3: Discrete Cosine Transform

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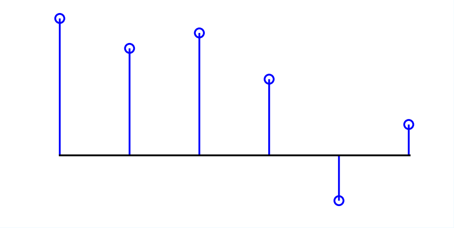

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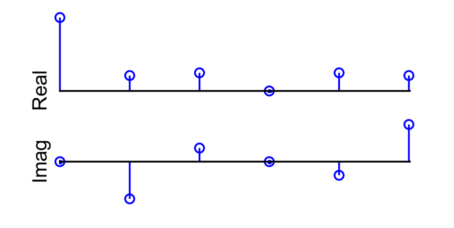

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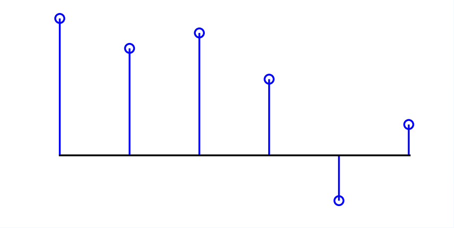

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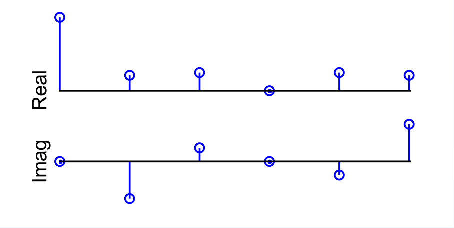

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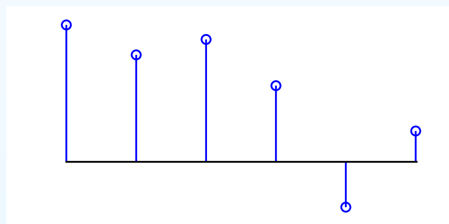

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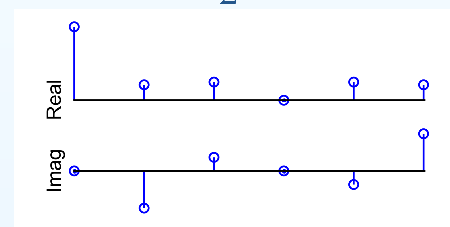

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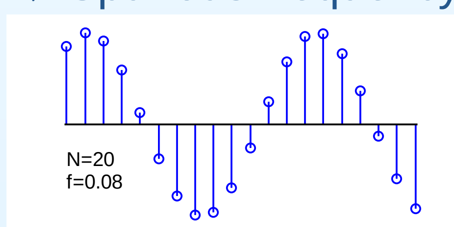

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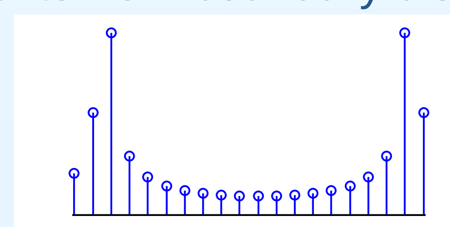

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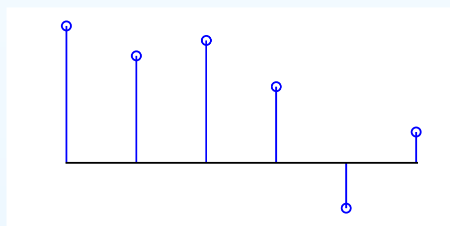

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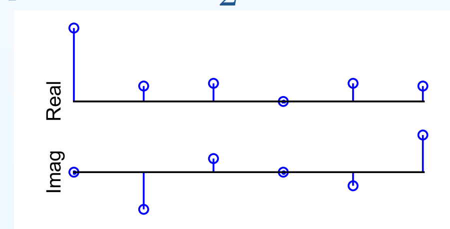

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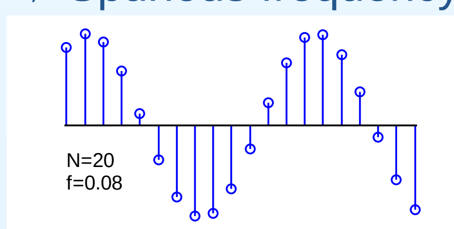

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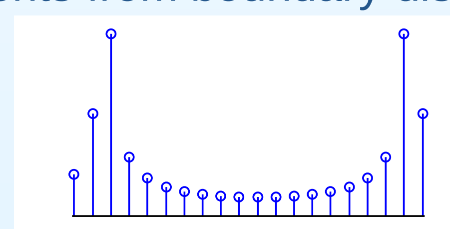

The Discrete Cosine Transform (DCT) overcomes these problems.

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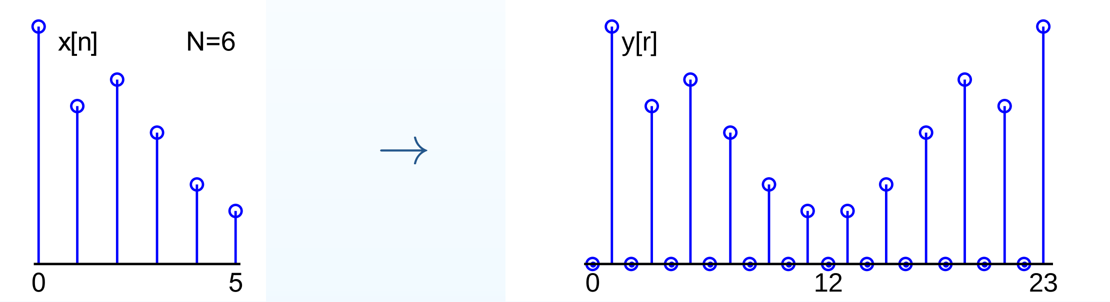

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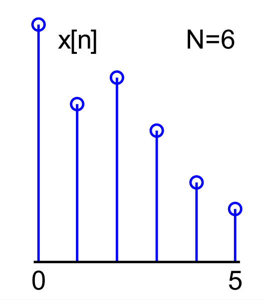

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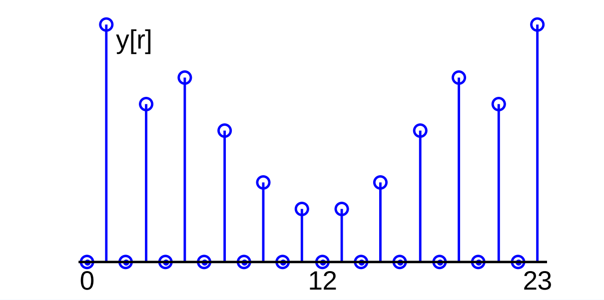

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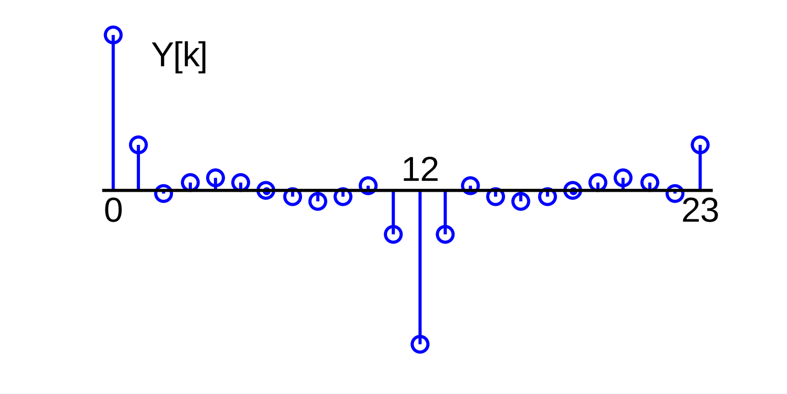
Result is real, symmetric and anti-periodic:



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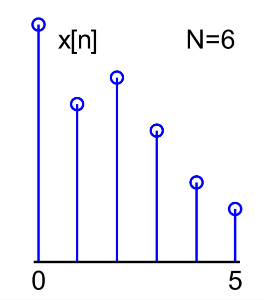

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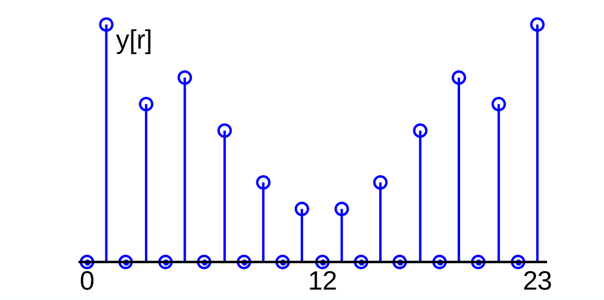

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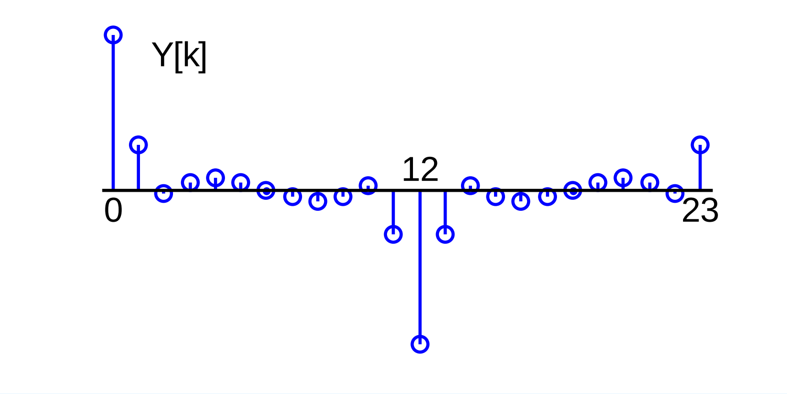

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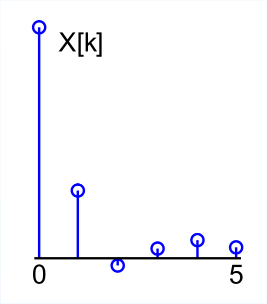

