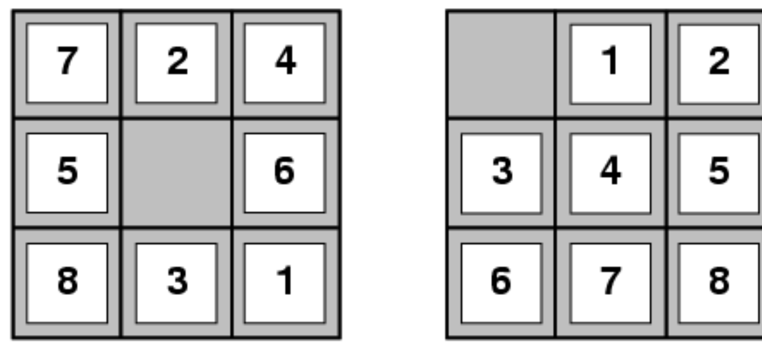
Admissible Heuristics

- A heuristic h is *admissible* (optimistic) if:

$$0 \leq h(n) \leq h^*(n)$$

where $h^*(n)$ is the true cost to a nearest goal

- Is Manhattan Distance admissible?



Start State Goal State

- Coming up with admissible heuristics is most of what's involved in using A* in practice.

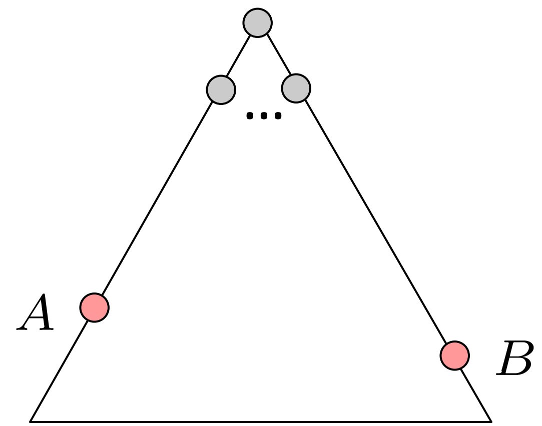
Optimality of A* Tree Search

Assume:

- **A is an optimal goal node**
- **B is a suboptimal goal node**
- **h is admissible**

Claim:

- **A will exit the fringe before B**



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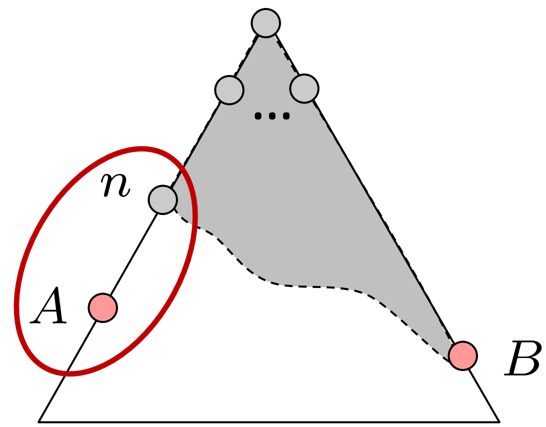
Optimality of A* Tree Search: Blocking

Proof:

- **Imagine B is on the fringe**
- **Some ancestor *n* of A is on the fringe, too (maybe A!)**
- **Claim: *n* will be expanded before B**
  1. f(n) is less or equal to f(A)

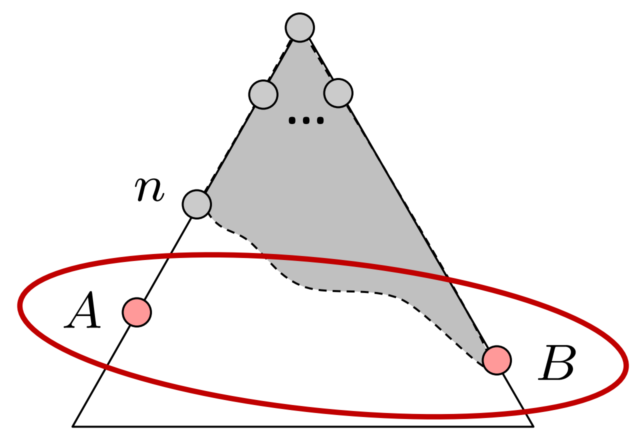


$f(n) = g(n) + h(n)$	Definition of f-cost
$f(n) \leq g(A)$	Admissibility of h
$g(A) = f(A)$	h = 0 at a goal

Optimality of A* Tree Search: Blocking

Proof:

- **Imagine B is on the fringe**
- **Some ancestor *n* of A is on the fringe, too (maybe A!)**
- **Claim: *n* will be expanded before B**
  1. f(n) is less or equal to f(A)
  2. f(A) is less than f(B)



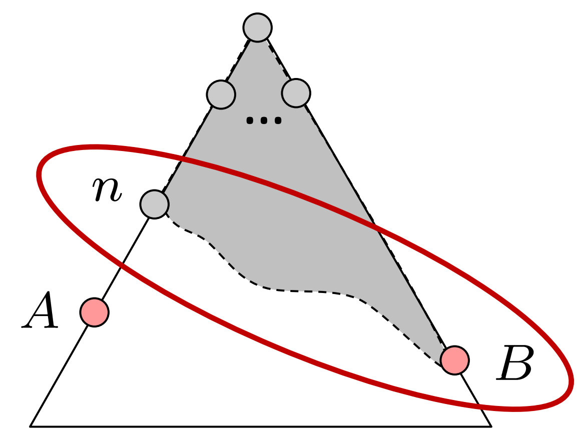
$g(A) < g(B)$ B is suboptimal

$f(A) < f(B)$ h = 0 at a goal

Optimality of A* Tree Search: Blocking

Proof:

- **Imagine B is on the fringe**
- **Some ancestor *n* of A is on the fringe, too (maybe A!)**
- **Claim: *n* will be expanded before B**
  1. f(n) is less or equal to f(A)
  2. f(A) is less than f(B)
  3. *n* expands before B
- **All ancestors of A expand before B**
- **A expands before B**
- **A* search is optimal**



$$f(n) \leq f(A) < f(B)$$

Slide credit: Dan Klein and Pieter Abbeel
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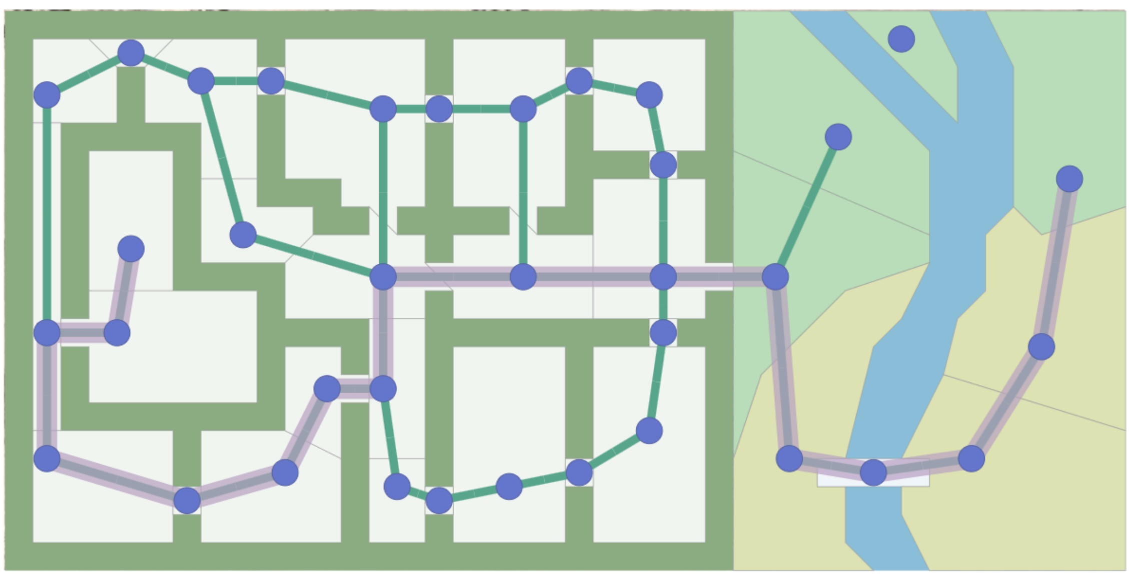
A* Applications

- Pathing / routing problems (A* is in your GPS!)
- Video games
- Robot motion planning
- Resource planning problems
- …



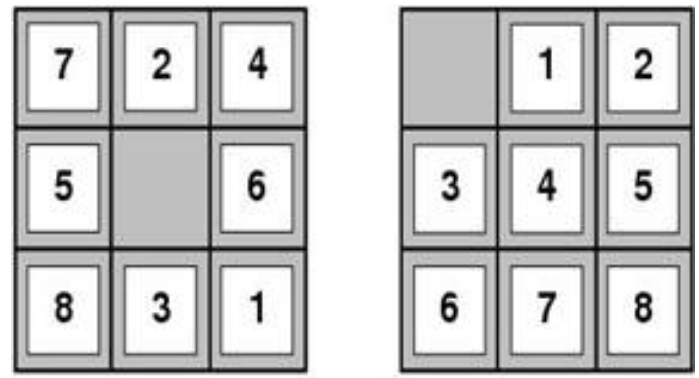
Constraint Satisfaction Problems

AIMA: Chapter 6

What is Search For?

- **Assumptions about the world: a single agent, deterministic actions, fully observed state, discrete state space**

- **Planning: sequences of actions**
  - The path to the goal is the important thing
  - Paths have various costs, depths
  - Heuristics give problem-specific guidance



- **Identification: assignments to variables**
  - The goal itself is important, not the path
  - All paths at the same depth (for some formulations)
  - CSPs are specialized for identification problems

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

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Big idea

- **Represent the *constraints* that solutions must satisfy in a uniform *declarative* language**
- **Find solutions by *GENERAL PURPOSE* search algorithms with no changes from problem to problem**
  - No hand built transition functions
  - No hand built heuristics

- **Just specify the problem in a formal declarative language, and a general purpose algorithm does everything else!**

Constraint Satisfaction Problems

A CSP consists of:

- *Finite set of variables* $X_1, X_2, \ldots, X_n$
- *Nonempty* ***domain*** *of possible values* for each variable $D_1, D_2, \ldots D_n$ *where* $D_i = \{v_1, \ldots, v_k\}$
- *Finite set of constraints* $C_1, C_2, \ldots, C_m$
  - Each *constraint* C_i limits the values that variables can take, e.g., $X_1 \neq X_2$ A *state* is defined as an *assignment* of values to some or all variables.

