BOUNDARY LAYER OF TRANSPORT EQUATION WITH IN-FLOW BOUNDARY

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Abstract. Consider the steady neutron transport equation in 2D convex domains with in-flow boundary condition. In this paper, we establish the diffusive limit while the boundary layers are present. Our contribution relies on a delicate decomposition of boundary data to separate the regular and singular boundary layers, novel weighted $W^{1,\infty}$ estimates for the Milne problem with geometric correction in convex domains, as well as an $L^{2m} - L^{\infty}$ framework which yields stronger remainder estimates.
Keywords: Boundary layer decomposition; geometric correction; $W^{1,\infty}$ estimates; $L^{2m} - L^{\infty}$ framework.

Contents

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1. Introduction

We consider the steady neutron transport equation in a two-dimensional bounded convex domain with in-flow boundary. In the spacial domain $\vec{x}=(x_1,x_2)\in\Omega$ where $\partial\Omega\in C^3$ and the velocity domain $\vec{w}=(w_1,w_2)\in\mathbb{S}^1$, the neutron density $u^\epsilon(\vec{x},\vec{w})$ satisfies

$$\left\{\begin{array}{rcl}\epsilon\vec{w}\cdot\nabla_x u^\epsilon+u^\epsilon-\bar{u}^\epsilon&=&0\ \text{ in }\ \Omega,\\ u^\epsilon(\vec{x}_0,\vec{w})&=&g(\vec{x}_0,\vec{w})\ \text{ for }\ \vec{w}\cdot\vec{\nu}<0\ \text{ and }\ \vec{x}_0\in\partial\Omega,\end{array}\right. \tag{1.1}$$

where

$$\bar{u}^\epsilon(\vec{x})=\frac{1}{2\pi}\int_{\mathbb{S}^1}u^\epsilon(\vec{x},\vec{w})\mathrm{d}\vec{w}, \tag{1.2}$$

$\vec{\nu}$ is the outward unit normal vector, with the Knudsen number $0<\epsilon<<1$. We intend to study the behavior of u^ϵ as $\epsilon\to0$.

Based on the flow direction, we can divide the boundary $\Gamma=\{(\vec{x},\vec{w}):\vec{x}\in\partial\Omega\}$ into the in-flow boundary Γ^-, the out-flow boundary Γ^+ and the grazing set Γ^0 as

$$\Gamma^-=\{(\vec{x},\vec{w}):\vec{x}\in\partial\Omega,\ \vec{w}\cdot\vec{\nu}<0\} \tag{1.3}$$
$$\Gamma^+=\{(\vec{x},\vec{w}):\vec{x}\in\partial\Omega,\ \vec{w}\cdot\vec{\nu}>0\} \tag{1.4}$$
$$\Gamma^0=\{(\vec{x},\vec{w}):\vec{x}\in\partial\Omega,\ \vec{w}\cdot\vec{\nu}=0\} \tag{1.5}$$

It is easy to see that $\Gamma=\Gamma^+\cup\Gamma^-\cup\Gamma^0$. Hence, the boundary condition is only given for Γ^-.

Diffusive limits, or more general hydrodynamic limits, are central to connecting the kinetic theory and fluid mechanics. The basic idea is to consider the asymptotic behaviors of the solutions to Boltzmann equation, transport equation, or Vlasov systems. Since early 20th century, this type of problems have been extensively studied in many different settings: steady or unsteady, linear or nonlinear, strong solution or weak solution, etc.

Among all these variations, one of the simplest but most important models — neutron transport equation in bounded domains, has attracted a lot of attention since the dawn of atomic age. The neutron transport equation is usually regarded as a linear prototype of the more complicated nonlinear Boltzmann equation, and thus, is an ideal starting point to develop new theories and techniques. We refer to [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17] for more details.

For steady neutron transport equation, the exact solution can be approximated by the sum of an interior solution U and a boundary layer $\mathscr{U}$. The interior solution satisfies certain fluid equations or thermodynamic equations, and the boundary layer satisfies a half-space kinetic equation, which decays rapidly when it is away from the boundary.

The justification of diffusive limit usually involves two steps:

(1) Expanding $U=\displaystyle\sum_{k=0}^{\infty}\epsilon^kU_k$ and $\mathscr{U}=\displaystyle\sum_{k=0}^{\infty}\epsilon^k\mathscr{U}_k$ as power series of ϵ and proving the coefficients U_k and $\mathscr{U}_k$ are well-defined. Traditionally, the estimates of interior solutions U_k are relatively straightforward. On the other hand, boundary layers $\mathscr{U}_k$ satisfy one-dimensional half-space problems which lose some key structures of the original equations. The well-posedness of boundary layer equations are sometimes extremely difficult and it is possible that they are actually ill-posed (e.g. certain type of Prandtl layers).

(2) Proving that $R=u^\epsilon-U_0-\mathscr{U}_0=o(1)$ as $\epsilon\to0$. Ideally, this should be done just by expanding to the leading-order level U_0 and $\mathscr{U}_0$. However, in singular perturbation problems, the estimates of the remainder R usually involves negative powers of ϵ, which requires expansion to higher order

terms U_N and $\mathcal{U}_N$ for $N \geq 1$ such that we have sufficient power of ϵ. In other words, we define $R = u^\epsilon - \sum_{k=0}^{N} \epsilon^k U_k - \sum_{k=0}^{N} \epsilon^k \mathcal{U}_k$ for $N \geq 1$ instead of $R = u^\epsilon - U_0 - \mathcal{U}_0$ to get better estimate of R.

The construction of kinetic boundary layers has long been believed to be satisfactorily solved since Bensoussan, Lions and Papanicolaou published their remarkable paper [1] in 1979. Their formulation, based on the flat Milne problem, was later extended to treat the nonlinear Boltzmann equation (see [19] and [20]).

In detail, in Ω, let $\eta \in [0, \infty)$ denote the rescaled normal variable with respect to the boundary, $\tau \in [-\pi, \pi)$ the tangential variable, and $\phi \in [-\pi, \pi)$ the velocity variable defined in (2.22), (2.26), (2.30), and (2.34). The boundary layer $\mathcal{U}_0$ satisfies the flat Milne problem,

$$\sin\phi \frac{\partial \mathcal{U}_0}{\partial \eta} + \mathcal{U}_0 - \bar{\mathcal{U}}_0 = 0. \tag{1.6}$$

Unfortunately, in [21], we demonstrated that both the proof and results of this formulation are invalid due to a lack of regularity in estimating $\dfrac{\partial \mathcal{U}_0}{\partial \tau}$. This pulls the whole research back to the starting point, and any later results based on this type of boundary layers should be reexamined.

To be more specific, the remainder estimates require that $\mathcal{U}_1 \in L^\infty$ which needs $\dfrac{\partial \mathcal{U}_0}{\partial \tau} \in L^\infty$. However, though [1] shows that $\mathcal{U}_0 \in L^\infty$, it does not necessarily mean that $\dfrac{\partial \mathcal{U}_0}{\partial \eta} \in L^\infty$. Furthermore, this singularity $\dfrac{\partial \mathcal{U}_0}{\partial \eta} \notin L^\infty$ will be transferred to $\dfrac{\partial \mathcal{U}_0}{\partial \tau} \notin L^\infty$. A careful construction of boundary data justifies this invalidity, i.e. the chain of estimates

$$R = o(1) \;\Leftarrow\; \mathcal{U}_1 \in L^\infty \;\Leftarrow\; \frac{\partial \mathcal{U}_0}{\partial \tau} \in L^\infty \;\Leftarrow\; \frac{\partial \mathcal{U}_0}{\partial \eta} \in L^\infty, \tag{1.7}$$

is broken since the rightmost estimate is wrong.

While the classical method breaks down, a new approach with geometric correction to the boundary layer construction has been developed to ensure regularity in the cases of disk and annulus in [21] and [22]. The new boundary layer $\mathcal{U}_0$ satisfies the ϵ-Milne problem with geometric correction,

$$\sin\phi \frac{\partial \mathcal{U}_0}{\partial \eta} + \frac{\epsilon}{R_\kappa - \epsilon\eta} \cos\phi \frac{\partial \mathcal{U}_0}{\partial \phi} + \mathcal{U}_0 - \bar{\mathcal{U}}_0 = 0, \tag{1.8}$$

where R_κ is the radius of curvature at boundary. We proved that the solution recovers the well-posedness and exponential decay as in flat Milne problem, and the regularity in τ is indeed improved, i.e. $\dfrac{\partial \mathcal{U}_0}{\partial \tau} \in L^\infty$.

However, this new method fails to treat more general domains. Roughly speaking, we have two contradictory goals to achieve:

- To prove diffusive limits, the remainder estimates require higher-order regularity estimates of the boundary layer.
- The geometric correction $\dfrac{\epsilon}{R_\kappa - \epsilon\eta} \cos\phi \dfrac{\partial \mathcal{U}_0}{\partial \phi}$ in the boundary layer equation is related to the curvature of the boundary curve, which prevents higher-order regularity estimates.

In other words, the improvement of regularity is still not enough to close the proof. We may analyze the effects of different domains and formulations as follows:

- In the absence of the geometric correction $\dfrac{\epsilon}{R_\kappa - \epsilon\eta} \cos\phi \dfrac{\partial \mathcal{U}_0}{\partial \phi}$, which is the flat Milne problem as in [1], the key tangential derivative $\dfrac{\partial \mathcal{U}_0}{\partial \tau}$ is not bounded. Therefore, the expansion breaks down.

- In the domain of disk or annulus, when R_κ is constant, as in [21] and [22], $\dfrac{\partial \mathscr{U}_0}{\partial\tau}$ is bounded, since the tangential derivative $\dfrac{\partial}{\partial\tau}$ commutes with the equation, and thus we do not even need to estimate $\dfrac{\partial \mathscr{U}_0}{\partial\eta}$.
- For general smooth convex domains, when R_κ is a function of τ, $\dfrac{\partial \mathscr{U}_0}{\partial\tau}$ relates to the normal derivative $\dfrac{\partial \mathscr{U}_0}{\partial\eta}$, which has been shown possibly unbounded in [21]. Therefore, we get stuck again at the regularity estimates.

In [5] and [6], for the case of diffusive boundary, the above argument is pushed from both sides, i.e. improvements in remainder estimates and boundary layer regularity.

In detail, consider the boundary layer expansion

$$\mathscr{U}(\eta,\tau,\vec{w}) \sim \mathscr{U}_0(\eta,\tau,\vec{w}) + \epsilon\mathscr{U}_1(\eta,\tau,\vec{w}). \tag{1.9}$$

The diffusive boundary condition leads to an important simplification that $\mathscr{U}_0 = 0$. As [21] stated, the next-order boundary layer $\mathscr{U}_1$ must formally satisfy

$$\sin\phi\frac{\partial \mathscr{U}_1}{\partial\eta} + \frac{\epsilon}{R_\kappa - \epsilon\eta}\cos\phi\frac{\partial \mathscr{U}_1}{\partial\phi} + \mathscr{U}_1 - \bar{\mathscr{U}}_1 = 0. \tag{1.10}$$

Naturally, the diffusive limit requires the estimate of $\dfrac{\partial \mathscr{U}_1}{\partial\tau}$. Here, a key observation is that $W = \dfrac{\partial \mathscr{U}_1}{\partial\tau}$ satisfies

$$\sin\phi\frac{\partial W}{\partial\eta} + \frac{\epsilon}{R_\kappa - \epsilon\eta}\cos\phi\frac{\partial W}{\partial\phi} + W - \bar{W} = -\frac{\partial_\tau R_\kappa}{R_\kappa - \epsilon\eta}\left(\frac{\epsilon}{R_\kappa - \epsilon\eta}\cos\phi\frac{\partial \mathscr{U}_1}{\partial\phi}\right). \tag{1.11}$$

Note that the right-hand side is part of the $\mathscr{U}_1$ equation and its estimate depends on $\sin\phi\dfrac{\partial \mathscr{U}_1}{\partial\eta}$. In other words, the estimate of $\dfrac{\partial \mathscr{U}_1}{\partial\tau}$ depends on $\sin\phi\dfrac{\partial \mathscr{U}_1}{\partial\eta}$, not just $\dfrac{\partial \mathscr{U}_1}{\partial\eta}$ which is possibly unbounded. The $\sin\phi$ is crucial to eliminate the singularity. This forms the major proof in [5] and [6], i.e. the weighted regularity of $\mathscr{U}_1$.

Our main idea is to delicate track $\mathscr{U}_1$ along the characteristics in the mild formulation, and prove the weighted $W^{1,\infty}$ estimates of the boundary layer. In particular, we showed that $\dfrac{\partial \mathscr{U}_1}{\partial\tau}$ is bounded even when R_κ is not constant for general convex domains.

Furthermore, with a novel $L^{2m} - L^\infty$ framework, we justified an almost optimal remainder estimate to reduce the further regularity requirement of $\mathscr{U}_1$.

In summary, in [5] and [6], we proved the diffusive limit that u^ϵ converges to the solution of a Laplace's equation with Neumann boundary condition.

It is notable that, for the case of in-flow boundary as equation (1.1), the situation is much worse. The leading-order boundary layer $\mathscr{U}_0$ is no longer zero.

$$\sin\phi\frac{\partial \mathscr{U}_0}{\partial\eta} + \frac{\epsilon}{R_\kappa - \epsilon\eta}\cos\phi\frac{\partial \mathscr{U}_0}{\partial\phi} + \mathscr{U}_0 - \bar{\mathscr{U}}_0 = 0, \tag{1.12}$$

$$\sin\phi\frac{\partial \mathscr{U}_1}{\partial\eta} + \frac{\epsilon}{R_\kappa - \epsilon\eta}\cos\phi\frac{\partial \mathscr{U}_1}{\partial\phi} + \mathscr{U}_1 - \bar{\mathscr{U}}_1 = -\cos\phi\frac{\partial \mathscr{U}_0}{\partial\tau}. \tag{1.13}$$

The remainder contains the term $\dfrac{\partial \mathcal{U}_1}{\partial\tau}$, which depends on the estimate of $\dfrac{\partial^2\mathcal{U}_0}{\partial\tau^2}$. Then we must prove $W^{2,\infty}$ estimates in the boundary layer equation. In principle, this is impossible for general kinetic equations as [4] pointed out.

On the other hand, we have a key observation that actually the singularity that prevents higher-order regularity concentrates in the neighborhood of the grazing set, so it is natural to isolate the singular part from the whole solution and tackle them in different methods.

Inspired by [18], we introduce a new regularization argument. Instead of trying different weighted norms, we may also modify the boundary data and smoothen the boundary layer in this modified problem.

To be precise, we decompose the boundary data $g = \mathscr{G} + \mathfrak{G}$, such that

- the boundary layer $\mathscr{U}$ with data $\mathscr{G}$, which we call regular boundary layer, attains second-order regularity in the tangential direction, i.e. $\dfrac{\partial^2\mathscr{U}}{\partial\tau^2} \in L^\infty$; $\mathscr{G} = g$ in most of the region except a small neighborhood of the grazing set in order to strengthen the smoothness of $\mathscr{U}$;
- the boundary layer $\mathfrak{U}$ with data $\mathfrak{G}$, which we call singular boundary layer, attains only first-order regularity in the tangential direction i.e. $\dfrac{\partial\mathfrak{U}}{\partial\tau} \in L^\infty$, but the support of $\mathfrak{G}$ is restricted to a very small neighborhood of the grazing set with diameter ϵ^α for some $0 < \alpha < 1$.

In other words, for the remainder estimates, the extra power of ϵ comes from two sources: $\mathscr{U}$ gains power by expanding to the higher order, and $\mathfrak{U}$ gains power through a small support ϵ^α.

Definitely, this decomposition comes with a price. Even if we assume $\dfrac{\partial g}{\partial\phi} = O(1)$, after the decomposition, we can at most have $\dfrac{\partial\mathscr{G}}{\partial\phi} = O(\epsilon^{-\alpha})$ and $\dfrac{\partial\mathfrak{G}}{\partial\phi} = O(\epsilon^{-\alpha})$. We have to prove a much stronger weighted $W^{1,\infty}$ estimates to suppress such loss of power in ϵ. Moreover, this decomposition introduces two contradictory goals in the estimates:

- to obtain $W^{2,\infty}$ estimate of $\mathscr{U}$ with data $\mathscr{G}$, we want α to be as small as possible; the smaller α is (better smoothness of $\mathscr{G}$), the better estimates we get;
- to obtain $W^{1,\infty}$ estimate of $\mathfrak{U}$ with data $\mathfrak{G}$, we want α to be as large as possible; the larger α is (smaller support of $\mathfrak{G}$), the better estimates we get.

This balance is quite delicate and the estimates for the ϵ-Milne problem with geometric correction in [21], [22], [5] and [6] are not sufficient. We have to start from scratch and prove the stronger version.

To fully solve such a problem, we need an intricate synthesis of previously developed methods, and the fresh regularization argument stated as above.

We inherit and modify the following ideas and techniques, which can be considered the minor contribution:

- **Geometric Correction:**
  The ϵ-Milne problem with geometric correction for $f = \mathscr{U}$ or $\mathfrak{U}$,
  $$\sin\phi\frac{\partial f}{\partial\eta} + \frac{\epsilon}{R_\kappa - \epsilon\eta}\cos\phi\frac{\partial f}{\partial\phi} + f - \bar{f} = S, \tag{1.14}$$
  has been shown to be the correct formulation to describe kinetic boundary layers (see [21]). In this paper, we start from scratch and justify the detailed dependence on the source term S. In particular, we isolate the contribution of $\bar{S}$ and $S - \bar{S}$.
- **Canonical Weighted $W^{1,\infty}$ Estimates of Boundary Layers:**
  The weighted $W^{1,\infty}$ estimates in ϵ-Milne problem with geometric correction is the key to estimate $\dfrac{\partial f}{\partial\tau}$ (see [5]). In this paper, we highlight the dependence on the characteristic curves and the boundary data. The convexity and the kinetic distance
  $$\zeta(\eta,\phi) = \left(1 - \left(\frac{R_\kappa - \epsilon\eta}{R_\kappa}\cos\phi\right)^2\right)^{\frac{1}{2}}, \tag{1.15}$$

is key to this proof.

- **Remainder Estimates:**
This is the key step to reduce the regularity requirement in boundary layers. It is originally developed in [21] and later strengthened in [5]. In the remainder equation for $R(\vec{x},\vec{w}) = u^\epsilon - U - \mathcal{U}$,

$$\epsilon\vec{w}\cdot\nabla_x R + R - \bar{R} = S, \tag{1.16}$$

the estimate justified in [21] using $L^2 - L^\infty$ framework is

$$\|R\|_{L^\infty} \lesssim \frac{1}{\epsilon^3}\|S\|_{L^2} + \text{higher order terms}. \tag{1.17}$$

We intend to show that $\|R\|_{L^\infty} = o(1)$ as $\epsilon \to 0$. Since we cannot expand to higher-order boundary layers to further improve S, the coefficients ϵ^{-3} is too singularity. A key improvement in [5] for diffusive boundary case is to develop the $L^{2m} - L^\infty$ framework to prove a stronger estimate for $m \geq 2$,

$$\|R\|_{L^\infty} \lesssim \frac{1}{\epsilon^{2+\frac{1}{m}}}\|S\|_{L^{\frac{2m}{2m-1}}} + \text{higher order terms}. \tag{1.18}$$

In this paper, we adapt it to treat the in-flow boundary case with a modified $L^{2m} - L^\infty$ framework. The main idea is to introduce a special test function in the weak formulation to treat $\bar{R}$ and $R - \bar{R}$ separately, and further to bootstrap to improve the L^∞ estimate by a modified double Duhamel's principle. The proof relies on a delicate analysis using interpolation and Young's inequality.

The key novelty of this paper lies in the innovative regularization argument and the corresponding regularity estimates, which constitute the major contribution:

- **Improved Weighted $W^{1,\infty}$ Estimates of Boundary Layers:**
We combine several different formulations to track the characteristics and justify that the solution of (1.14) satisfies

$$\begin{aligned}&\left\|\zeta\frac{\partial f}{\partial\eta}\right\|_{L^\infty L^\infty} + \left\|\frac{\epsilon}{R_\kappa - \epsilon\eta}\cos\phi\frac{\partial f}{\partial\phi}\right\|_{L^\infty L^\infty} \\ &\leq C\left|\ln(\epsilon)\right|^8\left(\|p\|_{L^\infty} + \left\|(\epsilon+\zeta)\frac{\partial p}{\partial\phi}\right\|_{L^\infty} + \|S\|_{L^\infty L^\infty} + \left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty} + \|f\|_{L^\infty L^\infty}\right).\end{aligned} \tag{1.19}$$

where the boundary data $p = \mathscr{G}$ or $\mathfrak{G}$. The extra weight $\epsilon + \zeta$ suppresses the singularity in $\dfrac{\partial\mathscr{G}}{\partial\phi}$ and $\dfrac{\partial\mathfrak{G}}{\partial\phi}$. In particular, the estimate does not depend on $\dfrac{\partial S}{\partial\phi}$. This is the key step to isolate the contribution of $\sin\dfrac{\partial f}{\partial\eta}$ and $\dfrac{\epsilon}{R_\kappa - \epsilon\eta}\cos\phi\dfrac{\partial f}{\partial\phi}$, which is crucial to later $W^{2,\infty}$ estimates.

The estimate is obtained through a delicate absorbing argument and novel characteristic analysis in half-space kinetic equations.

- $\dfrac{\partial^2}{\partial\tau^2}$ **Estimate of Regular Boundary Layer**:
[4] pointed out that weighted $W^{2,\infty}$ estimates of general kinetic equations is not available. This is true even for $\mathscr{U}$ with modified boundary data. In principle, we cannot bound $\dfrac{\partial^2\mathscr{U}_0}{\partial\eta^2}$ and $\dfrac{\partial^2\mathscr{U}_0}{\partial\phi^2}$. Instead, we propose a delicate analysis to show that we can estimate $\dfrac{\partial\mathscr{U}_1}{\partial\tau}$ without referring to the other second-order derivatives. This is quite unusual and cannot be done in a direct fashion.

Roughly speaking, we need a chain of estimates

$$
\begin{aligned}
\left\|\frac{\partial\mathscr{U}_1}{\partial\tau}\right\|_{L^\infty L^\infty} &\Leftarrow \left\|\frac{\partial}{\partial\tau}\left(\frac{\partial\mathscr{U}_0}{\partial\tau}\right)\right\|_{L^\infty L^\infty} \\
&\Leftarrow \left\|\zeta\frac{\partial}{\partial\eta}\left(\frac{\partial\mathscr{U}_0}{\partial\tau}\right)\right\|_{L^\infty L^\infty} + \left\|\frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial}{\partial\phi}\left(\frac{\partial\mathscr{U}_0}{\partial\tau}\right)\right\|_{L^\infty L^\infty} \\
&\Leftarrow \left\|\frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial}{\partial\phi}\left(\frac{\partial\mathscr{U}_0}{\partial\eta}\right)\right\|_{L^\infty L^\infty} \\
&\Leftarrow \left\|\frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right\|_{L^\infty L^\infty}.
\end{aligned}
\tag{1.20}
$$

Here, none of these steps are direct application of above improved weighted $W^{1,\infty}$ estimates. Instead, we need careful arrangement of dependent terms and utilize absorbing argument in a delicate way. Eventually, we can justify that

$$
\left\|\frac{\partial\mathscr{U}_1}{\partial\tau}\right\|_{L^\infty L^\infty} \sim \epsilon^{-\alpha}. \tag{1.21}
$$

- $\dfrac{\partial}{\partial\tau}$ **Estimate with Smallness of Singular Boundary Layer**:
Here, the major difficulty is how to preserve the smallness of boundary data. The key observation is that in our proof of well-posedness and $W^{1,\infty}$ estimates, we only use two types of quantities: the integral in ϕ and the value along the characteristics. Therefore, we introduce a domain decomposition as $\chi_1 : \zeta \leq \epsilon^\alpha$ and $\chi_2 : \zeta \geq \epsilon^\alpha$, and estimate $\mathfrak{U}$ in each domain separately.
  (1) χ_1: since $\mathfrak{G} = O(1)$, we know $\mathfrak{U} = O(1)$ whose major contribution is from the boundary data, so it is relatively large but is only restricted to a small domain for $\alpha > 0$.
  (2) χ_2: since $\mathfrak{G} = 0$, we know $\mathfrak{U} = O(\epsilon^\alpha)$ whose major contribution is from the non-local operator $\bar{\mathfrak{U}}$, so it is relatively small and spread most of the domain.

  In the remainder estimate, the estimates of $\mathfrak{U}$ is in $L^{\frac{2m}{2m-1}}$, so we can combine these two contribution in an integral to obtain smallness

$$
\left\|\frac{\partial\mathfrak{U}_0}{\partial\tau}\right\|_{L^{\frac{2m}{2m-1}}} \sim \epsilon^{1-\frac{1}{2m}+\frac{(2m-1)\alpha}{2m}}. \tag{1.22}
$$

Applying these new techniques, we successfully obtain the diffusive limit that u^ϵ converges to the solution of a Laplace's equation with Dirichlet boundary condition.

Theorem 1.1. *Assume $g(\vec{x}_0,\vec{w}) \in C^3(\Gamma^-)$. Then for the steady neutron transport equation (1.1), there exists a unique solution $u^\epsilon(\vec{x},\vec{w}) \in L^\infty(\Omega\times\mathbb{S}^1)$. Moreover,*

$$
\lim_{\epsilon\to 0}\|u^\epsilon - U - \mathcal{U}\|_{L^\infty(\Omega\times\mathbb{S}^1)} = 0, \tag{1.23}
$$

where $U(\vec{x})$ satisfies the Laplace equation with Dirichlet boundary condition

$$
\begin{cases}
\Delta_x U(\vec{x}) = 0 \quad in \quad \Omega,\\
U(\vec{x}_0) = D(\vec{x}_0) \quad on \quad \partial\Omega,
\end{cases} \tag{1.24}
$$

and $\mathcal{U}(\eta,\tau,\phi)$ satisfies the ϵ-Milne problem with geometric correction

$$
\begin{cases}
\sin\phi\dfrac{\partial\mathcal{U}}{\partial\eta}-\dfrac{\epsilon}{R_\kappa(\tau)-\epsilon\eta}\cos\phi\dfrac{\partial\mathcal{U}}{\partial\phi}+\mathcal{U}-\bar{\mathcal{U}}=0,\\
\mathcal{U}(0,\tau,\phi)=g(\tau,\phi)-D(\tau)\quad for\quad \sin\phi>0,\\
\mathcal{U}(L,\tau,\phi)=\mathcal{U}(L,\tau,\mathscr{R}[\phi]),
\end{cases}
\tag{1.25}
$$

for $L=\epsilon^{-\frac{1}{2}}$, $\mathscr{R}[\phi]=-\phi$, η the rescaled normal variable, τ the tangential variable, and ϕ the velocity variable.

Remark 1.2. *Note that the effects of the boundary layer decays very fast when it is away from the boundary. Roughly speaking, this theorem states that for $\vec{x}$ not very close to the boundary, $u^\epsilon(\vec{x},\vec{w})$ can be approximated by the solution of a Laplace equation with Dirichlet boundary condition.*

Throughout this paper, $C>0$ denotes a constant that only depends on the parameter Ω, but does not depend on the data. It is referred as universal and can change from one inequality to another. When we write $C(z)$, it means a certain positive constant depending on the quantity z. We write $a\lesssim b$ to denote $a\leq Cb$.

This paper is organized as follows: in Section 2, we present the asymptotic analysis of the equation (2.1) and introduce the decomposition of boundary layers; in Section 3, we establish the L^∞ well-posedness of the remainder equation; in Section 4, we prove the well-posedness and decay of the ϵ-Milne problem with geometric correction; in Section 5, we study the weighted regularity of the ϵ-Milne problem with geometric correction; finally, in Section 6, we give a detailed analysis of the asymptotic expansion and prove the main theorem.

Remark 1.3. *The general structure of this paper is very similar to that of* [5] *and* [6]*. In particular, Section 3, 4 and 5 seem to be an obvious adaption of the corresponding theorems there. However, we introduce new techniques to delicately improve the results in* [5]*, so it needs a careful handling and a fresh start from scratch.*

2. Asymptotic Analysis

In this section, we will present the asymptotic expansions of the neutron transport equation

$$\begin{cases} \epsilon\vec{w}\cdot\nabla_x u^\epsilon + u^\epsilon - \bar{u}^\epsilon = 0 \ \text{ in } \ \Omega, \\ u^\epsilon(\vec{x}_0,\vec{w}) = g(\vec{x}_0,\vec{w}) \ \text{ for } \ \vec{w}\cdot\vec{\nu}<0 \ \text{ and } \ \vec{x}_0\in\partial\Omega. \end{cases} \tag{2.1}$$

We define the interior expansion as follows:

$$U(\vec{x},\vec{w}) \sim U_0(\vec{x},\vec{w}) + \epsilon U_1(\vec{x},\vec{w}) + \epsilon^2 U_2(\vec{x},\vec{w}), \tag{2.2}$$

where U_k can be determined by comparing the order of ϵ by plugging (2.2) into the equation (2.1). Thus we have

$$U_0 - \bar{U}_0 = 0, \tag{2.3}$$
$$U_1 - \bar{U}_1 = -\vec{w}\cdot\nabla_x U_0, \tag{2.4}$$
$$U_2 - \bar{U}_2 = -\vec{w}\cdot\nabla_x U_1. \tag{2.5}$$

Plugging (2.3) into (2.4), we obtain

$$U_1 = \bar{U}_1 - \vec{w}\cdot\nabla_x\bar{U}_0. \tag{2.6}$$

Plugging (2.6) into (2.5), we get

$$\begin{aligned} U_2 - \bar{U}_2 =& -\vec{w}\cdot\nabla_x(\bar{U}_1 - \vec{w}\cdot\nabla_x\bar{U}_0) \\ =& -\vec{w}\cdot\nabla_x\bar{U}_1 + w_1^2\partial_{x_1x_1}\bar{U}_0 + w_2^2\partial_{x_2x_2}\bar{U}_0 + 2w_1w_2\partial_{x_1x_2}\bar{U}_0. \end{aligned} \tag{2.7}$$

Integrating (2.7) over $\vec{w}\in\mathbb{S}^1$, we achieve the final form

$$\Delta_x\bar{U}_0 = 0. \tag{2.8}$$

which further implies $U_0(\vec{x},\vec{w})$ satisfies the equation

$$\begin{cases} U_0 &= \bar{U}_0, \\ \Delta_x\bar{U}_0 &= 0. \end{cases} \tag{2.9}$$

In a similar fashion, for $k=1,2$, U_k satisfies

$$\begin{cases} U_k &= \bar{U}_k - \vec{w}\cdot\nabla_x U_{k-1}, \\ \Delta_x\bar{U}_k &= -\displaystyle\int_{\mathbb{S}^1}\vec{w}\cdot\nabla_x U_{k-1}\mathrm{d}\vec{w}. \end{cases} \tag{2.10}$$

It is easy to see that $\bar{U}_k$ satisfies an elliptic equation. However, the boundary condition of $\bar{U}_k$ is unknown at this stage, since generally U_k does not necessarily satisfy the diffusive boundary condition of (2.1). Therefore, we have to resort to boundary layers.

Besides the Cartesian coordinate system for interior solutions, we need a local coordinate system in a neighborhood of the boundary to describe boundary layers.

Assume the Cartesian coordinate system is $\vec{x}=(x_1,x_2)$. Using polar coordinates system $(r,\theta)\in[0,\infty)\times[-\pi,\pi)$ and choosing pole in Ω, we assume $\vec{x}_0\in\partial\Omega$ is

$$\left\{\begin{array}{rcl} x_{1,0} &=& r(\theta)\cos\theta,\\ x_{2,0} &=& r(\theta)\sin\theta, \end{array}\right. \tag{2.11}$$

where $r(\theta)>0$ is a given function. Our local coordinate system is similar to the polar coordinate system, but varies to satisfy the specific requirements.

In a neighborhood of the boundary, for each θ, we have the outward unit normal vector

$$\vec{\nu}=\left(\frac{r(\theta)\cos\theta+r'(\theta)\sin\theta}{\sqrt{r(\theta)^2+r'(\theta)^2}},\frac{r(\theta)\sin\theta-r'(\theta)\cos\theta}{\sqrt{r(\theta)^2+r'(\theta)^2}}\right). \tag{2.12}$$

We can determine each point $\vec{x}\in\bar{\Omega}$ as $\vec{x}=\vec{x}_0-\mu\vec{\nu}$ where μ is the normal distance to a boundary point $\vec{x}_0$. In detail, this means

$$\left\{\begin{array}{rcl} x_1 &=& r(\theta)\cos\theta-\mu\dfrac{r(\theta)\cos\theta+r'(\theta)\sin\theta}{\sqrt{r(\theta)^2+r'(\theta)^2}},\\ x_2 &=& r(\theta)\sin\theta-\mu\dfrac{r(\theta)\sin\theta-r'(\theta)\cos\theta}{\sqrt{r(\theta)^2+r'(\theta)^2}}, \end{array}\right. \tag{2.13}$$

where $r'(\theta)=\dfrac{\mathrm{d}r}{\mathrm{d}\theta}$. It is easy to see that $\mu=0$ denotes the boundary $\partial\Omega$ and $\mu>0$ denotes the interior of Ω. (μ,θ) is the desired local coordinate system.

By chain rule (see [5]), we may deduce that

$$\frac{\partial\theta}{\partial x_1}=\frac{MP}{P^3+Q\mu},\quad \frac{\partial\mu}{\partial x_1}=-\frac{N}{P}, \tag{2.14}$$

$$\frac{\partial\theta}{\partial x_2}=\frac{NP}{P^3+Q\mu},\quad \frac{\partial\mu}{\partial x_2}=\frac{M}{P}, \tag{2.15}$$

where

$$P=(r^2+r'^2)^{\frac{1}{2}}, \tag{2.16}$$

$$Q=rr''-r^2-2r'^2, \tag{2.17}$$

$$M=-r\sin\theta+r'\cos\theta, \tag{2.18}$$

$$N=r\cos\theta+r'\sin\theta. \tag{2.19}$$

Therefore, note the fact that for C^2 convex domains, the curvature

$$\kappa(\theta)=\frac{r^2+2r'^2-rr''}{(r^2+r'^2)^{\frac{3}{2}}}, \tag{2.20}$$

and the radius of curvature

$$R_\kappa(\theta)=\frac{1}{\kappa(\theta)}=\frac{(r^2+r'^2)^{\frac{3}{2}}}{r^2+2r'^2-rr''}. \tag{2.21}$$

We define substitutions as follows:

Substitution 1:
Let $(x_1, x_2) \to (\mu, \theta)$ with $(\mu, \theta) \in [0, R_{\min}) \times [-\pi, \pi)$ for $R_{\min} = \min_\theta R_\kappa$ as

$$\begin{cases} x_1 = r(\theta)\cos\theta - \mu\dfrac{r(\theta)\cos\theta + r'(\theta)\sin\theta}{\sqrt{r(\theta)^2 + r'(\theta)^2}}, \\ x_2 = r(\theta)\sin\theta - \mu\dfrac{r(\theta)\sin\theta - r'(\theta)\cos\theta}{\sqrt{r(\theta)^2 + r'(\theta)^2}}, \end{cases} \tag{2.22}$$

and then the equation (2.1) is transformed into

$$\begin{cases} \epsilon\left(w_1\dfrac{-r\cos\theta - r'\sin\theta}{(r^2 + r'^2)^{\frac{1}{2}}} + w_2\dfrac{-r\sin\theta + r'\cos\theta}{(r^2 + r'^2)^{\frac{1}{2}}}\right)\dfrac{\partial u^\epsilon}{\partial \mu} \\ +\epsilon\left(w_1\dfrac{-r\sin\theta + r'\cos\theta}{(r^2 + r'^2)(1-\kappa\mu)} + w_2\dfrac{r\cos\theta + r'\sin\theta}{(r^2 + r'^2)(1-\kappa\mu)}\right)\dfrac{\partial u^\epsilon}{\partial \theta} + u^\epsilon - \bar{u}^\epsilon = 0, \\ u^\epsilon(0, \theta, \vec{w}) = g(\theta, \vec{w}) \quad \text{for} \quad \vec{w}\cdot\vec{\nu} < 0, \end{cases} \tag{2.23}$$

where

$$\vec{w}\cdot\vec{\nu} = w_1\frac{r\cos\theta + r'\sin\theta}{(r^2 + r'^2)^{\frac{1}{2}}} + w_2\frac{r\sin\theta - r'\cos\theta}{(r^2 + r'^2)^{\frac{1}{2}}}. \tag{2.24}$$

Noting the fact that

$$\left(\frac{M}{P}\right)^2 + \left(\frac{N}{P}\right)^2 = \left(\frac{-r\cos\theta - r'\sin\theta}{(r^2 + r'^2)^{\frac{1}{2}}}\right)^2 + \left(\frac{-r\sin\theta + r'\cos\theta}{(r^2 + r'^2)^{\frac{1}{2}}}\right)^2 = 1, \tag{2.25}$$

we can further simplify (2.23).

Substitution 2:
Let $\theta \to \tau$ with $\tau \in [-\pi, \pi)$ as

$$\begin{cases} \sin\tau = \dfrac{r\sin\theta - r'\cos\theta}{(r^2 + r'^2)^{\frac{1}{2}}}, \\ \cos\tau = \dfrac{r\cos\theta + r'\sin\theta}{(r^2 + r'^2)^{\frac{1}{2}}}, \end{cases} \tag{2.26}$$

which implies

$$\frac{d\tau}{d\theta} = \kappa(r^2 + r'^2)^{\frac{1}{2}} > 0. \tag{2.27}$$

Then the equation (2.1) is transformed into

$$\begin{cases} -\epsilon\left(w_1\cos\tau + w_2\sin\tau\right)\dfrac{\partial u^\epsilon}{\partial \mu} - \dfrac{\epsilon}{R_\kappa - \mu}\left(w_1\sin\tau - w_2\cos\tau\right)\dfrac{\partial u^\epsilon}{\partial \tau} + u^\epsilon - \bar{u}^\epsilon = 0, \\ u^\epsilon(0, \tau, \vec{w}) = g(\tau, \vec{w}) \quad \text{for} \quad \vec{w}\cdot\vec{\nu} < 0, \end{cases} \tag{2.28}$$

where

$$\vec{w}\cdot\vec{\nu}=w_1\cos\tau+w_2\sin\tau. \tag{2.29}$$

Substitution 3:
We further make the scaling transform for $\mu\rightarrow\eta$ with $\eta\in\left[0,\dfrac{R_{\min}}{\epsilon}\right)$ as

$$\eta=\frac{\mu}{\epsilon}, \tag{2.30}$$

which implies

$$\frac{\partial u^{\epsilon}}{\partial\mu}=\frac{1}{\epsilon}\frac{\partial u^{\epsilon}}{\partial\eta}. \tag{2.31}$$

Then the equation (2.1) is transformed into

$$\left\{\begin{array}{l} -\bigg(w_1\cos\tau+w_2\sin\tau\bigg)\dfrac{\partial u^{\epsilon}}{\partial\eta}-\dfrac{\epsilon}{R_{\kappa}-\epsilon\eta}\bigg(w_1\sin\tau-w_2\cos\tau\bigg)\dfrac{\partial u^{\epsilon}}{\partial\tau}+u^{\epsilon}-\bar{u}^{\epsilon}=0,\\ \\ u^{\epsilon}(0,\tau,\vec{w})=g(\tau,\vec{w})\ \ \text{for}\ \ \vec{w}\cdot\vec{\nu}<0, \end{array}\right. \tag{2.32}$$

where

$$\vec{w}\cdot\vec{\nu}=w_1\cos\tau+w_2\sin\tau. \tag{2.33}$$

Substitution 4:
Define the velocity substitution for $(w_1,w_2)\rightarrow\xi$ with $\xi\in[-\pi,\pi)$ as

$$\left\{\begin{array}{rcl} w_1&=&-\sin\xi\\ w_2&=&-\cos\xi \end{array}\right. \tag{2.34}$$

We have the succinct form of the equation (2.1) as

$$\left\{\begin{array}{l} \sin(\tau+\xi)\dfrac{\partial u^{\epsilon}}{\partial\eta}-\dfrac{\epsilon}{R_{\kappa}-\epsilon\xi}\cos(\tau+\xi)\dfrac{\partial u^{\epsilon}}{\partial\tau}+u^{\epsilon}-\bar{u}^{\epsilon}=0,\\ \\ u^{\epsilon}(0,\tau,\xi)=g(\tau,\xi)\ \ \text{for}\ \ \sin(\tau+\xi)>0. \end{array}\right. \tag{2.35}$$

Substitution 5:
As [21] and [5] reveal, we need a further rotational substitution for $\xi\rightarrow\phi$ with $\phi\in[-\pi,\pi)$ as

$$\phi=\tau+\xi \tag{2.36}$$

and achieve the form

$$\left\{\begin{array}{l} \sin\phi\dfrac{\partial u^{\epsilon}}{\partial\eta}-\dfrac{\epsilon}{R_{\kappa}-\epsilon\eta}\cos\phi\bigg(\dfrac{\partial u^{\epsilon}}{\partial\phi}+\dfrac{\partial u^{\epsilon}}{\partial\tau}\bigg)+u^{\epsilon}-\bar{u}^{\epsilon}=0\\ \\ u^{\epsilon}(0,\tau,\phi)=g(\tau,\phi)\ \ \text{for}\ \ \sin\phi>0. \end{array}\right. \tag{2.37}$$

This step is trying to compensate the variants of the normal vector ν along the boundary. A bi-product of such substitution is that we decompose the tangential derivative and introduce a new velocity derivative.