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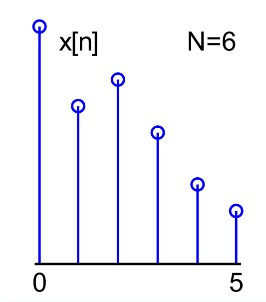

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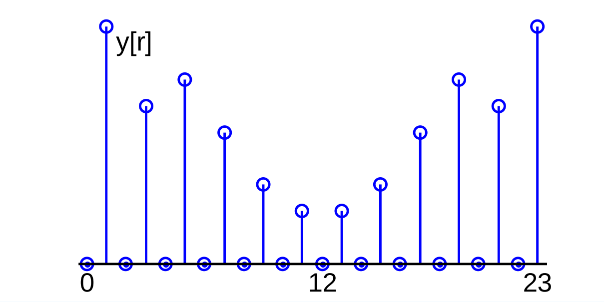

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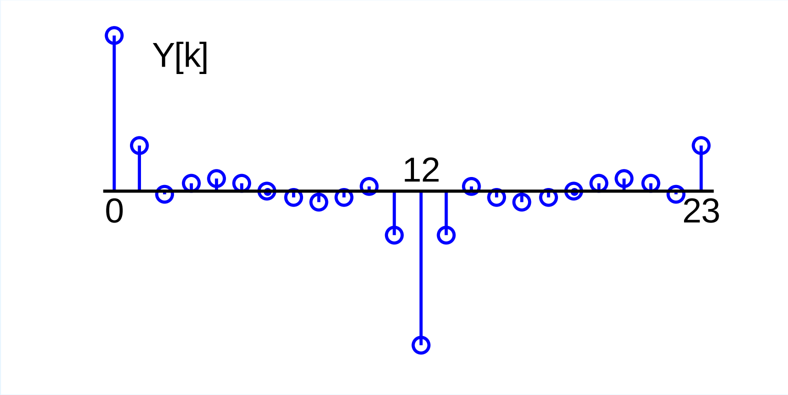

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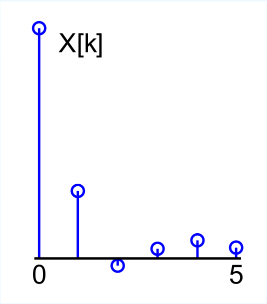

Forward DCT: $X_C[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$ for $k = 0 : N-1$

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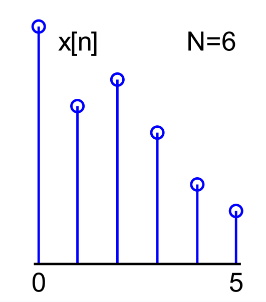

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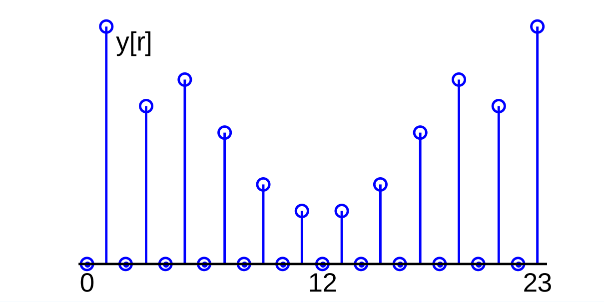

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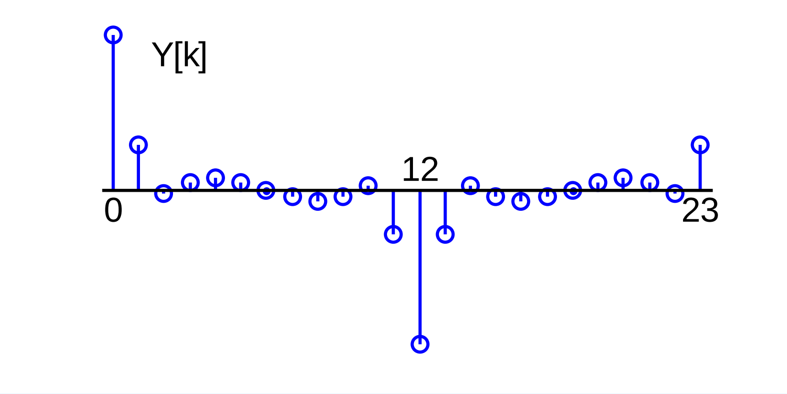

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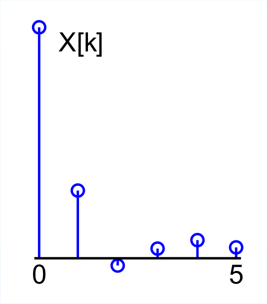

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Compare DFT: $X_F[k] = \sum_{n=0}^{N-1} x[n] \exp \frac{-j2\pi(4n+0)k}{4N}$

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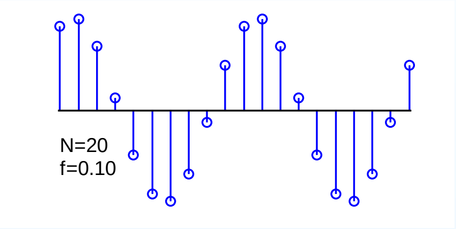

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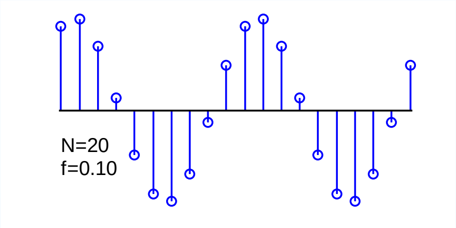

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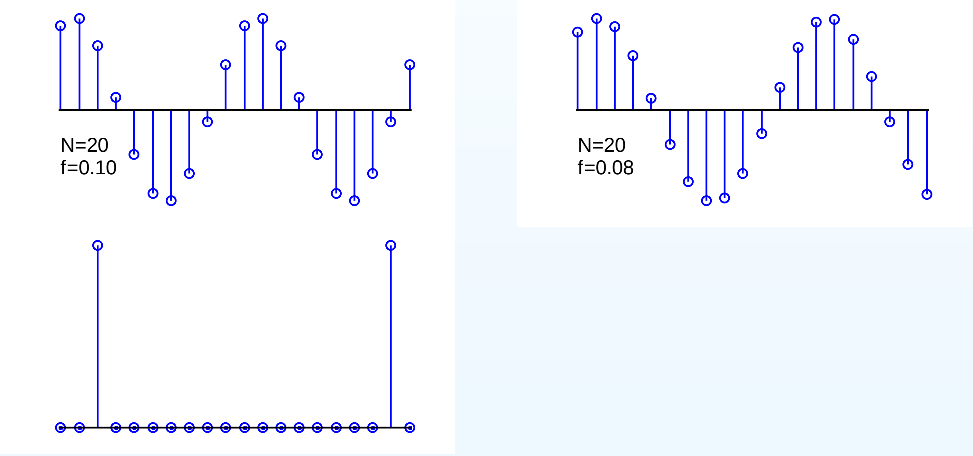

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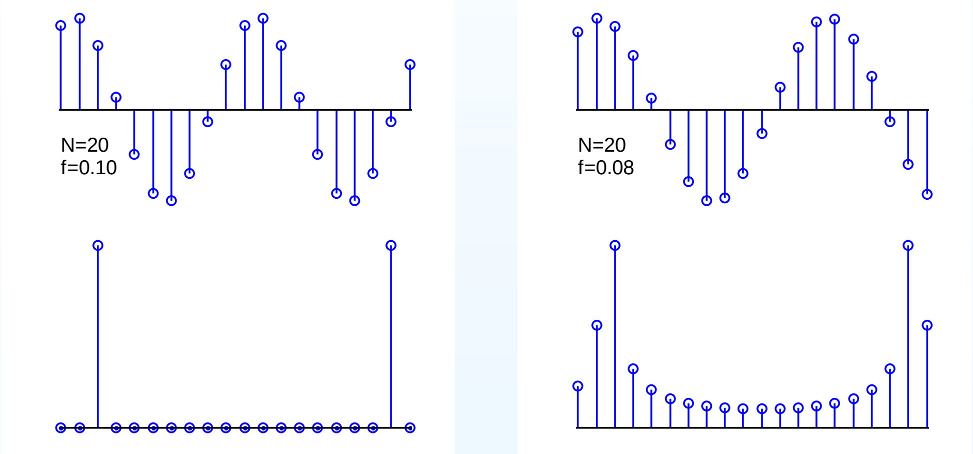

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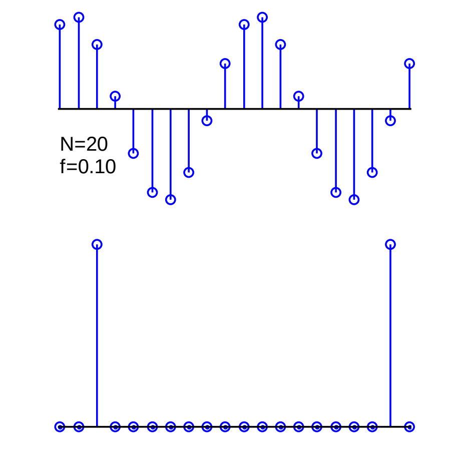

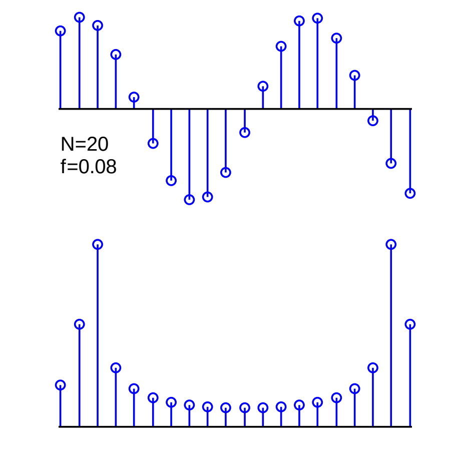

DFT: Real→Complex; Freq range [0, 1]; Poorly localized unless $f = \frac{m}{N}$; $|X_F[k]| \propto k^{-1}$ for $Nf < k \ll \frac{N}{2}$

