FAST APPROXIMATION OF THE AFFINITY DIMENSION FOR DOMINATED AFFINE ITERATED FUNCTION SYSTEMS

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Abstract. In 1988 K. Falconer introduced a formula which predicts the value of the Hausdorff dimension of the attractor of an affine iterated function system. The value given by this formula – sometimes referred to as the *affinity dimension* – is known to agree with the Hausdorff dimension both generically and in an increasing range of explicit cases. It is however a nontrivial problem to estimate the numerical value of the affinity dimension for specific iterated function systems. In this article we substantially extend an earlier result of M. Pollicott and P. Vytnova on the computation of the affinity dimension. Pollicott and Vytnova's work applies to planar invertible affine contractions with positive linear parts under several additional conditions which among other things constrain the affinity dimension to be between 0 and 1. We extend this result by passing from planar self-affine sets to self-affine sets in arbitrary dimensions, relaxing the positivity hypothesis to a domination condition, and removing all other constraints including that on the range of values of the affinity dimension. We provide some explicit examples of two- and three-dimensional affine iterated function systems for which the affinity dimension can be calculated to more than 30 decimal places.

1. Introduction

If $T_1, \ldots, T_N \colon \mathbb{R}^d \to \mathbb{R}^d$ are contractions it is well-known that there exists a unique nonempty compact set $X \subset \mathbb{R}^d$ such that $X = \bigcup_{i=1}^N T_i X$. In this case $(T_1, \ldots, T_N)$ is called an *iterated function system* and the set X its *attractor*. When each transformation T_i is a similitude with contraction ratio $r_i \in (0,1)$ and the distinct images $T_i X \cap T_j X$ do not overlap too strongly it is classical that the box dimension and Hausdorff dimension of the attractor are both equal to the unique real number $s > 0$ such that $\sum_{i=1}^N r_i^s = 1$ (see for example [17, Theorem 9.3] or the original article [31]). In the case where each T_i is instead an affine map $T_i x = A_i x + v_i$ the Hausdorff dimension and box dimension of the attractor X – which in this context we call a *self-affine set* – are more challenging to calculate. The problem of determining the Hausdorff dimension of such sets, even implicitly, has been an active topic of research since the 1980s and has received particularly intense research interest within the last decade (see for example the classic articles [10, 16, 21, 22, 30, 43] and more recent contributions such as [4, 5, 12, 13, 19, 23, 24, 39, 46]). In the landmark article [21] K. Falconer defined an implicit formula which is known to give the correct value for the Hausdorff dimension of a wide variety of self-affine sets. The subject of this article is the numerical estimation of the value predicted by Falconer's formula.

In order to define Falconer's formula we require a few preliminary definitions. Let $M_d(\mathbb{R})$ denote the set of all real $d \times d$ matrices. If $A \in M_d(\mathbb{R})$ we recall that the *singular values* of A are defined to be the square roots of the eigenvalues of the positive semidefinite matrix $A^\top A$. We denote the singular values of $A \in M_d(\mathbb{R})$ by $\sigma_1(A), \ldots, \sigma_d(A)$ in decreasing order of absolute value. For each $A \in M_d(\mathbb{R})$ and $s \geq 0$ let us define

$$\varphi^s(A) := \begin{cases} \sigma_1(A)\cdots\sigma_{\lfloor s\rfloor}(A)\sigma_{\lceil s\rceil}(A)^{s-\lfloor s\rfloor} & \text{if } 0 \leq s \leq d, \\ |\det A|^{\frac{s}{d}} & \text{if } s \geq d. \end{cases}$$

It was shown in [21] that for each $s \geq 0$ we have $\varphi^s(AB) \leq \varphi^s(A)\varphi^s(B)$ for all $A, B \in M_d(\mathbb{R})$. The *affinity dimension* of the iterated function system $T_i x := A_i x = v_i$, where $1 \leq i \leq N$, is then defined to be the quantity

$$\dim_{\mathsf{aff}}(T_1, \ldots, T_N) := \inf\left\{ s > 0\colon \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=1}^{N} \varphi^s(A_{i_1}\cdots A_{i_n}) < \infty \right\}.$$

Since $\dim_{\mathsf{aff}}(T_1, \ldots, T_N)$ depends only on $A_1, \ldots, A_N$ and not on the additive part of the transformations T_i we will also denote it by $\dim_{\mathsf{aff}}(A_1, \ldots, A_N)$. If the matrices $A_1, \ldots, A_N$ are assumed to be invertible and contracting with respect to some norm on $\mathbb{R}^d$ then the affinity dimension is the unique $s > 0$ such that the quantity

$$P(A_1, \ldots, A_N; s) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1,\ldots,i_n=1}^{N} \varphi^s(A_{i_1}\cdots A_{i_n})$$

is equal to zero.

Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^d$. It was shown in [21] that when $\max_{1\leq i\leq N}\|A_i\| < 1$ the affinity dimension $\dim_{\mathsf{aff}}(A_1, \ldots, A_N)$ is well-defined and is an upper bound for the box dimension of the attractor. (This argument may easily be adapted to the case where $\max_{1\leq i\leq N}|||A_i||| < 1$ in the operator norm induced by some norm $|||\cdot|||$ on $\mathbb{R}^d$.) It was additionally shown that when matrices $A_1, \ldots, A_N$ satisfying $\max_{1\leq i\leq N}\|A_i\| < \frac{1}{3}$ are fixed, then for Lebesgue-a.e. choice of $(v_1, \ldots, v_N) \in (\mathbb{R}^d)^N$ the attractor of the affine transformations $T_1, \ldots, T_N$ given by $T_i x := A_i x + v_i$ has Hausdorff dimension equal to $\min\{d, \dim_{\mathsf{aff}}(A_1, \ldots, A_N)\}$. Subsequent research focused on providing explicit examples for which the Hausdorff dimension of the attractor equals the affinity dimension of the defining iterated function system, with explicit special cases being given in articles such as [19, 24, 30, 46]. Very recently, B. Barańy, M. Hochman and A. Rapaport have shown that the Hausdorff dimension of a planar self-affine set is always equal to the affinity dimension of the defining iterated function system as long as the matrices A_i are invertible, the affine transformations satisfy the strong open set condition, and the matrices $|\det A_i|^{-1/2}A_i$ neither belong to a compact subgroup of $GL_2(\mathbb{R})$ nor preserve a finite subset of $\mathbb{RP}^1$. At the present time, however, results on higher-dimensional self-affine sets additional to that of Falconer are essentially unavailable.

Despite its prominent rôle in the dimension theory of self-affine sets, the properties of the affinity dimension itself have been investigated only very recently. In the 2014 article [23] D.-J. Feng and P. Shmerkin showed for the first time that the affinity dimension $\dim_{\mathsf{aff}}(A_1, \ldots, A_N)$ depends continuously on the entries of the matrices $A_1, \ldots, A_N$, and in [44] it was shown that the affinity dimension is computable in principle in the sense that for any given $\varepsilon > 0$ we may algorithmically compute an explicit approximation to $\dim_{\mathsf{aff}}(A_1, \ldots, A_N)$ which is guaranteed to be accurate to within the prescribed error ε and which requires only finitely many arithmetical operations to calculate. However, the method of [44] does not result in an algorithm which is fast enough to be useful in practical computations. Further general properties of the affinity dimension were investigated in [12, 38].

At the present time there are very few practical techniques available for the computation of the affinity dimension. In the article [45] the author gave a simple closed-form expression for the affinity dimension in the very special case where the matrices A_i are generalised permutation matrices, that is, matrices having exactly one nonzero entry in every row and column. Closed-form expressions are also available in the case of diagonal and upper-triangular matrices [20, 38]. To the best of the author's knowledge there so far exists only one result in the literature which is powerful enough to be able to estimate the affinity dimension for a nonempty open set of examples in a practicable time frame. The following result was proved by M. Pollicott and P. Vytnova in [52]. Here and throughout this article $\rho(A)$ denotes the spectral radius of the matrix or linear operator A.

Theorem 1. *Let $A_1, \ldots, A_N$ be 2×2 matrices which satisfy the following conditions:*

(i) We have $\sigma_1(A_i)^2 < \sigma_2(A_i) < 1$ for all $i = 1, \ldots, N$.

(ii) If $\mathcal{Q}_2$ is defined to be the open second quadrant $\{(x, y) \in \mathbb{R}^2 : x < 0 < y\}$, then the sets $A_1^{-1}\mathcal{Q}_2, \ldots, A_N^{-1}\mathcal{Q}_2$ are subsets of $\mathcal{Q}_2$ and have pairwise disjoint closures in $\mathcal{Q}_2$.

(iii) All entries of the matrices A_i are strictly positive[1].

For each $n \geq 1$ and $s \in \mathbb{C}$ define

$$t_n(s) = \sum_{i_1, \ldots, i_n = 1}^{N} \frac{\rho(A_{i_1} \cdots A_{i_n})^{2+s}}{\rho(A_{i_1} \cdots A_{i_n})^2 - \det A_{i_1} \cdots A_{i_n}},$$

$$a_n(s) := \sum_{k=1}^{n} \frac{(-1)^k}{k!} \sum_{\substack{(n_1, \ldots, n_k) \in \mathbb{N}^k \\ \sum_{i=1}^k n_i = n}} \prod_{i=1}^{k} \frac{t_{n_i}(s)}{n_i}$$

and $a_0(s) := 1$, and for each $n \geq 1$ let $s_n \in \mathbb{R}$ denote the smallest positive real number s such that $\sum_{i=0}^n a_i(s) = 0$. Then $\dim_{\mathsf{aff}}(A_1, \ldots, A_N) \in (0, 1)$, s_n is well-defined for all sufficiently large n, and there exists $\gamma > 0$ such that

$$|\dim_{\mathsf{aff}}(A_1, \ldots, A_N) - s_n| = O\left(\exp(-\gamma n^2)\right).$$

Remark. The quantity $a_n(s)$ may be alternatively characterised as

$$\frac{(-1)^n}{n!} \det \begin{pmatrix} t_1(s) & n-1 & 0 & \cdots & 0 & 0 \\ t_2(s) & t_1(s) & n-2 & \cdots & 0 & 0 \\ t_3(s) & t_2(s) & t_1(s) & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ t_{n-1}(s) & t_{n-2}(s) & t_{n-3}(s) & \cdots & t_1(s) & 1 \\ t_n(s) & t_{n-1}(s) & t_{n-2}(s) & \cdots & t_2(s) & t_1(s) \end{pmatrix},$$

and we will prefer this format in our exposition.

[1] This hypothesis is invoked in Pollicott and Vytnova's section 3 but is not explicitly stated in their introduction. It does not follow automatically from the other hypotheses unless the determinants are assumed positive.

The methods underlying the proof of Theorem 1 will be described in more detail in the following section. We remark that condition (i) above implies that the matrices are invertible, and the combination of the three conditions implies $0 < \dim_{\mathsf{aff}}(A_1, \ldots, A_N) < 1$ (see [30] for details).

In fact the only condition which is really essential to Pollicott and Vytnova's argument is that the matrix entries are positive, although in cases where we have $\dim_{\mathsf{aff}}(A_1, \ldots, A_N) \in (1, 2)$ the formula for $t_n(s)$ instead becomes

$$t_n(s) := \sum_{i_1,\ldots,i_n=1}^{N} \frac{\rho(A_{i_1}\cdots A_{i_n})^{4-s}|\det A_{i_1}\cdots A_{i_n}|^{s-1}}{\rho(A_{i_1}\cdots A_{i_n})^2 - \det A_{i_1}\cdots A_{i_n}}.$$

In this article we aim to prove as comprehensive as possible an extension of Theorem 1. In particular, as well as removing hypotheses (i)–(ii) from Theorem 1 we will establish a version of that theorem which is valid for affine iterated function systems in dimensions higher than two, in which $\dim_{\mathsf{aff}}(A_1, \ldots, A_N)$ may take any value in the range $(0, d)$, and in which the hypothesis of positivity is weakened to one of domination. In order to state our results in full we will require a number of definitions, which relate to multilinear algebra, to positivity and to domination.

In extending Theorem 1 one of our concerns will be to allow matrices of arbitrary dimension. Whereas in two dimensions the function $\varphi^s(A)$ admits the simple characterisation

$$\varphi^s(A) = \begin{cases} \|A\|^s & \text{if } 0 \leq s \leq 1, \\ |\det A|^{s-1}\|A\|^{2-s} & \text{if } 1 \leq s \leq 2, \end{cases}$$

when $s > 1$ and $d > 2$ the analogous formula involves exterior powers of the matrix A. In order to study the singular value function φ^s in dimensions higher than two we therefore need to recall some concepts and notation from multilinear algebra.

Recall that when $1 \leq k \leq d$ the real vector space $\wedge^k\mathbb{R}^d$ is the vector space spanned by the formal expressions $\{v_1 \wedge v_2 \wedge \cdots \wedge v_k : v_1, \ldots v_k \in \mathbb{R}^d\}$ subject to the identifications

$$\lambda(v_1 \wedge v_2 \wedge \cdots \wedge v_k) = (\lambda v_1) \wedge v_2 \wedge \cdots \wedge v_k,$$
$$(u_1 \wedge v_2 \wedge \cdots \wedge v_k) + (v_1 \wedge v_2 \wedge \cdots \wedge v_k) = (u_1 + v_1) \wedge v_2 \wedge \cdots \wedge v_k,$$
$$v_1 \wedge v_2 \wedge \cdots \wedge v_k = (-1)^{\mathrm{sign}(\pi)} v_{\pi(1)} \wedge v_{\pi(2)} \wedge \cdots v_{\pi(k)}$$

for all $v_1, \ldots, v_k, u_1 \in \mathbb{R}^d$, $\lambda \in \mathbb{R}$ and permutations $\pi \colon \{1, \ldots, k\} \to \{1, \ldots, k\}$. The vector space $\wedge^k\mathbb{R}^d$ is $\binom{d}{k}$-dimensional and if $v_1, \ldots, v_d$ is any basis for $\mathbb{R}^d$ then $\{v_{i_1} \wedge \cdots \wedge v_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq d\}$ is a basis for $\wedge^k\mathbb{R}^d$. The $\binom{d}{k}$-dimensional vector space $\wedge^k\mathbb{C}^d$ may be constructed analogously.

The space $\wedge^k\mathbb{R}^d$ inherits an inner product $\langle\cdot,\cdot\rangle_{\wedge^k\mathbb{R}^d}$ from the standard inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^d$ which satisfies

$$\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k\rangle_{\wedge^k\mathbb{R}^d} = \det\left(\left[\langle u_i, v_j\rangle\right]_{i,j=1}^k\right).$$

If $A \in M_d(\mathbb{R})$ then we may define a linear map $A^{\wedge k} \colon \wedge^k\mathbb{R}^d \to \wedge^k\mathbb{R}^d$ by $A^{\wedge k}(v_1 \wedge \cdots \wedge v_k) = Av_1 \wedge \cdots \wedge Av_k$. If $v_1, \ldots, v_d$ is a basis for $\mathbb{C}^d$ consisting of eigenvectors and generalised eigenvectors for A then the vectors $v_{i_1} \wedge \cdots \wedge v_{i_k}$ form a basis for $\wedge^k\mathbb{C}^d$ and it is not hard to see that if $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of A then the eigenvalues of $A^{\wedge k}$ are precisely the $\binom{d}{k}$ different products $\lambda_{i_1}\cdots\lambda_{i_k}$ with $1 \leq i_1 <$

$\cdots < i_k \leq d$. It is clear from the definition of the inner product on $\wedge^k\mathbb{R}^d$ that $(A^{\wedge k})^\top = (A^\top)^{\wedge k}$. Combining these observations we may easily see that

$$\left\|A^{\wedge k}\right\|_{\wedge^k\mathbb{R}^d} = \rho\left(\left(A^\top A\right)^{\wedge k}\right)^{\frac{1}{2}} = \sigma_1(A)\cdots\sigma_k(A)$$

for all $A \in M_d(\mathbb{R})$. By convention we also define $\wedge^0\mathbb{R}^d = \mathbb{R}$ and $A^{\wedge 0} = 1$. It follows easily that we may write

$$\varphi^s(A) = \left\|A^{\wedge\lfloor s\rfloor}\right\|^{1+s-\lfloor s\rfloor}\left\|A^{\wedge\lceil s\rceil}\right\|^{\lceil s\rceil - s}$$

for all $A \in M_d(\mathbb{R})$ and $s \in [0,d]$.

As well as increasing the dimension of the matrices to be considered in our extension of Theorem 1 we would like to weaken as much as possible the hypothesis that the matrices have positive entries. To this end we introduce the following definition:

Definition 1.1. *Let $\mathsf{A} \subset M_d(\mathbb{R})$ be nonempty. We say that $(\mathcal{K}_1,\ldots,\mathcal{K}_m)$ is a* multicone *for A if the following properties hold:*

(i) Each $\mathcal{K}_j$ is a closed, convex subset of $\mathbb{R}^d$ with nonempty interior such that $\lambda\mathcal{K}_j \subseteq \mathcal{K}_j$ for every non-negative real number λ.
(ii) There exists a unit vector $w \in \mathbb{R}^d$ such that $\langle u, w\rangle > 0$ for all nonzero vectors $u \in \bigcup_{j=1}^m \mathcal{K}_j$.
(iii) For every $A \in \mathsf{A}$ and $j \in \{1,\ldots,m\}$ there exists $\ell = \ell(j,A) \in \{1,\ldots,m\}$ such that $A(\mathcal{K}_j \setminus \{0\}) \subset (\operatorname{Int}\mathcal{K}_\ell) \cup (-\operatorname{Int}\mathcal{K}_\ell)$.
(iv) For all distinct $j_1, j_2 \in \{1,\ldots,m\}$ we have $\mathcal{K}_{j_1} \cap \mathcal{K}_{j_2} = \{0\}$.

When (ii) holds we say that w is a transverse-defining vector *for $(\mathcal{K}_1,\ldots,\mathcal{K}_m)$ since the hyperplane normal to w is transverse to $\bigcup_{j=1}^m \mathcal{K}_j$. If a multicone for A exists then we say that A is* multipositive.

We will say that $(A_1,\ldots,A_N) \in M_d(\mathbb{R})^N$ is *k-multipositive* if and only if the set $\{A_1^{\wedge k},\ldots,A_N^{\wedge k}\}$ is multipositive in the sense defined above. (By abuse of notation we shall say that a tuple of matrices is k-multipositive if and only if the corresponding set is.) We observe that a tuple of $d \times d$ matrices with all entries positive is multipositive since we may take $m = 1$ and $\mathcal{K}_1$ to be the closed positive orthant in $\mathbb{R}^d$. We also observe that every tuple of $d \times d$ matrices is 0-multipositive, and every tuple of $d \times d$ invertible matrices is d-multipositive.

If $1 \leq k < d$ then a tuple of invertible matrices $(A_1,\ldots,A_N) \in GL_d(\mathbb{R})^N$ is called *k-dominated* if there exist $C, \gamma > 0$ such that

$$\sigma_{k+1}(A_{i_1}\cdots A_{i_n}) \leq Ce^{-\gamma n}\sigma_k(A_{i_n}\cdots A_{i_1})$$

for all $i_1,\ldots,i_n \in \{1,\ldots,n\}$ and $n \geq 1$. By convention we will say that every $(A_1,\ldots,A_N) \in GL_d(\mathbb{R})^N$ is both 0- and d-dominated. It is not difficult to show using the previous observations that $(A_1,\ldots,A_N)$ is k-dominated if and only if $(A_1^{\wedge k},\ldots,A_N^{\wedge k})$ is 1-dominated. For $0 < k < d$ let $\operatorname{Gr}(k,d)$ denote the Grassmannian manifold of k-dimensional linear subspaces of $\mathbb{R}^d$. In the case $k = 1$ we will prefer the notation $\operatorname{Gr}(1,d) = \mathbb{RP}^{d-1}$. If $E \subset \mathbb{R}^d$ then we shall write $PE := \{V \in \mathbb{RP}^{d-1} \colon E \cap (V \setminus \{0\}) \neq \emptyset\}$. The property of k-domination (which is based on the earlier concept of dominated splittings in smooth ergodic theory) has numerous equivalent formulations which were explored in [11]:

Theorem 2 (Bochi-Gourmelon)**.** *Let $\mathsf{A} \subset M_d(\mathbb{R})$ be a nonempty compact set of invertible matrices and let $k \in \{1, \ldots, d-1\}$. Then the following statements are equivalent:*

(i) There exist $C, \gamma > 0$ such that

$$\sup_{A_1,\ldots,A_n \in \mathsf{A}} \frac{\sigma_{k+1}(A_n \cdots A_1)}{\sigma_k(A_n \cdots A_1)} \leq Ce^{-\gamma n} \tag{1}$$

for every integer $n \geq 1$.

(ii) There exists a nonempty set $\mathcal{C} \subset \mathrm{Gr}(k,d)$ such that the closure of $\bigcup_{A \in \mathsf{A}} A\mathcal{C}$ is a subset of the interior of $\mathcal{C}$, and such that there exists a $(d-k)$-dimensional linear subspace of $\mathbb{R}^d$ which is transverse to every element of $\mathcal{C}$.

(iii) There exists a nonempty subset $\mathcal{C}$ of real projective space $\mathbb{RP}^{d-1}$ such that $PE \subset \mathcal{C}$ for some k-dimensional subspace E of $\mathbb{R}^d$, such that $PF \cap \mathcal{C} = \emptyset$ for some $(d-k)$-dimensional subspace F of $\mathbb{R}^d$, and such that the closure of $\bigcup_{A \in \mathsf{A}} A\mathcal{C}$ is a subset of the interior of $\mathcal{C}$.

(iv) For every nonempty compact metric space X, homeomorphism $T \colon X \to X$ and continuous function $\mathcal{A} \colon X \to \mathsf{A}$ there exist continuous functions $\mathcal{U} \colon X \to \mathrm{Gr}(k,d)$ and $\mathcal{V} \colon X \to \mathrm{Gr}(d-k,d)$ such that $\mathcal{A}(x)\mathcal{U}(x) = \mathcal{U}(Tx)$, $\mathcal{A}(x)\mathcal{V}(x) = \mathcal{V}(Tx)$ and $\mathbb{R}^d = \mathcal{U}(x) \oplus \mathcal{V}(x)$ for all $x \in X$, and such that for some constants $C, \gamma > 0$ depending on $\mathcal{A}$ we have for all $x \in X$ and $n \geq 1$

$$\|\mathcal{A}(T^{n-1}x) \cdots \mathcal{A}(x)u\| \geq Ce^{\gamma n} \|\mathcal{A}(T^{n-1}x) \cdots \mathcal{A}(x)v\|$$

for all unit vectors $u \in \mathcal{U}(x)$ and $v \in \mathcal{V}(x)$.

Moreover the sets $\mathcal{C}$ in (ii) and (iii) may without loss of generality be taken to be closed and to have finitely many connected components.

If $(A_1, \ldots, A_N) \in M_d(\mathbb{R})^N$ is a tuple of invertible matrices then $(A_1, \ldots, A_N)$ is k-dominated if and only if it is k-multipositive. This result was obtained in [9] where it is expressed in quite different language; to save the reader the labour of translating that argument into the present article's terminology we provide a proof of this result in the appendix, where it is stated as Proposition A.1.

In order to state our main theorem we require just a few more items of notation. For each $N \geq 1$ let us define

$$\Sigma_N^* := \bigcup_{n=1}^{\infty} \{1, \ldots, N\}^n.$$

If $\mathtt{i} = (i_k)_{k=1}^n \in \Sigma_N^*$ we write $|\mathtt{i}| = n$ and refer to $|\mathtt{i}|$ as the *length* of $\mathtt{i}$. If $\mathtt{i}, \mathtt{j} \in \Sigma_N^*$ we let $\mathtt{ij} \in \Sigma_N^*$ denote the sequence of length $|\mathtt{i}| + |\mathtt{j}|$ obtained by running first through the symbols of $\mathtt{i}$ and then through those of $\mathtt{j}$ in the obvious fashion. Clearly Σ_N^* is a semigroup with respect to the operation $(\mathtt{i}, \mathtt{j}) \mapsto \mathtt{ij}$. If $A_1, \ldots, A_N \in M_d(\mathbb{R})$ and $\mathtt{i} = (i_k)_{k=1}^n \in \Sigma_N^*$ then we write $A_\mathtt{i} := A_{i_n} \cdots A_{i_1}$. We observe that $A_\mathtt{i} A_\mathtt{j} = A_{\mathtt{ji}}$ for all $\mathtt{i}, \mathtt{j} \in \Sigma_N^*$.