We define the boundary layer expansion as follows:

$$\mathcal{U}(\eta,\tau,\phi)\sim\mathcal{U}_0(\eta,\tau,\phi)+\epsilon\mathcal{U}_1(\eta,\tau,\phi), \tag{2.38}$$

where $\mathscr{U}_k$ can be determined by comparing the order of ϵ via plugging (2.38) into the equation (2.37). Thus, in a neighborhood of the boundary, we have

$$\sin\phi\frac{\partial\mathcal{U}_0}{\partial\eta}-\frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathcal{U}_0}{\partial\phi}+\mathcal{U}_0-\bar{\mathcal{U}}_0=0, \tag{2.39}$$

$$\sin\phi\frac{\partial\mathcal{U}_1}{\partial\eta}-\frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathcal{U}_1}{\partial\phi}+\mathcal{U}_1-\bar{\mathcal{U}}_1=\frac{1}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathcal{U}_0}{\partial\tau}, \tag{2.40}$$

where

$$\bar{\mathcal{U}}_k(\eta,\tau)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathcal{U}_k(\eta,\tau,\phi)\mathrm{d}\phi. \tag{2.41}$$

We call this type of equations the ϵ-Milne problem with geometric correction.

In this section, we prove the important decomposition of boundary data, which can greatly improve the regularity.

Consider the ϵ-Milne problem with geometric correction with $L=\epsilon^{-\frac{1}{2}}$ and $\mathscr{R}[\phi]=-\phi$,

$$\begin{cases}\sin\phi\dfrac{\partial f}{\partial\eta}-\dfrac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\dfrac{\partial f}{\partial\phi}+f-\bar{f}=0,\\ f(0,\phi)=g(\phi)\quad\text{for}\quad\sin\phi>0,\\ f(L,\phi)=f(L,\mathscr{R}[\phi]).\end{cases} \tag{2.42}$$

We assume that $g(\phi)$ is not a constant and $0\leq g(\phi)\leq 1$. This is always achievable and we do not lose the generality since the equation is linear. For some $\alpha>0$ which will be determined later, define two C^∞ auxiliary functions

$$g_1(\phi)=\begin{cases}0&\text{for}\quad\phi\in(0,\epsilon^\alpha]\cup[\pi-\epsilon^\alpha,\pi),\\ g(\phi)&\text{for}\quad\phi\in[2\epsilon^\alpha,\pi-2\epsilon^\alpha],\end{cases} \tag{2.43}$$

and

$$g_2(\phi)=\begin{cases}1&\text{for}\quad\phi\in(0,\epsilon^\alpha]\cup[\pi-\epsilon^\alpha,\pi),\\ g(\phi)&\text{for}\quad\phi\in[2\epsilon^\alpha,\pi-2\epsilon^\alpha].\end{cases} \tag{2.44}$$

Let $f_1(\eta,\phi)$ and $f_2(\eta,\phi)$ be the solutions to the equation (2.42) with in-flow data $g_1(\phi)$ and $g_2(\phi)$ respectively. Then by Theorem 4.8, we know f_1 and f_2 are well-defined in L^∞. By Theorem 4.10, they satisfy the maximum principle, which means

$$f_1(0,0^+)-\bar{f}_1(0)=f_1(0,\pi^-)-\bar{f}_1(0)=-\bar{f}_1(0)<0, \tag{2.45}$$

$$f_2(0,0^+)-\bar{f}_2(0)=f_2(0,\pi^-)-\bar{f}_2(0)=1-\bar{f}_2(0)>0. \tag{2.46}$$

Therefore, there exists a constant $0<\lambda<1$ such that

$$\lambda\Big(f_1(0,0^+)-\bar f_1(0)\Big)+(1-\lambda)\Big(f_2(0,0^+)-\bar f_2(0)\Big)=0, \tag{2.47}$$

$$\lambda\Big(f_1(0,\pi^-)-\bar f_1(0)\Big)+(1-\lambda)\Big(f_2(0,\pi^-)-\bar f_2(0)\Big)=0. \tag{2.48}$$

Let $g_\lambda(\phi)=\lambda g_1(\phi)+(1-\lambda)g_2(\phi)$ and the corresponding solution to the equation (2.42) is $f_\lambda(\eta,\phi)$. We have

$$f_\lambda(0,0^+)-\bar f_\lambda(0)=f_\lambda(0,\pi^-)-\bar f_\lambda(0)=0. \tag{2.49}$$

Since for $\phi\in(0,\epsilon^\alpha]\cup[\pi-\epsilon^\alpha,\pi)$, $g_\lambda=1-\lambda$ is a constant, we naturally have $\dfrac{\partial g_\lambda}{\partial\phi}=0$. We may solve from the equation (2.42) that

$$\frac{\partial f_\lambda}{\partial\eta}\bigg|_{\eta=0,\phi\in(0,\epsilon^\alpha]\cup[\pi-\epsilon^\alpha,\pi)}=\frac{1}{\sin\phi}\left(\frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial g_\lambda}{\partial\phi}\bigg|_{\phi\in(0,\epsilon^\alpha]\cup[\pi-\epsilon^\alpha,\pi)}-\Big(f_\lambda-\bar f_\lambda\Big)\bigg|_{\eta=0,\phi\in(0,\epsilon^\alpha]\cup[\pi-\epsilon^\alpha,\pi)}\right)=0. \tag{2.50}$$

Note that $g_\lambda(\phi)=g(\phi)$ for $\phi\in[2\epsilon^\alpha,\pi-2\epsilon^\alpha]$, so our modification is restricted to a small region near the grazing set and we can smoothen the normal derivative at the boundary.

This method can be easily generalized to treat other $g(\phi)$. In principle, for $g(\phi)\in C^1$, we can define a decomposition

$$g(\phi)=\mathscr{G}(\phi)+\mathfrak{G}(\phi), \tag{2.51}$$

such that $\mathfrak{G}(\phi)=0$ for $\sin\phi\geq 2\epsilon^\alpha$, and the solution to the equation (2.42) with in-flow data $\mathscr{G}(\phi)$ has L^∞ normal derivative at η=0. Such a decomposition comes with a price. Originally, we have $\left\|\dfrac{\partial g}{\partial\phi}\right\|_{L^\infty}\leq C$. However, now we only have $\left\|\dfrac{\partial\mathscr{G}}{\partial\phi}\right\|_{L^\infty}\leq C\epsilon^{-\alpha}$ and $\left\|\dfrac{\partial\mathfrak{G}}{\partial\phi}\right\|_{L^\infty}\leq C\epsilon^{-\alpha}$ due to the short-ranged cut-off function.

The bridge between the interior solution and boundary layer is the boundary condition of (2.1), so we first consider the boundary expansion:

$$U_0+\mathscr{U}_0+\mathfrak{U}_0=g, \tag{2.52}$$

$$U_1+\mathscr{U}_1=0. \tag{2.53}$$

Here $\mathscr{U}_0$ and $\mathfrak{U}_0$ are boundary layers with corresponding decomposed boundary data $\mathscr{G}$ and $\mathfrak{G}$. We call $\mathscr{U}$ the regular boundary layer and $\mathfrak{U}$ the singular boundary layer. They should both satisfy the ϵ-Milne problem with geometric correction.

Step 0: Preliminaries.
Define the weight function

$$\zeta(\eta,\phi)=\left(1-\left(\frac{R_\kappa-\epsilon\eta}{R_\kappa}\cos\phi\right)^2\right)^{\frac12}. \tag{2.54}$$

Define the force as

$$F(\epsilon;\eta,\tau)=-\frac{\epsilon}{R_\kappa(\tau)-\epsilon\eta}, \tag{2.55}$$

and the length for ϵ-Milne problem as $L=\epsilon^{-\frac{1}{2}}$. For $\phi\in[-\pi,\pi]$, denote $\mathscr{R}[\phi]=-\phi$.

Step 1: Construction of $\mathscr{U}_0$, $\mathfrak{U}_0$ and U_0.
Define the zeroth-order boundary layer as

$$\begin{cases}\mathscr{U}_0(\eta,\tau,\phi)=\mathscr{F}_0(\eta,\tau,\phi)-\mathscr{F}_{0,L}(\tau),\\ \sin\phi\dfrac{\partial\mathscr{F}_0}{\partial\eta}+F(\epsilon;\eta,\tau)\cos\phi\dfrac{\partial\mathscr{F}_0}{\partial\phi}+\mathscr{F}_0-\bar{\mathscr{F}}_0=0,\\ \mathscr{F}_0(0,\tau,\phi)=\mathscr{G}(\tau,\phi)\ \ \text{for}\ \ \sin\phi>0,\\ \mathscr{F}_0(L,\tau,\phi)=\mathscr{F}_0(L,\tau,\mathscr{R}[\phi]),\end{cases} \tag{2.56}$$

with $\mathscr{F}_{0,L}(\tau)$ is defined in Theorem 4.8, and

$$\begin{cases}\mathfrak{U}_0(\eta,\tau,\phi)=\mathfrak{F}_0(\eta,\tau,\phi)-\mathfrak{F}_{0,L}(\tau),\\ \sin\phi\dfrac{\partial\mathfrak{F}_0}{\partial\eta}+F(\epsilon;\eta,\tau)\cos\phi\dfrac{\partial\mathfrak{F}_0}{\partial\phi}+\mathfrak{F}_0-\bar{\mathfrak{F}}_0=0,\\ \mathfrak{F}_0(0,\tau,\phi)=\mathfrak{G}(\tau,\phi)\ \ \text{for}\ \ \sin\phi>0,\\ \mathfrak{F}_0(L,\tau,\phi)=\mathfrak{F}_0(L,\tau,\mathscr{R}[\phi]),\end{cases} \tag{2.57}$$

with $\mathfrak{F}_{0,L}(\tau)$ is defined in Theorem 4.8. Also, we define the zeroth-order interior solution $U_0(\vec{x},\vec{w})$ as

$$\begin{cases}U_0(\vec{x},\vec{w})=\bar{U}_0(\vec{x}),\\ \Delta_x\bar{U}_0(\vec{x})=0\ \ \text{in}\ \ \Omega,\\ \bar{U}_0(\vec{x}_0)=\mathscr{F}_{0,L}(\tau)+\mathfrak{F}_{0,L}(\tau)\ \ \text{on}\ \ \partial\Omega.\end{cases} \tag{2.58}$$

Step 2: Construction of $\mathscr{U}_1$ and U_1.
Define the first-order boundary layer as

$$\begin{cases}\mathscr{U}_1(\eta,\tau,\phi)=\mathscr{F}_1(\eta,\tau,\phi)-\mathscr{F}_{1,L}(\tau),\\ \sin\phi\dfrac{\partial\mathscr{F}_1}{\partial\eta}+F(\epsilon;\eta,\tau)\cos\phi\dfrac{\partial\mathscr{F}_1}{\partial\phi}+\mathscr{F}_1-\bar{\mathscr{F}}_1=\dfrac{1}{R_\kappa-\epsilon\eta}\cos\phi\dfrac{\partial\mathscr{U}_0}{\partial\tau},\\ \mathscr{F}_1(0,\tau,\phi)=\vec{w}\cdot\nabla_xU_0(0,\tau,\vec{w})\ \ \text{for}\ \ \sin\phi>0,\\ \mathscr{F}_1(L,\tau,\phi)=\mathscr{F}_1(L,\tau,\mathscr{R}[\phi]),\end{cases} \tag{2.59}$$

with $\mathscr{F}_{1,L}(\tau)$ is defined in Theorem 4.8. Then we define the first-order interior solution $U_1(\vec{x},\vec{w})$ as

$$\begin{cases}U_1(\vec{x},\vec{w})&=&\bar{U}_1(\vec{x})-\vec{w}\cdot\nabla_xU_0(\vec{x},\vec{w}),\\ \Delta_x\bar{U}_1(\vec{x})&=&-\displaystyle\int_{\mathbb{S}^1}\Big(\vec{w}\cdot\nabla_xU_0(\vec{x},\vec{w})\Big)\mathrm{d}\vec{w}\ \ \text{in}\ \ \Omega,\\ \bar{U}_1(\vec{x}_0)&=&f_{1,L}(\tau)\ \ \text{on}\ \ \partial\Omega.\end{cases} \tag{2.60}$$

Note that we do not define $\mathfrak{U}_1$ here.

Step 3: Construction of U_2.
Since we do not expand to $\mathscr{U}_2$ and $\mathfrak{U}_2$, we define the second-order interior solution as

$$\left\{\begin{array}{rcl}U_2(\vec{x},\vec{w}) &=& \bar{U}_2(\vec{x})-\vec{w}\cdot\nabla_xU_1(\vec{x},\vec{w}),\\ \Delta_x\bar{U}_2(\vec{x}) &=& -\displaystyle\int_{\mathbb{S}^1}\Big(\vec{w}\cdot\nabla_xU_1(\vec{x},\vec{w})\Big)\mathrm{d}\vec{w}\ \ \text{in}\ \ \Omega,\\ \bar{U}_2(\vec{x}_0) &=& 0\ \ \text{on}\ \ \partial\Omega.\end{array}\right. \tag{2.61}$$

Here, we might have $O(\epsilon^3)$ error in this step due to the trivial boundary data. Thanks to the remainder estimate, it will not affect the diffusive limit.

3. Remainder Estimate

In this section, we consider the remainder equation for $u(\vec{x},\vec{w})$ as

$$\begin{cases} \epsilon\vec{w}\cdot\nabla_x u+u-\bar{u}=f(\vec{x},\vec{w}) \ \text{ in } \ \Omega,\\ u(\vec{x}_0,\vec{w})=h(\vec{x}_0,\vec{w}) \ \text{ for } \ \vec{w}\cdot\vec{\nu}<0 \ \text{ and } \ \vec{x}_0\in\partial\Omega, \end{cases} \tag{3.1}$$

where

$$\bar{u}(\vec{x})=\frac{1}{2\pi}\int_{\mathbb{S}^1}u(\vec{x},\vec{w})\mathrm{d}\vec{w}, \tag{3.2}$$

$\vec{\nu}$ is the outward unit normal vector, with the Knudsen number $0<\epsilon<<1$.
We define the L^p norm with $1\leq p<\infty$ and L^∞ norms in $\Omega\times\mathbb{S}^1$ as usual:

$$\begin{aligned} \|f\|_{L^p(\Omega\times\mathbb{S}^1)} &= \bigg(\int_\Omega\int_{\mathbb{S}^1}|f(\vec{x},\vec{w})|^p\,\mathrm{d}\vec{w}\mathrm{d}\vec{x}\bigg)^{\frac{1}{p}}, && (3.3)\\ \|f\|_{L^\infty(\Omega\times\mathbb{S}^1)} &= \sup_{(\vec{x},\vec{w})\in\Omega\times\mathbb{S}^1}|f(\vec{x},\vec{w})|. && (3.4) \end{aligned}$$

Define the L^p norm with $1\leq p<\infty$ and L^∞ norms on the boundary as follows:

$$\begin{aligned} \|f\|_{L^p(\Gamma)} &= \bigg(\iint_\Gamma|f(\vec{x},\vec{w})|^p\,|\vec{w}\cdot\vec{\nu}|\,\mathrm{d}\vec{w}\mathrm{d}\vec{x}\bigg)^{\frac{1}{p}}, && (3.5)\\ \|f\|_{L^p(\Gamma^\pm)} &= \bigg(\iint_{\Gamma^\pm}|f(\vec{x},\vec{w})|^p\,|\vec{w}\cdot\vec{\nu}|\,\mathrm{d}\vec{w}\mathrm{d}\vec{x}\bigg)^{\frac{1}{p}}, && (3.6)\\ \|f\|_{L^\infty(\Gamma)} &= \sup_{(\vec{x},\vec{w})\in\Gamma}|f(\vec{x},\vec{w})|, && (3.7)\\ \|f\|_{L^\infty(\Gamma^\pm)} &= \sup_{(\vec{x},\vec{w})\in\Gamma^\pm}|f(\vec{x},\vec{w})|. && (3.8) \end{aligned}$$

In particular, we denote $\mathrm{d}\gamma=(\vec{w}\cdot\vec{\nu})\mathrm{d}\vec{w}\mathrm{d}\vec{x}_0$ on the boundary.

The remainder estimates for neutron transport equation with diffusive boundary was proved in [5] and [6]. Here, the case with in-flow boundary was first shown in [21], so here we will focus on the a priori estimates and prove an improved version.

Lemma 3.1. *(Green's Identity) Assume $u(\vec{x},\vec{w}),\ v(\vec{x},\vec{w})\in L^2(\Omega\times\mathbb{S}^1)$ and $\vec{w}\cdot\nabla_x u,\ \vec{w}\cdot\nabla_x v\in L^2(\Omega\times\mathbb{S}^1)$ with $u,\ v\in L^2(\Gamma)$. Then*

$$\iint_{\Omega\times\mathbb{S}^1}\bigg((\vec{w}\cdot\nabla_x u)v+(\vec{w}\cdot\nabla_x u)v\bigg)\mathrm{d}\vec{x}\mathrm{d}\vec{w}=\int_\Gamma uv\mathrm{d}\gamma. \tag{3.9}$$

Proof. See [2, Chapter 9] and [3]. □

Lemma 3.2. *The unique solution $u(\vec{x},\vec{w})$ to the equation (3.1) satisfies*

$$\frac{1}{\epsilon^{\frac{1}{2}}}\|u\|_{L^2(\Gamma^+)}+\|u\|_{L^2(\Omega\times\mathbb{S}^1)}\leq C\bigg(\frac{1}{\epsilon^2}\|f\|_{L^2(\Omega\times\mathbb{S}^1)}+\frac{1}{\epsilon^{\frac{1}{2}}}\|h\|_{L^2(\Gamma^-)}\bigg). \tag{3.10}$$

Proof. We divide the proof into several steps:

Step 1: Kernel Estimate.
Applying Lemma 3.1 to the equation (3.1). Then for any $\phi\in L^2(\Omega\times\mathbb{S}^1)$ satisfying $\vec w\cdot\nabla_x\phi\in L^2(\Omega\times\mathbb{S}^1)$ and $\phi\in L^2(\Gamma)$, we have

$$\epsilon\int_{\Gamma}u\phi\mathrm{d}\gamma-\epsilon\iint_{\Omega\times\mathbb{S}^1}(\vec w\cdot\nabla_x\phi)u+\iint_{\Omega\times\mathbb{S}^1}(u-\bar u)\phi=\iint_{\Omega\times\mathbb{S}^1}f\phi. \tag{3.11}$$

Our goal is to choose a particular test function ϕ. We first construct an auxiliary function ξ. Since $u\in L^2(\Omega\times\mathbb{S}^1)$, it naturally implies $\bar u\in L^2(\Omega)$. We define $\xi(\vec x)$ on Ω satisfying

$$\left\{\begin{array}{rcl}\Delta\xi&=&\bar u\ \text{ in }\ \Omega,\\ \xi&=&0\ \text{ on }\ \partial\Omega.\end{array}\right. \tag{3.12}$$

Hence, in the bounded domain Ω, based on the standard elliptic estimate, there exists a unique $\xi\in H^2(\Omega)$ such that

$$\|\xi\|_{H^2(\Omega)}\leq C\|\bar u\|_{L^2(\Omega)}. \tag{3.13}$$

We plug the test function

$$\phi=-\vec w\cdot\nabla_x\xi \tag{3.14}$$

into the weak formulation (3.11) and estimate each term there. Naturally, we have

$$\|\phi\|_{L^2(\Omega)}\leq C\|\xi\|_{H^1(\Omega)}\leq C\|\bar u\|_{L^2(\Omega)}. \tag{3.15}$$

Easily we can decompose

$$-\epsilon\iint_{\Omega\times\mathbb{S}^1}(\vec w\cdot\nabla_x\phi)u=-\epsilon\iint_{\Omega\times\mathbb{S}^1}(\vec w\cdot\nabla_x\phi)\bar u-\epsilon\iint_{\Omega\times\mathbb{S}^1}(\vec w\cdot\nabla_x\phi)(u-\bar u). \tag{3.16}$$

We estimate the two term on the right-hand side separately. By (3.12) and (3.14), we have

$$\begin{aligned}-\epsilon\iint_{\Omega\times\mathbb{S}^1}(\vec w\cdot\nabla_x\phi)\bar u&=\epsilon\iint_{\Omega\times\mathbb{S}^1}\bar u\Big(w_1(w_1\partial_{11}\xi+w_2\partial_{12}\xi)+w_2(w_1\partial_{12}\xi+w_2\partial_{22}\xi)\Big)\\&=\epsilon\iint_{\Omega\times\mathbb{S}^1}\bar u\Big(w_1^2\partial_{11}\xi+w_2^2\partial_{22}\xi\Big)\\&=2\epsilon\pi\int_{\Omega}\bar u(\partial_{11}\xi+\partial_{22}\xi)\\&=2\epsilon\pi\|\bar u\|_{L^2(\Omega)}^2\\&=\epsilon\|\bar u\|_{L^2(\Omega\times\mathbb{S}^1)}^2.\end{aligned} \tag{3.17}$$

In the second equality, above cross terms vanish due to the symmetry of the integral over $\mathbb{S}^1$. On the other hand, for the second term in (3.16), Hölder's inequality and the elliptic estimate imply

$$-\epsilon\iint_{\Omega\times\mathbb{S}^1}(\vec{w}\cdot\nabla_x\phi)(u-\bar{u})\leq C\epsilon\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}\left\|\xi\right\|_{H^2(\Omega)} \tag{3.18}$$
$$\leq C\epsilon\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}\left\|\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}.$$

Using the trace theorem, we have

$$\epsilon\int_\Gamma u\phi\mathrm{d}\gamma=\epsilon\int_{\Gamma^+}u\phi\mathrm{d}\gamma+\epsilon\int_{\Gamma^-}u\phi\mathrm{d}\gamma\leq C\epsilon\left\|\phi\right\|_{L^2(\Gamma)}\bigg(\left\|u\right\|_{L^2(\Gamma^+)}+\left\|h\right\|_{L^2(\Gamma^-)}\bigg) \tag{3.19}$$
$$\leq C\epsilon\left\|\phi\right\|_{H^1(\Omega\times\mathbb{S}^1)}\bigg(\left\|u\right\|_{L^2(\Gamma^+)}+\left\|h\right\|_{L^2(\Gamma^-)}\bigg)\leq C\epsilon\left\|\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}\bigg(\left\|u\right\|_{L^2(\Gamma^+)}+\left\|h\right\|_{L^2(\Gamma^-)}\bigg).$$

Also, we obtain

$$\iint_{\Omega\times\mathbb{S}^1}(u-\bar{u})\phi\leq C\left\|\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}, \tag{3.20}$$

$$\iint_{\Omega\times\mathbb{S}^1}f\phi\leq C\left\|\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}\left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)}. \tag{3.21}$$

Collecting terms in (3.17), (3.18), (3.19), (3.20) and (3.21), we obtain

$$\epsilon\left\|\bar{u}\right\|^2_{L^2(\Omega\times\mathbb{S}^1)}\leq C\left\|\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}\bigg(\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}+\epsilon\left\|u\right\|_{L^2(\Gamma^+)}+\left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)}+\epsilon\left\|h\right\|_{L^2(\Gamma^-)}\bigg). \tag{3.22}$$

Then this naturally implies that

$$\epsilon\left\|\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}\leq C\bigg(\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}+\epsilon\left\|u\right\|_{L^2(\Gamma^+)}+\left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)}+\epsilon\left\|h\right\|_{L^2(\Gamma^-)}\bigg). \tag{3.23}$$

Step 2: Energy Estimate.
In the weak formulation (3.11), we may take the test function $\phi=u$ to get the energy estimate

$$\frac{1}{2}\epsilon\int_\Gamma\left|u\right|^2\mathrm{d}\gamma+\left\|u-\bar{u}\right\|^2_{L^2(\Omega\times\mathbb{S}^1)}=\iint_{\Omega\times\mathbb{S}^1}fu. \tag{3.24}$$

Then we have

$$\frac{1}{2}\epsilon\left\|u\right\|^2_{L^2(\Gamma^+)}+\left\|u-\bar{u}\right\|^2_{L^2(\Omega\times\mathbb{S}^1)}=\iint_{\Omega\times\mathbb{S}^1}fu+\epsilon\left\|h\right\|^2_{L^2(\Gamma^-)}. \tag{3.25}$$

On the other hand, we can square on both sides of (3.36) to obtain

$$\epsilon^2\left\|\bar{u}\right\|^2_{L^2(\Omega\times\mathbb{S}^1)}\leq C\bigg(\left\|u-\bar{u}\right\|^2_{L^2(\Omega\times\mathbb{S}^1)}+\epsilon^2\left\|u\right\|^2_{L^2(\Gamma^+)}+\left\|f\right\|^2_{L^2(\Omega\times\mathbb{S}^1)}+\epsilon^2\left\|h\right\|^2_{L^2(\Gamma^-)}\bigg). \tag{3.26}$$

Multiplying a sufficiently small constant on both sides of (3.26) and adding it to (3.25) to absorb $\left\|u\right\|^2_{L^2(\Gamma^+)}$ and $\left\|u-\bar{u}\right\|^2_{L^2(\Omega\times\mathbb{S}^1)}$, we deduce

$$\epsilon\left\|u\right\|^2_{L^2(\Gamma^+)}+\epsilon^2\left\|\bar{u}\right\|^2_{L^2(\Omega\times\mathbb{S}^1)}+\left\|u-\bar{u}\right\|^2_{L^2(\Omega\times\mathbb{S}^1)}\leq C\bigg(\left\|f\right\|^2_{L^2(\Omega\times\mathbb{S}^1)}+\iint_{\Omega\times\mathbb{S}^1}fu+\epsilon\left\|h\right\|^2_{L^2(\Gamma^-)}\bigg). \tag{3.27}$$

Hence, we have

$$\epsilon\left\|u\right\|_{L^2(\Gamma^+)}^2+\epsilon^2\left\|u\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2\leq C\bigg(\left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2+\iint_{\Omega\times\mathbb{S}^1}fu+\epsilon\left\|h\right\|_{L^2(\Gamma^-)}^2\bigg). \tag{3.28}$$

A simple application of Cauchy's inequality leads to

$$\iint_{\Omega\times\mathbb{S}^1}fu\leq\frac{1}{4C\epsilon^2}\left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2+C\epsilon^2\left\|u\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2. \tag{3.29}$$

Taking C sufficiently small to absorb $C\epsilon^2\left\|u\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2$, we obtain

$$\epsilon\left\|u\right\|_{L^2(\Gamma^+)}^2+\epsilon^2\left\|u\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2\leq C\bigg(\frac{1}{\epsilon^2}\left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2+\epsilon\left\|h\right\|_{L^2(\Gamma^-)}^2\bigg). \tag{3.30}$$

Then we can divide ϵ^2 on both sides of (3.30) to obtain

$$\frac{1}{\epsilon}\left\|u\right\|_{L^2(\Gamma^+)}^2+\left\|u\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2\leq C\bigg(\frac{1}{\epsilon^4}\left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2+\frac{1}{\epsilon}\left\|h\right\|_{L^2(\Gamma^-)}^2\bigg). \tag{3.31}$$

Hence, we naturally have

$$\frac{1}{\epsilon^{\frac{1}{2}}}\left\|u\right\|_{L^2(\Gamma^+)}+\left\|u\right\|_{L^2(\Omega\times\mathbb{S}^1)}\leq C\bigg(\frac{1}{\epsilon^2}\left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)}+\frac{1}{\epsilon^{\frac{1}{2}}}\left\|h\right\|_{L^2(\Gamma^-)}\bigg). \tag{3.32}$$

□

Theorem 3.3. *The unique solution $u(\vec x,\vec w)$ to the equation (3.1) satisfies*

$$\left\|u\right\|_{L^\infty(\Omega\times\mathbb{S}^1)}\leq C\bigg(\frac{1}{\epsilon^3}\left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)}+\left\|f\right\|_{L^\infty(\Omega\times\mathbb{S}^1)}+\frac{1}{\epsilon^{\frac{3}{2}}}\left\|h\right\|_{L^2(\Gamma^-)}+\left\|h\right\|_{L^\infty(\Gamma^-)}\bigg). \tag{3.33}$$

Proof. We divide the proof into several steps:

Step 1: Double Duhamel iterations.
We can rewrite the equation (3.1) along the characteristics as

$$\begin{aligned}u(\vec x,\vec w)={}&h(\vec x-\epsilon t_b\vec w,\vec w)\mathrm{e}^{-t_b}+\int_0^{t_b}f\Big(\vec x-\epsilon(t_b-s)\vec w,\vec w\Big)\mathrm{e}^{-(t_b-s)}\mathrm{d}s\\&+\frac{1}{2\pi}\int_0^{t_b}\bigg(\int_{\mathbb{S}^1}u\Big(\vec x-\epsilon(t_b-s)\vec w,\vec w_t\Big)\mathrm{d}\vec w_t\bigg)\mathrm{e}^{-(t_b-s)}\mathrm{d}s.\end{aligned} \tag{3.34}$$

where the backward exit time t_b is defined as

$$t_b(\vec x,\vec w)=\inf\{t\geq0:(\vec x-\epsilon t\vec w,\vec w)\in\Gamma^-\}, \tag{3.35}$$

which represents the first time that the characteristics track back and hit the in-flow boundary. Note we have replaced $\bar{u}$ by the integral of u over the dummy velocity variable $\vec{w}_t$. For the last term in this formulation, we apply the Duhamel's principle again to $u\Big(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w}_t\Big)$ and obtain

$$\begin{aligned}u(\vec{x},\vec{w}) = {}& h(\vec{x}-\epsilon t_b\vec{w},\vec{w})\mathrm{e}^{-t_b} + \int_0^{t_b} f\Big(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w}\Big)\mathrm{e}^{-(t_b-s)}\mathrm{d}s \\ & + \frac{1}{2\pi}\int_0^{t_b}\bigg(\int_{\mathbb{S}^1} h\Big(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon s_b\vec{w}_t,\vec{w}_t\Big)\mathrm{e}^{-s_b}\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-(t_b-s)}\mathrm{d}s \\ & + \frac{1}{2\pi}\int_0^{t_b}\bigg(\int_{\mathbb{S}^1}\bigg(\int_0^{s_b} f\Big(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t,\vec{w}_t\Big)\mathrm{e}^{-(s_b-r)}\mathrm{d}r\bigg)\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-(t_b-s)}\mathrm{d}s \\ & + \frac{1}{2\pi}\int_0^{t_b}\bigg(\int_{\mathbb{S}^1}\bigg(\int_0^{s_b} \bar{u}\Big(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t\Big)\mathrm{e}^{-(s_b-r)}\mathrm{d}r\bigg)\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-(t_b-s)}\mathrm{d}s,\end{aligned} \tag{3.36}$$

where the exiting time from $\Big(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w}_t\Big)$ is defined as

$$s_b(\vec{x},\vec{w},s,\vec{w}_t) = \inf\{r\geq 0 : \Big(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon r\vec{w}_t,\vec{w}_t\Big)\in\Gamma^-\}. \tag{3.37}$$

Step 2: Estimates of all but the last term in (3.36).
We can directly estimate as follows:

$$\Big|h(\vec{x}-\epsilon t_b\vec{w},\vec{w})\mathrm{e}^{-t_b}\Big| \leq \|h\|_{L^\infty(\Gamma^-)}, \tag{3.38}$$

$$\left|\frac{1}{2\pi}\int_0^{t_b}\bigg(\int_{\mathbb{S}^1} h\Big(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon s_b\vec{w}_t,\vec{w}_t\Big)\mathrm{e}^{-s_b}\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-(t_b-s)}\mathrm{d}s\right| \leq \|h\|_{L^\infty(\Gamma^-)}, \tag{3.39}$$

$$\left|\int_0^{t_b} f\Big(\vec{x}-\epsilon(t_b-s)\vec{w},\vec{w}\Big)\mathrm{e}^{-(t_b-s)}\mathrm{d}s\right| \leq \|f\|_{L^\infty(\Omega\times\mathbb{S}^1)}, \tag{3.40}$$

$$\left|\frac{1}{2\pi}\int_0^{t_b}\bigg(\int_{\mathbb{S}^1}\bigg(\int_0^{s_b} f(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t,\vec{w}_t)\mathrm{e}^{-(s_b-r)}\mathrm{d}r\bigg)\mathrm{d}\vec{w}_t\bigg)\mathrm{e}^{-(t_b-s)}\mathrm{d}s\right| \leq \|f\|_{L^\infty(\Omega\times\mathbb{S}^1)}. \tag{3.41}$$

Step 3: Estimates of the last term in (3.36).
Now we decompose the last term in (3.36) as

$$\int_0^{t_b}\int_{\mathbb{S}^1}\int_0^{s_b} = \int_0^{t_b}\int_{\mathbb{S}^1}\int_{s_b-r\leq\delta} + \int_0^{t_b}\int_{\mathbb{S}^1}\int_{s_b-r\geq\delta} = I_1+I_2, \tag{3.42}$$

for some $\delta>0$. We can estimate I_1 directly as

$$I_1 \leq \int_0^{t_b}\mathrm{e}^{-(t_b-s)}\bigg(\int_{\max(0,s_b-\delta)}^{s_b}\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)}\mathrm{d}r\bigg)\mathrm{d}s \leq \delta\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)}. \tag{3.43}$$

Then we can bound I_2 as

$$I_2 \leq C\int_0^{t_b}\int_{\mathbb{S}^1}\int_0^{\max(0,s_b-\delta)} \left|\bar{u}\Big(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t\Big)\right| \mathrm{e}^{-(t_b-s)}\mathrm{d}r\mathrm{d}\vec{w}_t\mathrm{d}s. \tag{3.44}$$

By the definition of t_b and s_b, we always have $\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t\in\bar{\Omega}$. Hence, we may interchange the order of integration and apply Hölder's inequality to obtain

$$\begin{aligned} I_2 \leq{}& C\int_0^{t_b}\Bigg(\int_0^{\max(0,s_b-\delta)}\int_{\mathbb{S}^1}\mathbf{1}_\Omega\Big(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t\Big) \\ &\left|\bar{u}\Big(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t\Big)\right|\mathrm{d}\vec{w}_t\mathrm{d}r\Bigg)\mathrm{e}^{-(t_b-s)}\mathrm{d}s \\ \leq{}& C\int_0^{t_b}\Bigg(\Bigg(\int_{\mathbb{S}^1}\int_0^{\max(0,s_b-\delta)}\mathbf{1}_\Omega\Big(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t\Big) \\ &\left|\bar{u}\Big(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t\Big)\right|^2\mathrm{d}\vec{w}_t\mathrm{d}r\Bigg)^{\frac{1}{2}} \\ &\times\Bigg(\int_{\mathbb{S}^1}\int_0^{\max(0,s_b-\delta)}\mathbf{1}_\Omega\Big(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t\Big)\mathrm{d}\vec{w}_t\mathrm{d}r\Bigg)^{\frac{1}{2}}\Bigg)\mathrm{e}^{-(t_b-s)}\mathrm{d}s \end{aligned} \tag{3.45}$$

Note $\vec{w}_t\in\mathbb{S}^1$, which is essentially a one-dimensional variable. Thus, we may write it in a new variable ψ as $\vec{w}_t=(\cos\psi,\sin\psi)$. Then we define the change of variable $[-\pi,\pi)\times\mathbb{R}\to\Omega:(\psi,r)\to(y_1,y_2)=\vec{y}=\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t$, i.e.

$$\left\{\begin{array}{rcl} y_1 &=& x_1-\epsilon(t_b-s)w_1-\epsilon(s_b-r)\cos\psi, \\ y_2 &=& x_2-\epsilon(t_b-s)w_2-\epsilon(s_b-r)\sin\psi. \end{array}\right. \tag{3.46}$$

Therefore, for $s_b-r\geq\delta$, we can directly compute the Jacobian

$$\left|\frac{\partial(y_1,y_2)}{\partial(\psi,r)}\right| = \left\|\begin{array}{cc} -\epsilon(s_b-r)\sin\psi & \epsilon\cos\psi \\ \epsilon(s_b-r)\cos\psi & \epsilon\sin\psi \end{array}\right\| = \epsilon^2(s_b-r)\geq\epsilon^2\delta. \tag{3.47}$$

Hence, we may simplify (3.45) as

$$\begin{aligned} I_2 \leq{}& C\int_0^{t_b}\left(\int_\Omega\frac{1}{\epsilon^2\delta}\left|\bar{u}(\vec{y})\right|^2\mathrm{d}\vec{y}\right)^{\frac{1}{2}}\mathrm{e}^{-(t_b-s)}\mathrm{d}s \\ \leq{}& \frac{C}{\epsilon\sqrt{\delta}}\int_0^{t_b}\left(\int_\Omega\left|\bar{u}(\vec{y})\right|^2\mathrm{d}\vec{y}\right)^{\frac{1}{2}}\mathrm{e}^{-(t_b-s)}\mathrm{d}s \\ \leq{}& \frac{C}{\epsilon\delta^{\frac{1}{2}}}\left\|\bar{u}\right\|_{L^2(\Omega)}. \end{aligned} \tag{3.48}$$

Step 4: Synthesis.
In summary, collecting (3.38), (3.39), (3.40), (3.41), (3.43) and (3.48), for fixed $0<\delta<1$, we have

$$\left\|u\right\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leq \delta\left\|u\right\|_{L^\infty(\Omega\times\mathbb{S}^1)}+\frac{C}{\epsilon\delta^{\frac{1}{2}}}\left\|\bar{u}\right\|_{L^2(\Omega)}+C\bigg(\left\|f\right\|_{L^\infty(\Omega\times\mathbb{S}^1)}+\left\|h\right\|_{L^\infty(\Gamma^-)}\bigg). \tag{3.49}$$

Then taking δ small to absorb $\delta\left\|u\right\|_{L^\infty(\Omega\times\mathbb{S}^1)}$ into the left-hand side to get

$$\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leq \frac{C}{\epsilon}\|\bar{u}\|_{L^2(\Omega)} + C\bigg(\|f\|_{L^\infty(\Omega\times\mathbb{S}^1)} + \|h\|_{L^\infty(\Gamma^-)}\bigg). \tag{3.50}$$

Using Theorem 3.2, we get

$$\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leq C\bigg(\frac{1}{\epsilon^3}\|f\|_{L^2(\Omega\times\mathbb{S}^1)} + \|f\|_{L^\infty(\Omega\times\mathbb{S}^1)} + \frac{1}{\epsilon^{\frac{3}{2}}}\|h\|_{L^2(\Gamma^-)} + \|h\|_{L^\infty(\Gamma^-)}\bigg). \tag{3.51}$$

$\square$

In this subsection, we try to improve previous estimates. In the following, we assume $m > 2$ is an integer and let $o(1)$ denote a sufficiently small constant.