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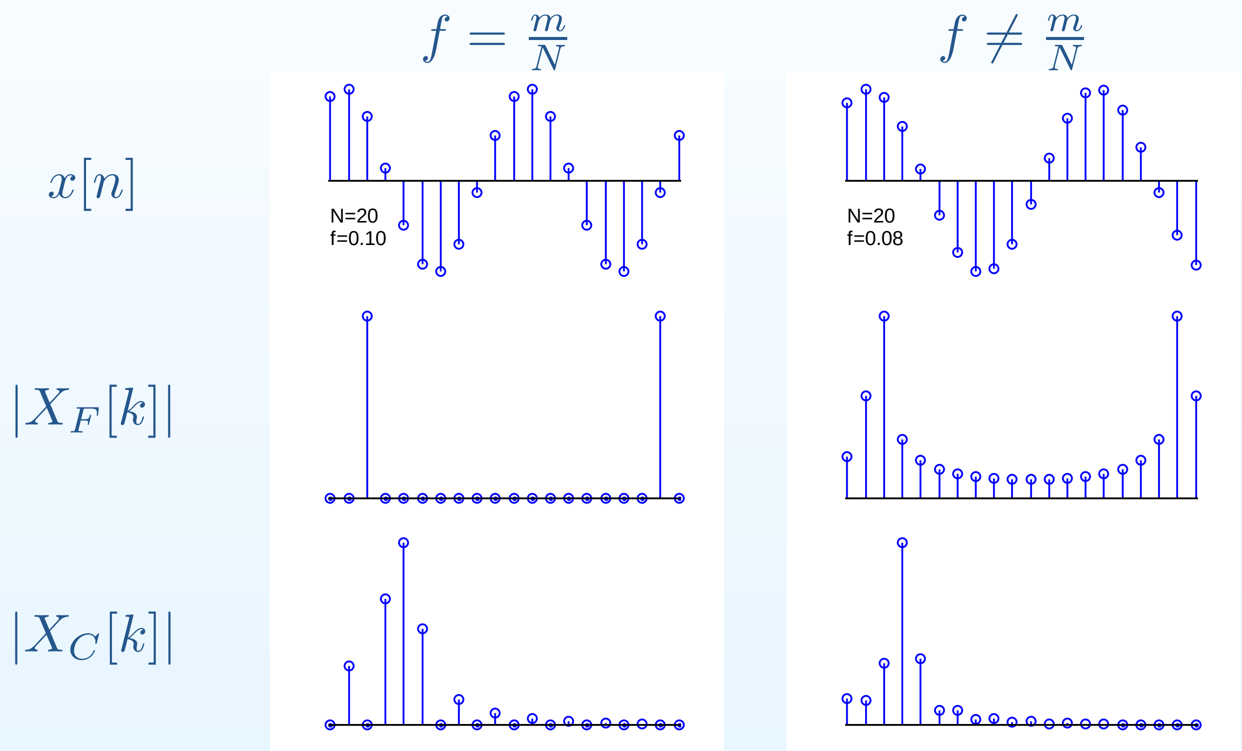

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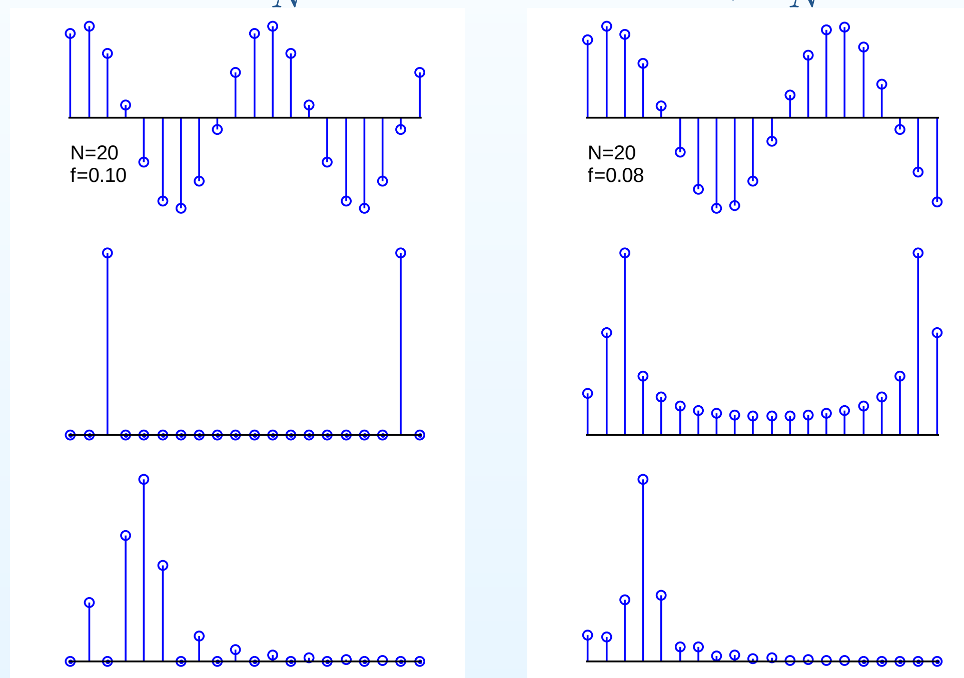

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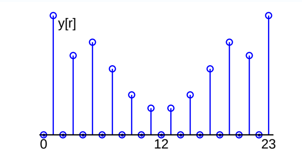
DCT: Real→Real; Freq range [0, 0.5]; Well localized $\forall f$; $|X_C[k]| \propto k^{-2}$ for $2Nf < k < N$

DCT/DFT Equivalence

$$X_C[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$$



$$\text{Define } y[r] = \begin{cases} 0 & r \text{ even} \\ x\left[\frac{r-1}{2}\right] & r = 1:2:2N-1 \\ x\left[\frac{4N-1-r}{2}\right] & r = 2N+1:2:4N-1 \end{cases}$$

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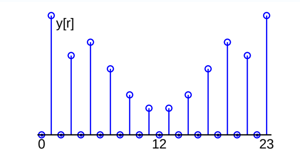

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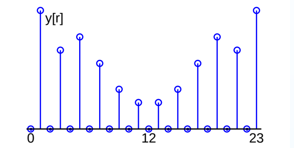

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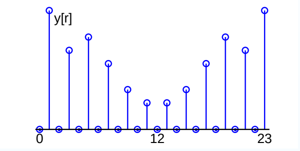

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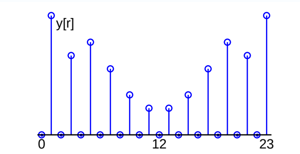

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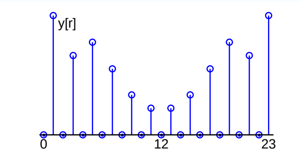

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$$\begin{aligned} Y_F[k] &= \sum_{r=0}^{4N-1} y[r] W_{4N}^{kr} \qquad \text{where } W_M = e^{\frac{-j2\pi}{M}} \\ &\overset{\text{(i)}}{=} \sum_{n=0}^{2N-1} y[2n+1] W_{4N}^{(2n+1)k} \\ &\overset{\text{(ii)}}{=} \sum_{n=0}^{N-1} y[2n+1] W_{4N}^{(2n+1)k} \\ &\qquad + \sum_{m=0}^{N-1} y[4N-2m-1] W_{4N}^{(4N-2m-1)k} \\ &\overset{\text{(iii)}}{=} \sum x[n] W_{4N}^{(2n+1)k} + \sum_{m=0}^{N-1} x[m] W_{4N}^{-(2m+1)k} \\ &= 2\sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N} = 2X_C[k] \end{aligned}$$

(i) odd r only: $r = 2n+1$

(ii) reverse order for $n \geq N$: $m = 2N-1-n$

(iii) substitute y definition & $W_{4N}^{4Nk} = e^{-j2\pi\frac{4Nk}{4N}} \equiv 1$

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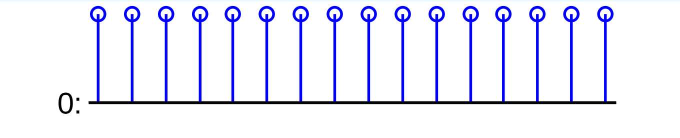

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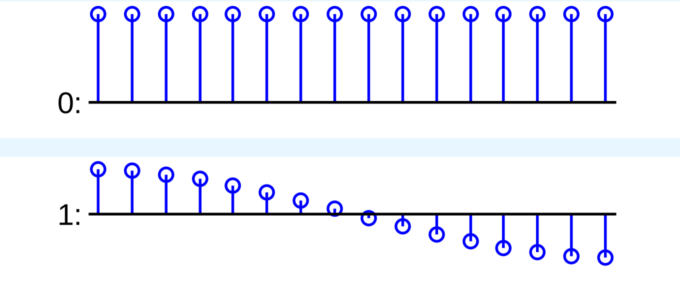

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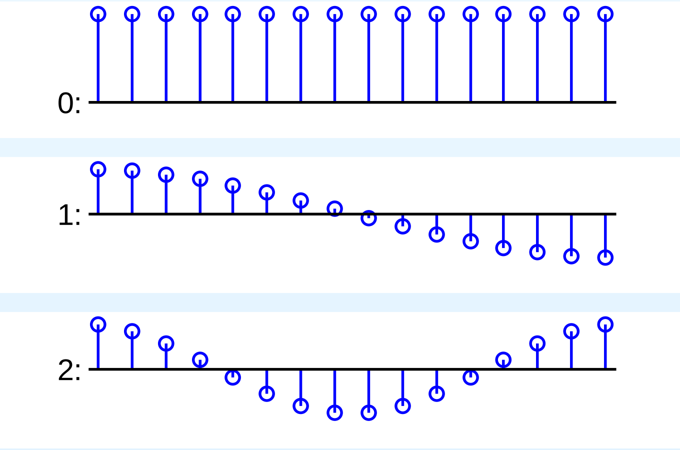

