MAP511 Project
Model-Free Hedging

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Abstract

In this work, we aim to treat the problem of pricing and hedging of options by means of a robust or "model-free" approach. First of all, we introduce the problem of super-replication strategies, by regarding the case of Vanilla options in a theoretical approach, then by doing numerical simulations with real market data. In a second moment, we attack the problem of pricing and hedging of multi-asset options in the case of Optimal Transport, doing once again numerical simulations in the dual and primal versions of the problem. Finally, the problem of Exotic Options in the case of Martingale Optimal Transport is also studied, in its primal and dual version as well.

Keywords : *Model-free, super-replication, numerical simulations, multi-asset option, exotic option, martingale optimal transport*

1 Introduction

Let us consider an option with payoff $F_T = F(S_T)$ depending on the price of a asset S at time T (the initial time is equal to zero). We are interested in finding the optimal super-replication strategy using static positions on the underlying. We are also interested in finding a representation of the market price of vanilla options.

1.1 Super-replication of Vanilla Options

Our first problem is to determine the sell price of a vanilla option, using a static hedging strategy. Let us denote F_T an European payoff at maturity T and depending on the value S_T of an asset. We want to determine the price C and hedging strategy for this option at time $t = 0$.
Let us assume that we have sold an option (at $t = 0$) at price C, and bought at the same time a quantity H of assets at price S_0. The portfolio value at time 0 is then

$$\pi_0 = -HS_0 + C$$

At the maturity of the contract (time T) the portfolio value is

$$\pi_T = (-HS_0 + C)e^{rT} + HS_T - F_T$$

which is equivalent to

$$e^{-rT}\pi_T = H(S_T e^{-rT} - S_0) + C - e^{-rT}F_T \tag{1.1}$$

To price this option we need some information about the distribution of the price S_T. That is why we make the following

Assumption 1. *The random variable S_T is well-modeled by an historical probability $\mathbb{P}^{hist}$.*

As the seller of the option, we want to hedge it in such a way that we have a non-negative payoff at maturity, but we also want to sell it by a reasonable price. Following these premises we define the seller's super replication price by

Definition 1.1. ***Seller's super-replication price***

$$C_{sel} \equiv inf\{C : \exists H \quad s.t. \quad \pi_T \geq 0, \quad \mathbb{P}^{hist} - a.s\} \tag{1.2}$$

where the abbreviation a.s. means almost surely. C and H are chosen such that the portfolio value at T is non negative for all realization of S_T distributed according to the law $\mathbb{P}^{hist}$.
Here, note that our definition depends weakly on our modeling assumption only through the negligible sets of $\mathbb{P}^{hist}$. $\mathbb{P}^{hist}$ can be replaced by any probability $\mathbb{Q}$ equivalent (see definition below) to $\mathbb{P}^{hist}$.

Definition 1.2. ***Equivalent Probabilities*** $\mathbb{P} \sim \mathbb{Q}$ *- we say* $\mathbb{P}$ *and* $\mathbb{Q}$ *are equivalent on a sigma field* $\mathcal{F}$ *- if* $\mathbb{P}$ *and* $\mathbb{Q}$ *have the same negligible sets:* $\mathbb{P}(A) = 0$ *if and only if* $\mathbb{Q}(A) = 0$ *for all* $A \in \mathcal{F}$*.*

Using this definition for C_{sel}, we can find the Monte-Kantorovich dual formulation of C_{sel}

Theorem 1.1. ***Seller's super-replication price***
Let $F_T \in L^\infty(\mathbb{P}^{hist})$*. Then,*

$$C_{sel} = \sup_{\mathbb{Q} \in \mathcal{M}_1} \mathbb{E}^{\mathbb{Q}}[e^{-rT} F_T] \tag{1.3}$$

with $\mathcal{M}_1 \equiv \{\mathbb{Q} : \mathbb{Q} \sim \mathbb{P}^{hist}, \quad \mathbb{E}^{\mathbb{Q}}[e^{-rT} S_T] = S_0\}$.

The elements of $\mathcal{M}_1$ are called "risk-neutral" probabilities and they are equivalent to $\mathbb{P}^{hist}$ in the following sense

Proof. Consult [1] □

We are going to study the super-replication problem for more complicated options: multi-asset options (optimal transport problem) and exotic option (martingale optimal transport). And we will see that this dual theorem is also present in those formulations

1.2 Options prices and Marginal distributions

We suppose from now on that the interest rate is null and that prices of call options $(S_T - K)^+$ and put options $(K - S_T)^+$ are available for all strikes K. Let us denote those prices by $C(K)$ and $P(K)$ respectively. We then introduce the pricing operator $\Pi[.]$, used to evaluate Vanillas. By the no arbitrage condition we have that $\Pi[(S_T - K)^+] = C(K)$ is non-increasing, convex, $C'(\infty) = 0$ and $C'(-\infty) = 1$. Moreover, we have the put/call parity

$$C(K) - P(K) = S_0 - K \tag{1.4}$$

We thus conclude that $\mathbb{P}^{mkt} = C''$ is a probability measure and that $C''(K) = P''(K), \forall K$.
Finally, we should also have that

$$\Pi[1] = 1, \qquad \Pi[S_T] = S_0 \tag{1.5}$$

All Vanilla options can be replicated by a strip of put/call options on every strike. In fact, for any function $\lambda \in \mathcal{C}^2(\mathbb{R})$ we can write [3]

$$\lambda(S_T) = \lambda(S_0) + \lambda'(S_0)(S_T - S_0) + \int_0^{S_0} \lambda''(K)(K - S_T)^+ dK + \int_{S_0}^{\infty} \lambda''(K)(S_T - K)^+ dK \tag{1.6}$$

Furthermore, let us assume that $\Pi[.]$ is linear and continuous. Hence, for all λ of class $\mathcal{C}^2$ we have

$$\begin{aligned}
\Pi[\lambda(S_T)] &= \lambda(S_0) + \lambda'(S_0)(\Pi[S_T] - S_0) + \int_0^{S_0} \lambda''(K)\Pi[(K - S_T)^+]dK \\
&\quad + \int_{S_0}^{\infty} \lambda''(K)\Pi[(S_T - K)^+]dK \\
&= \lambda(S_0) + \int_0^{S_0} \lambda''(K)P(K)dK + \int_{S_0}^{\infty} \lambda''(K)C(K)dK \\
&= \int \lambda(K)C''(K)dK, \qquad \text{(integration by parts)}
\end{aligned}$$

Meaning that we can write the price of a Vanilla as an expectation over the market measure implied from the call prices. Resuming, we have that

$$\Pi[\lambda(S_T)] = \mathbb{E}^{\mathbb{P}^{mkt}}[\lambda(S_T)], \quad \textit{where} \quad \mathbb{P}^{mkt}(S_T = K) = C''(K).$$

2 Multi-asset Options and Optimal Transport

Here we are interested in the super-replication problem for multi-asset options (we are going to study the case of 2 asset options only). Let us consider two assets S_1 and S_2 evaluated at the same maturity T and an option of payoff $c(S_1, S_2)$ depending on these underlyings. We denote by $\mathbb{P}^1$ and $\mathbb{P}^2$ the distributions implied from Vanilla options on S_1 and S_2 respectively.

2.1 Super-replication and Monge-Kantorovich duality

The following assumption on the payoff is made

Assumption 2. *Let $c : \mathbb{R}_+^2 \to [-\infty, \infty)$ be a continuous function such that*

$$c^+(s_1, s_2) \leq K \cdot (1 + s_1 + s_2)$$

on $(\mathbb{R}_+)^2$ for some constant K.

The super-replication price is defined as

Definition 2.1.

$$MK_2 \equiv \inf_{\mathcal{P}^*(\mathbb{P}^1, \mathbb{P}^2)} \mathbb{E}^{\mathbb{P}^1}[\lambda_1(S_1)] + \mathbb{E}^{\mathbb{P}^2}[\lambda_2(S_2)] \tag{2.1}$$

where $\mathcal{P}^*(\mathbb{P}^1, \mathbb{P}^2)$ is the set of all functions $(\lambda_1, \lambda_2) \in L^1(\mathbb{P}^1) \times L^2(\mathbb{P}^2)$ such that

$$\lambda_1(s_1) + \lambda_2(s_2) \geq c(s_1, s_2) \tag{2.2}$$

for $\mathbb{P}^1$-almost all $s_1 \in \mathbb{R}_+$ and $\mathbb{P}^2$-almost all $s_2 \in \mathbb{R}_+$.