If B is a linear transformation of a finite-dimensional real vector space we let $\lambda_1(B),\ldots,\lambda_d(B)$ denote the eigenvalues of B listed with repetition according to multiplicity and listed in decreasing order of absolute value. While this notation *a priori* introduces ambiguities when distinct eigenvalues of the same modulus exist, we will see that this consideration does not affect the statements of our results.

We may now present the following generalisation of Pollicott and Vytnova's result:

Theorem 3. *Let $d,N\geq 2$, let $(A_1,\ldots,A_N)\in M_d(\mathbb{R})^N$ and let $0\leq k<d$. Suppose that $(A_1,\ldots,A_N)$ is both k-multipositive and $(k+1)$-multipositive. For each integer $n\geq 1$ and $s\in\mathbb{R}$ define*

$$t_n(s):=\sum_{|\mathtt{i}|=n}\frac{\lambda_1\left(A_{\mathtt{i}}^{\wedge k}\right)^{\binom{d}{k}-1}\lambda_1\left(A_{\mathtt{i}}^{\wedge(k+1)}\right)^{\binom{d}{k+1}-1}\rho\left(A_{\mathtt{i}}^{\wedge k}\right)^{k+1-s}\rho\left(A_{\mathtt{i}}^{\wedge(k+1)}\right)^{s-k}}{p'_{A_{\mathtt{i}}^{\wedge k}}\left(\lambda_1\left(A_{\mathtt{i}}^{\wedge k}\right)\right)p'_{A_{\mathtt{i}}^{\wedge(k+1)}}\left(\lambda_1\left(A_{\mathtt{i}}^{\wedge(k+1)}\right)\right)}$$

where $p'_B(x_0)$ denotes the first derivative of the characteristic polynomial $p_B(x):=\det(xI-B)$ evaluated at the point x_0. Define also

$$a_n(s):=\frac{(-1)^n}{n!}\det\begin{pmatrix} t_1(s) & n-1 & 0 & \cdots & 0 & 0\\ t_2(s) & t_1(s) & n-2 & \cdots & 0 & 0\\ t_3(s) & t_2(s) & t_1(s) & \ddots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\ t_{n-1}(s) & t_{n-2}(s) & t_{n-3}(s) & \cdots & t_1(s) & 1\\ t_n(s) & t_{n-1}(s) & t_{n-2}(s) & \cdots & t_2(s) & t_1(s)\end{pmatrix}$$

for all $n\geq 1$, and $a_0(s):=1$. For each $s\in[k,k+1]$ let $r_n(s)$ denote the smallest positive real root of the polynomial $p_{n,s}(x):=\sum_{i=0}^n a_n(s)x^i$. Then there exists $n_0\in\mathbb{N}$ such that $r_n(s)$ is well-defined for all $s\in[k,k+1]$ and $n\geq n_0$, and we have

$$\left|e^{P(A_1,\ldots,A_N;s)}-\frac{1}{r_n(s)}\right|\leq K\exp\left(-\gamma n^{\alpha}\right)$$

for some constants $K,\gamma>0$ not depending on $s\in[k,k+1]$, where

$$\alpha:=\frac{\binom{d+1}{k+1}-1}{\binom{d+1}{k+1}-2}>1.$$

Suppose additionally that there is a norm $|||\cdot|||$ on $\mathbb{R}^d$ such that $\max_{1\leq i\leq N}|||A_i|||<1$, and that $\dim_{\mathsf{aff}}(A_1,\ldots,A_N)\in(k,k+1)$. Then for all for all sufficiently large n the function $s\mapsto 1/r_n(s)$ is a strictly decreasing convex function $[k,k+1]\to\mathbb{R}$ and there exists a unique $s_n\in[k,k+1]$ such that $r_n(s_n)=1$. There exist constants $K',\gamma'>0$ depending on $A_1,\ldots,A_N$ such that for all such n we have

$$|\dim_{\mathsf{aff}}(A_1,\ldots,A_N)-s_n|\leq K'\exp\left(-\gamma' n^{\alpha}\right).$$

Since every matrix tuple is 0-multipositive, in the case $k=0$ the hypothesis of Theorem 3 reduces to the requirement that $\dim_{\mathsf{aff}}(A_1,\ldots,A_N)$ is 1-multipositive and $\dim_{\mathsf{aff}}(A_1,\ldots,A_N)\in(0,1)$. Since $B^{\wedge 0}$ is the identity map on $\mathbb{R}$ the expressions involving $A_{\mathtt{i}}^{\wedge k}$ reduce to 1 in the case $k=0$, resulting in the formula

$$t_n(s) := \sum_{|\mathtt{i}|=n} \frac{\lambda_1\left(A_{\mathtt{i}}\right)^{d-1}\rho\left(A_{\mathtt{i}}\right)^s}{p'_{A_{\mathtt{i}}}\left(\lambda_1\left(A_{\mathtt{i}}\right)\right)}.$$

In particular when $d=2$, $k=0$ and the matrices A_i have positive entries we may recover the conclusion of Theorem 1. Similarly, since every tuple in $M_d(\mathbb{R})^N$ is d-multipositive and $B^{\wedge d} = \det B$, the expressions involving $A_{\mathtt{i}}^{\wedge(k+1)}$ simplify when $k = d-1$ yielding

$$t_n(s) := \sum_{|\mathtt{i}|=n} \frac{\lambda_1\left(A_{\mathtt{i}}^{\wedge(d-1)}\right)^{d-1}\rho\left(A_{\mathtt{i}}^{\wedge(d-1)}\right)^{d-s}|\det A_{\mathtt{i}}|^{s+1-d}}{p'_{A_{\mathtt{i}}^{\wedge(d-1)}}\left(\lambda_1\left(A_{\mathtt{i}}^{\wedge(d-1)}\right)\right)}.$$

and the hypotheses are reduced to the requirement that $(A_1,\ldots,A_N)$ is $(d-1)$-multipositive and $\dim_{\mathsf{aff}}(A_1,\ldots,A_N) \in (d-1,d)$. We remark that hypotheses of domination and positivity analogous to those in Theorem 3 have been a feature of numerous recent works on affine iterated function systems such as [6, 7, 8, 18, 19] as well as the older article [30].

If it is known that the tuple $(A_1^{\wedge k},\ldots,A_N^{\wedge k})$ preserves a single cone in $\wedge^k\mathbb{R}^d$ and similarly $(A_1^{\wedge(k+1)},\ldots,A_N^{\wedge(k+1)})$ preserves a single cone in $\wedge^{k+1}\mathbb{R}^d$ then the condition $\dim_{\mathsf{aff}}(A_1,\ldots,A_N) \in (k,k+1)$ may be easily checked. A theorem of V. Yu. Protasov [54] implies that if $B_1,\ldots,B_N$ preserve a cone then

$$\lim_{n\to\infty}\left(\sum_{i_1,\ldots,i_n=1}^N \|B_{i_1}\cdots B_{i_n}\|\right)^{\frac{1}{n}} = \rho\left(\sum_{i=1}^N B_i\right),$$

and so in this case

$$\lim_{n\to\infty}\left(\sum_{i_1,\ldots,i_n=1}^N \varphi^k\left(A_{i_1}\cdots A_{i_n}\right)\right)^{\frac{1}{n}} = \rho\left(\sum_{i=1}^N A_i^{\wedge k}\right),$$

$$\lim_{n\to\infty}\left(\sum_{i_1,\ldots,i_n=1}^N \varphi^{k+1}\left(A_{i_1}\cdots A_{i_n}\right)\right)^{\frac{1}{n}} = \rho\left(\sum_{i=1}^N A_i^{\wedge(k+1)}\right)$$

using the identity $\varphi^\ell(B) = \|B^{\wedge\ell}\|$ for $\ell = 0,\ldots,d$. It follows that in this situation Theorem 3 is applicable if

$$\rho\left(\sum_{i=1}^N A_i^{\wedge(k+1)}\right) < 1 < \rho\left(\sum_{i=1}^N A_i^{\wedge k}\right).$$

An example of this situation is presented in § below.

In the situation where $(A_1,\ldots,A_N)$ fails to be both k- and $(k+1)$-multipositive we believe it to be unlikely that any analogue of Theorem 3 can be proved. The precise role of the multipositive hypothesis is discussed in more detail in the following section, and in the final section §.

2. Overview of the method and statement of the main technical theorem

The method underlying Theorem 3 is, like Theorem 1, based on Fredholm determinants of transfer operators, and in broad terms resembles many other arguments of this type such as [33, 34, 36, 47, 50, 52, 53]. Both in order to give a sense of the organisation of this article and to indicate the difficulties present in the proof of Theorem 3 which do not occur in the context of Theorem 1 let us briefly describe this strategy. We recall that an operator $\mathscr{L}$ on an infinite-dimensional Hilbert space is called trace-class if the sequence of approximation numbers

$$\mathfrak{s}_n(\mathscr{L}) := \inf \{\|\mathscr{L} - \mathscr{F}\| \colon \operatorname{rank} \mathscr{F} < n\}$$

is summable; we observe in particular that such an operator is compact (being a limit in the norm topology of a sequence of finite-rank operators) and cannot be invertible. We also observe that clearly $\mathfrak{s}_n(\mathscr{L}^\ell) \leq \|\mathscr{L}^{\ell-1}\|\mathfrak{s}_n(\mathscr{L})$ for every $n, \ell \geq 1$ and consequently every power of a trace-class operator is also trace-class. The notion of trace-class operator is reviewed in detail for the reader's convenience in §. Suppose then that $\mathscr{H}$ is a separable complex Hilbert space and $\mathscr{L} \colon \mathscr{H} \to \mathscr{H}$ a trace-class linear operator, and let $(\lambda_\ell)_{\ell=1}^\infty$ be the sequence of nonzero eigenvalues of $\mathscr{L}$ listed with repetition according to their algebraic multiplicity. (If only $M < \infty$ nonzero eigenvalues exist then define $\lambda_\ell = 0$ for $\ell > M$.) It is a classical fact that the function $z \mapsto \det(I - z\mathscr{L})$ defined by

$$\det(I - z\mathscr{L}) := \prod_{\ell=1}^{M} (1 - z\lambda_\ell)$$

is an entire function from $\mathbb{C}$ to $\mathbb{C}$, and moreover one may show that in the power series $\det(I - z\mathscr{L}) = \sum_{n=0}^\infty a_\ell z^\ell$ the coefficients are given by $a_0 = 1$ and

$$a_\ell = \frac{(-1)^\ell}{\ell!} \det \begin{pmatrix} \operatorname{tr} \mathscr{L} & \ell - 1 & 0 & \cdots & 0 & 0 \\ \operatorname{tr} \mathscr{L}^2 & \operatorname{tr} \mathscr{L} & \ell - 2 & \cdots & 0 & 0 \\ \operatorname{tr} \mathscr{L}^3 & \operatorname{tr} \mathscr{L}^2 & \operatorname{tr} \mathscr{L} & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \operatorname{tr} \mathscr{L}^{\ell-1} & \operatorname{tr} \mathscr{L}^{\ell-2} & \operatorname{tr} \mathscr{L}^{\ell-3} & \cdots & \operatorname{tr} \mathscr{L} & 1 \\ \operatorname{tr} \mathscr{L}^{\ell} & \operatorname{tr} \mathscr{L}^{\ell-1} & \operatorname{tr} \mathscr{L}^{\ell-2} & \cdots & \operatorname{tr} \mathscr{L}^2 & \operatorname{tr} \mathscr{L} \end{pmatrix}$$

for $\ell \geq 1$. If we write

$$\sum_{\ell=0}^\infty a_\ell z^\ell = \det(I - z\mathscr{L}) = \prod_{\ell=1}^M (1 - z\lambda_\ell)$$

then by equating coefficients of z^n we find (at least informally) that also

$$a_n = (-1)^n \sum_{i_1 < i_2 < \cdots < i_n} \lambda_{i_1} \cdots \lambda_{i_n}. \tag{2}$$

for each $n \geq 1$. Suppose now that we wished to calculate the spectral radius $\rho(\mathscr{L})$, knowing the values of the traces $\mathscr{L}^\ell$ for $\ell = 1, \ldots, n$, say, and knowing also that the spectral radius is an eigenvalue of $\mathscr{L}$. The roots of $\det(I - z\mathscr{L})$ are precisely the reciprocals of the eigenvalues of $\mathscr{L}$ and therefore $\rho(\mathscr{L})^{-1}$ is the smallest positive root of $\sum_{\ell=0}^\infty a_\ell z^\ell$. In particular, the smallest positive root of $\sum_{\ell=0}^n a_\ell z^\ell$

should be a good approximation to $\rho(\mathscr{L})^{-1}$ as long as $\sum_{\ell=n+1}^{\infty}|a_\ell|$ is small. But if we are able to show that the eigenvalues (λ_n) decay exponentially (or even just stretched-exponentially) in n, then the expression (2) implies a super-exponential decay estimate for the coefficients a_n. Such an estimate will hold in particular if the approximation numbers of $\mathscr{L}$ decay stretched-exponentially. In such a situation we may therefore reasonably hope that the approximation procedure just outlined provides an estimate which becomes super-exponentially more accurate as n increases.

In order to implement this line of reasoning we need therefore to construct, for each $s \in [k, k+1]$, a trace-class operator $\mathscr{L}_s$ on a Hilbert space $\mathscr{H}$ such that $e^{P(A_1,\ldots,A_N;s)}$ is an eigenvalue of $\mathscr{L}_s$ and is equal to the spectral radius of $\mathscr{L}_s$, such that $\mathscr{L}_s$ is trace-class, such that the sequence of approximation numbers of $\mathscr{L}_s$ decays rapidly to zero, and such that the sequence of traces $\operatorname{tr}\mathscr{L}_s^n$ is easy to compute. Once such a family of operators has been constructed the result follows by relatively straightforward manipulations which, while they do not correspond precisely to any prior work, share a degree of familial resemblance with calculations occurring in numerous earlier articles such as [3, 32, 33, 34, 35, 36, 37, 40, 48, 49, 50, 51, 52, 53].

If V is a finite-dimensional real vector space let PV denote the real projective space of lines through the origin in V. Intuitively, in order to construct an operator $\mathscr{L}_s$ with spectral radius $e^{P(A_1,\ldots,A_N;s)}$, we might consider an operator acting on some space of continuous functions $P(\wedge^k\mathbb{R}^d)\times P(\wedge^{k+1}\mathbb{R}^d)\to\mathbb{C}$ defined by

$$\left(\mathscr{L}_s f\right)(\overline{u},\overline{v}) = \sum_{i=1}^{N}\left(\frac{\left\|A_i^{\wedge k}u\right\|}{\|u\|}\right)^{k+1-s}\left(\frac{\left\|A_i^{\wedge(k+1)}v\right\|}{\|v\|}\right)^{s-k} f\left(\overline{A_i^{\wedge k}u},\overline{A_i^{\wedge(k+1)}v}\right)$$

where for $v\in V$ the notation $\overline{v}$ represents the one-dimensional subspace spanned by the vector v. Since we would then have

$$\left(\mathscr{L}_s^n f\right)(\overline{u},\overline{v}) = \sum_{|\mathtt{i}|=n}\left(\frac{\left\|A_{\mathtt{i}}^{\wedge k}u\right\|}{\|u\|}\right)^{k+1-s}\left(\frac{\left\|A_{\mathtt{i}}^{\wedge(k+1)}v\right\|}{\|v\|}\right)^{s-k} f\left(\overline{A_{\mathtt{i}}^{\wedge k}u},\overline{A_{\mathtt{i}}^{\wedge(k+1)}v}\right)$$

for each $n\geq 1$ we might then reasonably expect that

$$\lim_{n\to\infty}\|\mathscr{L}_s^n\|^{\frac{1}{n}} = \lim_{n\to\infty}\left(\sum_{|\mathtt{i}|=n}\left\|A_{\mathtt{i}}^{\wedge k}\right\|^{k+1-s}\left\|A_{\mathtt{i}}^{\wedge(k+1)}\right\|^{s-k}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\sum_{|\mathtt{i}|=n}\varphi^s(A_{\mathtt{i}})\right)^{\frac{1}{n}}$$

so that $e^{P(A_1,\ldots,A_N;s)}$ is equal to the spectral radius of $\mathscr{L}_s$. Indeed, such operators were successfully constructed by Guivarc'h and Le Page on spaces of Hölder continuous functions $P(\wedge^k\mathbb{R}^d)\times P(\wedge^{k+1}\mathbb{R}^d)\to\mathbb{C}$ in the article [29].

However, notwithstanding the minor additional complications posed by the fact that the spaces defined above are not Hilbert, there is no reason to believe that $\mathscr{L}_s$ acting on such a space should have a summable sequence of approximation numbers $\mathfrak{s}_n(\mathscr{L}_s)$. Indeed, $\mathscr{L}_s$ as constructed is equal to a sum of weighted composition operators $f\mapsto g\cdot f\circ T$ where T is an invertible transformation of $P(\wedge^k\mathbb{R}^d)\times P(\wedge^{k+1}\mathbb{R}^d)$ and g is nowhere zero. Such an operator might reasonably be expected to be invertible, and there is certainly no reason to believe that $\mathscr{L}_s$ should be trace-class.

The problem is thus to define $\mathscr{L}_s$ approximately as above in such a way that it is a sum of trace-class, non-invertible operators. It is here that the hypothesis of k- and $(k+1)$-multipositivity becomes relevant: this hypothesis implies that for $\ell = k, k+1$ the matrices $A_1^{\wedge \ell}, \ldots, A_N^{\wedge \ell}$ map a finite union of patches of $P(\wedge^\ell \mathbb{R}^d)$ strictly inside itself. By taking $\mathscr{H}$ to be a suitable Hilbert space of functions defined only on the patches, composition with the projective action of the matrices should then induce an operator which is non-invertible and hopefully trace-class. It transpires that composition operators on spaces of holomorphic functions are reliably trace-class subject to moderate geometrical conditions, and as such our strategy will involve passing to a space of holomorphic functions defined on complex extensions of the patches in real projective space. Once we have verified that such an extension can be constructed in such a way that the operator $\mathscr{L}_s$ is well-defined on the patches we may proceed to prove Theorem 3 along the lines outlined above. The principal task arising in this article is therefore to construct the complex patches in such a way that that the operator $\mathscr{L}_s$ is well-defined and has the aforementioned necessary properties.

In the two-dimensional context of Theorem 1 the construction of these patches is straightforward. Since Theorem 1 is restricted to affine transformations whose linear parts contract the positive cone in $\mathbb{R}^2$, it is sufficient to consider the projective action of those linear maps on the interval $\{(x, 1-x) \colon x \in [0,1]\}$, which is an action by linear fractional transformations. A finite collection of linear fractional transformations each of which maps an interval strictly inside itself can easily be shown to also map a corresponding complex disc inside itself, and this complex disc can be used as the domain of the holomorphic functions on which the operator $\mathscr{L}_s$ acts. In the two-dimensional case the construction of $\mathscr{L}_s$ and the space on which it acts is thus rather trivial. In higher dimensions and using multicones instead of cones, the corresponding problem is to understand (in place of one-dimensional intervals) a family of $(d-1)$-dimensional sections of cones in $\mathbb{R}^d$ – in effect, a finite collection of arbitrary $(d-1)$-dimensional convex bodies – and a collection of linear fractional transformations between them, and to contrive a system of extensions of those convex bodies into $\mathbb{C}^{d-1}$ which is also preserved by the same family of linear fractional transformations. This rather technical procedure, undertaken in §, is responsible for much of the length of the present article.

The outcome of the constructions outlined above is summarised in the following technical theorem:

Theorem 4. *Let $d, N \geq 2$ and let $A_1, \ldots, A_N$ be real $d \times d$ matrices and suppose that $(A_1, \ldots, A_N)$ is both k-multipositive and $(k+1)$-multipositive, where $0 \leq k < d$. There exist a separable complex Hilbert space $\mathscr{H}$ and a family of bounded linear operators $\mathscr{L}_s \colon \mathscr{H} \to \mathscr{H}$ defined for all $s \in \mathbb{C}$ with the following properties:*

(i) There exist $C, \kappa, \gamma > 0$ such that for all $s \in \mathbb{C}$ and $n \geq 1$ we have $\mathfrak{s}_n(\mathscr{L}) \leq C \exp\left(\kappa |s| - \gamma n^\beta\right)$, where

$$\beta := \frac{1}{\binom{d+1}{k+1} - 2} \in (0, 1].$$

In particular each $\mathscr{L}_s$ is trace-class.

(ii) For every $s \in \mathbb{C}$ and $n \geq 1$ we have

$$\operatorname{tr} \mathscr{L}_s^n = \sum_{|\mathtt{i}|=n} \frac{\lambda_1\left(A_{\mathtt{i}}^{\wedge k}\right)^{\binom{d}{k}-1} \lambda_1\left(A_{\mathtt{i}}^{\wedge(k+1)}\right)^{\binom{d}{k+1}-1} \rho\left(A_{\mathtt{i}}^{\wedge k}\right)^{k+1-s} \rho\left(A_{\mathtt{i}}^{\wedge(k+1)}\right)^{s-k}}{p'_{A_{\mathtt{i}}^{\wedge k}}\left(\lambda_1\left(A_{\mathtt{i}}^{\wedge k}\right)\right) p'_{A_{\mathtt{i}}^{\wedge(k+1)}}\left(\lambda_1\left(A_{\mathtt{i}}^{\wedge(k+1)}\right)\right)}$$

where $p_B(x) := \det(xI - B)$ denotes the characteristic polynomial of B and $p'_B(x_0)$ its derivative evaluated at x_0.

(iii) For every $s \in \mathbb{R}$ the spectral radius of $\mathscr{L}_s$ is equal to

$$\lim_{n\to\infty} \frac{1}{n} \log \sum_{|\mathtt{i}|=n} \left\|A_{\mathtt{i}}^{\wedge k}\right\|^{k+1-s} \left\|A_{\mathtt{i}}^{\wedge(k+1)}\right\|^{s-k}.$$

In particular the above limit exists for all $s \in \mathbb{R}$, and for every $s \in [k, k+1]$ the spectral radius of $\mathscr{L}_s$ is equal to $e^{P(A_1,\ldots,A_N;s)}$. For all $s \in \mathbb{R}$ the spectral radius of $\mathscr{L}_s$ is a simple eigenvalue of $\mathscr{L}_s$ and there are no other eigenvalues of the same modulus.

Theorem 4 is a special case of a slightly more general result, Theorem 12, which will be proved later.

The remainder of this article is structured as follows. In § we undertake the construction of the complex extensions of the patches in real projective space. It is essentially immediate from the hypotheses of Theorem 3 that the projective action of each $A_{\mathtt{i}}^{\wedge k}$ and $A_{\mathtt{i}}^{\wedge(k+1)}$ maps an open region of the appropriate real projective space (corresponding to the projectivisation of the real cones witnessing the multipositivity of the matrices A_i) inside a compact subset of that region, but it does not automatically follow that there exists a complex region with the same property, which *a priori* is a much stronger requirement. (For example, the real map $f(x) := \frac{1}{2}\sin 100x$ maps the real interval $(-1, 1)$ analytically inside a compact subset of $(-1, 1)$, but there can be no bounded open subset U of $\mathbb{C}$ which contains $(-1, 1)$ and is mapped inside a compact subset of itself by f in the same manner: if such a set U existed then by the Earle-Hamilton fixed point theorem f would have a unique fixed point in U, which is manifestly false.) In order to obtain this stronger property we apply the theory of complex cones and gauges introduced by H. H. Rugh in [56] and extended by L. Dubois in [15].

Once the system of complex neighbourhoods underlying the domain of $\mathscr{L}_s$ has been constructed, we review in § the properties of trace-class operators which will be needed in this article, and prove suitable extensions of some standard results in view of the fact that we will be working with space of holomorphic functions defined on a non-connected region. We then proceed in § to establish the properties of the operator $\mathscr{L}_s$ and deduce Theorem 4. In § we derive Theorem 3 from Theorem 4 above. Some examples of the application of Theorem 3 are presented in §. In § we consider the problem of calculating the affinity dimension in situations where the hypotheses of Theorem 3 do not apply. We remark that sections – depend only on the statement of Theorem 4 and the material presented in sections 1 and 2 and as such may be read independently of sections – in which the proof of Theorem 4 is prepared for and presented.

3. The projective action on complex cones

Our first task in proving Theorem 3 is to translate the matter from the context of linear maps between cones to the context of holomorphic maps between complex domains. For this task we will use the machinery of complex cones and gauges introduced by H. H. Rugh and extended by L. Dubois [15, 56]. Given a nonempty compact set $\mathsf{A} \subset M_d(\mathbb{R})$ we let $\mathcal{S}(\mathsf{A})$ denote the semigroup generated by A. In the applications considered in this article we will consider only the finitely-generated semigroup $\mathcal{S}(\{A_1^{\wedge\ell}, \ldots, A_N^{\wedge\ell}\}) = \{A_{\mathtt{i}}^{\wedge\ell} \colon \mathtt{i} \in \Sigma_N^*\}$ for some $\ell \in \{1, \ldots, d-1\}$, but it is almost as easy to study the compactly-generated case and we include this case for possible use in future research.

