GNR-638
Machine Learning for Remote Sensing - II
Homework - 1

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1 3-bit Majority

1.1 Truth Table

$$\mathrm{MAJ}_3(x_1, x_2, x_3) \triangleq \mathbb{I}[x_1 + x_2 + x_3 \geq 2]$$

x_1	x_2	x_3	MAJ_3
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

Table 1: Truth Table for MAJ_3

1.2 Linear Separability

The majority function

$$\mathrm{MAJ}_3(x_1, x_2, x_3) = \mathbb{I}[x_1 + x_2 + x_3 \geq 2]$$

is linearly separable.
Consider a perceptron with weight vector and bias:

$$w = [1, 1, 1]^\top \qquad b = -2$$

Let the activation function be the Heaviside step function defined as

$$\sigma(z) = \begin{cases} 1, & z \geq 0, \\ 0, & z < 0. \end{cases}$$

The perceptron computes

$$y = \sigma(w^\top x + b) = \sigma(x_1 + x_2 + x_3 - 2)$$

This decision rule outputs 1 exactly when at least two of the three inputs are equal to 1, and outputs 0 otherwise. Hence, MAJ_3 is realizable by a single perceptron.

1.3 Verification of (w, b) on all 8 input patterns

(x_1, x_2, x_3)	$x_1 + x_2 + x_3$	$w^\top x + b$	y
$(0,0,0)$	0	-2	0
$(0,0,1)$	1	-1	0
$(0,1,0)$	1	-1	0
$(0,1,1)$	2	0	1
$(1,0,0)$	1	-1	0
$(1,0,1)$	2	0	1
$(1,1,0)$	2	0	1
$(1,1,1)$	3	1	1

The perceptron output matches MAJ_3 for all eight input patterns, thereby verifying the correctness of the chosen parameters.

2 Exactly-1-of-3

2.1 Truth Table

$$\text{EXACT}_3(x_1, x_2, x_3) \triangleq \mathbb{I}[x_1 + x_2 + x_3 = 1]$$

x_1	x_2	x_3	EXACT_3
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

Table 2: Truth Table for EXACT_3

2.2 Linearly Separable

Proof. We prove that the function EXACT_3 is not linearly separable by contradiction.
Assume that EXACT_3 is linearly separable. Then there exists a weight vector

$$w = [w_1, w_2, w_3]^\top$$

and a bias $b \in \mathbb{R}$ such that the perceptron output

$$f(x) = \sigma(w^\top x + b)$$

where

$$x = [x_1, x_2, x_3]^\top$$

correctly computes EXACT_3, and the activation function σ is the Heaviside step function

$$\sigma(z) = \begin{cases} 1, & z \geq 0 \\ 0, & z < 0 \end{cases}$$

From the definition of EXACT_3, we obtain the following constraints:

$$\begin{aligned} b < 0 &\quad \text{for } x = (0,0,0) \Rightarrow f(x) = 0 \\ w_3 + b \geq 0 &\quad \text{for } x = (0,0,1) \Rightarrow f(x) = 1 \\ w_2 + b \geq 0 &\quad \text{for } x = (0,1,0) \Rightarrow f(x) = 1 \end{aligned}$$

