A remark on the Gaussian lower bound for the Neumann heat kernel of the Laplace-Beltrami operator

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Abstract. We adapt in the present note the perturbation method introduced in [3] to get a lower Gaussian bound for the Neumann heat kernel of the Laplace-Beltrami operator on an open subset of a compact Riemannian manifold.

1. Introduction

The study of heat kernels is an important problem in the theory of parabolic PDE's. The properties of heat kernels give an efficient tool to answer to some central questions both in analysis and probability theory. One of the main questions is to know whether a heat kernel admits Gaussian bounds. An upper Gaussian bound is for instance an useful tool for getting L^p-L^q estimates, the analyticity of the corresponding semigroups in L^p for any finite $p \geq 1$ or bounded functional calculus, whereas one can get a strong maximum principle or a Harnack inequality from a lower Gaussian bound. We refer to the textbooks [5], [11] and [12] and references therein for more details on the subject.

In the preceding work [3], starting from the classical parametrix method, we constructed the Neumann heat kernel of a general parabolic operator as a perturbation of the fundamental solution of the same operator by a single-layer potential. From this construction, the two-sided Gaussian bounds for the fundamental solution and taking into account the smoothing effect in time of the single-layer potential, we succeeded in proving a lower Gaussian bound for the Neumann Green function. We adapt in the present note this

method to establish a lower Gaussian bound for the Neumann heat kernel of Laplace-Beltrami operator.

In this text $\mathcal{M} = (\mathcal{M}, g)$ is a n-dimensional compact connected Riemannian manifold without boundary and Ω is a domain in $\mathcal{M}$ so that its boundary Σ is an $(n-1)$-dimensional Riemannian submanifold of M when it is equipped with the metric induced by g.

The Riemannian measure on $\mathcal{M}$ is denoted by dV while the density measure on Σ is denoted by dA. The geodesic ball of center $x \in \mathcal{M}$ and radius $r > 0$ is denoted by $B(x, r)$.

2. Neumann heat kernel

Let d be the Riemannian distance function and
$$\mathscr{E}(x, y, t) = (4\pi t)^{-n/2} e^{-\frac{d^2(x,y)}{4t}}.$$
In general $\mathscr{E}$ is not a heat kernel on $\mathcal{M}$. However the parametrix method by S. Minakshisundaram and Å.Pleijel shows that any compact Riemannian manifold has an almost euclidien heat kernel (e.g. [1, 2]). In particular, the heat kernel p of $\mathcal{M}$, satisfies
$$p(x, y, t) \sim \mathscr{E}(x, y, t),$$
locally uniformly in (x, y) as $t \downarrow 0$. We have a similar statement with the first order derivatives of p and $\mathscr{E}$.

This estimate is not true in general for distant x and y. Indeed the counter example in [10, Example 3.1, page 23] shows that if $\mathscr{M} = \mathbb{S}^2$, the 2-dimensional sphere equipped with the round metric, x is the north pole and y is the south pole, then
$$p(x, y, t) \sim ct^{-3/2} e^{-\frac{d^2(x,y)}{4t}},$$
for some constant $c > 0$.

From [5, Theorem 5.5.11 and Theorem 5.6.1, page 173], any complete Riemannian manifold with non negative Ricci curvature satisfies the following two-sided Gaussian bounds.
$$\mathscr{E}(x, y, t) \le p(x, y, t) \le \frac{c}{V(B(x, \sqrt{t}))} e^{-\kappa \frac{d^2(x,y)}{4t}}, \quad x, y \in \mathcal{M}, \ t > 0.$$
Here $c > 0$ and $\kappa > 0$ are some constants.

Let $\Delta = \Delta_g$ be the Laplace-Beltrami operator associated to the metric g and denote by ν the outward normal vector field to Σ. Following the idea in [3], we construct the Green function of the Neumann problem
$$\begin{cases} (\partial_t - \Delta)u = 0 & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \Sigma \times (0, +\infty), \end{cases} \tag{2.1}$$

as a perturbation of the heat kernel p by a single-layer potential. As a first step, we seek the solution, in $C^{2,1}(\Omega \times (0, +\infty)) \cap C^{0,1}(\overline{\Omega} \times [0, +\infty))$, of the following IBVP

$$\begin{cases} (\partial_t - \Delta)u = 0 & \text{in } \Omega \times (0, +\infty), \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \Sigma \times (0, +\infty), \\ u(\cdot, 0) = \psi \in C_0^\infty(\Omega) \end{cases} \tag{2.2}$$

of the form

$$u(x,t) = \int_0^t \int_\Sigma p(x,y,s)\varphi(y,t-s)dA(y)ds + \int_\Omega p(x,y,t)\psi(y)dV(y),$$
$$(x,t) \in \Omega \times (0, +\infty).$$