- **A *consistent* assignment does not violate the constraints.**
- **Example problem: Sudoku**

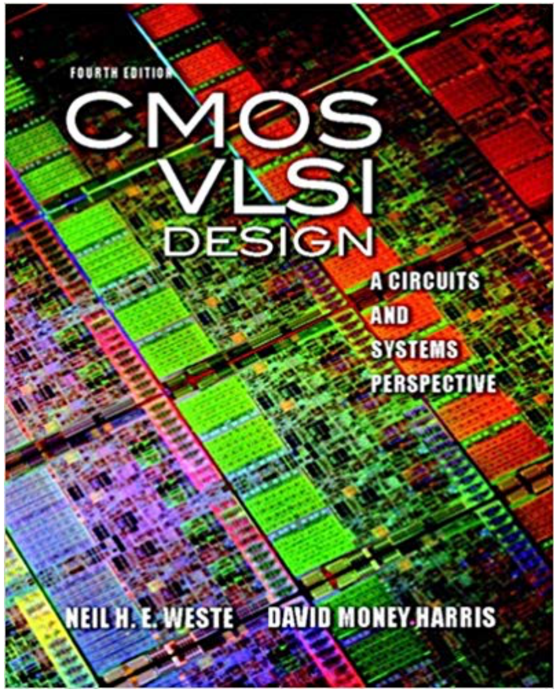
Constraint satisfaction problems

- **An assignment is *complete* when every variable is assigned a value.**
- **A *solution* to a CSP is a *complete, consistent* assignment.**
- **Solutions to CSPs can be found by a completely *general purpose* algorithm, given only the formal specification of the CSP.**

- **Beyond our scope: CSPs that require a solution that maximizes an *objective function*.**

Applications

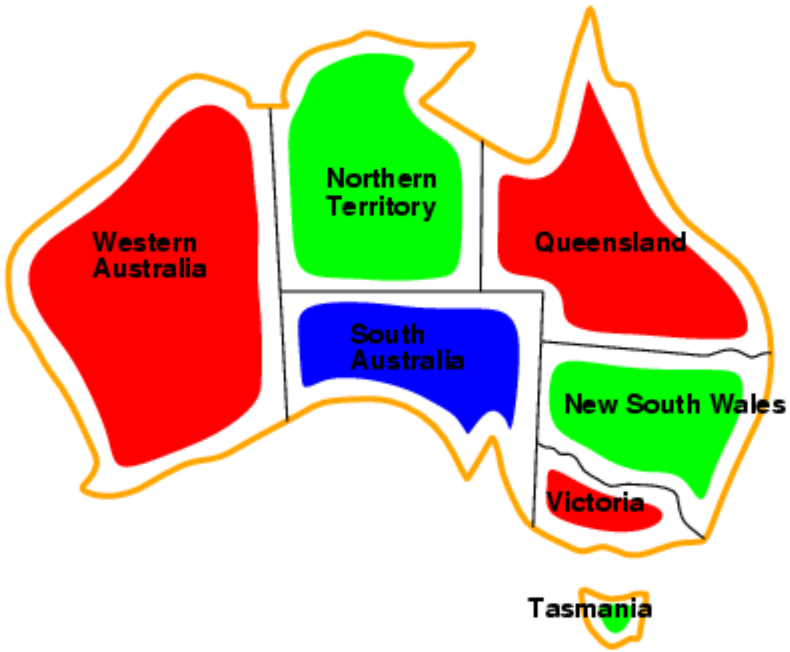
- **Map coloring**
- **Scheduling problems**
  - Job shop scheduling
  - Scheduling the Hubble Space Telescope
- **Floor planning for VLSI**
- **Sudoku**
- **…**



Example: Map-coloring

- **Variables:** ***WA, NT, Q, NSW, V, SA, T***
- **Domains:** **D_i = {red,green,blue}**
- **Constraints: adjacent regions must have different colors**
  - e.g., WA ≠ NT
    - —So (WA,NT) must be in {(red,green),(red,blue),(green,red), …}

Example: Map-coloring



Solutions: complete and consistent assignments

- e.g., WA = red, NT = green,Q = red, NSW = green,
  V = red, SA = blue, T = green

Benefits of CSP

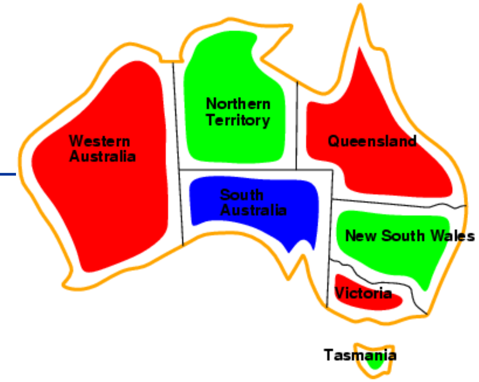
- **Clean specification of many problems, generic goal, successor function & heuristics**
  - Just represent problem as a CSP & solve with general package

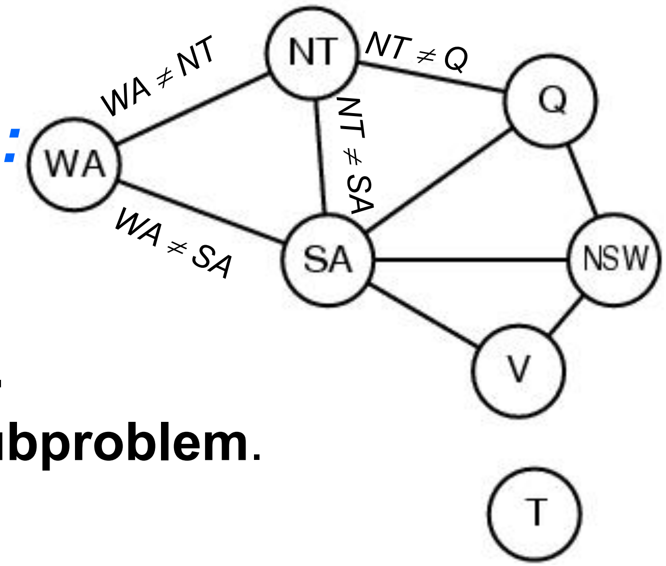
- **CSP "knows" which variables violate a constraint**
  - And hence where to focus the search

- ***CSPs:* Automatically prune off all branches that violate constraints**
  - (State space search could do this only by *hand-building constraints into the successor function)*

CSP Representations

- ***Constraint graph:***
  - *nodes* are variables
  - *arcs* are (binary) constraints
- ***Standard representation pattern:***
  - variables with values
- ***Constraint graph* simplifies search.**
  - **e.g. Tasmania is an independent subproblem**.
- ***This problem: A binary CSP:***
  - each constraint relates two variables





Varieties of CSPs

- ***Discrete variables***
  - finite domains:
    - —n variables, domain size d → $O(d^n)$ complete assignments
    - —e.g., Boolean CSPs, includes Boolean satisfiability (NP-complete)
  - infinite domains:
    - —integers, strings, etc.
    - —e.g., job scheduling, variables are start/end days for each job
    - —need a constraint language, e.g., $StartJob_1 + 5 \leq StartJob_3$

- ***Continuous variables***
  - e.g., start/end times for Hubble Space Telescope observations
  - linear constraints solvable in polynomial time by linear programming

Varieties of constraints

- ***Unary*** **constraints involve a single variable,**
  - e.g., SA ≠ green
- ***Binary*** **constraints involve pairs of variables,**
  - e.g., SA ≠ WA
- ***Higher-order*** **constraints involve 3 or more variables**
  - e.g., crypt-arithmetic column constraints
- ***Preference*** **(soft constraints) e.g.** ***red*** **is better than** ***green*** **can be represented by a cost for each variable assignment**
  - Constrained optimization problems.

Idea 1: CSP as a search problem

- **A CSP can easily be expressed as a search problem**
  - *Initial State:* the empty assignment {}.
  - *Successor function:* Assign value to any unassigned variable *provided that there is not a constraint conflict*.
  - *Goal test:* the current assignment is complete.
  - *Path cost:* a constant cost for every step.

- **Solution is always found at depth *n*, for *n* variables**
  - Hence Depth First Search can be used

Backtracking search

- **Note that variable assignments are *commutative***
  - Eg *[step 1: **WA = red**; step 2: **NT = green**]*
    equivalent to *[step 1: **NT = green**; step 2: **WA = red**]*
  - Therefore, a *tree search,* not a *graph search*

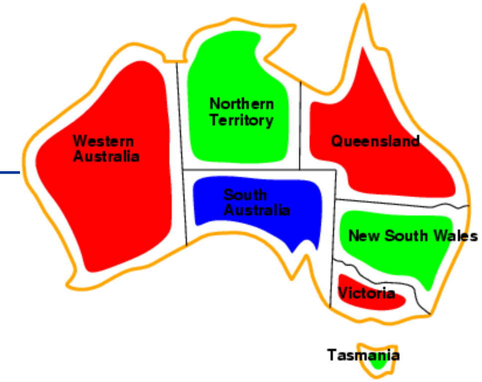
- **Only need to consider assignments to a single variable at each node**
  - $b = d$ and there are d^n leaves *(n variables, domain size d)*

- **Depth-first search for CSPs with single-variable assignments is called *backtracking* search**

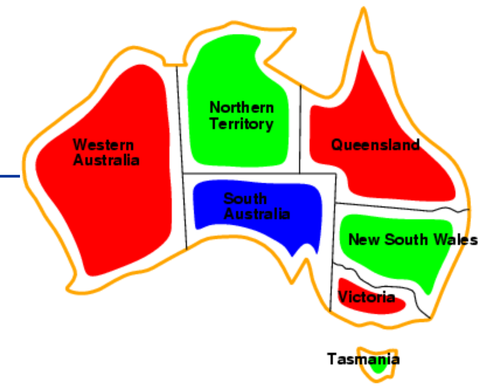
- **Backtracking search is the basic *uninformed* algorithm for CSPs**

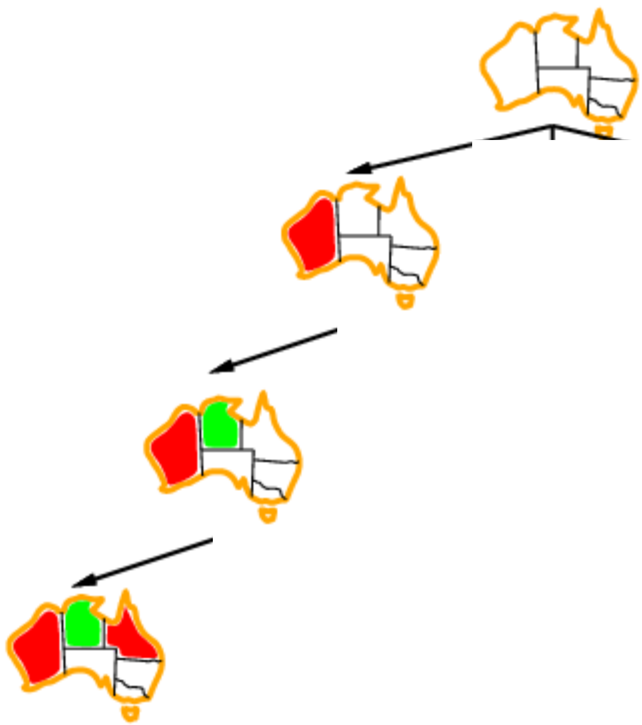
- **Can solve *n*-queens for $n \approx 25$**

Backtracking example



Backtracking example





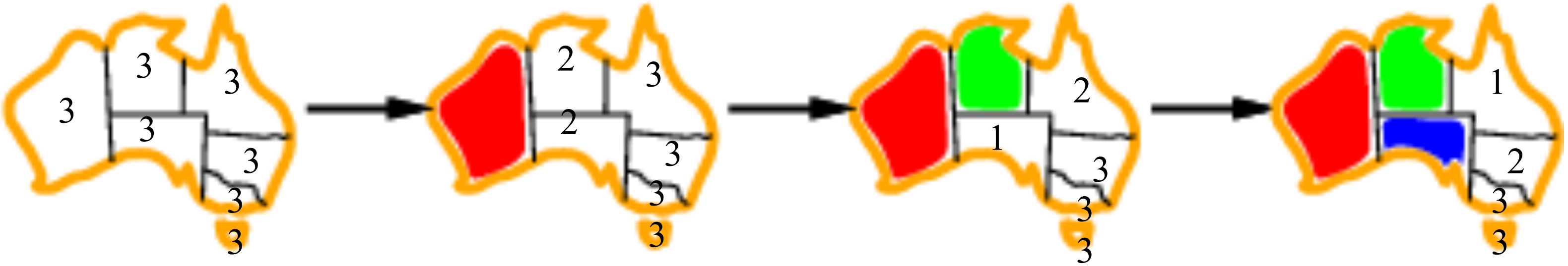
And so on….