Adding the last two inequalities gives

$$w_2 + w_3 + 2b \geq 0$$

Since $b < 0$, we have $2b < b$, and hence

$$w_2 + w_3 + b > 0$$

However, for the input

$$x = (1, 1, 1),$$

the function EXACT_3 evaluates to 0. Therefore, linear separability requires

$$w_2 + w_3 + b < 0$$

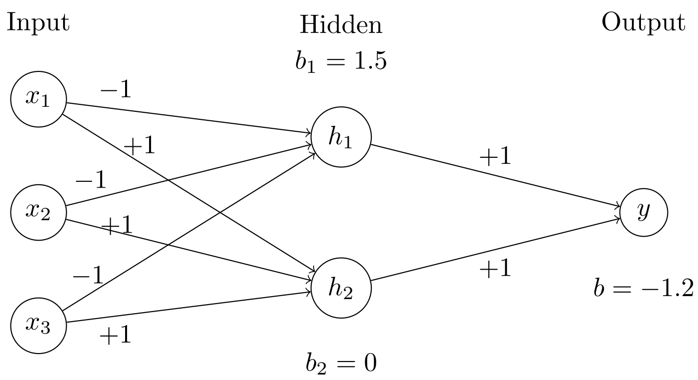
This contradicts the inequality derived above. Hence, our assumption that EXACT_3 is linearly separable is false.

2.3 Design of 2-layer network

We consider a two-layer neural network with three inputs, two hidden neurons, and one output neuron. All neurons use the Heaviside step activation function

$$\sigma(z) = \begin{cases} 1, & z \geq 0, \\ 0, & z < 0. \end{cases}$$

Network Architecture



Hidden Layer Computation

Let $x = (x_1, x_2, x_3) \in \{0, 1\}^3$.

The first hidden neuron computes

$$h_1(x) = \sigma(-x_1 - x_2 - x_3 + 1.5).$$

The second hidden neuron computes

$$h_2(x) = \sigma(x_1 + x_2 + x_3).$$

Evaluating the hidden layer:

(x_1, x_2, x_3)	(h_1, h_2)
000	$(1, 0)$
001	$(1, 1)$
010	$(1, 1)$
011	$(0, 1)$
100	$(1, 1)$
101	$(0, 1)$
110	$(0, 1)$
111	$(0, 1)$

Output Layer Computation

The output neuron computes

$$y(x) = \sigma(h_1(x) + h_2(x) - 1.2).$$

Evaluating the output:

(x_1, x_2, x_3)	y
000	0
001	1
010	1
011	0
100	1
101	0
110	0
111	0

Conclusion

The network computes the Boolean function

$$\mathrm{EXACT}_1^3(x_1, x_2, x_3) = \mathbb{I}[x_1 + x_2 + x_3 = 1].$$

Thus, EXACT_1^3 is realizable using a two-layer neural network with step activation functions.

3 3-bit Parity

3.1 Truth Table

$$\mathrm{PARITY}_3(x_1, x_2, x_3) \triangleq x_1 \oplus x_2 \oplus x_3$$

x_1	x_2	x_3	PARITY_3
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Table 3: Truth Table for PARITY_3

3.2 Linear Separability

3.3 Linearly Separable

Proof. We prove that the function PARITY_3 is not linearly separable by contradiction.
Assume that PARITY_3 is linearly separable. Then there exists a weight vector

$$w = [w_1, w_2, w_3]^\top$$

and a bias $b \in \mathbb{R}$ such that the perceptron output

$$f(x) = \sigma(w^\top x + b)$$

where

$$x = [x_1, x_2, x_3]^\top$$

correctly computes PARITY_3, and the activation function σ is the Heaviside step function

$$\sigma(z) = \begin{cases} 1, & z \geq 0 \\ 0, & z < 0 \end{cases}$$

From the truth table of PARITY_3, we obtain the following constraints:

$$\begin{aligned}
b < 0 &\quad \text{for } x = (0,0,0) \Rightarrow f(x) = 0 \\
w_1 + b \geq 0 &\quad \text{for } x = (1,0,0) \Rightarrow f(x) = 1 \\
w_2 + b \geq 0 &\quad \text{for } x = (0,1,0) \Rightarrow f(x) = 1 \\
w_3 + b \geq 0 &\quad \text{for } x = (0,0,1) \Rightarrow f(x) = 1 \\
w_1 + w_2 + w_3 + b \geq 0 &\quad \text{for } x = (1,1,1) \Rightarrow f(x) = 1
\end{aligned}$$

From the second, third, and fourth inequalities, we see that $w_1, w_2, w_3 > 0$.
However, consider the input