We obtain from the jump relation in [2, Theorem 2, page 161] that φ must be the solution of the following integral equation

$$\varphi(x,t) = -2\int_0^t \int_\Sigma \frac{\partial p}{\partial \nu_x}(x,y,s)\varphi(y,t-s)dA(y)ds$$
$$- 2\int_\Omega \frac{\partial p}{\partial \nu_x}(x,y,t)\psi(y)dV(y), \quad (x,t) \in \Sigma \times (0, +\infty).$$

This integral equation is solved by successive approximations. We get

$$\varphi(x,t) = \varphi_0(x,t) + \int_0^t \int_\Sigma r(x,y,s)\varphi_0(y,t-s)dA(y)ds, \quad (x,t) \in \Sigma \times (0, +\infty).$$

Here

$$\varphi_0(x,t) = -2\int_\Omega \frac{\partial p}{\partial \nu_x}(x,y,t)\psi(y)dV(y), \quad (x,t) \in \Sigma \times (0, +\infty),$$

$$r(x,y,t) = \sum_{j \geq 1} r^j(x,y,t), \quad (x,t) \in \Sigma \times (0, +\infty), \ y \in \overline{\Omega},$$

with

$$r^1(x,y,t) = -2\frac{\partial p}{\partial \nu_x}(x,y,t), \quad (x,t) \in \Sigma \times (0, +\infty), \ y \in \overline{\Omega},$$

$$r^{j+1}(x,y,t) = -2\int_0^t \int_\Sigma r^1(x,z,t-s)r^j(z,y,s)dA(z)ds, \quad j \geq 1,$$
$$(x,t) \in \Sigma \times (0, +\infty), \ y \in \overline{\Omega}.$$

In other words

$$r(x,y,t) = -2\frac{\partial p}{\partial \nu_x}(x,y,t) - 2\sum_{j \geq 1} \int_0^t \int_\Sigma r^j(x,z,s)\frac{\partial p}{\partial \nu_x}(z,y,t-s)dA(z)ds$$
$$(x,t) \in \Sigma \times (0, +\infty), \ y \in \overline{\Omega}.$$

The following two inequalities will be useful in the sequel. They are taken from [2].

For any $\mu > 0$, there is a constant C_0 such that

$$\left| \frac{\partial p}{\partial \nu_x}(x,y,t) \right| \leq C_0 t^{-\mu} d^{-n+2\mu}(x,y), \tag{2.3}$$

for any $x \in \Sigma$, $y \in \overline{\Omega}$ and $t \in (0,T]$.

There exists a constant C_1 such that, for any $\alpha, \beta \in (0, n-1)$,

$$\int_\Sigma d^{-\alpha}(x,z)d^{-\beta}(z,y)dA(z) \le C_1 \left\{ \begin{array}{ll} d^{n-1-(\alpha+\beta)}(x,y) & \text{if } \alpha+\beta > n-1, \\ 1 & \text{if } \alpha+\beta < n-1. \end{array} \right.$$
(2.4)

for all $x, y \in \overline{\Omega}$, $x \ne y$.

With the help of inequality (2.3), we prove similarly to [3, (3.6)]

$$\varphi(x,t) = \int_\Omega r(x,y,t)\psi(y)dV(y).$$

Then

$$u(x,t) = \int_\Omega q(x,y,t)\psi(y)dV(y),$$

where

$$q(x,y,t) = p(x,y,t) + \int_0^t \int_\Sigma p(x,z,s)r(z,y,t-s)dA(z)ds.$$

We call this function the Neumann heat kernel for the problem (2.1). We leave to the reader to verify that q satisfies the following reproducing property

$$q(x,y,t) = \int_\Omega q(x,z,t-s)q(z,y,s)dV(z), \quad x, y \in \Omega, \ 0 < s < t,$$

and

$$\int_\Omega q(x,y,t)dV(y) = 1, \quad x \in \Omega, \ t > 0.$$

3. Gaussian lower bound

We set $v(x,r) = V(B(x,r) \cap \Omega)$, $x \in \Omega$, and we consider the following three assumptions.