Idea 2: Improving backtracking efficiency

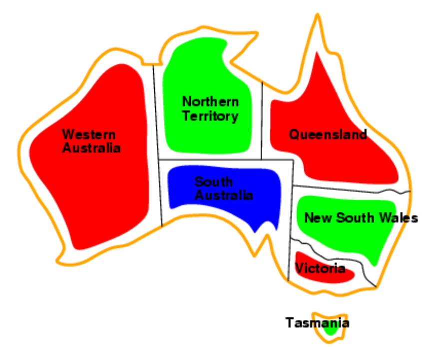
- ***General-purpose*** **methods &** ***general-purpose*** **heuristics can give huge gains in speed,** ***on average***
- **Heuristics:**
  - Q: Which variable should be assigned next?
    1. **Most constrain*ed* variable**
    2. **(if ties:) Most constrain*ing* variable**
  - Q: In what order should that variable's values be tried?
    3. **Least constraining *value***
  - Q: Can we detect inevitable failure early?
    4. **Forward checking**

Heuristic 1: Most constrain*ed* variable

- **Choose a variable with the *fewest legal values***

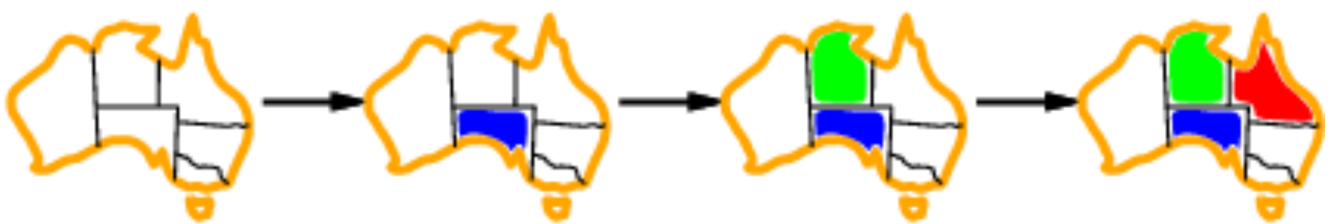


- **a.k.a. *minimum remaining values (MRV)* heuristic**

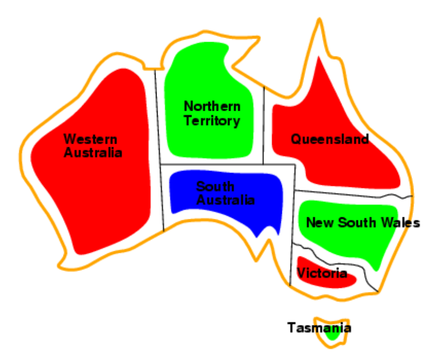


Heuristic 2: Most constrain*ing* variable

- **Tie-breaker among most constrained variables**
- **Choose the variable with the *most constraints on remaining variables***

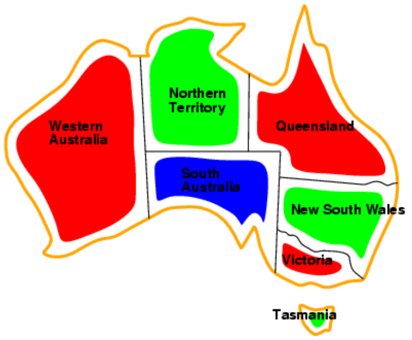


These two heuristics together lead to immediate solution of our example problem



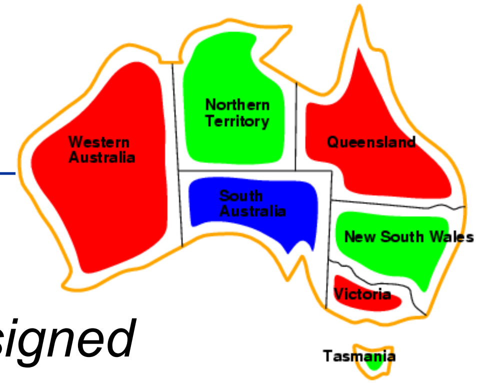
Heuristic 3: Least constraining *value*

- **Given a variable, *choose the least constraining value:***
  - the one that rules out the fewest values in the remaining variables



Note: demonstrated here independent of the other heuristics

Heuristic 4: Forward checking

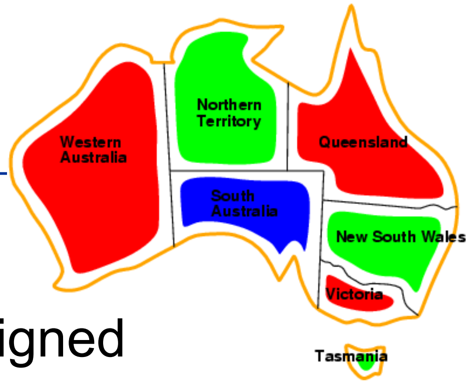


- **Idea**:
  - Keep track of *remaining* legal values for *unassigned* variables
  - Terminate search when any unassigned variable has no remaining legal values

New data structure

(A first step towards Arc Consistency & AC-3)

Forward checking



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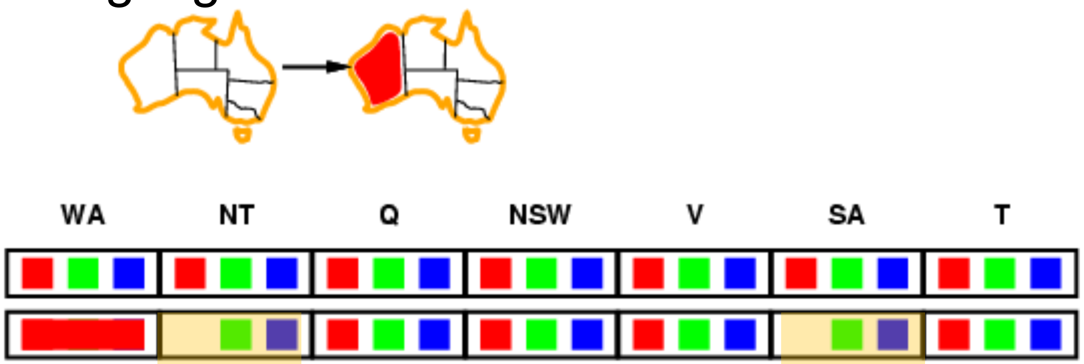
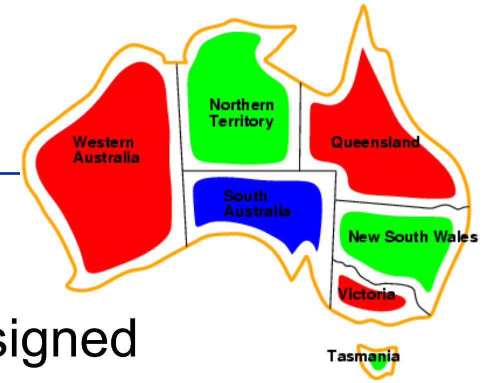


Forward checking



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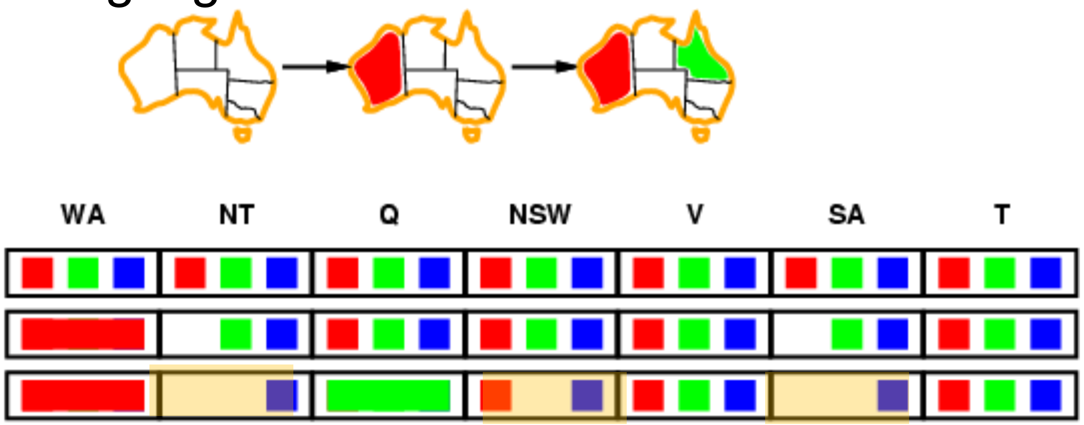
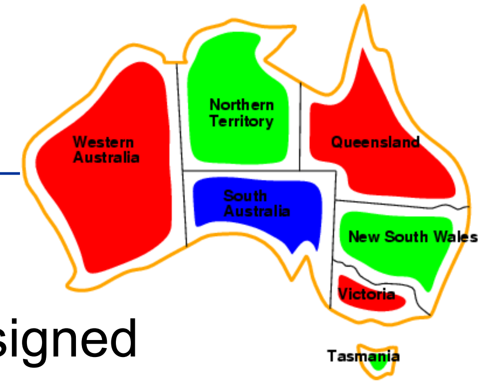