This static super-replication strategy consists in holding Vanilla payoffs $\lambda_1(s_1)$ and $\lambda_2(s_2)$ with $(t = 0)$ market prices $\mathbb{E}^{\mathbb{P}^1}[\lambda_1(S_1)]$, $\mathbb{E}^{\mathbb{P}^2}[\lambda_2(S_2)]$ such that the value of this portfolio $\lambda_1(s_1) + \lambda_2(s_2)$ at maturity is greater than or equal to the payoff $c(s_1, s_2)$.
The definition above introduces the primal problem, involving calculating the optimal hedging strategy. The following theorem states this problem in its dual formulation, which is the problem of calculating the optimal super-replication price.

Theorem 2.1.

$$MK_2 \equiv \sup_{\mathbb{P} \in \mathcal{P}(\mathbb{P}^1, \mathbb{P}^1)} \mathbb{E}^{\mathbb{P}}[c(S_1, S_2)] \tag{2.3}$$

with $\mathcal{P}(\mathbb{P}^1, \mathbb{P}^1) = \{\mathbb{P} : S_1 \overset{\mathbb{P}^1}{\sim} \mathbb{P}^1, \overset{\mathbb{P}^2}{\sim} \mathbb{P}^2\}$.

Proof. Consult [1] □

It is shown in [1] that the supremum is over $\mathcal{P}(\mathbb{P}^1, \mathbb{P}^1)$ is attained. That is, there exists a probability measure $\mathbb{P}^* \in \mathcal{P}(\mathbb{P}^1, \mathbb{P}^1)$ such that $MK_2 = \mathbb{E}^{\mathbb{P}^*}[c(S_1, S_2)]$. The next proposition proves that the infimum in the primal problem is also attained by a pair (λ, λ^*) of bounded continuous c-concave functions.

Proposition 2.1.

$$MK_2 = \inf_{\lambda \in C_b} \mathbb{E}^{\mathbb{P}^1}[\lambda^*(S_1)] + \mathbb{E}^{\mathbb{P}^2}[\lambda(S_2)] \tag{2.4}$$

with $\lambda^* \equiv \sup_{s_2 \in \mathbb{R}_+} \{c(s_1, s_2) - \lambda(s_2)\}$ the c-concave transform of λ.

2.2 Fréchet-Hoeffding solution

Under some conditions on the payoff, this super-replication problem has an exact solution for which the optimal transport measure does not depend on the payoff.

Theorem 2.2. *Under $c_{12} > 0$,*
***(i)**: The optimal measure $\mathbb{P}^*$ has the form*

$$\mathbb{P}^*(ds_1, ds_2) = \delta_{T(s_1)}(ds_2)\mathbb{P}^1(ds_1)$$

with T the forward image of the measure $\mathbb{P}^1$ onto $\mathbb{P}^2$: $T(x) = F_2^{-1} \circ F_1(x)$.
***(ii)**: The optimal upper bound is given by*

$$MK_2 = \int_0^1 c(F_1^{-1}(u), F_2^{-1}(u))du$$

This optimal bound can be attained by a static hedging strategy consisting in holding European payoffs $(\lambda_1, \lambda_2) \in L^1(\mathbb{P}^1) \times L^2(\mathbb{P}^2)$ with market prices $\mathbb{E}^{\mathbb{P}^1}[\lambda_1(S_1)]$ and $\mathbb{E}^{\mathbb{P}^2}[\lambda_2(S_2)]$:

$$MK_2 = \mathbb{E}^{\mathbb{P}^1}[\lambda_1(S_1)] + \mathbb{E}^{\mathbb{P}^2}[\lambda_2(S_2)]$$

with

$$\lambda_2(x) = \int_0^x c_2(T^{-1}(y), y)dy, \quad \lambda_1(x) = c(x, T(x)) - \lambda_2(T(x))$$

Example 1. $c(s_1, s_2) = s_1 s_2$.
The payoff satisfies the Spence-Mirrlees condition, $c_{12} = 1 > 0$. By applying Fréchet-Hoeffding solution, the upper bound is attained by

$$MK_2 = \int_0^1 c(F_1^{-1}(u), F_2^{-1}(u))du = \int_0^1 F_1^{-1}(u) \cdot F_2^{-1}(u)du$$

The optimal hedging is

$$\lambda_2(x) = \int_0^x T^{-1}(y)dy = \int_0^x F_1^{-1}(F_2(y))dy$$

$$\lambda_1(x) = xF_2^{-1}(F_1(x)) - \int_0^x F_1^{-1}(F_2(y))dy$$

In the special case of $F_1 = F_2$ we have

$$\lambda_2(x) = \lambda_1(x) = \frac{x^2}{2} \quad \text{and} \quad MK_2 = \mathbb{E}^{\mathbb{P}^1}[S_1^2] = \mathbb{E}^{\mathbb{P}^2}[S_2^2]$$

Example 2. $c(s_1, s_2) = s_1 s_2^2$.
The payoff satisfies the Spence-Mirrlees condition, $c_{12} = 2s_2 > 0$. By applying Fréchet-Hoeffding solution, the upper bound is attained by

$$MK_2 = \int_0^1 c(F_1^{-1}(u), F_2^{-1}(u))du = \int_0^1 F_1^{-1}(u) \cdot (F_2^{-1}(u))^2du$$

The optimal hedging is

$$\lambda_2(x) = \int_0^x 2yT^{-1}(y)dy = \int_0^x 2yF_1^{-1}(F_2(y))dy$$

$$\lambda_1(x) = x(F_2^{-1}(F_1(x)))^2 - \int_0^x 2yF_1^{-1}(F_2(y))dy$$

In the special case of $F_1 = F_2$ we have

$$\lambda_2(x) = 2x^3/3,\ \lambda_1(x) = x^3/3 \text{ and } MK_2 = \mathbb{E}^{\mathbb{P}^1}[S_1^3] = \mathbb{E}^{\mathbb{P}^2}[S_2^3]$$

Example 3. $c(s_1, s_2) = (s_1 - K_1)^+ \mathbb{1}_{s_2 > K_2}$.
This is Example 2.2 of [1]. The upper bound is attained by

$$MK_2 = \int_{max(F_1(K_1), F_2(K_2))}^1 (F_1^{-1}(u) - K_1) du$$

with

$$\begin{aligned}
\lambda_2(x) &= (F_1^{-1} \circ F_2(K_2) - K_1)^+ \mathbb{1}_{x > K_2} \\
\lambda_1(x) &= (x - K_1)^+ \mathbb{1}_{F_2^{-1} \circ F_1(x) > K_2} - (F_1^{-1} \circ F_2(K_2) - K_1)^+ \mathbb{1}_{F_2^{-1} \circ F_1(x) > K_2}
\end{aligned}$$

In the particular case where $F_1 = F_2$, we have

$$\begin{aligned}
\lambda_2(x) &= (K_2 - K_1)^+ \mathbb{1}_{x > K_2} \\
\lambda_1(x) &= (x - K_1)^+ \mathbb{1}_{x > K_2} - (K_2 - K_1)^+ \mathbb{1}_{x > K_2}
\end{aligned}$$

3 Exotic Options and Martingale Optimal Transport

When considering Exotic Options (path dependent payoffs) and a replication strategy that consists of a finite number of vanilla options on different maturities and a discrete hedging on the underlying we face a discrete time martingale optimal transport (MOT) problem. When the position on the underlying is continuously changed we have a continuous time MOT problem. Let us consider two periods options. Let S_1 and S_2 denote the price of an asset at times t_1 and t_2 respectively and an option with payoff $c(s_1, s_2)$. The super-replication problem is then written as

Definition 3.1.

$$\widetilde{MK}_2 \equiv \inf_{\mathcal{M}^*(\mathbb{P}^1, \mathbb{P}^2)} \mathbb{E}^{\mathbb{P}^1}[\lambda_1(S_1)] + \mathbb{E}^{\mathbb{P}^2}[\lambda_2(S_2)] \tag{3.1}$$

where $\mathcal{M}^*(\mathbb{P}^1, \mathbb{P}^2)$ is the set of functions $\lambda_1 \in L^1(\mathbb{P}^1)$, $\lambda_2 \in L^1(\mathbb{P}^2)$ and H a bounded continuous function on $\mathbb{R}_+$ such that

$$\lambda_1(s_1) + \lambda_2(s_2) + H(s_1)(s_2 - s_1) \geq c(s_1, s_2), \qquad \forall (s_1, s_2) \in \mathbb{R}_+^2 \tag{3.2}$$