Theorem 3.4. *The unique solution $u(\vec{x},\vec{w})$ to the equation (3.1) satisfies*

$$\begin{aligned}&\frac{1}{\epsilon^{\frac{1}{2}}}\|u\|_{L^2(\Gamma^+)} + \|\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)} + \frac{1}{\epsilon}\|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)} \\ \leq{}& C\bigg(o(1)\epsilon^{\frac{1}{m}}\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)} + \frac{1}{\epsilon}\|f\|_{L^2(\Omega\times\mathbb{S}^1)} + \frac{1}{\epsilon^2}\|f\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)} + \frac{1}{\epsilon^{\frac{1}{2}}}\|h\|_{L^2(\Gamma^-)} + \|h\|_{L^m(\Gamma^-)}\bigg).\end{aligned} \tag{3.52}$$

Proof. We divide the proof into several steps:

Step 1: Kernel Estimate.
Applying Green's identity to the equation (3.1). Then for any $\phi \in L^2(\Omega\times\mathbb{S}^1)$ satisfying $\vec{w}\cdot\nabla_x\phi \in L^2(\Omega\times\mathbb{S}^1)$ and $\phi \in L^2(\Gamma)$, we have

$$\epsilon\int_\Gamma u\phi \mathrm{d}\gamma - \epsilon\iint_{\Omega\times\mathbb{S}^1}(\vec{w}\cdot\nabla_x\phi)u + \iint_{\Omega\times\mathbb{S}^1}(u-\bar{u})\phi = \iint_{\Omega\times\mathbb{S}^1} f\phi. \tag{3.53}$$

Our goal is to choose a particular test function ϕ. We first construct an auxiliary function ξ. Naturally $u \in L^\infty(\Omega\times\mathbb{S}^1)$ implies $\bar{u} \in L^{2m}(\Omega)$ which further leads to $(\bar{u})^{2m-1} \in L^{\frac{2m}{2m-1}}(\Omega)$. We define $\xi(\vec{x})$ on Ω satisfying

$$\left\{\begin{array}{rcl}\Delta\xi &=& (\bar{u})^{2m-1} \ \text{ in } \ \Omega,\\ \xi &=& 0 \ \text{ on } \ \partial\Omega.\end{array}\right. \tag{3.54}$$

In the bounded domain Ω, based on the standard elliptic estimates, we have a unique ξ satisfying

$$\|\xi\|_{W^{2,\frac{2m}{2m-1}}(\Omega)} \leq C\left\|(\bar{u})^{2m-1}\right\|_{L^{\frac{2m}{2m-1}}(\Omega)} = C\|\bar{u}\|_{L^{2m}(\Omega)}^{2m-1}. \tag{3.55}$$

We plug the test function

$$\phi = -\vec{w}\cdot\nabla_x\xi \tag{3.56}$$

into the weak formulation (3.53) and estimate each term there. By Sobolev embedding theorem, we have

$$\|\phi\|_{L^2(\Omega)} \leq C\|\xi\|_{H^1(\Omega)} \leq C\|\xi\|_{W^{2,\frac{2m}{2m-1}}(\Omega)} \leq C\|\bar{u}\|_{L^{2m}(\Omega)}^{2m-1}, \tag{3.57}$$

$$\|\phi\|_{L^{\frac{2m}{2m-1}}(\Omega)} \leq C\|\xi\|_{W^{1,\frac{2m}{2m-1}}(\Omega)} \leq C\|\bar{u}\|_{L^{2m}(\Omega)}^{2m-1}. \tag{3.58}$$

Easily we can decompose

$$-\epsilon\iint_{\Omega\times\mathbb{S}^1}(\vec{w}\cdot\nabla_x\phi)u = -\epsilon\iint_{\Omega\times\mathbb{S}^1}(\vec{w}\cdot\nabla_x\phi)\bar{u} - \epsilon\iint_{\Omega\times\mathbb{S}^1}(\vec{w}\cdot\nabla_x\phi)(u-\bar{u}). \tag{3.59}$$

We estimate the two term on the right-hand side of (3.59) separately. By (3.54) and (3.56), we have

$$\begin{aligned}
-\epsilon\iint_{\Omega\times\mathbb{S}^1}(\vec{w}\cdot\nabla_x\phi)\bar{u} &= \epsilon\iint_{\Omega\times\mathbb{S}^1}\bar{u}\bigg(w_1(w_1\partial_{11}\xi+w_2\partial_{12}\xi)+w_2(w_1\partial_{12}\xi+w_2\partial_{22}\xi)\bigg) \\
&= \epsilon\iint_{\Omega\times\mathbb{S}^1}\bar{u}\bigg(w_1^2\partial_{11}\xi+w_2^2\partial_{22}\xi\bigg) \\
&= 2\epsilon\pi\int_\Omega\bar{u}(\partial_{11}\xi+\partial_{22}\xi) \\
&= \epsilon\|\bar{u}\|_{L^{2m}(\Omega)}^{2m}.
\end{aligned} \tag{3.60}$$

In the second equality, the cross terms vanish due to the symmetry of the integral over $\mathbb{S}^1$. On the other hand, for the second term in (3.59), Hölder's inequality and the elliptic estimate imply

$$\begin{aligned}
-\epsilon\iint_{\Omega\times\mathbb{S}^1}(\vec{w}\cdot\nabla_x\phi)(u-\bar{u}) &\leq C\epsilon\|u-\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)}\|\nabla_x\phi\|_{L^{\frac{2m}{2m-1}}(\Omega)} \\
&\leq C\epsilon\|u-\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)}\|\xi\|_{W^{2,\frac{2m}{2m-1}}(\Omega)} \\
&\leq C\epsilon\|u-\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)}\|\bar{u}\|_{L^{2m}(\Omega)}^{2m-1}.
\end{aligned} \tag{3.61}$$

Based on (3.55), (3.57), (3.58), Sobolev embedding theorem and the trace theorem, we have

$$\|\nabla_x\xi\|_{L^{\frac{m}{m-1}}(\Gamma)} \leq C\|\nabla_x\xi\|_{W^{\frac{1}{2m},\frac{2m}{2m-1}}(\Gamma)} \leq C\|\nabla_x\xi\|_{W^{1,\frac{2m}{2m-1}}(\Omega)} \leq C\|\xi\|_{W^{2,\frac{2m}{2m-1}}(\Omega)} \leq C\|\bar{u}\|_{L^{2m}(\Omega)}^{2m-1}. \tag{3.62}$$

Based on (3.55), (3.58) and Hölder's inequality, we have

$$\begin{aligned}
\epsilon\int_\Gamma u\phi\mathrm{d}\gamma &= \epsilon\int_{\Gamma^+}u\phi\mathrm{d}\gamma + \epsilon\int_{\Gamma^-}u\phi\mathrm{d}\gamma \\
&\leq C\epsilon\|\nabla_x\xi\|_{L^{\frac{m}{m-1}}(\Gamma)}\bigg(\|u\|_{L^m(\Gamma^+)}+\|h\|_{L^m(\Gamma^-)}\bigg) \\
&\leq C\epsilon\|\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)}^{2m-1}\bigg(\|u\|_{L^m(\Gamma^+)}+\|h\|_{L^m(\Gamma^-)}\bigg).
\end{aligned} \tag{3.63}$$

Also, we have

$$\iint_{\Omega\times\mathbb{S}^1}(u-\bar{u})\phi \leq C\|\phi\|_{L^2(\Omega\times\mathbb{S}^1)}\|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)} \leq C\|\bar{u}\|_{L^{2m}(\Omega)}^{2m-1}\|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)}, \tag{3.64}$$

$$\iint_{\Omega\times\mathbb{S}^1} f\phi \leq C\left\|\phi\right\|_{L^2(\Omega\times\mathbb{S}^1)}\left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)} \leq C\left\|\bar{u}\right\|_{L^{2m}(\Omega)}^{2m-1}\left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)}. \tag{3.65}$$

Collecting terms in (3.60), (3.61), (3.63), (3.64) and (3.65), we obtain

$$\begin{aligned}\epsilon\left\|\bar{u}\right\|_{L^{2m}(\Omega\times\mathbb{S}^1)} \leq C\bigg(&\epsilon\left\|u-\bar{u}\right\|_{L^{2m}(\Omega\times\mathbb{S}^1)}+\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}+\epsilon\left\|u\right\|_{L^m(\Gamma^+)} \\ &+\left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)}+\epsilon\left\|h\right\|_{L^m(\Gamma^-)}\bigg),\end{aligned} \tag{3.66}$$

Step 2: Energy Estimate.
In the weak formulation (3.53), we may take the test function $\phi=u$ to get the energy estimate

$$\frac{1}{2}\epsilon\int_\Gamma\left|u\right|^2\mathrm{d}\gamma+\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2=\iint_{\Omega\times\mathbb{S}^1} fu. \tag{3.67}$$

Hence, as in L^2 estimates, this naturally implies

$$\epsilon\left\|u\right\|_{L^2(\Gamma^+)}^2+\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2=\iint_{\Omega\times\mathbb{S}^1} fu+\epsilon\left\|h\right\|_{L^2(\Gamma^-)}^2. \tag{3.68}$$

On the other hand, we can square on both sides of (3.66) to obtain

$$\begin{aligned}\epsilon^2\left\|\bar{u}\right\|_{L^{2m}(\Omega\times\mathbb{S}^1)}^2 \leq C\bigg(&\epsilon^2\left\|u-\bar{u}\right\|_{L^{2m}(\Omega\times\mathbb{S}^1)}^2+\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2+\epsilon^2\left\|u\right\|_{L^m(\Gamma^+)} \\ &+\left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2+\epsilon^2\left\|h\right\|_{L^m(\Gamma^-)}^2\bigg),\end{aligned} \tag{3.69}$$

Multiplying a sufficiently small constant on both sides of (3.69) and adding it to (3.68) to absorb $\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2$, we deduce

$$\begin{aligned}&\epsilon\left\|u\right\|_{L^2(\Gamma^+)}^2+\epsilon^2\left\|\bar{u}\right\|_{L^{2m}(\Omega\times\mathbb{S}^1)}^2+\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2 \\ \leq{}& C\bigg(\epsilon^2\left\|u-\bar{u}\right\|_{L^{2m}(\Omega\times\mathbb{S}^1)}^2+\epsilon^2\left\|u\right\|_{L^m(\Gamma^+)}+\left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)}^2+\iint_{\Omega\times\mathbb{S}^1} fu+\epsilon\left\|h\right\|_{L^2(\Gamma^-)}^2+\epsilon^2\left\|h\right\|_{L^m(\Gamma^-)}^2\bigg).\end{aligned} \tag{3.70}$$

By interpolation estimate and Young's inequality, we have

$$\begin{aligned}\left\|u\right\|_{L^m(\Gamma^+)} &\leq \left\|u\right\|_{L^2(\Gamma^+)}^{\frac{2}{m}}\left\|u\right\|_{L^\infty(\Gamma^+)}^{\frac{m-2}{m}} \\ &= \left(\frac{1}{\epsilon^{\frac{m-2}{m^2}}}\left\|u\right\|_{L^2(\Gamma^+)}^{\frac{2}{m}}\right)\left(\epsilon^{\frac{m-2}{m^2}}\left\|u\right\|_{L^\infty(\Gamma^+)}^{\frac{m-2}{m}}\right) \\ &\leq C\left(\frac{1}{\epsilon^{\frac{m-2}{m^2}}}\left\|u\right\|_{L^2(\Gamma^+)}^{\frac{2}{m}}\right)^{\frac{m}{2}}+o(1)\left(\epsilon^{\frac{m-2}{m^2}}\left\|u\right\|_{L^\infty(\Gamma^+)}^{\frac{m-2}{m}}\right)^{\frac{m}{m-2}} \\ &\leq \frac{C}{\epsilon^{\frac{m-2}{2m}}}\left\|u\right\|_{L^2(\Gamma^+)}+o(1)\epsilon^{\frac{1}{m}}\left\|u\right\|_{L^\infty(\Gamma^+)} \\ &\leq \frac{C}{\epsilon^{\frac{m-2}{2m}}}\left\|u\right\|_{L^2(\Gamma^+)}+o(1)\epsilon^{\frac{1}{m}}\left\|u\right\|_{L^\infty(\Omega\times\mathbb{S}^1)}.\end{aligned} \tag{3.71}$$

Similarly, we have

$$
\begin{aligned}
\|u-\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)} &\leq \|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)}^{\frac{1}{m}}\|u-\bar{u}\|_{L^\infty(\Omega\times\mathbb{S}^1)}^{\frac{m-1}{m}} \\
&= \left(\frac{1}{\epsilon^{\frac{m-1}{m^2}}}\|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)}^{\frac{1}{m}}\right)\left(\epsilon^{\frac{m-1}{m^2}}\|u-\bar{u}\|_{L^\infty(\Omega\times\mathbb{S}^1)}^{\frac{m-1}{m}}\right) \\
&\leq C\left(\frac{1}{\epsilon^{\frac{m-1}{m^2}}}\|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)}^{\frac{1}{m}}\right)^m + o(1)\left(\epsilon^{\frac{m-1}{m^2}}\|u-\bar{u}\|_{L^\infty(\Omega\times\mathbb{S}^1)}^{\frac{m-1}{m}}\right)^{\frac{m}{m-1}} \\
&\leq \frac{C}{\epsilon^{\frac{m-1}{m}}}\|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)} + o(1)\epsilon^{\frac{1}{m}}\|u-\bar{u}\|_{L^\infty(\Omega\times\mathbb{S}^1)}.
\end{aligned} \tag{3.72}
$$

We need this extra $\epsilon^{\frac{1}{m}}$ for the convenience of L^∞ estimate. Then we know for sufficiently small ϵ,

$$
\begin{aligned}
\epsilon^2\|u\|_{L^m(\Gamma^+)}^2 &\leq C\epsilon^{2-\frac{m-2}{m}}\|u\|_{L^2(\Gamma^+)}^2 + o(1)\epsilon^{2+\frac{2}{m}}\|u\|_{L^\infty(\Gamma^+)}^2 \\
&\leq o(1)\epsilon\|u\|_{L^2(\Gamma^+)}^2 + o(1)\epsilon^{2+\frac{2}{m}}\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)}^2.
\end{aligned} \tag{3.73}
$$

Similarly, we have

$$
\begin{aligned}
\epsilon^2\|u-\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)}^2 &\leq \epsilon^{2-\frac{2m-2}{m}}\|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)}^2 + o(1)\epsilon^{2+\frac{2}{m}}\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)}^2 \\
&\leq o(1)\|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)}^2 + o(1)\epsilon^{2+\frac{2}{m}}\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)}^2.
\end{aligned} \tag{3.74}
$$

In (3.70), we can absorb $\|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)}$ and $\epsilon\|u\|_{L^2(\Gamma^+)}^2$ into left-hand side to obtain

$$
\begin{aligned}
&\epsilon\|u\|_{L^2(\Gamma^+)}^2 + \epsilon^2\|\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)}^2 + \|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)}^2 \\
\leq{}& C\bigg(o(1)\epsilon^{2+\frac{2}{m}}\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)}^2 + \|f\|_{L^2(\Omega\times\mathbb{S}^1)}^2 + \iint_{\Omega\times\mathbb{S}^1} fu + \epsilon\|h\|_{L^2(\Gamma^-)}^2 + \epsilon^2\|h\|_{L^m(\Gamma^-)}^2\bigg).
\end{aligned} \tag{3.75}
$$

We can decompose

$$
\iint_{\Omega\times\mathbb{S}^1} fu = \iint_{\Omega\times\mathbb{S}^1} f\bar{u} + \iint_{\Omega\times\mathbb{S}^1} f(u-\bar{u}). \tag{3.76}
$$

Hölder's inequality and Cauchy's inequality imply

$$
\iint_{\Omega\times\mathbb{S}^1} f\bar{u} \leq \|f\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)}\|\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)} \leq \frac{C}{\epsilon^2}\|f\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)}^2 + o(1)\epsilon^2\|\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)}^2, \tag{3.77}
$$

and

$$
\iint_{\Omega\times\mathbb{S}^1} f(u-\bar{u}) \leq C\|f\|_{L^2(\Omega\times\mathbb{S}^1)}^2 + o(1)\|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)}^2. \tag{3.78}
$$

Hence, absorbing $\epsilon^2\|\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)}^2$ and $\|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)}^2$ into left-hand side of (3.75), we get

$$
\begin{aligned}
&\epsilon\|u\|_{L^2(\Gamma^+)}^2 + \epsilon^2\|\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)}^2 + \|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)}^2 \\
\leq{}& C\bigg(o(1)\epsilon^{2+\frac{2}{m}}\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)}^2 + \|f\|_{L^2(\Omega\times\mathbb{S}^1)}^2 + \frac{1}{\epsilon^2}\|f\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)}^2 + \epsilon\|h\|_{L^2(\Gamma^-)}^2 + \epsilon^2\|h\|_{L^m(\Gamma^-)}^2\bigg),
\end{aligned} \tag{3.79}
$$

which implies

$$
\begin{aligned}
&\frac{1}{\epsilon^{\frac{1}{2}}}\|u\|_{L^2(\Gamma^+)}+\|\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)}+\frac{1}{\epsilon}\|u-\bar{u}\|_{L^2(\Omega\times\mathbb{S}^1)} \qquad (3.80)\\
\leq\ & C\bigg(o(1)\epsilon^{\frac{1}{m}}\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)}+\frac{1}{\epsilon}\|f\|_{L^2(\Omega\times\mathbb{S}^1)}+\frac{1}{\epsilon^2}\|f\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)}+\frac{1}{\epsilon^{\frac{1}{2}}}\|h\|_{L^2(\Gamma^-)}+\|h\|_{L^m(\Gamma^-)}\bigg).
\end{aligned}
$$

□

Theorem 3.5. *The unique solution $u(\vec{x},\vec{w})$ to the equation (3.1) satisfies*

$$
\begin{aligned}
\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)}\leq\ & C\bigg(\frac{1}{\epsilon^{1+\frac{1}{m}}}\|f\|_{L^2(\Omega\times\mathbb{S}^1)}+\frac{1}{\epsilon^{2+\frac{1}{m}}}\|f\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)}+\|f\|_{L^\infty(\Omega\times\mathbb{S}^1)} \qquad (3.81)\\
&+\frac{1}{\epsilon^{\frac{1}{2}+\frac{1}{m}}}\|h\|_{L^2(\Gamma^-)}+\frac{1}{\epsilon^{\frac{1}{m}}}\|h\|_{L^m(\Gamma^-)}+\|h\|_{L^\infty(\Gamma^-)}\bigg).
\end{aligned}
$$

Proof. Following the argument in the proof of Theorem 3.3, by double Duhamel's principle along the characteristics, we may apply Hölder's inequality to obtain

$$
\begin{aligned}
I_2\leq C\int_0^{t_b}&\Bigg(\bigg(\Big(\int_{\mathbb{S}^1}\int_0^{\max(0,s_b-\delta)}\mathbf{1}_\Omega\Big(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t\Big) \qquad (3.82)\\
&\left|\bar{u}\Big(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t\Big)\right|^{2m}\mathrm{d}\vec{w}_t\mathrm{d}r\Big)^{\frac{1}{2m}}\\
&\times\Big(\int_{\mathbb{S}^1}\int_0^{\max(0,s_b-\delta)}\mathbf{1}_\Omega\Big(\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t\Big)\mathrm{d}\vec{w}_t\mathrm{d}r\Big)^{\frac{2m-1}{2m}}\Bigg)\mathrm{e}^{-(t_b-s)}\mathrm{d}s
\end{aligned}
$$

Then, using the same substitution, for $\vec{w}_t=(\cos\psi,\sin\psi)$, we define the change of variable $[-\pi,\pi)\times\mathbb{R}\to\Omega:(\psi,r)\to(y_1,y_2)=\vec{y}=\vec{x}-\epsilon(t_b-s)\vec{w}-\epsilon(s_b-r)\vec{w}_t$, which, for $s_b-r\geq\delta$, implies the Jacobian

$$
\left|\frac{\partial(y_1,y_2)}{\partial(\psi,r)}\right|=\left|\left|\begin{matrix}-\epsilon(s_b-r)\sin\psi & \epsilon\cos\psi\\ \epsilon(s_b-r)\cos\psi & \epsilon\sin\psi\end{matrix}\right|\right|=\epsilon^2(s_b-r)\geq\epsilon^2\delta. \qquad (3.83)
$$

Hence, we may simplify (3.82) as

$$
\begin{aligned}
I_2&\leq C\int_0^{t_b}\bigg(\int_\Omega\frac{1}{\epsilon^2\delta}\left|\bar{u}(\vec{y})\right|^{2m}\mathrm{d}\vec{y}\bigg)^{\frac{1}{2m}}\mathrm{e}^{-(t_b-s)}\mathrm{d}s \qquad (3.84)\\
&\leq\frac{C}{\epsilon^{\frac{1}{m}}\delta^{\frac{1}{2m}}}\int_0^{t_b}\bigg(\int_\Omega\left|\bar{u}(\vec{y})\right|^{2m}\mathrm{d}\vec{y}\bigg)^{\frac{1}{2m}}\mathrm{e}^{-(t_b-s)}\mathrm{d}s\\
&\leq\frac{C}{\epsilon^{\frac{1}{m}}\delta^{\frac{1}{2m}}}\|\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)}.
\end{aligned}
$$

Hence, for fixed $0<\delta<1$, we have

$$
\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)}\leq\delta\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)}+\frac{C}{\epsilon^{\frac{1}{m}}\delta^{\frac{1}{2m}}}\|\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)}+C\bigg(\|f\|_{L^\infty(\Omega\times\mathbb{S}^1)}+\|h\|_{L^\infty(\Gamma^-)}\bigg). \qquad (3.85)
$$

Then taking δ small to absorb $\delta\|u\|_{L^\infty(\Omega\times\mathbb{S}^1)}$ into the left-hand side to get

$$\|u\|_{L^{\infty}(\Omega\times\mathbb{S}^1)}\leq\frac{C}{\epsilon^{\frac{1}{m}}\delta^{\frac{1}{2m}}}\|\bar{u}\|_{L^{2m}(\Omega\times\mathbb{S}^1)}+C\bigg(\|f\|_{L^{\infty}(\Omega\times\mathbb{S}^1)}+\|h\|_{L^{\infty}(\Gamma^-)}\bigg). \tag{3.86}$$

Using Theorem 3.4, we get

$$\begin{aligned}\|u\|_{L^{\infty}(\Omega\times\mathbb{S}^1)}\leq C\bigg(&\frac{1}{\epsilon^{1+\frac{1}{m}}}\|f\|_{L^2(\Omega\times\mathbb{S}^1)}+\frac{1}{\epsilon^{2+\frac{1}{m}}}\|f\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)}+\|f\|_{L^{\infty}(\Omega\times\mathbb{S}^1)}\\&+\frac{1}{\epsilon^{\frac{1}{2}+\frac{1}{m}}}\|h\|_{L^2(\Gamma^-)}+\frac{1}{\epsilon^{\frac{1}{m}}}\|h\|_{L^m(\Gamma^-)}+\|h\|_{L^{\infty}(\Gamma^-)}\bigg)+o(1)\|u\|_{L^{\infty}(\Omega\times\mathbb{S}^1)}.\end{aligned} \tag{3.87}$$

Absorbing $\|u\|_{L^{\infty}(\Omega\times\mathbb{S}^1)}$ into the left-hand side, we obtain

$$\begin{aligned}\|u\|_{L^{\infty}(\Omega\times\mathbb{S}^1)}\leq C\bigg(&\frac{1}{\epsilon^{1+\frac{1}{m}}}\|f\|_{L^2(\Omega\times\mathbb{S}^1)}+\frac{1}{\epsilon^{2+\frac{1}{m}}}\|f\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)}+\|f\|_{L^{\infty}(\Omega\times\mathbb{S}^1)}\\&+\frac{1}{\epsilon^{\frac{1}{2}+\frac{1}{m}}}\|h\|_{L^2(\Gamma^-)}+\frac{1}{\epsilon^{\frac{1}{m}}}\|h\|_{L^m(\Gamma^-)}+\|h\|_{L^{\infty}(\Gamma^-)}\bigg).\end{aligned} \tag{3.88}$$

$\square$

4. Well-Posedness of ϵ-Milne Problem with Geometric Correction

We consider the ϵ-Milne problem with geometric correction for $f^{\epsilon}(\eta,\tau,\phi)$ in the domain $(\eta,\tau,\phi)\in[0,L]\times[-\pi,\pi)\times[-\pi,\pi)$ where $L=\epsilon^{-\frac{1}{2}}$ as

$$\left\{\begin{array}{l}\sin\phi\dfrac{\partial f^{\epsilon}}{\partial\eta}+F(\epsilon;\eta,\tau)\cos\phi\dfrac{\partial f^{\epsilon}}{\partial\phi}+f^{\epsilon}-\bar{f}^{\epsilon}=S^{\epsilon}(\eta,\tau,\phi),\\ f^{\epsilon}(0,\tau,\phi)=h^{\epsilon}(\tau,\phi)\ \ \text{for}\ \ \sin\phi>0,\\ f^{\epsilon}(L,\tau,\phi)=f^{\epsilon}(L,\tau,\mathscr{R}[\phi]),\end{array}\right. \tag{4.1}$$

where $\mathscr{R}[\phi]=-\phi$ and

$$F(\epsilon;\eta,\tau)=-\frac{\epsilon}{R_{\kappa}(\tau)-\epsilon\eta}, \tag{4.2}$$

for the radius of curvature R_{κ}. In this section, for convenience, we temporarily ignore the superscript on ϵ and τ. In other words, we will study

$$\left\{\begin{array}{l}\sin\phi\dfrac{\partial f}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial f}{\partial\phi}+f-\bar{f}=S(\eta,\phi),\\ f(0,\phi)=h(\phi)\ \ \text{for}\ \ \sin\phi>0,\\ f(L,\phi)=f(L,\mathscr{R}[\phi]).\end{array}\right. \tag{4.3}$$

Define potential function $V(\eta)$ satisfying $V(0)=0$ and $\dfrac{\partial V}{\partial\eta}=-F(\eta)$. Then we can direct compute

$$V(\eta)=\ln\left(\frac{R_{\kappa}}{R_{\kappa}-\epsilon\eta}\right). \tag{4.4}$$

Define the weight function

$$\zeta(\eta,\phi)=\left(1-\left(\frac{R_{\kappa}-\epsilon\eta}{R_{\kappa}}\cos\phi\right)^2\right)^{\frac{1}{2}}. \tag{4.5}$$

We can easily show that

$$\sin\phi\frac{\partial\zeta}{\partial\eta}+F(\eta)\cos\phi\frac{\partial\zeta}{\partial\phi}=0. \tag{4.6}$$

We define the norms in the space $(\eta,\phi)\in[0,\infty)\times[-\pi,\pi)$ as follows:

$$\|f\|_{L^2L^2}=\left(\int_0^L\int_{-\pi}^{\pi}\left|f(\eta,\phi)\right|^2\mathrm{d}\phi\mathrm{d}\eta\right)^{\frac{1}{2}}, \tag{4.7}$$

$$\|f\|_{L^\infty L^\infty}=\sup_{(\eta,\phi)\in[0,L]\times[-\pi,\pi)}\left|f(\eta,\phi)\right|. \tag{4.8}$$

Similarly,

$$\|f(\eta)\|_{L^2}=\left(\int_{-\pi}^{\pi}\left|f(\eta,\phi)\right|^2\mathrm{d}\phi\right)^{\frac{1}{2}}, \tag{4.9}$$

$$\|f(\eta)\|_{L^\infty}=\sup_{\phi\in[-\pi,\pi)}\left|f(\eta,\phi)\right|. \tag{4.10}$$

Also, we define the weighted norms at in-flow boundary as

$$\|h\|_{L^2_-} = \left(\int_{\sin\phi>0} |h(\phi)|^2 \sin\phi \mathrm{d}\phi\right)^{\frac{1}{2}}, \tag{4.11}$$

$$\|h\|_{L^\infty_-} = \sup_{\sin\phi>0} |h(\phi)|. \tag{4.12}$$

Also define

$$\langle f, g\rangle_\phi(\eta) = \int_{-\pi}^{\pi} f(\eta,\phi)g(\eta,\phi)\mathrm{d}\phi, \tag{4.13}$$

as the L^2 inner product in ϕ.

In the following, we will always assume that for some $K>0$,

$$\|h\|_{L^\infty_-} + \left\|\mathrm{e}^{K\eta}S\right\|_{L^\infty L^\infty} \leq C. \tag{4.14}$$

The well-posedness, exponential decay and maximum principle of the equation (4.3) has been well studied in [21]. Here we will focus on the a priori estimates and present detail structure of the dependence of the boundary data h and the source term S.

Assume that S satisfies $\bar{S}(\eta)=0$ for any η. We may decompose the solution

$$f(\eta,\phi) = q_f(\eta) + r_f(\eta,\phi), \tag{4.15}$$

where the hydrodynamical part q_f is in the null space of the operator $f-\bar{f}$, and the microscopic part r_f is the orthogonal complement, i.e.

$$q_f(\eta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\eta,\phi)\mathrm{d}\phi = \bar{f}, \quad r_f(\eta,\phi) = f(\eta,\phi) - q_f(\eta). \tag{4.16}$$

In the following, when there is no confusion, we simply write $f = q + r$.

Lemma 4.1. *Assume $\bar{S}(\eta)=0$ for any $\eta\in[0,L]$. Then the unique solution $f(\eta,\phi)$ to the equation (4.3) satisfies*

$$\|r\|_{L^2L^2} \leq C\bigg(\|h\|_{L^2_-} + \|S\|_{L^2L^2}\bigg), \tag{4.17}$$

and there exists $q_L\in\mathbb{R}$ such that

$$|q_L| \leq C\bigg(\|h\|_{L^2_-} + \|S\|_{L^2L^2}\bigg) + C\left|\int_0^L \langle\sin\phi, S\rangle_\phi(y)\mathrm{d}y\right|, \tag{4.18}$$

$$\|q-q_L\|_{L^2L^2} \leq C\bigg(\|h\|_{L^2_-} + \|S\|_{L^2L^2}\bigg) + C\left(\int_0^L\left(\int_\eta^L \langle\sin\phi, S\rangle_\phi(y)\mathrm{d}y\right)^2\mathrm{d}\eta\right)^{\frac{1}{2}}. \tag{4.19}$$

Also, for any $\eta\in[0,L]$,

$$\langle\sin\phi, r\rangle_\phi(\eta) = 0. \tag{4.20}$$

Proof. We divide the proof into several steps:

Step 1: Estimate of r.
Multiplying f on both sides of (4.3) and integrating over $\phi \in [-\pi, \pi)$, we get the energy estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta}\langle f, f\sin\phi\rangle_\phi(\eta) + F(\eta)\left\langle \frac{\partial f}{\partial \phi}, f\cos\phi\right\rangle_\phi(\eta) + \|r(\eta)\|_{L^2}^2 = \langle S, f\rangle_\phi(\eta). \tag{4.21}$$

An integration by parts reveals

$$F(\eta)\left\langle \frac{\partial f}{\partial \phi}, f\cos\phi\right\rangle_\phi(\eta) = \frac{1}{2}F(\eta)\langle f, f\sin\phi\rangle_\phi(\eta). \tag{4.22}$$

Also, the assumption $\bar{S}(\eta) = 0$ leads to

$$\langle S, f\rangle_\phi(\eta) = \langle S, q\rangle_\phi(\eta) + \langle S, r\rangle_\phi(\eta) = \langle S, r\rangle_\phi(\eta). \tag{4.23}$$

Hence, we have the simplified form of (4.21) as follows:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta}\langle f, f\sin\phi\rangle_\phi(\eta) + \frac{1}{2}F(\eta)\langle f, f\sin\phi\rangle_\phi(\eta) + \|r(\eta)\|_{L^2}^2 = \langle S, r\rangle_\phi(\eta). \tag{4.24}$$

Define

$$\alpha(\eta) = \frac{1}{2}\langle f, f\sin\phi\rangle_\phi(\eta). \tag{4.25}$$

Then (4.24) can be rewritten as follows:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\eta} + F(\eta)\alpha(\eta) + \|r(\eta)\|_{L^2}^2 = \langle S, r\rangle_\phi(\eta). \tag{4.26}$$

We can solve this differential equation for α on $[\eta, L]$ and $[0, \eta]$ respectively to obtain

$$\alpha(\eta) = \alpha(L)\exp\left(\int_\eta^L F(y)\mathrm{d}(y)\right) + \int_\eta^L \exp\left(\int_\eta^y F(z)\mathrm{d}z\right)\left(\|r(y)\|_{L^2}^2 - \langle S, r\rangle_\phi(y)\right)\mathrm{d}y, \tag{4.27}$$

$$\alpha(\eta) = \alpha(0)\exp\left(-\int_0^\eta F(y)\mathrm{d}(y)\right) + \int_0^\eta \exp\left(-\int_y^\eta F(z)\mathrm{d}z\right)\left(-\|r(y)\|_{L^2}^2 + \langle S, r\rangle_\phi(y)\right)\mathrm{d}y. \tag{4.28}$$

The specular reflexive boundary $f(L, \phi) = f(L, \mathscr{R}[\phi])$ ensures $\alpha(L) = 0$. Hence, based on (4.27), we have

$$\alpha(\eta) \geq \int_\eta^L \exp\left(\int_\eta^y F(z)\mathrm{d}z\right)\left(-\langle S, r\rangle_\phi(y)\right)\mathrm{d}y \geq -C\int_\eta^L \langle S, r\rangle_\phi(y)\mathrm{d}y. \tag{4.29}$$

Also, (4.28) implies

$$\begin{aligned}\alpha(\eta) &\leq \alpha(0)\exp\bigg(-\int_0^\eta F(y)\mathrm{d}(y)\bigg) + \int_0^\eta \exp\bigg(-\int_y^\eta F(z)\mathrm{d}z\bigg)\bigg(\langle S, r\rangle_\phi(y)\bigg)\mathrm{d}y \\ &\leq C\left\|h\right\|_{L^2_-}^2 + C\int_0^\eta \bigg(\langle S, r\rangle_\phi(y)\bigg)\mathrm{d}y,\end{aligned} \tag{4.30}$$

due to the fact

$$\alpha(0) = \frac{1}{2}\left\langle \sin\phi f, f\right\rangle_\phi(0) \leq \frac{1}{2}\bigg(\int_{\sin\phi>0} h^2(\phi)\sin\phi\mathrm{d}\phi\bigg) \leq C\left\|h\right\|_{L^2_-}^2. \tag{4.31}$$

Then in (4.28) taking $\eta = L$, from $\alpha(L) = 0$, we have

$$\begin{aligned}\int_0^L \exp\bigg(\int_0^y F(z)\mathrm{d}z\bigg)\left\|r(y)\right\|_{L^2}^2\mathrm{d}y &\leq \alpha(0) + \int_0^L \exp\bigg(\int_0^y F(z)\mathrm{d}z\bigg)\langle S, r\rangle_\phi(y)\mathrm{d}y \\ &\leq C\left\|h\right\|_{L^2_-}^2 + C\int_0^L \langle S, r\rangle_\phi(y)\mathrm{d}y.\end{aligned} \tag{4.32}$$

On the other hand, we can directly estimate as follows:

$$\int_0^L \exp\bigg(\int_0^y F(z)\mathrm{d}z\bigg)\left\|r(y)\right\|_{L^2}^2\mathrm{d}y \geq C\int_0^L \left\|r(y)\right\|_{L^2}^2\mathrm{d}y. \tag{4.33}$$

Combining (4.32) and (4.33) yields

$$\int_0^L \left\|r(\eta)\right\|_{L^2}^2\mathrm{d}\eta \leq C\left\|h\right\|_{L^2_-}^2 + C\int_0^L \langle S, r\rangle_\phi(y)\mathrm{d}y. \tag{4.34}$$

By Cauchy's inequality, we have

$$\left|\int_0^L \langle S, r\rangle_\phi(y)\mathrm{d}y\right| \leq C_0\int_0^L \left\|r(\eta)\right\|_{L^2}^2\mathrm{d}\eta + \frac{4}{C_0}\int_0^L \left\|S(\eta)\right\|_{L^2}^2\mathrm{d}\eta, \tag{4.35}$$

for $C_0 > 0$ small. Therefore, absorbing $\displaystyle\int_0^L \left\|r(\eta)\right\|_{L^2}^2\mathrm{d}\eta$ and summarizing (4.34) and (4.35), we deduce

$$\int_0^L \left\|r(\eta)\right\|_{L^2}^2\mathrm{d}\eta \leq C\bigg(\left\|h\right\|_{L^2_-}^2 + \int_0^L \left\|S(\eta)\right\|_{L^2}^2\mathrm{d}\eta\bigg). \tag{4.36}$$

Step 2: Orthogonality relation.
A direct integration over $\phi \in [-\pi, \pi)$ in (4.3) implies

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left\langle \sin\phi, f\right\rangle_\phi(\eta) = -F\left\langle \cos\phi, \frac{\mathrm{d}f}{\mathrm{d}\phi}\right\rangle_\phi(\eta) + \bar{S}(\eta) = -F\left\langle \sin\phi, f\right\rangle_\phi(\eta), \tag{4.37}$$

due to $\bar{S} = 0$. The specular reflexive boundary $f(L, \phi) = f(L, \mathscr{R}[\phi])$ implies $\langle \sin\phi, f\rangle_\phi(L) = 0$. Then we have

$$\left\langle \sin\phi, f\right\rangle_\phi(\eta) = 0. \tag{4.38}$$

It is easy to see

$$\langle \sin\phi, q \rangle_\phi (\eta) = 0. \tag{4.39}$$

Hence, we may derive

$$\langle \sin\phi, r \rangle_\phi (\eta) = 0. \tag{4.40}$$

This leads to orthogonal relation (4.20).