Forward checking



- **Idea**:
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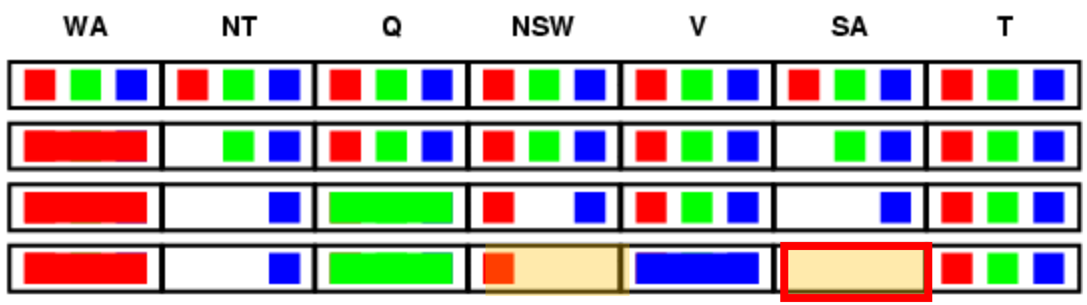
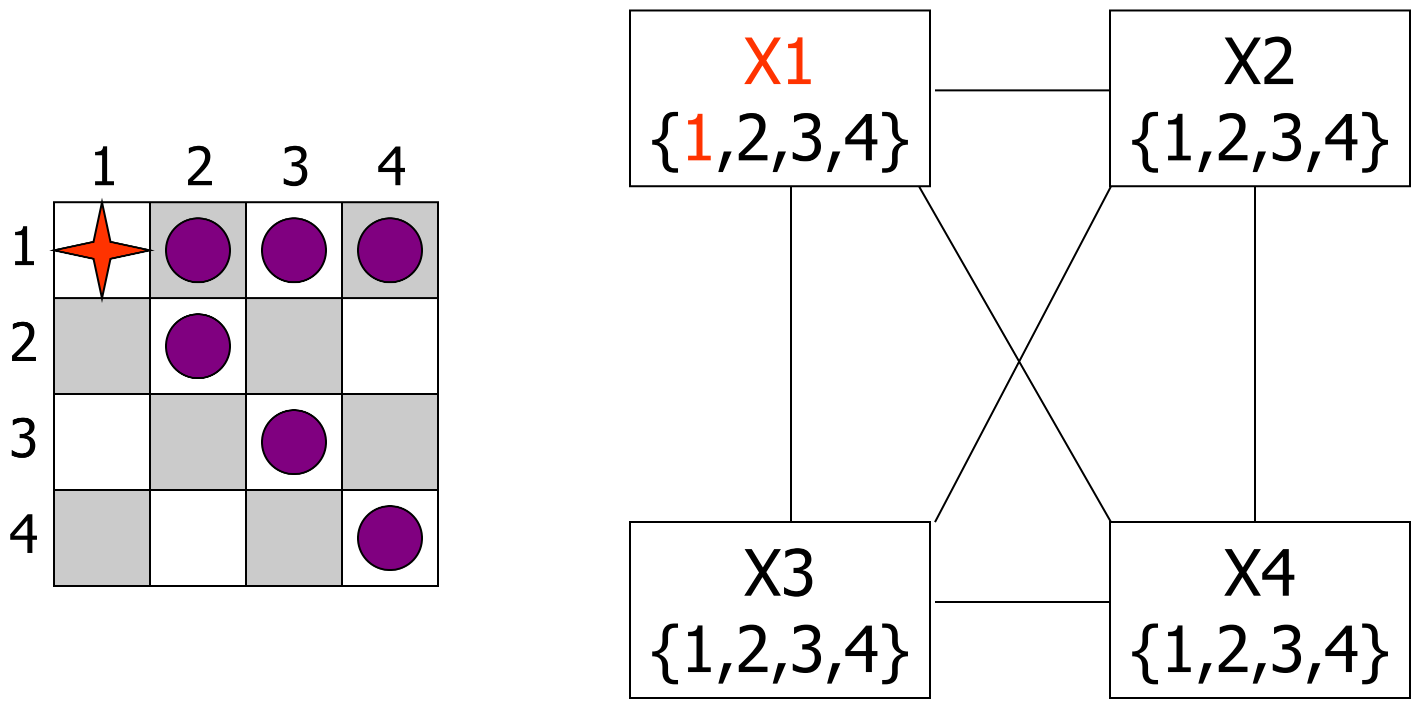
  - Terminate search when any unassigned variable has no remaining legal values



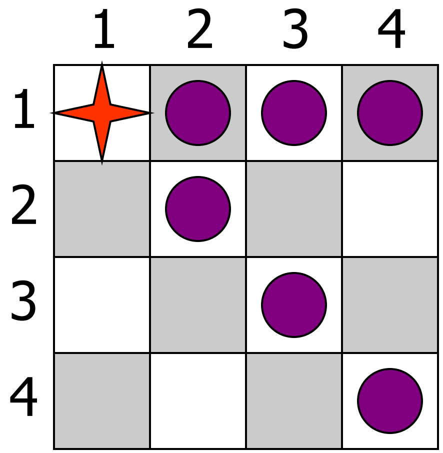
Terminate! No possible value for SA

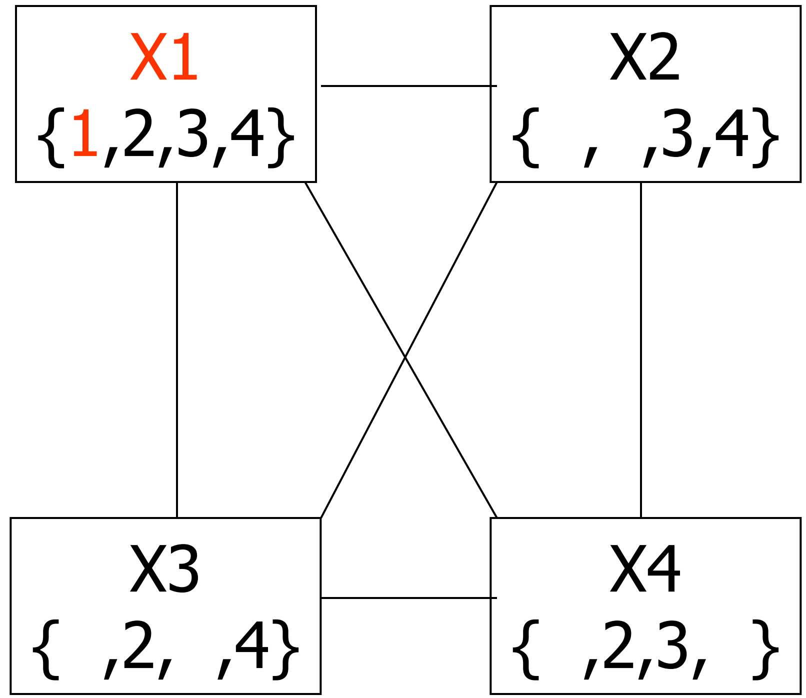
Example: 4-Queens Problem



Assign value to unassigned variable

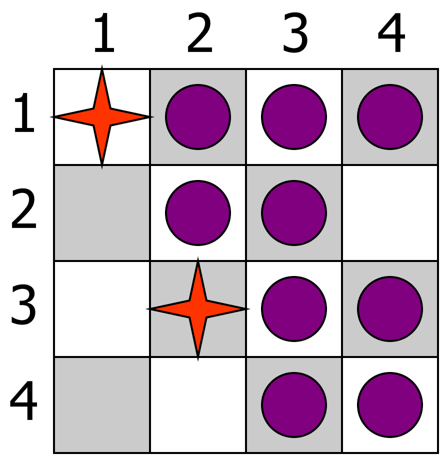
Example: 4-Queens Problem

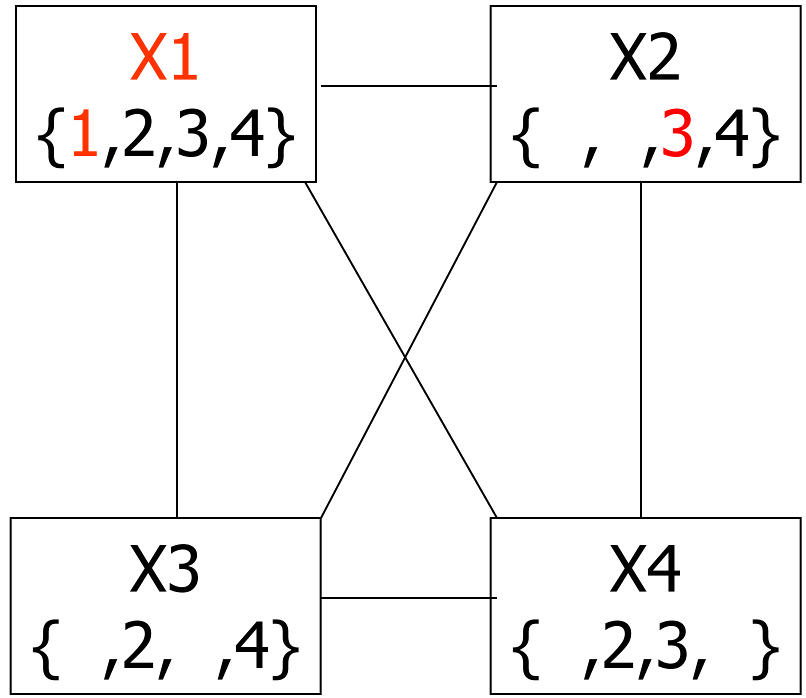




Forward check!

Example: 4-Queens Problem





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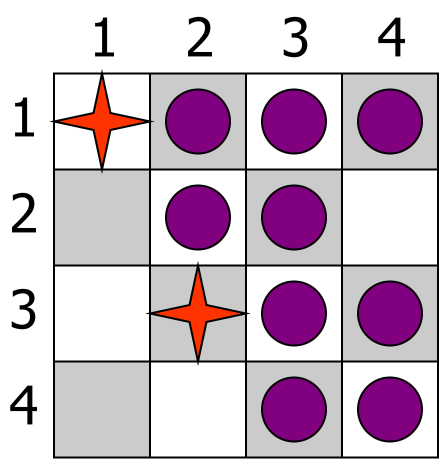
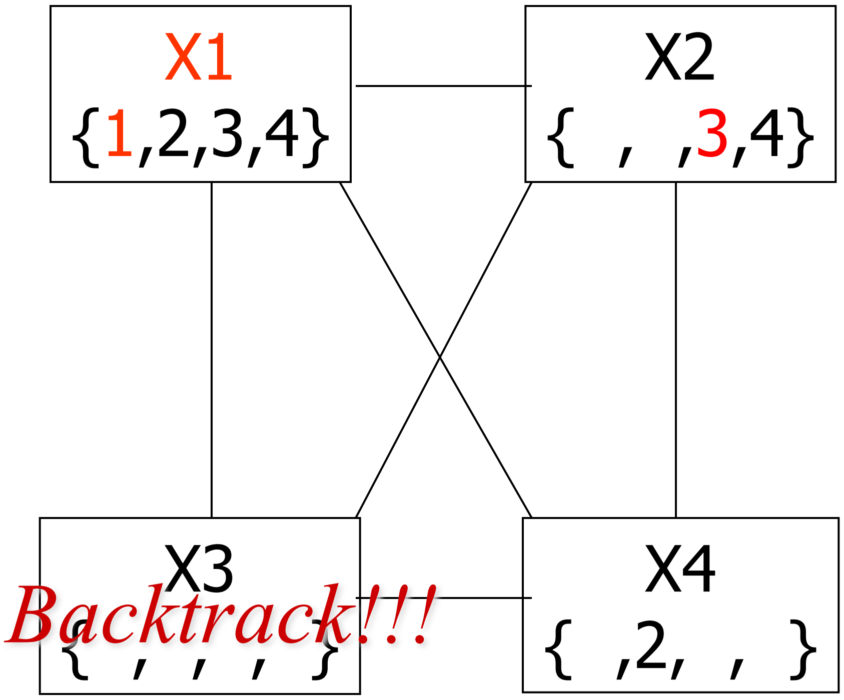
Example: 4-Queens Problem