Here and throughout this article we shall say that if U_1, U_2 are subsets of a metric space X then U_1 is *compactly contained in* U_2 if the closure of U_1 is a compact subset of the interior of U_2. We express this relation by writing $U_1 \Subset U_2$.

The results to be proved in this section are summarised in the following theorem:

Theorem 5. *Let $d \geq 1$, let $\mathsf{A} \subset M_d(\mathbb{R})$ be compact and nonempty and suppose that $(\mathcal{K}_1, \ldots, \mathcal{K}_m)$ is a multicone for A with transverse-defining vector w. For each $j = 1, \ldots, m$ define*

$$\mathcal{K}_j^{\mathbb{C}} := \{\lambda((u+v) + i(u-v)) \colon \lambda \in \mathbb{C} \text{ and } u, v \in \mathcal{K}_j\},$$

and let

$$\Omega := \left\{ z \in \mathbb{C}^d \colon z \in \bigcup_{j=1}^m \operatorname{Int} \mathcal{K}_j^{\mathbb{C}} \text{ and } \langle z, w \rangle = 1 \right\}.$$

For each $A \in \mathcal{S}(\mathsf{A})$ and $z \in \Omega$ let us write $\overline{A}z := \langle Az, w\rangle^{-1} Az$. Then:

(i) Every $A \in \mathcal{S}(\mathsf{A})$ has a simple leading eigenvalue $\lambda_1(A)$ which is real and is strictly larger in modulus than all of the other eigenvalues of A.

(ii) There is a constant $\tau > 0$ such that $\|A_1A_2\| \geq \tau\|A_1\| \cdot \|A_2\|$ for every $A_1, A_2 \in \mathcal{S}(\mathsf{A})$.

(iii) For every $z \in \Omega$ and $A \in \mathcal{S}(\mathsf{A})$ we have $\Re(\langle Az, w\rangle) \neq 0$.

(iv) Ω is a nonempty, open, bounded subset of the complex hyperplane $\{z \in \mathbb{C}^d \colon \langle z, w\rangle = 1\}$, and for every $A \in \mathcal{S}(\mathsf{A})$ the map $\overline{A} \colon \Omega \to \Omega$ is well-defined.

(v) There exist constants $C, \gamma > 0$ such that

$$\sup_{A_1, \ldots, A_n \in \mathsf{A}} \operatorname{diam} \overline{A_1 \cdots A_n}(\Omega) \leq Ce^{-\gamma n}$$

for every $n \geq 1$.

(vi) For each $A \in \mathcal{S}(\mathsf{A})$ the map $\overline{A} \colon \Omega \to \Omega$ has a unique fixed point $z_A \in \Omega$. We have $z_A \in \Omega \cap \mathbb{R}^d$ and $\langle Az_A, w\rangle = \lambda_1(A)$. The eigenvalues of the derivative $D_{z_A}\overline{A}$ are precisely the numbers $\lambda_j(A)/\lambda_1(A)$ for $j = 2, \ldots, d$, and in particular

$$\det(I - D_{z_A}\overline{A}) = \frac{p_A'(\lambda_1(A))}{\lambda_1(A)^{d-1}} \neq 0$$

where $p_A(x) := \det(xI - A)$ denotes the characteristic polynomial of A and p_A' its first derivative.

(vii) There is a constant $C > 0$ such that for each $A \in \mathcal{S}(\mathsf{A})$ and $z \in \Omega$ we have $C^{-1}\|A\| \leq |\langle Az, w\rangle| \leq C\|A\|$.

(viii) The set $\bigcup_{A \in \mathcal{S}(\mathsf{A})} \overline{A}(\Omega)$ is compactly contained in Ω.

Theorem 5 is trivial in the case $d = 1$ and for the remainder of this section we shall ignore this case, assuming at all times that $d \geq 2$. (When $d = 1$ the determinant in (vi) above will be interpreted as being equal to 1.) Here and throughout the remainder of this article we use the notation z^* to denote the complex conjugate of $z \in \mathbb{C}$ and reserve the notation $\overline{z}$ and $\overline{A}$ for projective quantities. We will establish some preliminary results towards the proof of Theorem 5 in the following three subsections and then combine them to give the proof itself at the end of this section.

Let us define an $\mathbb{R}$-*cone* to be a closed, convex set $\mathcal{K} \subseteq \mathbb{R}^d$ with nonempty interior such that $\lambda\mathcal{K} = \mathcal{K}$ for all $\lambda > 0$, and such that $\mathcal{K} \cap -\mathcal{K} = \{0\}$. (This definition is slightly more restrictive than that used in Rugh's article [56] in that we require $\mathbb{R}$-cones to have interior and Rugh does not.) If $\mathcal{K} \subseteq \mathbb{R}^d$ is an $\mathbb{R}$-cone, we recall that the *dual cone* is defined to be the set

$$\mathcal{K}' := \left\{v \in \mathbb{R}^d \colon \langle u, v\rangle \geq 0 \text{ for all } u \in \mathcal{K}\right\}.$$

Obviously $\mathcal{K}'$ is closed and convex and satisfies $\lambda\mathcal{K}' = \mathcal{K}'$ for all real $\lambda > 0$. We remark that the dual cone of an $\mathbb{R}$-cone (in our sense of an $\mathbb{R}$-cone) is also an $\mathbb{R}$-cone, but this fact will not be required.

It is easier to provide a proof of the following elementary separation lemma than to find a crisp reference:

Lemma 3.1. *Let $\mathcal{K} \subset \mathbb{R}^d$ be an $\mathbb{R}$-cone and $w \in \mathbb{R}^d \setminus \mathcal{K}$. Then there exists $v \in \mathcal{K}'$ such that $\langle w, v\rangle < 0$.*

Proof. By a trivial compactness argument there exists $u_0 \in \mathcal{K}$ which minimises the distance $\|u_0 - w\|$. We claim that $\langle u_0, u_0 - w\rangle = 0$. The definitions of u_0 and of $\mathcal{K}$ together imply that the function $\lambda \mapsto \|\lambda u_0 - w\|^2$ defined for $\lambda \geq 0$ has a local minimum at $\lambda = 1$. Differentiating $\|\lambda u_0 - w\|^2 = \lambda^2\|u_0\|^2 - 2\lambda\langle u_0, w\rangle + \|w\|^2$ at $\lambda = 1$ yields $2\|u_0\|^2 - 2\langle u_0, w\rangle = 0$ since the derivative must be zero at a local minimum. It follows that $\langle u_0, u_0 - w\rangle = \|u_0\|^2 - \langle u_0, w\rangle = 0$ as claimed. Since $\langle u_0 - w, u_0 - w\rangle > 0$ we deduce that $\langle w, u_0 - w\rangle < 0$.

If $u \in \mathcal{K}$, $\lambda \in (0, 1)$ are arbitrary then $u_0 + \lambda(u - u_0) \in \mathcal{K}$ and therefore $\|u_0 - w\|^2 \leq \|u_0 + \lambda(u - u_0) - w\|^2 = \|u_0 - w\|^2 + \lambda\langle u - u_0, u_0 - w\rangle + \lambda^2\|u - u_0\|^2$. Hence $\langle u - u_0, u_0 - w\rangle + \lambda\|u - u_0\|^2 \geq 0$ for all $\lambda \in (0, 1)$ and it follows that $\langle u - u_0, u_0 - w\rangle \geq 0$ for all $u \in \mathcal{K}$. Applying $\langle u_0, u_0 - w\rangle = 0$ yields $\langle u, u_0 - w\rangle \geq 0 > \langle w, u_0 - w\rangle$ for all $u \in \mathcal{K}$ and the result follows by taking $v := u_0 - w$. □

The following result is elementary, but its proof illuminates the version for complex cones which will shortly follow.

Lemma 3.2. *Let $\mathcal{K} \subseteq \mathbb{R}^d$ be a closed convex cone. Then*

$$\operatorname{Int}\mathcal{K} = \left\{u \in \mathbb{R}^d \colon \langle u, v\rangle > 0 \textit{ for all nonzero } v \in \mathcal{K}'\right\}.$$

Proof. If $\langle u,v\rangle > 0$ for all nonzero $v \in \mathcal{K}'$ then $u \in \mathcal{K}$ since otherwise Lemma 3.1 would be contradicted. Clearly $\langle u,v\rangle > 0$ for all nonzero $v \in \mathcal{K}'$ if and only if $\langle u,v\rangle > 0$ for all $v \in \mathcal{K}'$ such that $\|v\| = 1$, if and only if

$$\inf_{\substack{v\in\mathcal{K}' \\ \|v\|=1}} \langle u,v\rangle > 0;$$

but this quantity is locally Lipschitz continuous in u, so the set of all $u \in \mathbb{R}^d$ such that $\langle u,v\rangle > 0$ for all nonzero $v \in \mathcal{K}'$ is an open subset of $\mathcal{K}$ and in particular is contained in $\operatorname{Int}\mathcal{K}$. We now claim that every point *not* in this set does not belong to $\operatorname{Int}\mathcal{K}$. Indeed, suppose $u \in \mathbb{R}^d$ and $\langle u,v\rangle \le 0$ for some nonzero $v \in \mathcal{K}'$. We have $\langle u-\varepsilon v, v\rangle = \langle u,v\rangle - \varepsilon\|v\|^2 < 0$ for every $\varepsilon > 0$, so u is an accumulation point of the complement of $\mathcal{K}$ and hence is not in $\operatorname{Int}\mathcal{K}$. □

Proposition 3.3. *Let $\mathsf{A} \subset M_d(\mathbb{R})^N$ be compact and nonempty and suppose that $(\mathcal{K}_1,\ldots,\mathcal{K}_m)$ is a multicone for A with transverse-defining vector w. Then there exist constants $\tau_1,\tau_2,\tau_3 > 0$ such that:*

(i) For every $u \in \bigcup_{j=1}^m \mathcal{K}_j$ and every $A \in \mathcal{S}(\mathsf{A})$ we have $\|Au\| \ge \tau_1\|A\|\cdot\|u\|$.
(ii) For every $u \in \bigcup_{j=1}^m \mathcal{K}_j$ and every $A \in \mathcal{S}(\mathsf{A})$ we have $|\langle Au,w\rangle| \ge \tau_2\|A\|\cdot\|u\|$.
(iii) For every $A_1,A_2 \in \mathcal{S}(\mathsf{A})$ we have $\|A_1A_2\| \ge \tau_3\|A_1\|\cdot\|A_2\|$.

Proof. We will prove (i) first and then deduce (ii) and (iii). Since A is compact and each $A \in \mathsf{A}$ maps $\bigcup_{j=1}^m \mathcal{K}_j \setminus \{0\}$ into $\bigcup_{j=1}^m(\operatorname{Int}\mathcal{K}_j \cup -\operatorname{Int}\mathcal{K}_j)$, by slightly enlarging each $\mathcal{K}_j$ we may find new $\mathbb{R}$-cones $\mathcal{K}_1^+,\ldots,\mathcal{K}_m^+$ such that we still have $A\left(\bigcup_{j=1}^m \mathcal{K}_j^+\right) \subseteq \bigcup_{j=1}^m(\mathcal{K}_j \cup -\mathcal{K}_j)$ for all $A \in \mathsf{A}$, but such that additionally $\bigcup_{j=1}^m \mathcal{K}_j \setminus \{0\} \subseteq \bigcup_{j=1}^m \operatorname{Int}\mathcal{K}_j^+$. Define

$$\mathfrak{S} := \left\{ B \in M_d(\mathbb{R})\colon B\left(\bigcup_{j=1}^m \mathcal{K}_j^+\right) \subseteq \bigcup_{j=1}^m (\mathcal{K}_j \cup -\mathcal{K}_j) \right\}.$$

We note that $\mathfrak{S}$ is a closed subsemigroup of $M_d(\mathbb{R})$ which includes A, so in particular it contains $\mathcal{S}(\mathsf{A})$.

We claim that there exists $\tau_1 > 0$ such that $\|Bu\| \ge \tau_1\|B\|\cdot\|u\|$ for every $B \in \mathfrak{S}$ and $u \in \bigcup_{k=1}^m \mathcal{K}_j$. By homogeneity it is sufficient to consider only the case where $\|B\| = \|u\| = 1$. By compactness it is in turn sufficient to show that $\|Bu\| > 0$ whenever $B \in \mathfrak{S}$, $u \in \bigcup_{k=1}^m \mathcal{K}_j$ and $\|B\| = \|u\| = 1$. Suppose for a contradiction that there exist $B \in \mathfrak{S}$ and $u \in \bigcup_{j=1}^m \mathcal{K}_j$ such that $\|B\| = \|u\| = 1$ and $Bu = 0$. Let $u \in \mathcal{K}_{j_1}$, say. Since $\|B\| \ne 0$ there exists $v \in \mathbb{R}^d$ such that $Bv \ne 0$. Note that since u is nonzero we have $u \in \operatorname{Int}\mathcal{K}_{j_1}^+$, and in particular there exists $\varepsilon > 0$ such that $u+\varepsilon v$ and $u-\varepsilon v$ both belong to $\mathcal{K}_{j_1}^+$.

Since $B\mathcal{K}_{j_1}^+ \subseteq \bigcup_{j=1}^m(\mathcal{K}_j \cup -\mathcal{K}_j)$ and the sets $(\mathcal{K}_j \cup -\mathcal{K}_j) \setminus \{0\}$ are disjoint for distinct values of j, the nonzero vector Bv belongs to $\mathcal{K}_j \cup -\mathcal{K}_j$ for some unique value of j, say j_2. In particular $B\mathcal{K}_{j_1}^+ \cap ((\mathcal{K}_{j_2} \cup -\mathcal{K}_{j_2}) \setminus \{0\})$ is nonempty since it contains $B(u+\varepsilon v) = \varepsilon Bv$. Since $B\mathcal{K}_{j_1}^+$ is connected it can intersect at most one of the pairwise disjoint sets $(\mathcal{K}_j \cup -\mathcal{K}_j) \setminus \{0\}$, so we have either $B\mathcal{K}_{j_1}^+ \subseteq \mathcal{K}_{j_2}$ or $B\mathcal{K}_{j_1}^+ \subseteq -\mathcal{K}_{j_2}$. If the former, we find that

$$\varepsilon Bv = B(u + \varepsilon v) \in B\mathcal{K}_{j_1}^+ \subseteq \mathcal{K}_{j_2}$$

and

$$-\varepsilon Bv = B(u - \varepsilon v) \in B\mathcal{K}_{j_1}^+ \subseteq \mathcal{K}_{j_2}$$

which implies that $\mathcal{K}_{j_2} \cap -\mathcal{K}_{j_2} \ni \varepsilon Bv \neq 0$, contradicting the statement that $\mathcal{K}_{j_2}$ is an $\mathbb{R}$-cone; if the latter, the same result holds by the same argument modulo some appropriate changes of sign. We conclude that there must exist $\tau_1 > 0$ such that $\|Bu\| \geq \tau_1 \|B\| \cdot \|u\|$ for every $B \in \mathfrak{S}$ and $u \in \bigcup_{k=1}^m \mathcal{K}_j$, and since $\mathcal{S}(\mathsf{A}) \subseteq \mathfrak{S}$ this completes the proof of (i).

To deduce (ii), we note that by (i) it is sufficient to find a constant $\tilde{\tau} > 0$ such that $|\langle Au, w\rangle| \geq \tilde{\tau}\|Au\|$ for every $A \in \mathcal{S}(\mathsf{A})$ and $u \in \bigcup_{j=1}^m \mathcal{K}_j$. To obtain this inequality it clearly in turn suffices to show that $|\langle v, w\rangle| \geq \tilde{\tau}\|v\|$ for all $v \in \bigcup_{j=1}^m \mathcal{K}_j$. By homogeneity it is sufficient to consider only unit vectors $v \in \bigcup_{j=1}^m \mathcal{K}_j$. Since the intersection of the unit sphere of $\mathbb{R}^d$ with $\bigcup_{j=1}^m \mathcal{K}_j$ is compact we may achieve this by showing that $|\langle v, w\rangle| \neq 0$ for every unit vector $v \in \bigcup_{j=1}^m \mathcal{K}_j$, but this is true by Definition 1.1(ii). We deduce (ii). To see that (iii) also follows, let $u_0 \in \mathcal{K}_1$ be an arbitrary unit vector. We have

$$\|A_1A_2\| \geq \|A_1A_2u_0\| \geq \tau_1\|A_1\| \cdot \|A_2u_0\| \geq \tau_1^2\|A_1\| \cdot \|A_2\| \cdot \|u_0\| = \tau_1^2\|A_1\| \cdot \|A_2\|$$

which yields (iii). □

Following H. H. Rugh [56] we shall say that a set $\mathsf{K} \subseteq \mathbb{C}^d$ is a $\mathbb{C}$-*cone* if it is nonempty and closed, satisfies $\lambda\mathsf{K} = \mathsf{K}$ for all nonzero $\lambda \in \mathbb{C}$, and contains no 2-dimensional subspace of $\mathbb{C}^d$. Following [56, §5] we define the *complexification* of an $\mathbb{R}$-cone $\mathcal{K} \subseteq \mathbb{R}^d$ to be the set

$$\mathcal{K}^{\mathbb{C}} := \left\{z \in \mathbb{C}^d \colon \Re\left(\langle z, w_1\rangle\langle z, w_2\rangle^*\right) \geq 0 \text{ for all } w_1, w_2 \in \mathcal{K}'\right\}$$

where ω^* denotes the complex conjugate of ω. The set $\mathcal{K}^{\mathbb{C}}$ admits the alternative characterisation

$$\mathcal{K}^{\mathbb{C}} = \{\lambda((u+v) + i(u-v)) \colon \lambda \in \mathbb{C} \text{ and } u, v \in \mathcal{K}\}$$

which is proved in [56, Proposition 5.2]. By replacing u and v with $|\lambda|u$ and $|\lambda|v$ if necessary we may clearly assume if desired that $\lambda = e^{i\theta}$ for some $\theta \in \mathbb{R}$. The complexification $\mathcal{K}^{\mathbb{C}}$ of an $\mathbb{R}$-cone is always a $\mathbb{C}$-cone [56, Theorem 5.5].

In this subsection we will prove some elementary geometric and topological properties of complexifications which will be useful in the following two subsections.

Lemma 3.4. *Let $\mathcal{K} \subseteq \mathbb{R}^d$ be an $\mathbb{R}$-cone and suppose that $w \in \mathbb{R}^d$ satisfies $\langle u, w\rangle > 0$ for all nonzero $u \in \mathcal{K}$. Then $\langle z, w\rangle \neq 0$ for all nonzero $z \in \mathcal{K}^{\mathbb{C}}$.*

Proof. For a contradiction take $z = \lambda((u+v)+i(u-v)) \in \mathcal{K}^{\mathbb{C}}$ where $u, v \in \mathcal{K}$, $z \neq 0$ and $\langle z, w\rangle = 0$. We have $\langle \lambda^{-1}z, w\rangle = 0$ which is to say $\langle (u+v) + i(u-v), w\rangle = 0$, but then $\langle u+v, w\rangle = 0$ which implies that $\langle u, w\rangle = -\langle v, w\rangle$ with $u, v \in \mathcal{K}$. If $u \neq 0$ then $\langle u, w\rangle > 0$ which implies that $\langle v, w\rangle < 0$ with $v \in \mathcal{K}$, a contradiction.

It follows that $u = 0$ and therefore $\langle v, w\rangle = 0$ which implies $v = 0$ and therefore $z = 0$, which is also a contradiction. The proof is complete. $\square$

Lemma 3.5. *Let $\mathsf{K} \subseteq \mathbb{C}^d$ be a $\mathbb{C}$-cone and let $w \in \mathbb{C}^d$ be a unit vector such that $\langle z, w\rangle \neq 0$ for all nonzero $z \in \mathsf{K}$. Then there exists $C > 0$ such that $\|z\| \leq C|\langle z, w\rangle|$ for all $z \in \mathsf{K}$.*

Proof. We will show that there exists $\tau > 0$ such that $|\langle z, w\rangle| \geq \tau\|z\|$ for all $z \in \mathsf{K}$. By homogeneity it is clearly sufficient to consider only those cases where $\|z\| = 1$. By compactness it in turn suffices to show that we cannot simultaneously have $\|z\| = 1$, $z \in \mathsf{K}$ and $|\langle z, w\rangle| = 0$, but this is true by hypothesis. The result follows. $\square$

The following complex version of Lemma 3.2 seems not to have been previously remarked.

Lemma 3.6. *Let $\mathcal{K} \subseteq \mathbb{R}^d$ be an $\mathbb{R}$-cone. Then*

$$\operatorname{Int}\mathcal{K}^{\mathbb{C}} = \left\{z \in \mathbb{C}^d \colon \Re(\langle z, w_1\rangle\langle z, w_2\rangle^*) > 0 \text{ for all nonzero } w_1, w_2 \in \mathcal{K}'\right\}.$$

Proof. It is clear that if z belongs to the set

$$\left\{z \in \mathbb{C}^d \colon \Re(\langle z, w_1\rangle\langle z, w_2\rangle^*) > 0 \text{ for all nonzero } w_1, w_2 \in \mathcal{K}'\right\} \tag{3}$$

then

$$\inf_{\substack{w_1, w_2 \in \mathcal{K}' \\ \|w_1\| = \|w_2\| = 1}} \Re(\langle z, w_1\rangle\langle z, w_2\rangle^*) > 0$$

by continuity and compactness. It follows easily that every sufficiently small perturbation of z belongs to the same set and in particular belongs to $\mathcal{K}^{\mathbb{C}}$, so the set (3) is contained in the interior of $\mathcal{K}^{\mathbb{C}}$.

We now wish to show that if $z \in \mathcal{K}^{\mathbb{C}}$ and $\Re(\langle z, w_1\rangle\langle z, w_2\rangle^*) = 0$ for some nonzero $w_1, w_2 \in \mathcal{K}'$ then z is not an interior point of $\mathcal{K}^{\mathbb{C}}$. We consider three cases. Suppose firstly that $z \in \mathcal{K}^{\mathbb{C}}$ is equal to 0. If $z = 0 \in \operatorname{Int}\mathcal{K}^{\mathbb{C}}$ then by homogeneity $\mathbb{C}^d \subseteq \mathcal{K}^{\mathbb{C}}$ and therefore $\mathcal{K}^{\mathbb{C}}$ contains a two-dimensional subspace of $\mathbb{C}^d$, which is impossible since $\mathcal{K}^{\mathbb{C}}$ is a $\mathbb{C}$-cone. It follows that $0 \notin \operatorname{Int}\mathcal{K}^{\mathbb{C}}$.

Suppose secondly that $z \in \mathcal{K}^{\mathbb{C}}$ and $\langle z, w_1\rangle = 0$ for some nonzero $w_1 \in \mathcal{K}'$, but $z \neq 0$. Without loss of generality we assume $\|w_1\| = 1$. We claim that there exists nonzero $w_2 \in \mathcal{K}'$ such that $\langle z, w_2\rangle \neq 0$. Let us write $z_2 = \lambda((u+v) + i(u-v))$ where $\lambda \in \mathbb{C}$ and $u, v \in \mathcal{K}$. If we have $u + v = 0$ then $u = -v \in \mathcal{K} \cap -\mathcal{K} = \{0\}$ which implies $z = 0$, so $u + v \in \mathcal{K}$ must be nonzero. It follows that $-u - v \notin \mathcal{K}$ and therefore $\langle -u - v, w_2\rangle < 0$ for some $w_2 \in \mathcal{K}'$ by Lemma 3.1, where we may freely assume $\|w_2\| = 1$. In particular $\Re(\langle \lambda^{-1}z, w_2\rangle) = \langle u + v, w_2\rangle \neq 0$ which suffices to prove the claim. Now choose $\theta \in \mathbb{R}$ such that $e^{i\theta}\langle z, w_2\rangle^* = -|\langle z, w_2\rangle|$. For every $\varepsilon > 0$ we have

$$\Re(\langle z + \varepsilon e^{i\theta}w_1, w_1\rangle\langle z + \varepsilon e^{i\theta}w_1, w_2\rangle^*) = -\varepsilon|\langle z, w_2\rangle| + \varepsilon^2\langle w_1, w_2\rangle$$

and this implies that $z+\varepsilon e^{i\theta}w_1 \notin \mathcal{K}^{\mathbb{C}}$ for all sufficiently small $\varepsilon > 0$, so in particular $z \notin \operatorname{Int}\mathcal{K}^{\mathbb{C}}$.