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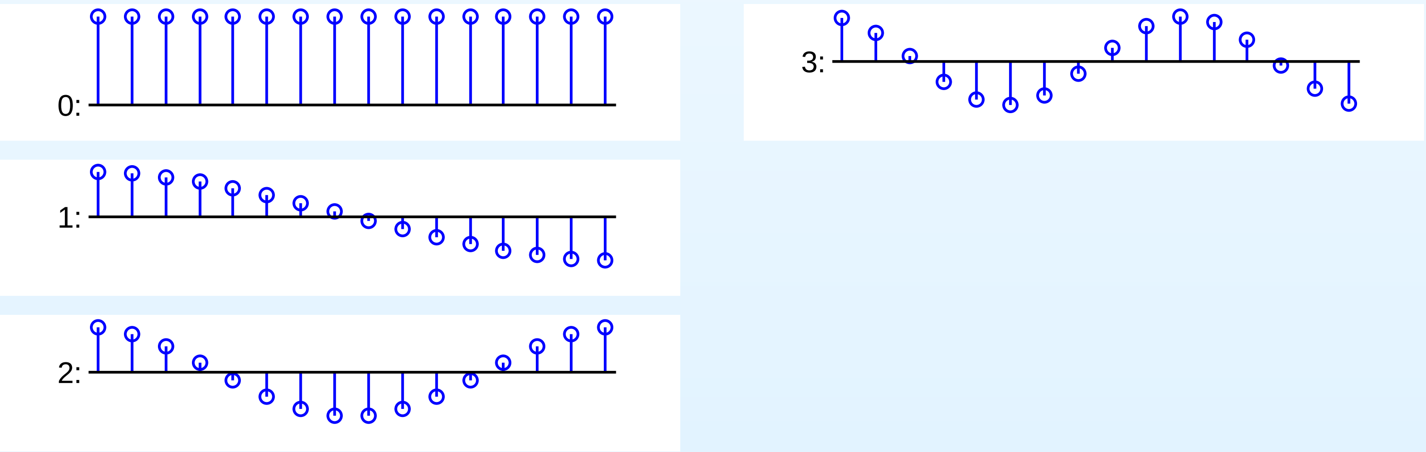

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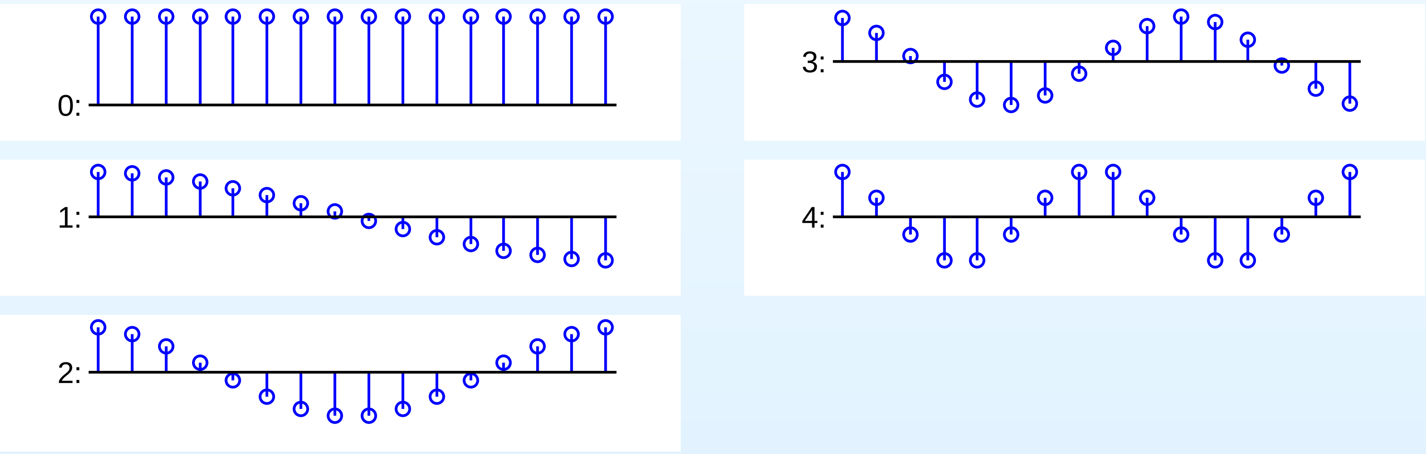

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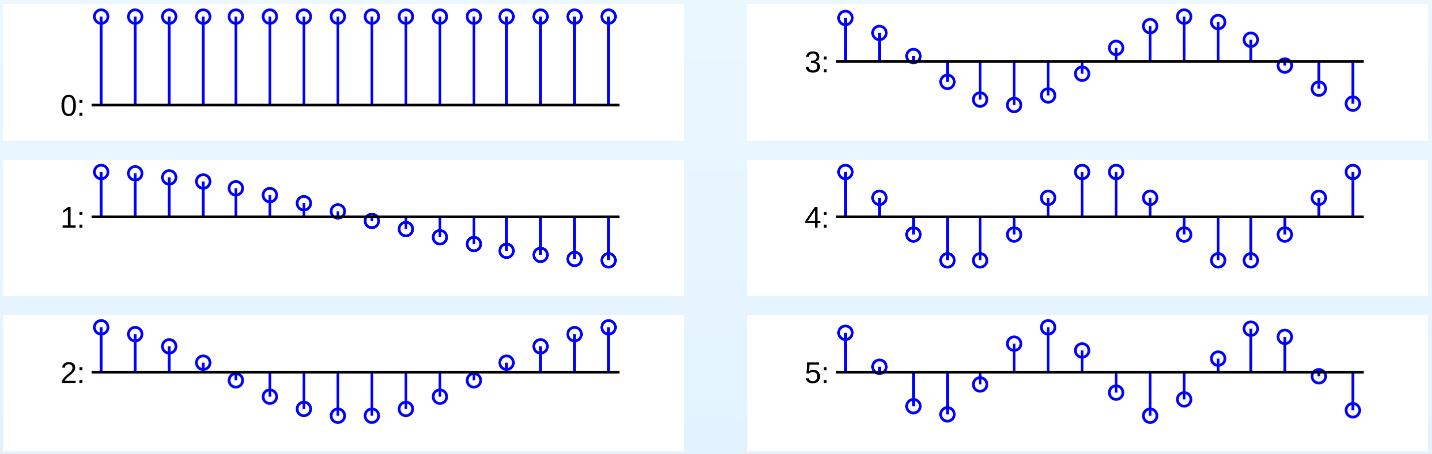

IDCT

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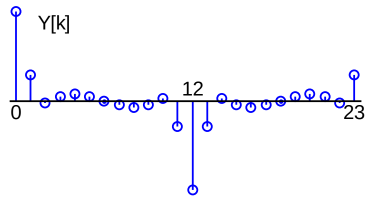

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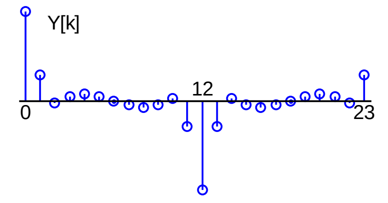

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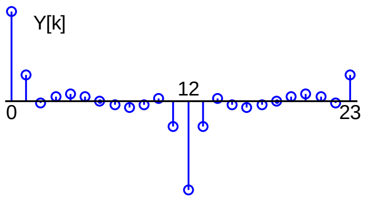

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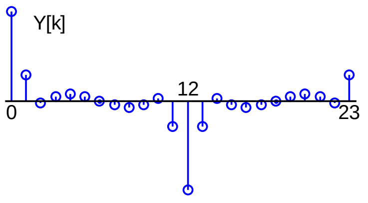

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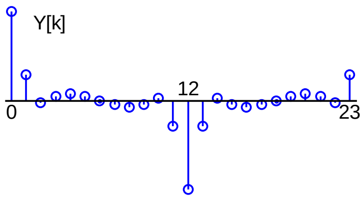

$$\overset{(i)}{=} \frac{1}{2N}\sum_{k=0}^{2N-1} X[k]W_{4N}^{-(2n+1)k} - \frac{1}{2N}\sum_{l=0}^{2N-1} X[l]W_{4N}^{-(2n+1)(l+2N)}$$

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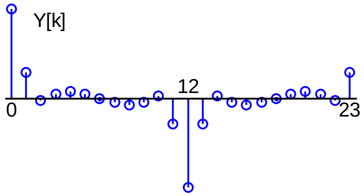

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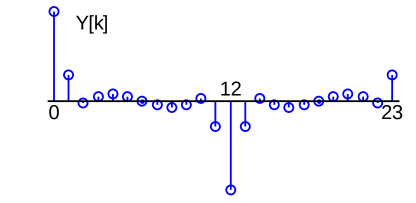

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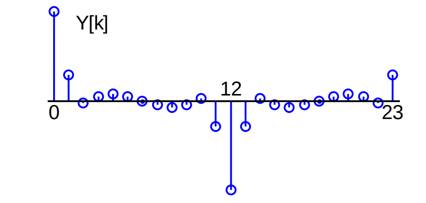

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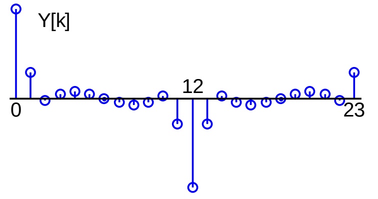