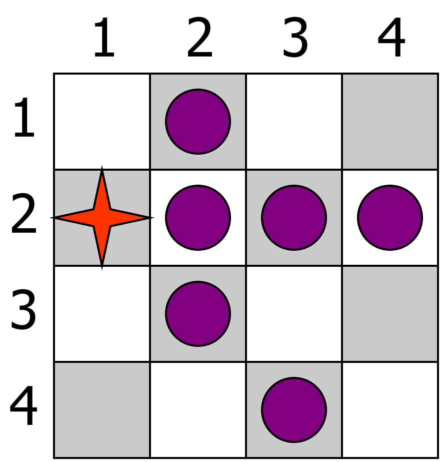


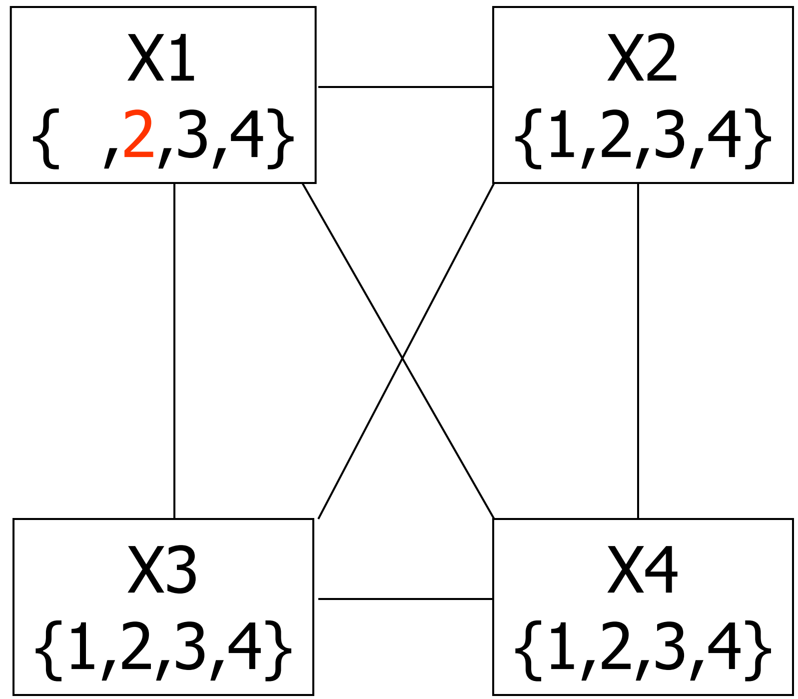


Forward check!

Example: 4-Queens Problem

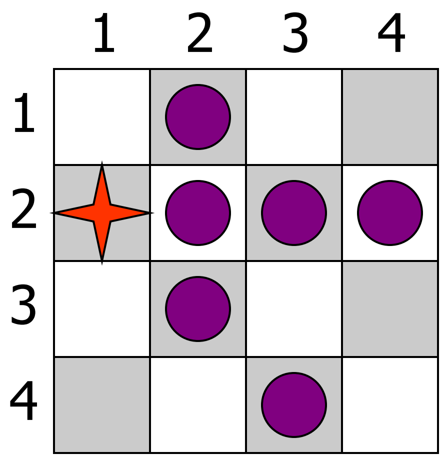
Picking up a little later after two steps of backtracking….

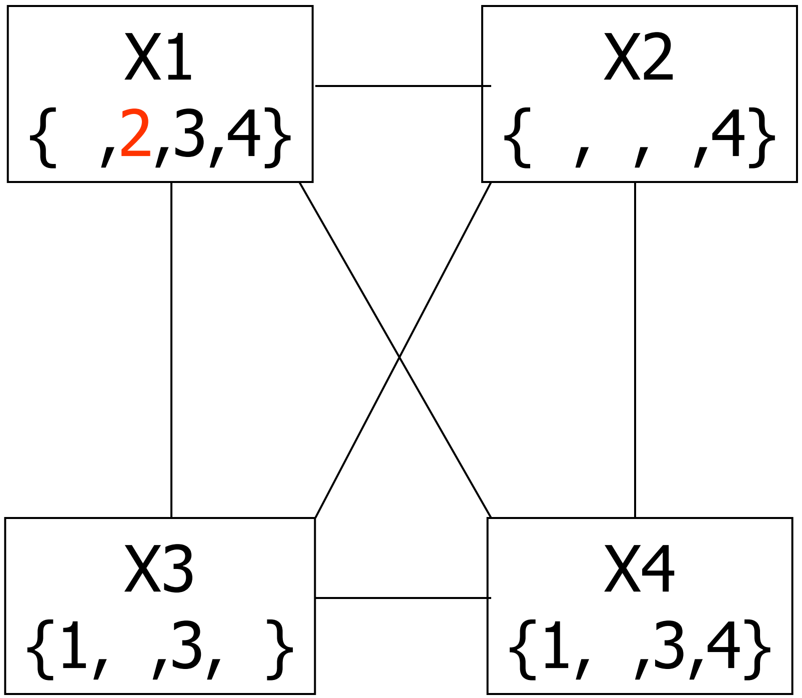




Assign value to unassigned variable

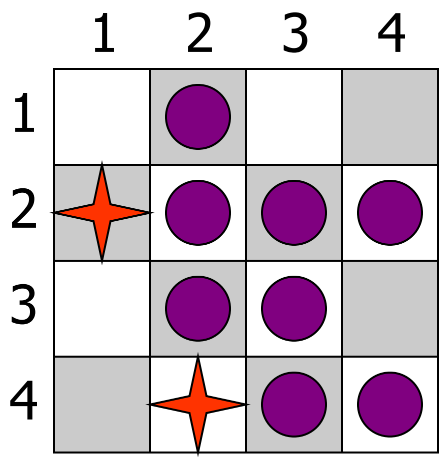
Example: 4-Queens Problem

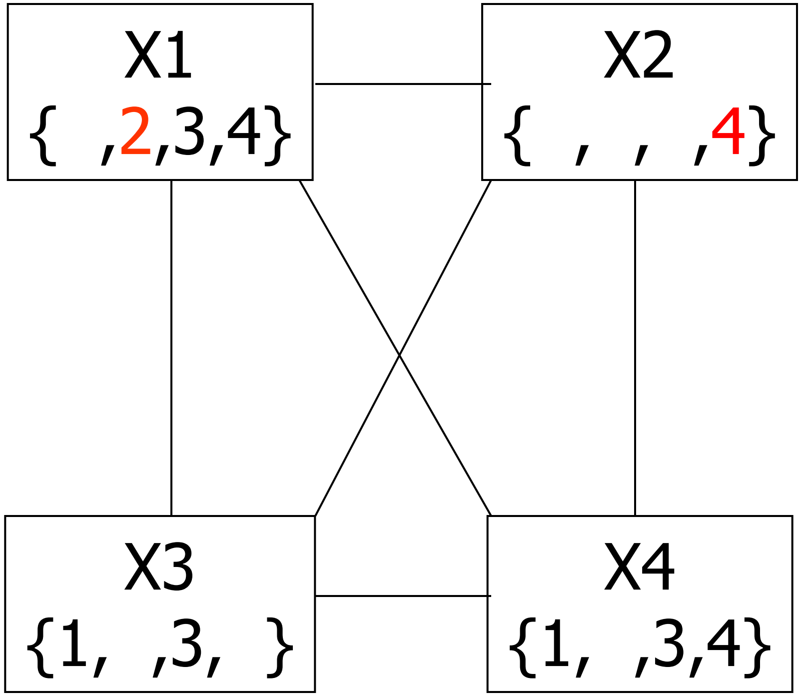




Forward check!

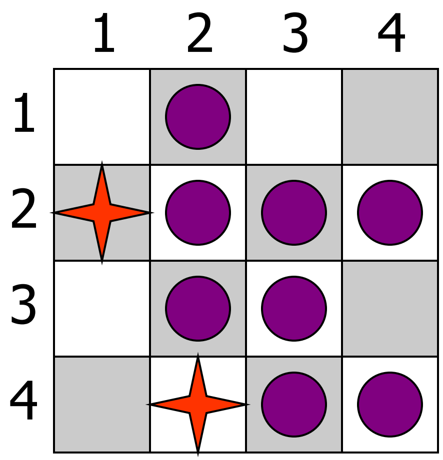
Example: 4-Queens Problem

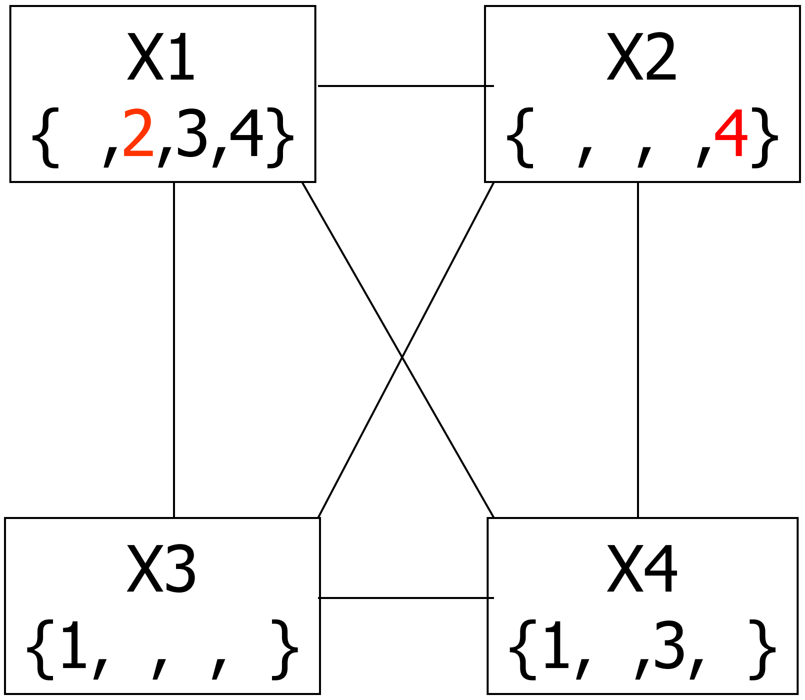




Assign value to unassigned variable

Example: 4-Queens Problem





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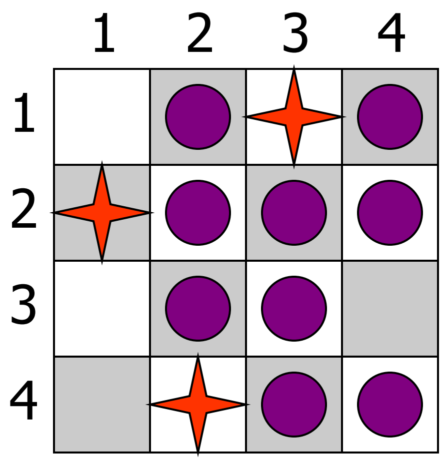