Consider finally the case in which $z \in \mathcal{K}^{\mathbb{C}}$, $\langle z, w\rangle \neq 0$ for all nonzero $w \in \mathcal{K}'$, and $\Re(\langle z, w_1\rangle\langle z, w_2\rangle^*) = 0$ for some $w_1, w_2 \in \mathcal{K}'$ where we assume that $\|w_1\| = \|w_2\| = 1$. We in particular have $\langle z, w_1\rangle\langle z, w_2\rangle^* \neq 0$. By interchanging w_1 with w_2 if necessary we assume that $|\langle z, w_2\rangle| \geq |\langle z, w_1\rangle| > 0$. If $|\langle w_1, w_2\rangle| = 1$ then $w_1 = \pm w_2$ so that $\Re(\langle z, w_1\rangle\langle z, w_2\rangle^*) = \pm|\langle z, w_1\rangle|^2 \neq 0$, a contradiction, so necessarily $|\langle w_1, w_2\rangle| < 1$. Choose $\theta \in \mathbb{R}$ such that $e^{i\theta}\langle z, w_2\rangle^* = -|\langle z, w_2\rangle|$. We have

$$\begin{aligned}
&\Re\left(\left\langle z+\varepsilon e^{i\theta}w_1, w_1\right\rangle \left\langle z+\varepsilon e^{i\theta}w_1, w_2\right\rangle^*\right)\\
=\;& \Re\left(\left\langle \varepsilon e^{i\theta}w_1, w_1\right\rangle \langle z, w_2\rangle^*\right) + \Re\left(\langle z, w_1\rangle \left\langle \varepsilon e^{i\theta}w_1, w_2\right\rangle^*\right)\\
&+\Re\left(\left\langle \varepsilon e^{i\theta}w_1, w_1\right\rangle \left\langle \varepsilon e^{i\theta}w_1, w_2\right\rangle^*\right)\\
=\;& -\varepsilon\left|\langle z, w_2\rangle\right| + \varepsilon\Re\left(e^{-i\theta}\langle z, w_1\rangle\right)\langle w_1, w_2\rangle + \varepsilon^2\langle w_1, w_2\rangle.
\end{aligned}$$

Since $|\langle z, w_2\rangle| \geq |\langle z, w_1\rangle| > |e^{i\theta}\langle z, w_1\rangle\langle w_1, w_2\rangle|$ this quantity is negative for all sufficiently small $\varepsilon > 0$. It follows that z is an accumulation point of the complement of $\mathcal{K}^{\mathbb{C}}$ and as claimed is not interior to $\mathcal{K}^{\mathbb{C}}$. The proof is complete. □

Corollary 3.7. *Let $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathbb{R}^d$ be $\mathbb{R}$-cones and let $A \in M_d(\mathbb{R})$ such that $A(\mathcal{K}_1 \setminus \{0\}) \subseteq \operatorname{Int}\mathcal{K}_2 \cup -\operatorname{Int}\mathcal{K}_2$. Then $A(\mathcal{K}_1^{\mathbb{C}} \setminus \{0\}) \subseteq \operatorname{Int}\mathcal{K}_2^{\mathbb{C}}$.*

Proof. By connectedness of $\mathcal{K}_1 \setminus \{0\}$ we either have $A(\mathcal{K}_1 \setminus \{0\}) \subseteq \operatorname{Int}\mathcal{K}_2$ or $A(\mathcal{K}_1 \setminus \{0\}) \subseteq -\operatorname{Int}\mathcal{K}_2$. In the former case we have $\langle Au, w\rangle > 0$ for every nonzero $u \in \mathcal{K}_1$ and $w \in \mathcal{K}_2'$ by Lemma 3.2, and in the latter case we have $\langle Au, w\rangle < 0$ for every nonzero $u \in \mathcal{K}_1$ and $w \in \mathcal{K}_2'$. In either case the sign of $\langle Au, w\rangle$ is constant with respect to the choice of nonzero $u \in \mathcal{K}_1$ and $w \in \mathcal{K}_2'$, and is never zero for any such u and w.

If $z \in \mathcal{K}_1^{\mathbb{C}}$ is nonzero and $w_1, w_2 \in \mathcal{K}_2'$ are arbitrary nonzero vectors, we must by Lemma 3.6 show that $\Re(\langle Az, w_1\rangle\langle Az, w_2\rangle^*) > 0$. Since this condition is unaffected by the substitution $z \mapsto e^{i\theta}z$ we may assume that $z \in \mathcal{K}_1^{\mathbb{C}}$ has the form $z = u + v + i(u - v)$ where $u, v \in \mathcal{K}_1$. We then have

$$\begin{aligned}
\Re(\langle Az, w_1\rangle\langle Az, w_2\rangle^*) =&\langle Au + Av, w_1\rangle\langle Au + Av, w_2\rangle\\
&+ \langle Au - Av, w_1\rangle\langle Au - Av, w_2\rangle\\
=&2\langle Au, w_1\rangle\langle Au, w_2\rangle + 2\langle Av, w_1\rangle\langle Av, w_2\rangle > 0
\end{aligned}$$

since at least one of u and v is nonzero and since $\langle Au, w_1\rangle$, $\langle Au, w_2\rangle$, $\langle Av, w_1\rangle$ and $\langle Av, w_2\rangle$ cannot have mixed positive and negative signs. It follows that $Az \in \operatorname{Int}\mathcal{K}_2^{\mathbb{C}}$ as required. □

Lemma 3.8. *Let $\mathcal{K}_1, \mathcal{K}_2 \subseteq \mathbb{R}^d$ be $\mathbb{R}$-cones such that $\mathcal{K}_1 \cap \mathcal{K}_2 = \{0\}$, and suppose that there exists $w \in \mathbb{R}^d$ such that $\langle u, w\rangle > 0$ for all nonzero $u \in \mathcal{K}_1 \cup \mathcal{K}_2$. Then $\mathcal{K}_1^{\mathbb{C}} \cap \mathcal{K}_2^{\mathbb{C}} = \{0\}$.*

Proof. We observe that the hypothesis directly implies $\mathcal{K}_2 \cap (\mathcal{K}_1 \cup -\mathcal{K}_1) = \{0\}$. Suppose that $z \in \mathcal{K}_1^{\mathbb{C}} \cap \mathcal{K}_2^{\mathbb{C}}$; we wish to show that $z = 0$. Write $z = \lambda_1((u_1 + v_1) + i(u_1 - v_1)) = \lambda_2((u_2 + v_2) + i(u_2 - v_2))$ where $\lambda_1, \lambda_2 \in \mathbb{C}$ and $u_1, v_1 \in \mathcal{K}_1$, $u_2, v_2 \in \mathcal{K}_2$. If $\lambda_2 = 0$ or $\lambda_1 = 0$ then we are done; otherwise, let $\lambda_2^{-1}\lambda_1 = a + ib \neq 0$ with $a, b \in \mathbb{R}$. We have

$$(a + ib)((u_1 + v_1) + i(u_1 - v_1)) = u_2 + v_2 + i(u_2 - v_2)$$

which is to say

$$(a - b)u_1 + (a + b)v_1 = u_2 + v_2, \qquad (a + b)u_1 - (a - b)v_1 = u_2 - v_2$$

by separating real and imaginary parts, and therefore

$$u_2 = au_1 + bv_1, \qquad v_2 = -bu_1 + av_1.$$

Consider first the case in which one of a and b is strictly positive and the other strictly negative. In this case $v_2 = -bu_1 + av_1 \in \mathcal{K}_2 \cap (\mathcal{K}_1 \cup -\mathcal{K}_1) = \{0\}$ so that $v_2 = 0$. It follows that $bu_1 = av_1$, but since a and b have opposite signs this implies $bu_1, av_1 \in \mathcal{K}_1 \cap -\mathcal{K}_1 = \{0\}$. Since neither a nor b is zero it follows that $u_1 = v_1 = 0$ and therefore $z = 0$ as required.

On the other hand, if a and b are both non-negative then $au_1 + bv_1 \in \mathcal{K}_1$, and if a and b are both non-positive then $au_1 + bv_1 \in -\mathcal{K}_1$. In either case $u_2 = au_1 + bv_1 \in \mathcal{K}_2 \cap (\mathcal{K}_1 \cup -\mathcal{K}_1) = \{0\}$ and so $u_2 = 0$. This implies $au_1 + bv_1 = 0$ so $au_1 = -bv_1 \in \mathcal{K}_1 \cap -\mathcal{K}_1 = \{0\}$ and therefore $au_1 = bv_1 = 0$. If $a = 0$ then $b \neq 0$ which implies $v_1 = 0$, so $v_2 = -bu_1 \in \mathcal{K}_2 \cap (\mathcal{K}_1 \cup -\mathcal{K}_1) = \{0\}$ and we have $u_2 = v_2 = 0$ so that $z = 0$. If $a \neq 0$ then $u_1 = 0$ and therefore $v_2 = av_1 \in \mathcal{K}_2 \cap (\mathcal{K}_1 \cup -\mathcal{K}_1) = \{0\}$. We again have $u_2 = v_2 = 0$ so that $z = 0$. The proof is complete. □

In this subsection we will apply a projective version of Hilbert's metric to obtain contraction results for the complexified cones $\mathcal{K}_j^{\mathbb{C}}$. We will prefer the technique of L. Dubois [15] to that of H. H. Rugh [56] since Dubois' projective distance is more easily defined.

If $\mathsf{K} \subseteq \mathbb{C}^d$ is a $\mathbb{C}$-cone then for all $z_1, z_2 \in \mathsf{K}$ we define

$$E_{\mathsf{K}}(z_1, z_2) := \{\lambda \in \mathbb{C} \colon \lambda z_1 - z_2 \notin \mathsf{K}\}.$$

It is easy to see that $E_{\mathsf{K}}(\lambda z_1, z_2) = \lambda^{-1} E_{\mathsf{K}}(z_1, z_2)$ and $E_{\mathsf{K}}(z_1, \lambda z_2) = \lambda E_{\mathsf{K}}(z_1, z_2)$ for every nonzero $\lambda \in \mathbb{C}$, and that $\lambda \in E_{\mathsf{K}}(z_1, z_2)$ if and only if $\lambda^{-1} \in E_{\mathsf{K}}(z_2, z_1)$. Clearly we always have $0 \notin E_{\mathsf{K}}(z_1, z_2)$. Moreover, $\operatorname{Int} E_{\mathsf{K}}(z_1, z_2)$ can never be empty when $\{z_1, z_2\}$ is linearly independent: if this were possible for some z_1, z_2 then $\lambda z_1 - z_2$ would belong to K for a dense set of $\lambda \in \mathbb{C}$, which since K is closed would imply that the span of z_1 and z_2 is contained in K, contradicting the definition of a $\mathbb{C}$-cone. In particular $E_{\mathsf{K}}(z_1, z_2)$ itself is never empty when z_1 and z_2 are not colinear.

If $z \in \mathbb{C}^d$ is nonzero, let $\overline{z}$ denote the (complex) one-dimensional subspace spanned by z (which should not be confused with the complex conjugate of z, denoted in this article by z^*). Following [15] we define a notion of distance between two one-dimensional subspaces of a $\mathbb{C}$-cone $\mathsf{K} \subseteq \mathbb{C}^d$ by

$$d_{\mathsf{K}}(\overline{z_1}, \overline{z_2}) := \begin{cases} 0 & \text{if } \overline{z_1} = \overline{z_2}, \\ \log\left(\frac{\sup\{|\lambda|\colon \lambda \in E_{\mathsf{K}}(z_1,z_2)\}}{\inf\{|\lambda|\colon \lambda \in E_{\mathsf{K}}(z_1,z_2)\}}\right) & \text{otherwise.} \end{cases}$$

Note that $d_{\mathsf{K}}(\overline{z_1}, \overline{z_2})$ may take the value $+\infty$. It is easy to see that $d_{\mathsf{K}}(\overline{z_1}, \overline{z_2})$ depends only on the subspaces $\overline{z_1}, \overline{z_2}$ and not on the choice of spanning vector z_1 or z_2 within those subspaces. The function d_{K} is easily seen to satisfy $d_{\mathsf{K}}(\overline{z_1}, \overline{z_2}) = d_{\mathsf{K}}(\overline{z_2}, \overline{z_1})$, and $d_{\mathsf{K}}(\overline{z_1}, \overline{z_2}) \geq 0$ with equality if and only if $\overline{z_1} = \overline{z_2}$. (If we had $d_{\mathsf{K}}(\overline{z_1}, \overline{z_2}) = 0$ with $\overline{z_1} \neq \overline{z_2}$ then $E_{\mathsf{K}}(z_1, z_2)$ would be a subset of a circle in $\mathbb{C}$ and hence have empty interior, which we noted previously to be impossible.) However, even where it takes finite values d_{K} is not in general a metric, since the triangle inequality is not satisfied for certain $\mathbb{C}$-cones K (see [15, Remark 6]). We shall nonetheless write $\operatorname{diam}_{\mathsf{K}} X := \sup\{d_{\mathsf{K}}(\overline{z_1}, \overline{z_2})\colon \overline{z_1}, \overline{z_2} \in X\}$ in the same manner as we would if d_{K} were a *bona fide* metric. By abuse of notation if $X \subseteq \mathbb{C}^d$ is an arbitrary set then we shall also write $\operatorname{diam}_{\mathsf{K}} X$ for the diameter of the set of all one-dimensional subspaces generated by nonzero elements of X.

The following is a modification of part of [15, Lemma 2.2]:

Lemma 3.9. *Let $\mathsf{K} \subseteq \mathbb{C}^d$ be a $\mathbb{C}$-cone and $w \in \mathbb{C}^d$ a unit vector such that $\langle z, w\rangle \neq 0$ for all nonzero $z \in \mathsf{K}$. Then there exists $C > 0$ such that for all nonzero $z_1, z_2 \in \mathsf{K}$*

$$\left\| \frac{z_1}{\langle z_1, w\rangle} - \frac{z_2}{\langle z_2, w\rangle} \right\| \leq C d_{\mathsf{K}}(\overline{z_1}, \overline{z_2}).$$

Proof. Let $z_1, z_2 \in \mathsf{K}$ be nonzero. If z_1 and z_2 are colinear or if $d_{\mathsf{K}}(\overline{z_1}, \overline{z_2}) = +\infty$ then the result is trivial, so we assume otherwise. By homogeneity it is clearly sufficient to consider the case in which $\langle z_1, w\rangle = \langle z_2, w\rangle = 1$. Choose $a \in (0, \inf E_{\mathsf{K}}(z_1, z_2))$ and $b \in (\sup E_{\mathsf{K}}(z_1, z_2), +\infty)$ which is possible since $d_{\mathsf{K}}(\overline{z_1}, \overline{z_2}) < +\infty$. Since $z_1 - z_2 \neq 0$ and $\langle z_1 - z_2, w\rangle = 0$ we have $z_1 - z_2 \notin \mathsf{K}$ so in particular $1 \in E_{\mathsf{K}}(z_1, z_2)$ and therefore $a < 1 < b$. Let $C_1 > 0$ be the constant given by Lemma 3.5. Since $bz_1 - z_2, az_1 - z_2 \in \mathsf{K}$ we may estimate

$$\begin{aligned} \|z_1 - z_2\| &= \left\| \frac{1-a}{b-a}(bz_1 - z_2) - \frac{1-b}{b-a}(az_1 - z_2) \right\| \\ &\leq 2C_1 \frac{(1-a)(b-1)}{b-a} = 2C_1 \frac{b + a - 1 - ab}{b-a} \\ &\leq 2C_1 \frac{b + a - 2\sqrt{ab}}{b-a} = 2C_1 \frac{\sqrt{b} - \sqrt{a}}{\sqrt{b} + \sqrt{a}} \\ &= 2C_1 \frac{\sqrt{b/a} - 1}{\sqrt{b/a} + 1} = 2C_1 \tanh\left(\frac{\log b/a}{4}\right) < \frac{C_1}{2} \log b/a \end{aligned}$$

using the arithmetic-geometric mean inequality and the elementary estimate $\tanh t < t$ for real $t > 0$. Since $a < \inf E_{\mathsf{K}}(z_1, z_2)$ and $b > \sup E_{\mathsf{K}}(z_1, z_2)$ were arbitrary the result follows. $\square$

The following result is a special case of [15, Theorem 2.3]:

Proposition 3.10 ([15]). *Let $\mathsf{K}_1, \mathsf{K}_2 \subseteq \mathbb{C}^d$ be $\mathbb{C}$-cones, let $A \in M_d(\mathbb{C})$ and suppose that $A(\mathsf{K}_1 \setminus \{0\}) \subseteq \mathsf{K}_2 \setminus \{0\}$ and that $\Delta := \operatorname{diam}_{\mathsf{K}_2} A\mathsf{K}_1$ is finite. Then*

$$d_{\mathsf{K}_2}(\overline{Az_1}, \overline{Az_2}) \le \left(\tanh \frac{\Delta}{4}\right) d_{\mathsf{K}_1}(\overline{z_1}, \overline{z_2})$$

for all nonzero $z_1, z_2 \in \mathsf{K}_2$.

The key result of this subsection is the following:

Proposition 3.11. *Let $\mathsf{A} \subset M_d(\mathbb{R})$ be compact and nonempty and suppose that $(\mathcal{K}_1, \ldots, \mathcal{K}_m)$ is a multicone for A with transverse-defining vector w. Then there exist $C, \gamma > 0$ such that for every $n \ge 1$*

$$\left\| \frac{A_n \cdots A_1 z_1}{\langle A_n \cdots A_1 z_1, w\rangle} - \frac{A_n \cdots A_1 z_2}{\langle A_n \cdots A_1 z_2, w\rangle} \right\| \le Ce^{-\gamma n}$$

for every nonzero $z_1, z_2 \in \bigcup_{j=1}^m \mathcal{K}_j^{\mathbb{C}}$ and every $A_1, \ldots, A_n \in \mathsf{A}$.

Proof. For each $A \in \mathsf{A}$ and $j = 1, \ldots, m$ there exists $\ell = \ell(A, j)$ such that $A(\mathcal{K}_j \setminus \{0\}) \subseteq (\operatorname{Int} \mathcal{K}_\ell) \cup (-\operatorname{Int} \mathcal{K}_\ell)$, and in particular by Corollary 3.7 we have $A(\mathcal{K}_j^{\mathbb{C}} \setminus \{0\}) \subset \operatorname{Int} \mathcal{K}_\ell^{\mathbb{C}}$ for this value of ℓ. We claim that there exists $\Delta > 0$ such that

$$\operatorname{diam}_{\mathcal{K}^{\mathbb{C}}_{\ell(A,j)}} A\mathcal{K}_j^{\mathbb{C}} \le \Delta < \infty$$

independently of $A \in \mathsf{A}$ and $j \in \{1, \ldots, m\}$.

To this end fix $j, \ell \in \{1, \ldots, m\}$ and define

$$X_{j,\ell} := \left\{ Az \colon z \in \mathcal{K}_j^{\mathbb{C}},\ Az \in \mathcal{K}_\ell^{\mathbb{C}},\ A \in \mathsf{A} \text{ and } \langle Az, w\rangle = 1 \right\}.$$

We assert that $X_{j,\ell}$ is compact. If $X_{j,\ell} = \emptyset$ then this is trivial. Otherwise, let (A_n) be a sequence of elements of A and (z_n) a sequence of elements of $\mathcal{K}_j$ such that $A_n z_n \in X_{j,\ell}$ for every $n \ge 1$, and let us show that $(A_n z_n)$ has a limit point in $X_{j,\ell}$. By passing to a subsequence we may clearly assume that (A_n) converges to some $A \in \mathsf{A}$, say, and it is clear that $A\mathcal{K}_j^{\mathbb{C}} \subseteq \mathcal{K}_\ell^{\mathbb{C}}$ since every A_n has this property. By passing to a further subsequence we assume that $\|z_n\|^{-1} z_n$ converges to a nonzero limit $z \in \mathcal{K}_j^{\mathbb{C}}$. It is clear that $\lim_{n\to\infty} \|z_n\|^{-1} A_n z_n = Az$ and therefore $\lim_{n\to\infty} \|z_n\|^{-1} = \langle Az, w\rangle$. If $\langle Az, w\rangle = 0$ then by Lemma 3.4 we have $Az = 0$ contradicting Corollary 3.7, so we have $\lim_{n\to\infty} \|z_n\|^{-1} > 0$. In particular (z_n) is bounded, and by passing to a subsequence we assume it to converge to a limit $z' \in \mathcal{K}_j^{\mathbb{C}}$. We note that $\lim_{n\to\infty} A_n z_n = Az'$ and consequently $\langle Az', w\rangle = \lim_{n\to\infty} \langle A_n z_n, w\rangle = 1$. In particular $z' \in X_{j,\ell}$ as required to prove the assertion.

By Corollary 3.7 the compact set $X_{j,\ell}$ is a subset of $\operatorname{Int} \mathcal{K}_\ell^{\mathbb{C}}$. By Lemma 3.5 there exists $C_1 > 0$ such that $\|z\| \le C_1$ for every $z \in X_{j,\ell}$. Choose $r = r(j, \ell) > 0$ such that for every $z \in X_{j,\ell}$ the open r-ball centred at z is a subset of $\mathcal{K}_\ell^{\mathbb{C}}$. If $z_1, z_2 \in X_{j,\ell}$ it follows that $z_1 + \lambda z_2 \in \mathcal{K}_\ell^{\mathbb{C}}$ whenever $\|\lambda z_2\| < r$, which is true in particular whenever $|\lambda| < C_1^{-1} r$. Equally if $|\lambda| > C_1 r^{-1}$ then $z_2 + \lambda^{-1} z_1 \in \mathcal{K}_\ell^{\mathbb{C}}$ and therefore $z_1 + \lambda z_2 = \lambda(z_2 + \lambda^{-1} z_1) \in \lambda \mathcal{K}_\ell^{\mathbb{C}} = \mathcal{K}_\ell^{\mathbb{C}}$. It follows that $E_{\mathcal{K}_\ell^{\mathbb{C}}}(z_1, z_2) \subseteq [C_1^{-1} r, C_1 r^{-1}]$

and therefore $d_{\mathcal{K}_\ell^{\mathbb{C}}}(\overline{z_1},\overline{z_2}) \leq \log C_1^2/r^2$. Hence $\operatorname{diam}_{\mathcal{K}_\ell^{\mathbb{C}}} X_{j,\ell} \leq \log C_1^2/r^2 < \infty$. To prove the claim we define

$$\Delta := \max_{1\leq j,\ell\leq m} \operatorname{diam}_{\mathcal{K}_\ell^{\mathbb{C}}} X_{j,\ell} < \infty.$$

By Lemma 3.9 there exists $C_2 > 0$ such that if $j \in \{1,\ldots,m\}$ and $z_1, z_2 \in \mathcal{K}_j^{\mathbb{C}} \setminus \{0\}$ then

$$\left\| \frac{z_1}{\langle z_1, w\rangle} - \frac{z_2}{\langle z_2, w\rangle} \right\| \leq C_2 d_{\mathcal{K}_j^{\mathbb{C}}}(\overline{z_1},\overline{z_2}).$$

We claim next that if $j \in \{1,\ldots,m\}$, $n \geq 1$, $A_1,\ldots,A_n \in \mathsf{A}$ and $z_1, z_2 \in \mathcal{K}_j^{\mathbb{C}} \setminus \{0\}$ then

$$\left\| \frac{A_n\cdots A_1 z_1}{\langle A_n\cdots A_1 z_1, w\rangle} - \frac{A_n\cdots A_1 z_2}{\langle A_n\cdots A_1 z_2, w\rangle} \right\| \leq C_2\Delta \left(\tanh\frac{\Delta}{4}\right)^{n-1}.$$

To see this let $A_1,\ldots,A_n \in \mathsf{A}$ and $z_1, z_2 \in \mathcal{K}_j^{\mathbb{C}} \setminus \{0\}$ and choose integers $\ell_1,\ldots,\ell_n$ such that $A_k\cdots A_1(\mathcal{K}_j\setminus\{0\}) \subseteq \operatorname{Int}\mathcal{K}_{\ell_k}$ for each $k=1,\ldots,n$. We have $A_k\cdots A_1(\mathcal{K}_j^{\mathbb{C}}\setminus\{0\}) \subseteq \operatorname{Int}\mathcal{K}_{\ell_k}^{\mathbb{C}}$ for each $k=1,\ldots,n$ by Corollary 3.7. By inductive application of Proposition 3.10 we have for each $k=1,\ldots,n$

$$\begin{aligned} d_{\mathcal{K}_{\ell_k}^{\mathbb{C}}}\left(\overline{A_k\cdots A_1 z_1}, \overline{A_k\cdots A_1 z_2}\right) &\leq \left(\tanh\frac{\Delta}{4}\right)^{k-1} d_{\mathcal{K}_{\ell_1}^{\mathbb{C}}}\left(\overline{A_1 z_1}, \overline{A_1 z_2}\right)\\ &\leq \Delta\left(\tanh\frac{\Delta}{4}\right)^{k-1} \end{aligned}$$

using the definition of Δ, so in particular by Lemma 3.9

$$\begin{aligned} \left\| \frac{A_n\cdots A_1 z_1}{\langle A_n\cdots A_1 z_1, w\rangle} - \frac{A_n\cdots A_1 z_2}{\langle A_n\cdots A_1 z_2, w\rangle} \right\| &\leq C_2 d_{\mathcal{K}_{\ell_n}^{\mathbb{C}}}\left(\overline{A_n\cdots A_1 z_1}, \overline{A_n\cdots A_1 z_2}\right)\\ &\leq C_2\Delta\left(\tanh\frac{\Delta}{4}\right)^{n-1} \end{aligned}$$

as required to prove the claim.