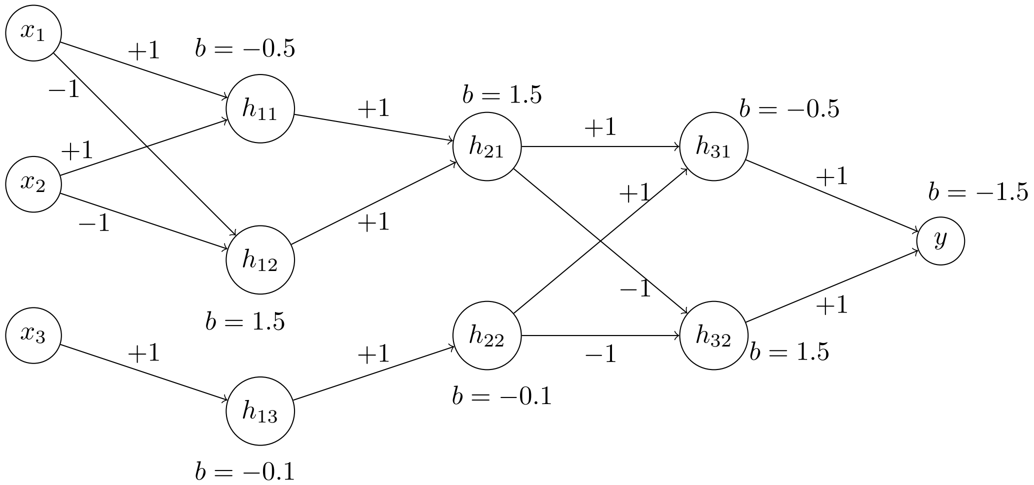
$$x = (1, 1, 0),$$

for which $\text{PARITY}_3(x) = 0$. This requires

$$w_1 + w_2 + b < 0$$

But we already have $w_1 > 0$ and $w_2 + b \geq 0$ from earlier inequalities, which makes $w_1 + w_2 + b > 0$. This is a contradiction.
Hence, the assumption that PARITY_3 is linearly separable is false.

3.4 Multilayer Perceptron



Note: All connections not explicitly shown in the diagram are assumed to have weight 0. This multilayer perceptron implements the 3-bit parity function, i.e., it computes

$$y = x_1 \oplus x_2 \oplus x_3,$$

using a combination of threshold units that realize XOR operations in a layered fashion.

4 4-to-1 Multiplexer

4.1 Boolean Formula

$$\text{MUX}_4 = (\neg s_1 \wedge \neg s_0 \wedge d_0) \vee (\neg s_1 \wedge s_0 \wedge d_1) \vee (s_1 \wedge \neg s_0 \wedge d_2) \vee (s_1 \wedge s_0 \wedge d_3)$$

4.2 Truth Table

s_0	s_1	MUX_4
0	0	d_0
0	1	d_1
1	0	d_2
1	1	d_3

Table 4: Compressed Truth Table for MUX_4

4.3 Linear Separability

Proof. We prove that the function MUX_4 is not linearly separable by contradiction.
Assume that MUX_4 is linearly separable. Then there exists a perceptron characterized by a weight vector

$$w = [w_0, w_1, w_2, w_3, w_4, w_5]^\top \in \mathbb{R}^6,$$

a bias term $b \in \mathbb{R}$, and an input vector

$$x = [s_0, s_1, d_0, d_1, d_2, d_3]^\top \in \{0, 1\}^6,$$

such that the perceptron correctly computes MUX_4. The perceptron output is given by

$$f(x) = \sigma(w^\top x + b)$$

where $\sigma(\cdot)$ denotes the Heaviside step function,

$$\sigma(z) = \begin{cases} 1, & z \geq 0 \\ 0, & z < 0 \end{cases}$$

Then

$$\text{MUX}_4 = \sigma(w^\top x + b)$$

Using the truth table of MUX_4, we obtain the following constraints on the parameters w and b:

$$\begin{aligned} w_2 + b &\geq 0 \quad \text{since } x = [0,0,1,0,0,0] \text{ and } \text{MUX}_4(x) = 1 \\ w_4 + b &< 0 \quad \text{since } x = [0,0,0,0,1,0] \text{ and } \text{MUX}_4(x) = 0 \\ w_1 + w_4 + b &\geq 0 \quad \text{since } x = [0,1,0,0,1,0] \text{ and } \text{MUX}_4(x) = 1 \end{aligned}$$