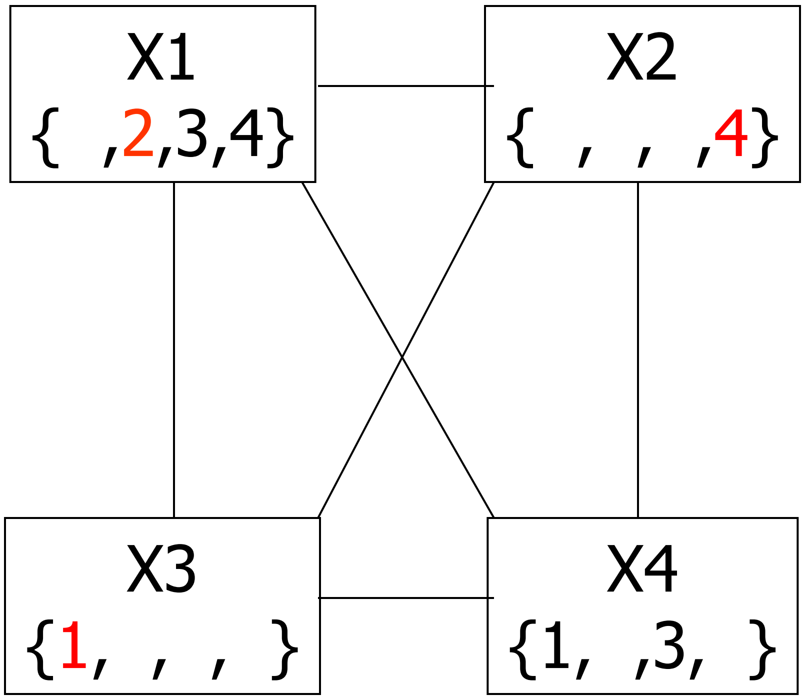

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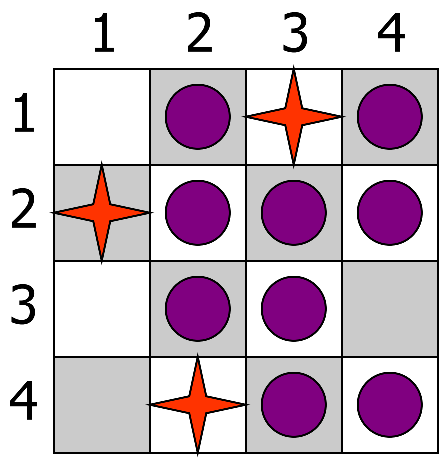

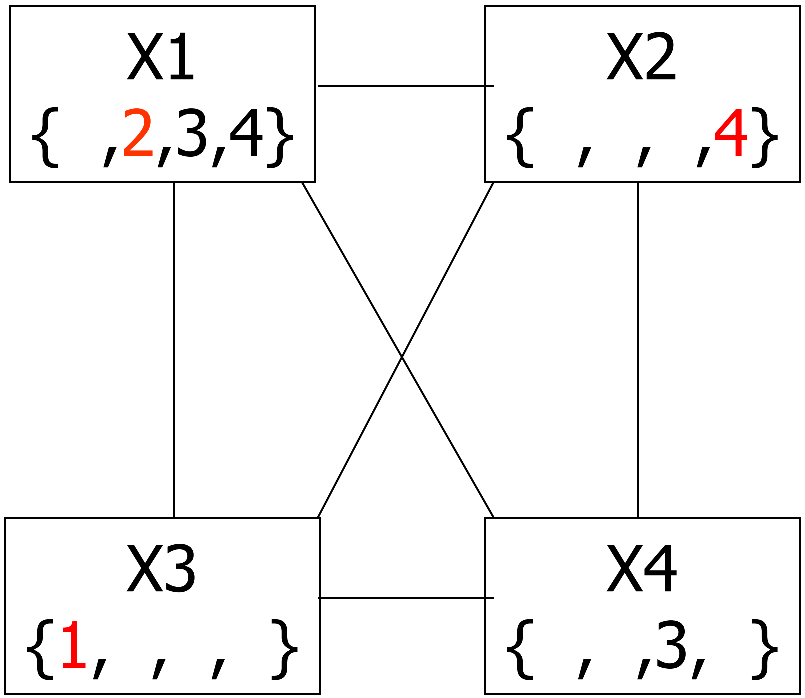

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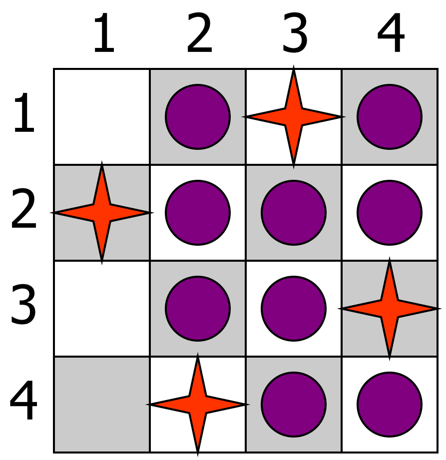
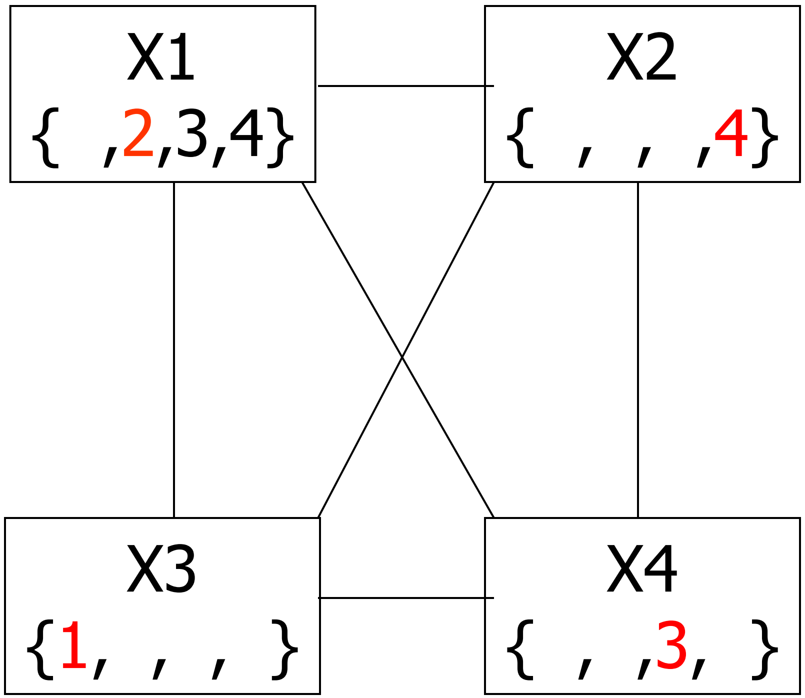
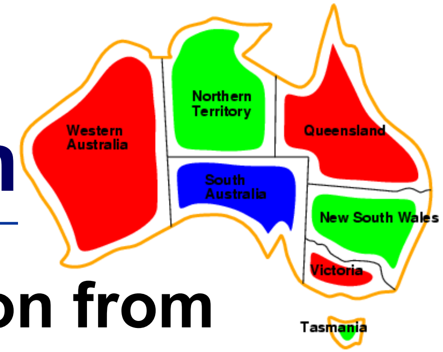


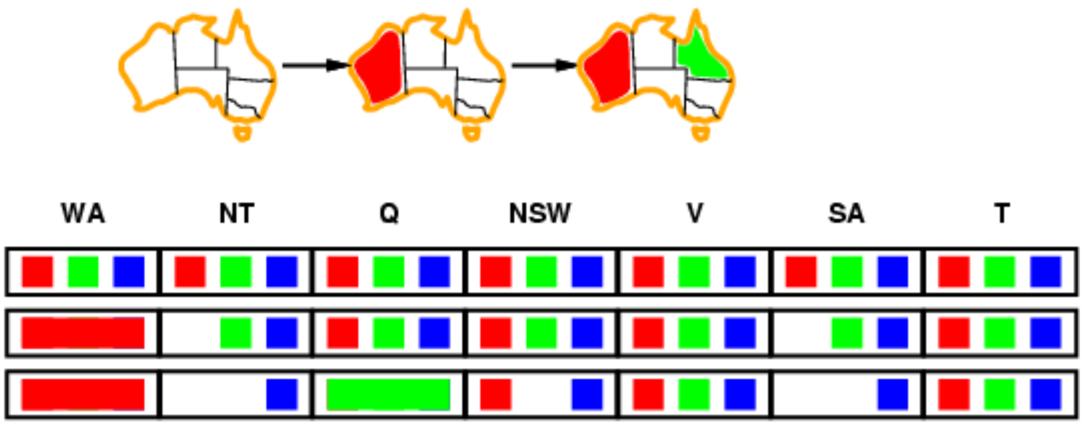


Assign value to unassigned variable

Towards Constraint propagation



- **Forward checking propagates information from *assigned* to *unassigned* variables, but doesn't provide early detection for all failures:**



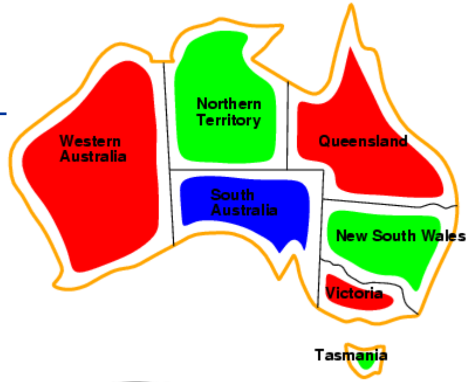
- **NT and SA cannot both be blue!**
- **Constraint propagation goes beyond forward checking & repeatedly enforces constraints locally**