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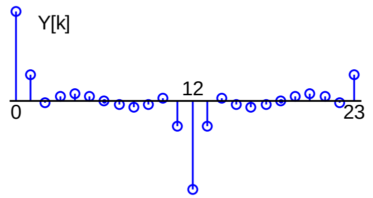

$$\stackrel{\text{(i)}}{=} \frac{1}{2N}\sum_{k=0}^{2N-1} X[k]W_{4N}^{-(2n+1)k} - \frac{1}{2N}\sum_{l=0}^{2N-1} X[l]W_{4N}^{-(2n+1)(l+2N)}$$

$$\stackrel{\text{(ii)}}{=} \frac{1}{N}\sum_{k=0}^{2N-1} X[k]W_{4N}^{-(2n+1)k} \qquad W_a^b = e^{-j\frac{2\pi b}{a}}$$

$$\stackrel{\text{(iii)}}{=} \frac{1}{N}X[0] + \frac{1}{N}\sum_{k=1}^{N-1} X[k]W_{4N}^{-(2n+1)k} + \frac{1}{N}X[N]W_{4N}^{-(2n+1)N} + \frac{1}{N}\sum_{r=1}^{N-1} X[2N-r]W_{4N}^{-(2n+1)(2N-r)}$$

$$\stackrel{\text{(iv)}}{=} \frac{1}{N}X[0] + \frac{1}{N}\sum_{k=1}^{N-1} X[k]W_{4N}^{-(2n+1)k} + \frac{1}{N}\sum_{r=1}^{N-1} -X[r]W_{4N}^{(2n+1)r+2N}$$

$$x[n] = \frac{1}{N}X[0] + \frac{2}{N}\sum_{k=1}^{N-1} X[k]\cos\frac{2\pi(2n+1)k}{4N}$$ = Inverse DCT

(i) $k = l + 2N$ for $k \ge 2N$ and $X[k+2N] = -X[k]$

(ii) $\frac{(2n+1)(l+2N)}{4N} = \frac{(2n+1)l}{4N} + n + \frac{1}{2}$

(iii) $k = 2N - r$ for $k > N$ (iv) $X[N] = 0$ and $X[2N-r] = -X[r]$

Energy Conservation

DCT: $X[k] = \sum_{n=0}^{N-1} x[n] \cos \frac{2\pi(2n+1)k}{4N}$

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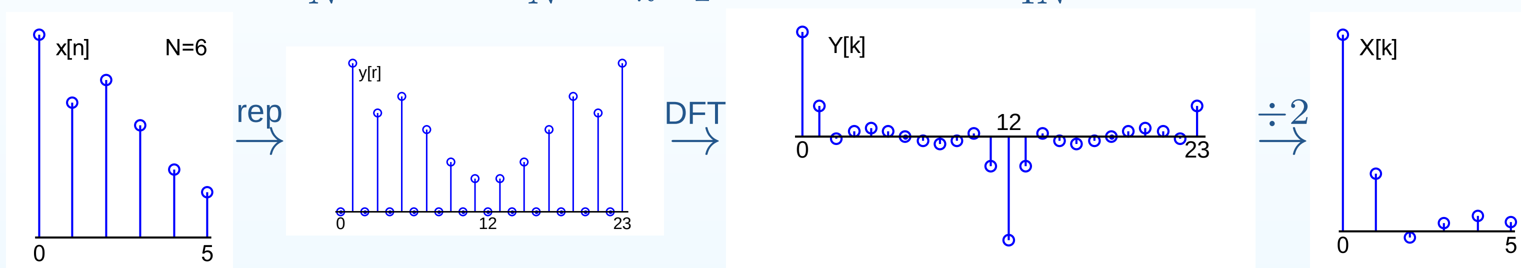

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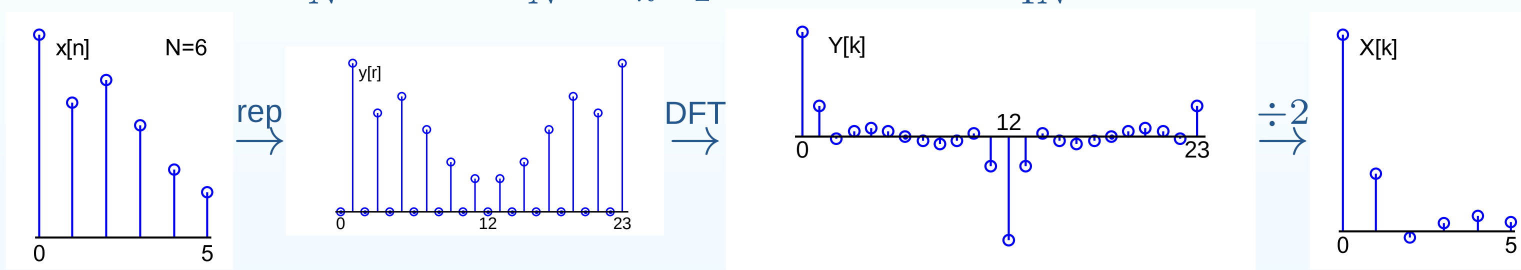

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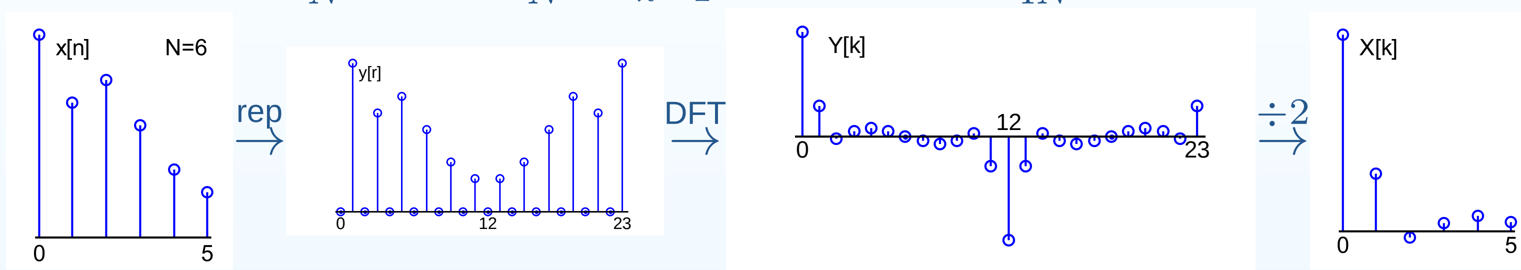

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Define: $c[k] = \sqrt{\frac{2-\delta_k}{N}} \Rightarrow c[0] = \sqrt{\frac{1}{N}},\ c[k \neq 0] = \sqrt{\frac{2}{N}}$

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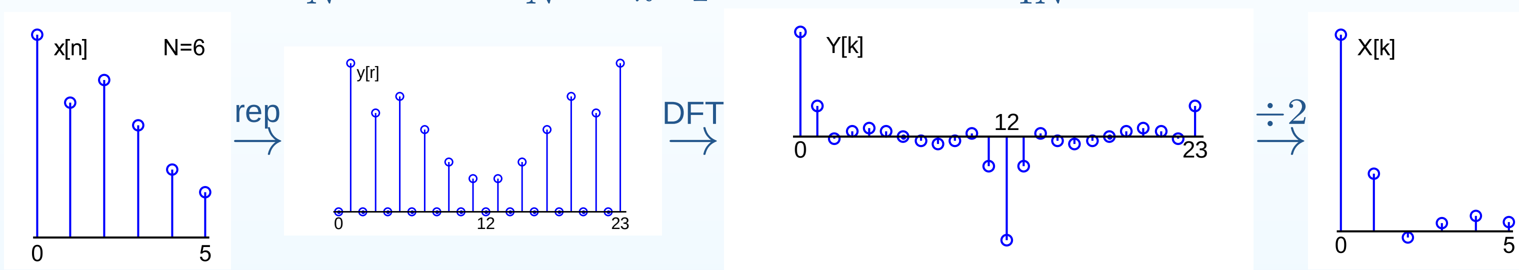

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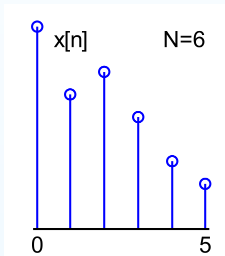
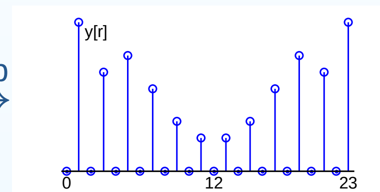
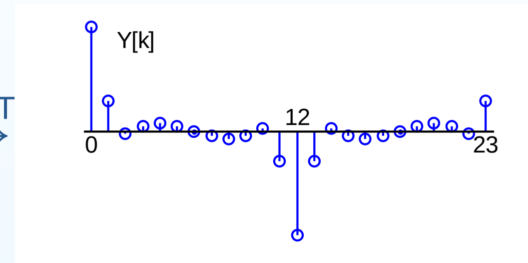


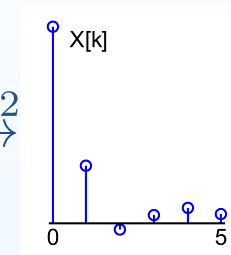
$\xrightarrow{\text{rep}}$



$\xrightarrow{\text{DFT}}$



$\xrightarrow{\div 2}$



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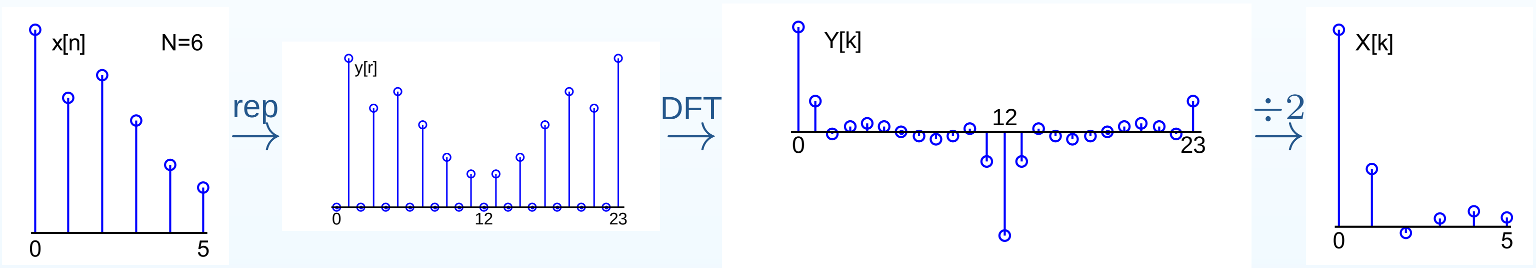

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Note: MATLAB dct() calculates the ODCT

Energy Compaction

If consecutive $x[n]$ are positively correlated, DCT concentrates energy in a few $X[k]$ and decorrelates them.