Step 3: Estimate of q.
Multiplying $\sin\phi$ on both sides of (4.3) and integrating over $\phi \in [-\pi, \pi)$ lead to

$$\frac{\mathrm{d}}{\mathrm{d}\eta} \left\langle \sin^2\phi, f \right\rangle_\phi (\eta) = - \langle \sin\phi, r \rangle_\phi (\eta) - F(\eta) \left\langle \sin\phi\cos\phi, \frac{\partial f}{\partial \phi} \right\rangle_\phi (\eta) + \langle \sin\phi, S \rangle_\phi (\eta). \tag{4.41}$$

We can further integrate by parts as follows:

$$-F(\eta) \left\langle \sin\phi\cos\phi, \frac{\partial f}{\partial \phi} \right\rangle_\phi (\eta) = F(\eta) \langle \cos(2\phi), f \rangle_\phi (\eta) = F(\eta) \langle \cos(2\phi), r \rangle_\phi (\eta). \tag{4.42}$$

Using the orthogonal relation (4.20), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\eta} \left\langle \sin^2\phi, f \right\rangle_\phi (\eta) = F(\eta) \langle \cos(2\phi), r \rangle_\phi (\eta) + \langle \sin\phi, S \rangle_\phi (\eta).$$

Define

$$\beta(\eta) = \left\langle \sin^2\phi, f \right\rangle_\phi (\eta), \tag{4.43}$$

and

$$\frac{\mathrm{d}\beta}{\mathrm{d}\eta} = D(\eta, \phi), \tag{4.44}$$

where

$$D(\eta, \phi) = F(\eta) \langle \cos(2\phi), r \rangle_\phi + \langle \sin\phi, S \rangle_\phi (\eta). \tag{4.45}$$

Hence, we can integrate (4.44) over $[0, \eta]$ to get that

$$\beta(\eta) - \beta(0) = \int_0^\eta F(y) \langle \cos(2\phi), r \rangle_\phi (y)\mathrm{d}y + \int_0^\eta \langle \sin\phi, S \rangle_\phi (y)\mathrm{d}y. \tag{4.46}$$

Then the initial data

$$\beta(0) = \left\langle \sin^2\phi, f \right\rangle_\phi (0) \leq \bigg(\langle f, f\,|\sin\phi| \rangle_\phi (0) \bigg)^{\frac{1}{2}} \|\sin\phi\|_{L^2}^{3/2} \leq C\bigg(\langle f, f\,|\sin\phi| \rangle_\phi (0) \bigg)^{\frac{1}{2}}. \tag{4.47}$$

Obviously, we have

$$\langle f, f\,|\sin\phi|\rangle_{\phi}(0) = \int_{\sin\phi>0} h^2(\phi)\sin\phi \mathrm{d}\phi - \int_{\sin\phi<0} \Big(f(0,\phi)\Big)^2 \sin\phi \mathrm{d}\phi. \tag{4.48}$$

However, based on the definition of $\alpha(\eta)$ and (4.29), we can obtain

$$\int_{\sin\phi>0} h^2(\phi)\sin\phi \mathrm{d}\phi + \int_{\sin\phi<0} \Big(f(0,\phi)\Big)^2 \sin\phi \mathrm{d}\phi = 2\alpha(0) \geq\ -C\int_0^L \langle S, r\rangle_{\phi}(y)\mathrm{d}y.$$

Hence, we can deduce

$$\begin{aligned} -\int_{\sin\phi<0} \Big(f(0,\phi)\Big)^2 \sin\phi \mathrm{d}\phi &\leq \int_{\sin\phi>0} h^2(\phi)\sin\phi \mathrm{d}\phi + C\int_0^L \langle S, r\rangle_{\phi}(y)\mathrm{d}y \\ &\leq C\bigg(\|h\|_{L^2_-}^2 + \int_0^L \|S(\eta)\|_{L^2}^2 \,\mathrm{d}\eta\bigg). \end{aligned} \tag{4.49}$$

From (4.36), we can deduce

$$\beta(0) \leq C\bigg(\|h\|_{L^2_-} + \|S\|_{L^2L^2}\bigg). \tag{4.50}$$

Since $F \in L^1[0,L]\cap L^2[0,L]$, $r \in L^2([0,L]\times[-\pi,\pi))$, by (4.50) and (4.17), we have

$$\begin{aligned} |\beta(L)| \ &\leq\ |\beta(0)| + \left|\int_0^L F(y)\langle\cos(2\phi), r\rangle_{\phi}(y)\mathrm{d}y\right| + \left|\int_0^L \langle\sin\phi, S\rangle_{\phi}(y)\mathrm{d}y\right| \\ &\leq C\bigg(\|h\|_{L^2_-} + \|S\|_{L^2L^2}\bigg) + C\|F\|_{L^2L^2}\|r\|_{L^2L^2} + \left|\int_0^L \langle\sin\phi, S\rangle_{\phi}(y)\mathrm{d}y\right| \\ &\leq C\bigg(\|h\|_{L^2_-} + \|S\|_{L^2L^2}\bigg) + \left|\int_0^L \langle\sin\phi, S\rangle_{\phi}(y)\mathrm{d}y\right|. \end{aligned} \tag{4.51}$$

We define

$$q_L = \frac{\beta(L)}{\|\sin\phi\|_{L^2}^2}. \tag{4.52}$$

Naturally, we have

$$|q_L|\ \leq C\bigg(\|h\|_{L^2_-} + \|S\|_{L^2L^2}\bigg) + C\left|\int_0^L \langle\sin\phi, S\rangle_{\phi}(y)\mathrm{d}y\right|. \tag{4.53}$$

Note that q_L is not necessarily $q(L)$. Moreover,

$$\beta(L) - \beta(\eta) = \int_{\eta}^L D(y)\mathrm{d}y = \int_{\eta}^L F(y)\langle\cos(2\phi), r\rangle_{\phi}(y)\mathrm{d}y + \int_{\eta}^L \langle\sin\phi, S\rangle_{\phi}(y)\mathrm{d}y. \tag{4.54}$$

Note

$$\beta(\eta) = \left\langle\sin^2\phi, f\right\rangle_{\phi}(\eta) = \left\langle\sin^2\phi, q\right\rangle_{\phi}(\eta) + \left\langle\sin^2\phi, r\right\rangle_{\phi}(\eta) = q(\eta)\,\|\sin\phi\|_{L^2}^2 + \left\langle\sin^2\phi, r\right\rangle_{\phi}(\eta). \tag{4.55}$$

Thus we can estimate

$$
\begin{aligned}
&\|\sin\phi\|_{L^2}^2 \|q(\eta)-q_L\|_{L^2} \\
&= \beta(L)-\beta(\eta)+\left\langle \sin^2\phi, r\right\rangle_\phi(\eta) \\
&\leq C\bigg(\int_\eta^L \left|F(y)\left\langle \cos(2\phi), r(y)\right\rangle_\phi \mathrm{d}y\right|\mathrm{d}\eta + \left|\int_\eta^L \left\langle \sin\phi, S\right\rangle_\phi(y)\mathrm{d}y\right| + \left|\left\langle \sin^2\phi, r\right\rangle_\phi(\eta)\right|\bigg) \\
&\leq C\bigg(\|r(\eta)\|_{L^2} + \int_\eta^L |F(y)|\,\|r(y)\|_{L^2}\,\mathrm{d}y + \left|\int_\eta^L \left\langle \sin\phi, S\right\rangle_\phi(y)\mathrm{d}y\right|\bigg).
\end{aligned} \tag{4.56}
$$

Then we integrate (4.56) over $\eta\in[0,L]$. Cauchy's inequality implies

$$
\int_0^L\left(\int_\eta^L |F(y)|\,\|r(y)\|_{L^2}\,\mathrm{d}y\right)^2\mathrm{d}\eta \leq \|r\|_{L^2L^2}^2\int_0^L\int_\eta^L |F(y)|^2\,\mathrm{d}y\mathrm{d}\eta \leq C\|r\|_{L^2L^2}^2. \tag{4.57}
$$

Hence, we have

$$
\|q-q_L\|_{L^2L^2} \leq C\Big(\|h\|_{L^2_-}+\|S\|_{L^2L^2}\Big) + C\bigg(\int_0^L\left(\int_\eta^L \left\langle \sin\phi, S\right\rangle_\phi(y)\mathrm{d}y\right)^2\mathrm{d}\eta\bigg)^{\frac{1}{2}}. \tag{4.58}
$$

$\square$

For general S, we define $S=\bar{S}+(S-\bar{S})=S_Q+S_R$.

Lemma 4.2. *The unique solution $f(\eta,\phi)$ to the equation (4.3) satisfies*

$$
\|r\|_{L^2L^2} \leq C\Big(\|h\|_{L^2_-}+\|S\|_{L^2L^2}\Big) + C\bigg(\int_0^L\left(\int_\eta^L |S_Q(y)|\,\mathrm{d}y\right)^2\mathrm{d}\eta\bigg)^{\frac{1}{2}}, \tag{4.59}
$$

and there exists $q_L\in\mathbb{R}$ such that

$$
\begin{aligned}
|q_L| &\leq C\Big(\|h\|_{L^2_-}+\|S\|_{L^2L^2}\Big) \\
&\quad + C\left|\int_0^L \left\langle \sin\phi, S_R\right\rangle_\phi(y)\mathrm{d}y\right| + C\left|\int_0^L\int_\eta^L |S_Q(y)|\,\mathrm{d}y\mathrm{d}\eta\right|,
\end{aligned} \tag{4.60}
$$

$$
\begin{aligned}
\|q-q_L\|_{L^2L^2} &\leq C\Big(\|h\|_{L^2_-}+\|S\|_{L^2L^2}\Big) \\
&\quad + C\bigg(\int_0^L\left(\int_\eta^L \left\langle \sin\phi, S_R\right\rangle_\phi(y)\mathrm{d}y\right)^2\mathrm{d}\eta\bigg)^{\frac{1}{2}} + C\bigg(\int_0^L\left(\int_\eta^L\int_y^L |S_Q(z)|\,\mathrm{d}z\mathrm{d}y\right)^2\mathrm{d}\eta\bigg)^{\frac{1}{2}}.
\end{aligned} \tag{4.61}
$$

Also, for any $\eta\in[0,L]$,

$$
\left\langle \sin\phi, r\right\rangle_\phi(\eta) = -\int_\eta^L \mathrm{e}^{V(\eta)-V(y)}S_Q(y)\mathrm{d}y. \tag{4.62}
$$

Proof. We can apply superposition property for this linear problem. For simplicity, we just above estimates as the L^2 estimates.

Step 1: Construction of auxiliary function f^1.
We first solve f^1 as the solution to

$$\begin{cases} \sin\phi\dfrac{\partial f^1}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial f^1}{\partial\phi}+f^1-\bar{f}^1=S_R(\eta,\phi),\\ f^1(0,\phi)=h(\phi)\quad\text{for}\quad\sin\phi>0,\\ f^1(L,\phi)=f^1(L,\mathscr{R}[\phi]). \end{cases} \tag{4.63}$$

Since $\bar{S}_R=0$, by Lemma 4.1, we know there exists a unique solution f^1 satisfying the L^2 estimate.

Step 2: Construction of auxiliary function f^2.
We seek a function f^2 satisfying

$$-\frac{1}{2\pi}\int_{-\pi}^{\pi}\bigg(\sin\phi\frac{\partial f^2}{\partial\eta}+F(\eta)\cos\phi\frac{\partial f^2}{\partial\phi}\bigg)\mathrm{d}\phi+S_Q=0. \tag{4.64}$$

The following analysis shows this type of function can always be found. An integration by parts transforms the equation (4.64) into

$$-\int_{-\pi}^{\pi}\sin\phi\frac{\partial f^2}{\partial\eta}\mathrm{d}\phi-\int_{-\pi}^{\pi}F(\eta)\sin\phi f^2\mathrm{d}\phi+2\pi S_Q=0. \tag{4.65}$$

Setting

$$f^2(\phi,\eta)=a(\eta)\sin\phi. \tag{4.66}$$

and plugging this ansatz into (4.65), we have

$$-\frac{\mathrm{d}a}{\mathrm{d}\eta}\int_{-\pi}^{\pi}\sin^2\phi\mathrm{d}\phi-F(\eta)a(\eta)\int_{-\pi}^{\pi}\sin^2\phi\mathrm{d}\phi+2\pi S_Q=0. \tag{4.67}$$

Hence, we have

$$-\frac{\mathrm{d}a}{\mathrm{d}\eta}-F(\eta)a(\eta)+2S_Q=0. \tag{4.68}$$

This is a first order linear ordinary differential equation, which possesses infinite solutions. We can directly solve it to obtain

$$a(\eta)=\exp\bigg(-\int_0^{\eta}F(y)\mathrm{d}y\bigg)\bigg(a(0)+\int_0^{\eta}\exp\bigg(\int_0^{y}F(z)\mathrm{d}z\bigg)2S_Q(y)\mathrm{d}y\bigg). \tag{4.69}$$

We may take

$$a(0)=-\int_0^{L}\exp\bigg(\int_0^{y}F(z)\mathrm{d}z\bigg)2S_Q(y)\mathrm{d}y. \tag{4.70}$$

Then, we can directly verify

$$|a(\eta)| \leq C\int_{\eta}^{L}|S_Q(y)|\,\mathrm{d}y, \tag{4.71}$$

and f^2 satisfies the L^2 estimate.

Step 3: Construction of auxiliary function f^3.
Based on above construction, we can directly verify that

$$\int_{-\pi}^{\pi}\bigg(-\sin\phi\frac{\partial f^2}{\partial\eta}-F(\eta)\cos\phi\frac{\partial f^2}{\partial\phi}-f^2+\bar{f}^2+S_Q\bigg)\mathrm{d}\phi=0. \tag{4.72}$$

Then we can solve f^3 as the solution to

$$\begin{cases}\sin\phi\dfrac{\partial f^3}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial f^3}{\partial\phi}+f^3-\bar{f}^3=-\sin\phi\dfrac{\partial f^2}{\partial\eta}-F(\eta)\cos\phi\dfrac{\partial f^2}{\partial\phi}-f^2+\bar{f}^2+S_Q,\\ f^3(0,\phi)=-a(0)\sin\phi\quad\text{for}\quad\sin\phi>0,\\ f^3(L,\phi)=f^3(L,\mathscr{R}[\phi]).\end{cases} \tag{4.73}$$

By (4.72), we can apply Lemma 4.1 to obtain a unique solution f^3 satisfying the L^2 estimate.

Step 4: Construction of auxiliary function f^4.
We now define $f^4=f^2+f^3$ and an explicit verification shows

$$\begin{cases}\sin\phi\dfrac{\partial f^4}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial f^4}{\partial\phi}+f^4-\bar{f}^4=S_Q(\eta,\phi),\\ f^4(0,\phi)=0\quad\text{for}\quad\sin\phi>0,\\ f^4(L,\phi)=f^4(L,\mathscr{R}[\phi]),\end{cases} \tag{4.74}$$

and f^4 satisfies the L^2 estimate.

In summary, we deduce that f^1+f^4 is the solution of (4.3) and satisfies the L^2 estimate. □

Combining all above, we have the following theorem.

Theorem 4.3. *The unique solution $f(\eta,\phi)$ to the equation (4.3) satisfies*

$$\begin{aligned}\|f-f_L\|_{L^2L^2}\leq{}& C\bigg(\|h\|_{L^2_-}+\|S\|_{L^2L^2}\bigg)+C\bigg(\int_0^L\bigg(\int_\eta^L\langle\sin\phi,S_R\rangle_\phi(y)\mathrm{d}y\bigg)^2\mathrm{d}\eta\bigg)^{\frac{1}{2}}\\ &+C\bigg(\int_0^L\bigg(\int_\eta^L|S_Q(y)|\,\mathrm{d}y\bigg)^2\mathrm{d}\eta\bigg)^{\frac{1}{2}}+C\bigg(\int_0^L\bigg(\int_\eta^L\int_y^L|S_Q(z)|\,\mathrm{d}z\mathrm{d}y\bigg)^2\mathrm{d}\eta\bigg)^{\frac{1}{2}},\end{aligned} \tag{4.75}$$

for some $f_L\in\mathbb{R}$ satisfying

$$|f_L|\leq C\bigg(\|h\|_{L^2_-}+\|S\|_{L^2L^2}\bigg)+C\left|\int_0^L\langle\sin\phi,S_R\rangle_\phi(y)\mathrm{d}y\right|+C\left|\int_0^L\int_\eta^L|S_Q(y)|\,\mathrm{d}y\mathrm{d}\eta\right|. \tag{4.76}$$

Consider the ϵ-transport problem for $f(\eta,\phi)$ in $(\eta,\phi)\in[0,L]\times[-\pi,\pi)$

$$
\begin{cases}
\sin\phi\dfrac{\partial f}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial f}{\partial\phi}+f=H(\eta,\phi),\\
f(0,\phi)=h(\phi)\ \ \text{for}\ \ \sin\phi>0,\\
f(L,\phi)=f(L,\mathscr{R}[\phi]).
\end{cases}
\tag{4.77}
$$

Define the energy as follows:

$$
E(\eta,\phi)=\mathrm{e}^{-V(\eta)}\cos\phi. \tag{4.78}
$$

Along the characteristics, this energy is conserved and the equation can be simplified as follows:

$$
\sin\phi\frac{\mathrm{d}f}{\mathrm{d}\eta}+f=H. \tag{4.79}
$$

An implicit function $\eta^+(\eta,\phi)$ can be determined through

$$
|E(\eta,\phi)|=\mathrm{e}^{-V(\eta^+)}. \tag{4.80}
$$

which means (η^+,ϕ_0) with $\sin\phi_0=0$ is on the same characteristics as (η,ϕ). Define the quantities for $0\leq\eta'\leq\eta^+$ as follows:

$$
\phi'(\eta,\phi;\eta')=\cos^{-1}\left(\mathrm{e}^{V(\eta')-V(\eta)}\cos\phi\right), \tag{4.81}
$$

$$
\mathscr{R}[\phi'(\eta,\phi;\eta')]=-\cos^{-1}\left(\mathrm{e}^{V(\eta')-V(\eta)}\cos\phi\right)=-\phi'(\eta,\phi;\eta'), \tag{4.82}
$$

where the inverse trigonometric function can be defined single-valued in the domain $[0,\pi)$ and the quantities are always well-defined due to the monotonicity of V. Finally we put

$$
G_{\eta,\eta'}(\phi)=\int_{\eta'}^{\eta}\frac{1}{\sin\Big(\phi'(\eta,\phi;\xi)\Big)}\mathrm{d}\xi. \tag{4.83}
$$

We can rewrite the solution to the equation (4.77) along the characteristics as

$$
f(\eta,\phi)=\mathcal{K}[h](\phi)+\mathcal{T}[H](\eta,\phi), \tag{4.84}
$$

where

Region I:
For $\sin\phi>0$,

$$
\mathcal{K}[h](\phi)=h\Big(\phi'(\eta,\phi;0)\Big)\exp(-G_{\eta,0}), \tag{4.85}
$$

$$
\mathcal{T}[H](\eta,\phi)=\int_0^{\eta}\frac{H\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'. \tag{4.86}
$$

Region II:
For $\sin\phi<0$ and $|E(\eta,\phi)|\leq \mathrm{e}^{-V(L)}$,

$$\mathcal{K}[h](\phi)=h(\phi'(\eta,\phi;0))\exp(-G_{L,0}-G_{L,\eta}) \tag{4.87}$$

$$\begin{aligned}\mathcal{T}[H](\eta,\phi)=&\int_0^L\frac{H\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{L,\eta'}-G_{L,\eta})\mathrm{d}\eta' \\ &+\int_\eta^L\frac{H\Big(\eta',\mathscr{R}[\phi'(\eta,\phi;\eta')]\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(G_{\eta,\eta'})\mathrm{d}\eta'.\end{aligned} \tag{4.88}$$

Region III:
For $\sin\phi<0$ and $|E(\eta,\phi)|\geq \mathrm{e}^{-V(L)}$,

$$\mathcal{K}[h](\phi)=h\Big(\phi'(\eta,\phi;0)\Big)\exp(-G_{\eta^+,0}-G_{\eta^+,\eta}) \tag{4.89}$$

$$\begin{aligned}\mathcal{T}[H](\eta,\phi)=&\int_0^{\eta^+}\frac{H\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta^+,\eta'}-G_{\eta^+,\eta})\mathrm{d}\eta' \\ &+\int_\eta^{\eta^+}\frac{H\Big(\eta',\mathscr{R}[\phi'(\eta,\phi;\eta')]\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(G_{\eta,\eta'})\mathrm{d}\eta'.\end{aligned} \tag{4.90}$$

In order to achieve the estimate of f, we need to control $\mathcal{K}[h]$ and $\mathcal{T}[H]$.

We first give several technical lemmas to be used for proving L^∞ estimates of f.

Lemma 4.4. *For any $0\leq\beta\leq1$, we have*

$$\left\|\mathrm{e}^{\beta\eta}\mathcal{K}[h]\right\|_{L^\infty}\leq\left\|h\right\|_{L^\infty}. \tag{4.91}$$

In particular,

$$\left\|\mathcal{K}[h]\right\|_{L^\infty}\leq\left\|h\right\|_{L^\infty}. \tag{4.92}$$

Proof. Since ϕ' is always in the domain $[0,\pi)$, we naturally have

$$0\leq\sin\Big(\phi'(\eta,\phi;\xi)\Big)\leq1, \tag{4.93}$$

which further implies

$$\frac{1}{\sin\Big(\phi'(\eta,\phi;\xi)\Big)}\geq1. \tag{4.94}$$

Combined with the fact $L\geq\eta^+\geq\eta$, this implies

$$\exp(-G_{\eta,0}) \leq \mathrm{e}^{-\eta}, \tag{4.95}$$

$$\exp(-G_{L,0} - G_{L,\eta}) \leq \mathrm{e}^{-\eta}, \tag{4.96}$$

$$\exp(-G_{\eta^+,0} - G_{\eta^+,\eta}) \leq \exp(-G_{\eta^+,0}) \leq \exp(-G_{\eta,0}) \leq \mathrm{e}^{-\eta}. \tag{4.97}$$

Hence, our result easily follows. □

Lemma 4.5. *The integral operator $\mathcal{T}$ satisfies*

$$\|\mathcal{T}[H]\|_{L^\infty L^\infty} \leq \|H\|_{L^\infty L^\infty}, \tag{4.98}$$

and for any $0 \leq \beta \leq \dfrac{1}{2}$

$$\left\|\mathrm{e}^{\beta\eta}\mathcal{T}[H]\right\|_{L^\infty L^\infty} \leq \left\|\mathrm{e}^{\beta\eta}H\right\|_{L^\infty L^\infty}. \tag{4.99}$$

Proof. For (4.98), when $\sin\phi > 0$

$$\begin{aligned}
|\mathcal{T}[H]| &\leq \int_0^\eta \left|H\Big(\eta', \phi'(\eta,\phi;\eta')\Big)\right| \frac{1}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)} \exp(-G_{\eta,\eta'})\mathrm{d}\eta' \\
&\leq \|H\|_{L^\infty L^\infty} \int_0^\eta \frac{1}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)} \exp(-G_{\eta,\eta'})\mathrm{d}\eta'.
\end{aligned} \tag{4.100}$$

We can directly estimate

$$\int_0^\eta \frac{1}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)} \exp(-G_{\eta,\eta'})\mathrm{d}\eta' \leq \int_0^\infty \mathrm{e}^{-z}\mathrm{d}z = 1, \tag{4.101}$$

and (4.98) naturally follows. For $\sin\phi < 0$ and $|E(\eta,\phi)| \leq \mathrm{e}^{-V(L)}$,

$$\begin{aligned}
|\mathcal{T}[H]| &\leq \int_\eta^\infty \left|H\Big(\eta', \phi'(\eta,\phi;\eta')\Big)\right| \frac{1}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)} \exp(G_{\eta,\eta'})\mathrm{d}\eta' \\
&\leq \|H\|_{L^\infty L^\infty} \int_\eta^\infty \frac{1}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)} \exp(G_{\eta,\eta'})\mathrm{d}\eta'.
\end{aligned} \tag{4.102}$$

we have

$$\int_\eta^\infty \frac{1}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)} \exp(G_{\eta,\eta'})\mathrm{d}\eta' \leq \int_{-\infty}^0 \mathrm{e}^{z}\mathrm{d}z = 1, \tag{4.103}$$

and (4.98) easily follows. The region $\sin\phi < 0$ and $|E(\eta,\phi)| \geq \mathrm{e}^{-V(L)}$ can be proved combining above two techniques, so we omit it here.

For (4.99), when $\sin\phi > 0$, $\eta \geq \eta'$ and $\beta < \dfrac{1}{2}$, since $G_{\eta,\eta'} \geq \eta - \eta'$, we have

$$\beta(\eta-\eta')-G_{\eta,\eta'}\leq\beta(\eta-\eta')-\frac{1}{2}(\eta-\eta')-\frac{1}{2}G_{\eta,\eta'}\leq-\frac{1}{2}G_{\eta,\eta'}. \tag{4.104}$$

Then it is natural that

$$\begin{aligned}\int_0^\eta\frac{1}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp\Big(\beta(\eta-\eta')-G_{\eta,\eta'}\Big)\mathrm{d}\eta'&\leq\int_0^\eta\frac{1}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp\left(-\frac{G_{\eta,\eta'}}{2}\right)\mathrm{d}\eta'\\&\leq\int_0^\infty\mathrm{e}^{-\frac{z}{2}}\mathrm{d}z=2.\end{aligned} \tag{4.105}$$

This leads to

$$\begin{aligned}\left|\mathrm{e}^{\beta\eta}\mathcal{T}[H]\right|&\leq\mathrm{e}^{\beta\eta}\int_0^\eta\left|H\Big(\eta',\phi'(\eta,\phi;\eta')\Big)\right|\frac{1}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'\\&\leq\left\|\mathrm{e}^{\beta\eta}H\right\|_{L^\infty L^\infty}\int_0^\eta\frac{1}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp\Big(\beta(\eta-\eta')-G_{\eta,\eta'}\Big)\mathrm{d}\eta'\\&\leq C\left\|\mathrm{e}^{\beta\eta}H\right\|_{L^\infty L^\infty},\end{aligned} \tag{4.106}$$

and (4.99) naturally follows. For $\sin\phi<0$ and $|E(\eta,\phi)|\leq\mathrm{e}^{-V(L)}$, note that $-G_{L,\eta'}-G_{L,\eta}\leq-G_{\eta,\eta'}$ and for $\eta'\geq\eta$

$$\beta(\eta-\eta')+G_{\eta,\eta'}\leq\beta(\eta-\eta')+\frac{1}{2}(\eta-\eta')+\frac{1}{2}G_{\eta,\eta'}\leq\frac{1}{2}G_{\eta,\eta'}. \tag{4.107}$$

Then (4.99) holds by obvious modifications of $\sin\phi>0$ region. The case $\sin\phi<0$ and $|E(\eta,\phi)|\geq\mathrm{e}^{-V(L)}$ can be shown by combining above two regions, so we omit it here. □

Lemma 4.6. *For any $\delta>0$ there is a constant $C(\delta)>0$ independent of data such that*

$$\|\mathcal{T}[H]\|_{L^\infty L^2}\leq C(\delta)\|H\|_{L^2L^2}+\delta\|H\|_{L^\infty L^\infty}. \tag{4.108}$$

Proof. In the following, we use χ_i to represent certain indicator functions. Also, we let $m>0$ and $\sigma>0$ be some constants that are determined later.

Region I: $\sin\phi>0$.
We have

$$\mathcal{T}[H](\eta,\phi)=\int_0^\eta\frac{H\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'. \tag{4.109}$$

We consider

$$
\begin{aligned}
I=&\int_{\sin\phi>0}\left|\mathcal{T}[H](\eta,\phi)\right|^2\mathrm{d}\phi=\int_{\sin\phi>0}\bigg(\int_0^{\eta}\frac{H\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'\bigg)^2\mathrm{d}\phi\\
=&\,I_1+I_2.
\end{aligned}\tag{4.110}
$$

Region I - Case I: $\chi_1:\sin(\phi'(\eta,\phi;\eta'))\geq m$.
By Cauchy's inequality and (4.101), we get

$$
\begin{aligned}
I_1\leq&\int_{\sin\phi>0}\bigg(\int_0^{\eta}\left|H\Big(\eta',\phi'(\eta,\phi;\eta')\Big)\right|^2\mathrm{d}\eta'\bigg)\bigg(\int_0^{\eta}\chi_1\frac{\exp(-2G_{\eta,\eta'})}{\sin^2\Big(\phi'(\eta,\phi;\eta')\Big)}\mathrm{d}\eta'\bigg)\mathrm{d}\phi\\
\leq&\,\frac{1}{m}\int_{\sin\phi>0}\bigg(\int_0^{\eta}\left|H\Big(\eta',\phi'(\eta,\phi;\eta')\Big)\right|^2\mathrm{d}\eta'\bigg)\bigg(\int_0^{\eta}\chi_1\frac{\exp(-2G_{\eta,\eta'})}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\mathrm{d}\eta'\bigg)\mathrm{d}\phi\\
\leq&\,\frac{1}{m}\|H\|_{L^2L^2}^2\bigg(\int_{\sin\phi>0}\bigg(\int_0^{\eta}\chi_1\frac{\exp(-2G_{\eta,\eta'})}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\mathrm{d}\eta'\bigg)^2\mathrm{d}\phi\bigg)^{\frac{1}{2}}\\
\leq&\,\frac{C}{m}\|H\|_{L^2L^2}^2.
\end{aligned}\tag{4.111}
$$

due to

$$
\int_0^{\eta}\frac{1}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-2G_{\eta,\eta'})\mathrm{d}\eta'\leq\int_0^{\infty}\mathrm{e}^{-2z}\mathrm{d}z=\frac{1}{2}.\tag{4.112}
$$

Region I - Case II: $\chi_2:\sin\Big(\phi'(\eta,\phi;\eta')\Big)\leq m$.
For $\eta'\leq\eta$, we can directly estimate $\phi'(\eta,\phi;\eta')\geq\phi$. Hence, we have the relation

$$
\sin\phi\leq\sin\Big(\phi'(\eta,\phi;\eta')\Big).\tag{4.113}
$$

Therefore, we can directly estimate I_2 as follows:

$$
\begin{aligned}
I_2\leq&\,\|H\|_{L^\infty L^\infty}^2\int_{\sin\phi>0}\bigg(\int_0^{\eta}\chi_2\frac{1}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'\bigg)^2\mathrm{d}\phi\\
\leq&\,\|H\|_{L^\infty L^\infty}^2\int_{\sin\phi>0}\chi_2\mathrm{d}\phi\\
\leq&\,Cm\|H\|_{L^\infty L^\infty}^2,
\end{aligned}\tag{4.114}
$$

due to

$$
\int_0^{\eta}\frac{1}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'\leq\int_0^{\infty}\mathrm{e}^{-z}\mathrm{d}z=1.\tag{4.115}
$$

Summing up (4.111) and (4.114), for m sufficiently small, we deduce (4.108).

Region II: $\sin\phi<0$ and $|E(\eta,\phi)|\leq\mathrm{e}^{-V(L)}$.
We have

$$\begin{aligned}\mathcal{T}[H](\eta,\phi)= &\int_0^L \frac{H\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{L,\eta'}-G_{L,\eta})\mathrm{d}\eta' \\ &+\int_\eta^L \frac{H\Big(\eta',\mathscr{R}[\phi'(\eta,\phi;\eta')]\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(G_{\eta,\eta'})\mathrm{d}\eta'.\end{aligned}\tag{4.116}$$

Since $-G_{L,\eta'}-G_{L,\eta}\leq -G_{\eta,\eta'}$, it suffices to estimate

$$\begin{aligned}II=&\int_{\sin\phi<0}\mathbf{1}_{\{|E(\eta,\phi)|\leq \mathrm{e}^{-V(L)}\}}\bigg(\int_\eta^L \frac{H\Big(\eta',\mathscr{R}[\phi'(\eta,\phi;\eta')]\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(G_{\eta,\eta'})\mathrm{d}\eta'\bigg)^2\mathrm{d}\phi\\ =&II_1+II_2+II_3.\end{aligned}$$

Region II - Case I: $\chi_1:\sin(\phi'(\eta,\phi;\eta'))>m$.
We can directly estimate II_1 as follows:

$$\begin{aligned}II_1\leq&\int_{\sin\phi<0}\bigg(\int_\eta^L\Big|H\Big(\eta',\mathscr{R}[\phi'(\eta,\phi;\eta')]\Big)\Big|^2\mathrm{d}\eta'\bigg)\bigg(\int_\eta^L\chi_1\frac{\exp(2G_{\eta,\eta'})}{\sin^2\Big(\phi'(\eta,\phi;\eta')\Big)}\mathrm{d}\eta'\bigg)\mathrm{d}\phi\\ \leq&\frac{1}{m}\int_{\sin\phi<0}\bigg(\int_\eta^L\Big|H\Big(\eta',\mathscr{R}[\phi'(\eta,\phi;\eta')]\Big)\Big|^2\mathrm{d}\eta'\bigg)\bigg(\int_\eta^L\chi_1\frac{\exp(2G_{\eta,\eta'})}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\mathrm{d}\eta'\bigg)\mathrm{d}\phi\\ \leq&\frac{1}{m}\|H\|_{L^2L^2}^2\int_{\sin\phi<0}\bigg(\int_\eta^L\chi_1\frac{\exp(2G_{\eta,\eta'})}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\mathrm{d}\eta'\bigg)\mathrm{d}\phi\\ \leq&\frac{C}{m}\|H\|_{L^2L^2}^2,\end{aligned}\tag{4.117}$$

due to

$$\int_\eta^L\frac{1}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(2G_{\eta,\eta'})\mathrm{d}\eta'\leq\int_{-\infty}^0\mathrm{e}^{2z}\mathrm{d}z=\frac{1}{2}.\tag{4.118}$$

Region II - Case II: $\chi_2:\sin\Big(\phi'(\eta,\phi;\eta')\Big)>m,\ \eta'-\eta\geq\sigma$.
We have

$$II_2\leq\|H\|_{L^\infty L^\infty}^2\int_{\sin\phi<0}\bigg(\int_\eta^L\chi_2\frac{\exp(G_{\eta,\eta'})}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\mathrm{d}\eta'\bigg)^2\mathrm{d}\phi.\tag{4.119}$$

Note

$$G_{\eta,\eta'}=\int_{\eta'}^\eta\frac{1}{\sin\Big(\phi'(\eta,\phi;y)\Big)}\mathrm{d}y\leq-\frac{\eta'-\eta}{m}=-\frac{\sigma}{m}.\tag{4.120}$$

Then we can obtain

$$II_2 \leq \|H\|^2_{L^\infty L^\infty} \int_{\sin\phi<0} \left(\int_{-\infty}^{-\frac{\sigma}{m}} \mathrm{e}^z \mathrm{d}z \right)^2 \mathrm{d}\phi \leq C\mathrm{e}^{-\frac{\sigma}{m}} \|H\|^2_{L^\infty L^\infty}. \tag{4.121}$$

Region II - Case III: $\chi_3 : \sin\Big(\phi'(\eta,\phi;\eta')\Big) > m,\ \eta'-\eta \leq \sigma$
For II_3, we can estimate as follows:

$$\begin{aligned} I_3 &\leq \|H\|^2_{L^\infty L^\infty} \int_{\sin\phi<0} \left(\int_\eta^L \chi_3 \frac{\exp(G_{\eta,\eta'})}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)} \mathrm{d}\eta' \right)^2 \mathrm{d}\phi \\ &\leq \|H\|^2_{L^\infty L^\infty} \int_{\sin\phi<0} \chi_3 \left(\int_\eta^{\eta+\sigma} \frac{\exp(G_{\eta,\eta'})}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)} \mathrm{d}\eta' \right)^2 \mathrm{d}\phi. \end{aligned} \tag{4.122}$$

Note that

$$\int_\eta^L \frac{1}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)} \exp(G_{\eta,\eta'})\mathrm{d}\eta' \leq \int_{-\infty}^0 \mathrm{e}^z \mathrm{d}z = 1. \tag{4.123}$$

Then $1 \leq \alpha = \mathrm{e}^{V(\eta')-V(\eta)} \leq \mathrm{e}^{V(\eta+\sigma)-V(\eta)} \leq 1+4\sigma$, and for $\eta' \in [\eta,\eta+\sigma]$, $\sin\Big(\phi'(\eta,\phi;\eta')\Big) = \sin\bigg(\cos^{-1}(\alpha\cos\phi)\bigg)$, $\sin\Big(\phi'(\eta,\phi;\eta')\Big) < m$ lead to

$$\begin{aligned} |\sin\phi| &= \sqrt{1-\cos^2\phi} = \sqrt{1 - \frac{\cos^2\Big(\phi'(\eta,\phi;\eta')\Big)}{\alpha^2}} = \frac{\sqrt{\alpha^2 - \bigg(1-\sin^2\Big(\phi'(\eta,\phi;\eta')\Big)\bigg)}}{\alpha} \\ &\leq \frac{\sqrt{\alpha^2-1+m^2}}{\alpha} \leq \frac{\sqrt{(1+4\sigma)^2-1+m^2}}{\alpha} \leq \sqrt{9\sigma+m^2}. \end{aligned} \tag{4.124}$$

Hence, we can obtain

$$I_3 \leq \|H\|^2_{L^\infty L^\infty} \int_{\sin\phi<0} \chi_3 \mathrm{d}\phi \leq \|H\|^2_{L^\infty L^\infty} \int_{\sin\phi<0} \mathbf{1}_{\{|\sin\phi| \leq \sqrt{9\sigma+m^2}\}} \mathrm{d}\phi \leq C\sqrt{\sigma+m^2}. \tag{4.125}$$

Summarizing (4.117), (4.121) and (4.125), for sufficiently small σ, we can always choose $m << \sigma$ small enough to guarantee the relation (4.108).

Region III: $\sin\phi < 0$ and $|E(\eta,\phi)| \geq \mathrm{e}^{-V(L)}$.
We have

$$\begin{aligned} \mathcal{T}[H](\eta,\phi) = &\int_0^{\eta^+} \frac{H\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)} \exp(-G_{\eta^+,\eta'} - G_{\eta^+,\eta})\mathrm{d}\eta' \\ &+ \int_\eta^{\eta^+} \frac{H\Big(\eta',\mathscr{R}[\phi'(\eta,\phi;\eta')]\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)} \exp(G_{\eta,\eta'})\mathrm{d}\eta'. \end{aligned} \tag{4.126}$$

We can decompose $\mathcal{T}[H]$. For the integral on $[0,\eta]$, we can apply a similar argument as in Region 1 and for the integral on $[\eta,\eta^+]$, a similar argument as in Region 2 concludes the proof. $\square$

Consider the equation satisfied by $\mathscr{V}=f-f_L$ as follows:

$$\begin{cases} \sin\phi\dfrac{\partial\mathscr{V}}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial\mathscr{V}}{\partial\phi}+\mathscr{V}=\bar{\mathscr{V}}+S,\\ \mathscr{V}(0,\phi)=p(\phi)=h(\phi)-f_L \quad\text{for}\quad \sin\phi>0,\\ \mathscr{V}(L,\phi)=\mathscr{V}(L,\mathscr{R}[\phi]). \end{cases} \tag{4.127}$$

Theorem 4.7. *The unique solution $f(\eta,\phi)$ to the equation (4.3) satisfies*

$$\|f-f_L\|_{L^\infty L^\infty}\leq C\bigg(|f_L|+\|h\|_{L^\infty}+\|S\|_{L^\infty L^\infty}+\|f-f_L\|_{L^2L^2}\bigg). \tag{4.128}$$

Proof. We first show the following important facts:

$$\big\|\bar{\mathscr{V}}\big\|_{L^2L^2}\leq\|\mathscr{V}\|_{L^2L^2}, \tag{4.129}$$

$$\big\|\bar{\mathscr{V}}\big\|_{L^\infty L^\infty}\leq\|\mathscr{V}\|_{L^\infty L^2}. \tag{4.130}$$

We can directly derive them by Cauchy's inequality as follows:

$$\begin{aligned}\|\bar{\mathscr{V}}\|_{L^2L^2}^2 &= \int_0^\infty\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\right)^2\left(\int_{-\pi}^{\pi}\mathscr{V}(\eta,\phi)\mathrm{d}\phi\right)^2\mathrm{d}\phi\mathrm{d}\eta\leq\int_0^\infty\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\right)\left(\int_{-\pi}^{\pi}\mathscr{V}^2(\eta,\phi)\mathrm{d}\phi\right)\mathrm{d}\phi\mathrm{d}\eta\\ &= \int_0^\infty\left(\int_{-\pi}^{\pi}\mathscr{V}^2(\eta,\phi)\mathrm{d}\phi\right)\mathrm{d}\eta=\|\mathscr{V}\|_{L^2L^2}^2.\end{aligned} \tag{4.131}$$

$$\begin{aligned}\|\bar{\mathscr{V}}\|_{L^\infty L^\infty}^2 &= \sup_\eta\bar{\mathscr{V}}^2(\eta)=\sup_\eta\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathscr{V}(\eta,\phi)\mathrm{d}\phi\right)^2\leq\sup_\eta\left(\frac{1}{2\pi}\right)^2\left(\int_{-\pi}^{\pi}\mathscr{V}^2(\eta,\phi)\mathrm{d}\phi\right)\left(\int_{-\pi}^{\pi}1^2\mathrm{d}\phi\right)\\ &= \sup_\eta\left(\int_{-\pi}^{\pi}\mathscr{V}^2(\eta,\phi)\mathrm{d}\phi\right)=\|\mathscr{V}\|_{L^\infty L^2}^2.\end{aligned} \tag{4.132}$$

By (4.127), $\mathscr{V}=\mathcal{K}[p]+\mathcal{T}[\bar{\mathscr{V}}]+\mathcal{T}[S]$ leads to

$$\mathcal{T}[\bar{\mathscr{V}}]=\mathscr{V}-\mathcal{K}[p]-\mathcal{T}[S], \tag{4.133}$$

Then by Lemma 4.6, (4.129) and (4.130), we can show

$$\|\mathscr{V}-\mathcal{K}[p]-\mathcal{T}[S]\|_{L^\infty L^2}\leq C(\delta)\big\|\bar{\mathscr{V}}\big\|_{L^2L^2}+\delta\big\|\bar{\mathscr{V}}\big\|_{L^\infty L^\infty}\leq C(\delta)\|\mathscr{V}\|_{L^2L^2}+\delta\|\mathscr{V}\|_{L^\infty L^2}. \tag{4.134}$$

Therefore, based on Lemma 4.4, Lemma 4.5 and (4.134), we can directly estimate

$$\begin{aligned}\|\mathscr{V}\|_{L^\infty L^2} &\leq \|\mathcal{K}[p]\|_{L^\infty}+\|\mathcal{T}[S]\|_{L^\infty L^2}+C(\delta)\|\mathscr{V}\|_{L^2L^2}+\delta\|\mathscr{V}\|_{L^\infty L^2}\\ &\leq \|p\|_{L^\infty}+\|S\|_{L^\infty L^\infty}+C(\delta)\|\mathscr{V}\|_{L^2L^2}+\delta\|\mathscr{V}\|_{L^\infty L^2}.\end{aligned} \tag{4.135}$$

We can take $\delta=\dfrac{1}{2}$ to obtain

$$\|\mathscr{V}\|_{L^\infty L^2}\leq C\bigg(\|\mathscr{V}\|_{L^2L^2}+\|p\|_{L^\infty}+\|S\|_{L^\infty L^\infty}\bigg). \tag{4.136}$$

Therefore, based on Lemma 4.5, (4.136) and (4.130), we can achieve

$$\begin{aligned}\|\mathscr{V}\|_{L^\infty L^\infty}&\leq \|\mathcal{K}[p]\|_{L^\infty L^\infty}+\|\mathcal{T}[S]\|_{L^\infty L^\infty}+\left\|\mathcal{T}[\bar{\mathscr{V}}]\right\|_{L^\infty L^\infty}\\&\leq C\bigg(\|p\|_{L^\infty}+\|S\|_{L^\infty L^\infty}+\left\|\bar{\mathscr{V}}\right\|_{L^\infty L^\infty}\bigg)\\&\leq C\bigg(\|p\|_{L^\infty}+\|S\|_{L^\infty L^\infty}+\|\mathscr{V}\|_{L^\infty L^2}\bigg)\\&\leq C\bigg(\|p\|_{L^\infty}+\|S\|_{L^\infty L^\infty}+\|\mathscr{V}\|_{L^2L^2}\bigg).\end{aligned} \tag{4.137}$$

□

Combining Theorem 4.7 and Theorem 4.3, we deduce the main theorem.