Arc Consistency, Constraint Propagation & AC-3

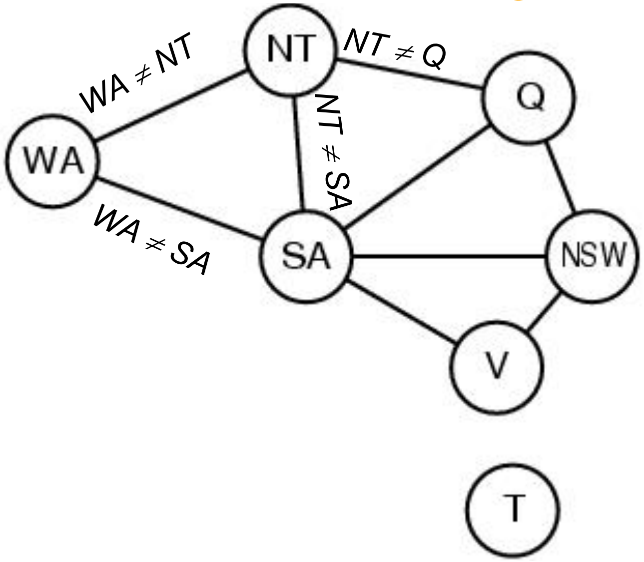
Idea 3 *(big* idea)*: Inference* in CSPs

- **CSP solvers combine search *and inference***
  - Search
    - —assigning a value to a variable
  - *Constraint propagation (inference)*
    - —Eliminates possible values for a variable if the value would violate local consistency
  - *Can do inference first, or intertwine it with search*
    - —You'll investigate this in the Sudoku homework
- **Local consistency**
  - Node consistency: satisfies unary constraints
    - —This is trivial!
  - *Arc consistency*: satisfies binary constraints
    - —(X_i is arc-consistent w.r.t. X_j if for every value v in D_i, there is some value w in D_j that satisfies the binary constraint on the arc between X_i and X_j)

Review: CSP Representations

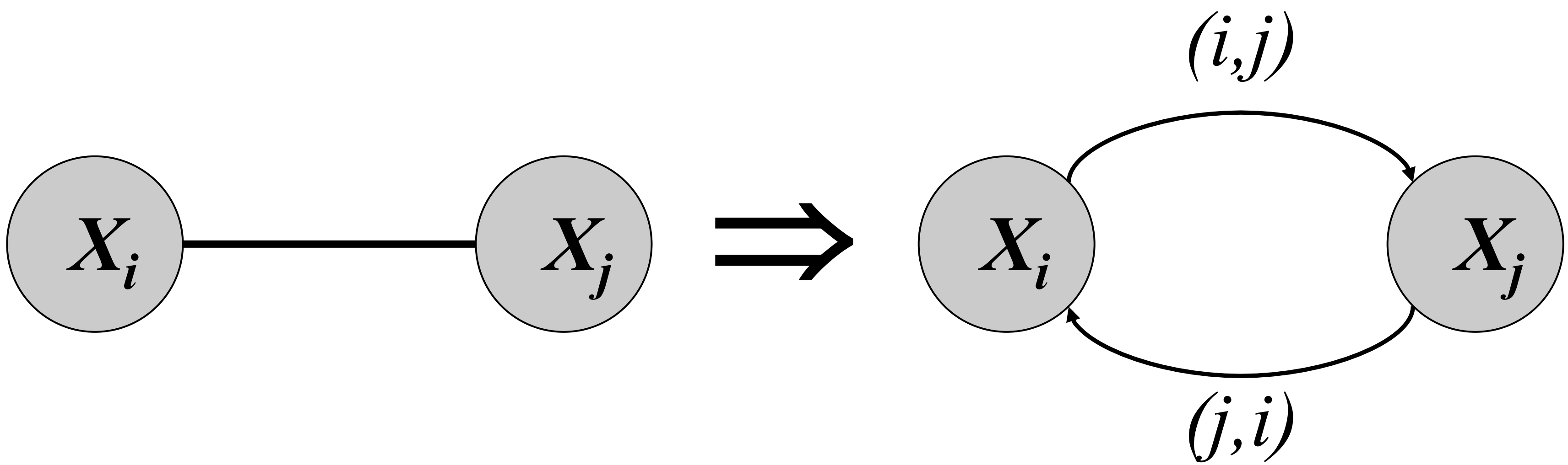


- ***Constraint graph:***
  - *nodes* are variables
  - *edges are constraints*



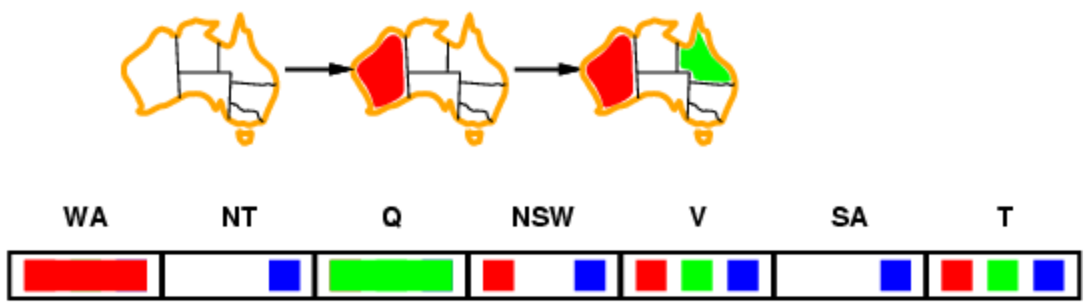
Edges to Arcs: From Constraint Graph to DAG

- **Given a pair of nodes X_i and X_j connected by a constraint *edge*, we represent this not by a single undirected edge, but a *pair of directed arcs*.**
  - **For a connected pair of nodes X_i and X_j, there are *two* arcs that connect them: *(i,j)* and *(j,i)*.**



Arc consistency

- **Simplest form of propagation makes each arc consistent**
- ***X* → *Y* is consistent iff**
  for every value *x* of *X* there is some allowed *y*



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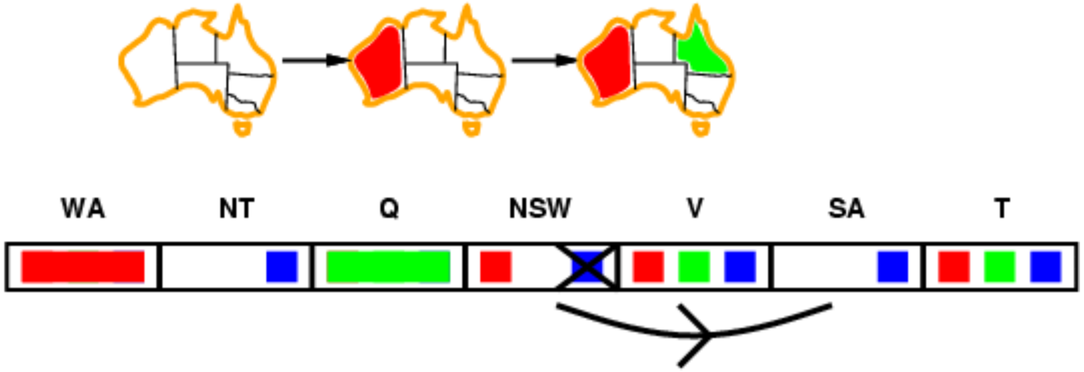

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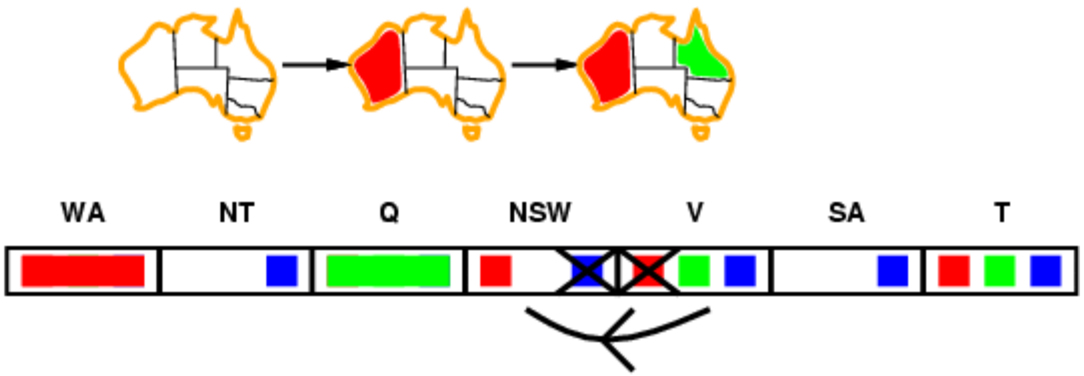
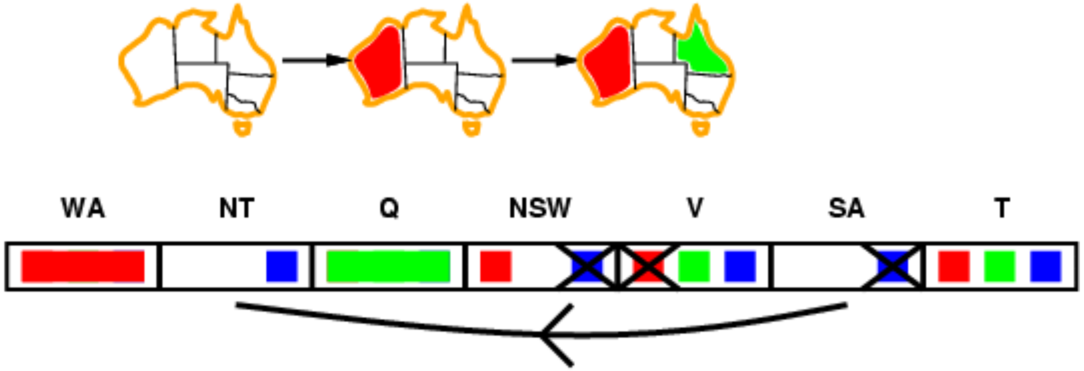


- **If *X* loses a value, recheck neighbors of *X***

Arc consistency

- **Simplest form of propagation makes each arc consistent**
- **$X \rightarrow Y$ is consistent iff**
  for every value x of X there is some allowed y



- **If X loses a value, we need to recheck neighbors of X**
- **Detects failure earlier than forward checking**
- **Can be run as a preprocessor or after each assignment**

Arc Consistency

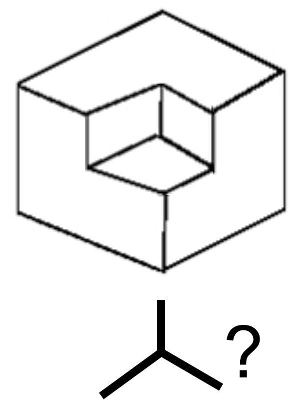
An arc *(i,j) is* arc consistent if and only if every value v on X_i is consistent with some label on X_j.