The dual version problem above introduces the martingale measure. We set

$$\widetilde{MK}_2^* \equiv \sup_{\mathbb{P} \in \mathcal{M}(\mathbb{P}^1, \mathbb{P}^2)} \mathbb{E}^{\mathbb{P}}[c(S_1, S_2)] \tag{3.3}$$

where $\mathcal{M}(\mathbb{P}^1, \mathbb{P}^2) = \{\mathbb{P} : \mathbb{E}^{\mathbb{P}}[S_2|S_1] = S_1, S_1 \overset{\mathbb{P}}{\sim} \mathbb{P}^1, S_2 \overset{\mathbb{P}}{\sim} \mathbb{P}^2\}$ denotes the set of (discrete) martingale measures on $\mathbb{R}_+^2$ with marginals $\mathbb{P}^1$ and $\mathbb{P}^2$.
One can show [1] that the set $\mathcal{M}(\mathbb{P}^1, \mathbb{P}^2)$ is convex and weakly compact, the dual $\widetilde{MK}_2^*$ is attained by an extremal point. The following proposition gives a necessary and sufficient condition to ensure that $\mathcal{M}(\mathbb{P}^1, \mathbb{P}^2)$ is non-empty:

Proposition 3.1. ***see Kellerer [2] and [1].***
The set $\mathcal{M}(\mathbb{P}^1, \mathbb{P}^2)$ is feasible (non-empty) if and only if $\mathbb{P}^1, \mathbb{P}^2$ have the same mean S_0 and $\mathbb{P}^1 \leq \mathbb{P}^2$ are in convex order.

Definition 3.2. ***Convex order.*** *$\mathbb{P}^1 \leq \mathbb{P}^2$ are in convex order if and only if*

$$\mathbb{E}^{\mathbb{P}^1}[(S_1 - K)^+] \leq \mathbb{E}^{\mathbb{P}^2}[(S_2 - K)^+], \quad \forall K \in \mathbb{R}_+ \tag{3.4}$$

This condition excludes calendar spread arbitrage opportunities on call options.

Theorem 3.1. *Assume that $\mathbb{P}^1 \leq \mathbb{P}^2$ are probability measures on $\mathbb{R}_+$ with first moments S_0 and Assumption 3 holds. Then there is no duality gap, i.e., $\widetilde{MK}_2^* = \widetilde{MK}_2$. Moreover, the dual value $\widetilde{MK}_2^*$ is attained. In general, the primal $\widetilde{MK}_2$ is not attained (see [4] for a counter-example).*

A *quasi-sure* formulation has been developed [5] and shows a similar result in which the primal solution is attained whenever the super-replication price is finite.

4 Numerical Experiments and Computational Methods

4.1 Extracting the Risk-Neutral Distribution from Call prices

We have seem that we can imply the risk neutral distribution from the price of Call options on all strikes. But in practice we only have a limited amount of strikes available. There are a lot of different methods of extracting the RND (risk-neutral distribution).
One of them is developed in [1], and consists in finding the distribution with minimal distance from another fixed distribution (like log-normal) that satisfies the market prices (The price of every listed call under this distribution coincides with the market price).
Another simpler alternative [6] consists in interpolating the implied volatility as a function of strike, using clamped (zero first order derivatives at the data bounds) cubic splines and extrapolate of the volatility outside the bound of available market strikes. This is the method we used here.

4.1.1 Numerical Experiments

We investigated this method on GOOGL (Google) stock options. The data we used was extracted from `https://finance.yahoo.com/quote/GOOG/options?p=GOOG`. We extracted the RND from call options with maturity on 17 January 2020. Since GOOGL does not yield dividends, even though it is an American options it can be treated as an European option with no dividends, this means that we can price it using the basic Black-Scholes model for calls. **In our tests we did not use a clamped spline**.
On the maturity used there are 113 strikes available on the market, we can observe the price on the figure 1 (left). The implied volatility is then calculated, resulting in the figure 1 (right).

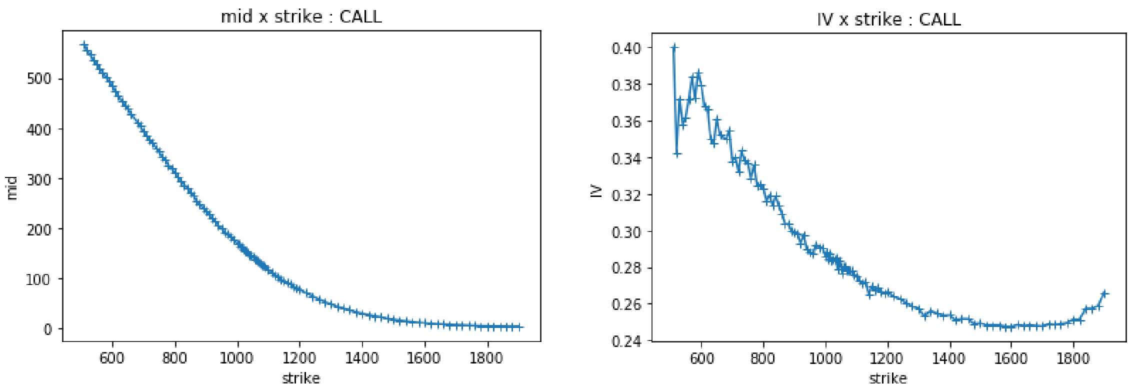

Figure 1: Market Data for GOOGL 2020-01-17 Call Options

The next step is interpolating the implied volatility by a cubic spline. To have a non-negative RND it was necessary to use smoothing when calculating the spline. This raises the approximation error but also grants that the price will be convex on the strike. We tried two different interpolations methods: (1) first find a smoothing spline for the market data and then extrapolate the implied volatility for strikes outside of market bounds - The result is in figure 2 (left); (2) first extrapolate the volatility for some points outside of the market strike bounds and then smoothly interpolate the spline - result in figure 2 (right).
The method (1) has less error due to smoothing but will present some artifacts in the RND in the strikes corresponding to the transitions from extrapolated to interpolated data. On the other hand, the method (2) will have a smoother RND but will present more error due to smoothing. The RND's generated by these two methods are in

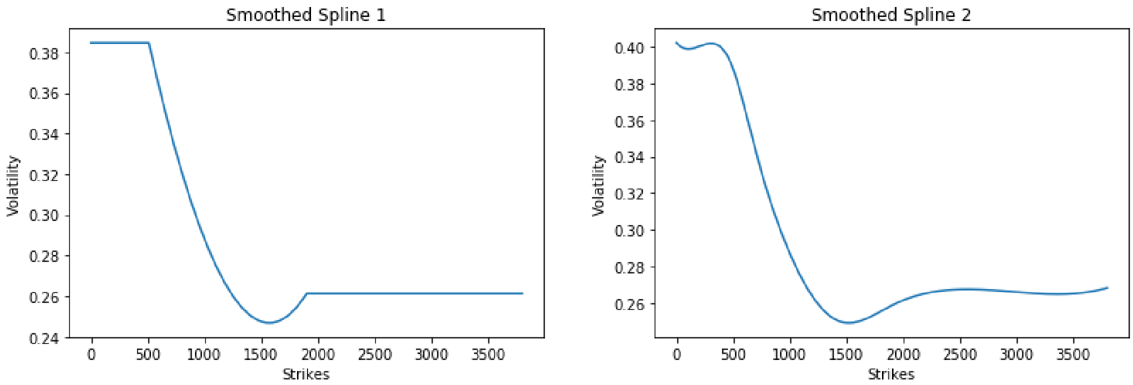

Figure 2: Volatility Spline using two different methods: (1) left, (2) right

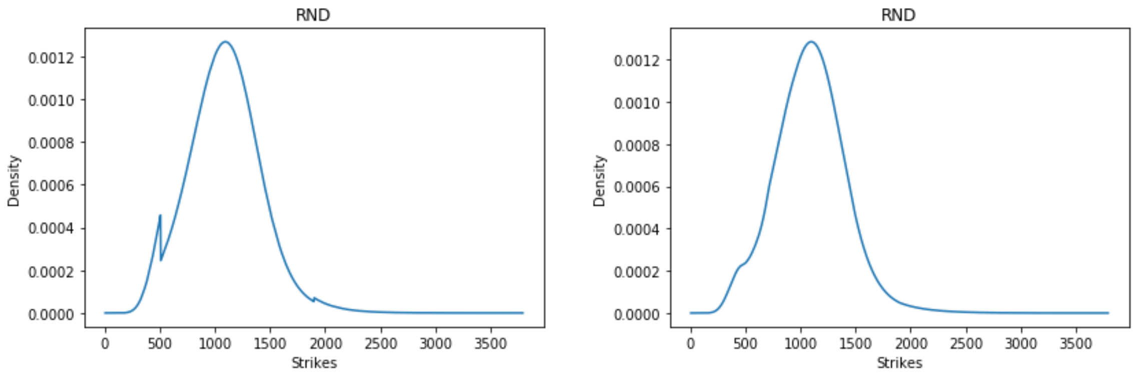

Figure 3: RND extracted from GOOGL call options using methods: (1) on the left; and (2) on the right.

