Exact Solitary-wave Solutions for the General Fifth-order Shallow Water-wave Models

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We perform a computerized symbolic computation to find some general solitonic solutions for the general fifth-order shallow water-wave models. Applying the tanh-typed method, we have found certain new exact solitary wave solutions. The previously published solutions turn out to be special cases with restricted model parameters.

We investigate the generalized fifth-order shallow water-wave model which describes certain physically-interesting $(1+1)$-dimensional waves but is not integrable by the conventional methods [1]:

$$v_t + \alpha v_{xxxxx} + \mu v_{xxx} + \gamma \partial_x [2 v v_{xx} + v_x^2] + 2 q v v_x + 3 r v^2 v_x = 0 , \tag{1}$$

where α, μ, γ, q, and r are model parameters, v is a real scalar function of the two independent variables, x and t, and the subscripts denote partial derivatives.

We apply the tanh-typed method [2–4] with symbolic computation to (1) and assume that the physical field $v(x, t)$ has the form

$$v(x,t) = \sum_{n=0}^{N} \mathcal{A}_n(t) \cdot \tanh^n [\mathcal{G}(t) x + \mathcal{H}(t)], \tag{2}$$

where N is an integer determined via the balance of the highest-order contributions from the linear and nonlinear terms of (1) as $N=2$, while $\mathcal{A}_N(t)$, $\mathcal{G}(t)$, and $\mathcal{H}(t)$ are the non-trivial differentiable functions to be determined.

With the symbolic computation package *Maple* we introduce Ansatz (2) together with the above conditions into (1) and collect the coefficients of like powers of tanh:

$$(\tanh^7){:}\ -6\mathcal{A}_2(t)\mathcal{G}(t) \cdot [r\mathcal{A}_2(t)^2 + 16\gamma\mathcal{G}(t)^2\mathcal{A}_2(t) + 120\alpha\mathcal{G}(t)^4] \tag{3}$$

$$(\tanh^6){:}\ -5\mathcal{A}_1(t)\mathcal{G}(t) \cdot [24\alpha\mathcal{G}(t)^4 + 20\gamma\mathcal{G}(t)^2\mathcal{A}_2(t) + 3r\mathcal{A}_2(t)^2] \tag{4}$$

$$\begin{aligned}(\tanh^5){:}\ & -2\mathcal{G}(t)\,[-840\alpha\mathcal{A}_2(t)\mathcal{G}(t)^4 \\ & + 10\mathcal{G}(t)^2\gamma\mathcal{A}_1(t)^2 + 12\mathcal{G}(t)^2\alpha\mathcal{A}_2(t) \\ & + 24\mathcal{G}(t)^2\gamma\mathcal{A}_0(t)\mathcal{A}_2(t) - 96\gamma\mathcal{A}_2(t)^2\mathcal{G}(t)^2 \\ & - 3r\mathcal{A}_2(t)^3 + 6r\mathcal{A}_1(t)^2\mathcal{A}_2(t) \\ & + 6r\mathcal{A}_0(t)\mathcal{A}_2(t)^2 + 2q\mathcal{A}_2(t)^2]\end{aligned} \tag{5}$$

$$\begin{aligned}(\tanh^4){:}\ & -\mathcal{A}_1(t)\mathcal{G}(t)\,[3r\mathcal{A}_1(t)^2 - 240\alpha\mathcal{G}(t)^4 \\ & + 6\mathcal{G}(t)^2\mu - 184\gamma\mathcal{G}(t)^2\mathcal{A}_2(t) \\ & + 12\mathcal{G}(t)^2\gamma\mathcal{A}_0(t) + 18r\mathcal{A}_0(t)\mathcal{A}_2(t) \\ & + 6q\mathcal{A}_2(t) - 15r\mathcal{A}_2(t)^2]\end{aligned} \tag{6}$$

$$\begin{aligned}(\tanh^3){:}\ & -2\mathcal{A}_2(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t) - 1232\alpha\mathcal{A}_2(t)\mathcal{G}(t)^5 \\ & + 12r\mathcal{A}_1(t)^2\mathcal{A}_2(t)\mathcal{G}(t) \\ & - 2\mathcal{A}_2(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(t)x + 12r\mathcal{A}_0(t)\mathcal{A}_2(t)^2\mathcal{G}(t) \\ & - 6r\mathcal{A}_0(t)\mathcal{A}_1(t)^2\mathcal{G}(t) - 6r\mathcal{A}_0(t)^2\mathcal{A}_2(t)\mathcal{G}(t) \\ & - 2q\mathcal{A}_1(t)^2\mathcal{G}(t) - 4q\mathcal{A}_0(t)\mathcal{A}_2(t)\mathcal{G}(t) \\ & + 80\gamma\mathcal{A}_0(t)\mathcal{A}_2(t)\mathcal{G}(t)^3 + 40\mu\mathcal{A}_2(t)\mathcal{G}(t)^3 \\ & - 112\gamma\mathcal{A}_2(t)^2\mathcal{G}(t)^3 + 32\gamma\mathcal{A}_1(t)^2\mathcal{G}(t)^3 \\ & + 4q\mathcal{A}_2(t)^2\mathcal{G}(t)\end{aligned} \tag{7}$$