(VLB) (volume lower bound) There exist two constants C and r_0 so that

$$v(x,r) \ge Cr^n, \quad x \in \Omega, \ 0 < r \le r_0.$$

(DP) (doubling property) There exist two constants $r_1 > 0$ and $C > 0$ so that

$$v(x,s) \le C \left(\frac{s}{r}\right)^n v(x,r),$$

for all $0 < r \le s \le r_1$ and $x \in \Omega$.

(CC) (chain condition) There exists a constant $C > 0$ such that for any $x, y \in \Omega$ and $k \in \mathbb{N}$, we find a sequence of points $(x_i)_{0 \le i \le k}$ so that $x_0 = x$, $x_k = y$ and

$$d(x_i, x_{i+1}) \le C \frac{d(x,y)}{k}, \quad 0 \le i \le k-1.$$

In the flat case, (VLC) and (DP) are true for any Lipschitz domain and (CC) holds for instance for a convex domain.

Let $\mathbb{S}^1$ be the unit sphere of $\mathbb{R}^2$ equipped with the round metric, that is the metric induced by the Euclidean metric on $\mathbb{R}^2$. Then any segment of

length strictly less than π possesses the the three conditions (VLB), (DP) and (CC). Let $\mathbb{S}^2$ be the unit sphere of $\mathbb{R}^3$ equipped with the round metric. Then it is not hard to show that the sub-domain of $\mathbb{S}^2$ given by $z > \delta > 0$ satisfies also the three conditions (VLB), (DP) and (CC).

Conditions (DP) and (CC) are usual (see for instance [11, Theorem 7.29, page 248]), while condition (VLB) guarantees that the near diagonal lower bound (3.10) above holds.

We aim to sketch the proof of the following theorem

Theorem 3.1. *Fix $T > 0$ and assume that Ω satisfies (VLB), (DP) and (CC). Then*

$$q(x,y,t) \geq \frac{c}{v(x,\sqrt{t})} e^{-\frac{d^2(x,y)}{ct}}, \quad x, y \in \Omega, \ 0 < t \leq T. \tag{3.5}$$

Sketch of the proof. Let $1/2 < \mu < n/2$. In light of (2.3) and (2.4), reasoning as in the proof of [3, Lemma 3.1], we obtain

$$|r(x,y,t)| \leq Ct^{-\mu}d^{-n+2\mu}(x,y),$$

for any $x \in \Sigma$, $y \in \overline{\Omega}$, $x \neq y$, $t \in (0,T]$.

Let

$$q_0(x,y,t) = \int_0^t \int_\Sigma p(x,z,s)r(z,y,t-s)dA(z)ds, \quad x,y \in \Omega, \ t > 0,$$

and $0 < \alpha < 1/2$. We proceed as in the beginning of the proof of [3, Theorem 3.1] to get

$$|q_0(x,y,t)| \leq Ct^{-n/2+\alpha} \quad x,y \in \Omega, \ t > 0. \tag{3.6}$$

Let $\epsilon := \operatorname{inj}(\mathcal{M})/4$, where $\operatorname{inj}(\mathcal{M})$ is the injectivity radius of $\mathcal{M}$. It follows from [2, formula (45), page 154] that there exists $\eta > 0$ so that

$$p(x,y,t) \geq \mathscr{E}(x,y,t), \quad 0 < t \leq \eta, \ x,y \in \mathcal{M}, \ d(x,y) \leq \epsilon. \tag{3.7}$$

Hence,

$$p(x,y,t) \geq Ct^{-n/2}, \quad 0 < t \leq \inf(\eta, \epsilon^2), \ x,y \in \mathcal{M}, \ d(x,y) \leq \sqrt{t}. \tag{3.8}$$

Now a combination of (3.6) and (3.8) leads

$$q(x,y,t) \geq p(x,y,t) - |q_0(x,y,y)| \geq Ct^{-n/2}(1 - ct^\alpha),$$
$$0 < t \leq \inf(\eta, \epsilon^2), \ x,y \in \Omega, \ d(x,y) \leq \sqrt{t}.$$