4.2 Numerical Solution for 2-Asset Options

Here we solve the optimal transport problem for finite measures $\mathbb{P}^1$ and $\mathbb{P}^2$. In practice, we will discretize continuous measures in order to numerically solve the optimal transport problem.
If we use the definition 2.1, the primal discrete problem can be written as:

$$MK_2^{(m,n)} = \inf_{(\lambda_1^i)_{1\le i\le m},(\lambda_2^j)_{1\le j\le n}} \sum_{i=1}^{m} \lambda_1^i \cdot \alpha_i + \sum_{j=1}^{n} \lambda_2^j \cdot \beta_j \tag{4.1}$$

s.t. $\lambda_1^i + \lambda_2^j \ge c(x_i, y_j), \quad \forall\, 1 \le i \le m, 1 \le j \le n$. Where $(\alpha_i)_{1\le i\le m}$ is the vector of probabilities of x and $(\beta_j)_{1\le j\le n}$ is the vector of probabilities of y.
Analogously, by the theorem 2.1, the dual problem is written as

$$MK_2^{(m,n)} = \sup_{(p_{ij})_{1\le i\le m,1\le j\le n}} \sum_{1\le i\le m,1\le j\le n} p_{ij} \cdot c(x_i, y_j) \tag{4.2}$$

s.t.

$$\sum_{1\le j\le n} p_{ij} = \alpha_i, \qquad i = 1, 2, \ldots, m. \tag{4.3}$$

$$\sum_{1\le i\le m} p_{ij} = \beta_j, \qquad j = 1, 2, \ldots, n. \tag{4.4}$$

Both of these problems are linear programming problems that can be solved by algorithms such as "Simplex" and "Interior Point". Thus, in order to numerically solve the super-replication problem for continuous measures we need to discretise the measures of the underlyings.

4.2.1 The discretisation scheme

To solve numerically the super-replication problem, for a measure $\mathbb{P}$ (we suppose that $\mathbb{P}$ has no mass points) on $\mathbb{R}$ one should restrict this measure to a compact interval $[a - R, a + R]$. The value of R may be chosen

as the radius of a confidence interval centered at the mean value a of the measure.
Once the radius is chosen, one can partition the interval $[-R, R]$ (suppose that $a = 0$) in uniformly distributed points $-R = x_0 < x_1 < \ldots < x_n = R$ with $x_i = -R + \frac{2iR}{n}$.
It is possible to define the discrete measure $\mathbb{P}_n[X = x_i] = \mathbb{P}[|X - x_i| < \frac{R}{n}]$. We assume that, as $R \to \infty$ and $n \to \infty$ the measure $\mathbb{P}_n$ converges to $\mathbb{P}$.

Uniform distribution

For an uniform variable in the interval $[0, S_0]$ where $S_0 \in \mathbb{R}$ and $S_0 \geq 0$, we chose to do a partition with only one parameter, the number of points N. These points are chosen to be equally distributed in $[0, S_0]$.

Log-Normal distribution

A log-normal variable has the form $X = S_0 * exp(\sigma \cdot G - \frac{\sigma^2}{2})$, where G is a standard gaussian variable. To discretise this variable we chose two parameters: m and N. The parameter N represents the number of points in our discretisation and m denotes the number of standard deviations of G we are going to consider when constructing the interval described above. Thus, for given m, N we will consider the interval

$$I = [exp(-\sigma \cdot m - \frac{\sigma^2}{2}), \quad exp(\sigma \cdot m - \frac{\sigma^2}{2})]$$

and a subdivision of N equally distributed points.

4.2.2 Numerical Experiments

In this section, we aim to treat 2 out of the 3 examples discussed in section 2.2. Particularly, we'll show the results obtained when the discretisations of the primal and dual problems are done and we'll be doing a comparison between these results and the results of the "exact" solution [1].
For each different cost function, we'll be interested in analyzing some of the following densities of probability :

- $S_1 \sim Unif[0, 10]$ and $S_2 \sim Unif[0, 5]$
- $S_1 \sim LogN(100, 0.5)$ and $S_2 \sim LogN(50, 0.3)$
- $S_1 \sim LogN(100, 0.2)$ and $S_2 \sim LogN(100, 0.2)$

Since we have the theorem 2.2, we know that the support function - or the support, in the numerical case - is independent of the cost functions : it depends only of the density of probability in each case[2]. The reader will find in the figures 4, 5 and 6 a comparison between the obtained graphics for the support, in the numerical and in the exact approach for the mentioned densities. Figures 4, 5, 6 refer respectively to the first, second and third pairs of densities of probability above.

[1]By "exact" solution, we understand the numerical determination of $MK2$ in its integral form :

$$MK_2 = \int_0^1 c(F_1^{-1}(u), F_2^{-1}(u))du$$

[2]Nonetheless, the reader should keep in mind that the theorem 2.2 assures that under $c_{12} > 0$, which is not the case of the cost function in the example 3

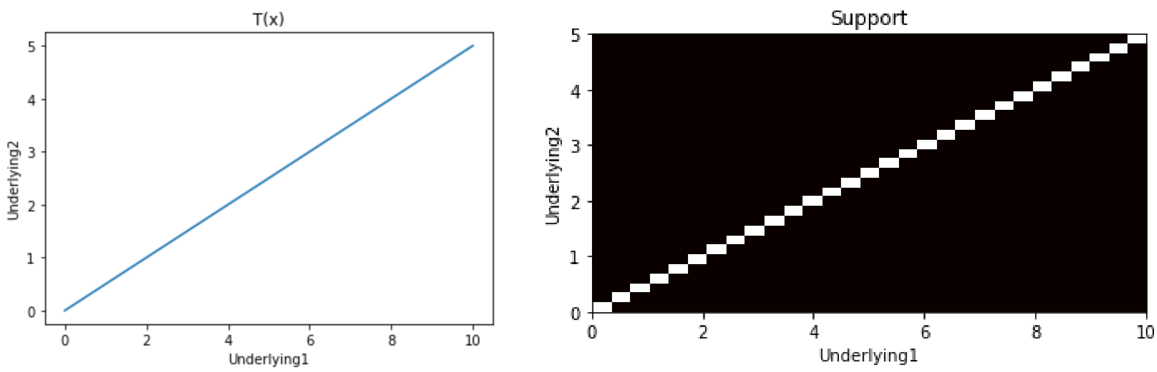

Figure 4: Function support and support using two different solutions: (1) "exact", (2) numerical in the case where $S_1 \sim Unif[0, 10]$ and $S_2 \sim Unif[0, 5]$

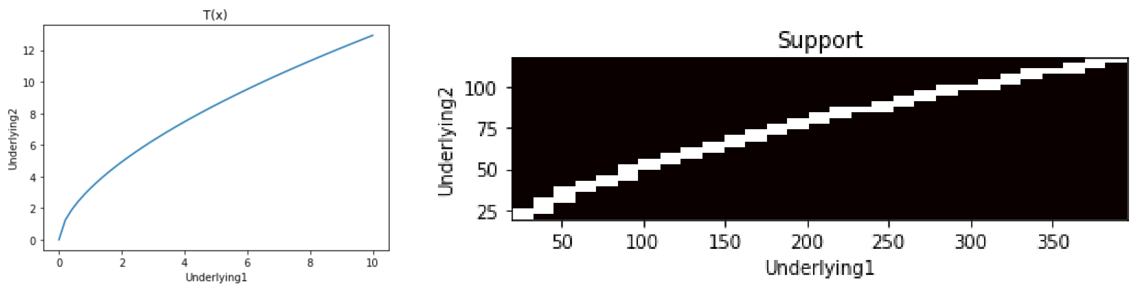

Figure 5: Function support and support using two different solutions: (1) "exact", (2) numerical in the case where $S_1 \sim LogN(100, 0.5)$ and $S_2 \sim LogN(50, 0.3)$