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Suppose ODCT of $\mathbf{x}$ is $\mathbf{Cx}$ and DFT is $\mathbf{Fx}$.
Covariance of $\mathbf{Cx}$ is $\left\langle \mathbf{C}\mathbf{x}\mathbf{x}^H\mathbf{C}^H \right\rangle = \mathbf{CSC}^H$ (similarly $\mathbf{FSF}^H$)
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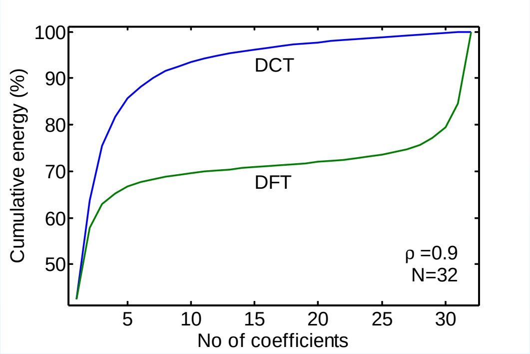

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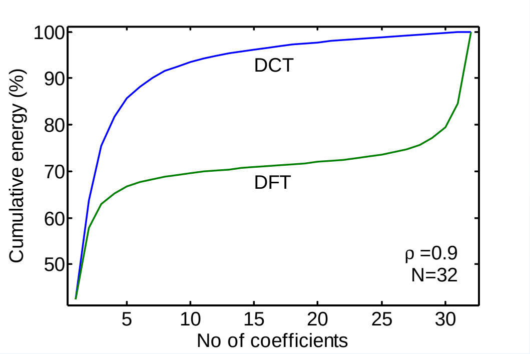
Diagonal elements give mean coefficient energy.



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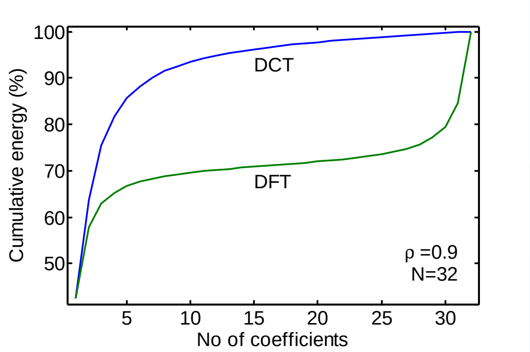

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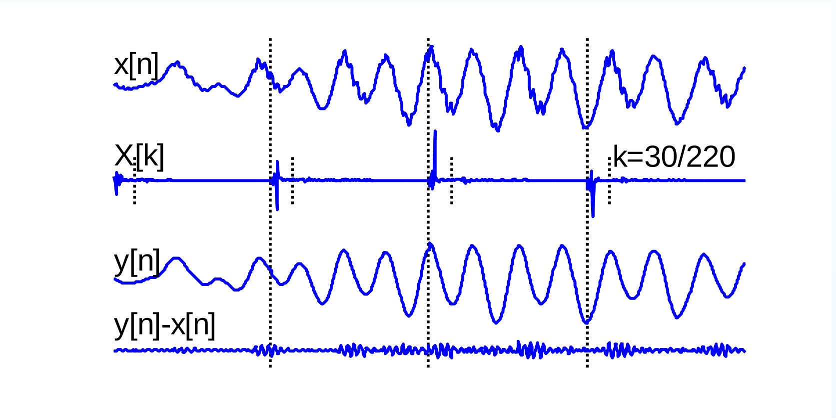
Energy compaction good for coding (low-valued coefficients can be set to 0)
Decorrelation good for coding and for probability modelling

Frame-based coding

- Divide continuous signal into frames



Frame-based coding

- Divide continuous signal into frames
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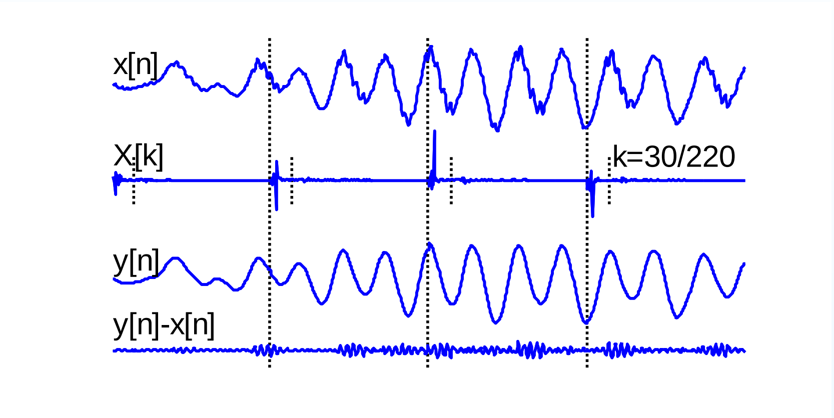

Frame-based coding

- Divide continuous signal into frames
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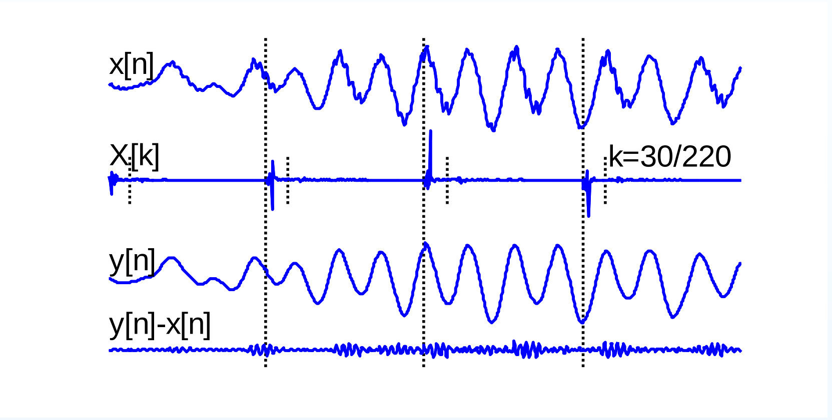
- Encode DCT
  - e.g. keep only 30 $X[k]$



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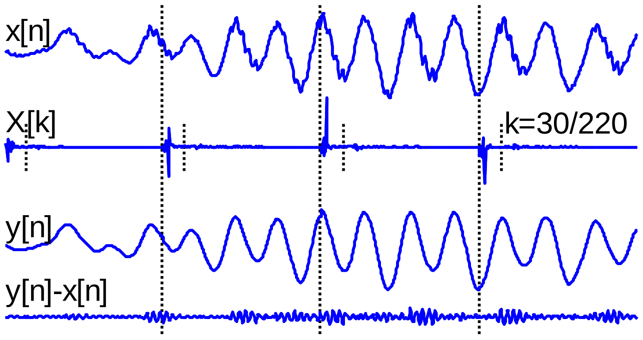

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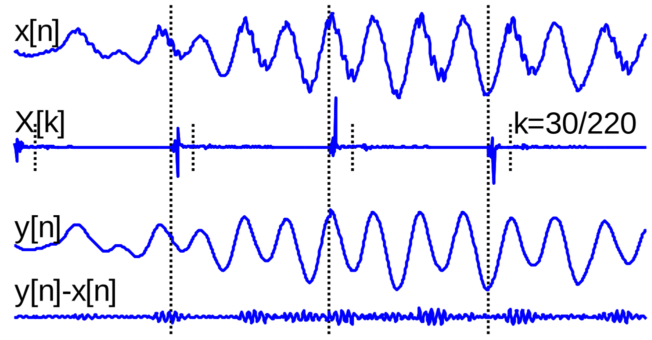

Problem: Coding may create discontinuities at frame boundaries

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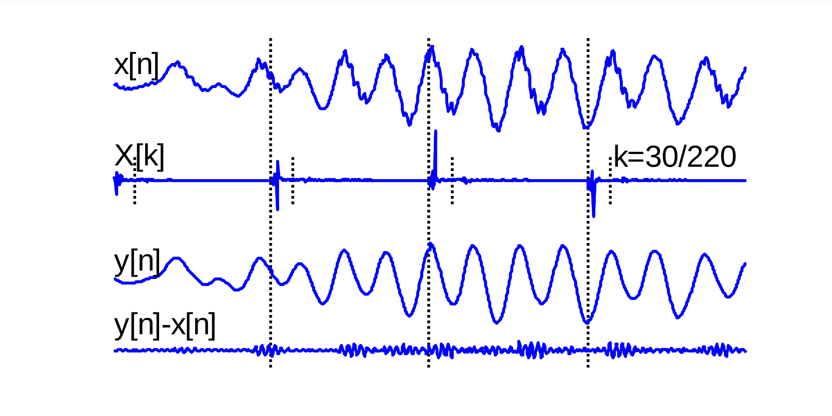
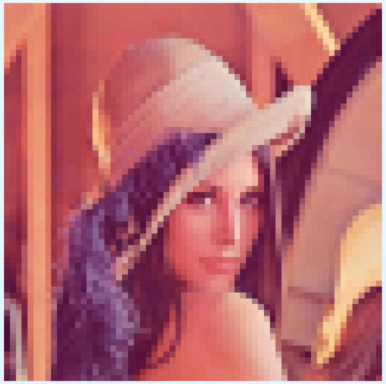
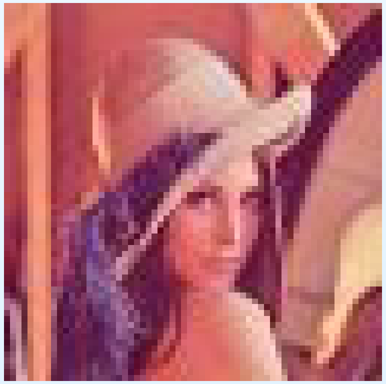


Problem: Coding may create discontinuities at frame boundaries
e.g. JPEG, MPEG use 8×8 pixel blocks



8.3 kB (PNG)