Theorem 4.8. *The unique solution $f(\eta,\phi)$ to the equation (4.3) satisfies*

$$\begin{aligned}\|f-f_L\|_{L^\infty L^\infty}\leq{}& C\bigg(\|h\|_{L^2_-}+\|S\|_{L^2L^2}+\|h\|_{L^\infty}+\|S\|_{L^\infty L^\infty}\bigg)\\&+C\bigg(\int_0^L\bigg(\int_\eta^L\langle\sin\phi,S_R\rangle_\phi(y)\mathrm{d}y\bigg)^2\mathrm{d}\eta\bigg)^{\frac{1}{2}}+C\left|\int_0^L\langle\sin\phi,S_R\rangle_\phi(y)\mathrm{d}y\right|\\&+C\left|\int_0^L\int_\eta^L|S_Q(y)|\,\mathrm{d}y\mathrm{d}\eta\right|+C\bigg(\int_0^L\bigg(\int_\eta^L|S_Q(y)|\,\mathrm{d}y\bigg)^2\mathrm{d}\eta\bigg)^{\frac{1}{2}}\\&+C\bigg(\int_0^L\bigg(\int_\eta^L\int_y^L|S_Q(z)|\,\mathrm{d}z\mathrm{d}y\bigg)^2\mathrm{d}\eta\bigg)^{\frac{1}{2}},\end{aligned} \tag{4.138}$$

for some $f_L\in\mathbb{R}$ satisfying

$$|f_L|\leq C\bigg(\|h\|_{L^2_-}+\|S\|_{L^2L^2}\bigg)+C\left|\int_0^L\langle\sin\phi,S_R\rangle_\phi(y)\mathrm{d}y\right|+C\left|\int_0^L\int_\eta^L|S_Q(y)|\,\mathrm{d}y\mathrm{d}\eta\right|. \tag{4.139}$$

In this section, we prove the spatial decay of the solution to the Milne problem.

Theorem 4.9. *For $K_0>0$ sufficiently small, the unique solution $f(\eta,\phi)$ to the equation*

$$
\begin{aligned}
\|f-f_L\|_{L^\infty L^\infty} \leq\ & C\bigg(\|h\|_{L^2_-}+\left\|\mathrm{e}^{K_0\eta}S\right\|_{L^2L^2}+\|h\|_{L^\infty}+\left\|\mathrm{e}^{K_0\eta}S\right\|_{L^\infty L^\infty}\bigg) \\
&+C\bigg(\int_0^L \mathrm{e}^{2K_0\eta}\bigg(\int_\eta^L \langle\sin\phi, S_R\rangle_\phi(y)\mathrm{d}y\bigg)^2\mathrm{d}\eta\bigg)^{\frac{1}{2}}+C\left|\int_0^L \langle\sin\phi, S_R\rangle_\phi(y)\mathrm{d}y\right| \\
&+C\left|\int_0^L\int_\eta^L |S_Q(y)|\,\mathrm{d}y\mathrm{d}\eta\right|+C\bigg(\int_0^L \mathrm{e}^{2K_0\eta}\bigg(\int_\eta^L |S_Q(y)|\,\mathrm{d}y\bigg)^2\mathrm{d}\eta\bigg)^{\frac{1}{2}} \\
&+C\bigg(\int_0^L \mathrm{e}^{2K_0\eta}\bigg(\int_\eta^L\int_y^L |S_Q(z)|\,\mathrm{d}z\mathrm{d}y\bigg)^2\mathrm{d}\eta\bigg)^{\frac{1}{2}},
\end{aligned}
\tag{4.140}
$$

for some $f_L\in\mathbb{R}$ satisfying

$$
|f_L|\leq C\bigg(\|h\|_{L^2_-}+\|S\|_{L^2L^2}\bigg)+C\left|\int_0^L \langle\sin\phi, S_R\rangle_\phi(y)\mathrm{d}y\right|+C\left|\int_0^L\int_\eta^L |S_Q(y)|\,\mathrm{d}y\mathrm{d}\eta\right|. \tag{4.141}
$$

Proof. Define $Z=\mathrm{e}^{K_0\eta}\mathscr{V}$ for $\mathscr{V}=f-f_L$. We divide the analysis into several steps:

Step 1: L^2 estimates.
The orthogonal property reveals

$$
\langle f, f\sin\phi\rangle_\phi(\eta)=\langle r, r\sin\phi\rangle_\phi(\eta). \tag{4.142}
$$

Multiplying $\mathrm{e}^{2K_0\eta}f$ on both sides of equation (4.3) and integrating over $\phi\in[-\pi,\pi)$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\eta}\bigg(\mathrm{e}^{2K_0\eta}\langle r, r\sin\phi\rangle_\phi(\eta)\bigg)+\frac{1}{2}F(\eta)\bigg(\mathrm{e}^{2K_0\eta}\langle r, r\sin\phi\rangle_\phi(\eta)\bigg) \\
&\qquad -\mathrm{e}^{2K_0\eta}\bigg(K_0\langle r, r\sin\phi\rangle_\phi(\eta)-\langle r, r\rangle_\phi(\eta)\bigg)=\mathrm{e}^{2K_0\eta}\langle S, f\rangle_\phi(\eta).
\end{aligned}
\tag{4.143}
$$

For $K_0<\min\left\{\dfrac{1}{2}, K\right\}$, we have

$$
\frac{3}{2}\|r(\eta)\|_{L^2}^2\geq -K_0\langle r, r\sin\phi\rangle_\phi(\eta)+\langle r, r\rangle_\phi(\eta)\geq\frac{1}{2}\|r(\eta)\|_{L^2}^2. \tag{4.144}
$$

Similar to the proof of Lemma 4.1, formula as (4.143) and (4.144) imply

$$
\left\|\mathrm{e}^{K_0\eta}r\right\|_{L^2L^2}^2=\int_0^L \mathrm{e}^{2K_0\eta}\langle r, r\rangle_\phi(\eta)\mathrm{d}\eta\leq C\bigg(\|h\|_{L^2_-}^2+\left\|\mathrm{e}^{K_0\eta}S\right\|_{L^2L^2}^2\bigg). \tag{4.145}
$$

From the proof of Lemma 4.1 and Cauchy's inequality, we can deduce

$$
\begin{aligned}
&\int_0^L \mathrm{e}^{2K_0\eta}\bigg(\int_{-\pi}^{\pi}(f(\eta,\phi)-f_L)^2\mathrm{d}\phi\bigg)\mathrm{d}\eta \\
\leq\;& \int_0^L \mathrm{e}^{2K_0\eta}\bigg(\int_{-\pi}^{\pi}r^2(\eta,\phi)\mathrm{d}\phi\bigg)\mathrm{d}\eta+\int_0^L \mathrm{e}^{2K_0\eta}\bigg(\int_{-\pi}^{\pi}\Big(q(\eta)-q_L\Big)^2\mathrm{d}\phi\bigg)\mathrm{d}\eta \\
\leq\;& \int_0^L \mathrm{e}^{2K_0\eta}\left\|r(\eta)\right\|_{L^2}^2\mathrm{d}\eta \\
&+\int_0^L \mathrm{e}^{2K_0\eta}\bigg(\int_{\eta}^{L}\left|F(y)\right|\left\|r(y)\right\|_{L^2}\mathrm{d}y\bigg)^2\mathrm{d}\eta+\int_0^L \mathrm{e}^{2K_0\eta}\bigg(\int_{\eta}^{L}\left\langle\sin\phi,S\right\rangle_{\phi}(y)\mathrm{d}y\bigg)^2\mathrm{d}\eta \\
\leq\;& C\bigg(\left\|h\right\|_{L^2_-}^2+\left\|\mathrm{e}^{K_0\eta}S\right\|_{L^2L^2}^2\bigg) \\
&+C\bigg(\int_0^L \mathrm{e}^{2K_0\eta}\left\|r(\eta)\right\|_{L^2}^2\mathrm{d}\eta\bigg)\bigg(\int_0^L\int_{\eta}^{L}\mathrm{e}^{2K_0(\eta-y)}F^2(y)\mathrm{d}y\mathrm{d}\eta\bigg)+\int_0^L \mathrm{e}^{2K_0\eta}\bigg(\int_{\eta}^{L}\left\langle\sin\phi,S\right\rangle_{\phi}(y)\mathrm{d}y\bigg)^2\mathrm{d}\eta \\
\leq\;& C\bigg(\left\|h\right\|_{L^2_-}^2+\left\|\mathrm{e}^{K_0\eta}S\right\|_{L^2L^2}^2\bigg) \\
&+C\bigg(\int_0^L \mathrm{e}^{2K_0\eta}\left\|r(\eta)\right\|_{L^2}^2\mathrm{d}\eta\bigg)\bigg(\int_0^L\int_{\eta}^{L}F^2(y)\mathrm{d}y\mathrm{d}\eta\bigg)+\int_0^L \mathrm{e}^{2K_0\eta}\bigg(\int_{\eta}^{L}\left\langle\sin\phi,S\right\rangle_{\phi}(y)\mathrm{d}y\bigg)^2\mathrm{d}\eta \\
\leq\;& C\bigg(\left\|h\right\|_{L^2_-}^2+\left\|\mathrm{e}^{K_0\eta}S\right\|_{L^2L^2}^2\bigg)+\int_0^L \mathrm{e}^{2K_0\eta}\bigg(\int_{\eta}^{L}\left\langle\sin\phi,S\right\rangle_{\phi}(y)\mathrm{d}y\bigg)^2\mathrm{d}\eta.
\end{aligned}
\tag{4.146}
$$

This completes the proof of L^2 estimate when $\bar{S}=0$. By the method introduced in Lemma 4.2, we can extend above L^2 estimates to the general S case. Note all the auxiliary functions constructed in Lemma 4.2 satisfy the estimates. We have

$$
\begin{aligned}
\left\|Z\right\|_{L^2L^2}\leq\;& C\bigg(\left\|h\right\|_{L^2_-}+\left\|S\right\|_{L^2L^2}\bigg)+C\Bigg(\int_0^L \mathrm{e}^{2K_0\eta}\bigg(\int_{\eta}^{L}\left\langle\sin\phi,S_R\right\rangle_{\phi}(y)\mathrm{d}y\bigg)^2\mathrm{d}\eta\Bigg)^{\frac{1}{2}} \\
&+C\Bigg(\int_0^L \mathrm{e}^{2K_0\eta}\bigg(\int_{\eta}^{L}\left|S_Q(y)\right|\mathrm{d}y\bigg)^2\mathrm{d}\eta\Bigg)^{\frac{1}{2}}+C\Bigg(\int_0^L \mathrm{e}^{2K_0\eta}\bigg(\int_{\eta}^{L}\int_{y}^{L}\left|S_Q(z)\right|\mathrm{d}z\mathrm{d}y\bigg)^2\mathrm{d}\eta\Bigg)^{\frac{1}{2}},
\end{aligned}
\tag{4.147}
$$

Step 2: L^∞ estimates.
Z satisfies the equation

$$
\left\{
\begin{aligned}
&\sin\phi\frac{\partial Z}{\partial\eta}+F(\eta)\cos\phi\frac{\partial Z}{\partial\phi}+Z=\bar{Z}+\mathrm{e}^{K_0\eta}S+K_0\sin\phi Z,\\
&Z(0,\phi)=p(\phi)=h(\phi)-f_L\quad\text{for}\quad\sin\phi>0\\
&Z(L,\phi)=Z(L,\mathscr{R}[\phi]).
\end{aligned}
\right.
\tag{4.148}
$$

Since we know $Z=\mathcal{K}[p]+\mathcal{T}[\bar{Z}+\mathrm{e}^{K_0\eta}S+K_0\sin\phi Z]$ leads to

$$
\mathcal{T}[\bar{Z}]=Z-\mathcal{K}[p]-\mathcal{T}[\mathrm{e}^{K_0\eta}S]-\mathcal{T}[K_0\sin\phi Z],
\tag{4.149}
$$

then by Lemma 4.6, (4.129) and (4.130), we can show

$$
\begin{aligned}
\left\|Z-\mathcal{K}[p]-\mathcal{T}[\mathrm{e}^{K_0\eta}S]-\mathcal{T}[K_0\sin\phi Z]\right\|_{L^\infty L^2}&\leq C(\delta)\left\|\bar{Z}\right\|_{L^2L^2}+\delta\left\|\bar{Z}\right\|_{L^\infty L^\infty}\\
&\leq C(\delta)\left\|Z\right\|_{L^2L^2}+\delta\left\|Z\right\|_{L^\infty L^2}.
\end{aligned}
\tag{4.150}
$$

Therefore, based on Lemma 4.4 and (4.134), we can directly estimate

$$
\begin{aligned}
\|Z\|_{L^\infty L^2} \leq\; & \|\mathcal{K}[p]\|_{L^\infty} + \left\|\mathcal{T}[\mathrm{e}^{K_0\eta}S]\right\|_{L^\infty L^\infty} + \|\mathcal{T}[K_0\sin\phi Z]\|_{L^\infty L^\infty} + C(\delta)\|Z\|_{L^2L^2} + \delta\|Z\|_{L^\infty L^2} \\
\leq\; & \|p\|_{L^\infty} + \left\|\mathrm{e}^{K_0\eta}S\right\|_{L^\infty L^\infty} + K_0\|Z\|_{L^\infty L^\infty} + C(\delta)\|Z\|_{L^2L^2} + \delta\|Z\|_{L^\infty L^2}.
\end{aligned}
\tag{4.151}
$$

We can take $\delta = \dfrac{1}{2}$ to obtain

$$
\|Z\|_{L^\infty L^2} \leq C\bigg(\|p\|_{L^\infty} + \left\|\mathrm{e}^{K_0\eta}S\right\|_{L^\infty L^\infty} + K_0\|Z\|_{L^\infty L^\infty} + \|Z\|_{L^2L^2}\bigg). \tag{4.152}
$$

Then based on Lemma 4.4, Lemma 4.5 and Lemma 4.6, we can deduce

$$
\begin{aligned}
\|Z\|_{L^\infty L^\infty} \leq\; & \left\|\mathrm{e}^{K_0\eta}\mathcal{K}[p]\right\|_{L^\infty} + \left\|\mathrm{e}^{K_0\eta}\mathcal{T}[S]\right\|_{L^\infty L^\infty} + \left\|\bar{Z}\right\|_{L^\infty L^\infty} + \|K_0\sin\phi Z\|_{L^\infty L^\infty} \\
\leq\; & \|p\|_{L^\infty} + \left\|\mathrm{e}^{K_0\eta}S\right\|_{L^\infty L^\infty} + \left\|\bar{Z}\right\|_{L^\infty L^\infty} + K_0\|Z\|_{L^\infty L^\infty} \\
\leq\; & \|p\|_{L^\infty} + \left\|\mathrm{e}^{K_0\eta}S\right\|_{L^\infty L^\infty} + \|Z\|_{L^\infty L^2} + K_0\|Z\|_{L^\infty L^\infty} \\
\leq\; & C\bigg(\|Z\|_{L^2L^2} + \left\|\mathrm{e}^{K_0\eta}S\right\|_{L^2L^2} + \left\|\mathrm{e}^{K_0\eta}S\right\|_{L^\infty L^\infty} + \|p\|_{L^\infty} + K_0\|Z\|_{L^\infty L^\infty}\bigg).
\end{aligned}
\tag{4.153}
$$

Taking K_0 sufficiently small, we absorb $K_0\|Z\|_{L^\infty L^\infty}$ to the left-hand side and obtain

$$
\|Z\|_{L^\infty L^\infty} \leq C\bigg(\|Z\|_{L^2L^2} + \left\|\mathrm{e}^{K_0\eta}S\right\|_{L^2L^2} + \left\|\mathrm{e}^{K_0\eta}S\right\|_{L^\infty L^\infty} + \|p\|_{L^\infty}\bigg). \tag{4.154}
$$

Then the final result is obvious. □

In [21], the author proved the maximum principle.

Theorem 4.10. *The unique solution $f(\eta,\phi)$ to the equation with $S=0$ satisfies the maximum principle, i.e.*

$$
\min_{\sin\phi>0} h(\phi) \leq f(\eta,\phi) \leq \max_{\sin\phi>0} h(\phi). \tag{4.155}
$$

5. Regularity of ϵ-Milne Problem with Geometric Correction

We consider the ϵ-Milne problem with geometric correction for $f^{\epsilon}(\eta,\tau,\phi)$ in the domain $(\eta,\tau,\phi)\in[0,L]\times[-\pi,\pi)\times[-\pi,\pi)$ where $L=\epsilon^{-\frac{1}{2}}$ as

$$\begin{cases} \sin\phi\dfrac{\partial f^{\epsilon}}{\partial\eta}+F(\epsilon;\eta,\tau)\cos\phi\dfrac{\partial f^{\epsilon}}{\partial\phi}+f^{\epsilon}-\bar{f}^{\epsilon}=S^{\epsilon}(\eta,\tau,\phi),\\ f^{\epsilon}(0,\tau,\phi)=h^{\epsilon}(\tau,\phi)\quad\text{for}\quad\sin\phi>0,\\ f^{\epsilon}(L,\tau,\phi)=f^{\epsilon}(L,\tau,\mathscr{R}[\phi]), \end{cases} \tag{5.1}$$

where $\mathscr{R}[\phi]=-\phi$ and

$$F(\epsilon;\eta,\tau)=-\frac{\epsilon}{R_{\kappa}(\tau)-\epsilon\eta}, \tag{5.2}$$

for the radius of curvature R_{κ}. In this section, for convenience, we temporarily ignore the superscript on ϵ and τ. In other words, we will study

$$\begin{cases} \sin\phi\dfrac{\partial f}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial f}{\partial\phi}+f-\bar{f}=S(\eta,\phi),\\ f(0,\phi)=h(\phi)\quad\text{for}\quad\sin\phi>0,\\ f(L,\phi)=f(L,\mathscr{R}[\phi]). \end{cases} \tag{5.3}$$

Define potential function $V(\eta)$ satisfying $V(0)=0$ and $\dfrac{\partial V}{\partial\eta}=-F(\eta)$. Then we can direct compute

$$V(\eta)=\ln\left(\frac{R_{\kappa}}{R_{\kappa}-\epsilon\eta}\right). \tag{5.4}$$

Define the weight function

$$\zeta(\eta,\phi)=\left(1-\left(\frac{R_{\kappa}-\epsilon\eta}{R_{\kappa}}\cos\phi\right)^{2}\right)^{\frac{1}{2}}. \tag{5.5}$$

We can easily show that

$$\sin\phi\frac{\partial\zeta}{\partial\eta}+F(\eta)\cos\phi\frac{\partial\zeta}{\partial\phi}=0. \tag{5.6}$$

It is easy to see $\mathscr{V}(\eta,\tau,\phi)=f(\eta,\tau,\phi)-f_L(\tau)$ satisfies the equation

$$\begin{cases} \sin\phi\dfrac{\partial\mathscr{V}}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial\mathscr{V}}{\partial\phi}+\mathscr{V}=\bar{\mathscr{V}}+S,\\ \mathscr{V}(0,\phi)=p(\phi)=h(\phi)-f_L\quad\text{for}\quad\sin\phi>0,\\ \mathscr{V}(L,\phi)=\mathscr{V}(L,\mathscr{R}[\phi]). \end{cases} \tag{5.7}$$

The regularity has been thoroughly studied in [5]. However, here we will focus on the a priori estimates and prove an improved version of the regularity theorem. The major upshot is that we can avoid using the information of $\dfrac{\partial S}{\partial\phi}$.

Consider the ϵ-transport problem for $\mathscr{A}=\zeta\dfrac{\partial\mathscr{V}}{\partial\eta}$ as

$$\begin{cases} \sin\phi\dfrac{\partial\mathscr{A}}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial\mathscr{A}}{\partial\phi}+\mathscr{A}=\tilde{\mathscr{A}}+S_{\mathscr{A}}, \\ \mathscr{A}(0,\phi)=p_{\mathscr{A}}(\phi)\quad\text{for}\quad\sin\phi>0, \\ \mathscr{A}(L,\phi)=\mathscr{A}(L,R\phi), \end{cases} \tag{5.8}$$

where $p_{\mathscr{A}}$ and $S_{\mathscr{A}}$ will be specified later with

$$\tilde{\mathscr{A}}(\eta,\phi)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\zeta(\eta,\phi)}{\zeta(\eta,\phi_*)}\mathscr{A}(\eta,\phi_*)\mathrm{d}\phi_*. \tag{5.9}$$

Define the energy as before

$$E(\eta,\phi)=\mathrm{e}^{-V(\eta)}\cos\phi=\cos\phi\frac{R_\kappa-\epsilon\eta}{R_\kappa}. \tag{5.10}$$

Along the characteristics, where this energy is conserved and ζ is a constant, the equation can be simplified as follows:

$$\sin\phi\frac{\mathrm{d}\mathscr{A}}{\mathrm{d}\eta}+\mathscr{A}=\tilde{\mathscr{A}}+S_{\mathscr{A}}. \tag{5.11}$$

An implicit function $\eta^+(\eta,\phi)$ can be determined through

$$\lvert E(\eta,\phi)\rvert=\mathrm{e}^{-V(\eta^+)}. \tag{5.12}$$

which means (η^+,ϕ_0) with $\sin\phi_0=0$ is on the same characteristics as (η,ϕ). Define the quantities for $0\leq\eta'\leq\eta^+$ as follows:

$$\phi'(\eta,\phi;\eta')=\cos^{-1}\left(\mathrm{e}^{V(\eta')-V(\eta)}\cos\phi\right), \tag{5.13}$$

$$\mathscr{R}[\phi'(\eta,\phi;\eta')]=-\cos^{-1}\left(\mathrm{e}^{V(\eta')-V(\eta)}\cos\phi\right)=-\phi'(\eta,\phi;\eta'), \tag{5.14}$$

where the inverse trigonometric function can be defined single-valued in the domain $[0,\pi)$ and the quantities are always well-defined due to the monotonicity of V. Note that $\sin\phi'\geq0$, even if $\sin\phi<0$. Finally we put

$$G_{\eta,\eta'}(\phi)=\int_{\eta'}^{\eta}\frac{1}{\sin\Big(\phi'(\eta,\phi;\xi)\Big)}\mathrm{d}\xi. \tag{5.15}$$

Similar to ϵ-Milne problem, we can define the solution along the characteristics as follows:

$$\mathscr{A}(\eta,\phi)=\mathcal{K}[p_{\mathscr{A}}]+\mathcal{T}[\tilde{\mathscr{A}}+S_{\mathscr{A}}], \tag{5.16}$$

where

Region I:
For $\sin\phi>0$,

$$\mathcal{K}[p_{\mathscr{A}}] = p_{\mathscr{A}}\Big(\phi'(\eta,\phi;0)\Big)\exp(-G_{\eta,0}) \tag{5.17}$$

$$\mathcal{T}[\tilde{\mathscr{A}}+S_{\mathscr{A}}] = \int_0^{\eta}\frac{(\tilde{\mathscr{A}}+S_{\mathscr{A}})\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'. \tag{5.18}$$

Region II:
For $\sin\phi<0$ and $|E(\eta,\phi)|\leq \mathrm{e}^{-V(L)}$,

$$\mathcal{K}[p_{\mathscr{A}}] = p_{\mathscr{A}}\Big(\phi'(\eta,\phi;0)\Big)\exp(-G_{L,0}-G_{L,\eta}) \tag{5.19}$$

$$\begin{aligned}\mathcal{T}[\tilde{\mathscr{A}}+S_{\mathscr{A}}] = &\int_0^{L}\frac{(\tilde{\mathscr{A}}+S)\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{L,\eta'}-G_{L,\eta})\mathrm{d}\eta' \\ &+\int_{\eta}^{L}\frac{(\tilde{\mathscr{A}}+S)\Big(\eta',\mathscr{R}[\phi'(\eta,\phi;\eta')]\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta',\eta})\mathrm{d}\eta'.\end{aligned} \tag{5.20}$$

Region III:
For $\sin\phi<0$ and $|E(\eta,\phi)|\geq \mathrm{e}^{-V(L)}$,

$$\mathcal{K}[p_{\mathscr{A}}] = p_{\mathscr{A}}\Big(\phi'(\eta,\phi;0)\Big)\exp(-G_{\eta^+,0}-G_{\eta^+,\eta}) \tag{5.21}$$

$$\begin{aligned}\mathcal{T}[\tilde{\mathscr{A}}+S_{\mathscr{A}}] = &\int_0^{\eta^+}\frac{(\tilde{\mathscr{A}}+S_{\mathscr{A}})\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta^+,\eta'}-G_{\eta^+,\eta})\mathrm{d}\eta' \\ &+\int_{\eta}^{\eta^+}\frac{(\tilde{\mathscr{A}}+S_{\mathscr{A}})\Big(\eta',\mathscr{R}[\phi'(\eta,\phi;\eta')]\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta',\eta})\mathrm{d}\eta'.\end{aligned} \tag{5.22}$$

Then we need to estimate $\mathcal{K}[p_{\mathscr{A}}]$ and $\mathcal{T}[\tilde{\mathscr{A}}+S_{\mathscr{A}}]$ in each region. We assume $0<\delta<<1$ and $0<\delta_0<<1$ are small quantities which will be determined later. Since we always assume that (η,ϕ) and (η',ϕ') are on the same characteristics, when there is no confusion, we simply write ϕ' or $\phi'(\eta')$ instead of $\phi'(\eta,\phi;\eta')$.

We consider

$$\mathcal{K}[p_{\mathscr{A}}] = p_{\mathscr{A}}\Big(\phi'(\eta,\phi;0)\Big)\exp(-G_{\eta,0}) \tag{5.23}$$

$$\mathcal{T}[\tilde{\mathscr{A}}+S_{\mathscr{A}}] = \int_0^{\eta}\frac{(\tilde{\mathscr{A}}+S_{\mathscr{A}})\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'. \tag{5.24}$$

Based on [21, Lemma 4.7, Lemma 4.8], we can directly obtain

$$\left\|\mathcal{K}[p_{\mathscr{A}}]\right\|_{L^\infty} \leq \left\|p_{\mathscr{A}}\right\|_{L^\infty}, \tag{5.25}$$

$$\left\|\mathcal{T}[S_{\mathscr{A}}]\right\|_{L^\infty L^\infty} \leq \left\|S_{\mathscr{A}}\right\|_{L^\infty L^\infty}. \tag{5.26}$$

Hence, we only need to estimate

$$I=\mathcal{T}[\tilde{\mathscr{A}}]=\int_0^{\eta}\frac{\tilde{\mathscr{A}}\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'. \tag{5.27}$$

We divide it into several steps:

Step 0: Preliminaries.
We have

$$E(\eta',\phi')=\frac{R_\kappa-\epsilon\eta'}{R_\kappa}\cos\phi'. \tag{5.28}$$

We can directly obtain

$$\begin{aligned}\zeta(\eta',\phi')&=\frac{1}{R_\kappa}\sqrt{R_\kappa^2-\Big((R_\kappa-\epsilon\eta')\cos\phi'\Big)^2}=\frac{1}{R_\kappa}\sqrt{R_\kappa^2-(R_\kappa-\epsilon\eta')^2+(R_\kappa-\epsilon\eta')^2\sin^2\phi'}, \\ &\leq\sqrt{R_\kappa^2-(R_\kappa-\epsilon\eta')^2}+\sqrt{(R_\kappa-\epsilon\eta')^2\sin^2\phi'}\leq C\Big(\sqrt{\epsilon\eta'}+\sin\phi'\Big),\end{aligned} \tag{5.29}$$

and

$$\zeta(\eta',\phi')\geq\frac{1}{R_\kappa}\sqrt{R_\kappa^2-(R_\kappa-\epsilon\eta')^2}\geq C\sqrt{\epsilon\eta'}. \tag{5.30}$$

Also, we know for $0\leq\eta'\leq\eta$,

$$\sin\phi'=\sqrt{1-\cos^2\phi'}=\sqrt{1-\bigg(\frac{R_\kappa-\epsilon\eta}{R_\kappa-\epsilon\eta'}\bigg)^2\cos^2\phi} \tag{5.31}$$

$$=\frac{\sqrt{(R_\kappa-\epsilon\eta')^2\sin^2\phi+(2R_\kappa-\epsilon\eta-\epsilon\eta')(\epsilon\eta-\epsilon\eta')\cos^2\phi}}{R_\kappa-\epsilon\eta'}. \tag{5.32}$$

Since

$$0\leq(2R_\kappa-\epsilon\eta-\epsilon\eta')(\epsilon\eta-\epsilon\eta')\cos^2\phi\leq 2R_\kappa\epsilon(\eta-\eta'), \tag{5.33}$$

we have

$$\sin\phi\leq\sin\phi'\leq 2\sqrt{\sin^2\phi+\epsilon(\eta-\eta')}, \tag{5.34}$$

which means

$$\frac{1}{2\sqrt{\sin^2\phi+\epsilon(\eta-\eta')}}\leq\frac{1}{\sin\phi'}\leq\frac{1}{\sin\phi}. \tag{5.35}$$

Therefore,

$$
\begin{aligned}
-\int_{\eta'}^{\eta}\frac{1}{\sin\phi'(y)}\mathrm{d}y \leq &\; -\int_{\eta'}^{\eta}\frac{1}{2\sqrt{\sin^2\phi+\epsilon(\eta-y)}}\mathrm{d}y \\
=&\; \frac{1}{\epsilon}\bigg(\sin\phi-\sqrt{\sin^2\phi+\epsilon(\eta-\eta')}\bigg) \\
=&\; -\frac{\eta-\eta'}{\sin\phi+\sqrt{\sin^2\phi+\epsilon(\eta-\eta')}} \\
\leq&\; -\frac{\eta-\eta'}{2\sqrt{\sin^2\phi+\epsilon(\eta-\eta')}}.
\end{aligned}
\tag{5.36}
$$

Define a cut-off function $\chi\in C^{\infty}[-\pi,\pi]$ satisfying

$$
\chi(\phi)=\left\{\begin{array}{ll}1 & \text{for}\quad |\sin\phi|\leq\delta,\\ 0 & \text{for}\quad |\sin\phi|\geq 2\delta,\end{array}\right.
\tag{5.37}
$$

In the following, we will divide the estimate of I into several cases based on the value of $\sin\phi$, $|\cos\phi|$, $\sin\phi'$, $\epsilon\eta'$ and $\epsilon(\eta-\eta')$. Let $\mathbf{1}$ denote the indicator function. We write

$$
\begin{aligned}
I=&\int_0^{\eta}\mathbf{1}_{\{\sin\phi\geq\delta_0\}}\mathbf{1}_{\{|\cos\phi|\geq\delta_0\}}+\int_0^{\eta}\mathbf{1}_{\{0\leq\sin\phi\leq\delta_0\}}\mathbf{1}_{\{\chi(\phi_*)<1\}}\\
&+\int_0^{\eta}\mathbf{1}_{\{0\leq\sin\phi\leq\delta_0\}}\mathbf{1}_{\{\chi(\phi_*)=1\}}\mathbf{1}_{\{\sqrt{\epsilon\eta'}\geq\sin\phi'\}}\\
&+\int_0^{\eta}\mathbf{1}_{\{0\leq\sin\phi\leq\delta_0\}}\mathbf{1}_{\{\chi(\phi_*)=1\}}\mathbf{1}_{\{\sqrt{\epsilon\eta'}\leq\sin\phi'\}}\mathbf{1}_{\{\sin^2\phi\leq\epsilon(\eta-\eta')\}}\\
&+\int_0^{\eta}\mathbf{1}_{\{0\leq\sin\phi\leq\delta_0\}}\mathbf{1}_{\{\chi(\phi_*)=1\}}\mathbf{1}_{\{\sqrt{\epsilon\eta'}\leq\sin\phi'\}}\mathbf{1}_{\{\sin^2\phi\geq\epsilon(\eta-\eta')\}}\\
&+\int_0^{\eta}\mathbf{1}_{\{|\cos\phi|\leq\delta_0\}}\\
=&\,I_1+I_2+I_3+I_4+I_5+I_6.
\end{aligned}
\tag{5.38}
$$

Step 1: Estimate of I_1 for $\sin\phi\geq\delta_0$ and $|\cos\phi|\geq\delta_0$.
For $\sin\phi\geq\delta_0$ and $|\cos\phi|\geq\delta_0$, we do not need the mild formulation of $\mathscr{A}$. Instead, we directly estimate

$$
|\mathscr{A}|\leq\left|\frac{\partial\mathscr{V}}{\partial\eta}\right|.
\tag{5.39}
$$

We will estimate I_1 based on the characteristics of $\mathscr{V}$ itself instead of the derivative. Here, we will use two formulations of the equation (5.7) along the characteristics

- Formulation I: η is the principal variable, $\phi=\phi(\eta)$, and the equation can be rewritten as

$$
\sin\phi\frac{\mathrm{d}\mathscr{V}}{\mathrm{d}\eta}+\mathscr{V}=\bar{\mathscr{V}}+S.
\tag{5.40}
$$

- Formulation II: ϕ is the principal variable, $\eta=\eta(\phi)$ and the equation can be rewritten as

$$
F(\eta)\cos\phi\frac{\mathrm{d}\mathscr{V}}{\mathrm{d}\phi}+\mathscr{V}=\bar{\mathscr{V}}+S.
\tag{5.41}
$$

These two formulations are equivalent and can be applied to different regions of the domain.

We may decompose $\mathscr{V} = \mathscr{V}_1 + \mathscr{V}_2$ where $\mathscr{V}_1$ satisfies

$$\begin{cases} \sin\phi\dfrac{\partial \mathscr{V}_1}{\partial\eta} + F(\eta)\cos\phi\dfrac{\partial \mathscr{V}_1}{\partial\phi} + \mathscr{V}_1 = \bar{\mathscr{V}}, \\ \mathscr{V}_1(0,\phi) = p(\phi) \quad \text{for} \quad \sin\phi > 0, \\ \mathscr{V}_1(L,\phi) = \mathscr{V}_1(L,\mathscr{R}[\phi]), \end{cases} \tag{5.42}$$

and $\mathscr{V}_2$ satisfies

$$\begin{cases} \sin\phi\dfrac{\partial \mathscr{V}_2}{\partial\eta} + F(\eta)\cos\phi\dfrac{\partial \mathscr{V}_2}{\partial\phi} + \mathscr{V}_2 = S, \\ \mathscr{V}_2(0,\phi) = 0 \quad \text{for} \quad \sin\phi > 0, \\ \mathscr{V}_2(L,\phi) = \mathscr{V}_2(L,\mathscr{R}[\phi]). \end{cases} \tag{5.43}$$

Assume $\mathscr{V}$ is well-defined. Then we can easily see that $\mathscr{V}_1$ and $\mathscr{V}_2$ are well-defined.