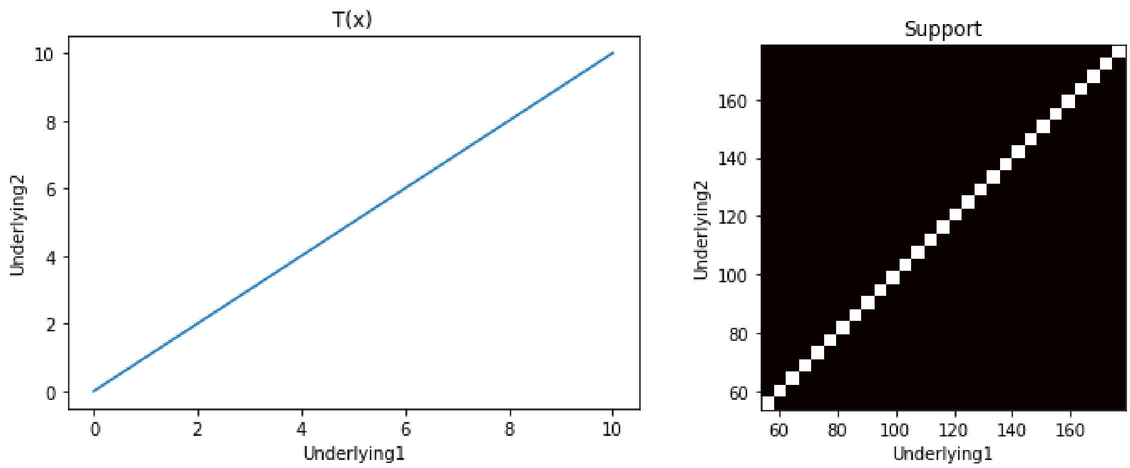

Figure 6: Function support and support using two different solutions: (1) "exact", (2) numerical in the case where $S_1 \sim LogN(100, 0.2)$ and $S_2 \sim LogN(100, 0.2)$

By direct comparison between each pair of figures, we may develop an intuition that the numerical approach converges indeed to the function represented by the exact solution, if the conditions of the theorem 2.2 are held.
As a result of the discretisation scheme mentioned in section 4.2.1, we have the figure 7 that shows the density distribution in the partition's points.

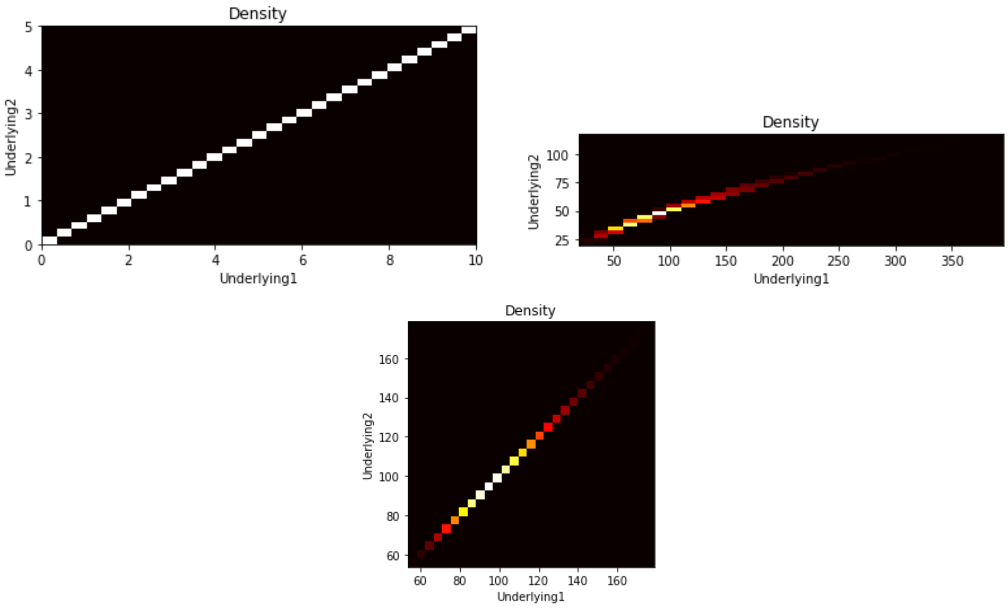

Figure 7: Density distribution in the partition points for the 3 pairs of density of probability studied

At the following part of this section, we'll present the functions that represent the hedging strategies for 2 different cost functions. Then, we'll also present the obtained result for the "support" in the numerical approach of a cost function that doesn't held and the conditions of the theorem 2.2.

Case 01 : $c(s_1, s_2) = s_1 s_2$

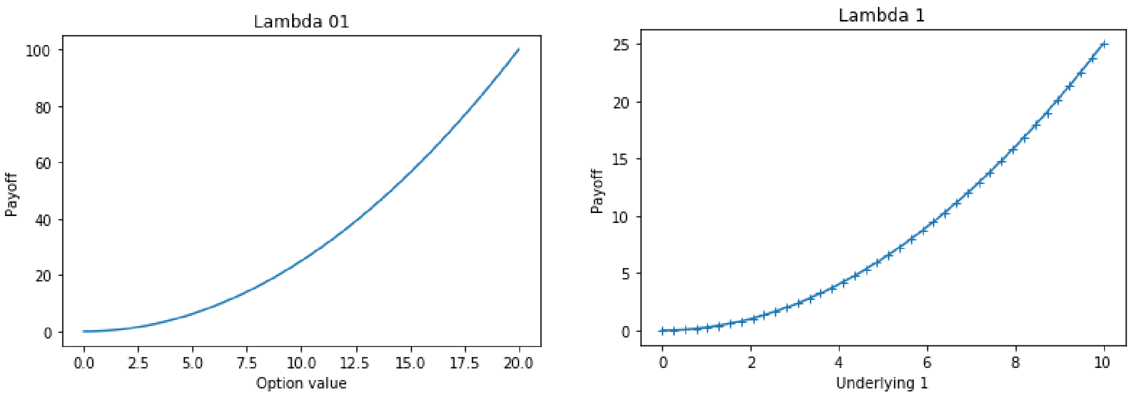

Figure 8: Function $\lambda_1(x)$ using two different solutions: (1) "exact", (2) numerical in the case where $S_1 \sim Unif[0, 10]$ and $S_2 \sim Unif[0, 5]$

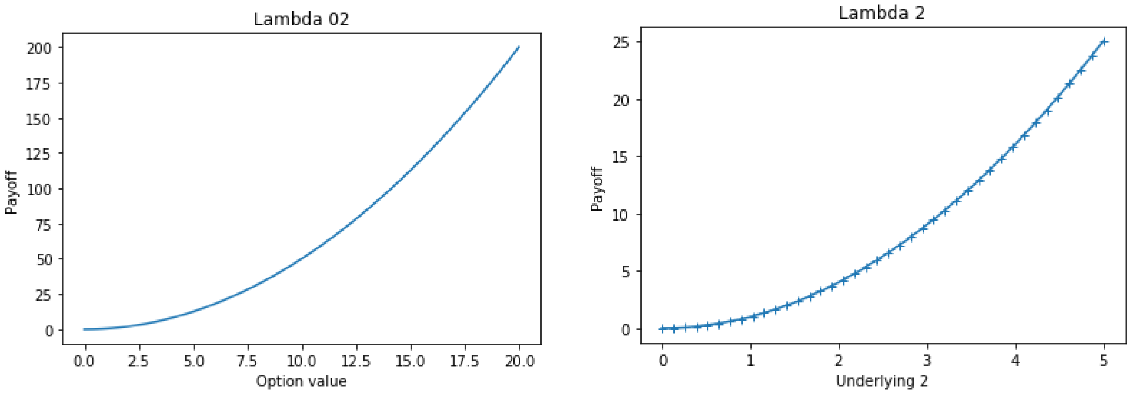

Figure 9: Function $\lambda_2(x)$ using two different solutions: (1) "exact", (2) numerical in the case where $S_1 \sim Unif[0, 10]$ and $S_2 \sim Unif[0, 5]$