We claim lastly that there exists $N \geq 1$ such that for every $A_1,\ldots,A_N \in \mathsf{A}$ there exists $\ell = \ell(A_1,\ldots,A_N)$ such that $A_N\cdots A_1\left(\bigcup_{j=1}^m \mathcal{K}_j^{\mathbb{C}}\right) \subseteq \mathcal{K}_\ell^{\mathbb{C}}$. If this is not the case then there exist a strictly increasing sequence of natural numbers (n_k), a sequence of matrices (B_k) of the form $B_k = A_{n_k}\cdots A_1$ for some $A_1,\ldots,A_{n_k} \in \mathsf{A}$ depending on k, and four sequences $(j_{1,k})$, $(j_{2,k})$, $(\ell_{1,k})$, $(\ell_{2,k})$ of integers in $\{1,\ldots,m\}$ such that $B_k\mathcal{K}_{j_{1,k}}^{\mathbb{C}} \subseteq \mathcal{K}_{\ell_{1,k}}^{\mathbb{C}}$ and $B_k\mathcal{K}_{j_{2,k}}^{\mathbb{C}} \subseteq \mathcal{K}_{\ell_{2,k}}^{\mathbb{C}}$ for each $k \geq 1$, and such that $\ell_{1,k} \neq \ell_{2,k}$ for every $k \geq 1$. By passing to a subsequence if necessary we may assume that $\|B_k\|^{-1}B_k$ converges to a limit matrix B with $\|B\| = 1$, that $(j_{1,n})$ and $(j_{2,n})$ take constant values j_1 and j_2, and that $(\ell_{1,n})$ and $(\ell_{2,n})$ take constant values ℓ_1 and ℓ_2 with $\ell_1 \neq \ell_2$. Let $z \in \mathcal{K}_{j_1}^{\mathbb{C}}$ be nonzero and write $z = e^{i\theta}((u+v)+i(u-v))$ with $u, v \in \mathcal{K}_{j_1}$ and $\theta \in \mathbb{R}$. We note that for each $k \geq 1$

$$\begin{aligned} |\langle B_k z, w\rangle|^2 = |e^{-i\theta}\langle B_k z, w\rangle|^2 &= \langle B_k(u+v), w\rangle^2 + \langle B_k(u-v), w\rangle^2\\ &= 2\langle B_k u, w\rangle^2 + 2\langle B_k v, w\rangle^2\\ &\geq \tau_2\|B_k\|^2(2\|u\|^2 + 2\|v\|^2)\\ &= \tau_2\|B_k\|^2(\|u+v\|^2 + \|u-v\|^2)\\ &= \tau_2\|B_k\|^2\|e^{-i\theta}z\|^2 = \tau_2\|B_k\|^2\|z\|^2 \end{aligned}$$

using Proposition 3.3(ii) and the parallelogram law, and it follows in particular that the limit $\langle Bz, w\rangle = \lim_{k\to\infty}\langle \|B_k\|^{-1}B_k z, w\rangle$ cannot be zero. Hence

$$\lim_{k\to\infty}\langle B_k z, w\rangle^{-1}B_k z = \lim_{k\to\infty}\langle \|B_k\|^{-1}B_k z, w\rangle^{-1}\|B_k\|^{-1}B_k z = \langle Bz, w\rangle^{-1}Bz \in \mathcal{K}_{\ell_1}^{\mathbb{C}}$$

for every nonzero $z \in \mathcal{K}_{j_1}^{\mathbb{C}}$. Applying the previous claim we notice that the limit vector $\langle Bz, w\rangle^{-1}Bz$ does not depend on the choice of $z \in \mathcal{K}_{j_1}^{\mathbb{C}}$ and thus B maps the whole $\mathbb{C}$-cone $\mathcal{K}_{j_1}^{\mathbb{C}}$ to a one-dimensional subspace of $\mathcal{K}_{\ell_1}^{\mathbb{C}}$. Similarly B maps the $\mathbb{C}$-cone $\mathcal{K}_{j_2}^{\mathbb{C}}$ to a one-dimensional subspace of $\mathcal{K}_{\ell_2}^{\mathbb{C}}$. Since $\mathcal{K}_{\ell_1}^{\mathbb{C}} \cap \mathcal{K}_{\ell_2}^{\mathbb{C}} = \{0\}$ by Lemma 3.8, the image subspace $B\mathcal{K}_{j_1}^{\mathbb{C}}$ is distinct from the image subspace $B\mathcal{K}_{j_2}^{\mathbb{C}}$. But it is impossible for a linear map to take two sets with disjoint nonempty interior to distinct one-dimensional subspaces. This contradiction proves the existence of the desired constant N.

We may now prove the full statement of the proposition. Let $n \geq 1$ and $A \in \mathsf{A}_n$. If $n \leq N$ then we have

$$\left\|\frac{Az_1}{\langle Az_1, w\rangle} - \frac{Az_2}{\langle Az_2, w\rangle}\right\| \leq \sup_{z\in\bigcup_{j=1}^m \mathcal{K}_j^{\mathbb{C}}\setminus\{0\}} \frac{2\|z\|}{|\langle z, w\rangle|} \leq C_3,$$

say, by Lemma 3.5. Otherwise write $A = A_2A_1$ where $A_1 \in \mathsf{A}_N$ and $A_2 \in \mathsf{A}_{n-N}$. We have

$$\begin{aligned}\left\|\frac{Az_1}{\langle Az_1, w\rangle} - \frac{Az_2}{\langle Az_2, w\rangle}\right\| &= \left\|\frac{A_2A_1z_1}{\langle A_2A_1z_1, w\rangle} - \frac{A_2A_1z_2}{\langle A_2A_1z_2, w\rangle}\right\| \\ &\leq C_2\Delta\left(\tanh\frac{\Delta}{4}\right)^{n-N}\end{aligned}$$

since A_1z_1 and A_1z_2 belong to the same cone $\mathcal{K}_\ell^{\mathbb{C}}$. Since $\tanh(\Delta/4) < 1$ this suffices to complete the proof of the proposition. □

(i). Let $A \in \mathcal{S}(\mathsf{A})$. For each $n \geq 1$ let $\ell_n \in \{1, \ldots, m\}$ be the unique integer such that $A^n(\mathcal{K}_1 \setminus \{0\}) \subseteq \operatorname{Int}\mathcal{K}_{\ell_n} \cup -\operatorname{Int}\mathcal{K}_{\ell_n}$. By the pigeonhole principle there exist integers n_1, n_2 such that $n_2 > n_1 \geq 1$ and $\ell_{n_2} = \ell_{n_1} = \ell$, say, so we have $A^{n_2-n_1}(\mathcal{K}_\ell \setminus \{0\}) \subseteq \operatorname{Int}\mathcal{K}_\ell \cup -\operatorname{Int}\mathcal{K}_\ell$. In particular $A^{2(n_2-n_1)}$ maps $\mathcal{K}_\ell \setminus \{0\}$ to its interior, so by the Perron-Frobenius Theorem we have $|\lambda_1(A^{2(n_2-n_1)})| > |\lambda_2(A^{2(n_2-n_1)})|$ and the eigenvector corresponding to this eigenvalue belongs to $\operatorname{Int}\mathcal{K}_\ell$. It follows that $|\lambda_1(A)| > |\lambda_2(A)|$ as required, and since the associated eigenvector belongs to $\operatorname{Int}\mathcal{K}_\ell$ and is in particular real, the associated eigenvalue of $\lambda_1(A)$ must be real also.

Part (ii) is given directly by Proposition 3.3(iii). Let us now prove (iii), let $z \in \Omega$ and $A \in \mathcal{S}(\mathsf{A})$ and write

$$z = (a + ib)((u + v) + i(u - v)) = (a - b)u + (a + b)v + i((a + b)u - (a - b)v)$$

where $u, v \in \mathcal{K}_j$ and $a, b \in \mathbb{R}$. We have $\langle u, w\rangle \geq 0$ and $\langle v, w\rangle \geq 0$, and since for some $\ell \in \{1, \ldots, m\}$ we have either $A\mathcal{K}_j \subseteq \mathcal{K}_\ell$ or $A\mathcal{K}_j \subseteq -\mathcal{K}_\ell$, the real numbers $\langle Au, w\rangle$ and $\langle Av, w\rangle$ either are both greater than or equal to zero, or are both less than or equal to zero. Since Az cannot be zero we note that Au and Av cannot both be zero, and consequently $\langle Au, w\rangle$ and $\langle Av, w\rangle$ cannot both be zero. We observe that

$$\Re(\langle Az, w\rangle) = (a - b)\langle Au, w\rangle + (a + b)\langle Av, w\rangle$$

and the equation $\langle z,w\rangle=1$ implies

$$(a-b)\langle u,w\rangle+(a+b)\langle v,w\rangle=1,\qquad (a+b)\langle u,w\rangle-(a-b)\langle v,w\rangle=0$$

by separating real and imaginary parts.

If $v=0$ then we have $(a-b)\langle u,w\rangle=1$ and it follows that $\Re(\langle Az,w\rangle)=(a-b)\langle Au,w\rangle\neq 0$ since otherwise $\langle Au,w\rangle$ and $\langle Av,w\rangle$ would both be zero. If $v\neq 0$ but $a-b=0$ then it follows that $Av\neq 0$ and $(a+b)\langle v,w\rangle=1$, so $\langle Av,w\rangle\neq 0$ and $a+b\neq 0$ and we obtain $\Re(\langle Az,w\rangle)=(a+b)\langle Av,w\rangle\neq 0$ as desired. If v and $a-b$ are both nonzero the equation $(a+b)\langle u,w\rangle-(a-b)\langle v,w\rangle=0$ implies that $a+b$, $\langle u,w\rangle\neq 0$. Since necessarily $\langle u,w\rangle,\langle v,w\rangle>0$ it follows that the real numbers $a+b$ and $a-b$ have the same sign, and since $\langle Au,w\rangle$ and $\langle Av,w\rangle$ agree with one another in sign we deduce that $\Re(\langle Az,w\rangle)=(a-b)\langle Au,w\rangle+(a+b)\langle Av,w\rangle$ is the sum of two nonzero real numbers with matching signs, hence nonzero. In all cases we find that $\Re(\langle Az,w\rangle)\neq 0$ and this completes the proof of (iii).

To prove (iv), we first note that by hypothesis each of the real cones $\mathcal{K}_j$ has nonempty interior as a subset of $\mathbb{R}^d$. If $u\in\operatorname{Int}\mathcal{K}_j$ then $\langle u,v\rangle>0$ for every $v\in\mathcal{K}_j'$ by Lemma 3.2, so in particular $\Re(\langle u,\ell\rangle\langle u,m\rangle^*)>0$ for every $\ell,m\in\mathcal{K}_j'$ and hence by Lemma 3.6 u belongs to the interior of the complexification $\mathcal{K}_j^{\mathbb{C}}$. In particular $\operatorname{Int}\mathcal{K}_j^{\mathbb{C}}$ is nonempty for each j and it follows by homogeneity that Ω is nonempty. It is clear that Ω is open as a subset of the hyperplane $\{z\in\mathbb{C}^d\colon\langle z,w\rangle=1\}$. If $z\in\Omega$ and $A\in\mathcal{S}(A)$ then $z\in\operatorname{Int}\mathcal{K}_j^{\mathbb{C}}$ for some $j\in\{1,\ldots,m\}$ and it follows easily via Corollary 3.7 that $Az\in\operatorname{Int}\mathcal{K}_\ell^{\mathbb{C}}$ for some $\ell\in\{1,\ldots,m\}$. By (iii) we have $\langle Az,w\rangle\neq 0$ and therefore $\overline{A}z$ is well-defined, and it is clear that $\overline{A}z\in\Omega$. In particular $\overline{A}\colon\Omega\to\Omega$ is well-defined and is clearly holomorphic for every $A\in\mathcal{S}(\mathsf{A})$. To see that Ω is bounded we note that each of the sets

$$\left\{z\in\mathbb{C}^d\colon z\in\mathcal{K}_j^{\mathbb{C}}\text{ and }\langle z,w\rangle=1\right\}$$

is bounded by Lemma 3.5. This completes the proof of (iv).

Part (v) follows directly from Proposition 3.11 together with the definition of Ω. Let us now prove (vi). Fix $A\in\mathcal{S}(\mathsf{A})$. We showed when proving (i) that there exists an eigenvector $v_A\in\bigcup_{j=1}^m\operatorname{Int}\mathcal{K}_j$ such that $Av_A=\lambda_1(A)v_A$. In previous arguments we have seen that $\bigcup_{j=1}^m\operatorname{Int}\mathcal{K}_j\subseteq\bigcup_{j=1}^m\operatorname{Int}\mathcal{K}_j^{\mathbb{C}}$ and it follows that $z_A:=\langle v_A,w\rangle^{-1}v_A$ belongs to Ω, and indeed to $\Omega\cap\mathbb{R}^d$. It is clear that $\overline{A}z_A=z_A$ and that $\langle Az_A,w\rangle=\lambda_1(A)$. It follows from (v) that $\lim_{n\to\infty}\operatorname{diam}\overline{A^n}(\Omega)=0$ and this implies that $\overline{A}$ cannot have a second fixed point in Ω which is distinct from z_A.

Let us now calculate the eigenvalues of the derivative $D_{z_A}\overline{A}$. Let $u_1,\ldots,u_d\in\mathbb{C}^d$ be a Jordan basis for A with basis elements listed in descending order of the absolute value of the corresponding eigenvalue, and with $u_1=z_A$. Since $|\lambda_1(A)|>|\lambda_2(A)|$ we have $Au_1=\lambda_1(A)u_1$ and $Au_2=\lambda_2(A)u_2$. For each $j\in\{3,\ldots,d\}$, let $\delta_j\in\{0,1\}$ such that $Au_j=\lambda_j(A)u_j+\delta_ju_{j-1}$.

For every v in the tangent space $\{v \in \mathbb{C}^d \colon \langle v, w\rangle = 0\}$ to Ω at z_A we have

$$\begin{aligned}
\left(D_{z_A}\overline{A}\right) v &:= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\overline{A}(u_1+\varepsilon v) - \overline{A}u_1\right)\\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\frac{A(u_1+\varepsilon v)}{\langle A(u_1+\varepsilon v), w\rangle} - \frac{Au_1}{\langle Au_1, w\rangle}\right)\\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\frac{\langle Au_1, w\rangle \cdot A(u_1+\varepsilon v) - \langle A(u_1+\varepsilon v), w\rangle \cdot Au_1}{\langle A(u_1+\varepsilon v), w\rangle\langle Au_1, w\rangle}\right)\\
&= \lim_{\varepsilon\to 0} \frac{\langle Au_1, w\rangle \cdot Av - \langle Av, w\rangle \cdot Au_1}{\langle A(u_1+\varepsilon v), w\rangle\langle Au_1, w\rangle}\\
&= \frac{\langle Au_1, w\rangle \cdot Av - \langle Av, w\rangle \cdot Au_1}{\langle Au_1, w\rangle\langle Au_1, w\rangle}\\
&= \frac{1}{\lambda_1(A)}\left(Av - \langle Av, w\rangle u_1\right).
\end{aligned}$$

Clearly the vectors $v_j := u_j - \langle u_j, w\rangle u_1$, where j runs from 2 to d, form a basis of the tangent space $\{z \in \mathbb{C}^d \colon \langle z, w\rangle = 0\}$. We have

$$\begin{aligned}
\left(D_{z_A}\overline{A}\right) v_2 &= \frac{1}{\lambda_1(A)}\left(Av_2 - \langle Av_2, w\rangle u_1\right)\\
&= \frac{1}{\lambda_1(A)}\left(\lambda_2(A)u_2 - \lambda_1(A)\langle u_2, w\rangle u_1 - \lambda_2(A)\langle u_2, w\rangle u_1 + \lambda_1(A)\langle u_2, w\rangle u_1\right)\\
&= \frac{1}{\lambda_1(A)}\left(\lambda_2(A)u_2 - \lambda_2(A)\langle u_2, w\rangle u_1\right)\\
&= \frac{\lambda_2(A)}{\lambda_1(A)} v_2,
\end{aligned}$$

and for $j = 3, \ldots, d$ we similarly have

$$\begin{aligned}
\left(D_{z_A}\overline{A}\right) v_j =& \frac{1}{\lambda_1(A)}\left(Av_j - \langle Av_j, w\rangle u_1\right)\\
=& \frac{1}{\lambda_1(A)}\big(\lambda_j(A)u_j + \delta_j u_{j-1} - \lambda_1(A)\langle u_j, w\rangle u_1\\
&- \lambda_j(A)\langle u_j, w\rangle u_1 - \delta_j\langle u_{j-1}, w\rangle u_1 + \lambda_1(A)\langle u_j, w\rangle u_1\big)\\
=& \frac{1}{\lambda_1(A)}\left(\lambda_j(A)u_j - \lambda_j(A)\langle u_j, w\rangle u_1 + \delta_j u_{j-1} - \delta_j\langle u_{j-1}, w\rangle u_1\right)\\
=& \frac{\lambda_j(A)}{\lambda_1(A)} v_j + \frac{\delta_j}{\lambda_1(A)} v_{j-1}.
\end{aligned}$$

It follows that with respect to the basis $v_2, \ldots, v_d$ the matrix of $D_{z_A}\overline{A}$ is upper triangular with the values $\lambda_j(A)/\lambda_1(A)$ along the diagonal. In particular its eigenvalues are precisely the numbers $\lambda_j(A)/\lambda_1(A)$ for $j = 2, \ldots, d$ as claimed. Since $p_A(x) = \det(xI - A) = \prod_{j=1}^d (x - \lambda_j(A))$ we have

$$p_A'(x) = \sum_{\ell=1}^d \prod_{\substack{1\leq j\leq d\\ j\neq \ell}} (x - \lambda_j(A))$$

and therefore

$$\frac{p_A'(\lambda_1(A))}{\lambda_1(A)^{d-1}} = \frac{\prod_{j=2}^d (\lambda_1(A) - \lambda_j(A))}{\lambda_1(A)^{d-1}} = \prod_{j=2}^d \left(1 - \frac{\lambda_j(A)}{\lambda_1(A)}\right) = \det(I - D_{z_A}\overline{A}).$$

Since $1 - \lambda_j(A)/\lambda_1(A)$ is nonzero for all $j = 2, \ldots, d$ this quantity is nonzero. This completes the proof of (vi).

In order to prove (vii) we first claim that there exists $\tau > 0$ such that $\|Az\| \geq \tau\|A\| \cdot \|z\|$ for all $z \in \bigcup_{j=1}^m \mathcal{K}_j^{\mathbb{C}}$ and $A \in \mathcal{S}(\mathsf{A})$. To this end let $z \in \bigcup_{j=1}^m \mathcal{K}_j^{\mathbb{C}}$ and $A \in \mathcal{S}(\mathsf{A})$ and write $z = e^{i\theta}((u+v)+i(u-v))$ where $u, v \in \mathcal{K}_j$, say. By Proposition 3.3(i) there exists $\tau > 0$ not depending on A or z such that $\|Au\| \geq \tau\|A\| \cdot \|u\|$ and $\|Av\| \geq \tau\|A\| \cdot \|v\|$. We therefore have

$$\begin{aligned}
\|Az\|^2 &= \|e^{i\theta}A((u+v)+i(u-v))\|^2 \\
&= \|A(u+v)\|^2 + \|A(u-v)\|^2 \\
&= 2\|Au\|^2 + 2\|Av\|^2 \\
&\geq 2\tau^2\|A\|^2 \left(\|u\|^2 + \|v\|^2\right) \\
&= \tau^2\|A\|^2 \left(\|u+v\|^2 + \|u-v\|^2\right) = \tau^2\|A\|^2\|z\|^2
\end{aligned}$$

using the parallelogram law, which proves the claim. Since each $\mathcal{K}_j^{\mathbb{C}}$ is a $\mathbb{C}$-cone and $\langle z, w\rangle \neq 0$ for all nonzero $z \in \mathcal{K}_j^{\mathbb{C}}$, by Lemma 3.5 there exists $\delta > 0$ such that $|\langle z, w\rangle| \geq \delta\|z\|$ for every $z \in \bigcup_{j=1}^m \mathcal{K}_j^{\mathbb{C}}$. In particular

$$|\langle Az, w\rangle| \geq \delta\|Az\| \geq \delta\tau\|A\| \cdot \|z\| \geq \delta\tau\|A\| \cdot |\langle z, w\rangle| = \delta\tau\|A\|$$

for every $z \in \Omega$ and $A \in \mathcal{S}(\mathsf{A})$ as required. The opposing inequality $|\langle Az, w\rangle| \leq C\|A\|$ follows by the Cauchy-Schwarz inequality and the boundedness of Ω. This completes the proof of (vii).

To prove (viii) we note that $\bigcup_{A \in \mathcal{S}(\mathsf{A})} \overline{A}(\Omega) \subseteq \bigcup_{A \in \mathsf{A}} \overline{A}(\Omega)$ since for each $A \in \mathcal{S}(\mathsf{A})$ we may write $\overline{A} = \overline{A_1} \circ \cdots \circ \overline{A_n}$ for some $A_1, \ldots, A_n \in \mathsf{A}$ and therefore $\overline{A}(\Omega) \subseteq \overline{A_1}(\Omega)$. It is therefore sufficient to show that the set $\bigcup_{A \in \mathsf{A}} \overline{A}(\Omega)$ is compactly contained in Ω. By (iv) this set is a subset of Ω, so it is sufficient to show that its closure in $\mathbb{C}^d$ – which is a closed, bounded subset of $\mathbb{C}^d$, and therefore compact – is a subset of Ω.

Suppose that z_0 belongs to this closure. We may write $z_0 = \lim_{n\to\infty} \langle A_n z_n, w\rangle^{-1} A_n z_n$ for some sequence (z_n) of elements of Ω and some sequence (A_n) of elements of A. By passing to subsequences if necessary we may assume that (A_n) converges to $A \in \mathsf{A}$, that each z_n belongs to $\operatorname{Int} \mathcal{K}_j^{\mathbb{C}}$ for a particular constant value of $j \in \{1, \ldots, m\}$, and that (z_n) converges to $z \in \mathcal{K}_j^{\mathbb{C}}$, since A is compact by hypothesis and Ω is bounded by (iv). It follows that $\lim_{n\to\infty} A_n z_n = Az$ and therefore $\langle A_n z_n, w\rangle = \langle Az, w\rangle$. By (vii) we have $C^{-1}\|A_n\| \leq |\langle A_n z_n, w\rangle| \leq C\|A_n\|$ for all $n \geq 1$ and therefore $\langle Az, w\rangle \neq 0$. Hence $\lim_{n\to\infty} \langle A_n z_n, w\rangle^{-1} A_n z_n = \langle Az, w\rangle^{-1} Az$. Clearly there exists $\ell \in \{1, \ldots, m\}$ such that $Az \in \mathcal{K}_\ell^{\mathbb{C}}$. By Corollary 3.7 we have $A(\mathcal{K}_j^{\mathbb{C}} \setminus \{0\}) \subseteq \operatorname{Int} \mathcal{K}_\ell^{\mathbb{C}}$ and therefore $z_0 = \langle Az, w\rangle^{-1} Az \in \operatorname{Int} \mathcal{K}_\ell^{\mathbb{C}}$. Since $\langle z_0, w\rangle = 1$ we find that $z_0 \in \Omega$ as required. This completes the proof of (viii) and hence of the theorem.