From the second and third inequalities, we obtain

$$w_1 \geq -(w_4 + b)$$

Since $w_4 + b < 0$, it follows that

$$w_1 > 0$$

Now consider the input

$$x = [0,1,1,0,0,0]$$

for which $\text{MUX}_4(x) = 0$. This yields the constraint

$$w_1 + w_2 + b < 0$$

However, from the previously derived inequalities, we have

$$w_1 > 0 \quad \text{and} \quad w_2 + b \geq 0$$

which together imply

$$w_1 + (w_2 + b) > 0$$

This contradicts the inequality $w_1 + w_2 + b < 0$. Therefore, our assumption that MUX_4 is linearly separable is false. Since the function is not linearly separable, it cannot be implemented by a single perceptron.

5 Parity of Two 3-bit Majorities

5.1 Explanation

The function f evaluates to 1 if and only if exactly one of m_A or m_B equals 1. Equivalently,

$$f = 1 \iff \Big(x_1 + x_2 + x_3 \geq 2 \ \wedge \ x_3 + x_4 + x_5 < 2\Big) \ \vee \ \Big(x_1 + x_2 + x_3 < 2 \ \wedge \ x_3 + x_4 + x_5 \geq 2\Big)$$

In other words, the function is true if and only if exactly one set of variables i.e either (x_1, x_2, x_3) or (x_4, x_5, x_6) has two or more variables equal to 1. In all other cases, $f = 0$.

5.2 Truth Table

$m_A(x_1, x_2, x_3)$	$m_b(x_4, x_5, x_6)$	$f(x_1, x_2, x_3, x_4, x_5, x_6)$
0	0	0
0	1	1
1	0	1
1	1	0

Table 5: Truth Table for $f(x_1, x_2, x_3, x_4, x_5, x_6)$

5.3 Linear Separability

Proof. We prove that the function f is not linearly separable by contradiction.
Assume that f is linearly separable. Then there exists a weight vector

$$w = [w_1, w_2, w_3, w_4, w_5, w_6]^\top$$

and a bias $b \in \mathbb{R}$ such that the perceptron output

$$f(x) = \sigma(w^\top x + b)$$

where

$$x = [x_1, x_2, x_3, x_4, x_5, x_6]^\top$$

correctly computes f, and the activation function σ is the Heaviside step function

$$\sigma(z) = \begin{cases} 1, & z \geq 0 \\ 0, & z < 0 \end{cases}$$

From the definition of f, we obtain the following constraints:

$$\begin{aligned} w_1 + w_2 + w_3 + b \geq 0 &\quad \text{for } x = (1,1,1,0,0,0) \Rightarrow f(x) = 1 \\ w_4 + w_5 + w_6 + b \geq 0 &\quad \text{for } x = (0,0,0,1,1,1) \Rightarrow f(x) = 1 \\ b < 0 &\quad \text{for } x = (0,0,0,0,0,0) \Rightarrow f(x) = 0 \end{aligned}$$

Adding the first two inequalities gives

$$w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + 2b \geq 0$$

Since $b < 0$, we have $2b < b$, and hence

$$w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + b > 0$$

However, for the input

$$x = (1,1,1,1,1,1),$$

the function f evaluates to 0. Therefore, linear separability requires

$$w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + b < 0$$

This contradicts the inequality derived above. Hence, our assumption that f is linearly separable is false.

Since the function is not linearly separable, it cannot be implemented by a single perceptron.