$$\begin{aligned}(\tanh^2){:}\ & -\mathcal{A}_1(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t) + 3r\mathcal{A}_1(t)^3\mathcal{G}(t) \\ & + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_2(t) + 8\mu\mathcal{A}_1(t)\mathcal{G}(t)^3 \\ & + 16\gamma\mathcal{A}_0(t)\mathcal{A}_1(t)\mathcal{G}(t)^3 - \mathcal{A}_1(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(t)x \\ & - 136\alpha\mathcal{A}_1(t)\mathcal{G}(t)^5 \\ & + 18r\mathcal{A}_0(t)\mathcal{A}_1(t)\mathcal{A}_2(t)\mathcal{G}(t) \\ & - 3r\mathcal{A}_0(t)^2\mathcal{A}_1(t)\mathcal{G}(t) + 6q\mathcal{A}_1(t)\mathcal{A}_2(t)\mathcal{G}(t) \\ & - 2q\mathcal{A}_0(t)\mathcal{A}_1(t)\mathcal{G}(t) - 92\gamma\mathcal{A}_1(t)\mathcal{G}(t)^3\mathcal{A}_2(t)\end{aligned} \tag{8}$$

$$\begin{aligned}(\tanh^1){:}\ & 2\mathcal{A}_2(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t) + 272\alpha\mathcal{A}_2(t)\mathcal{G}(t)^5 \\ & - 32\gamma\mathcal{A}_0(t)\mathcal{A}_2(t)\mathcal{G}(t)^3 + 2\mathcal{A}_2(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(t)x \\ & + 6r\mathcal{A}_0(t)\mathcal{A}_1(t)^2\mathcal{G}(t) + 6r\mathcal{A}_0(t)^2\mathcal{A}_2(t)\mathcal{G}(t) \\ & + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_1(t) - 16\mu\mathcal{A}_2(t)\mathcal{G}(t)^3 + 2q\mathcal{A}_1(t)^2\mathcal{G}(t) \\ & + 16\gamma\mathcal{A}_2(t)^2\mathcal{G}(t)^3 + 4q\mathcal{A}_0(t)\mathcal{A}_2(t)\mathcal{G}(t) \\ & - 12\gamma\mathcal{A}_1(t)^2\mathcal{G}(t)^3\end{aligned} \tag{9}$$

$$\begin{aligned}(\tanh^0){:}\ & \mathcal{A}_1(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(t)x + \mathcal{A}_1(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t) \\ & - 2\mu\mathcal{A}_1(t)\mathcal{G}(t)^3 + 16\alpha\mathcal{A}_1(t)\mathcal{G}(t)^5 \\ & + 3r\mathcal{A}_0(t)^2\mathcal{A}_1(t)\mathcal{G}(t) + 2q\mathcal{A}_0(t)\mathcal{A}_1(t)\mathcal{G}(t) \\ & + 8\gamma\mathcal{A}_1(t)\mathcal{G}(t)^3\mathcal{A}_2(t) - 4\gamma\mathcal{A}_0(t)\mathcal{A}_1(t)\mathcal{G}(t)^3 \\ & + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_0(t) .\end{aligned} \tag{10}$$

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We find the conditions for $\mathcal{A}_N(t)$, $\mathcal{G}(t)$, and $\mathcal{H}(t)$ which simultaneously cause the above terms to become zero:

$$(\tanh^1 \cdot x):\ \mathcal{A}_2(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(t)x = 0 \tag{11}$$

$$\rightarrow\ \mathcal{G}(t) = \mathcal{G} = \text{non} - \text{zero constant},$$

$$(\tanh^4):\ \mathcal{A}_0(t) = -\frac{38\gamma^2\mathcal{G}^2 + r\mu}{2r\gamma}, \tag{12}$$

$$(\tanh^7 \text{ and } \tanh^6):\ \mathcal{A}_2(t) = -6\frac{\gamma\mathcal{G}^2}{r}, \tag{13}$$