Using Formulation I, we rewrite the equation (5.42) along the characteristics as

$$\mathscr{V}_1(\eta,\phi) = \exp\left(-G_{\eta,0}\right)\left(p\Big(\phi'(0)\Big) + \int_0^\eta \frac{\bar{\mathscr{V}}(\eta')}{\sin\Big(\phi'(\eta')\Big)}\exp\left(G_{\eta',0}\right)\mathrm{d}\eta'\right) \tag{5.44}$$

where (η',ϕ'), $(0,\phi'(0))$ and (η,ϕ) are on the same characteristic with $\sin\phi' \geq 0$, and

$$G_{t,s} = \int_s^t \frac{1}{\sin\Big(\phi'(\xi)\Big)}\mathrm{d}\xi. \tag{5.45}$$

Taking η derivative on both sides of (5.44), we have

$$\frac{\partial \mathscr{V}_1}{\partial\eta} = X_1 + X_2 + X_3 + X_4 + X_5, \tag{5.46}$$

where

$$X_1 = -\exp\left(-G_{\eta,0}\right)\frac{\partial G_{\eta,0}}{\partial\eta}\left(p\Big(\phi'(0)\Big) + \int_0^\eta \frac{\bar{\mathscr{V}}(\eta')}{\sin\Big(\phi'(\eta')\Big)}\exp\left(G_{\eta',0}\right)\mathrm{d}\eta'\right),$$

$$X_2 = \exp\left(-G_{\eta,0}\right)\frac{\partial p\Big(\phi'(0)\Big)}{\partial\eta}, \tag{5.47}$$

$$X_3 = \frac{\bar{\mathscr{V}}(\eta)}{\sin\phi}, \tag{5.48}$$

$$X_4 = -\exp\left(-G_{\eta,0}\right)\int_0^\eta \bar{\mathscr{V}}(\eta')\exp\left(G_{\eta',0}\right)\frac{\cos\Big(\phi'(\eta')\Big)}{\sin^2\Big(\phi'(\eta')\Big)}\frac{\partial\phi'(\eta')}{\partial\eta}\mathrm{d}\eta',$$

$$X_5 = \exp\left(-G_{\eta,0}\right)\int_0^\eta \frac{\bar{\mathscr{V}}(\eta')}{\sin\Big(\phi'(\eta')\Big)}\exp\left(G_{\eta',0}\right)\frac{\partial G_{\eta',0}}{\partial\eta}\mathrm{d}\eta'. \tag{5.49}$$

Then we need to estimate each term. This procedure is standard, so we omit the details. Note that fact that for $0 \leq \eta' \leq \eta$, we have $\sin\phi' \geq \sin\phi \geq \delta_0$ and

$$\int_0^{\eta}\frac{1}{\sin\Big(\phi'(\eta')\Big)}\exp\left(-G_{\eta,\eta'}\right)\mathrm{d}\eta' \leq \int_0^{\infty}\mathrm{e}^{-y}\mathrm{d}y = 1, \tag{5.50}$$

with the substitution $y = G_{\eta,\eta'}$. The estimates can be listed as below:

$$|X_1| \leq \frac{C}{\delta_0}\|\mathscr{V}\|_{L^\infty L^\infty}, \tag{5.51}$$

$$|X_2| \leq \frac{C}{\delta_0}\left\|\frac{\partial p}{\partial\phi}\right\|_{L^\infty}, \tag{5.52}$$

$$|X_3| \leq \frac{C}{\delta_0}\|\mathscr{V}\|_{L^\infty L^\infty}, \tag{5.53}$$

$$|X_4| \leq \frac{C}{\delta_0}\|\mathscr{V}\|_{L^\infty L^\infty}, \tag{5.54}$$

$$|X_5| \leq \frac{C}{\delta_0}\|\mathscr{V}\|_{L^\infty L^\infty}. \tag{5.55}$$

In total, we have

$$\left|\frac{\partial\mathscr{V}_1}{\partial\eta}\right| \leq \frac{C}{\delta_0}\left(\left\|\frac{\partial p}{\partial\phi}\right\|_{L^\infty} + \|\mathscr{V}\|_{L^\infty L^\infty}\right). \tag{5.56}$$

Using Formulation II, we rewrite the equation (5.43) along the characteristics as

$$\mathscr{V}_2(\eta,\phi) = \exp\left(-H_{\phi,\phi_*}\right)\int_{\phi_*}^{\phi}\frac{S\Big(\eta'(\phi'),\phi'\Big)}{F\Big(\eta'(\phi')\Big)\cos\phi'}\exp\left(H_{\phi',\phi_*}\right)\mathrm{d}\phi'. \tag{5.57}$$

where (η',ϕ'), $(0,\phi_*)$ and (η,ϕ) are on the same characteristic with $\sin\phi' \geq 0$, and

$$H_{t,s} = \int_s^t \frac{1}{F\Big(\eta'(\xi)\Big)\cos\xi'}\mathrm{d}\xi. \tag{5.58}$$

Taking η derivative on both sides of (5.57), we have

$$\frac{\partial\mathscr{V}_2}{\partial\eta} = Y_1 + Y_2 + Y_3 + Y_4 + Y_5, \tag{5.59}$$

where

$$
\begin{aligned}
Y_1 =& -\exp\left(-H_{\phi,\phi_*}\right)\frac{\partial H_{\phi,\phi_*}}{\partial\eta}\int_{\phi_*}^{\phi}\frac{S\Big(\eta'(\phi'),\phi'\Big)}{F\Big(\eta'(\phi')\Big)\cos\phi'}\exp\left(H_{\phi',\phi_*}\right)\mathrm{d}\phi',\\
Y_2 =& \frac{S(0,\phi_*)}{F(0)\cos\phi_*}\frac{\partial\phi_*}{\partial\eta}, \qquad (5.60)\\
Y_3 =& -\exp\left(-H_{\phi,\phi_*}\right)\int_{\phi_*}^{\phi}S\Big(\eta'(\phi'),\phi'\Big)\frac{1}{F^2\Big(\eta'(\phi')\Big)\cos\phi'}\frac{\partial F\Big(\eta'(\phi')\Big)}{\partial\eta}\exp\left(H_{\phi',\phi_*}\right)\mathrm{d}\phi',\\
Y_4 =& \exp\left(-H_{\phi,\phi_*}\right)\int_{\phi_*}^{\phi}\frac{S\Big(\eta'(\phi'),\phi'\Big)}{F\Big(\eta'(\phi')\Big)\cos\phi'}\exp\left(H_{\phi',\phi_*}\right)\frac{\partial H_{\phi',\phi_*}}{\partial\eta}\mathrm{d}\phi', \qquad (5.61)\\
Y_5 =& \exp\left(-H_{\phi,\phi_*}\right)\int_{\phi_*}^{\phi}\frac{\partial_{\eta'}S\Big(\eta'(\phi'),\phi'\Big)}{F\Big(\eta'(\phi')\Big)\cos\phi'}\frac{\partial\eta'(\phi')}{\partial\eta}\exp\left(H_{\phi',\phi_*}\right)\mathrm{d}\phi'. \qquad (5.62)
\end{aligned}
$$

Then we just need to estimate each term. Along the characteristics, we know

$$
\mathrm{e}^{-V(\eta')}\cos\phi' = \mathrm{e}^{-V(\eta)}\cos\phi, \tag{5.63}
$$

which implies

$$
\cos\phi' = \mathrm{e}^{V(\eta')-V(\eta)}\cos\phi \geq \mathrm{e}^{V(0)-V(L)}\cos\phi \geq \mathrm{e}^{V(0)-V(L)}\delta_0. \tag{5.64}
$$

We can further deduce that

$$
\cos\phi' \geq \left(1-\frac{\epsilon^{\frac{1}{2}}}{R_\kappa}\right)\delta_0 \geq \frac{\delta_0}{2}, \tag{5.65}
$$

when ϵ is sufficiently small. Also, we have

$$
\int_{\phi_*}^{\phi}\frac{1}{F\Big(\eta'(\phi')\Big)\cos\phi'}\exp\left(H_{\phi,\phi'}\right)\mathrm{d}\phi' \leq \int_0^\infty \mathrm{e}^{-y}\mathrm{d}y = 1, \tag{5.66}
$$

with the substitution $y = H_{\phi,\phi'}$. Similar to X_i estimates, we may directly obtain

$$
\begin{aligned}
|Y_1| &\leq \frac{C}{\delta_0}\|S\|_{L^\infty L^\infty}, && (5.67)\\
|Y_2| &\leq \frac{C}{\delta_0}\|S\|_{L^\infty L^\infty}, && (5.68)\\
|Y_3| &\leq \frac{C}{\delta_0}\|S\|_{L^\infty L^\infty}, && (5.69)\\
|Y_4| &\leq \frac{C}{\delta_0}\|S\|_{L^\infty L^\infty}, && (5.70)\\
|Y_5| &\leq \frac{C}{\delta_0}\left\|\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}. && (5.71)
\end{aligned}
$$

In total, we have

$$\left|\frac{\partial\mathscr{V}_2}{\partial\eta}\right|\leq\frac{C}{\delta_0}\bigg(\|S\|_{L^\infty L^\infty}+\left\|\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}\bigg). \tag{5.72}$$

Combining all above, we have

$$\left|\frac{\partial\mathscr{V}}{\partial\eta}\right| \leq \left|\frac{\partial\mathscr{V}_1}{\partial\eta}\right|+\left|\frac{\partial\mathscr{V}_2}{\partial\eta}\right|\leq\frac{C}{\delta_0}\bigg(\left\|\frac{\partial p}{\partial\phi}\right\|_{L^\infty}+\|S\|_{L^\infty L^\infty}+\left\|\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}+\|\mathscr{V}\|_{L^\infty L^\infty}\bigg). \tag{5.73}$$

Hence, noting that $\zeta\geq\delta_0$, we know

$$I_1\leq\frac{C}{\delta_0^2}\bigg(\left\|\zeta\frac{\partial p}{\partial\phi}\right\|_{L^\infty}+\|S\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}+\|\mathscr{V}\|_{L^\infty L^\infty}\bigg). \tag{5.74}$$

Step 2: Estimate of I_2 for $0\leq\sin\phi\leq\delta_0$ and $\chi(\phi_*)<1$.
We have

$$\begin{aligned}I_2&=\frac{1}{2\pi}\int_0^\eta\bigg(\int_{-\pi}^{\pi}\frac{\zeta(\eta',\phi')}{\zeta(\eta',\phi_*)}\Big(1-\chi(\phi_*)\Big)\mathscr{A}(\eta',\phi_*)\mathrm{d}\phi_*\bigg)\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'\\&=\frac{1}{2\pi}\int_0^\eta\bigg(\int_{-\pi}^{\pi}\zeta(\eta',\phi')\Big(1-\chi(\phi_*)\Big)\frac{\mathscr{V}(\eta',\phi_*)}{\partial\eta'}\mathrm{d}\phi_*\bigg)\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'.\end{aligned} \tag{5.75}$$

Based on the ϵ-Milne problem of $\mathscr{V}$ as

$$\sin\phi_*\frac{\partial\mathscr{V}(\eta',\phi_*)}{\partial\eta'}+F(\eta')\cos\phi_*\frac{\partial\mathscr{V}(\eta',\phi_*)}{\partial\phi_*}+\mathscr{V}(\eta',\phi_*)-\bar{\mathscr{V}}(\eta')=S(\eta',\phi_*), \tag{5.76}$$

we have

$$\frac{\partial\mathscr{V}(\eta',\phi_*)}{\partial\eta'}=-\frac{1}{\sin\phi_*}\bigg(F(\eta')\cos\phi_*\frac{\partial\mathscr{V}(\eta',\phi_*)}{\partial\phi_*}+\mathscr{V}(\eta',\phi_*)-\bar{\mathscr{V}}(\eta')-S(\eta',\phi_*)\bigg) \tag{5.77}$$

Hence, we have

$$\begin{aligned}\tilde{\mathscr{A}}&=\int_{-\pi}^{\pi}\zeta(\eta',\phi')\Big(1-\chi(\phi_*)v\Big)\frac{\partial\mathscr{V}(\eta',\phi_*)}{\partial\eta'}\mathrm{d}\phi_*\\&=-\int_{-\pi}^{\pi}\zeta(\eta',\phi')\Big(1-\chi(\phi_*)\Big)\frac{1}{\sin\phi_*}\bigg(\mathscr{V}(\eta',\phi_*)-\bar{\mathscr{V}}(\eta')-S(\eta',\phi_*)\bigg)\mathrm{d}\phi_*\\&\quad-\int_{-\pi}^{\pi}\zeta(\eta',\phi')\Big(1-\chi(\phi_*)\Big)\frac{1}{\sin\phi_*}F(\eta')\cos\phi_*\frac{\partial\mathscr{V}(\eta',\phi_*)}{\partial\phi_*}\mathrm{d}\phi_*\\&=\tilde{\mathscr{A}_1}+\tilde{\mathscr{A}_2}.\end{aligned} \tag{5.78}$$

We may directly obtain

$$\begin{aligned}\left|\tilde{\mathscr{A}_1}\right|&\leq\int_{-\pi}^{\pi}\zeta(\eta',\phi')\Big(1-\chi(\phi_*)\Big)\frac{1}{\sin\phi_*}\bigg(\mathscr{V}(\eta',\phi_*)-\bar{\mathscr{V}}(\eta')-S(\eta',\phi_*)\bigg)\mathrm{d}\phi_*\\&\leq\frac{R_\kappa}{\delta}\left|\int_{-\pi}^{\pi}\bigg(\mathscr{V}(\eta',\phi_*)-\bar{\mathscr{V}}(\eta')-S(\eta',\phi_*)\bigg)\mathrm{d}\phi_*\right|\\&\leq\frac{C}{\delta}\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\bigg).\end{aligned} \tag{5.79}$$

On the other hand, an integration by parts yields

$$\tilde{\mathscr{A}_2} = \int_{-\pi}^{\pi} \frac{\partial}{\partial \phi_*}\bigg(\zeta(\eta',\phi')\Big(1-\chi(\phi_*)\Big)\frac{1}{\sin\phi_*}F(\eta')\cos\phi_*\bigg)\mathscr{V}(\eta',\phi_*)\mathrm{d}\phi_*, \tag{5.80}$$

which further implies

$$\left|\tilde{\mathscr{A}_2}\right| \leq \frac{C\epsilon}{\delta^2}\|\mathscr{V}\|_{L^\infty L^\infty}. \tag{5.81}$$

Since we can use substitution to show

$$\int_0^\eta \frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \leq 1, \tag{5.82}$$

we have

$$\begin{aligned}|I_2| &\leq C\bigg(\frac{1}{\delta}+\frac{\epsilon}{\delta^2}\bigg)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\bigg)\int_0^\eta \frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \\ &\leq C\bigg(\frac{1}{\delta}+\frac{\epsilon}{\delta^2}\bigg)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\bigg).\end{aligned} \tag{5.83}$$

Step 3: Estimate of I_3 for $0\leq\sin\phi\leq\delta_0$, $\chi(\phi_*)=1$ and $\sqrt{\epsilon\eta'}\geq\sin\phi'$.
Based on (5.29), this implies

$$\zeta(\eta',\phi') \leq C\sqrt{\epsilon\eta'}.$$

Then combining this with (5.30), we can directly obtain

$$\int_{-\pi}^{\pi}\frac{\zeta(\eta',\phi')}{\zeta(\eta',\phi_*)}\chi(\phi_*)\mathscr{A}(\eta',\phi_*)\mathrm{d}\phi_* \leq C\int_{-\delta}^{\delta}\mathscr{A}(\eta',\phi_*)\mathrm{d}\phi_* \leq C\delta\|\mathscr{A}\|_{L^\infty L^\infty}. \tag{5.84}$$

Hence, we have

$$|I_3| \leq C\delta\|\mathscr{A}\|_{L^\infty L^\infty}\int_0^\eta\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \leq C\delta\|\mathscr{A}\|_{L^\infty L^\infty}. \tag{5.85}$$

Step 4: Estimate of I_4 for $0\leq\sin\phi\leq\delta_0$, $\chi(\phi_*)=1$, $\sqrt{\epsilon\eta'}\leq\sin\phi'$ and $\sin^2\phi\leq\epsilon(\eta-\eta')$.
Based on (5.29), this implies

$$\zeta(\eta',\phi') \leq C\sin\phi'. \tag{5.86}$$

Based on (5.36), we have

$$-G_{\eta,\eta'} = -\int_{\eta'}^{\eta}\frac{1}{\sin\phi'(y)}\mathrm{d}y \leq -\frac{\eta-\eta'}{2\sqrt{\epsilon(\eta-\eta')}} \leq -C\sqrt{\frac{\eta-\eta'}{\epsilon}}. \tag{5.87}$$

Hence, considering $\zeta(\eta',\phi_*)\geq\sqrt{\epsilon\eta'}$, we know

$$
\begin{aligned}
|I_4| &\leq C\int_0^{\eta}\bigg(\int_{-\pi}^{\pi}\frac{\zeta(\eta',\phi')}{\zeta(\eta',\phi_*)}\chi(\phi_*)\mathscr{A}(\eta',\phi_*)\mathrm{d}\phi_*\bigg)\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \\
&\leq C\int_0^{\eta}\bigg(\int_{-\delta}^{\delta}\frac{1}{\zeta(\eta',\phi_*)}\mathscr{A}(\eta',\phi_*)\mathrm{d}\phi_*\bigg)\frac{\zeta(\eta',\phi')}{\sin\phi'}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \\
&\leq C\|\mathscr{A}\|_{L^\infty L^\infty}\int_0^{\eta}\bigg(\int_{-\delta}^{\delta}\frac{1}{\zeta(\eta',\phi_*)}\mathrm{d}\phi_*\bigg)\frac{\sin\phi'}{\sin\phi'}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \\
&\leq C\delta\|\mathscr{A}\|_{L^\infty L^\infty}\int_0^{\eta}\frac{1}{\sqrt{\epsilon\eta'}}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \\
&\leq C\delta\|\mathscr{A}\|_{L^\infty L^\infty}\int_0^{\eta}\frac{1}{\sqrt{\epsilon\eta'}}\exp\bigg(-C\sqrt{\frac{\eta-\eta'}{\epsilon}}\bigg)\mathrm{d}\eta'
\end{aligned}
\tag{5.88}
$$

Define $z=\dfrac{\eta'}{\epsilon}$, which implies $\mathrm{d}\eta'=\epsilon\mathrm{d}z$. Substituting this into above integral, we have

$$
\begin{aligned}
|I_4| &\leq C\delta\|\mathscr{A}\|_{L^\infty L^\infty}\int_0^{\frac{\eta}{\epsilon}}\frac{1}{\sqrt{z}}\exp\bigg(-C\sqrt{\frac{\eta}{\epsilon}-z}\bigg)\mathrm{d}z \\
&= C\delta\|\mathscr{A}\|_{L^\infty L^\infty}\Bigg(\int_0^{1}\frac{1}{\sqrt{z}}\exp\bigg(-C\sqrt{\frac{\eta}{\epsilon}-z}\bigg)\mathrm{d}z+\int_1^{\frac{\eta}{\epsilon}}\frac{1}{\sqrt{z}}\exp\bigg(-C\sqrt{\frac{\eta}{\epsilon}-z}\bigg)\mathrm{d}z\Bigg).
\end{aligned}
\tag{5.89}
$$

We can estimate these two terms separately.

$$
\int_0^{1}\frac{1}{\sqrt{z}}\exp\bigg(-C\sqrt{\frac{\eta}{\epsilon}-z}\bigg)\mathrm{d}z\leq\int_0^1\frac{1}{\sqrt{z}}\mathrm{d}z=2. \tag{5.90}
$$

$$
\int_1^{\frac{\eta}{\epsilon}}\frac{1}{\sqrt{z}}\exp\bigg(-C\sqrt{\frac{\eta}{\epsilon}-z}\bigg)\mathrm{d}z\leq\int_1^{\frac{\eta}{\epsilon}}\exp\bigg(-C\sqrt{\frac{\eta}{\epsilon}-z}\bigg)\mathrm{d}z\overset{t^2=\frac{\eta}{\epsilon}-z}{\leq}2\int_0^{\infty}t\mathrm{e}^{-Ct}\mathrm{d}t<\infty. \tag{5.91}
$$

Hence, we know

$$
|I_4|\leq C\delta\|\mathscr{A}\|_{L^\infty L^\infty}. \tag{5.92}
$$

Step 5: Estimate of I_5 for $0\leq\sin\phi\leq\delta_0$, $\chi(\phi_*)=1$, $\sqrt{\epsilon\eta'}\leq\sin\phi'$ and $\sin^2\phi\geq\epsilon(\eta-\eta')$. Based on (5.29), this implies

$$
\zeta(\eta',\phi')\leq C\sin\phi'.
$$

Based on (5.36), we have

$$
-G_{\eta,\eta'}=-\int_{\eta'}^{\eta}\frac{1}{\sin\phi'(y)}\mathrm{d}y\leq-\frac{C(\eta-\eta')}{\sin\phi}. \tag{5.93}
$$

Hence, we have

$$
|I_5|\leq C\|\mathscr{A}\|_{L^\infty L^\infty}\int_0^{\eta}\bigg(\int_{-\delta}^{\delta}\frac{1}{\zeta(\eta',\phi_*)}\mathrm{d}\phi_*\bigg)\exp\bigg(-\frac{C(\eta-\eta')}{\sin\phi}\bigg)\mathrm{d}\eta' \tag{5.94}
$$

Here, we use a different way to estimate the inner integral. We use substitution to find

$$
\begin{aligned}
\int_{-\delta}^{\delta}\frac{1}{\zeta(\eta',\phi_*)}\mathrm{d}\phi_* &= \int_{-\delta}^{\delta}\frac{1}{\bigg(R_\kappa^2-(R_\kappa-\epsilon\eta')^2\cos\phi_*^2\bigg)^{1/2}}\mathrm{d}\phi_* \\
&\overset{\sin\phi_*\ small}{\leq} C\int_{-\delta}^{\delta}\frac{\cos\phi_*}{\bigg(R_\kappa^2-(R_\kappa-\epsilon\eta')^2\cos\phi_*^2\bigg)^{1/2}}\mathrm{d}\phi_* \\
&= C\int_{-\delta}^{\delta}\frac{\cos\phi_*}{\bigg(R_\kappa^2-(R_\kappa-\epsilon\eta')^2+(R_\kappa-\epsilon\eta')^2\sin\phi_*^2\bigg)^{1/2}}\mathrm{d}\phi_* \\
&\overset{y=\sin\phi_*}{=} C\int_{-\delta}^{\delta}\frac{1}{\bigg(R_\kappa^2-(R_\kappa-\epsilon\eta')^2+(R_\kappa-\epsilon\eta')^2y^2\bigg)^{1/2}}\mathrm{d}y.
\end{aligned} \tag{5.95}
$$

Define

$$
p=\sqrt{R_\kappa^2-(R_\kappa-\epsilon\eta')^2}=\sqrt{2R_\kappa\epsilon\eta'-\epsilon^2\eta'^2}\leq C\sqrt{\epsilon\eta'}, \tag{5.96}
$$

$$
q=R_\kappa-\epsilon\eta'\geq C, \tag{5.97}
$$

$$
r=\frac{p}{q}\leq C\sqrt{\epsilon\eta'}. \tag{5.98}
$$

Then we have

$$
\begin{aligned}
\int_{-\delta}^{\delta}\frac{1}{\zeta(\eta',\phi_*)}\mathrm{d}\phi_* &\leq C\int_{-\delta}^{\delta}\frac{1}{(p^2+q^2y^2)^{1/2}}\mathrm{d}y \\
&\leq C\int_{-2}^{2}\frac{1}{(p^2+q^2y^2)^{1/2}}\mathrm{d}y\leq C\int_{-2}^{2}\frac{1}{(r^2+y^2)^{1/2}}\mathrm{d}y \\
&\leq C\int_{0}^{2}\frac{1}{(r^2+y^2)^{1/2}}\mathrm{d}y=\bigg(\ln(y+\sqrt{r^2+y^2})-\ln(r)\bigg)\bigg|_0^2 \\
&\leq C\bigg(\ln(2+\sqrt{r^2+4})-\ln r\bigg)\leq C\bigg(1+\ln(r)\bigg) \\
&\leq C\bigg(1+|\ln(\epsilon)|+|\ln(\eta')|\bigg).
\end{aligned} \tag{5.99}
$$

Hence, we know

$$
|I_5|\ \leq C\|\mathscr{A}\|_{L^\infty L^\infty}\int_0^\eta\bigg(1+|\ln(\epsilon)|+|\ln(\eta')|\bigg)\exp\bigg(-\frac{C(\eta-\eta')}{\sin\phi}\bigg)\mathrm{d}\eta' \tag{5.100}
$$

We may directly compute

$$
\left|\int_0^\eta\bigg(1+|\ln(\epsilon)|\bigg)\exp\bigg(-\frac{C(\eta-\eta')}{\sin\phi}\bigg)\mathrm{d}\eta'\right|\leq C\sin\phi(1+|\ln(\epsilon)|). \tag{5.101}
$$

Hence, we only need to estimate

$$
\left|\int_0^\eta|\ln(\eta')|\exp\bigg(-\frac{C(\eta-\eta')}{\sin\phi}\bigg)\mathrm{d}\eta'\right|. \tag{5.102}
$$

If $\eta\leq 2$, using Cauchy's inequality, we have

$$
\begin{aligned}
\left|\int_0^{\eta}|\ln(\eta')|\exp\left(-\frac{C(\eta-\eta')}{\sin\phi}\right)\mathrm{d}\eta'\right| &\leq \left(\int_0^{\eta}\ln^2(\eta')\mathrm{d}\eta'\right)^{\frac{1}{2}}\left(\int_0^{\eta}\exp\left(-\frac{2C(\eta-\eta')}{\sin\phi}\right)\mathrm{d}\eta'\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^{2}\ln^2(\eta')\mathrm{d}\eta'\right)^{\frac{1}{2}}\left(\int_0^{\eta}\exp\left(-\frac{2C(\eta-\eta')}{\sin\phi}\right)\mathrm{d}\eta'\right)^{\frac{1}{2}} \\
&\leq \sqrt{\sin\phi}.
\end{aligned}
\tag{5.103}
$$

If $\eta\geq 2$, we decompose and apply Cauchy's inequality to obtain

$$
\begin{aligned}
&\left|\int_0^{\eta}|\ln(\eta')|\exp\left(-\frac{C(\eta-\eta')}{\sin\phi}\right)\mathrm{d}\eta'\right| \\
\leq& \left|\int_0^{2}|\ln(\eta')|\exp\left(-\frac{C(\eta-\eta')}{\sin\phi}\right)\mathrm{d}\eta'\right|+\left|\int_2^{\eta}\ln(\eta')\exp\left(-\frac{C(\eta-\eta')}{\sin\phi}\right)\mathrm{d}\eta'\right| \\
\leq& \left(\int_0^{2}\ln^2(\eta')\mathrm{d}\eta'\right)^{\frac{1}{2}}\left(\int_0^{2}\exp\left(-\frac{2C(\eta-\eta')}{\sin\phi}\right)\mathrm{d}\eta'\right)^{\frac{1}{2}}+\ln(L)\left|\int_2^{\eta}\exp\left(-\frac{C(\eta-\eta')}{\sin\phi}\right)\mathrm{d}\eta'\right| \\
\leq& C\bigg(\sqrt{\sin\phi}+|\ln(\epsilon)|\sin\phi\bigg)\leq C\Big(1+|\ln(\epsilon)|\Big)\sqrt{\sin\phi}.
\end{aligned}
\tag{5.104}
$$

Hence, we have

$$
|I_5|\leq C\Big(1+|\ln(\epsilon)|\Big)\sqrt{\delta_0}\|\mathscr{A}\|_{L^\infty L^\infty}. \tag{5.105}
$$

Step 6: Estimate of I_6 for $|\cos\phi|<\delta_0$.
We have

$$
\begin{aligned}
I_6=&\frac{1}{2\pi}\int_0^{\eta}\left(\int_{-\pi}^{\pi}\frac{\zeta(\eta',\phi')}{\zeta(\eta',\phi_*)}\mathscr{A}(\eta',\phi_*)\mathrm{d}\phi_*\right)\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \\
=&\frac{1}{2\pi}\int_0^{\eta}\left(\int_{-\pi}^{\pi}\zeta(\eta',\phi')\Big(1-\chi(\phi_*)\Big)\frac{\mathscr{V}(\eta',\phi_*)}{\partial\eta'}\mathrm{d}\phi_*\right)\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \\
&+\frac{1}{2\pi}\int_0^{\eta}\left(\int_{-\pi}^{\pi}\chi(\phi_*)\frac{\zeta(\eta',\phi')}{\zeta(\eta',\phi_*)}\mathscr{A}(\eta',\phi_*)\mathrm{d}\phi_*\right)\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\mathrm{d}\eta'.
\end{aligned}
\tag{5.106}
$$

The first term can be estimated as I_2.

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{\eta}\left(\int_{-\pi}^{\pi}\zeta(\eta',\phi')\Big(1-\chi(\phi_*)\Big)\frac{\mathscr{V}(\eta',\phi_*)}{\partial\eta'}\mathrm{d}\phi_*\right)\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \\
\leq& C\bigg(\frac{1}{\delta}+\frac{\epsilon}{\delta^2}\bigg)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\bigg).
\end{aligned}
\tag{5.107}
$$

It is easy to check that $\sqrt{\epsilon\eta'}\leq\sin\phi\leq\sin\phi'$ and $\sin^2\phi\geq\epsilon(\eta-\eta')$, so the second term can be estimated as I_5.

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{\eta}\left(\int_{-\pi}^{\pi}\chi(\phi_*)\frac{\zeta(\eta',\phi')}{\zeta(\eta',\phi_*)}\mathscr{A}(\eta',\phi_*)\mathrm{d}\phi_*\right)\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \\
\leq& C\Big(1+|\ln(\epsilon)|\Big)\sqrt{\sin\phi}\sup_{|\sin\phi_*|\leq\delta}|\mathscr{A}(\eta,\phi_*)|\leq C\Big(1+|\ln(\epsilon)|\Big)\sup_{|\sin\phi_*|\leq\delta}|\mathscr{A}(\eta,\phi_*)|.
\end{aligned}
\tag{5.108}
$$

Note that now we lose the smallness since $\sin\phi \geq \dfrac{1}{2}$, so we need a more detailed analysis. Actually, the value of $|\mathscr{A}|$ for $|\sin\phi| \leq \delta$, is covered in I_2, I_3, I_4, I_5 and the following II_2, II_3, II_4. Therefore, in fact, we get the estimate

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{\eta}\left(\int_{-\pi}^{\pi}\chi(\phi_*)\frac{\zeta(\eta',\phi')}{\zeta(\eta',\phi_*)}\mathscr{A}(\eta',\phi_*)\mathrm{d}\phi_*\right)\frac{1}{\sin\phi'}\exp(-G_{\eta,\eta'})\mathrm{d}\eta' \qquad (5.109)\\
&\leq C\Big(1+|\ln(\epsilon)|\Big)\bigg(\|p_{\mathscr{A}}\|_{L^\infty}+\|S_{\mathscr{A}}\|_{L^\infty L^\infty}\bigg)+C\Big(1+|\ln(\epsilon)|\Big)\left(\frac{1}{\delta}+\frac{\epsilon}{\delta^2}\right)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\bigg)\\
&\quad+C\Big(1+|\ln(\epsilon)|\Big)\bigg(\delta+\Big(1+|\ln(\epsilon)|\Big)\sqrt{\delta_0}\bigg)\|\mathscr{A}\|_{L^\infty L^\infty}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
|I_6|\ &\leq C\Big(1+|\ln(\epsilon)|\Big)\bigg(\|p_{\mathscr{A}}\|_{L^\infty}+\|S_{\mathscr{A}}\|_{L^\infty L^\infty}\bigg)+C\Big(1+|\ln(\epsilon)|\Big)\left(\frac{1}{\delta}+\frac{\epsilon}{\delta^2}\right)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\bigg)\\
&\quad+C\Big(1+|\ln(\epsilon)|\Big)\bigg(\delta+\Big(1+|\ln(\epsilon)|\Big)\sqrt{\delta_0}\bigg)\|\mathscr{A}\|_{L^\infty L^\infty}.
\end{aligned} \tag{5.110}$$

Step 7: Synthesis.
Collecting all the terms in previous steps, we have proved

$$\begin{aligned}
|I|\ &\leq C\Big(1+|\ln(\epsilon)|\Big)\bigg(\|p_{\mathscr{A}}\|_{L^\infty}+\|S_{\mathscr{A}}\|_{L^\infty L^\infty}\bigg)\\
&\quad+\frac{C}{\delta_0^2}\left(\zeta\left\|\frac{\partial p}{\partial\phi}\right\|_{L^\infty}+\|S\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}+\|\mathscr{V}\|_{L^\infty L^\infty}\right)\\
&\quad+C\Big(1+|\ln(\epsilon)|\Big)\left(\frac{1}{\delta}+\frac{\epsilon}{\delta^2}\right)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\bigg)\\
&\quad+C\Big(1+|\ln(\epsilon)|\Big)\bigg(\delta+\Big(1+|\ln(\epsilon)|\Big)\sqrt{\delta_0}\bigg)\|\mathscr{A}\|_{L^\infty L^\infty}.
\end{aligned} \tag{5.111}$$

We consider

$$\mathcal{K}[p_{\mathscr{A}}]=p_{\mathscr{A}}\Big(\phi'(\eta,\phi;0)\Big)\exp(-G_{L,0}-G_{L,\eta}) \tag{5.112}$$

$$\begin{aligned}
\mathcal{T}[\tilde{\mathscr{A}}+S_{\mathscr{A}}]=&\int_0^L\frac{(\tilde{\mathscr{A}}+S)\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{L,\eta'}-G_{L,\eta})\mathrm{d}\eta' \qquad (5.113)\\
&+\int_\eta^L\frac{(\tilde{\mathscr{A}}+S)\Big(\eta',\mathscr{R}[\phi'(\eta,\phi;\eta')]\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta',\eta})\mathrm{d}\eta'.
\end{aligned}$$

Based on [21, Lemma 4.7, Lemma 4.8], we can directly obtain

$$\left|\mathcal{K}[p_{\mathscr{A}}]\right|\ \leq\ \|p_{\mathscr{A}}\|_{L^\infty}\,, \tag{5.114}$$

$$\left|\mathcal{T}[S_{\mathscr{A}}]\right|\ \leq\|S_{\mathscr{A}}\|_{L^\infty L^\infty}. \tag{5.115}$$

Hence, we only need to estimate

$$II = \mathcal{T}[\tilde{\mathscr{A}} + S_{\mathscr{A}}] = \int_0^L \frac{\tilde{\mathscr{A}}\Big(\eta', \phi'(\eta, \phi; \eta')\Big)}{\sin\Big(\phi'(\eta, \phi; \eta')\Big)} \exp(-G_{L,\eta'} - G_{L,\eta}) \mathrm{d}\eta' \qquad (5.116)$$

$$+ \int_\eta^L \frac{\tilde{\mathscr{A}}\Big(\eta', \mathscr{R}[\phi'(\eta, \phi; \eta')]\Big)}{\sin\Big(\phi'(\eta, \phi; \eta')\Big)} \exp(-G_{\eta',\eta}) \mathrm{d}\eta'.$$

In particular, since the integral $\displaystyle\int_0^\eta \cdots$ can be estimated as in Region I, so we only need to estimate the integral $\displaystyle\int_\eta^L \cdots$. Also, noting that fact that

$$\exp(-G_{L,\eta'} - G_{L,\eta}) \leq \exp(-G_{\eta',\eta}), \qquad (5.117)$$

we only need to estimate

$$\int_\eta^L \frac{\tilde{\mathscr{A}}\Big(\eta', \mathscr{R}[\phi'(\eta, \phi; \eta')]\Big)}{\sin\Big(\phi'(\eta, \phi; \eta')\Big)} \exp(-G_{\eta',\eta}) \mathrm{d}\eta'. \qquad (5.118)$$

Here the proof is almost identical to that in Region I, so we only point out the key differences.

Step 0: Preliminaries.
We need to update one key result. For $0 \leq \eta \leq \eta'$,

$$\begin{aligned} \sin\phi' &= \sqrt{1 - \cos^2\phi'} = \sqrt{1 - \left(\frac{R_\kappa - \epsilon\eta}{R_\kappa - \epsilon\eta'}\right)^2 \cos^2\phi} \qquad (5.119) \\ &= \frac{\sqrt{(R_\kappa - \epsilon\eta')^2 \sin^2\phi + (2R_\kappa - \epsilon\eta - \epsilon\eta')(\epsilon\eta' - \epsilon\eta)\cos^2\phi}}{R_\kappa - \epsilon\eta'} \\ &\leq \left|\sin\phi\right|. \end{aligned}$$

Then we have

$$-\int_\eta^{\eta'} \frac{1}{\sin\phi'(y)} \mathrm{d}y \leq -\frac{\eta' - \eta}{\left|\sin\phi\right|}. \qquad (5.120)$$

In the following, we will divide the estimate of II into several cases based on the value of $\sin\phi$, $\left|\cos\phi\right|$, $\sin\phi'$ and $\epsilon\eta'$. We write

$$\begin{aligned} II =& \int_\eta^L \mathbf{1}_{\{\sin\phi \leq -\delta_0\}} \mathbf{1}_{\{|\cos\phi| \geq \delta_0\}} + \int_\eta^L \mathbf{1}_{\{-\delta_0 \leq \sin\phi \leq 0\}} \mathbf{1}_{\{\chi(\phi_*) < 1\}} \qquad (5.121) \\ &+ \int_\eta^L \mathbf{1}_{\{-\delta_0 \leq \sin\phi \leq 0\}} \mathbf{1}_{\{\chi(\phi_*) = 1\}} \mathbf{1}_{\{\sqrt{\epsilon\eta'} \geq \sin\phi'\}} + \int_\eta^L \mathbf{1}_{\{-\delta_0 \leq \sin\phi \leq 0\}} \mathbf{1}_{\{\chi(\phi_*) = 1\}} \mathbf{1}_{\{\sqrt{\epsilon\eta'} \leq \sin\phi'\}} \\ &+ \int_\eta^L \mathbf{1}_{\{|\cos\phi| \leq \delta_0\}} \\ =& II_1 + II_2 + II_3 + II_4 + II_5. \end{aligned}$$

Step 1: Estimate of II_1 for $\sin\phi \leq -\delta_0$.
We first estimate $\sin\phi'$. Along the characteristics, we know

$$\mathrm{e}^{-V(\eta')}\cos\phi' = \mathrm{e}^{-V(\eta)}\cos\phi, \tag{5.122}$$

which implies

$$\cos\phi' = \mathrm{e}^{V(\eta')-V(\eta)}\cos\phi \leq \mathrm{e}^{V(L)-V(0)}\cos\phi = \mathrm{e}^{V(L)-V(0)}\sqrt{1-\delta_0^2}. \tag{5.123}$$

We can further deduce that

$$\cos\phi' \leq \left(1-\frac{\epsilon^{\frac{1}{2}}}{R_\kappa}\right)^{-1}\sqrt{1-\delta_0^2}. \tag{5.124}$$

Then we have

$$\sin\phi' \geq \sqrt{1-\left(1-\frac{\epsilon^{\frac{1}{2}}}{R_\kappa}\right)^{-2}(1-\delta_0^2)} \geq \delta_0 - \epsilon^{\frac{1}{4}} > \frac{\delta_0}{2}, \tag{5.125}$$

when ϵ is sufficiently small.
Similar to Region I, we will use two formulations to handle different terms and we will decompose $\mathscr{V} = \mathscr{V}_1 + \mathscr{V}_2$.
Using Formulation I, we rewrite the $\mathscr{V}_1$ equation along the characteristics as

$$\begin{aligned}\mathscr{V}_1(\eta,\phi) = {}& p\Big(\phi'(0)\Big)\exp(-G_{L,0}-G_{L,\eta}) \\ &+ \int_0^L \frac{\bar{\mathscr{V}}(\eta')}{\sin\Big(\phi'(\eta')\Big)}\exp(-G_{L,\eta'}-G_{L,\eta})\mathrm{d}\eta' + \int_\eta^L \frac{\bar{\mathscr{V}}(\eta')}{\sin\Big(\phi'(\eta')\Big)}\exp(-G_{\eta',\eta})\mathrm{d}\eta'\end{aligned} \tag{5.126}$$

where (η',ϕ') and (η,ϕ) are on the same characteristic with $\sin\phi' \geq 0$. Then taking η derivative on both sides of (5.126) yields

$$\frac{\partial\mathscr{V}_1}{\partial\eta} = X_1+X_2+X_3+X_4+X_5+X_6+X_7, \tag{5.127}$$

where

$$X_1 = \frac{\partial p\Big(\phi'(0)\Big)}{\partial\eta}\exp(-G_{L,0}-G_{L,\eta}), \tag{5.128}$$

$$X_2 = -p\Big(\phi'(0)\Big)\exp(-G_{L,0}-G_{L,\eta})\bigg(\frac{\partial G_{L,0}}{\partial\eta}+\frac{\partial G_{L,\eta}}{\partial\eta}\bigg), \tag{5.129}$$

$$X_3 = -\int_0^L \bar{\mathscr{V}}(\eta')\frac{\cos\Big(\phi'(\eta')\Big)}{\sin^2\Big(\phi'(\eta')\Big)}\frac{\partial\phi'(\eta')}{\partial\eta}\exp(-G_{L,\eta'}-G_{L,\eta})\mathrm{d}\eta', \tag{5.130}$$

$$X_4 = -\int_0^L \frac{\bar{\mathscr{V}}(\eta')}{\sin\Big(\phi'(\eta')\Big)}\exp(-G_{L,\eta'}-G_{L,\eta})\bigg(\frac{\partial G_{L,\eta'}}{\partial\eta}+\frac{\partial G_{L,\eta}}{\partial\eta}\bigg)\mathrm{d}\eta', \tag{5.131}$$

$$X_5 = -\int_\eta^L \bar{\mathscr{V}}(\eta')\frac{\cos\Big(\phi'(\eta')\Big)}{\sin^2\Big(\phi'(\eta')\Big)}\frac{\partial\phi'(\eta')}{\partial\eta}\exp(-G_{\eta',\eta})\mathrm{d}\eta', \tag{5.132}$$

$$X_6 = -\int_\eta^L \frac{\bar{\mathscr{V}}(\eta')}{\sin\Big(\phi'(\eta')\Big)}\exp(-G_{\eta',\eta})\frac{\partial G_{\eta',\eta}}{\partial\eta}\mathrm{d}\eta', \tag{5.133}$$

$$X_7 = -\frac{\bar{\mathscr{V}}(\eta)}{\sin(\phi)}. \tag{5.134}$$

We need to estimate each term. The estimates are standard, so we only list the results:

$$|X_1| \leq \frac{C}{\delta_0}\left\|\frac{\partial p}{\partial\phi}\right\|_{L^\infty}, \tag{5.135}$$

$$|X_2| \leq \frac{C}{\delta_0}\left\|p\right\|_{L^\infty}, \tag{5.136}$$

$$|X_3| \leq \frac{C}{\delta_0}\|\mathscr{V}\|_{L^\infty L^\infty}, \tag{5.137}$$

$$|X_4| \leq \frac{C}{\delta_0}\|\mathscr{V}\|_{L^\infty L^\infty}, \tag{5.138}$$

$$|X_5| \leq \frac{C}{\delta_0}\|\mathscr{V}\|_{L^\infty L^\infty}, \tag{5.139}$$

$$|X_6| \leq \frac{C}{\delta_0}\|\mathscr{V}\|_{L^\infty L^\infty}, \tag{5.140}$$

$$|X_7| \leq \frac{C}{\delta_0}\|\mathscr{V}\|_{L^\infty L^\infty}. \tag{5.141}$$