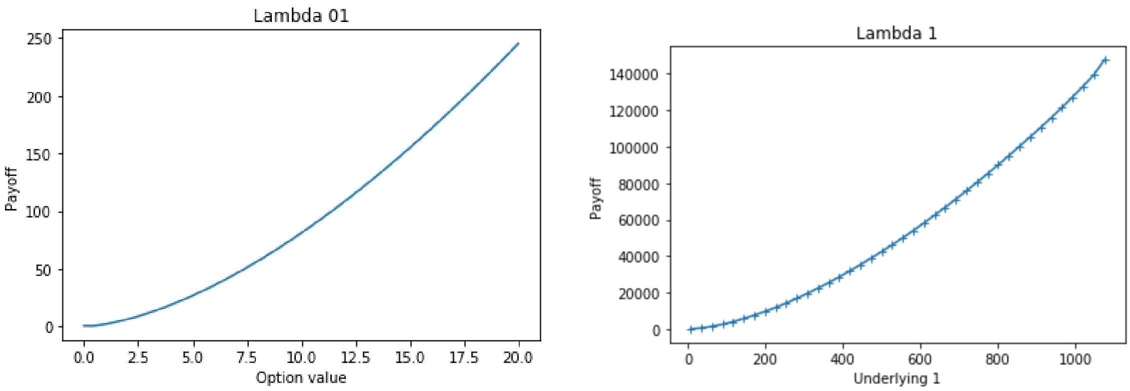

Figure 10: Function $\lambda_1(x)$ using two different solutions: (1) "exact", (2) numerical in the case where $S_1 \sim LogN(100, 0.5)$ and $S_2 \sim LogN(50, 0.3)$

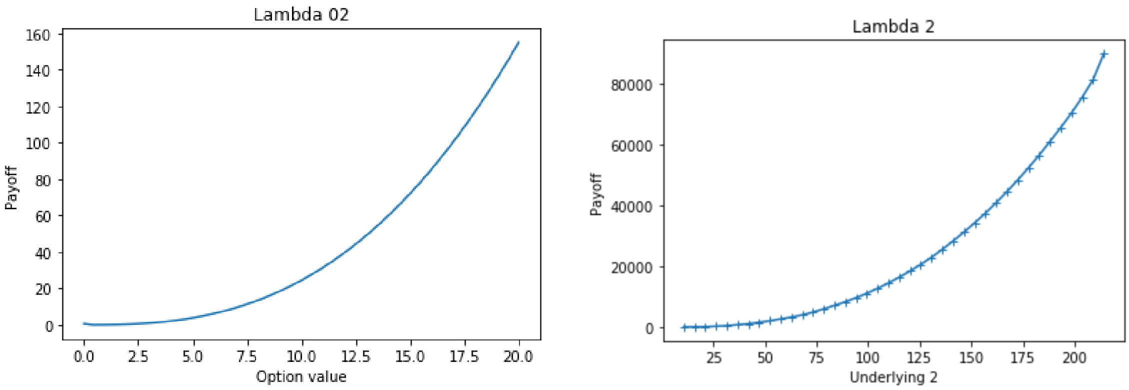

Figure 11: Function $\lambda_2(x)$ using two different solutions: (1) "exact", (2) numerical in the case where $S_1 \sim LogN(100, 0.5)$ and $S_2 \sim LogN(50, 0.3)$

We remark that in each pair of figures, the functions $\lambda_1(x)$ and $\lambda_2(x)$ seem to converge to the same values, as expected.

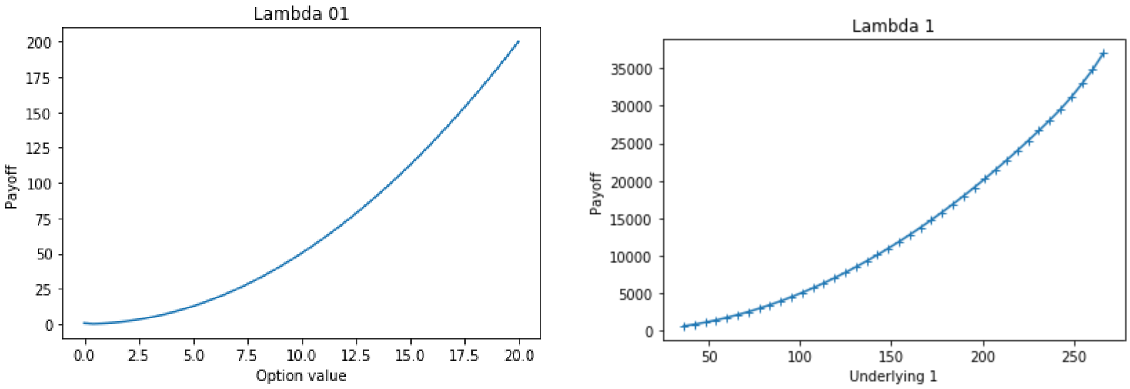

Figure 12: Function $\lambda_1(x)$ using two different solutions: (1) "exact", (2) numerical in the case where $S_1 \sim LogN(100, 0.2)$ and $S_2 \sim LogN(100, 0.2)$

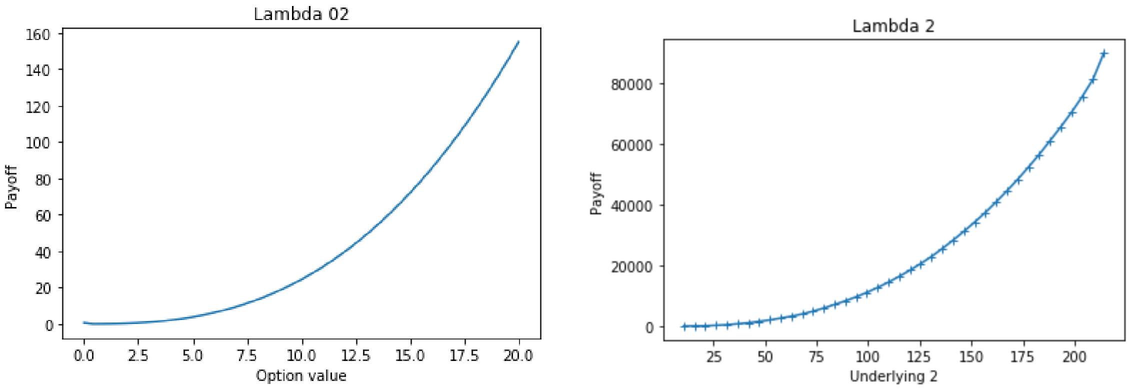

Figure 13: Function $\lambda_2(x)$ using two different solutions: (1) "exact", (2) numerical in the case where $S_1 \sim LogN(100, 0.5)$ and $S_2 \sim LogN(50, 0.3)$

Case 02 : $c(s_1, s_2) = (s_1 - K_1)^+ \mathbb{1}_{s_2 > K_2}$

In this particular case, before analyzing the graphics of $\lambda_1(x)$ and $\lambda 2_(x)$, the reader should remark that the cost function doesn't respect the condition $c_{12} > 0$.

We'll start our analysis of this case by the graphics of the "support" in the numerical approach.