$$(\tanh^5):\ \mathcal{A}_1(t) = \frac{12\gamma\,\mathrm{csgn}(\mathcal{G})\mathcal{G}^2}{r}, \tag{14}$$

where csgn(x) is defined by

$$\mathrm{csgn}(x) = \begin{cases} 1 & \text{if } \mathrm{Re}(x) > 0 \\ -1 & \text{if } \mathrm{Re}(x) < 0 \end{cases}, \tag{15}$$

$$(\tanh^0):\ \mathcal{H}(t) \tag{16}$$
$$= \frac{(3\mathcal{G}\mu^2r^2 + 300\mathcal{G}^3\mu\gamma^2r + 6924\mathcal{G}^5\gamma^4)t}{5\gamma^2r} + \frac{4}{5}\mathcal{C},$$

where $\mathcal{C}$ is an arbitrary constant. In addition to the above conditions, the following auxiliary condition is required for $v(x,t)$ to be the solution of (1):

$$q = \frac{3}{2}\frac{r\mu + 50\gamma^2\mathcal{G}^2}{\gamma}. \tag{17}$$

Thus, computerized symbolic computation helps us to obtain a new family of exact solitary-wave solutions of (1), i.e.

$$v(x,t) = \frac{12\gamma\mathcal{G}^2}{r}\cdot\tanh \left[\mathcal{G}x + \frac{(3\mathcal{G}\mu^2r^2+300\mathcal{G}^3\mu\gamma^2r+6924\mathcal{G}^5\gamma^4)t}{5\gamma^2r} + \frac{4}{5}\mathcal{C}\right]$$
$$-6\frac{\gamma\,\mathrm{csgn}(\mathcal{G})\mathcal{G}^2}{r}\cdot\tanh^2 \left[\mathcal{G}x + \frac{(3\mathcal{G}\mu^2r^2+300\mathcal{G}^3\mu\gamma^2r+6924\mathcal{G}^5\gamma^4)t}{5\gamma^2r} + \frac{4}{5}\mathcal{C}\right]$$
$$-\frac{38\gamma^2\mathcal{G}^2 + r\mu}{2r\gamma}. \tag{18}$$

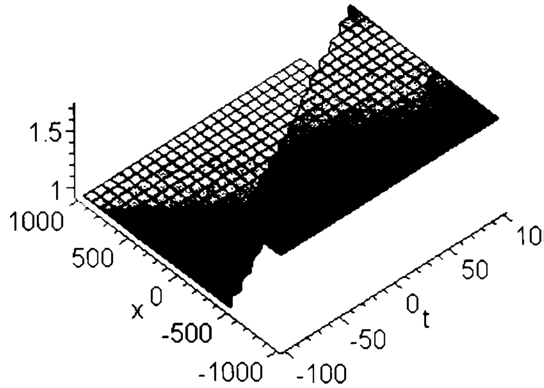

Fig. 1. A kink-typed solitary wave with the parameters $\mu=1$, $\gamma=1$, $q=1$, $\mathcal{C}=1$, $r=-1$, and $\mathcal{G}=\sqrt{-18r+12}/30$.

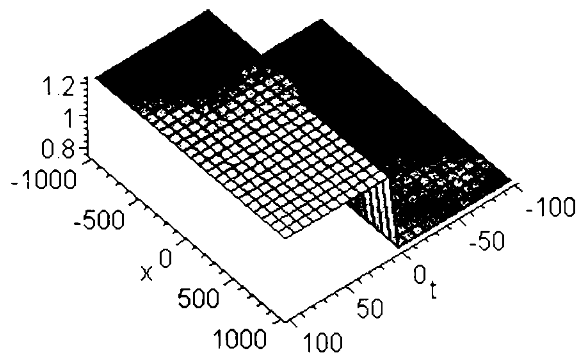

Fig. 2. A kink-typed solitary wave with the parameters $\mu=1$, $\gamma=1$, $q=1$, $\mathcal{C}=1$, $r=-1000$, and $\mathcal{G}=\sqrt{-18r+12}/30$.

Lastly we present figures with the some selected parameters. For simplicity we set $\mu=1$, $\gamma=1$, the auxiliary condition $q=\frac{3}{2}r+75\mathcal{G}^2=1$, and $\mathcal{C}=1$. In Figs. 1 and 2 we plot the kink-typed solitary waves with parameters: $r=-1, -1000$, $\mathcal{G}=\sqrt{-18r+12}/30$, respectively.

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