4. Operator-theoretic preliminaries

In this section we collect some preliminary results which will underpin the construction of the operators $\mathscr{L}_s$ defined in Theorem 4.

If $\Omega \subset \mathbb{C}^k$ is open and nonempty we define the Bergman space $\mathcal{A}^2(\Omega)$ to be the set of all holomorphic functions $f\colon \Omega \to \mathbb{C}$ such that the integral $\int_\Omega |f(z)|^2 dV(z)$ is finite, where V denotes $2k$-dimensional Lebesgue measure on $\mathbb{C}^k \simeq \mathbb{R}^{2k}$. The space $\mathcal{A}^2(\Omega)$ is a Hilbert space when equipped with the inner product $\langle f, g\rangle_{\mathcal{A}^2(\Omega)} := \int_\Omega f(z)g(z)^* dV(z)$. In particular it is a closed subspace of the Hilbert space $L^2(\Omega)$ and is therefore separable. We note the following elementary estimate:

Lemma 4.1. *Let $\Omega \subseteq \mathbb{C}^k$ be a nonempty open set and let $K \subseteq \Omega$ be compact. Then there exists $C_K > 0$ depending on K such that $\sup_{z\in K} |f(z)| \leq C_K \|f\|_{\mathcal{A}^2(\Omega)}$ for every $f \in \mathcal{A}^2(\Omega)$.*

Proof. Choose $\varepsilon > 0$ small enough that for every $z \in K$ the open ball $B_\varepsilon(z_0)$ is a subset of Ω. By harmonicity we have

$$\begin{aligned} |f(z_0)|^2 &= \left| \frac{1}{V(B_\varepsilon(z_0))} \int_{B_\varepsilon(z_0)} f(z)^2 dV(z) \right| \\ &\leq \frac{1}{V(B_\varepsilon(z_0))} \int_\Omega |f(z)|^2 dV(z) \\ &= \frac{1}{V(B_\varepsilon(z_0))} \|f\|^2_{\mathcal{A}(\Omega)} = \frac{k!}{\pi^k \varepsilon^k} \|f\|^2_{\mathcal{A}(\Omega)} \end{aligned}$$

for all $f \in \mathcal{A}^2(\Omega)$ and $z_0 \in K$. $\square$

We observe in particular that for every $z \in \Omega$ the evaluation map $f \mapsto f(z)$ is a continuous linear functional $\mathcal{A}^2(\Omega) \to \mathbb{C}$.

In practice we will be interested in the case where Ω is a bounded open subset of an affine subspace of $\mathbb{C}^d$ rather than of $\mathbb{C}^d$ itself. Clearly the results of this section will apply equally well in that context with k being equal to the dimension of the affine subspace of $\mathbb{C}^d$ of which Ω is an open subset.

We define the *singular values* or *approximation numbers* $\mathfrak{s}_n(L)$ of a bounded linear operator $L\colon H \to H$ acting on a separable complex Hilbert space H to be the quantities

$$\mathfrak{s}_n(L) := \inf \left\{ \|L - F\| \colon F\colon H \to H \text{ is bounded with rank at most } n-1 \right\},$$

where n ranges over the positive integers. If L is compact then the values $\mathfrak{s}_n(L)^2$ coincide with the sequence of eigenvalues of the positive self-adjoint operator L^*L (see e.g. [27, Theorem IV.2.5]). If L satisfies $\sum_{n=1}^\infty \mathfrak{s}_n(L) < \infty$ then L is called *trace-class*. Any trace-class operator is obviously the limit in the operator norm of a sequence of finite-rank operators and in particular is compact. It follows easily from the definition of $\mathfrak{s}_n$ that if L_1 and L_2 are bounded then $\mathfrak{s}_n(L_1L_2)$ and $\mathfrak{s}_n(L_2L_1)$ are both bounded by $\|L_1\|\mathfrak{s}_n(L_2)$ for every $n \geq 1$, and in particular the composition of a trace-class operator with a bounded operator is trace-class. In particular every power of a trace-class operator is trace-class.

The fundamental properties of the trace are summarised in the following result which combines several statements from [59, §3]:

Theorem 6 (Lidskii's theorem)**.** *Let L be a trace-class operator acting on a complex separable Hilbert space H and let $(\lambda_n)_{n=1}^M$ be a complete enumeration of the nonzero eigenvalues of L, listed with repetition according to algebraic multiplicity, where $M \in \mathbb{N} \cup \{0, +\infty\}$. Then for every orthonormal basis $(e_n)_{n=1}^\infty$ of H we have*

$$\sum_{n=1}^{\infty} \langle Le_n, e_n \rangle = \sum_{n=1}^{M} \lambda_n \tag{4}$$

with both series being absolutely convergent. The common value of (4) *is defined to be the* trace *of L and is denoted* $\operatorname{tr} L$.

It is clear from the definition that $\mathfrak{s}_{2n-1}(L_1 + L_2) \leq \mathfrak{s}_n(L_1) + \mathfrak{s}_n(L_2)$ for every pair of bounded linear operators $L_1, L_2 \colon H \to H$ and every $n \geq 1$. It follows easily that if $L_1, \ldots, L_k$ are trace-class operators on H then any finite linear combination $\sum_{i=1}^k a_i L_i$ is also trace-class and satisfies

$$\operatorname{tr} \sum_{i=1}^{k} a_i L_i = \sum_{i=1}^{k} a_i \operatorname{tr} L_i$$

as a consequence of (4).

The following result also combines several statements from [59, §3], with the exception of the determinant formula for a_n which may be found instead in, for example, [58, Theorem 6.8] or [27, Theorem IV.5.2].

Theorem 7. *Let L be a trace-class operator on a separable complex Hilbert space H and let $(\lambda_n)_{n=1}^\infty$ be an enumeration of the nonzero eigenvalues of L, repeated according to algebraic multiplicity. (If only $M < \infty$ nonzero eigenvalues exist then we define $\lambda_n := 0$ for all $n > M$.) For every $n \geq 1$ define*

$$a_n := (-1)^n \sum_{i_1 < \cdots < i_n} \lambda_{i_1}(L) \cdots \lambda_{i_n}(L)$$

and define also $a_0 := 1$. Then the function

$$\det(I - zL) := \sum_{n=0}^{\infty} a_n z^n$$

is well-defined and entire, and is equal to the absolutely convergent infinite product $\prod_{n=1}^\infty (1 - z\lambda_n)$. The zeros of $z \mapsto \det(I - zL)$ are precisely the reciprocals of the nonzero eigenvalues of L and the order of each zero is equal to the algebraic *multiplicity of the corresponding eigenvalue. The coefficients a_n satisfy*

$$a_n = \frac{(-1)^n}{n!} \det \begin{pmatrix} t_1(s) & n-1 & 0 & \cdots & 0 & 0 \\ t_2(s) & t_1(s) & n-2 & \cdots & 0 & 0 \\ t_3(s) & t_2(s) & t_1(s) & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ t_{n-1}(s) & t_{n-2}(s) & t_{n-3}(s) & \cdots & t_1(s) & 1 \\ t_n(s) & t_{n-1}(s) & t_{n-2}(s) & \cdots & t_2(s) & t_1(s) \end{pmatrix},$$

and

$$|a_n| \leq \sum_{i_1 < \cdots < i_n} \mathfrak{s}_{i_1}(L) \cdots \mathfrak{s}_{i_n}(L)$$

for all $n \geq 1$.

It has long been known that composition operators on Bergman spaces, and on other Banach spaces of holomorphic functions, are trace-class under mild conditions (see e.g. [28]). Historically most results in this context have assumed the set $\Omega \subset \mathbb{C}^k$ to be bounded and connected, but in this article we will need to work with sets having multiple connected components. The following is a special case of [1, Theorem 5.9].

Theorem 8. *Let $\Omega \subseteq \mathbb{C}^k$ be a nonempty open set, and let $\Omega_0 \Subset \Omega$ be nonempty and open. Suppose that $\phi_1, \ldots, \phi_m \colon \Omega \to \Omega_0$ are holomorphic and $\psi_1, \ldots, \psi_m \colon \Omega \to \mathbb{C}$ are holomorphic and bounded. Then the operator $\mathscr{L} \colon \mathcal{A}^2(\Omega) \to \mathcal{A}^2(\Omega)$ given by*

$$(\mathscr{L} f)(z) := \sum_{j=1}^{m} \psi_j(z) f(\phi_j(z))$$

is a well-defined bounded linear operator on $\mathcal{A}^2(\Omega)$, and there exist $C, \gamma > 0$ depending only on Ω and Ω_0 such that

$$\mathfrak{s}_n(\mathscr{L}) \leq C \left(\sum_{j=1}^{m} \sup_{z \in \Omega} |\psi_j(z)| \right) \exp\left(-\gamma n^{\frac{1}{k}}\right)$$

for every $n \geq 1$. In particular $\mathscr{L}$ is trace class.

In this article we will need to calculate explicitly the traces of a family of operators. The following result is a minor variation on a type of result appearing in work of D. Ruelle ([55, Lemma 1]), D. Mayer ([41, §III] and remark following [42, Corollary 7.11]), D. Fried ([25, Lemma 5]) and other authors. The result may be proved easily by following the second, third and fourth paragraphs of the proof of [2, Theorem 4.2].

Theorem 9. *Let $\Omega \subset \mathbb{C}^k$ be a bounded, connected, nonempty open set and suppose that $\phi \colon \Omega \to \Omega$ is a holomorphic function such that $\phi(\Omega) \Subset \Omega$. Let $\psi \colon \Omega \to \mathbb{C}$ be holomorphic and bounded. Then ϕ has a unique fixed point $z_0 \in \Omega$, the eigenvalues of the derivative $D_{z_0}\phi$ are all strictly less than 1 in modulus, and the operator $\mathcal{L} \colon \mathcal{A}^2(\Omega) \to \mathcal{A}^2(\Omega)$ defined by $(\mathcal{L} f)(z) := \psi(z) f(\phi(z))$ is trace-class and has trace equal to $\psi(z_0)/\det(I - D_{z_0}\phi)$.*

Since we will in general need to study operators on Bergman spaces $\mathcal{A}^2(\Omega)$ for which Ω is not connected, we prove the following extension of Theorem 9 which does not seem to have been previously stated elsewhere:

Theorem 10. *Let $\Omega \subseteq \mathbb{C}^k$ be a bounded nonempty open set and suppose that $\phi\colon \Omega \to \Omega$ is a holomorphic function such that $\phi(\Omega) \Subset \Omega$. Let $\psi\colon \Omega \to \mathbb{C}$ be holomorphic and bounded. Then the set of fixed points $\operatorname{Fix}\phi := \{z \in \Omega \colon \phi(z) = z\}$ is either finite or empty, and each connected component of Ω contains at most one fixed point of ϕ. At each fixed point $z \in \operatorname{Fix}\phi$ the eigenvalues of the derivative $D_z\phi$ are all strictly less than 1 in modulus. The operator $\mathcal{L}\colon \mathcal{A}^2(\Omega) \to \mathcal{A}^2(\Omega)$ defined by $(\mathcal{L}f)(z) := \psi(z)f(\phi(z))$ is trace-class and satisfies*

$$\operatorname{tr}\mathcal{L} = \sum_{z \in \operatorname{Fix}\phi} \frac{\psi(z)}{\det(I - D_z\phi)}. \tag{5}$$

Additionally, if Ω is connected then $\operatorname{Fix}\phi$ is a singleton.

Proof. The case in which Ω is connected is precisely Theorem 9, so we assume throughout that Ω is disconnected. The operator $\mathcal{L}$ meets the hypotheses of Theorem 8 with $m = 1$ and hence in particular is trace-class. Note that since each connected component of Ω has positive Lebesgue measure, and their union Ω, being bounded, has finite measure, Ω has at most countably many connected components. Let $M \in \mathbb{N} \cup \{\infty\}$ be the number of connected components of Ω and enumerate those components as $(\Omega_m)_{m=1}^M$.

For each integer m such that $1 \leq m \leq M$ let $(f_{m,n})_{n=1}^\infty$ be an orthonormal basis for $\mathcal{A}^2(\Omega_m)$. Extend each $f_{m,n}$ to a function $\tilde{f}_{m,n}\colon \Omega \to \mathbb{C}$ by defining $\tilde{f}_{m,n}(z) := f_{m,n}(z)$ when $z \in \Omega_m$ and $\tilde{f}_{m,n}(z) := 0$ otherwise. Clearly $(\tilde{f}_{m,n})$ is an orthonormal basis for $\mathcal{A}^2(\Omega)$, so by Theorem 6 we have

$$\operatorname{tr}\mathcal{L} = \sum_{m=1}^M \sum_{n=1}^\infty \langle \mathcal{L}\tilde{f}_{m,n}, \tilde{f}_{m,n}\rangle_{\mathcal{A}^2(\Omega)} = \sum_{m=1}^M \sum_{n=1}^\infty \int_\Omega \psi(z)\tilde{f}_{m,n}(\phi(z))\tilde{f}_{m,n}(z)^* dV(z)$$

and this series is absolutely convergent.

Fix an integer m such that $1 \leq m \leq M$ and consider the series

$$\sum_{n=1}^\infty \int_\Omega \psi(z)\tilde{f}_{m,n}(\phi(z))\tilde{f}_{m,n}(z)^* dV(z). \tag{6}$$

Since Ω_m is connected, either $\phi(\Omega_m)$ is a subset of Ω_m or it does not intersect Ω_m. If the latter holds then by definition $\tilde{f}_{m,n}(\phi(z))$ is zero for all $z \in \Omega_m$ and $n \geq 1$, and $\tilde{f}_{m,n}(z)^*$ is zero for all $z \in \Omega \setminus \Omega_m$ and $n \geq 1$. It follows that in the case where $\phi(\Omega_m) \not\subseteq \Omega_m$ all terms in the series (6) vanish and the series evaluates to zero.

In the case where $\phi(\Omega_m) \subseteq \Omega_m$, consider the operator $\mathcal{L}_m\colon \mathcal{A}^2(\Omega_m) \to \mathcal{A}^2(\Omega_m)$ defined by $(\mathcal{L}_m f)(z) := \psi(z)f(\phi(z))$. By Theorem 9 there is a unique fixed point

z_m of ϕ in Ω_m, the operator $\mathcal{L}_m$ is trace-class, the eigenvalues of $D_{z_m}\phi$ are all less than one in absolute value, and we have

$$\operatorname{tr}\mathcal{L}_m = \sum_{n=1}^{\infty}\langle \mathcal{L}_m f_{m,n}, f_{m,n}\rangle_{\mathcal{A}^2(\Omega_m)} = \frac{\psi(z_m)}{\det(I - D_{z_m}\phi)}$$

with this series being absolutely convergent. Since clearly

$$\begin{aligned}\sum_{n=1}^{\infty}\int_{\Omega}\psi(z)\tilde{f}_{m,n}(\phi(z))\tilde{f}_{m,n}(z)^* dV(z) &= \sum_{n=1}^{\infty}\int_{\Omega_m}\psi(z)f_{m,n}(\phi(z))f_{m,n}(z)^* dV(z)\\ &= \sum_{n=1}^{\infty}\langle \mathcal{L}_m f_{m,n}, f_{m,n}\rangle_{\mathcal{A}^2(\Omega_m)}\\ &= \frac{\psi(z_m)}{\det(I - D_{z_m}\phi)}\end{aligned}$$

by the definition of $\tilde{f}_{m,n}$, we conclude that the series (6) evaluates to $\psi(z_m)/\det(I - D_{z_m}\phi)$. In each of the two cases we have obtained

$$\sum_{n=1}^{\infty}\int_{\Omega}\psi(z)\tilde{f}_{m,n}(\phi(z))\tilde{f}_{m,n}(z)^* dV(z) = \sum_{z\in\Omega_m\cap\operatorname{Fix}\phi}\frac{\psi(z)}{\det(I - D_z\phi)}$$

and we have also shown that each Ω_m can contain at most one fixed point, and that the derivative of ϕ at each fixed point has all eigenvalues less than one in absolute value. We deduce

$$\begin{aligned}\operatorname{tr}\mathcal{L} &= \sum_{m=1}^{M}\sum_{n=1}^{\infty}\int_{\Omega}\psi(z)\tilde{f}_{m,n}(\phi(z))\tilde{f}_{m,n}(z)^* dV(z)\\ &= \sum_{m=1}^{M}\sum_{z\in\Omega_m\cap\operatorname{Fix}\phi}\frac{\psi(z)}{\det(I - D_z\phi)}\\ &= \sum_{z\in\operatorname{Fix}\phi}\frac{\psi(z)}{\det(I - D_z\phi)}\end{aligned}$$

as required.

It remains only to show that $\operatorname{Fix}\phi$ is finite. If it is infinite then since $\Omega_m \cap \operatorname{Fix}\phi$ can contain at most one point for each m, there must exist an infinite sequence $(z_\ell)_{\ell=1}^{\infty}$ such that $\phi(z_\ell) = z_\ell$ for each ℓ and such that each z_ℓ belongs to a distinct component Ω_m. In particular (z_ℓ) cannot have any accumulation points in Ω since if this were the case (z_ℓ) would have to return infinitely many times to the connected component containing the accumulation point; but (z_ℓ) necessarily has accumulation points in Ω because all its values belong to the compact set $\overline{\phi(\Omega)} \subset \Omega$. This contradiction shows that $\operatorname{Fix}\phi$ must be finite as claimed. □

The last general functional-analytic result which we will require is the following:

Theorem 11 (Kreĭn-Rutman). *Let $\mathcal{X}$ be a real Banach space and $\mathcal{C} \subseteq \mathcal{X}$ a subset such that:*

(i) $\mathcal{C}$ is closed and convex and satisfies $\lambda\mathcal{C} = \mathcal{C}$ for all real $\lambda > 0$,

(ii) $\mathcal{C} \cap -\mathcal{C} = \{0\}$,
(iii) *The span of* $\mathcal{C}$ *is dense in* $\mathcal{X}$.

If $\mathcal{L}\colon \mathcal{X} \to \mathcal{X}$ *is a compact linear operator such that* $\mathcal{L}\mathcal{C} \subseteq \mathcal{C}$ *and* $\rho(\mathcal{L}) \neq 0$, *then there exists nonzero* $x \in \mathcal{C}$ *such that* $\mathcal{L}x = \rho(\mathcal{L})x$.

A proof of Theorem 11 may be found in [14, Theorem 19.2], or in a somewhat different form [57, p.313].

5. Proof of Theorem 4

The following result, which will be proved in this section, clearly implies Theorem 4:

Theorem 12. *Let* $\mathsf{A} = (A_1, \ldots, A_N) \in M_d(\mathbb{R})^N$ *and* $k \in \{0, \ldots, d-1\}$. *Suppose that* $(A_1^{\wedge k}, \ldots, A_N^{\wedge k})$ *strictly preserves a multicone* $(\mathcal{K}_1^{(1)}, \ldots, \mathcal{K}_{m_1}^{(1)})$ *in* $\wedge^k\mathbb{R}^d$ *with transverse-defining vector* $w_1 \in \wedge^k\mathbb{R}^d$ *and that* $(A_1^{\wedge(k+1)}, \ldots, A_N^{\wedge(k+1)})$ *strictly preserves a multicone* $(\mathcal{K}_1^{(2)}, \ldots, \mathcal{K}_{m_2}^{(2)})$ *in* $\wedge^{k+1}\mathbb{R}^d$ *with transverse-defining vector* $w_2 \in \wedge^{k+1}\mathbb{R}^d$. *Define*

$$\Omega_1 := \left\{ z \in \wedge^k\mathbb{C}^d \colon z \in \bigcup_{j=1}^{m_1} \left(\operatorname{Int}\mathcal{K}_j^{(1)}\right)^{\mathbb{C}} \text{ and } \langle z, w_1\rangle = 1 \right\},$$

$$\Omega_2 := \left\{ z \in \wedge^{k+1}\mathbb{C}^d \colon z \in \bigcup_{j=1}^{m_2} \left(\operatorname{Int}\mathcal{K}_j^{(2)}\right)^{\mathbb{C}} \text{ and } \langle z, w_2\rangle = 1 \right\}$$

and for every $f \in \mathcal{A}^2(\Omega_1 \times \Omega_2)$, $t = (t_1, t_2) \in \mathbb{C}^2$ *and* $(z_1, z_2) \in \Omega_1 \times \Omega_2$ *define*

$$\left(\mathscr{L}_t f\right)(z) := \sum_{j=1}^{N} \left(\frac{\langle A_j^{\wedge k} z_1, w_1\rangle}{\operatorname{sign}\Re(\langle A_j^{\wedge k} z_1, w_1\rangle)} \right)^{t_1} \left(\frac{\langle A_j^{\wedge(k+1)} z_2, w_2\rangle}{\operatorname{sign}\Re(\langle A_j^{\wedge(k+1)} z_2, w_2\rangle)} \right)^{t_2}$$
$$\cdot f\left(\langle A_j^{\wedge k} z_1, w_1\rangle^{-1} A_j^{\wedge k} z_1, \langle A_j^{\wedge(k+1)} z_2, w_2\rangle^{-1} A_j^{\wedge(k+1)} z_2 \right).$$

Then $\mathscr{L}_t\colon \mathcal{A}^2(\Omega_1 \times \Omega_2) \to \mathcal{A}^2(\Omega_1 \times \Omega_2)$ *is a well-defined bounded linear operator, and:*

(i) *There exist constants* $C, \kappa, \gamma > 0$ *such that the approximation numbers* $\mathfrak{s}_n(\mathscr{L}_t)$ *satisfy* $\mathfrak{s}_n(\mathscr{L}_t) \leq C\exp(\kappa\|t\| - \gamma n^{1/(\hat{d}-2)})$ *for every* $n \geq 1$ *and* $t = (t_1, t_2) \in \mathbb{C}^2$, *where* $\hat{d} := \binom{d+1}{k+1}$.
(ii) *For each* $n \geq 1$ *the trace of the operator* $\mathscr{L}_t^n$ *is equal to*

$$\sum_{|\mathtt{i}|=n} \frac{\lambda_1\left(A_{\mathtt{i}}^{\wedge k}\right)^{\binom{d}{k}-1} \lambda_1\left(A_{\mathtt{i}}^{\wedge(k+1)}\right)^{\binom{d}{k+1}-1} \rho\left(A_{\mathtt{i}}^{\wedge k}\right)^{t_1} \rho\left(A_{\mathtt{i}}^{\wedge(k+1)}\right)^{t_2}}{p'_{A_{\mathtt{i}}^{\wedge k}}\left(\lambda_1\left(A_{\mathtt{i}}^{\wedge k}\right)\right) p'_{A_{\mathtt{i}}^{\wedge(k+1)}}\left(\lambda_1\left(A_{\mathtt{i}}^{\wedge(k+1)}\right)\right)} \tag{7}$$

where $p_B(x) := \det(xI - B)$ *denotes the characteristic polynomial of* B *and* $p'_B(x_0)$ *its derivative evaluated at* x_0.

(iii) If $t_1, t_2 \in \mathbb{R}$ then

$$\rho(\mathscr{L}_t) = \lim_{n\to\infty} \left(\sum_{|\mathtt{i}|=n} \left\| A_{\mathtt{i}}^{\wedge k} \right\|^{t_1} \left\| A_{\mathtt{i}}^{\wedge (k+1)} \right\|^{t_2} \right)^{\frac{1}{n}}$$

and in particular this limit exists. Furthermore in this case $\rho(\mathscr{L}_t)$ is a simple eigenvalue of $\mathscr{L}_t$, and $\mathscr{L}_t$ has no other eigenvalues with modulus equal to $\rho(\mathscr{L}_t)$.

If $k = 0$ then $\left(\mathcal{K}_j^{(1)}\right)^{\mathbb{C}}$ is equal to $\mathbb{C}$ so that Ω_1 is a single point and $\hat{d} - 2 = d - 1$, and the expression (7) simplifies considerably in the same manner as was noted subsequently to the statement of Theorem 3. Similarly if $k = d - 1$ then $\left(\mathcal{K}_j^{(2)}\right)^{\mathbb{C}} = \wedge^d \mathbb{C} \simeq \mathbb{C}$, Ω_2 is a point and the formula for $t_n(s)$ simplifies as described in §.