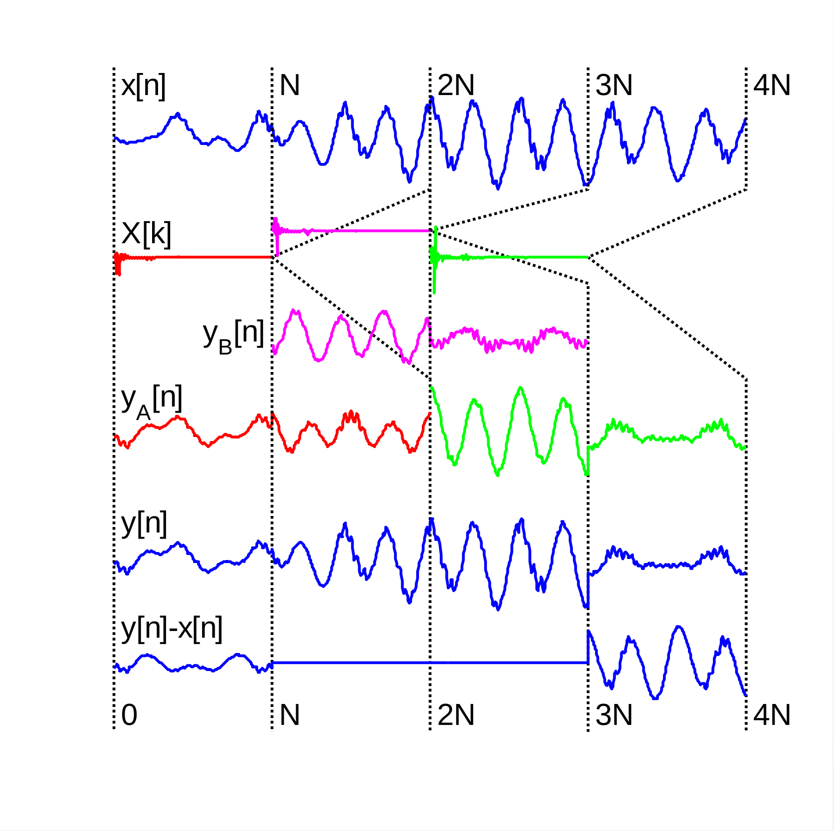
1.6 kB (JPEG)

0.5 kB (JPEG)

Lapped Transform

Modified Discrete Cosine Transform (MDCT): overlapping frames $2N$ long

$x[0:2N-1]$



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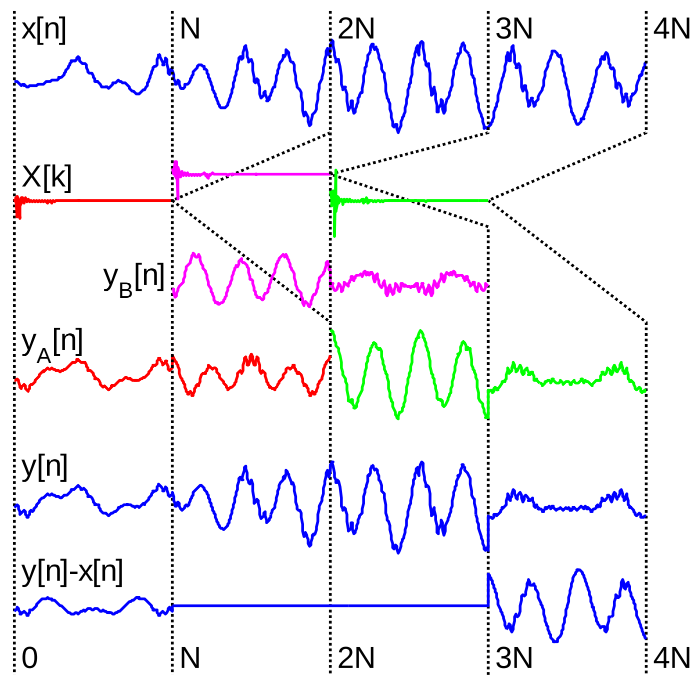
$$x[0:2N-1] \stackrel{\text{MDCT}}{\rightarrow} X[0:N-1]$$



MDCT: $2N \rightarrow N$ coefficients

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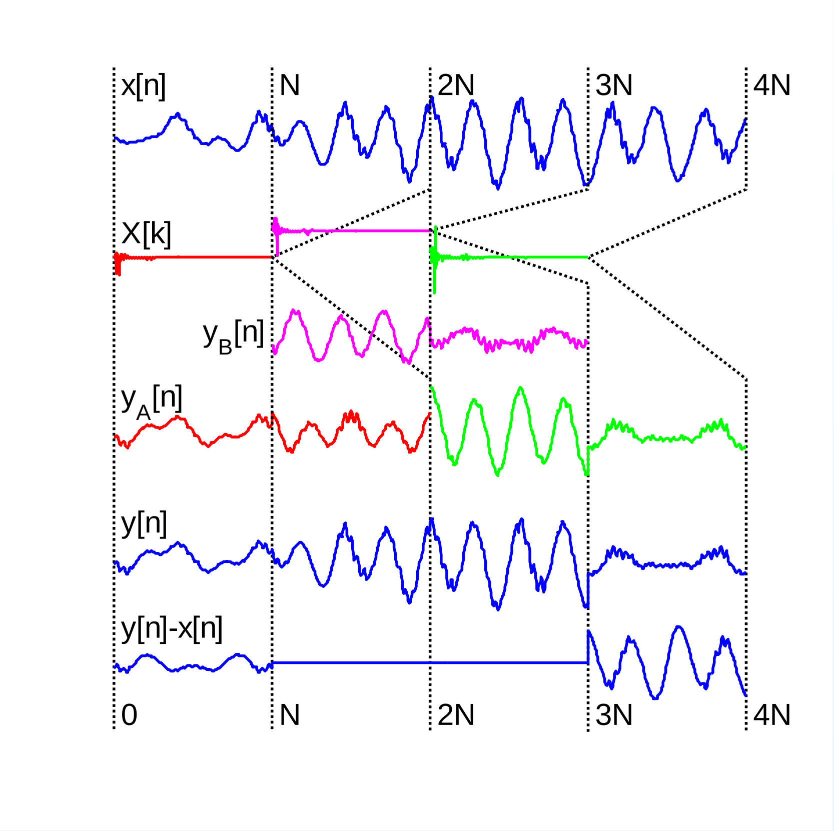
$$x[0:2N-1] \xrightarrow{\text{MDCT}} X[0:N-1] \xrightarrow{\text{IMDCT}} y_A[0:2N-1]$$



MDCT: $2N \rightarrow N$ coefficients, IMDCT: $N \rightarrow 2N$ samples

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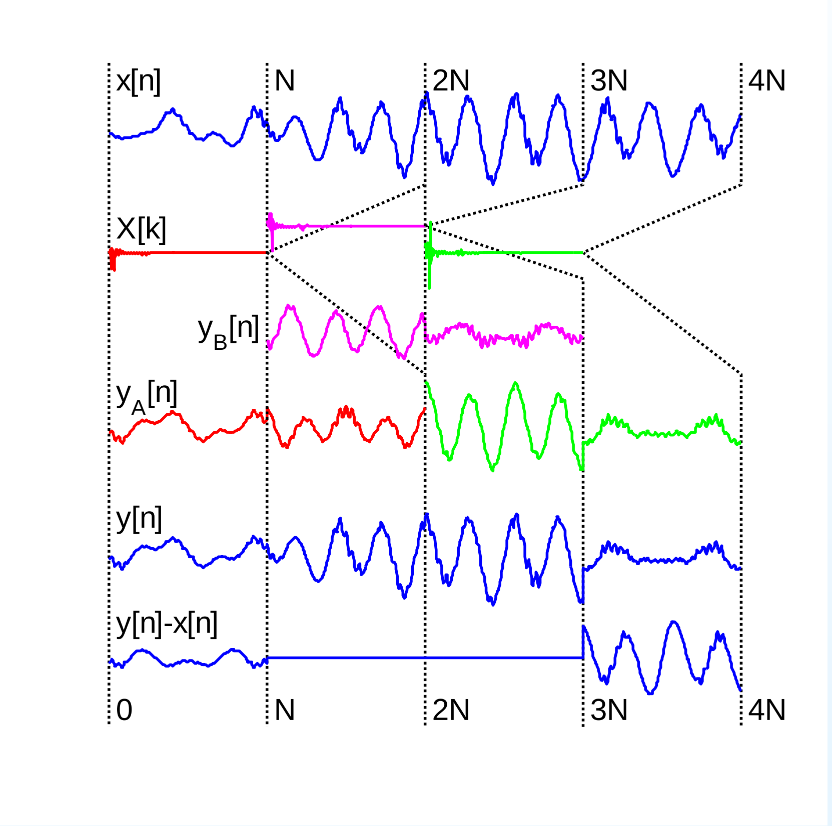
$$x[N:3N-1] \overset{\text{MDCT}}{\rightarrow} X[N:2N-1]$$



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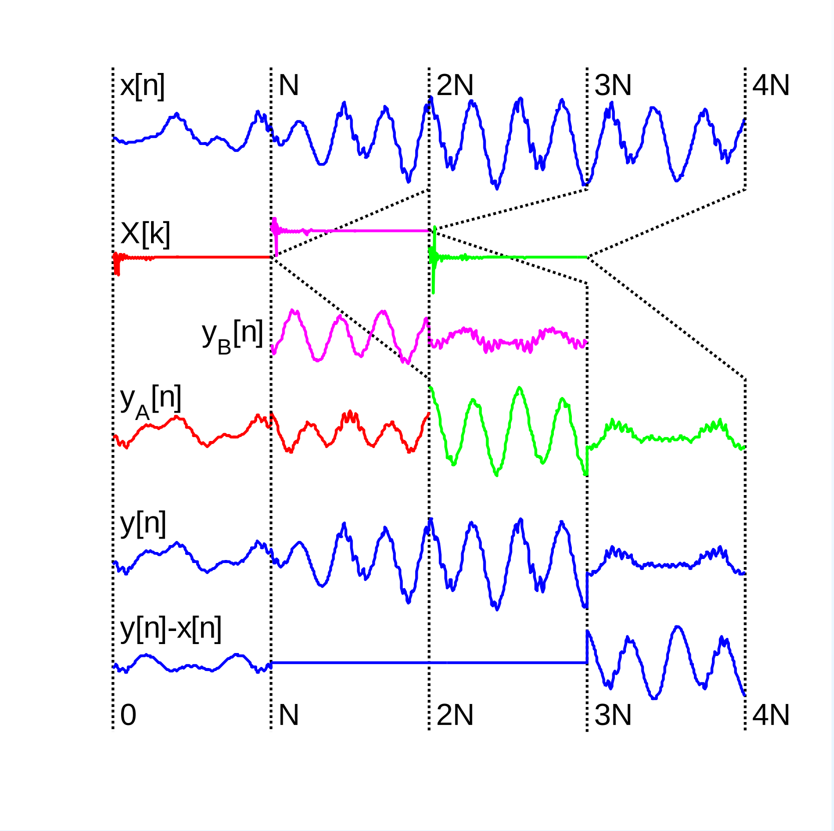