In total, we have

$$\left|\frac{\partial\mathscr{V}_1}{\partial\eta}\right| \leq \frac{C}{\delta_0}\bigg(\left\|p\right\|_{L^\infty}+\left\|\frac{\partial p}{\partial\phi}\right\|_{L^\infty}+\|\mathscr{V}\|_{L^\infty L^\infty}\bigg). \tag{5.142}$$

Using Formulation II, we rewrite the $\mathscr{V}_2$ equation along the characteristics as

$$\mathscr{V}_2(\eta,\phi) = \int_{\phi_*}^{\phi^*}\frac{S\Big(\eta'(\phi'),\phi'\Big)}{F\Big(\eta'(\phi')\Big)\cos(\phi')}\exp(-H_{\phi^*,\phi'}-H_{-\phi^*,\phi})\mathrm{d}\phi' + \int_{\phi}^{-\phi^*}\frac{S\Big(\eta'(\phi'),\phi'\Big)}{F\Big(\eta'(\phi')\Big)\cos(\phi')}\exp(-H_{\phi',\phi})\mathrm{d}\phi' \tag{5.143}$$

where (η',ϕ'), $(0,\phi_*)$, (L,ϕ^*), $(L,-\phi^*)$ and (η,ϕ) are on the same characteristic with $\sin\phi'\geq 0$ and $\phi^*\geq 0$. Then taking η derivative on both sides of (5.143) yields

$$\frac{\partial \mathscr{V}_2}{\partial\eta}=Y_1+Y_2+Y_3+Y_4+Y_5+Y_6+Y_7+Y_8, \tag{5.144}$$

where

$$
\begin{aligned}
Y_1&=\frac{S(L,\phi^*)}{F(L)\cos(\phi^*)}\exp(-H_{-\phi^*,\phi})\frac{\partial\phi^*}{\partial\eta}-\frac{S(0,\phi_*)}{F(0)\cos(\phi_*)}\exp(-H_{\phi^*,\phi_*}-H_{-\phi^*,\phi})\frac{\partial\phi_*}{\partial\eta},\\
Y_2&=-\int_{\phi_*}^{\phi^*}S\Big(\eta'(\phi'),\phi'\Big)\frac{1}{F^2\Big(\eta'(\phi')\Big)\cos(\phi')}\frac{\partial F\Big(\eta'(\phi')\Big)}{\partial\eta}\exp(-H_{\phi^*,\phi'}-H_{-\phi^*,\phi})\mathrm{d}\phi',\\
Y_3&=-\int_{\phi_*}^{\phi^*}\frac{S\Big(\eta'(\phi'),\phi'\Big)}{F\Big(\eta'(\phi')\Big)\cos(\phi')}\exp(-H_{\phi^*,\phi'}-H_{-\phi^*,\phi})\bigg(\frac{\partial H_{\phi^*,\phi'}}{\partial\eta}+\frac{\partial H_{-\phi^*,\phi}}{\partial\eta}\bigg)\mathrm{d}\phi', && (5.145)\\
Y_4&=\int_{\phi_*}^{\phi^*}\frac{\partial_{\eta'}S\Big(\eta'(\phi'),\phi'\Big)}{F\Big(\eta'(\phi')\Big)\cos(\phi')}\frac{\partial\eta'(\phi')}{\partial\eta}\exp(-H_{\phi^*,\phi'}-H_{-\phi^*,\phi})\mathrm{d}\phi', && (5.146)\\
Y_5&=-\frac{S(L,-\phi^*)}{F(L)\cos(-\phi^*)}\exp(-H_{-\phi^*,\phi})\frac{\partial\phi^*}{\partial\eta}, && (5.147)\\
Y_6&=-\int_{\phi}^{-\phi^*}S\Big(\eta'(\phi'),\phi'\Big)\frac{1}{F^2\Big(\eta'(\phi')\Big)\cos(\phi')}\frac{\partial F\Big(\eta'(\phi')\Big)}{\partial\eta}\exp(-H_{\phi',\phi})\mathrm{d}\phi', && (5.148)\\
Y_7&=-\int_{\phi}^{-\phi^*}\frac{S\Big(\eta'(\phi'),\phi'\Big)}{F\Big(\eta'(\phi')\Big)\cos(\phi')}\exp(-H_{\phi',\phi})\frac{\partial H_{\phi',\phi}}{\partial\eta}\mathrm{d}\phi', && (5.149)\\
Y_8&=\int_{\phi}^{-\phi^*}\frac{\partial_{\eta'}S\Big(\eta'(\phi'),\phi'\Big)}{F\Big(\eta'(\phi')\Big)\cos(\phi')}\frac{\partial\eta'(\phi')}{\partial\eta}\exp(-H_{\phi',\phi})\mathrm{d}\phi'. && (5.150)
\end{aligned}
$$

We need to estimate each term. The estimates are standard, so we only list the results:

$$|Y_1| \leq \frac{C}{\delta_0}\|S\|_{L^\infty L^\infty}, \tag{5.151}$$

$$|Y_2| \leq \frac{C}{\delta_0}\|S\|_{L^\infty L^\infty}, \tag{5.152}$$

$$|Y_3| \leq \frac{C}{\delta_0}\|S\|_{L^\infty L^\infty}, \tag{5.153}$$

$$|Y_4| \leq \frac{C}{\delta_0}\left\|\frac{\partial S}{\partial \eta}\right\|_{L^\infty L^\infty}, \tag{5.154}$$

$$|Y_5| \leq \frac{C}{\delta_0}\|S\|_{L^\infty L^\infty}, \tag{5.155}$$

$$|Y_6| \leq \frac{C}{\delta_0}\|S\|_{L^\infty L^\infty}, \tag{5.156}$$

$$|Y_7| \leq \frac{C}{\delta_0}\|S\|_{L^\infty L^\infty}, \tag{5.157}$$

$$|Y_8| \leq \frac{C}{\delta_0}\left\|\frac{\partial S}{\partial \eta}\right\|_{L^\infty L^\infty}, \tag{5.158}$$

$$\tag{5.159}$$

In total, we have

$$\left|\frac{\partial \mathscr{V}_2}{\partial \eta}\right| \leq \frac{C}{\delta_0}\bigg(\|S\|_{L^\infty L^\infty} + \left\|\frac{\partial S}{\partial \eta}\right\|_{L^\infty L^\infty}\bigg). \tag{5.160}$$

Combining all above, noting that $\zeta \geq \delta_0$, we have

$$|II_1| \leq \frac{C}{\delta_0^2}\bigg(\|p\|_{L^\infty} + \left\|\zeta\frac{\partial p}{\partial \phi}\right\|_{L^\infty} + \|S\|_{L^\infty L^\infty} + \left\|\zeta\frac{\partial S}{\partial \eta}\right\|_{L^\infty L^\infty} + \|\mathscr{V}\|_{L^\infty L^\infty}\bigg). \tag{5.161}$$

Step 2: Estimate of II_2 for $-\delta_0 \leq \sin\phi \leq 0$ and $\chi(\phi_*) < 1$.
This is similar to the estimate of I_2 based on the integral

$$\int_\eta^L \frac{1}{\sin\phi'}\exp(-G_{\eta',\eta})\mathrm{d}\eta' \leq 1. \tag{5.162}$$

Then we have

$$|II_2| \leq \left(\frac{1}{\delta} + \frac{\epsilon}{\delta^2}\right)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty}\bigg). \tag{5.163}$$

Step 3: Estimate of II_3 for $-\delta_0 \leq \sin\phi \leq 0$, $\chi(\phi_*) = 1$ and $\sqrt{\epsilon\eta'} \geq \sin\phi'$.
This is identical to the estimate of I_4, we have

$$|II_3| \leq C\delta\|\mathscr{A}\|_{L^\infty L^\infty}. \tag{5.164}$$

Step 4: Estimate of II_4 for $-\delta_0 \leq \sin\phi \leq 0$, $\chi(\phi_*) = 1$ and $\sqrt{\epsilon\eta'} \leq \sin\phi'$.
This step is different. We do not need to further decompose the cases. Based on (5.120), we have,

$$-G_{\eta,\eta'} \leq -\frac{\eta'-\eta}{|\sin\phi|}. \tag{5.165}$$

Then following the same argument in estimating I_5, we obtain

$$|II_4| \leq C\|\mathscr{A}\|_{L^\infty L^\infty}\int_\eta^L \Big(1+|\ln(\epsilon)|+|\ln(\eta')|\Big)\exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right)\mathrm{d}\eta' \tag{5.166}$$

If $\eta\geq 2$, we directly obtain

$$\begin{aligned}\left|\int_\eta^L |\ln(\eta')|\exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right)\mathrm{d}\eta'\right| &\leq \left|\int_2^L \ln(\eta')\exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right)\mathrm{d}\eta'\right| \\ &\leq \ln(2)\left|\int_2^L \exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right)\mathrm{d}\eta'\right| \\ &\leq C\sqrt{|\sin\phi|}.\end{aligned} \tag{5.167}$$

If $\eta\leq 2$, we decompose as

$$\begin{aligned}&\left|\int_\eta^L |\ln(\eta')|\exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right)\mathrm{d}\eta'\right| \\ \leq &\left|\int_\eta^2 |\ln(\eta')|\exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right)\mathrm{d}\eta'\right| + \left|\int_2^L |\ln(\eta')|\exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right)\mathrm{d}\eta'\right|.\end{aligned} \tag{5.168}$$

The second term is identical to the estimate in $\eta\geq 2$. We apply Cauchy's inequality to the first term

$$\begin{aligned}\left|\int_\eta^2 |\ln(\eta')|\exp\left(-\frac{\eta'-\eta}{|\sin\phi|}\right)\mathrm{d}\eta'\right| &\leq \left(\int_\eta^2 \ln^2(\eta')\mathrm{d}\eta'\right)^{\frac{1}{2}}\left(\int_\eta^2 \exp\left(-\frac{2(\eta'-\eta)}{|\sin\phi|}\right)\mathrm{d}\eta'\right)^{\frac{1}{2}} \\ &\leq \left(\int_0^2 \ln^2(\eta')\mathrm{d}\eta'\right)^{\frac{1}{2}}\left(\int_\eta^2 \exp\left(-\frac{2(\eta'-\eta)}{|\sin\phi|}\right)\mathrm{d}\eta'\right)^{\frac{1}{2}} \\ &\leq C\sqrt{|\sin\phi|}.\end{aligned} \tag{5.169}$$

Hence, we have

$$|II_4| \leq C(1+|\ln(\epsilon)|)\sqrt{\delta_0}\|\mathscr{A}\|_{L^\infty L^\infty}. \tag{5.170}$$

Step 5: Estimate of II_5 for $|\cos\phi|<\delta_0$.
This is identical to the estimate of I_6, we have

$$\begin{aligned}|II_5| \leq\ &C\Big(1+|\ln(\epsilon)|\Big)\bigg(\|p_{\mathscr{A}}\|_{L^\infty}+\|S_{\mathscr{A}}\|_{L^\infty L^\infty}\bigg)+C\Big(1+|\ln(\epsilon)|\Big)\left(\frac{1}{\delta}+\frac{\epsilon}{\delta^2}\right)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\bigg) \\ &+C\Big(1+|\ln(\epsilon)|\Big)\bigg(\delta+\Big(1+|\ln(\epsilon)|\Big)\sqrt{\delta_0}\bigg)\|\mathscr{A}\|_{L^\infty L^\infty}.\end{aligned} \tag{5.171}$$

Step 6: Synthesis.

$$
\begin{aligned}
|II| \leq\, & C\Big(1+|\ln(\epsilon)|\Big)\bigg(\, \|p_{\mathscr{A}}\|_{L^\infty}+\|S_{\mathscr{A}}\|_{L^\infty L^\infty}\bigg) \\
&+\frac{C}{\delta_0^2}\bigg(\, \|p\|_{L^\infty}+\left\|\zeta\frac{\partial p}{\partial\phi}\right\|_{L^\infty}+\|S\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}+\|\mathscr{V}\|_{L^\infty L^\infty}\bigg) \\
&+C\Big(1+|\ln(\epsilon)|\Big)\left(\frac{1}{\delta}+\frac{\epsilon}{\delta^2}\right)\bigg(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\bigg) \\
&+C\Big(1+|\ln(\epsilon)|\Big)\bigg(\delta+\Big(1+|\ln(\epsilon)|\Big)\sqrt{\delta_0}\bigg)\|\mathscr{A}\|_{L^\infty L^\infty}.
\end{aligned}
\tag{5.172}
$$

We consider

$$
\mathcal{K}[p_{\mathscr{A}}]=p_{\mathscr{A}}\Big(\phi'(\eta,\phi;0)\Big)\exp(-G_{\eta^+,0}-G_{\eta^+,\eta}) \tag{5.173}
$$

$$
\begin{aligned}
\mathcal{T}[\tilde{\mathscr{A}}+S_{\mathscr{A}}]=\, & \int_0^{\eta^+}\frac{(\tilde{\mathscr{A}}+S_{\mathscr{A}})\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta^+,\eta'}-G_{\eta^+,\eta})\mathrm{d}\eta' \\
&+\int_\eta^{\eta^+}\frac{(\tilde{\mathscr{A}}+S_{\mathscr{A}})\Big(\eta',\mathscr{R}[\phi'(\eta,\phi;\eta')]\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta',\eta})\mathrm{d}\eta'.
\end{aligned}
\tag{5.174}
$$

Based on [21, Lemma 4.7, Lemma 4.8], we still have

$$
\left|\mathcal{K}[p_{\mathscr{A}}]\right| \leq \|p_{\mathscr{A}}\|_{L^\infty}, \tag{5.175}
$$

$$
\left|\mathcal{T}[S_{\mathscr{A}}]\right| \leq \|S_{\mathscr{A}}\|_{L^\infty L^\infty}. \tag{5.176}
$$

Hence, we only need to estimate

$$
\begin{aligned}
III=\mathcal{T}[\tilde{\mathscr{A}}]=\, & \int_0^{\eta^+}\frac{\tilde{\mathscr{A}}\Big(\eta',\phi'(\eta,\phi;\eta')\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta^+,\eta'}-G_{\eta^+,\eta})\mathrm{d}\eta' \\
&+\int_\eta^{\eta^+}\frac{\tilde{\mathscr{A}}\Big(\eta',\mathscr{R}[\phi'(\eta,\phi;\eta')]\Big)}{\sin\Big(\phi'(\eta,\phi;\eta')\Big)}\exp(-G_{\eta',\eta})\mathrm{d}\eta'.
\end{aligned}
\tag{5.177}
$$

Note that $|E(\eta,\phi)|\geq \mathrm{e}^{-V(L)}$ implies

$$
\mathrm{e}^{-V(\eta)}\cos\phi\geq \mathrm{e}^{-V(L)}. \tag{5.178}
$$

Hence, we can further deduce that

$$
\cos\phi\geq \mathrm{e}^{V(\eta)-V(L)}\geq \mathrm{e}^{V(0)-V(L)}\geq\left(1-\frac{\epsilon^{\frac{1}{2}}}{R_\kappa}\right). \tag{5.179}
$$

Hence, we know

$$|\sin\phi| \leq \sqrt{1-\left(1-\frac{\epsilon^{\frac{1}{2}}}{R_\kappa}\right)^2} \leq \epsilon^{\frac{1}{4}}. \tag{5.180}$$

Hence, when ϵ is sufficiently small, we always have

$$|\sin\phi| \leq \epsilon^{\frac{1}{4}} \leq \delta_0. \tag{5.181}$$

This means we do not need to bother with the estimate of $\sin\phi \leq -\delta_0$ as Step 1 in estimating I and II. Also, it is not necessary to discuss the case $|\cos\phi| < \delta_0$.

Then the integral $\displaystyle\int_0^\eta(\cdots)$ is similar to the argument in Region I, and the integral $\displaystyle\int_\eta^{\eta^+}(\cdots)$ is similar to the argument in Region II. Hence, combining the methods in Region I and Region II, we can show the desired result, i.e.

$$\begin{aligned}|III| \leq\, & C\Big(1+|\ln(\epsilon)|\Big)\Big(\|p_{\mathscr{A}}\|_{L^\infty}+\|S_{\mathscr{A}}\|_{L^\infty L^\infty}\Big)\\ &+C\Big(1+|\ln(\epsilon)|\Big)\left(\frac{1}{\delta}+\frac{\epsilon}{\delta^2}\right)\Big(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\Big)\\ &+C\Big(1+|\ln(\epsilon)|\Big)\Big(\delta+\Big(1+|\ln(\epsilon)|\Big)\sqrt{\delta_0}\Big)\|\mathscr{A}\|_{L^\infty L^\infty}.\end{aligned} \tag{5.182}$$

Combining the analysis in these three regions, we have

$$\begin{aligned}|\mathscr{A}| \leq\, & C\Big(1+|\ln(\epsilon)|\Big)\Big(\|p_{\mathscr{A}}\|_{L^\infty}+\|S_{\mathscr{A}}\|_{L^\infty L^\infty}\Big)\\ &+\frac{C}{\delta_0^2}\left(\|p\|_{L^\infty}+\left\|\zeta\frac{\partial p}{\partial\phi}\right\|_{L^\infty}+\|S\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}+\|\mathscr{V}\|_{L^\infty L^\infty}\right)\\ &+C\Big(1+|\ln(\epsilon)|\Big)\left(\frac{1}{\delta}+\frac{\epsilon}{\delta^2}\right)\Big(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\Big)\\ &+C\Big(1+|\ln(\epsilon)|\Big)\Big(\delta+\Big(1+|\ln(\epsilon)|\Big)\sqrt{\delta_0}\Big)\|\mathscr{A}\|_{L^\infty L^\infty}.\end{aligned} \tag{5.183}$$

Taking supremum over all (η,ϕ), we have

$$\begin{aligned}\|\mathscr{A}\|_{L^\infty L^\infty} \leq\, & C\Big(1+|\ln(\epsilon)|\Big)\Big(\|p_{\mathscr{A}}\|_{L^\infty}+\|S_{\mathscr{A}}\|_{L^\infty L^\infty}\Big)\\ &+\frac{C}{\delta_0^2}\left(\|p\|_{L^\infty}+\left\|\zeta\frac{\partial p}{\partial\phi}\right\|_{L^\infty}+\|S\|_{L^\infty L^\infty}+\left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}+\|\mathscr{V}\|_{L^\infty L^\infty}\right)\\ &+C\Big(1+|\ln(\epsilon)|\Big)\left(\frac{1}{\delta}+\frac{\epsilon}{\delta^2}\right)\Big(\|\mathscr{V}\|_{L^\infty L^\infty}+\|S\|_{L^\infty L^\infty}\Big)\\ &+C\Big(1+|\ln(\epsilon)|\Big)\Big(\delta+\Big(1+|\ln(\epsilon)|\Big)\sqrt{\delta_0}\Big)\|\mathscr{A}\|_{L^\infty L^\infty}.\end{aligned} \tag{5.184}$$

Then we choose these constants to perform absorbing argument. First we choose $\delta = C_0\Big(1+|\ln(\epsilon)|\Big)^{-1}$ for $C_0>0$ sufficiently small such that

$$C\delta \leq \frac{1}{4}. \tag{5.185}$$

Then we take $\delta_0 = C_0\Big(1 + |\ln(\epsilon)|\Big)^{-4}$ such that

$$C\Big(1 + |\ln(\epsilon)|\Big)^2 \sqrt{\delta_0} \leq \frac{1}{4}. \tag{5.186}$$

for ϵ sufficiently small. Note that this mild decay of δ_0 with respect to ϵ also justifies the assumption in Case III that

$$\epsilon^{\frac{1}{4}} \leq \frac{\delta_0}{2}, \tag{5.187}$$

for ϵ sufficiently small. Hence, we can absorb all the term related to $\|\mathscr{A}\|_{L^\infty L^\infty}$ on the right-hand side of (5.184) to the left-hand side to obtain

$$\begin{aligned}\|\mathscr{A}\|_{L^\infty L^\infty} \leq{}& C\,|\ln(\epsilon)| \bigg(\|p_{\mathscr{A}}\|_{L^\infty} + \|S_{\mathscr{A}}\|_{L^\infty L^\infty} \bigg) \\ &+ C\,|\ln(\epsilon)|^8 \bigg(\|p\|_{L^\infty} + \left\|\zeta\frac{\partial p}{\partial\phi}\right\|_{L^\infty} + \|S\|_{L^\infty L^\infty} + \left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty} + \|\mathscr{V}\|_{L^\infty L^\infty} \bigg).\end{aligned} \tag{5.188}$$

In this subsection, we further estimate the normal and velocity derivatives.

Theorem 5.1. *We have*

$$\begin{aligned}&\left\|\zeta\frac{\partial\mathscr{V}}{\partial\eta}\right\|_{L^\infty L^\infty} + \left\|F(\eta)\cos\phi\frac{\partial\mathscr{V}}{\partial\phi}\right\|_{L^\infty L^\infty} \\ &\leq C\,|\ln(\epsilon)|^8 \bigg(\|p\|_{L^\infty} + \left\|(\epsilon+\zeta)\frac{\partial p}{\partial\phi}\right\|_{L^\infty} + \|S\|_{L^\infty L^\infty} + \left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty} + \|\mathscr{V}\|_{L^\infty L^\infty} \bigg).\end{aligned} \tag{5.189}$$

Proof. We have

$$\begin{aligned}\|\mathscr{A}\|_{L^\infty L^\infty} \leq{}& C\,|\ln(\epsilon)| \bigg(\|p_{\mathscr{A}}\|_{L^\infty} + \|S_{\mathscr{A}}\|_{L^\infty L^\infty} \bigg) \\ &+ C\,|\ln(\epsilon)|^8 \bigg(\|p\|_{L^\infty} + \left\|\zeta\frac{\partial p}{\partial\phi}\right\|_{L^\infty} + \|S\|_{L^\infty L^\infty} + \left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty} + \|\mathscr{V}\|_{L^\infty L^\infty} \bigg).\end{aligned} \tag{5.190}$$

Taking derivatives on both sides of (5.7) and multiplying ζ, we have

$$p_{\mathscr{A}} = \epsilon\cos\phi\frac{\partial p}{\partial\phi} + p - \bar{\mathscr{V}}(0) - S(0,\phi), \tag{5.191}$$

$$S_{\mathscr{A}} = \frac{\partial F}{\partial\eta}\zeta\cos\phi\frac{\partial\mathscr{V}}{\partial\phi} + \zeta\frac{\partial S}{\partial\eta}. \tag{5.192}$$

Since $|F(\eta)| \leq C\epsilon$ and $\left|\dfrac{\partial F}{\partial\eta}\right| \leq C\epsilon F$, we may directly estimate

$$\|p_{\mathscr{A}}\|_{L^\infty} \leq C\bigg(\|p\|_{L^\infty} + \epsilon\left\|\frac{\partial p}{\partial\phi}\right\|_{L^\infty} + \|S\|_{L^\infty L^\infty} + \|\mathscr{V}\|_{L^\infty L^\infty}\bigg), \tag{5.193}$$

$$\|S_{\mathscr{A}}\|_{L^\infty L^\infty} \leq C\bigg(\epsilon\left\|F(\eta)\cos\phi\frac{\partial\mathscr{V}}{\partial\phi}\right\|_{L^\infty L^\infty} + \left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty}\bigg) \tag{5.194}$$

Then we derive

$$\begin{aligned}\|\mathscr{A}\|_{L^\infty L^\infty} \leq{}& C\epsilon\left\|F(\eta)\cos\phi\frac{\partial\mathscr{V}}{\partial\phi}\right\|_{L^\infty L^\infty} \\ &+ C\left|\ln(\epsilon)\right|^8\bigg(\|p\|_{L^\infty} + \left\|(\epsilon+\zeta)\frac{\partial p}{\partial\phi}\right\|_{L^\infty} + \|S\|_{L^\infty L^\infty} + \left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty} + \|\mathscr{V}\|_{L^\infty L^\infty}\bigg).\end{aligned} \tag{5.195}$$

We know

$$\begin{aligned}\left\|\zeta\frac{\partial\mathscr{V}}{\partial\eta}\right\|_{L^\infty L^\infty} \leq{}& C\epsilon\left\|F(\eta)\cos\phi\frac{\partial\mathscr{V}}{\partial\phi}\right\|_{L^\infty L^\infty} \\ &+ C\left|\ln(\epsilon)\right|^8\bigg(\|p\|_{L^\infty} + \left\|(\epsilon+\zeta)\frac{\partial p}{\partial\phi}\right\|_{L^\infty} + \|S\|_{L^\infty L^\infty} + \left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty} + \|\mathscr{V}\|_{L^\infty L^\infty}\bigg).\end{aligned} \tag{5.196}$$

Considering the equation (5.7), since $\zeta(\eta,\phi) \geq |\sin\phi|$, we have

$$\begin{aligned}\left\|F(\eta)\cos\phi\frac{\partial\mathscr{V}}{\partial\phi}\right\|_{L^\infty L^\infty} \leq{}& \left\|\sin\phi\frac{\partial\mathscr{V}}{\partial\eta}\right\|_{L^\infty L^\infty} + \|\mathscr{V}\|_{L^\infty L^\infty} + \|\bar{\mathscr{V}}\|_{L^\infty L^\infty} + \|S\|_{L^\infty L^\infty} \\ \leq{}& C\epsilon\left\|F(\eta)\cos\phi\frac{\partial\mathscr{V}}{\partial\phi}\right\|_{L^\infty L^\infty} \\ &+ C\left|\ln(\epsilon)\right|^8\bigg(\|p\|_{L^\infty} + \left\|(\epsilon+\zeta)\frac{\partial p}{\partial\phi}\right\|_{L^\infty} + \|S\|_{L^\infty L^\infty} + \left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty} + \|\mathscr{V}\|_{L^\infty L^\infty}\bigg).\end{aligned} \tag{5.197}$$

Absorbing $\left\|F(\eta)\cos\phi\frac{\partial\mathscr{V}}{\partial\phi}\right\|_{L^\infty L^\infty}$ into the left-hand side, we obtain

$$\left\|F(\eta)\cos\phi\frac{\partial\mathscr{V}}{\partial\phi}\right\|_{L^\infty L^\infty} \leq C\left|\ln(\epsilon)\right|^8\bigg(\|p\|_{L^\infty} + \left\|(\epsilon+\zeta)\frac{\partial p}{\partial\phi}\right\|_{L^\infty} + \|S\|_{L^\infty L^\infty} + \left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty} + \|\mathscr{V}\|_{L^\infty L^\infty}\bigg). \tag{5.198}$$

Therefore, we further derive

$$\left\|\zeta\frac{\partial\mathscr{V}}{\partial\eta}\right\|_{L^\infty L^\infty} \leq C\left|\ln(\epsilon)\right|^8\bigg(\|p\|_{L^\infty} + \left\|(\epsilon+\zeta)\frac{\partial p}{\partial\phi}\right\|_{L^\infty} + \|S\|_{L^\infty L^\infty} + \left\|\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty} + \|\mathscr{V}\|_{L^\infty L^\infty}\bigg). \tag{5.199}$$

$\square$

Theorem 5.2. *For $K_0>0$ sufficiently small, we have*

$$\begin{aligned}&\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial\mathscr{V}}{\partial\eta}\right\|_{L^\infty L^\infty} + \left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial\mathscr{V}}{\partial\phi}\right\|_{L^\infty L^\infty} \\ &\leq C\left|\ln(\epsilon)\right|^8\bigg(\|p\|_{L^\infty} + \left\|(\epsilon+\zeta)\frac{\partial p}{\partial\phi}\right\|_{L^\infty} + \|\mathrm{e}^{K_0\eta}S\|_{L^\infty L^\infty} + \left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial S}{\partial\eta}\right\|_{L^\infty L^\infty} + \|\mathrm{e}^{K_0\eta}\mathscr{V}\|_{L^\infty L^\infty}\bigg).\end{aligned} \tag{5.200}$$

Proof. This proof is almost identical to Theorem 5.1. The only difference is that $S_{\mathscr{A}}$ is added by $K_0\mathscr{A}\sin\phi$. When K_0 is sufficiently small, we can also absorb them into the left-hand side. Hence, this is obvious. $\square$

6. Diffusive Limit

In this subsection, we will justify that the regular boundary layers are all well-defined. We divide it into several steps:

Step 1: Well-Posedness of $\mathscr{U}_0$.
$\mathscr{U}_0$ satisfies the ϵ-Milne problem with geometric correction

$$\begin{cases} \sin\phi\dfrac{\partial\mathscr{U}_0}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial\mathscr{U}_0}{\partial\phi}+\mathscr{U}_0-\bar{\mathscr{U}}_0=0,\\ \mathscr{U}_0(0,\tau,\phi)=\mathscr{G}(\tau,\phi)-\mathscr{F}_0(\tau)\ \ \text{for}\ \ \sin\phi>0,\\ \mathscr{U}_0(L,\tau,\phi)=\mathscr{U}_0(L,\tau,\mathscr{R}[\phi]), \end{cases} \tag{6.1}$$

Therefore, since $\|\mathscr{G}\|_{L^\infty}\leq C$, by Theorem 4.9, we know

$$\left\|\mathrm{e}^{K_0\eta}\mathscr{U}_0\right\|_{L^\infty L^\infty}\leq C. \tag{6.2}$$

Step 2: Tangential Derivatives of $\mathscr{U}_0$.
The τ derivative $W=\dfrac{\partial\mathscr{U}_0}{\partial\tau}$ satisfies

$$\begin{cases} \sin\phi\dfrac{\partial W}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial W}{\partial\phi}+W-\bar{W}=-\dfrac{R_\kappa'}{R_\kappa-\epsilon\eta}F(\eta)\cos\phi\dfrac{\partial\mathscr{U}_0}{\partial\phi},\\ W(0,\tau,\phi)=\dfrac{\partial\mathscr{G}}{\partial\tau}(\tau,\phi)-\dfrac{\partial\mathscr{F}_0}{\partial\tau}(\tau)\ \ \text{for}\ \ \sin\phi>0,\\ W(L,\tau,\phi)=W(L,\tau,\mathscr{R}[\phi]), \end{cases} \tag{6.3}$$

where R_κ' represents the θ derivative of R_κ. Here we need the regularity estimates of $\mathscr{U}_0$.
Based on Theorem 5.2, we know

$$\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right\|_{L^\infty L^\infty}\leq C\left|\ln(\epsilon)\right|^8\left(\|\mathscr{G}\|_{L^\infty}+\left\|(\epsilon+\zeta)\frac{\partial\mathscr{G}}{\partial\phi}\right\|_{L^\infty}+\left\|\mathrm{e}^{K_0\eta}\mathscr{U}_0\right\|_{L^\infty L^\infty}\right)\leq C\left|\ln(\epsilon)\right|^8. \tag{6.4}$$

Note that here although $\left\|\dfrac{\partial\mathscr{G}}{\partial\phi}\right\|_{L^\infty}\leq C\epsilon^{-\alpha}$, with the help of $\epsilon+\zeta$, we can get rid of this negative power. Therefore, by Theorem 4.9, we have

$$\left\|\mathrm{e}^{K_0\eta}W\right\|_{L^\infty L^\infty}\leq C\left|\ln(\epsilon)\right|^8. \tag{6.5}$$

Step 3: Well-Posedness of $\mathscr{U}_1$.
$\mathscr{U}_1$ satisfies the ϵ-Milne problem with geometric correction

$$\begin{cases} \sin\phi\dfrac{\partial\mathscr{U}_1}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial\mathscr{U}_1}{\partial\phi}+\mathscr{U}_1-\bar{\mathscr{U}}_1=\dfrac{W}{R_\kappa-\epsilon\eta}\cos\phi,\\ \mathscr{U}_1(0,\tau,\phi)=\vec{w}\cdot\nabla_xU_0(\vec{x}_0,\vec{w})-\mathscr{F}_{1,L}(\tau)\ \ \text{for}\ \ \sin\phi>0,\\ \mathscr{U}_1(L,\tau,\phi)=\mathscr{U}_1(L,\tau,\mathscr{R}[\phi]). \end{cases} \tag{6.6}$$

Therefore, by Theorem 4.9, we know

$$\left\|\mathrm{e}^{K_0\eta}\mathscr{U}_1\right\|_{L^\infty L^\infty}\leq C\left\|\mathrm{e}^{K_0\eta}W\right\|_{L^\infty L^\infty}\leq C\left|\ln(\epsilon)\right|^8. \tag{6.7}$$

Step 4: Tangential Derivatives of $\mathscr{U}_1$.
The τ derivative $V=\dfrac{\partial\mathscr{U}_1}{\partial\tau}$ satisfies

$$\begin{cases}\sin\phi\dfrac{\partial V}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial V}{\partial\phi}+V-\bar{V}=S_1+S_2+S_3,\\ V(0,\tau,\phi)=\dfrac{\partial}{\partial\tau}\bigg(\vec{w}\cdot\nabla_xU_0(\vec{x}_0,\vec{w})-\mathscr{F}_{1,L}(\tau)\bigg)\ \text{ for }\ \sin\phi>0,\\ V(L,\tau,\phi)=V(L,\tau,\mathscr{R}[\phi]),\end{cases} \tag{6.8}$$

where

$$S_1=-\frac{R_\kappa'}{R_\kappa-\epsilon\eta}F(\eta)\cos\phi\frac{\partial\mathscr{U}_1}{\partial\phi}, \tag{6.9}$$

$$S_2=-\frac{R_\kappa'}{(R_\kappa-\epsilon\eta)^2}W\cos\phi, \tag{6.10}$$

$$S_3=\frac{1}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial W}{\partial\tau}. \tag{6.11}$$

Based on Theorem 5.2, we have

$$\begin{aligned}\left\|\mathrm{e}^{K_0\eta}S_1\right\|_{L^\infty L^\infty}&\leq C\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial\mathscr{U}_1}{\partial\phi}\right\|_{L^\infty L^\infty}\\ &\leq C\left(\left\|\mathrm{e}^{K_0\eta}\frac{W}{R_\kappa-\epsilon\eta}\cos\phi\right\|_{L^\infty L^\infty}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial}{\partial\eta}\left(\frac{W}{R_\kappa-\epsilon\eta}\cos\phi\right)\right\|_{L^\infty L^\infty}\right)\\ &\leq C\left(\left\|\mathrm{e}^{K_0\eta}W\right\|_{L^\infty L^\infty}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial W}{\partial\eta}\right\|_{L^\infty L^\infty}\right),\end{aligned} \tag{6.12}$$

$$\begin{aligned}\left\|\mathrm{e}^{K_0\eta}S_2\right\|_{L^\infty L^\infty}&\leq C\left\|\mathrm{e}^{K_0\eta}\frac{R_\kappa'}{(R_\kappa-\epsilon\eta)^2}W\cos\phi\right\|_{L^\infty L^\infty}\\ &\leq C\left\|\mathrm{e}^{K_0\eta}W\right\|_{L^\infty L^\infty},\end{aligned} \tag{6.13}$$

$$\left\|\mathrm{e}^{K_0\eta}S_3\right\|_{L^\infty L^\infty}\leq C\left\|\mathrm{e}^{K_0\eta}\frac{\partial W}{\partial\tau}\right\|_{L^\infty L^\infty}. \tag{6.14}$$

Step 5: Tangential Derivatives of W.
The τ derivative $Z=\dfrac{\partial W}{\partial\tau}$ satisfies

$$\begin{cases}\sin\phi\dfrac{\partial Z}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial Z}{\partial\phi}+Z-\bar{Z}=T_1+T_2,\\ Z(0,\tau,\phi)=\dfrac{\partial^2\mathscr{G}}{\partial\tau^2}(\tau,\phi)-\dfrac{\partial^2\mathscr{F}_0}{\partial\tau^2}(\tau)\ \text{ for }\ \sin\phi>0,\\ Z(L,\tau,\phi)=Z(L,\tau,\mathscr{R}[\phi]),\end{cases} \tag{6.15}$$

where

$$T_1 = -\frac{R_\kappa'}{R_\kappa - \epsilon\eta} F(\eta)\cos\phi \frac{\partial W}{\partial\phi}, \tag{6.16}$$

$$T_2 = -\frac{\partial}{\partial\tau}\left(\frac{R_\kappa'}{R_\kappa - \epsilon\eta}\right) F(\eta)\cos\phi \frac{\partial \mathscr{U}_0}{\partial\phi}. \tag{6.17}$$

Based on Theorem 5.2, we have

$$\left\|\mathrm{e}^{K_0\eta} T_1\right\|_{L^\infty L^\infty} \leq C\left\|\mathrm{e}^{K_0\eta} F(\eta)\cos\phi \frac{\partial W}{\partial\phi}\right\|_{L^\infty L^\infty}, \tag{6.18}$$

$$\left\|\mathrm{e}^{K_0\eta} T_2\right\|_{L^\infty L^\infty} \leq C\left\|F(\eta)\cos\phi \frac{\partial \mathscr{U}_0}{\partial\phi}\right\|_{L^\infty L^\infty} \leq C\left|\ln(\epsilon)\right|^8. \tag{6.19}$$

Therefore, we have

$$\left\|\mathrm{e}^{K_0\eta} S_3\right\|_{L^\infty L^\infty} \leq \left\|\mathrm{e}^{K_0\eta} Z\right\|_{L^\infty L^\infty} \leq C\left|\ln(\epsilon)\right|^8 + C\left\|\mathrm{e}^{K_0\eta} F(\eta)\cos\phi \frac{\partial W}{\partial\phi}\right\|_{L^\infty L^\infty}. \tag{6.20}$$

In total, we have

$$\begin{aligned} &\left\|\mathrm{e}^{K_0\eta} S_1\right\|_{L^\infty L^\infty} + \left\|\mathrm{e}^{K_0\eta} S_2\right\|_{L^\infty L^\infty} + \left\|\mathrm{e}^{K_0\eta} S_3\right\|_{L^\infty L^\infty} \\ &\leq C\left|\ln(\epsilon)\right|^8 + C\left(\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial W}{\partial\eta}\right\|_{L^\infty L^\infty} + \left\|\mathrm{e}^{K_0\eta} F(\eta)\cos\phi \frac{\partial W}{\partial\phi}\right\|_{L^\infty L^\infty}\right). \end{aligned} \tag{6.21}$$

Hence, we need the regularity estimate of W. However, this cannot be done directly. We will first study the normal derivative of $\mathscr{U}_0$.