We emphasise that the proof of Theorem 12 will not in any respect use the fact that $A_{\mathtt{i}}^{\wedge k}$ and $A_{\mathtt{i}}^{\wedge (k+1)}$ are related by being different exterior powers of the same matrix. Indeed, one could as easily prove a more general result in which the two tuples $(A_1^{\wedge k}, \ldots, A_N^{\wedge k})$ and $(A_1^{\wedge (k+1)}, \ldots, A_N^{\wedge (k+1)})$ are replaced by m unrelated tuples $(A_1^{(i)}, \ldots, A_N^{(i)}) \in M_{d_i}(\mathbb{R})^N$ for $i = 1, \ldots, m$ with the property that each tuple individually is multipositive, resulting in a theorem describing an operator $\mathscr{L}_t$ which for $t = (t_1, \ldots, t_m) \in \mathbb{R}^m$ has spectral radius

$$\lim_{n\to\infty} \left(\sum_{|\mathtt{i}|=n} \prod_{i=1}^{m} \left\| A_{\mathtt{i}}^{(i)} \right\|^{t_i} \right)^{\frac{1}{n}}$$

and for $t \in \mathbb{C}^m$ has approximation numbers

$$\mathfrak{s}_n(\mathscr{L}_t) = O\left(\exp\left(\kappa\|t\| - \gamma n^{\beta}\right)\right)$$

and traces

$$\operatorname{tr} \mathscr{L}_t^n = \sum_{|\mathtt{i}|=n} \prod_{i=1}^{m} \frac{\lambda_1\left(A_{\mathtt{i}}^{(i)}\right)^{d_i - 1} \rho\left(A_{\mathtt{i}}^{(i)}\right)^{t_i}}{p'_{A_{\mathtt{i}}^{(i)}}\left(\lambda_1\left(A_{\mathtt{i}}^{(i)}\right)\right)},$$

where $\beta := \left(\sum_{i=1}^m (d_i - 1)\right)^{-1}$. This underlying mathematical attitude to the singular value function φ^s in Theorem 12 – namely, of treating it as a product of the powers of the norms of two essentially unrelated matrix products – is identical to that used in [12], and we suspect that other results of a similar character such as [23, 29] could in principle be rewritten in those terms. We leave to the reader the labour of constructing in detail the more general version of Theorem 12 just described.

The case $k = 0$ of Theorem 12 will be used in a subsequent article and we single out its statement for later convenience:

Corollary 5.1. *Let $\mathsf{A} = (A_1, \ldots, A_N) \in M_d(\mathbb{R})^N$ and suppose that A strictly preserves a multicone $(\mathcal{K}_1, \ldots, \mathcal{K}_m)$ in $\mathbb{R}^d$ with transverse-defining vector $w \in \mathbb{R}^d$.*

Define

$$\Omega := \left\{ z \in \mathbb{C}^d : z \in \bigcup_{j=1}^{m} \operatorname{Int} \mathcal{K}_j^{\mathbb{C}} \text{ and } \langle z, w \rangle = 1 \right\},$$

and for every $f \in \mathcal{A}^2(\Omega)$, $t \in \mathbb{C}$ and $z \in \Omega$ define

$$(\mathscr{L}_t f)(z) := \sum_{j=1}^{N} \left(\frac{\langle A_j z, w \rangle}{\operatorname{sign} \Re(\langle A_j z, w \rangle)} \right)^t f\left(\langle A_j z, w \rangle^{-1} A_j z \right).$$

Then $\mathscr{L}_t \colon \mathcal{A}^2(\Omega) \to \mathcal{A}^2(\Omega)$ is a well-defined bounded linear operator, and:

(i) There exist constants $C, \kappa, \gamma > 0$ such that the approximation numbers $\mathfrak{s}_n(\mathscr{L}_t)$ satisfy $\mathfrak{s}_n(\mathscr{L}_t) \leq C \exp(\kappa |t| - \gamma n^{1/(d-1)})$ for every $n \geq 1$ and $t \in \mathbb{C}$.

(ii) For each $n \geq 1$ the trace of the operator $\mathscr{L}_t^n$ is equal to

$$\sum_{|\mathtt{i}|=n} \frac{\lambda_1(A_{\mathtt{i}})^{d-1} \rho(A_{\mathtt{i}})^t}{p'_{A_{\mathtt{i}}}(\lambda_1(A_{\mathtt{i}}))}$$

where $p_B(x) := \det(xI - B)$ denotes the characteristic polynomial of B and $p'_B(x_0)$ its derivative evaluated at x_0.

(iii) If $t \in \mathbb{R}$ then

$$\rho(\mathscr{L}_t) = \lim_{n \to \infty} \left(\sum_{|\mathtt{i}|=n} \|A_{\mathtt{i}}\|^t \right)^{\frac{1}{n}}.$$

Furthermore in this case $\rho(\mathscr{L}_t)$ is a simple eigenvalue of $\mathscr{L}_t$, and $\mathscr{L}_t$ has no other eigenvalues with modulus equal to $\rho(\mathscr{L}_t)$.

Proof of Theorem 12. We observe that Theorem 5 applies to $(A_1^{\wedge k}, \ldots, A_N^{\wedge k})$, $(\mathcal{K}_1^{(1)}, \ldots, \mathcal{K}_{m_1}^{(1)})$ and w_1 and also applies to $(A_1^{\wedge(k+1)}, \ldots, A_N^{\wedge(k+1)})$, $(\mathcal{K}_1^{(2)}, \ldots, \mathcal{K}_{m_2}^{(2)})$ and w_2. Define $\Omega := \Omega_1 \times \Omega_2$ which, via Theorem 5(iv), is a bounded, nonempty open subset of a 2-codimensional affine subspace of $\wedge^k \mathbb{C}^d \oplus \wedge^{k+1} \mathbb{C}^d \simeq \mathbb{C}^{\hat{d}}$.

By Theorem 5(iii), for each $\mathtt{i} \in \Sigma_N^*$ the function $z_1 \mapsto \operatorname{sign} \Re(\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1 \rangle)$ is constant and nowhere zero on each connected component of Ω_1 and is therefore holomorphic on Ω_1. For $\mathtt{i} \in \Sigma_N^*$ define

$$\phi_{\mathtt{i}}^{(1)}(z_1) := \langle A_{\mathtt{i}}^{\wedge k} z_1, w_1 \rangle^{-1} A_{\mathtt{i}}^{\wedge k} z_1$$

and

$$\psi_{\mathtt{i},t}^{(1)}(z_1) := \left(\frac{\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1 \rangle}{\operatorname{sign} \Re(\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1 \rangle)} \right)^{t_1} := \exp\left(t_1 \log\left(\frac{\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1 \rangle}{\operatorname{sign} \Re(\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1 \rangle)} \right) \right)$$

for all $z_1 \in \Omega_1$. We observe that $\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1 \rangle / \operatorname{sign} \Re(\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1 \rangle)$ has positive real part for all $z_1 \in \Omega_1$ and therefore its logarithm is a well-defined holomorphic function of $z_1 \in \Omega_1$. These considerations also ensure that the definition of $\phi_{\mathtt{i}}^{(1)}$ does not constitute a division by zero. We also observe that the set $\Omega_1^0 := \bigcup_{i=1}^N \overline{\phi_i^{(1)}(\Omega_1)}$ is a compact subset of Ω_1 and that $\phi_{\mathtt{i}}^{(1)}(\Omega_1) \subseteq \Omega_1^0$ for every $\mathtt{i} \in \Sigma_N^*$ by Theorem 5(iv) and (viii).

Since for all $z_1 \in \Omega_1$

$$\left|\Re\left(\log\left(\frac{\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1\rangle}{\operatorname{sign}\Re(\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1\rangle)}\right)\right) - \log\left\|A_{\mathtt{i}}^{\wedge k}\right\|\right| \leq \log C_1$$

for some constant $C_1 > 1$ using Theorem 5(vii), and also

$$\Im\left(\log\left(\frac{\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1\rangle}{\operatorname{sign}\Re(\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1\rangle)}\right) = \arg\left(\frac{\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1\rangle}{\operatorname{sign}\Re(\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1\rangle)}\right)\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

by the preceding observations, we have

$$\begin{aligned}\Re\left(t_1 \log\left(\frac{\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1\rangle}{\operatorname{sign}\Re(\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1\rangle)}\right)\right) &= \Re(t_1)\Re\left(\log\left(\frac{\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1\rangle}{\operatorname{sign}\Re(\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1\rangle)}\right)\right) \\ &\quad - \Im(t_1)\Im\left(\log\left(\frac{\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1\rangle}{\operatorname{sign}\Re(\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1\rangle)}\right)\right) \\ &\leq \Re(t_1)\log\left\|A_{\mathtt{i}}^{\wedge k}\right\| + |\Re(t_1)|\log C_1 + \frac{\pi}{2}\,|\Im(t_1)|\end{aligned}$$

for all $z_1 \in \Omega_1$ and therefore

$$\sup_{z_1\in\Omega_1}\left|\psi_{\mathtt{i},t}^{(1)}(z_1)\right| \leq \left(C_1 e^{\pi/2}\right)^{|t_1|}\left\|A_{\mathtt{i}}^{\wedge k}\right\|^{\Re(t_1)}$$

for every $\mathtt{i} \in \Sigma_N^*$. In a similar manner we may define

$$\phi_{\mathtt{i}}^{(2)}(z_2) := \langle A_{\mathtt{i}}^{\wedge(k+1)} z_2, w_2\rangle^{-1} A_{\mathtt{i}}^{\wedge(k+1)} z_2$$

and

$$\psi_{\mathtt{i},t}^{(2)}(z_2) := \left(\frac{\langle A_{\mathtt{i}}^{\wedge(k+1)} z_2, w_2\rangle}{\operatorname{sign}\Re(\langle A_{\mathtt{i}}^{\wedge(k+1)} z_2, w_2\rangle)}\right)^{t_2} := \exp\left(t_2\log\left(\frac{\langle A_{\mathtt{i}}^{\wedge(k+1)} z_2, w_2\rangle}{\operatorname{sign}\Re(\langle A_{\mathtt{i}}^{\wedge(k+1)} z_2, w_2\rangle)}\right)\right)$$

for all $z_2 \in \Omega_2$ and obtain

$$\sup_{z_2\in\Omega_2}\left|\psi_{\mathtt{i},t}^{(2)}(z_2)\right| \leq \left(C_1 e^{\pi/2}\right)^{|t_2|}\left\|A_{\mathtt{i}}^{\wedge(k+1)}\right\|^{\Re(t_2)}$$

for every $\mathtt{i} \in \Sigma_N^*$, where the value of C_1 has been increased from its previous value if necessary. The analogously-defined set $\Omega_2^0 := \bigcup_{i=1}^N \overline{\phi_i^{(2)}(\Omega_2)}$ is a compact subset of Ω_2 and similarly satisfies $\phi_{\mathtt{i}}^{(2)}(\Omega_2) \subseteq \Omega_2^0$ for every $\mathtt{i} \in \Sigma_N^*$.

Define $\phi_{\mathtt{i}} \colon \Omega \to \Omega$ by $\phi_{\mathtt{i}}(z_1, z_2) := (\phi_{\mathtt{i}}^{(1)}(z_1), \phi_{\mathtt{i}}^{(2)}(z_2))$ for all $z = (z_1, z_2) \in \Omega$ and $\mathtt{i} \in \Sigma_N^*$. Obviously each $\phi_{\mathtt{i}}$ is holomorphic and $\phi_{\mathtt{i}}(\Omega)$ is contained in the compact set $\Omega_0 := \Omega_1^0 \times \Omega_2^0 \Subset \Omega$. For every $t = (t_1, t_2) \in \mathbb{C}^d$, $\mathtt{i} \in \Sigma_N^*$ and $z = (z_1, z_2) \in \Omega$ define $\psi_{\mathtt{i},t}(z) := \psi_{\mathtt{i},t}^{(1)}(z_1)\psi_{\mathtt{i},t}^{(2)}(z_2)$. Clearly we have

$$\sup_{z\in\Omega}|\psi_{\mathtt{i},t}(z)| \leq C_2^{\|t\|}\left\|A_{\mathtt{i}}^{\wedge k}\right\|^{\Re(t_1)}\left\|A_{\mathtt{i}}^{\wedge(k+1)}\right\|^{\Re(t_2)} \tag{8}$$

for every $\mathtt{i} \in \Sigma_N^*$ and $t \in \mathbb{C}^2$, where $C_2 := (C_1 e^{\pi/2})^{\sqrt{2}}$. In particular

$$\sum_{i=1}^N \sup_{z\in\Omega}|\psi_{i,t}(z)| \leq C_2^{\|t\|}\sum_{i=1}^N\left\|A_i^{\wedge k}\right\|^{\Re(t_1)}\left\|A_i^{\wedge(k+1)}\right\|^{\Re(t_2)} \leq NC_3^{\|t\|}, \tag{9}$$

say, for every $t \in \mathbb{C}^2$. We may now define the operator $\mathscr{L}_t$ by

$$(\mathscr{L}_t f)(z) := \sum_{i=1}^N \psi_{i,t}(z) f(\phi_i(z))$$

for all $f \in \mathcal{A}^2(\Omega)$ and $z \in \Omega$. The set Ω is a bounded, nonempty open subset of a 2-codimensional affine subspace of $\mathbb{C}^{\hat{d}}$, so it follows by Theorem 8 that $\mathscr{L}_t$ is a well-defined bounded linear operator acting on $\mathcal{A}^2(\Omega)$ and that there exist $C, \gamma > 0$ depending only on Ω_0 such that for all $t \in \mathbb{C}^2$ we have

$$\mathfrak{s}_n(\mathscr{L}_t) \leq C \left(\sum_{i=1}^N \sup_{z \in \Omega} |\psi_{i,t}(z)| \right) \exp\left(-\gamma n^{1/(\hat{d}-2)}\right) \leq CN \exp\left(\kappa \|t\| - \gamma n^{1/(\hat{d}-2)}\right)$$

as a consequence of (9), where $\kappa := \log C_3$. This proves (i).

It follows from (i) that $\mathscr{L}_t$ is a trace-class operator. For each $\mathtt{i} \in \Sigma_N^*$ and $t \in \mathbb{C}^2$ let us define an auxiliary operator $\mathscr{L}_{\mathtt{i},t}$ by

$$(\mathscr{L}_{\mathtt{i},t} f)(z) := \psi_{\mathtt{i},t}(z) f(\phi_{\mathtt{i}}(z)).$$

Theorem 8 shows in the same manner as before that each $\mathscr{L}_{\mathtt{i},t}$ is a well-defined trace-class operator on $\mathcal{A}^2(\Omega)$. The reader may easily verify the equations

$$\psi_{\mathtt{ji},t}(z) = \psi_{\mathtt{i},t}(\phi_{\mathtt{j}}(z))\psi_{\mathtt{j},t}(z), \qquad \phi_{\mathtt{ji}}(z) = \phi_{\mathtt{i}}(\phi_{\mathtt{j}}(z))$$

and therefore $\mathscr{L}_{\mathtt{ji},t} = \mathscr{L}_{\mathtt{i},t}\mathscr{L}_{\mathtt{j},t}$ for all $\mathtt{i},\mathtt{j} \in \Sigma_N^*$ and $t \in \mathbb{C}^2$. It follows by a simple inductive argument that $\mathscr{L}_t^n = \sum_{|\mathtt{i}|=n} \mathscr{L}_{\mathtt{i},t}$ for every $n \geq 1$ and $t \in \mathbb{C}^2$, so in particular we have

$$\operatorname{tr} \mathscr{L}_t = \operatorname{tr} \sum_{|\mathtt{i}|=n} \mathscr{L}_{\mathtt{i},t} = \sum_{|\mathtt{i}|=n} \operatorname{tr} \mathscr{L}_{\mathtt{i},t} \tag{10}$$

for every $n \geq 1$ and $t \in \mathbb{C}^2$ by the linearity of the trace.

Let $\mathtt{i} \in \Sigma_N^*$. By Theorem 5(vi) the map $\phi_{\mathtt{i}}^{(1)}$ has a unique fixed point $z_{\mathtt{i}}^{(1)} \in \Omega_1$. This fixed point satisfies $z_{\mathtt{i}}^{(1)} \in \Omega_1 \cap \wedge^k \mathbb{R}^d$ and $\langle A_{\mathtt{i}}^{\wedge k} z_{\mathtt{i}}^{(1)}, w_1 \rangle = \lambda_1(A_{\mathtt{i}}^{\wedge k})$ and it follows directly that $\psi_{\mathtt{i},t}^{(1)}(z_{\mathtt{i}}^{(1)}) = \rho(A_{\mathtt{i}}^{\wedge k})^{t_1}$ by inspection of the definitions. Similarly $\phi_{\mathtt{i}}^{(2)}$ has a unique fixed point $z_{\mathtt{i}}^{(2)} \in \Omega_2$ which satisfies $z_{\mathtt{i}}^{(2)} \in \Omega_2 \cap \wedge^{k+1}\mathbb{R}^d$ and $\psi_{\mathtt{i},t}^{(2)}(z_{\mathtt{i}}^{(2)}) = \rho(A_{\mathtt{i}}^{\wedge(k+1)})^{t_2}$. If we define $z_{\mathtt{i}} := (z_{\mathtt{i}}^{(1)}, z_{\mathtt{i}}^{(2)}) \in \Omega \cap (\wedge^k \mathbb{R}^d \oplus \wedge^{k+1}\mathbb{R}^d)$ then $z_{\mathtt{i}}$ is clearly the unique fixed point of $\phi_{\mathtt{i}}$ in Ω and we have $\psi_{\mathtt{i},t}(z_{\mathtt{i}}) = \rho(A_{\mathtt{i}}^{\wedge k})^{t_1} \rho(A_{\mathtt{i}}^{\wedge(k+1)})^{t_2}$. By Theorem 5(vi) the derivative $D_{z_{\mathtt{i}}^{(1)}} \phi_{\mathtt{i}}^{(1)}$ of $\phi_{\mathtt{i}}^{(1)}$ at $z_{\mathtt{i}}^{(1)}$ satisfies

$$\det\left(I - D_{z_{\mathtt{i}}^{(1)}} \phi_{\mathtt{i}}^{(1)}\right) = \frac{p'_{A_{\mathtt{i}}^{\wedge k}}\left(\lambda_1\left(A_{\mathtt{i}}^{\wedge k}\right)\right)}{\lambda_1\left(A_{\mathtt{i}}^{\wedge k}\right)^{\binom{d}{k}-1}}$$

and the derivative $D_{z_{\mathtt{i}}^{(2)}} \phi_{\mathtt{i}}^{(2)}$ of $\phi_{\mathtt{i}}^{(2)}$ at $z_{\mathtt{i}}^{(2)}$ satisfies

$$\det\left(I - D_{z_{\mathtt{i}}^{(2)}} \phi_{\mathtt{i}}^{(2)}\right) = \frac{p'_{A_{\mathtt{i}}^{\wedge(k+1)}}\left(\lambda_1\left(A_{\mathtt{i}}^{\wedge(k+1)}\right)\right)}{\lambda_1\left(A_{\mathtt{i}}^{\wedge(k+1)}\right)^{\binom{d}{k+1}-1}}.$$

Since clearly $D_{z_{\mathtt{i}}}\phi_{\mathtt{i}} = D_{z_{\mathtt{i}}^{(1)}}\phi_{\mathtt{i}}^{(1)} \oplus D_{z_{\mathtt{i}}^{(2)}}\phi_{\mathtt{i}}^{(2)}$ we easily obtain

$$\begin{aligned}\det\left(I - D_{z_{\mathtt{i}}}\phi_{\mathtt{i}}\right) &= \det\left(I - D_{z_{\mathtt{i}}^{(1)}}\phi_{\mathtt{i}}^{(1)}\right)\det\left(I - D_{z_{\mathtt{i}}^{(2)}}\phi_{\mathtt{i}}^{(2)}\right)\\ &= \frac{p'_{A_{\mathtt{i}}^{\wedge k}}\left(\lambda_1\left(A_{\mathtt{i}}^{\wedge k}\right)\right)p'_{A_{\mathtt{i}}^{\wedge(k+1)}}\left(\lambda_1\left(A_{\mathtt{i}}^{\wedge(k+1)}\right)\right)}{\lambda_1\left(A_{\mathtt{i}}^{\wedge k}\right)^{\binom{d}{k}-1}\lambda_1\left(A_{\mathtt{i}}^{\wedge(k+1)}\right)^{\binom{d}{k+1}-1}}.\end{aligned}$$

It follows by Theorem 10 that

$$\operatorname{tr}\mathscr{L}_{\mathtt{i},t} = \frac{\lambda_1\left(A_{\mathtt{i}}^{\wedge k}\right)^{\binom{d}{k}-1}\lambda_1\left(A_{\mathtt{i}}^{\wedge(k+1)}\right)^{\binom{d}{k+1}-1}\rho\left(A_{\mathtt{i}}^{\wedge k}\right)^{t_1}\rho\left(A_{\mathtt{i}}^{\wedge(k+1)}\right)^{t_2}}{p'_{A_{\mathtt{i}}^{\wedge k}}\left(\lambda_1\left(A_{\mathtt{i}}^{\wedge k}\right)\right)p'_{A_{\mathtt{i}}^{\wedge(k+1)}}\left(\lambda_1\left(A_{\mathtt{i}}^{\wedge(k+1)}\right)\right)} \tag{11}$$

for every $t = (t_1, t_2) \in \mathbb{C}^2$ and every $\mathtt{i} \in \Sigma_N^*$. We obtain (7) by combining (10) with (11) which proves (ii).

The proof of (iii) is by far the longest part of Theorem 12 and requires substantial groundwork. We begin this process by investigating the limit set $L := \bigcap_{n=1}^{\infty}\bigcup_{|\mathtt{i}|=n}\overline{\phi_{\mathtt{i}}(\Omega)}$. (Subsequently, we will analyse the spectrum of $\mathscr{L}_t$ by studying the restrictions of certain eigenfunctions to this limit set.) We claim that every nonempty closed set $Z \subseteq L$ with the property $\bigcup_{i=1}^{N}\phi_i(Z) \subseteq Z$ is equal to the set

$$L' := \overline{\{z_{\mathtt{i}} \colon \mathtt{i} \in \Sigma_N^*\}}.$$

Since L itself is such a set we in particular have $L' = L$, and therefore every nonempty closed set $Z \subseteq L$ with the property $\bigcup_{i=1}^{N}\phi_i(Z) \subseteq Z$ is in fact equal to L. The claim thus additionally yields $L \subseteq \Omega \cap (\wedge^k\mathbb{R}^d \oplus \wedge^{k+1}\mathbb{R}^d)$ since every $z_{\mathtt{i}}$ belongs to that set.

Let us prove the claim. Theorem 5(v) implies that

$$\lim_{n\to\infty}\max_{|\mathtt{i}|=n}\operatorname{diam}\phi_{\mathtt{i}}^{(1)}(\Omega_1) = \lim_{n\to\infty}\max_{|\mathtt{i}|=n}\operatorname{diam}\phi_{\mathtt{i}}^{(2)}(\Omega_2) = 0$$

and it follows immediately that

$$\lim_{n\to\infty}\max_{|\mathtt{i}|=n}\operatorname{diam}\phi_{\mathtt{i}}(\Omega) = 0. \tag{12}$$

Suppose that $Z \subseteq L$ has the property $\bigcup_{i=1}^{N}\phi_i(Z) \subseteq Z$ and is closed and nonempty. Let $z \in Z$ and $\mathtt{i} \in \Sigma_N^*$. It is clear that $\phi_{\mathtt{i}}^n(z) \in Z$ for every $n \geq 1$. By (12) we have $\lim_{n\to\infty}\operatorname{diam}\phi_{\mathtt{i}}^n(\Omega) = 0$ and clearly $z_{\mathtt{i}} \in \phi_{\mathtt{i}}^n(\Omega)$ for every $n \geq 1$, so necessarily $\lim_{n\to\infty} d(\phi_{\mathtt{i}}^n z, z_{\mathtt{i}}) = 0$ and therefore $z_{\mathtt{i}} = \lim_{n\to\infty}\phi_{\mathtt{i}}^n z \in Z$. We conclude that $\{z_{\mathtt{i}} \colon \mathtt{i} \in \Sigma_N^*\} \subseteq Z$ and consequently $L' \subseteq Z$. On the other hand, if $z \in Z$ then since $z \in L$ we may for every $n \geq 1$ find $\mathtt{i}(n) \in \Sigma_N^*$ such that $|\mathtt{i}(n)| = n$ and $z \in \overline{\phi_{\mathtt{i}(n)}(\Omega)}$. Since $z_{\mathtt{i}(n)} \in \phi_{\mathtt{i}(n)}(\Omega)$ we in particular have $d(z, z_{\mathtt{i}(n)}) \leq \operatorname{diam}\phi_{\mathtt{i}(n)}(\Omega)$ for every $n \to \infty$ which by (12) implies $\lim_{n\to\infty} d(z, z_{\mathtt{i}(n)}) = 0$. We conclude that $z \in \overline{\{z_{\mathtt{i}} \in \Sigma_N^*\}}$ and therefore $Z \subseteq L'$ as required to prove the claim.