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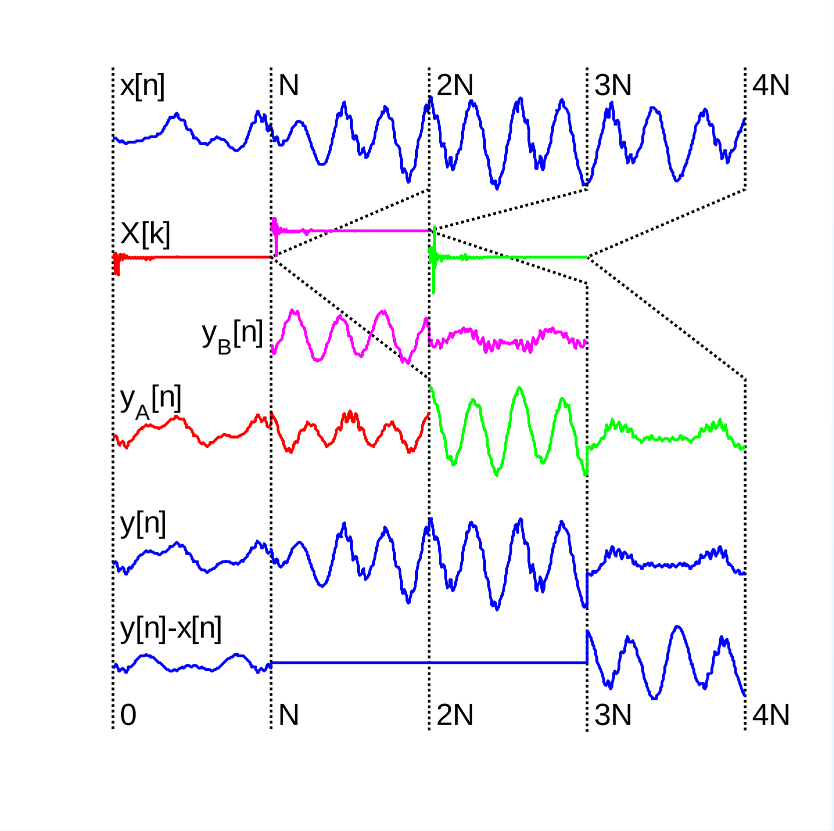

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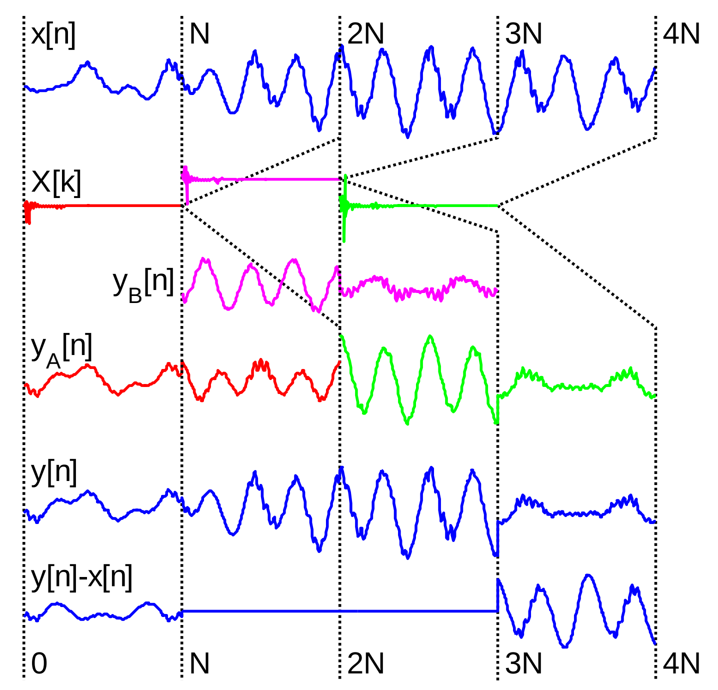

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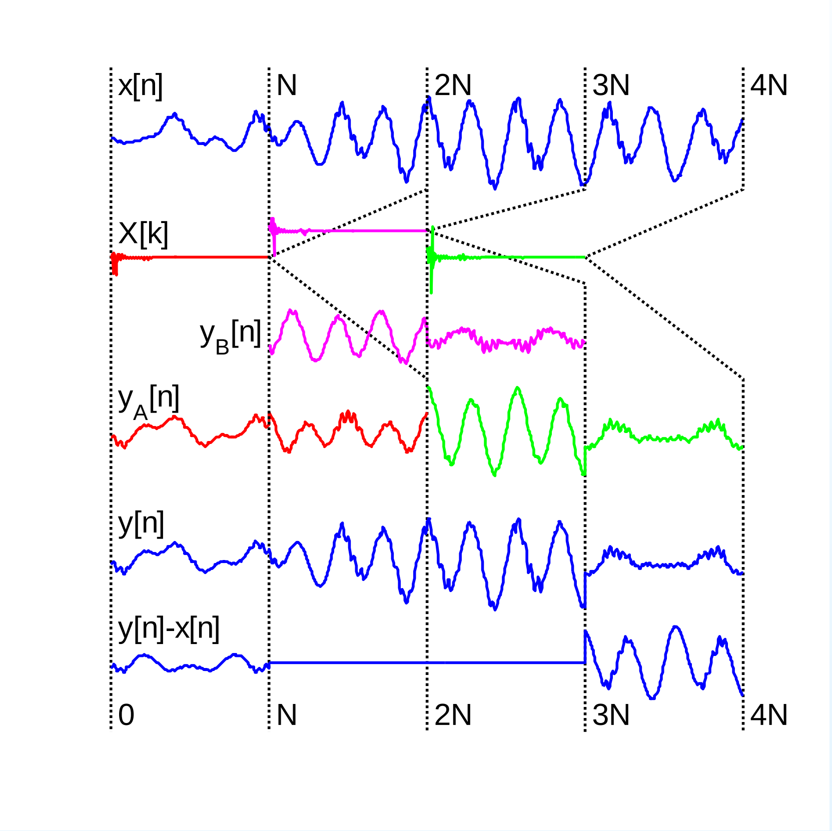
$$x[2N:4N-1] \xrightarrow{\text{MDCT}} X[2N:3N-1] \xrightarrow{\text{IMDCT}} y_A[2N:3N-1]$$

$$y[*] = y_A[*] + y_B[*]$$



MDCT: $2N \to N$ coefficients, IMDCT: $N \to 2N$ samples

Even frames $\to y_A$, Odd frames $\to y_B$ then $y = y_A + y_B$

Lapped Transform

Modified Discrete Cosine Transform (MDCT): overlapping frames $2N$ long

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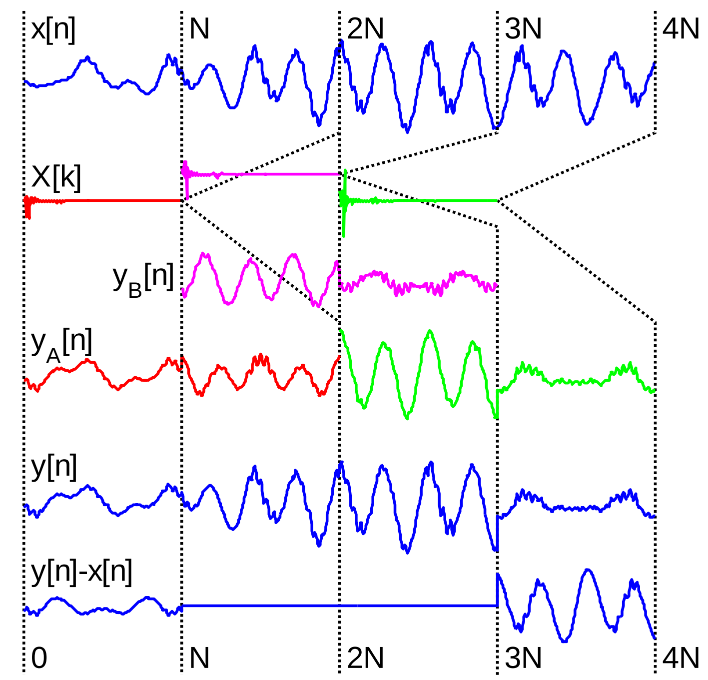

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Even frames $\rightarrow y_A$, Odd frames $\rightarrow y_B$ then $y = y_A + y_B$

Errors cancel exactly: Time-domain alias cancellation (TDAC)

MDCT

MDCT: $X_M[k] = \sum_{n=0}^{2N-1} x[n] \cos \frac{2\pi(2n+1+N)(2k+1)}{8N}$

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Next frame: $\mathbf{y}_B[n] = \begin{bmatrix} \mathbf{c} - \overleftarrow{\mathbf{d}} & \mathbf{d} - \overleftarrow{\mathbf{c}} & \mathbf{e} + \overleftarrow{\mathbf{f}} & \mathbf{f} + \overleftarrow{\mathbf{e}} \end{bmatrix}$

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Used in audio coding: MP3, WMA, AC-3, AAC, Vorbis, ATRAC

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For further details see Mitra: 5.

MATLAB routines

dct, idct	ODFT with optional zero-padding