Step 6: Regularity of Normal Derivative.
The normal derivative $A = \dfrac{\partial \mathscr{U}_0}{\partial\eta}$ satisfies

$$\begin{cases} \sin\phi \dfrac{\partial A}{\partial\eta} + F(\eta)\cos\phi \dfrac{\partial A}{\partial\phi} + A - \bar{A} = \dfrac{\epsilon}{R - \epsilon\eta} F(\eta)\cos\phi \dfrac{\partial \mathscr{U}_0}{\partial\phi}, \\ A(0,\tau,\phi) = \dfrac{1}{\sin\phi}\left(F(\eta)\cos\phi \dfrac{\partial \mathscr{G}}{\partial\phi}(\tau,\phi) - \mathscr{G}(0,\tau,\phi) + \bar{\mathscr{U}}_0(0,\tau,\phi)\right) \quad \text{for} \quad \sin\phi > 0, \\ A(L,\tau,\phi) = A(L,\tau,\mathscr{R}[\phi]), \end{cases} \tag{6.22}$$

This is where the cut-off in $\mathscr{G}$ plays a role. Based on the construction of $\mathscr{G}$, we know $\left\|A(0,\phi,\tau)\right\|_{L^\infty} \leq C\epsilon^{-\alpha}$ and $\left\|(\epsilon+\zeta)\dfrac{\partial A}{\partial\phi}(0,\phi,\tau)\right\|_{L^\infty} \leq C\epsilon^{-\alpha}$. Therefore, using Theorem 4.9, we have

$$\left\|\mathrm{e}^{K_0\eta} A\right\|_{L^\infty L^\infty} \leq C\left(\left\|A(0,\phi,\tau)\right\|_{L^\infty} + \left\|\mathrm{e}^{K_0\eta} F(\eta)\cos\phi \frac{\partial \mathscr{U}_0}{\partial\phi}\right\|_{L^\infty L^\infty}\right) \leq C\epsilon^{-\alpha}. \tag{6.23}$$

By Theorem 5.2, we know

$$\begin{aligned} &\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial A}{\partial\eta}\right\|_{L^\infty L^\infty} + \left\|\mathrm{e}^{K_0\eta} F(\eta)\cos\phi \frac{\partial A}{\partial\phi}\right\|_{L^\infty L^\infty} \\ &\leq C\left|\ln(\epsilon)\right|^8 \left(\epsilon^{-\alpha} + \left\|\mathrm{e}^{K_0\eta}\frac{\epsilon}{R-\epsilon\eta} F(\eta)\cos\phi \frac{\partial \mathscr{U}_0}{\partial\phi}\right\|_{L^\infty L^\infty} + \left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial}{\partial\eta}\left(\frac{\epsilon}{R-\epsilon\eta} F(\eta)\cos\phi \frac{\partial \mathscr{U}_0}{\partial\phi}\right)\right\|_{L^\infty L^\infty}\right) \\ &\leq C\left|\ln(\epsilon)\right|^8 \left(\epsilon^{-\alpha} + \epsilon\left\|\mathrm{e}^{K_0\eta} F(\eta)\cos\phi \frac{\partial \mathscr{U}_0}{\partial\phi}\right\|_{L^\infty L^\infty} + \epsilon\left\|\mathrm{e}^{K_0\eta} F(\eta)\cos\phi \frac{\partial A}{\partial\phi}\right\|_{L^\infty L^\infty}\right) \end{aligned} \tag{6.24}$$

Then we may absorb $\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\dfrac{\partial A}{\partial\phi}\right\|_{L^\infty L^\infty}$ into the left-hand side to obtain

$$\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial A}{\partial\eta}\right\|_{L^\infty L^\infty}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial A}{\partial\phi}\right\|_{L^\infty L^\infty}\leq C\epsilon^{-\alpha}\left|\ln(\epsilon)\right|^8. \tag{6.25}$$

Step 7: Regularity of Tangential Derivative.
We turn to the regularity of W. Based on Theorem 5.2, we have

$$\begin{aligned}
&\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial W}{\partial\eta}\right\|_{L^\infty L^\infty}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial W}{\partial\phi}\right\|_{L^\infty L^\infty} \qquad (6.26)\\
\leq{}& C\left|\ln(\epsilon)\right|^8\left(\epsilon^{-\alpha}+\left\|\mathrm{e}^{K_0\eta}\frac{R_\kappa'}{R_\kappa-\epsilon\eta}F(\eta)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right\|_{L^\infty L^\infty}+\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial}{\partial\eta}\left(\frac{R_\kappa'}{R_\kappa-\epsilon\eta}F(\eta)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right)\right\|_{L^\infty L^\infty}\right)\\
\leq{}& C\left|\ln(\epsilon)\right|^8\left(\epsilon^{-\alpha}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial\mathscr{U}_0}{\partial\phi}\right\|_{L^\infty L^\infty}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial A}{\partial\phi}\right\|_{L^\infty L^\infty}\right)\\
\leq{}& C\epsilon^{-\alpha}\left|\ln(\epsilon)\right|^{16}.
\end{aligned}$$

Step 8: Synthesis.
Using above estimates, we actually have shown that

$$\left\|\mathrm{e}^{K_0\eta}V\right\|_{L^\infty L^\infty}\leq C\epsilon^{-\alpha}\left|\ln(\epsilon)\right|^{16}. \tag{6.27}$$

In this subsection, we will justify that the singular boundary layers are all well-defined. We divide it into several steps:

Step 1: Well-Posedness of $\mathfrak{U}_0$.
$\mathfrak{U}_0$ satisfies the ϵ-Milne problem with geometric correction

$$\begin{cases}
\sin\phi\dfrac{\partial\mathfrak{U}_0}{\partial\eta}+F(\eta)\cos\phi\dfrac{\partial\mathfrak{U}_0}{\partial\phi}+\mathfrak{U}_0-\bar{\mathfrak{U}}_0=0,\\
\mathfrak{U}_0(0,\tau,\phi)=\mathfrak{G}(\tau,\phi)-\mathfrak{F}_{0,L}(\tau)\quad\text{for}\quad\sin\phi>0,\\
\mathfrak{U}_0(L,\tau,\phi)=\mathfrak{U}_0(L,\tau,\mathscr{R}[\phi]).
\end{cases} \tag{6.28}$$

Therefore, by Theorem 4.9, we know

$$\left\|\mathrm{e}^{K_0\eta}\mathfrak{U}_0\right\|_{L^\infty L^\infty}\leq C. \tag{6.29}$$

However, this is not sufficient for future use and we need more detailed analysis. We will divide the domain $(\eta,\phi)\in[0,L]\times[-\pi,\pi)$ into two regions:

- Region I χ_1: $0\leq\zeta<2\epsilon^\alpha$.
- Region II χ_2: $2\epsilon^\alpha\leq\zeta\leq1$.

Here we use χ_i to represent either the corresponding region or the indicator function. It is easy to see that $\mathfrak{G}=0$ in Region II. Similarly we decompose the solution $\mathfrak{U}_0=\chi_1\mathfrak{U}_0+\chi_2\mathfrak{U}_0=f_1+f_2$ in these two regions. In the following, the estimates for f_i will be restricted to the region χ_i for $i=1,2$. Using Theorem 4.3, we can easily show that

$$\left\|\mathrm{e}^{K_0\eta}\mathfrak{U}_0\right\|_{L^2L^2}\leq C\epsilon^{\alpha}. \tag{6.30}$$

The key to L^{∞} estimates in Theorem 4.10 is Lemma 4.6 and Lemma 4.7. Their proofs are basically tracking along the characteristics. Hence, we know

$$\begin{aligned}\left\|\mathrm{e}^{K_0\eta}\bar{\mathfrak{U}}_0\right\|_{L^\infty L^\infty}&\leq C\bigg(\epsilon^{\alpha}\left\|\mathrm{e}^{K_0\eta}f_1\right\|_{L^\infty L^2}+\left\|\mathrm{e}^{K_0\eta}f_2\right\|_{L^\infty L^2}\bigg)\\&\leq C\bigg(\left\|\mathrm{e}^{K_0\eta}\mathfrak{U}_0\right\|_{L^2L^2}+\delta\epsilon^{\alpha}\left\|\mathrm{e}^{K_0\eta}f_1\right\|_{L^\infty L^\infty}+\delta\left\|\mathrm{e}^{K_0\eta}f_2\right\|_{L^\infty L^\infty}\bigg).\end{aligned} \tag{6.31}$$

Thus, considering $\chi_1\mathfrak{G}=\mathfrak{G}$ and $\chi_2\mathfrak{G}=0$, we may directly obtain

$$\begin{aligned}\left\|\mathrm{e}^{K_0\eta}f_1\right\|_{L^\infty L^\infty}&\leq C\bigg(\left\|\chi_1\mathfrak{G}\right\|_{L^\infty}+\left\|\mathrm{e}^{K_0\eta}\bar{\mathfrak{U}}_0\right\|_{L^\infty L^\infty}\bigg)\\&\leq C\bigg(\left\|\chi_1\mathfrak{G}\right\|_{L^\infty}+\left\|\mathrm{e}^{K_0\eta}\mathfrak{U}_0\right\|_{L^2L^2}+\delta\epsilon^{\alpha}\left\|\mathrm{e}^{K_0\eta}f_1\right\|_{L^\infty L^\infty}+\delta\left\|\mathrm{e}^{K_0\eta}f_2\right\|_{L^\infty L^\infty}\bigg)\\&\leq C\bigg(1+\delta\epsilon^{\alpha}\left\|\mathrm{e}^{K_0\eta}f_1\right\|_{L^\infty L^\infty}+\delta\left\|\mathrm{e}^{K_0\eta}f_2\right\|_{L^\infty L^\infty}\bigg),\end{aligned} \tag{6.32}$$

$$\begin{aligned}\left\|\mathrm{e}^{K_0\eta}f_2\right\|_{L^\infty L^\infty}&\leq C\bigg(\left\|\chi_2\mathfrak{G}\right\|_{L^\infty}+\left\|\mathrm{e}^{K_0\eta}\bar{\mathfrak{U}}_0\right\|_{L^\infty L^\infty}\bigg)\\&\leq C\bigg(\left\|\chi_2\mathfrak{G}\right\|_{L^\infty}+\left\|\mathrm{e}^{K_0\eta}\mathfrak{U}_0\right\|_{L^2L^2}+\delta\epsilon^{\alpha}\left\|\mathrm{e}^{K_0\eta}f_1\right\|_{L^\infty L^\infty}+\delta\left\|\mathrm{e}^{K_0\eta}f_2\right\|_{L^\infty L^\infty}\bigg)\\&\leq C\bigg(\epsilon^{\alpha}+\delta\epsilon^{\alpha}\left\|\mathrm{e}^{K_0\eta}f_1\right\|_{L^\infty L^\infty}+\delta\left\|\mathrm{e}^{K_0\eta}f_2\right\|_{L^\infty L^\infty}\bigg).\end{aligned} \tag{6.33}$$

Letting δ small, absorbing $\left\|\mathrm{e}^{K_0\eta}f_1\right\|_{L^\infty L^\infty}$ and $\left\|\mathrm{e}^{K_0\eta}f_2\right\|_{L^\infty L^\infty}$, we know

$$\left\|\mathrm{e}^{K_0\eta}f_1\right\|_{L^\infty L^\infty}\leq C\bigg(1+\delta\left\|\mathrm{e}^{K_0\eta}f_2\right\|_{L^\infty L^\infty}\bigg), \tag{6.34}$$

$$\left\|\mathrm{e}^{K_0\eta}f_2\right\|_{L^\infty L^\infty}\leq C\bigg(\epsilon^{\alpha}+\delta\epsilon^{\alpha}\left\|\mathrm{e}^{K_0\eta}f_1\right\|_{L^\infty L^\infty}\bigg). \tag{6.35}$$

Combining them together, we can easily see that

$$\left\|\mathrm{e}^{K_0\eta}f_1\right\|_{L^\infty L^\infty}\leq C, \tag{6.36}$$

$$\left\|\mathrm{e}^{K_0\eta}f_2\right\|_{L^\infty L^\infty}\leq C\epsilon^{\alpha}. \tag{6.37}$$

In total, we can derive

$$\left\|\mathrm{e}^{K_0\eta}\bar{\mathfrak{U}}_0\right\|_{L^\infty L^\infty}\leq C\epsilon^{\alpha}. \tag{6.38}$$

Step 2: Regularity of $\mathfrak{U}_0$.
This is very similar to the well-posedness proof, we will also consider the regularity of $\mathfrak{U}_0$ in two regions. Note that in the proof of Theorem 5.2, the L^{∞} estimates relies on two kinds of quantities:

- $\left|\zeta\dfrac{\partial\mathfrak{U}_0}{\partial\eta}\right|$ on the same characteristics.

- $\displaystyle\int_{-\pi}^{\pi}\zeta\frac{\partial\mathfrak{U}_0}{\partial\eta}\mathrm{d}\phi$ for some $\eta>0$.

Correspondingly, we may handle them separately: for the first case, since ζ is preserved along the characteristics, we can directly separate the estimate of f_1 and f_2; for the second case, we may use the simple domain decomposition

$$\int_{-\pi}^{\pi}\zeta\frac{\partial\mathfrak{U}_0}{\partial\eta}(\eta,\phi)\mathrm{d}\phi=\int_{\chi_1}\zeta\frac{\partial f_1}{\partial\eta}\mathrm{d}\phi+\int_{\chi_2}\zeta\frac{\partial f_2}{\partial\eta}\mathrm{d}\phi\leq C\bigg(\epsilon^{\alpha}\left\|\zeta\frac{\partial f_1}{\partial\eta}\right\|_{L^\infty L^2}+\left\|\zeta\frac{\partial f_2}{\partial\eta}\right\|_{L^\infty L^2}\bigg). \tag{6.39}$$

Then following a similar absorbing argument as in above well-posedness proof, we have

$$\begin{aligned}&\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial f_1}{\partial\eta}\right\|_{L^\infty L^\infty}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial f_1}{\partial\phi}\right\|_{L^\infty L^\infty} &(6.40)\\ &\leq C\left|\ln(\epsilon)\right|^8\bigg(\left\|\mathfrak{G}\right\|_{L^\infty}+\left\|(\epsilon+\zeta)\frac{\partial\mathfrak{G}}{\partial\phi}\right\|_{L^\infty}+\left\|\mathrm{e}^{K_0\eta}\mathfrak{U}_0\right\|_{L^\infty L^\infty}\bigg)\leq C\left|\ln(\epsilon)\right|^8,\\ &\left\|\mathrm{e}^{K_0\eta}\zeta\frac{\partial f_2}{\partial\eta}\right\|_{L^\infty L^\infty}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial f_2}{\partial\phi}\right\|_{L^\infty L^\infty} &(6.41)\\ &\leq C\left|\ln(\epsilon)\right|^8\Big(\left\|\mathrm{e}^{K_0\eta}f_2\right\|_{L^\infty L^\infty}+\epsilon^{\alpha}\left\|\mathrm{e}^{K_0\eta}f_1\right\|_{L^\infty L^\infty}\Big)\leq C\epsilon^{\alpha}\left|\ln(\epsilon)\right|^8.\end{aligned}$$

Note that although $\left\|\dfrac{\partial\mathfrak{G}}{\partial\phi}\right\|_{L^\infty}\leq C\epsilon^{-\alpha}$, with the help of $\epsilon+\zeta$, we can get rid of this negative power.

Step 3: Tangential Derivatives of $\mathfrak{U}_0$.
The τ derivative $P=\dfrac{\partial\mathfrak{U}_0}{\partial\tau}$ satisfies

$$\left\{\begin{aligned}&\sin\phi\frac{\partial P}{\partial\eta}+F(\eta)\cos\phi\frac{\partial P}{\partial\phi}+P-\bar{P}=-\frac{R_\kappa'}{R_\kappa-\epsilon\eta}F(\eta)\cos\phi\frac{\partial\mathfrak{U}_0}{\partial\phi},\\ &P(0,\tau,\phi)=\frac{\partial\mathfrak{G}}{\partial\tau}(\tau,\phi)-\frac{\partial\mathfrak{F}_{0,L}}{\partial\tau}(\tau)\ \ \text{for}\ \ \sin\phi>0,\\ &P(L,\tau,\phi)=P(L,\tau,\mathscr{R}[\phi]).\end{aligned}\right. \tag{6.42}$$

It is easy to check that

$$\int_{-\pi}^{\pi}\cos\phi\frac{\partial\mathfrak{U}_0}{\partial\phi}\mathrm{d}\phi=\int_{-\pi}^{\pi}\mathfrak{U}_0\sin\phi\mathrm{d}\phi=0, \tag{6.43}$$

due to the orthogonal property. Hence, using Theorem 4.3 with $S_Q=0$, we have

$$\left\|\mathrm{e}^{K_0\eta}P\right\|_{L^2L^2}\leq C\epsilon^{\alpha}\left|\ln(\epsilon)\right|^8, \tag{6.44}$$

which further implies

$$\begin{aligned}\left\|\mathrm{e}^{K_0\eta}P_1\right\|_{L^\infty L^\infty}&\leq C\bigg(\left\|\frac{\partial\mathfrak{G}}{\partial\tau}\right\|_{L^\infty L^\infty}+\left\|\mathrm{e}^{K_0\eta}P\right\|_{L^2L^2}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial\mathfrak{U}_0}{\partial\phi}\right\|_{L^\infty L^\infty}\bigg) &(6.45)\\ &\leq C\left|\ln(\epsilon)\right|^8,\end{aligned}$$

$$\begin{aligned}\left\|\mathrm{e}^{K_0\eta}P_2\right\|_{L^\infty L^\infty}&\leq C\bigg(\mathrm{e}^{K_0\eta}\left\|P\right\|_{L^2L^2}+\epsilon^{\alpha}\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial f_1}{\partial\phi}\right\|_{L^\infty L^\infty}+\left\|\mathrm{e}^{K_0\eta}F(\eta)\cos\phi\frac{\partial f_2}{\partial\phi}\right\|_{L^\infty L^\infty}\bigg)\\ &\leq C\epsilon^{\alpha}\left|\ln(\epsilon)\right|^8.\end{aligned} \tag{6.46}$$

where $P_1 = \dfrac{\partial f_1}{\partial \tau}$ and $P_2 = \dfrac{\partial f_2}{\partial \tau}$.

In this subsection, we will justify that the interior solutions are all well-defined. We divide it into several steps:

Step 1: Well-Posedness of U_0.
U_0 satisfies an elliptic equation

$$\left\{\begin{array}{l} U_0(\vec x,\vec w)=\bar U_0(\vec x),\\ \Delta_x\bar U_0(\vec x)=0\ \text{ in }\ \Omega,\\ \bar U_0(\vec x_0)=\mathscr{F}_{0,L}(\tau)+\mathfrak{F}_{0,L}(\tau)\ \text{ on }\ \partial\Omega. \end{array}\right. \tag{6.47}$$

Based on standard elliptic theory, we have

$$\left\|\bar U_0\right\|_{H^3(\Omega)}\leq C\bigg(\left\|\mathscr{F}_{0,L}\right\|_{H^{\frac52}(\partial\Omega)}+\left\|\mathfrak{F}_{0,L}\right\|_{H^{\frac52}(\partial\Omega)}\bigg)\leq C. \tag{6.48}$$

Step 2: Well-Posedness of U_1.
U_1 satisfies an elliptic equation

$$\left\{\begin{array}{rcl} U_1(\vec x,\vec w)&=&\bar U_1(\vec x)-\vec w\cdot\nabla_xU_0(\vec x,\vec w),\\ \Delta_x\bar U_1(\vec x)&=&-\displaystyle\int_{\mathbb{S}^1}\Big(\vec w\cdot\nabla_xU_0(\vec x,\vec w)\Big)\mathrm{d}\vec w\ \text{ in }\ \Omega,\\ \bar U_1(\vec x_0)&=&f_{1,L}(\tau)\ \text{ on }\ \partial\Omega. \end{array}\right. \tag{6.49}$$

Based on standard elliptic theory, we have

$$\left\|\bar U_1\right\|_{H^3(\Omega)}\leq C\bigg(\left\|\mathscr{F}_{1,L}\right\|_{H^{\frac52}(\partial\Omega)}+\left\|U_0\right\|_{H^2(\Omega)}\bigg)\leq C\left|\ln(\epsilon)\right|^8. \tag{6.50}$$

Step 3: Well-Posedness of U_2.
U_2 satisfies an elliptic equation

$$\left\{\begin{array}{rcl} U_2(\vec x,\vec w)&=&\bar U_2(\vec x)-\vec w\cdot\nabla_xU_1(\vec x,\vec w),\\ \Delta_x\bar U_2(\vec x)&=&-\displaystyle\int_{\mathbb{S}^1}\Big(\vec w\cdot\nabla_xU_1(\vec x,\vec w)\Big)\mathrm{d}\vec w\ \text{ in }\ \Omega,\\ \bar U_2(\vec x_0)&=&0\ \text{ on }\ \partial\Omega. \end{array}\right. \tag{6.51}$$

Based on standard elliptic theory, we have

$$\left\|\bar U_2\right\|_{H^3(\Omega)}\leq C\bigg(\left\|\bar U_0\right\|_{H^3(\Omega)}+\left\|\bar U_1\right\|_{H^2(\Omega)}\bigg)\leq C\left|\ln(\epsilon)\right|^8. \tag{6.52}$$

Theorem 6.1. *Assume $g(\vec{x}_0,\vec{w})\in C^3(\Gamma^-)$. Then for the steady neutron transport equation (1.1), there exists a unique solution $u^\epsilon(\vec{x},\vec{w})\in L^\infty(\Omega\times\mathbb{S}^1)$. Moreover, for any $0<\delta<<1$, the solution obeys the estimate*

$$\|u^\epsilon-U-\mathcal{U}\|_{L^\infty(\Omega\times\mathbb{S}^1)}\leq C(\delta)\epsilon^{\frac{1}{2}-\delta}, \tag{6.53}$$

where $U(\vec{x})$ satisfies the Laplace equation with Dirichlet boundary condition

$$\begin{cases}\Delta_x U(\vec{x})=0 \quad \text{in} \quad \Omega,\\ U(\vec{x}_0)=D(\vec{x}_0) \quad \text{on} \quad \partial\Omega,\end{cases} \tag{6.54}$$

and $\mathcal{U}(\eta,\tau,\phi)$ satisfies the ϵ-Milne problem with geometric correction

$$\begin{cases}\sin\phi\dfrac{\partial\mathcal{U}}{\partial\eta}-\dfrac{\epsilon}{R_\kappa(\tau)-\epsilon\eta}\cos\phi\dfrac{\partial\mathcal{U}}{\partial\phi}+\mathcal{U}-\bar{\mathcal{U}}=0,\\ \mathcal{U}(0,\tau,\phi)=g(\tau,\phi)-D(\tau) \quad \text{for} \quad \sin\phi>0,\\ \mathcal{U}(L,\tau,\phi)=\mathcal{U}(L,\tau,\mathscr{R}[\phi]),\end{cases} \tag{6.55}$$

for $L=\epsilon^{-\frac{1}{2}}$, $\mathscr{R}[\phi]=-\phi$, η the rescaled normal variable, τ the tangential variable, and ϕ the velocity variable.

Proof. Based on Theorem 3.5, we know there exists a unique $u^\epsilon(\vec{x},\vec{w})\in L^\infty(\Omega\times\mathbb{S}^1)$, so we focus on the diffusive limit. We can divide the proof into several steps:

Step 1: Remainder definitions.
We define the remainder as

$$R=u^\epsilon-\sum_{k=0}^{2}\epsilon^kU_k-\sum_{k=0}^{1}\epsilon^k\mathscr{U}_k-\mathfrak{U}_0=u^\epsilon-Q-\mathscr{Q}-\mathfrak{Q}, \tag{6.56}$$

where

$$Q=U_0+\epsilon U_1+\epsilon^2U_2, \tag{6.57}$$
$$\mathscr{Q}=\mathscr{U}_0+\epsilon\mathscr{U}_1, \tag{6.58}$$
$$\mathfrak{Q}=\mathfrak{U}_0. \tag{6.59}$$

Noting the equation (2.37) is equivalent to the equation (1.1), we write $\mathcal{L}$ to denote the neutron transport operator as follows:

$$\begin{aligned}\mathcal{L}[u]&=\epsilon\vec{w}\cdot\nabla_xu+u-\bar{u}\\ &=\sin\phi\frac{\partial u}{\partial\eta}-\frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\left(\frac{\partial u}{\partial\phi}+\frac{\partial u}{\partial\tau}\right)+u-\bar{u}.\end{aligned} \tag{6.60}$$

Step 2: Estimates of $\mathcal{L}[Q]$.
The interior contribution can be estimated as

$$\mathcal{L}[Q] = \epsilon\vec{w}\cdot\nabla_x Q + Q - \bar{Q} = \epsilon^3\vec{w}\cdot\nabla_x U_2. \tag{6.61}$$

Based on classical elliptic estimates, we have

$$\left\|\mathcal{L}[Q]\right\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leq \left\|\epsilon^3\vec{w}\cdot\nabla_x U_2\right\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leq C\epsilon^3\left\|\nabla_x U_2\right\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leq C\epsilon^3\left|\ln(\epsilon)\right|^8. \tag{6.62}$$

This implies

$$\begin{aligned}
\left\|\mathcal{L}[Q]\right\|_{L^2(\Omega\times\mathbb{S}^1)} &\leq C\epsilon^3\left|\ln(\epsilon)\right|^8, && (6.63)\\
\left\|\mathcal{L}[Q]\right\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)} &\leq C\epsilon^3\left|\ln(\epsilon)\right|^8, && (6.64)\\
\left\|\mathcal{L}[Q]\right\|_{L^\infty(\Omega\times\mathbb{S}^1)} &\leq C\epsilon^3\left|\ln(\epsilon)\right|^8. && (6.65)
\end{aligned}$$

Step 3: Estimates of $\mathcal{L}\mathscr{Q}$.
We need to estimate $\mathscr{U}_0+\epsilon\mathscr{U}_1$. The boundary layer contribution can be estimated as

$$\begin{aligned}
\mathcal{L}[\mathscr{U}_0+\epsilon\mathscr{U}_1] =& \sin\phi\frac{\partial(\mathscr{U}_0+\epsilon\mathscr{U}_1)}{\partial\eta} - \frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\bigg(\frac{\partial(\mathscr{U}_0+\epsilon\mathscr{U}_1)}{\partial\phi}+\frac{\partial(\mathscr{U}_0+\epsilon\mathscr{U}_1)}{\partial\tau}\bigg)\\
&+(\mathscr{U}_0+\epsilon\mathscr{U}_1)-(\bar{\mathscr{U}}_0+\epsilon\bar{\mathscr{U}}_1)\\
=& -\epsilon^2\frac{1}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_1}{\partial\tau}.
\end{aligned} \tag{6.66}$$

By previous analysis, we have

$$\left\|-\epsilon^2\frac{1}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_1}{\partial\tau}\right\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leq C\epsilon^2\left\|\frac{\partial\mathscr{U}_1}{\partial\tau}\right\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leq C\epsilon^{2-\alpha}\left|\ln(\epsilon)\right|^8. \tag{6.67}$$

Also, the exponential decay of $\dfrac{\partial\mathscr{U}_1}{\partial\tau}$ and the rescaling $\eta=\frac{\mu}{\epsilon}$ implies

$$\begin{aligned}
\left\|-\epsilon^2\frac{1}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_1}{\partial\tau}\right\|_{L^2(\Omega\times\mathbb{S}^1)} &\leq \epsilon^2\left\|\frac{\partial\mathscr{U}_1}{\partial\tau}\right\|_{L^2(\Omega\times\mathbb{S}^1)}\\
&\leq \epsilon^2\bigg(\int_{-\pi}^{\pi}\int_0^{R_{\min}}(R_{\min}-\mu)\left\|\frac{\partial\mathscr{U}_1}{\partial\tau}(\mu,\tau)\right\|_{L^\infty}^2\mathrm{d}\mu\mathrm{d}\tau\bigg)^{\frac{1}{2}}\\
&\leq \epsilon^{\frac{5}{2}}\bigg(\int_{-\pi}^{\pi}\int_0^{\frac{R_{\min}}{\epsilon}}(R_{\min}-\epsilon\eta)\left\|\frac{\partial\mathscr{U}_1}{\partial\tau}(\eta,\tau)\right\|_{L^\infty}^2\mathrm{d}\eta\mathrm{d}\tau\bigg)^{\frac{1}{2}}\\
&\leq C\epsilon^{\frac{5}{2}-\alpha}\left|\ln(\epsilon)\right|^8\bigg(\int_{-\pi}^{\pi}\int_0^{\frac{R_{\min}}{\epsilon}}\mathrm{e}^{-2K_0\eta}\mathrm{d}\eta\mathrm{d}\tau\bigg)^{\frac{1}{2}}\\
&\leq C\epsilon^{\frac{5}{2}-\alpha}\left|\ln(\epsilon)\right|^8.
\end{aligned} \tag{6.68}$$

Similarly, we have

$$\left\|-\epsilon^2\frac{1}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathscr{U}_1}{\partial\tau}\right\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)} \leq C\epsilon^{3-\frac{1}{2m}-\alpha}\left|\ln(\epsilon)\right|^8. \tag{6.69}$$

In total, we have

$$
\begin{aligned}
\|\mathcal{L}[\mathscr{Q}]\|_{L^2(\Omega\times\mathbb{S}^1)} &\leq C\epsilon^{\frac{5}{2}-\alpha}\left|\ln(\epsilon)\right|^8, && (6.70)\\
\|\mathcal{L}[\mathscr{Q}]\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)} &\leq C\epsilon^{3-\frac{1}{2m}-\alpha}\left|\ln(\epsilon)\right|^8, && (6.71)\\
\|\mathcal{L}[\mathscr{Q}]\|_{L^\infty(\Omega\times\mathbb{S}^1)} &\leq C\epsilon^{2-\alpha}\left|\ln(\epsilon)\right|^8. && (6.72)
\end{aligned}
$$

Step 4: Estimates of $\mathcal{L}\mathfrak{Q}$.
We need to estimate $\mathfrak{U}_0$. The boundary layer contribution can be estimated as

$$
\begin{aligned}
\mathcal{L}[\mathfrak{U}_0] =&\ \sin\phi\frac{\partial\mathfrak{U}_0}{\partial\eta}-\frac{\epsilon}{R_\kappa-\epsilon\eta}\cos\phi\bigg(\frac{\partial\mathfrak{U}_0}{\partial\phi}+\frac{\partial\mathfrak{U}_0}{\partial\tau}\bigg)+\mathfrak{U}_0-\bar{\mathfrak{U}}_0 && (6.73)\\
=&\ -\epsilon\frac{1}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathfrak{U}_0}{\partial\tau}.
\end{aligned}
$$

By previous analysis, we have

$$
\left\|-\epsilon\frac{1}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathfrak{U}_0}{\partial\tau}\right\|_{L^\infty(\Omega\times\mathbb{S}^1)}\leq C\epsilon\left\|\frac{\partial\mathfrak{U}_0}{\partial\tau}\right\|_{L^\infty(\Omega\times\mathbb{S}^1)}\leq C\epsilon\left|\ln(\epsilon)\right|^8. \tag{6.74}
$$

Also, the exponential decay of $\dfrac{\partial\mathfrak{U}_0}{\partial\tau}$ and the rescaling $\eta=\dfrac{\mu}{\epsilon}$ implies

$$
\begin{aligned}
&\left\|-\epsilon\frac{1}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathfrak{U}_0}{\partial\tau}\right\|_{L^2(\Omega\times\mathbb{S}^1)}\leq\epsilon\left\|\frac{\partial\mathfrak{U}_0}{\partial\tau}\right\|_{L^2(\Omega\times\mathbb{S}^1)} && (6.75)\\
\leq&\ \epsilon\bigg(\int_{-\pi}^{\pi}\int_0^{R_{\min}}\int_{-\pi}^{\pi}\chi_1(R_{\min}-\mu)\left\|\frac{\partial P_1}{\partial\tau}(\mu,\tau)\right\|_{L^\infty}^2\mathrm{d}\phi\mathrm{d}\mu\mathrm{d}\tau\bigg)^{\frac{1}{2}}\\
&+\epsilon\bigg(\int_{-\pi}^{\pi}\int_0^{R_{\min}}\int_{-\pi}^{\pi}\chi_2(R_{\min}-\mu)\left\|\frac{\partial P_2}{\partial\tau}(\mu,\tau)\right\|_{L^\infty}^2\mathrm{d}\phi\mathrm{d}\mu\mathrm{d}\tau\bigg)^{\frac{1}{2}}\\
\leq&\ \epsilon^{\frac{3}{2}}\bigg(\int_{-\pi}^{\pi}\int_0^{\frac{R_{\min}}{\epsilon}}\int_{-\pi}^{\pi}\chi_1(R_{\min}-\epsilon\eta)\left\|\frac{\partial P_1}{\partial\tau}(\eta,\tau)\right\|_{L^\infty}^2\mathrm{d}\phi\mathrm{d}\eta\mathrm{d}\tau\bigg)^{\frac{1}{2}}\\
&+\epsilon^{\frac{3}{2}}\bigg(\int_{-\pi}^{\pi}\int_0^{\frac{R_{\min}}{\epsilon}}\int_{-\pi}^{\pi}\chi_2(R_{\min}-\epsilon\eta)\left\|\frac{\partial P_2}{\partial\tau}(\eta,\tau)\right\|_{L^\infty}^2\mathrm{d}\phi\mathrm{d}\eta\mathrm{d}\tau\bigg)^{\frac{1}{2}}\\
\leq&\ C\epsilon^{\frac{3}{2}+\frac{\alpha}{2}}\left|\ln(\epsilon)\right|^8\bigg(\int_{-\pi}^{\pi}\int_0^{\frac{R_{\min}}{\epsilon}}\mathrm{e}^{-2K_0\eta}\mathrm{d}\eta\mathrm{d}\tau\bigg)^{\frac{1}{2}}\\
\leq&\ C\epsilon^{\frac{3}{2}+\frac{\alpha}{2}}\left|\ln(\epsilon)\right|^8.
\end{aligned}
$$

Similarly, we have

$$
\left\|-\epsilon\frac{1}{R_\kappa-\epsilon\eta}\cos\phi\frac{\partial\mathfrak{U}_0}{\partial\tau}\right\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)}\leq C\epsilon^{2-\frac{1}{2m}+\alpha}\left|\ln(\epsilon)\right|^8. \tag{6.76}
$$

In total, we have

$$\|\mathcal{L}[\mathscr{Q}]\|_{L^2(\Omega\times\mathbb{S}^1)} \leq C\epsilon^{\frac{3}{2}+\frac{\alpha}{2}}\left|\ln(\epsilon)\right|^8, \tag{6.77}$$

$$\|\mathcal{L}[\mathscr{Q}]\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)} \leq C\epsilon^{2-\frac{1}{2m}+\frac{(2m-1)\alpha}{2m}}\left|\ln(\epsilon)\right|^8, \tag{6.78}$$

$$\|\mathcal{L}[\mathscr{Q}]\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leq C\epsilon\left|\ln(\epsilon)\right|^8. \tag{6.79}$$

Step 5: Source Term and Boundary Condition.
In summary, since $\mathcal{L}[u^\epsilon]=0$, collecting estimates in Step 2 to Step 4, we can prove

$$\|\mathcal{L}[R]\|_{L^2(\Omega\times\mathbb{S}^1)} \leq C\bigg(\epsilon^{\frac{5}{2}-\alpha}+\epsilon^{\frac{3}{2}+\frac{\alpha}{2}}\bigg)\left|\ln(\epsilon)\right|^8, \tag{6.80}$$

$$\|\mathcal{L}[R]\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)} \leq C\bigg(\epsilon^{3-\frac{1}{2m}-\alpha}+\epsilon^{2-\frac{1}{2m}+\frac{(2m-1)\alpha}{2m}}\bigg)\left|\ln(\epsilon)\right|^8, \tag{6.81}$$

$$\|\mathcal{L}[R]\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leq C\bigg(\epsilon^{2-\alpha}+\epsilon\bigg)\left|\ln(\epsilon)\right|^8. \tag{6.82}$$

We can directly obtain that the boundary data is satisfied up to $O(\epsilon)$, so we know

$$\|R\|_{L^2(\Gamma^-)} \leq C\epsilon^2, \tag{6.83}$$

$$\|R\|_{L^m(\Gamma^-)} \leq C\epsilon^2, \tag{6.84}$$

$$\|R\|_{L^\infty(\Gamma^-)} \leq C\epsilon^2 \tag{6.85}$$

Step 6: Diffusive Limit.
Hence, the remainder R satisfies the equation

$$\left\{\begin{array}{l} \epsilon\vec{w}\cdot\nabla_x R+R-\bar{R}=\mathcal{L}[R] \quad\text{for}\quad \vec{x}\in\Omega,\\ R=R \quad\text{for}\quad \vec{w}\cdot\vec{\nu}<0 \quad\text{and}\quad \vec{x}_0\in\partial\Omega. \end{array}\right. \tag{6.86}$$

By Theorem 3.5, we have for m sufficiently large,

$$\begin{aligned}
\|R\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leq\;& C\bigg(\frac{1}{\epsilon^{1+\frac{1}{m}}}\|\mathcal{L}[R]\|_{L^2(\Omega\times\mathbb{S}^1)}+\frac{1}{\epsilon^{2+\frac{1}{m}}}\|\mathcal{L}[R]\|_{L^{\frac{2m}{2m-1}}(\Omega\times\mathbb{S}^1)}+\|\mathcal{L}[R]\|_{L^\infty(\Omega\times\mathbb{S}^1)} \\
&+\frac{1}{\epsilon^{\frac{1}{2}+\frac{1}{m}}}\|R\|_{L^2(\Gamma^-)}+\frac{1}{\epsilon^{\frac{1}{m}}}\|R\|_{L^m(\Gamma^-)}+\|R\|_{L^\infty(\Gamma^-)}\bigg),\\
\leq\;& C\Bigg(\frac{1}{\epsilon^{1+\frac{1}{m}}}\bigg(\epsilon^{\frac{5}{2}-\alpha}+\epsilon^{\frac{3}{2}+\frac{\alpha}{2}}\bigg)\left|\ln(\epsilon)\right|^8+\frac{1}{\epsilon^{2+\frac{1}{m}}}\bigg(\epsilon^{3-\frac{1}{2m}-\alpha}+\epsilon^{2-\frac{1}{2m}+\frac{(2m-1)\alpha}{2m}}\bigg)\left|\ln(\epsilon)\right|^8+(\epsilon)\left|\ln(\epsilon)\right|^8\\
&+\frac{1}{\epsilon^{\frac{1}{2}+\frac{1}{m}}}(\epsilon^2)+\frac{1}{\epsilon^{\frac{1}{m}}}(\epsilon^2)+(\epsilon^2)\Bigg)\\
\leq\;& C\bigg(\epsilon^{1-\frac{3}{2m}-\alpha}+\epsilon^{\frac{(2m-1)\alpha}{2m}-\frac{3}{2m}}\bigg)\left|\ln(\epsilon)\right|^8.
\end{aligned} \tag{6.87}$$

Here, we need

$$1-\frac{3}{2m}-\alpha>0,\quad \frac{(2m-1)\alpha}{2m}-\frac{3}{2m}>0, \tag{6.88}$$

which means

$$\frac{3}{2m-1} < \alpha < 1 - \frac{3}{2m}. \tag{6.89}$$

For $m > 3$, this is always achievable. Also, we know

$$\min_{\alpha} \left\{ \epsilon^{1-\frac{3}{2m}-\alpha} + \epsilon^{\frac{(2m-1)\alpha}{2m}-\frac{3}{2m}} \right\} = 2\epsilon^{\frac{4m^2-14m+3}{8m^2-2m}} \leq C(\delta)\epsilon^{\frac{1}{2}-\delta}. \tag{6.90}$$

Note that the constant C might depend on m and thus depend on δ. Since it is easy to see

$$\left\| \sum_{k=1}^{2} \epsilon^k U_k + \sum_{k=1}^{1} \epsilon^k \mathscr{U}_k \right\|_{L^\infty(\Omega\times\mathbb{S}^1)} \leq C\epsilon, \tag{6.91}$$

our result naturally follows. We simply take $U = U_0$ and $\mathcal{U} = \mathscr{U}_0 + \mathfrak{U}_0$. It is obvious that $\mathcal{U}$ satisfies the ϵ-Milne problem with geometric correction with the full boundary data $g(\phi,\tau) - \mathscr{F}_{0,L}(\tau) - \mathfrak{F}_{0,L}(\tau)$. This completes the proof of main theorem. $\square$

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