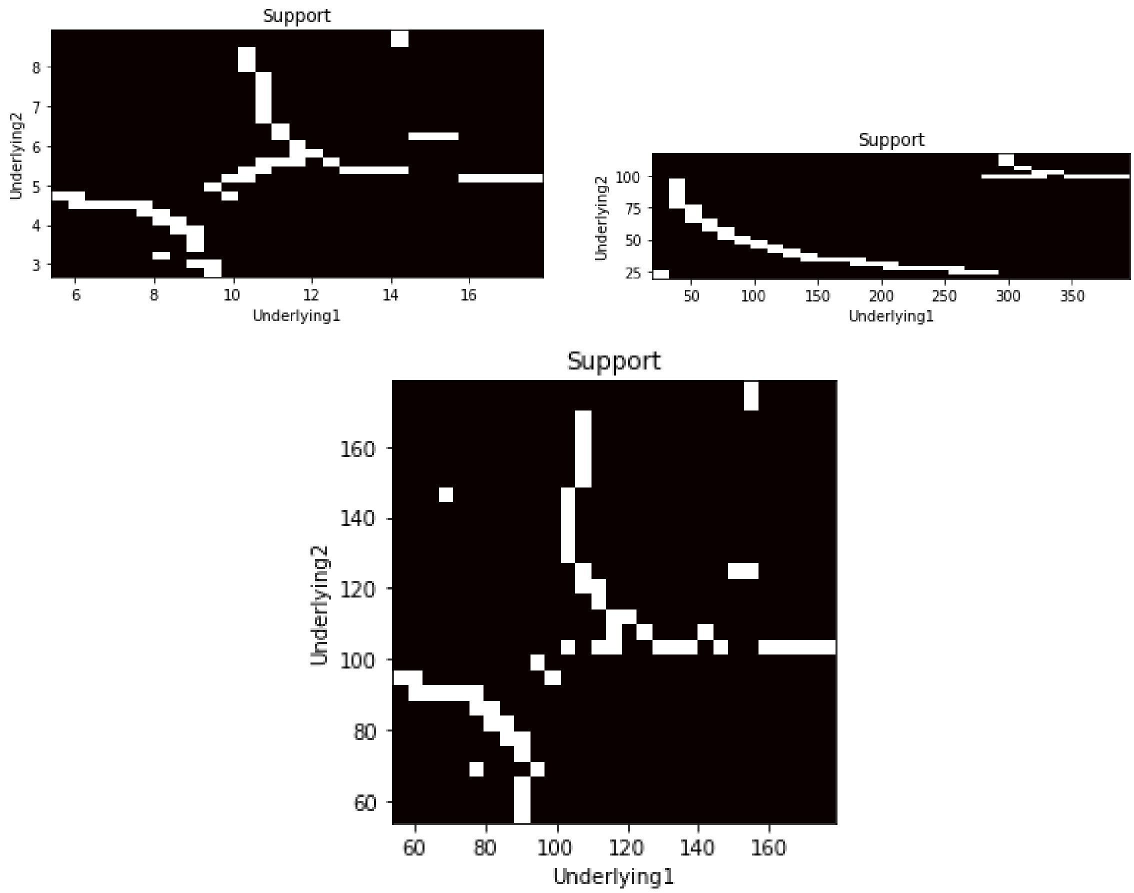

Figure 14: Graphics for the support in the numerical approach of the problem when the cost functions is $(s_1 - K_1)^+ \mathbb{1}_{s_2 > K_2}$. We took $K1 = 5$ and$K2 = 6$ for the uniform distribution and $K1 = K2 = 100$ for the Log-Normal distributions

As expected, since the cost function doesn't held the conditions of the theorem 2.2, the profile of the support's graphics is not similar or related to the formula $Support = F_2^{-1} \circ F_1(x)$. As for $\lambda_1(x)$ and $\lambda_2(x)$, the reader will find in the figures 15 and 16 the profiles of $\lambda_1(x)$ and $\lambda_2(x)$ for the numerical approach.

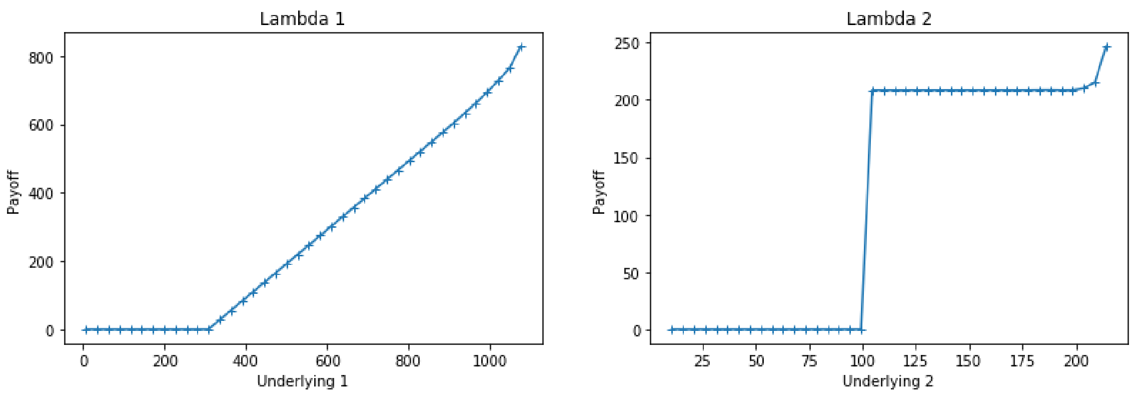

Figure 15: Graphics of the functions $\lambda_1(x)$ and $\lambda_2(x)$ obtained by the numerical approach in the case where $S_1 \sim LogN(100, 0.5)$ and $S_2 \sim LogN(50, 0.3)$. We took $K1 = K2 = 100$ to define the cost function.

We observe that the figures 15 and 16 have the profile very similar to what we should expect, if we consider the founded expressions in the example 3.

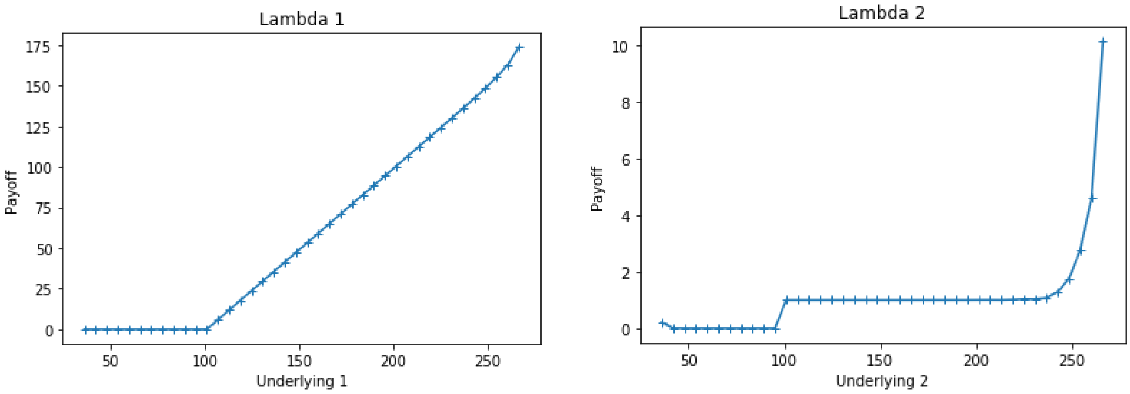

Figure 16: Graphics of the functions $\lambda_1(x)$ and $\lambda_2(x)$ obtained by the numerical approach in the case where $S_1 \sim LogN(100, 0.2)$ and $S_2 \sim LogN(100, 0.2)$. We took $K1 = K2 = 100$ to define the cost function.

Finally, to end this subsection, the reader will find in the table 1 a comparison between prices related to the analyzed distributions an cost functions.

$c(s_1, s_2)$	Distribution	"Exact" price	"Dual price"	"Primal price"
$s_1 s_2$	$Unif[0, 10]$	16.66	16.96	16.88
$s_1 s_2$	$LogN(100, 0.5)$, $LogN(50, 0.3)$	5809.17	5819.28	5809.12
$s_1 s_2$	$LogN(100, 0.2)$, $LogN(100, 0.2)$	10408.11	10414.42	10411.01
$(s_1 - K_1)^+ \mathbb{1}_{s_2 > K_2}$	$LogN(100, 0.5)$, $LogN(50, 0.3)$	1.80	1.44	1.56
$(s_1 - K_1)^+ \mathbb{1}_{s_2 > K_2}$	$LogN(100, 0.2)$, $LogN(50, 0.2)$	7.96	8.05	7.99

Table 1 : A comparison between prices of options determined by means of an "exact" approach, a numerical approach of the dual problem and a numerical approach of the primal problem

4.3 Numerical Results and Methods for Martingale Optimal Transport

We will focus on solving the dual problem. For discrete probability measures such as in 4.1 the problem is written as

$$\widetilde{MK}_2^{(m,n)} = \sup_{(p_{ij})_{1\le i\le m, 1\le j\le n}} \sum_{1\le i\le m, 1\le j\le n} p_{ij} \cdot c(x_i, y_j) \tag{4.5}$$

subject to 4.3, 4.4 and the martingale constraint

$$\sum_{1\le j\le n} p_{ij} y_j = \alpha_i x_i, \qquad i = 1, 2, \ldots, m. \tag{4.6}$$

A first approach when solving this problem is trying to solve the linear programming problem consisting in equations 4.5, 4.3, 4.4, 4.6. But this task proves to be rather difficult as some implemented algorithms (ex. scipy.optimize module) struggle to find a solution.
Hence, as affirmed by [7], it is beneficial to substitute the condition 4.6 by a relaxed constraint:

$$\sum_{1\le j\le n} p_{ij} y_j \quad \le \quad \alpha_i x_i + \epsilon, \qquad i = 1, 2, \ldots, m. \tag{4.7}$$

$$\sum_{1\le j\le n} p_{ij} y_j \quad \ge \quad \alpha_i x_i - \epsilon, \qquad i = 1, 2, \ldots, m. \tag{4.8}$$

for some ϵ fixed. Let us denote the relaxed dual problem as $\widetilde{MK}_\epsilon^{(m,n)}$. One can prove, see [7], that there exists a sequence $(\epsilon_n)_{n\in\mathbb{N}} \in \mathbb{R}_+$ and $(m(n)) \in \mathbb{N}$ such that

$$\widetilde{MK}_\epsilon^{(m,n)} \to \widetilde{MK}_2 \qquad \text{as} \qquad n \to \infty$$