In consequence, there is $\delta > 0$ such that

$$q(x,y,t) \geq Ct^{-n/2}, \quad \text{if } 0 < t \leq \delta, \ x,y \in \Omega, \ d(x,y) \leq \sqrt{t}. \tag{3.9}$$

In light of the volume lower bound (VLB), this estimate entails

$$q(x, y, t) \geq \frac{C}{v(x, \sqrt{t})}, \text{ if } 0 < t \leq \widetilde{\delta}, \ x, y \in \Omega \ d(x,y) \leq \sqrt{t}, \qquad (3.10)$$

for some constant $\widetilde{\delta}$.

We can now mimic the proof of [11, Theorem 7.29, page 248]. We get from (3.10) the following Gaussian lower bound

$$q(x, y, t) \geq \frac{c}{v(x, \sqrt{t})} e^{-\frac{d^2(x,y)}{ct}}, \ x, y \in \Omega, \ 0 < t \leq \widetilde{\delta}.$$

We finally use the argument as in [3, Theorem 3.1] to pass from $0 < t \leq \widetilde{\delta}$ to $0 < t \leq T$. □

It is worthwhile mentioning that one can establish a lower Gaussian bound by considering $\mathcal{N} = \overline{\Omega}$ itself as a compact Riemannian manifold with boundary. The structure of Riemannian manifold is the one inherited from $\mathcal{M}$. Obviously, (3.9) entails

$$q(x, y, t) \geq Ct^{-n/2}, \ 0 < t \leq \delta, \ x, y \in \mathcal{N}, \ d_{\mathcal{N}}(x,y) \leq \sqrt{t}. \qquad (3.11)$$

Assume that the Ricci curvature of $\mathcal{N}$ is such that $Ric \geq (n-1)\kappa g$ for some $\kappa \in \mathbb{R}$. Then $\mathcal{N}$ satisfies the a doubling property (DP) when V is substituted by $V_{\mathcal{N}}$, the volume measure over $\mathcal{N}$. This fact is an immediate consequence of [7, formula in the bottom of page 7]. Let $v_{\mathcal{N}}(x,r) = V_{\mathcal{N}}(B(x,r))$, where $B(x,r)$ is the geodesic ball in $\mathcal{N}$ of center $x \in \mathcal{N}$ and radius r. In that case we can proceed as in the proof of Theorem 3.1 to derive the following estimate.

$$q(x, y, t) \geq \frac{c}{v_{\mathcal{N}}(x, \sqrt{t})} e^{-\frac{d_{\mathcal{N}}^2(x,y)}{ct}}, \ x, y \in \Omega, \ 0 < t \leq T.$$

This estimate should be compared to the one obtained by Li and Yau in [9, Theorem 4.2, page 184]. Specifically, they established a lower Gaussian bound for the heat kernel of a compact Riemannian manifold with convex boundary and having non negative Ricci curvature.

When $\mathcal{M}$ is any complete Riemannian manifold with finite diameter and having volume doubling property, and Ω is Lipschitz domain in $\mathcal{M}$ with volume doubling property, the Neumann heat kernel of Ω, denoted here by h, satisfies the following upper Gaussian bound.

$$h(x, y, t) \leq \frac{C}{v(x, \sqrt{t})v(y, \sqrt{t})} e^{-\frac{d^2(x,y)}{8t}}, \ x, y \in \Omega, t > 0,$$

where $C > 0$ is some constant.

This estimate was recently established by the authors and E. M. Ouhabaz [4].

4. Comments on geometric assumptions

Chain condition: A subset $\mathcal{C}$ of $\mathcal{M}$ is called strongly convex if for any $x, y \in \mathcal{C}$, there exists a unique minimal geodesic $\gamma : [0,1] \to \mathcal{M}$ joining x to y, so that $\gamma([0,1]) \subset \mathcal{C}$. According to a theorem due to Whitehead (see for instance [6, pages 161 and 162]), there exists a positive continuous function $\epsilon : \mathcal{M} \to (0, \infty]$, the convexity radius, such that any open ball $B(x, r) \subset B(x, \epsilon(x))$ is strongly convex. It is straightforward to check that if Ω is strongly convex then it has the chain condition.