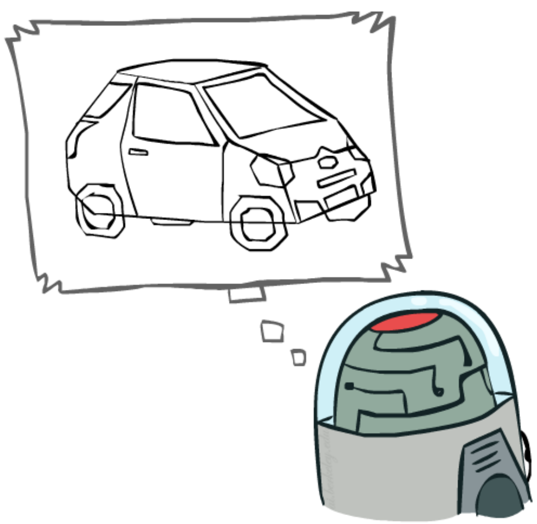
To make an arc *(i,j)* arc consistent,
for each value v on X_i ,
if there is no label on X_j consistent with v
then remove v from X_i

- **Given *d* values, checking arc (i,j) takes *O(d^2)* time worst case**

Example: The Waltz Algorithm

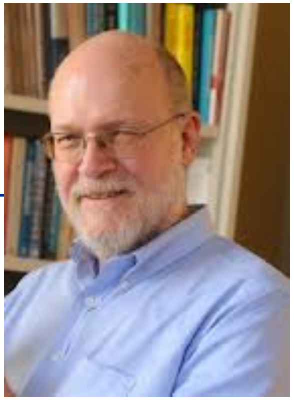
- The Waltz algorithm is for interpreting line drawings of solid polyhedra as 3D objects
- An early example of an AI computation posed as a CSP





- Approach:
  - Each intersection is a variable
  - Adjacent intersections impose constraints on each other
  - Solutions are physically realizable 3D interpretations

Replacing Search: Constraint Propagation Invented…



Dave Waltz's insight:

- **By *iterating* over the graph, the arc-consistency *constraints* can be *propagated* along arcs of the graph.**
- ***Search*: Use constraints to *add* labels to find *one solution***
- ***Constraint Propagation*: Use constraints to *eliminate labels* to simultaneously find *all solutions***

The Waltz/Mackworth Constraint Propagation Algorithm

1. **Assign *every* node in the constraint graph a set of *all* possible values**
2. **Repeat until there is no change in the set of values associated with any node:**
   3. For each node i:
      4. For each neighboring node j in the picture:
         5. Remove any value from i which is not arc consistent with j.

Inefficiencies: Towards AC-3

1. **At each iteration, we only need to examine those** X_i *where at least one neighbor of* X_i *has lost a value* **in the previous iteration.**
2. **If** X_i **loses a value only because of arc inconsistencies with** X_j**, we** *don't need to check* X_j **on the next iteration.**
3. **Removing a value on** X_i **can only make** X_j **arc-inconsistent with respect to** X_i **itself. Thus, we only need to check that** *(j,i)* **is still arc-consistent.**

These insights lead a much better algorithm...

AC-3

function **AC-3(*csp*)** return **the CSP, possibly with reduced domains**

inputs**:** ***csp*, a binary csp with variables *{X_1, X_2, …, X_n}***

local variables: ***queue,* a queue of arcs initially the arcs in *csp***

while **queue is not empty** do

(X_i, X_j) $\leftarrow$ **REMOVE-FIRST(*queue*)**

if **REMOVE-INCONSISTENT-VALUES(X_i, X_j)** then

for each X_k in **NEIGHBORS[X_i] – {X_j}** do

add (X_k, X_i) **to queue**

function **REMOVE-INCONSISTENT-VALUES(X_i, X_j)** return ***true*** **iff we remove a value**

removed $\leftarrow$ ***false***

for each ***x*** in **DOMAIN[X_i]** do

if **no value *y* in DOMAIN[X_j] allows (x,y) to satisfy the constraints between X_i and X_j**

then delete **x from DOMAIN[X_i];** ***removed*** $\leftarrow$ ***true***

return ***removed***

AC-3: Worst Case Complexity Analysis

- **All nodes can be connected to *every* other node,**
  - so each of ***n*** nodes must be compared against ***n-1*** other nodes,
  - so total # of arcs is ***2*n*(n-1), i.e. $O(n^2)$***
- **If there are *d* values, checking arc (i,j) takes *$O(d^2)$* time**
- **Each arc (i,j) can only be inserted into the queue *d* times**
- **Worst case complexity: *$O(n^2d^3)$***

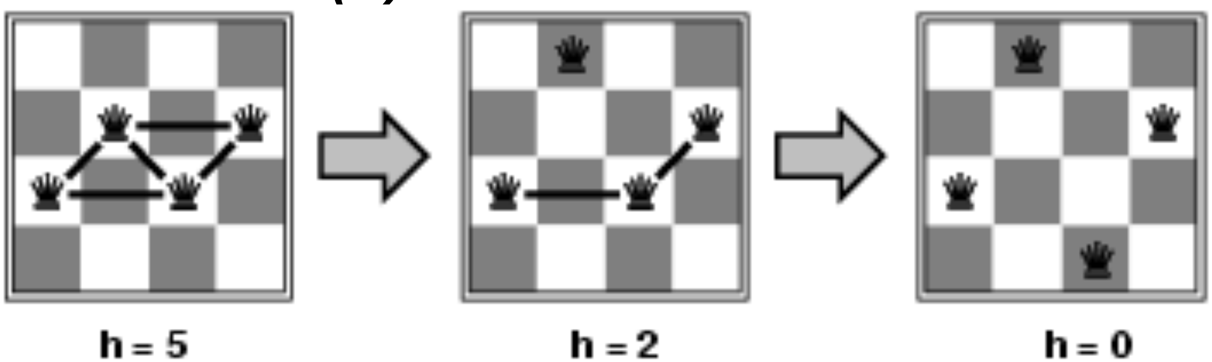
(For *planar* constraint graphs, the number of arcs can only be *linear in N and* the time complexity is only *$O(nd^3)$*)

Local search for CSPs

- **Hill-climbing, simulated annealing typically work with "complete" states, i.e., all variables assigned**
- **To apply to CSPs:**
  - allow states with unsatisfied constraints
  - operators reassign variable values
- **Variable selection: randomly select any conflicted variable**
- **Value selection by min-conflicts heuristic:**
  - choose value that violates the fewest constraints
  - i.e., hill-climb with *h(n)* = total number of violated constraints

Example: n-queens

- **States**: 4 queens in 4 columns (4^4 = 256 states)
- **Actions**: move queen in column
- **Goal test**: no attacks
- **Evaluation**: *h(n)* = number of attacks



- **Given random initial state, local min-conflicts can solve *n*-queens in almost constant time for arbitrary *n* with high probability (e.g., *n* = 10,000,000)**

Beyond binary constraints: Path consistency

- Generalizes arc-consistency from individual binary constraints to multiple constraints
- A pair of variables X_i, X_j is path-consistent w.r.t. X_m if for every assignment $X_i = a$, $X_j = b$ consistent with the constraints on X_i, X_j there is an assignment to X_m that satisfied the constraints on X_i, X_m and X_j, X_m

- **Global constraints**
  - Can apply to any number of variables
  - E.g., in Sudoko, all numbers in a row must be different
  - E.g., in cryptarithmetic, each letter must be a different digit
  - Example algorithm:
    - —If any variable has a single possible value, delete that variable from the domains of all other constrained variables
    - —If no values are left for any variable, you found a contradiction

Simple CSPs can be solved quickly

1. **Completely independent subproblems**
   - e.g. Australia & Tasmania
   - Easiest
2. **Constraint graph is a tree**
   - Any two variables are connected by only a single path
   - Permits solution in time linear in number of variables
   - Do a topological sort and just march down the list

```
A          E
 \        /
  B –  D
 /        \
C          F
```

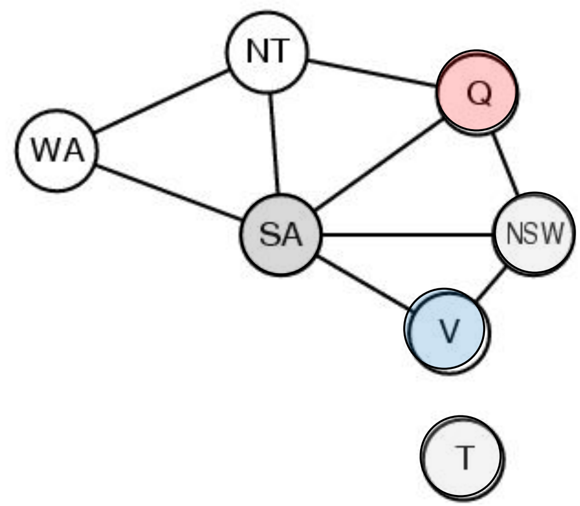
A → B → C → D → E → F

Simplifying hard CSPs: Cycle Cutsets

- **Constraint graph can be decomposed into a tree**
  - Collapse or remove nodes
  - *Cycle cutset* S of a graph G: any subset of vertices of G that, if removed, leaves G a tree
- **Cycle cutset algorithm**
  - Choose some cutset S
  - For each possible assignment to the variables in S that satisfies all constraints on S
    - —Remove any values for the domains of the remaining variables that are not consistent with S
    - —If the remaining CSP has a solution, then you have are done
  - For graph size n, domain size d
    - —Time complexity for cycle cutset of size c: $O(d^c * d^2(n-c)) = O(d^{c+2}(n-c))$

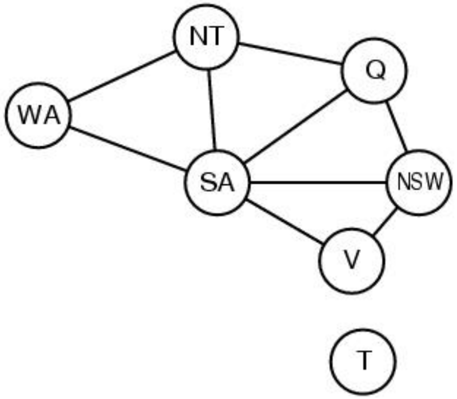
Chronological backtracking

- **DFS does Chronological backtracking**
  - If a branch of a search fails, backtrack to the most recent variable assignment and try something different
  - But this variable may not be related to the failure
- **Example: Map coloring of Australia**
  - Variable order
    - —Q, NSW, V, T, SA, WA, NT.
  - Current assignment:
    - —Q=red, NWS=green, V=blue, T= red
  - SA cannot be assigned anything
  - But reassigning T does not help!



Backjumping: Improved backtracking

- **Find "the conflict set"**
  - Those variable assignments that are in conflict
  - Conflict set for SA: {Q=red, NSW=green, V=blue}
- **Jump back to reassign one of those conflicting variables**
- **Forward checking can build the conflict set**
  - When a value is deleted from a variable's domain, add it to its conflict set
  - But backjumping finds the same conflicts that forward checking does
  - Fix using "conflict-directed backjumping"
    - —Go back to predecessors of conflict set



When to Iterate, When to Stop?

The crucial principle:

If a value is removed from a node X_i,
then the values on all of X_i's neighbors must be reexamined.

Why? *Removing* **a value from a node may result in one of the neighbors becoming arc** *inconsistent***, so we need to check…**

(but each neighbor X_j can only become inconsistent with respect to the removed values on X_i)