(we assumed that the discretized distributions converge to the continuous distributions).
An alternative to the classical linear programming methods is the method of iterative Bregman projections, see [7]. In our tests the method of Bregman projections proved to be more efficient (lower computational time) when dealing with uniform distributions, but our implementation was very unstable with respect to the choice of the parameter ϵ and use of other distributions, as this algorithms depends on finding the root of ah n dimensional function and any error on the calculation of the root leads to an error in the method.
We show some examples below:

Example 4. ***Monge's cost.*** *We consider the payoff* $c(s_1, s_2) = |s_1 - s_2|$*. We take* $\mathbb{P}^1 = U[-1, 1]$*,* $\mathbb{P}^2 = U[-2, 2]$*. This problem has solution* $\widetilde{MK}_2 = 1$*, demonstration in book. We will try to compute numerically this result using linear programming and Bregman projections.*

Linear Programming - Relaxed MOT.
Here we use a discretization on of $2*n$ *points on the x-axis and* $4n$ *points on the y-axis (*$x = s_1$ *and* $y = s_2$*). We tested the algorithm with* $n = 6, 8, 10, 12$ *and our results were* $Optimalprice = 1.105, 1.082, 1.065, 1.055$ *respectively. The optimal densities found are represented in figure 17.*

Iterative Bregman Projections
This method is very well described in [7]. When applying this method one should choose the number of points in the discretization, the number of iterations used in the algorithm and parameter ϵ *that relaxes the martingale condition. This method proves to be quicker than the linear programming, allowing the use of more fine discretisations.*
At first, we set 200 *iterations and the parameter* $\epsilon = 0.05$ *and, for each* n *we use discretisations of* n *and* $2n$ *points on* x *and* y *respectively. For* $n = 10, 50, 100$ *and* 200 *we got the prices* $1.00737, 0.9824, 0.979117$ *and* 0.9775 *and the following densities:*
We then set $\epsilon = 0.02$ *(intuitively, a smaller value should yield smaller error but that is not the case) and used the same discretisations than before (*$n = 10, 50, 100, 200$*). As result we had the following prices respectively* $1.00277, 1.3570, 1.3593, 1.3373$ *and the following densities 19.*
These results show that this method is not stable to changing in the parameters but it allows the use of better discretisations.

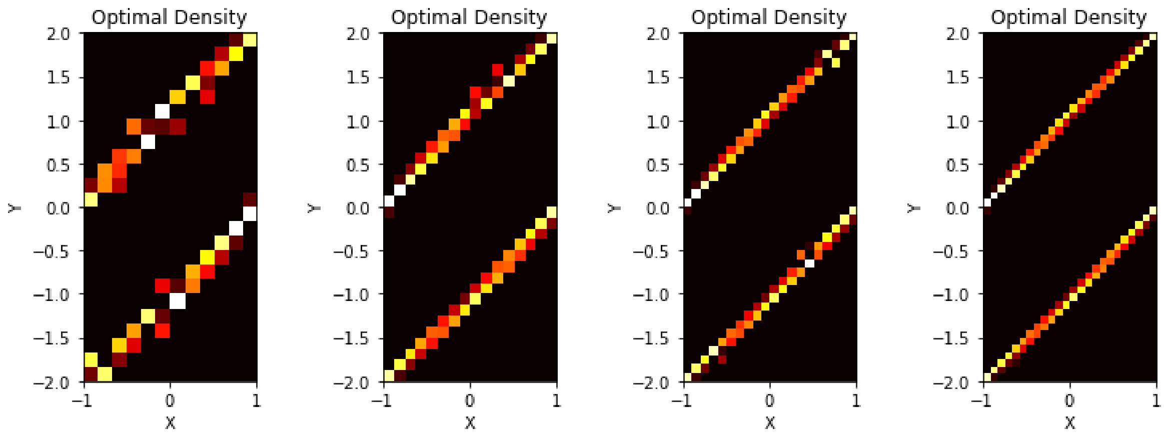

Figure 17: Solution of the Monge's cost relaxed MOT with uniform distributions. We used $\epsilon = 1/n$ (from left to right n=6, 8, 10, 12) and a linear programming approach.

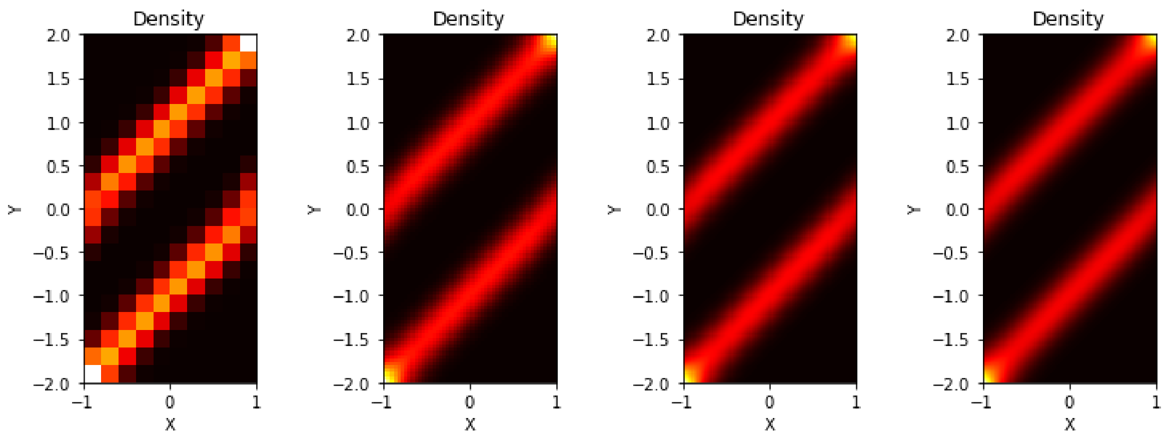

Figure 18: Solution of the Monge's cost relaxed MOT with uniform distributions. We used from left to right n=10, 50, 100, 200 for discretisations of n points on x and $2n$ points on y. The method used is Bregman projections with $\epsilon = 0.05$.

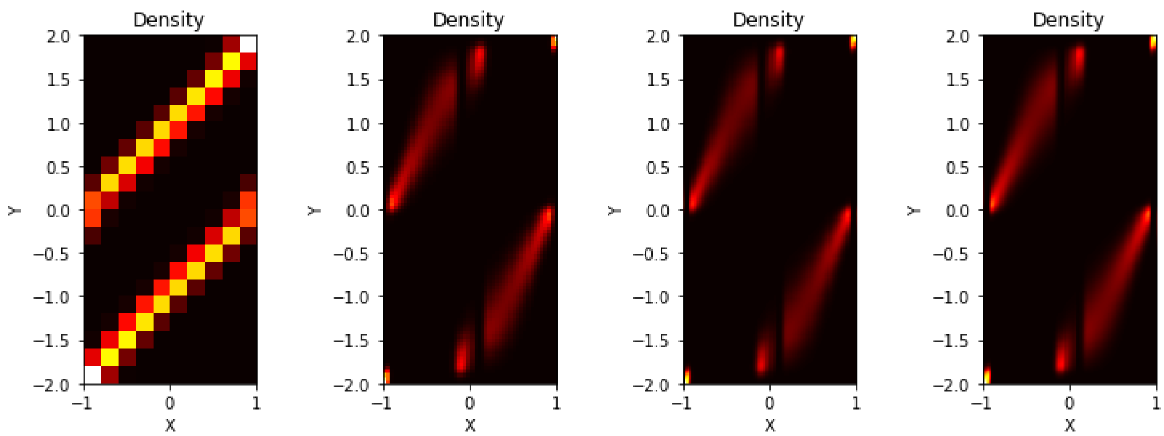

Figure 19: Solution of the Monge's cost relaxed MOT with uniform distributions. We used from left to right n=10, 50, 100, 200 for discretisations of n points on x and $2n$ points on y. The method used is Bregman projections with $\epsilon = 0.02$.

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