Volume lower bound: Let $T_x\mathcal{M}$ be the tangent space at $x \in \mathcal{M}$, $\mathbb{S}_x \subset T_x\mathcal{M}$ the unit tangent sphere and $S\mathcal{M}$ the unit tangent bundle. Let Φ_t be the geodesic flow with phase space $S\mathcal{M}$. That is, for any $t \geq 0$,

$$\Phi_t : S\mathcal{M} \to S\mathcal{M} : (x, \xi) \in S\mathcal{M} \to \Phi_t(x, \xi) = (\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t)).$$

Here $\gamma_{x,\xi} : [0, \infty) \to \mathcal{M}$ is the unit speed geodesic starting at x with tangent unit vector ξ and $\dot{\gamma}_{x,\xi}(t)$ is the unit tangent vector to $\gamma_{x,\xi}$ at $\gamma_{x,\xi}(t)$ in the forward t direction.

If $(x, \xi) \in S\mathcal{M}$, we denote by $r(x, \xi)$ the distance from x to the cutlocus in the direction of ξ:

$$r(x, \xi) = \inf\{t > 0; \ d(x, \Phi_t(x, \xi)) < t\}.$$

We fix $\delta \in (0, 1]$ and $r > 0$. Following [13], a (δ, r)-cone at $x \in \mathcal{M}$ is the set of the form

$$\mathscr{C}(x, \omega_x, r) = \{y = \gamma_{x,\xi}(s); \ \xi \in \omega_x, \ 0 \leq s < r\},$$

where ω_x is a subset of $\mathbb{S}_x$ so that $r < r(x, \xi)$ for all $\xi \in \omega_x$ and $|\omega_x| \geq \delta$ (here $|\omega_x|$ is the volume of ω_x with respect to the normalized measure on the sphere $\mathbb{S}_x$).

A domain D which contains an (δ, r)-cone at x for any $x \in D$ is said to satisfy the interior (δ, r)-cone condition.

We observe that if $\mathcal{C}$ is a closed strongly convex subset of $\mathcal{M}$, then $\Omega = \mathcal{M} \setminus \mathcal{C}$ has the $(1/2, r)$-cone condition, for some r (this fact follows from the same argument to that in [13, Example 8.1, page 370]).

Let

$$s_\kappa(r) = \begin{cases} \left(\frac{\sin(\sqrt{\kappa} r)}{\sqrt{\kappa}}\right)^{n-1} & \text{if } \kappa > 0, \\ r^{n-1} & \text{if } \kappa = 0, \\ \left(\frac{\sinh(\sqrt{-\kappa} r)}{\sqrt{-\kappa}}\right)^{n-1} & \text{if } \kappa < 0. \end{cases}$$

We make the assumption that the sectional curvature of $\mathcal{M}$ is is bounded above by a constant κ, $\kappa \in \mathbb{R}$, and Ω satisfies the interior (δ, r)-cone condition. Let $J(x, \xi, t)$ be the density of the volume element in geodesic coordinates around x:

$$dV(y) = J(x, \xi, t) d_{\mathbb{S}_x} dt, \ y = \gamma_{x,\xi}(t), \ t < r(x, \xi).$$

By an extension of Günther's comparison theorem (see for instance [8]), J satisfies the following uniform lower bound

$$J(x, \xi, t) \geq s_\kappa(t).$$

Consequently, shrinking r_0 if necessary, we have

$$v(x, r) \geq V(\mathscr{C}(x, \omega_x, r)) \geq c_0 r^n, \quad x \in \Omega, \ 0 < r \leq r_0, \qquad (4.12)$$

which means that v satisfy the volume lower bound (VLB).

Additionally, if $\mathcal{M}$ satisfies the following volume growth condition

$$V(x, r) \leq c_1 r^n, \quad 0 < r \leq r_0, \qquad (4.13)$$

for some constants c_1 and r_1 then v has the doubling property (DP).

As a consequence of Theorem 3.1, we have

Corollary 4.1. *Assume that the sectional curvature of $\mathcal{M}$ is bounded from above, the volume growth condition* (4.13) *is fulfilled and Ω is strongly convex and satisfies the interior (δ, r)-cone condition. Then*

$$q(x, y, t) \geq c \mathscr{E}(x, y, ct), \quad x, y \in \Omega, \ 0 < t \leq T. \qquad (4.14)$$

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