The next piece of groundwork is a lower estimate for the value of $\psi_{\mathtt{i},t}(z)$ when $z \in L$ and $t \in \mathbb{R}^2$. If $t_1 \in \mathbb{R}$ and $z_1 \in \Omega_1 \cap \wedge^k \mathbb{R}^d$ then we have

$$\left(\frac{\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1\rangle}{\operatorname{sign} \Re(\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1\rangle)}\right)^{t_1} = \left|\langle A_{\mathtt{i}}^{\wedge k} z_1, w_1\rangle\right|^{t_1} \geq C_1^{-|t_1|} \left\|A_{\mathtt{i}}^{\wedge k}\right\|^{t_1}$$

using Theorem 5(vii), and similarly if $t_2 \in \mathbb{R}$ and $z_2 \in \Omega_2 \cap \wedge^{k+1}\mathbb{R}^d$ then

$$\left(\frac{\langle A_{\mathtt{i}}^{\wedge (k+1)} z_2, w_2\rangle}{\operatorname{sign} \Re(\langle A_{\mathtt{i}}^{\wedge (k+1)} z_2, w_2\rangle)}\right)^{t_2} = \left|\langle A_{\mathtt{i}}^{\wedge (k+1)} z_2, w_2\rangle\right|^{t_2} \geq C_1^{-|t_2|} \left\|A_{\mathtt{i}}^{\wedge (k+1)}\right\|^{t_2}.$$

In particular the values of $\psi_{\mathtt{i},t}^{(1)}(z_1)$ and $\psi_{\mathtt{i},t}^{(2)}(z_2)$ in this case are real and positive. It follows that if $t \in \mathbb{R}^2$ and $z \in L \subset \Omega \cap (\wedge^k\mathbb{R}^d \oplus \wedge^{k+1}\mathbb{R}^d)$ then $\psi_{\mathtt{i},t}(z)$ is real and positive and satisfies

$$\psi_{\mathtt{i},t}(z) \geq C_1^{-|t_1|-|t_2|} \left\|A_{\mathtt{i}}^{\wedge k}\right\|^{t_1} \left\|A_{\mathtt{i}}^{\wedge (k+1)}\right\|^{t_2}, \tag{13}$$

an estimate which will be repeatedly found useful later.

The final piece of groundwork is the investigation of the rate of growth of the sums $\sum_{|\mathtt{i}|=n} \|A_{\mathtt{i}}^{\wedge k}\|^{t_1} \|A_{\mathtt{i}}^{\wedge (k+1)}\|^{t_2}$ for $t = (t_1, t_2) \in \mathbb{R}^2$. We claim that the limit

$$\mathscr{P}(t) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\mathtt{i}|=n} \left\|A_{\mathtt{i}}^{\wedge k}\right\|^{t_1} \left\|A_{\mathtt{i}}^{\wedge (k+1)}\right\|^{t_2}$$

exists and that there is $C_4 > 1$ such that

$$\sum_{|\mathtt{i}|=n} \left\|A_{\mathtt{i}}^{\wedge k}\right\|^{t_1} \left\|A_{\mathtt{i}}^{\wedge (k+1)}\right\|^{t_2} \leq C_4^{\|t\|} e^{n\mathscr{P}(t)} \tag{14}$$

for every $n \geq 1$ and $t \in \mathbb{R}^2$. By Theorem 5(ii) there are constants $\tau_1, \tau_2 \in (0, 1]$ such that

$$\tau_1 \left\|A_{\mathtt{i}}^{\wedge k}\right\| \cdot \left\|A_{\mathtt{j}}^{\wedge k}\right\| \leq \left\|A_{\mathtt{i}}^{\wedge k} A_{\mathtt{j}}^{\wedge k}\right\| \leq \left\|A_{\mathtt{i}}^{\wedge k}\right\| \cdot \left\|A_{\mathtt{j}}^{\wedge k}\right\|,$$

$$\tau_2 \left\|A_{\mathtt{i}}^{\wedge (k+1)}\right\| \cdot \left\|A_{\mathtt{j}}^{\wedge (k+1)}\right\| \leq \left\|A_{\mathtt{i}}^{\wedge (k+1)} A_{\mathtt{j}}^{\wedge (k+1)}\right\| \leq \left\|A_{\mathtt{i}}^{\wedge (k+1)}\right\| \cdot \left\|A_{\mathtt{j}}^{\wedge (k+1)}\right\|$$

for all $\mathtt{i}, \mathtt{j} \in \Sigma_N^*$. In particular we have

$$\left\|A_{\mathtt{ij}}^{\wedge k}\right\|^{t_1} \left\|A_{\mathtt{ij}}^{\wedge (k+1)}\right\|^{t_2} \geq \tau_1^{|t_1|} \tau_2^{|t_2|} \left\|A_{\mathtt{i}}^{\wedge k}\right\|^{t_1} \left\|A_{\mathtt{i}}^{\wedge (k+1)}\right\|^{t_2} \left\|A_{\mathtt{j}}^{\wedge k}\right\|^{t_1} \left\|A_{\mathtt{j}}^{\wedge (k+1)}\right\|^{t_2}$$

for all $\mathtt{i}, \mathtt{j} \in \Sigma_N^*$. It follows that the sequence

$$a_n := \sum_{|\mathtt{i}|=n} \left\|A_{\mathtt{i}}^{\wedge k}\right\|^{t_1} \left\|A_{\mathtt{i}}^{\wedge (k+1)}\right\|^{t_2}$$

satisfies $a_{n+m} \geq \tau_1^{|t_1|} \tau_2^{|t_2|} a_n a_m$ for every $n, m \geq 1$. In particular the sequence $\log(\tau_1^{|t_1|} \tau_2^{|t_2|} a_n)$ is superadditive and therefore

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \left(\tau_1^{|t_1|} \tau_2^{|t_2|} a_n\right)^{1/n} = \sup_{n\geq 1} \left(\tau_1^{|t_1|} \tau_2^{|t_2|} a_n\right)^{1/n}$$

which yields the existence of the limit

$$\mathscr{P}(t) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{|\mathtt{i}|=n} \left\|A_{\mathtt{i}}^{\wedge k}\right\|^{t_1} \left\|A_{\mathtt{i}}^{\wedge (k+1)}\right\|^{t_2} \right)$$

together with the inequality

$$\sum_{|\mathtt{i}|=n} \left\|A_{\mathtt{i}}^{\wedge k}\right\|^{t_1} \left\|A_{\mathtt{i}}^{\wedge (k+1)}\right\|^{t_2} \leq \tau_1^{-|t_1|}\tau_2^{-|t_2|} e^{n\mathscr{P}(t)}$$

as desired.

The groundwork being completed, we may now begin to analyse the spectrum of the operator $\mathscr{L}_t$ for $t \in \mathbb{R}^2$. In the first step in this process we show at once that $\rho(\mathscr{L}_t) \leq e^{\mathscr{P}(t)}$ and also that any eigenfunctions of $\mathscr{L}_t$ corresponding to sufficiently large eigenvalues are everywhere approximately bounded by their values on the limit set L. Specifically, we claim that there exists $C_5 > 1$ such that if $\mathscr{L}_t f = \lambda f$, $|\lambda| \geq e^{\mathscr{P}(t)}$ and $f \in \mathcal{A}^2(\Omega)$ then

$$\sup_{z\in\Omega} |f(z)| \leq C_5^{\|t\|} \sup_{z\in L} |f(z)|, \tag{15}$$

and if f is nonzero then additionally $|\lambda| = e^{\mathscr{P}(t)}$. To prove the claim let $z_0 \in \Omega$ be arbitrary: for each $n \geq 1$ we have

$$\begin{aligned} |\lambda^n| \cdot |f(z_0)| = |(\mathscr{L}_t^n f)(z_0)| &= \left| \sum_{|\mathtt{i}|=n} \psi_{\mathtt{i},t}(z_0) f(\phi_{\mathtt{i}}(z_0)) \right| \\ &\leq C_2^{\|t\|} \sum_{|\mathtt{i}|=n} \left\|A_{\mathtt{i}}^{\wedge k}\right\|^{t_1} \left\|A_{\mathtt{i}}^{\wedge (k+1)}\right\|^{t_2} |f(\phi_{\mathtt{i}}(z_0))| \\ &\leq C_4^{\|t\|} C_2^{\|t\|} e^{n\mathscr{P}(t)} \sup_{z\in\overline{\bigcup_{|\mathtt{i}|=n}\phi_{\mathtt{i}}(\Omega)}} |f(z)| \end{aligned}$$

using (8) and (14) so that

$$\sup_{z\in\Omega} |f(z)| \leq C_4^{\|t\|} C_2^{\|t\|} |\lambda|^{-n} e^{n\mathscr{P}(t)} \sup_{z\in\overline{\bigcup_{|\mathtt{i}|=n}\phi_{\mathtt{i}}(\Omega)}} |f(z)|$$

for all $n \geq 1$. If $|\lambda| = e^{\mathscr{P}(t)}$ then we obtain (15) by taking the infimum of the right-hand side with respect to $n \in \mathbb{N}$; if $|\lambda| > e^{\mathscr{P}(t)}$ the same mechanism yields $f = 0$. Since $\mathscr{L}_t$ is compact its spectral radius is the maximum of the absolute values of its eigenvalues and we conclude in particular that $\rho(\mathscr{L}_t) \leq e^{\mathscr{P}(t)}$.

Our next task is to construct an eigenfunction corresponding to the largest eigenvalue of $\mathscr{L}_t$, which will be needed in order to show that the leading eigenvalue is real and simple. Following the standard lines of a Ruelle-Perron-Frobenius theorem on a holomorphic function space (see e.g. [42]) the natural approach is to specialise to the real Hilbert space of elements of $\mathcal{A}^2(\Omega)$ which take real values throughout the limit set L, use the Kreĭn-Rutman theorem to construct an eigenfunction which is positive on L, and then use (15) to show that such a function must be unique. However, in general the limit set L may be so small that an element of $\mathcal{A}^2(\Omega)$ can be identically zero on L without necessarily being the zero function. (For example, if all of the matrices $A_i^{\wedge k}$ have a common leading eigenvector and so do all the matrices $A_i^{\wedge(k+1)}$, then L will be a singleton set.) When this is the case the set $\mathcal{C}$ of functions taking real non-negative values on L does not satisfy condition (ii) of Theorem 11 and the Kreĭn-Rutman theorem is inapplicable. This obstruction can be circumvented by working on a suitable quotient space, but this results in the construction of a function which is positive on L but satisfies the eigenfunction equation only in its restriction to the set L. These considerations are responsible for the length of the arguments which follow. Throughout the remainder of the argument we fix $t \in \mathbb{R}^2$.

Let us define $\mathcal{H} := \{f \in \mathcal{A}^2(\Omega) \colon f(z) \in \mathbb{R} \text{ for all } z \in L\}$ and note that $\mathcal{H}$ is a closed subset of $\mathcal{A}^2(\Omega)$ as a consequence of Lemma 4.1. Observe also that $\mathcal{H}$ is a real Hilbert space when equipped with the norm $\|\cdot\|_{\mathcal{A}^2(\Omega)}$. In proving the estimate (13) we showed that $\psi_{i,t}(z)$ is real and positive when $z \in L$ and $i \in \{1, \ldots, N\}$, and it follows from this that $\mathscr{L}_t$ preserves $\mathcal{H}$. Clearly $\mathscr{L}_t$ is a compact operator on $\mathcal{H}$ as a direct consequence of the compactness of $\mathscr{L}_t$ acting on $\mathcal{A}^2(\Omega)$. Define $\mathcal{Z} := \{f \in \mathcal{H} \colon f(z) = 0 \text{ for all } z \in L\}$ and note that $\mathcal{Z}$ is a vector subspace of $\mathcal{H}$ and is closed as a consequence of Lemma 4.1. Since $\bigcup_{i=1}^N \phi_i(L) \subseteq L$ it follows that $\mathscr{L}_t\mathcal{Z} \subseteq \mathcal{Z}$ by inspection of the definition of $\mathscr{L}_t$. The quotient space $\mathcal{H}/\mathcal{Z}$ is a Hilbert space when equipped with norm $\|[f]\|_{\mathcal{H}/\mathcal{Z}} := \inf\{\|f - g\|_{\mathcal{A}^2(\Omega)} \colon g \in \mathcal{Z}\}$, being isometrically isomorphic to the orthogonal complement of $\mathcal{Z}$ in $\mathcal{H}$. Since $\mathscr{L}_t\mathcal{Z} \subseteq \mathcal{Z}$ it is not difficult to see that the operator $\mathscr{L}_t$ induces a compact operator on the real Hilbert space $\mathcal{H}/\mathcal{Z}$ which we also denote by $\mathscr{L}_t$.

We observe that for each $z \in L$ the functional $[f] \mapsto f(z)$ is a well-defined continuous linear functional $\mathcal{H}/\mathcal{Z} \to \mathbb{R}$. Indeed, if $[f] \in \mathcal{H}/\mathcal{Z}$ and $g \in \mathcal{Z}$ then we have $f(z) = (f+g)(z)$ and

$$|f(z)| = |(f+g)(z)| \leq C_L\|f+g\|_{\mathcal{A}^2(\Omega)}$$

where $C_L > 0$ is the constant given by Lemma 4.1 in respect of the nonempty compact set L. In particular $f(z)$ is independent of the choice of representative $f \in [f]$. Since $g \in \mathcal{Z}$ was arbitary we obtain

$$|f(z)| \leq C_L \inf\{\|f+g\|_{\mathcal{A}^2(\Omega)} \colon g \in \mathcal{Z}\} = C_L\|[f]\|_{\mathcal{H}/\mathcal{Z}}$$

and the functional $[f] \mapsto f(z)$ is continuous as required. Now define

$$\mathcal{C} := \{[f] \in \mathcal{H}/\mathcal{Z} \colon f(z) \geq 0 \text{ for all } z \in L\} = \bigcap_{z \in L} \{[f] \in \mathcal{H}/\mathcal{Z} \colon f(z) \geq 0\}.$$

This set is clearly well-defined, positively homogenous, convex, and closed. The constant function $\mathbf{1}$ belongs to $\mathcal{A}^2(\Omega)$ since Ω is bounded, and therefore $\mathbf{1} \in \mathcal{H}$. Clearly also $[\mathbf{1}] \in \mathcal{C}$, and if $\|[f] - [\mathbf{1}]\|_{\mathcal{H}/\mathcal{Z}} < C_L^{-1}$ then $f(z) > 0$ for all $z \in L$ and therefore $[f] \in \mathcal{C}$. We conclude that $[\mathbf{1}]$ is an interior point of $\mathcal{C}$ and it follows that the span of $\mathcal{C}$ is dense in $\mathcal{H}/\mathcal{Z}$. If $[f] \in \mathcal{C} \cap -\mathcal{C}$ then $f(z) = 0$ for all $z \in L$ so that $f \in \mathcal{Z}$ and therefore the only element of $\mathcal{C} \cap -\mathcal{C}$ is $[0]$. In particular the set $\mathcal{C}$ satisfies conditions (i)–(iii) of Theorem 11. We observe also that $\mathscr{L}_t\mathcal{C} \subseteq \mathcal{C}$ as an easy consequence of (13).

Let $z_0 \in L$ be arbitary. For every $g \in \mathcal{Z}$ we have

$$\begin{aligned}|(\mathscr{L}_t^n \mathbf{1})(z_0) + g(z_0)| &= \left(\sum_{|\mathtt{i}|=n} \psi_{\mathtt{i},t}(z_0)\mathbf{1}(\phi_{\mathtt{i}}(z_0))\right) + g(z_0)\\ &= \sum_{|\mathtt{i}|=n} \psi_{\mathtt{i},t}(z_0)\\ &\geq C_1^{-|t_1|-|t_2|} \sum_{|\mathtt{i}|=n} \left\|A_{\mathtt{i}}^{\wedge k}\right\|^{t_1} \left\|A_{\mathtt{i}}^{\wedge(k+1)}\right\|^{t_2}\end{aligned}$$

using (13) so that

$$C_1^{-|t_1|-|t_2|} \sum_{|\mathtt{i}|=n} \left\|A_{\mathtt{i}}^{\wedge k}\right\|^{t_1} \left\|A_{\mathtt{i}}^{\wedge(k+1)}\right\|^{t_2} \leq |(\mathscr{L}_t^n \mathbf{1})(z_0) + g(z_0)| \leq C_L \|\mathscr{L}_t^n \mathbf{1} + g\|_{\mathcal{A}^2(\Omega)}$$

and by taking the infimum over all $g \in \mathcal{Z}$ we obtain

$$C_1^{-|t_1|-|t_2|} \sum_{|\mathtt{i}|=n} \left\|A_{\mathtt{i}}^{\wedge k}\right\|^{t_1} \left\|A_{\mathtt{i}}^{\wedge(k+1)}\right\|^{t_2} \leq C_L \|[\mathscr{L}_t^n \mathbf{1}]\|_{\mathcal{H}/\mathcal{Z}} \leq C_L \|\mathscr{L}_t^n\|_{\mathcal{H}/\mathcal{Z}} \|[\mathbf{1}]\|_{\mathcal{H}/\mathcal{Z}}$$

for every $n \geq 1$. It follows that the spectral radius of $\mathscr{L}_t$ acting on $\mathcal{H}/\mathcal{Z}$ is at least

$$\lim_{n\to\infty} \left(\sum_{|\mathtt{i}|=n} \left\|A_{\mathtt{i}}^{\wedge k}\right\|^{t_1} \left\|A_{\mathtt{i}}^{\wedge(k+1)}\right\|^{t_2}\right)^{\frac{1}{n}} = e^{\mathscr{P}(t)}.$$

Let $R \geq e^{\mathscr{P}(t)} > 0$ denote this spectral radius.

By Theorem 11 it follows that there exists $\xi_t \in \mathcal{H}$ such that $[\mathscr{L}_t \xi_t] = R[\xi_t]$, $[\xi_t] \in \mathcal{C}$ and $[\xi_t] \neq [0]$. In particular we have $\mathscr{L}_t^n \xi_t - R^n \xi_t \in \mathcal{Z}$ for all $n \geq 1$ and $\xi_t \notin \mathcal{Z}$. Since $[\xi_t] \in \mathcal{C}$ and $\xi_t \notin \mathcal{Z}$ there exists $z_0 \in L$ such that $\xi_t(z_0) > 0$. We have $(\mathscr{L}_t^n \xi_t)(z_0) = R^n \xi_t(z_0)$ for all $n \geq 1$ since $\mathscr{L}_t \xi_t - R^n \xi_t \in \mathcal{Z}$, so

$$0 < R^n |\xi_t(z_0)| = |(\mathscr{L}_t^n \xi_t)(z_0)| \leq C_L \|\mathscr{L}_t^n \xi_t\|_{\mathcal{A}^2(\Omega)} \leq C_L \|\mathscr{L}_t^n\|_{\mathcal{A}^2(\Omega)} \|\xi_t\|_{\mathcal{A}^2(\Omega)}$$

for all $n \geq 1$ and therefore the spectral radius of $\mathscr{L}_t$ acting on $\mathcal{A}^2(\Omega)$ must be at least $R \geq e^{\mathscr{P}(t)}$. On the other hand we showed earlier that the spectral radius of $\mathscr{L}_t$ acting on $\mathcal{A}^2(\Omega)$ is not greater than $e^{\mathscr{P}(t)}$, and therefore both that spectral radius and R must equal $e^{\mathscr{P}(t)}$. We conclude that $\mathscr{L}_t \xi_t - e^{\mathscr{P}(t)} \xi_t$ is identically zero on L and the spectral radius of $\mathscr{L}_t$ on $\mathcal{A}^2(\Omega)$ is precisely $e^{\mathscr{P}(t)}$.

We next claim that in fact $\xi_t(z) > 0$ for all $z \in L$. Since $[\xi_t] \in \mathcal{C}$ this is equivalent to the statement that $\xi_t(z) \neq 0$ for all $z \in L$. Let $Z := \{z \in L \colon \xi_t(z) = 0\}$. If Z contains a point z_0 then since

$$0 \leq e^{\mathscr{P}(t)} \sum_{i=1}^N \psi_{i,t}(z_0) \xi_t(\phi_i(z_0)) = \xi_t(z_0) = 0$$

and $\psi_{i,t}(z_0) > 0$ for all $i = 1, \ldots, N$ using (13), we have $\phi_1(z_0), \ldots, \phi_N(z_0) \in Z$. It follows that $\bigcup_{i=1}^N \phi_i(Z) \subseteq Z$. We showed earlier that the only closed nonempty subset of L with this property is L itself, so if $Z \neq \emptyset$ then necessarily $Z = L$. It follows that if $\xi_t(z) = 0$ for any $z \in L$ then ξ_t is identically zero on L, contradicting $\xi_t \notin \mathcal{Z}$. We conclude that $\xi_t(z)$ cannot be zero for any $z \in L$ which proves the claim.

We claim now that if $\mathscr{L}_t\eta = \lambda\eta$ with $|\lambda| = e^{\mathscr{P}(t)}$, $\eta \in \mathcal{A}^2(\Omega)$ and $\eta \neq 0$, then η must coincide on L with a scalar multiple of ξ_t and consequently $\lambda = e^{\mathscr{P}(t)}$. Let us fix such a function η and number λ and begin the proof of the claim. Since $\xi_t(z) > 0$ for all $z \in L$ and L is compact, we may multiply η by a real scalar in such a way that $\sup_{z\in L} |\eta(z)/\xi_t(z)| = 1$. Multiplying η in turn by a suitable complex unit we may assume that $\eta(z_0)/\xi_t(z_0) = 1$ for some $z_0 \in L$. Now, for all $n \geq 1$ we have

$$\begin{aligned}
\eta(z_0) = \lambda^{-n}(\mathscr{L}_t^n\eta)(z_0) &= \sum_{|\mathtt{i}|=n} \lambda^{-n}\psi_{\mathtt{i},t}(z_0)\eta(\phi_{\mathtt{i}}(z_0)) \\
&= \left|\sum_{|\mathtt{i}|=n} \lambda^{-n}\psi_{\mathtt{i},t}(z_0)\eta(\phi_{\mathtt{i}}(z_0))\right| \\
&= e^{-n\mathscr{P}(t)}\left|\sum_{|\mathtt{i}|=n} \psi_{\mathtt{i},t}(z_0)\eta(\phi_{\mathtt{i}}(z_0))\right| \\
&\leq e^{-n\mathscr{P}(t)}\sum_{|\mathtt{i}|=n} |\psi_{\mathtt{i},t}(z_0)\eta(\phi_{\mathtt{i}}(z_0))| \\
&\leq e^{-n\mathscr{P}(t)}\sum_{|\mathtt{i}|=n} \psi_{\mathtt{i},t}(z_0)\xi_t(\phi_{\mathtt{i}}(z_0)) \\
&= e^{-n\mathscr{P}(t)}\mathscr{L}_t^n\xi_t(z_0) = \xi_t(z_0) = \eta(z_0)
\end{aligned}$$

using the positivity of each $\psi_{\mathtt{i},t}$ on L. Since the first and last terms in this chain of inequalities are equal, none of the inequalities can be strict. It follows that the argument of $\psi_{\mathtt{i},t}(z_0)\eta(\psi_{\mathtt{i}}(z_0))$ is independent of $\mathtt{i}$ when the length of $\mathtt{i}$ is fixed, since otherwise the first inequality would be strict due to cancellation. Moreover we must have $|\eta(\phi_{\mathtt{i}}(z_0))| = \xi_t(\phi_{\mathtt{i}}(z_0))$ for all $\mathtt{i}$ with length n since $|\eta(\phi_{\mathtt{i}}(z_0))| \leq \xi_t(\phi_{\mathtt{i}}(z_0))$ for all such $\mathtt{i}$ and if $|\eta(\phi_{\mathtt{i}}(z_0))| < \xi_t(\phi_{\mathtt{i}}(z_0))$ for some $\mathtt{i}$ then the second inequality above would be strict, which it is not. We deduce that for every $n \geq 1$ the quantity $\eta(\phi_{\mathtt{i}}(z_0))/\xi_t(\phi_{\mathtt{i}}(z_0))$ takes a constant value on the complex unit circle for all $\mathtt{i}$ such that $|\mathtt{i}| = n$. Since

$$\sum_{|\mathtt{i}|=n} \lambda^{-n}\psi_{\mathtt{i},t}(z_0)\eta(\phi_{\mathtt{i}}(z_0)) = \sum_{|\mathtt{i}|=n} e^{-n\mathscr{P}(t)}\psi_{\mathtt{i},t}(z_0)\xi_t(\phi_{\mathtt{i}}(z_0))$$

it must be the case that $\lambda^{-n}\psi_{\mathtt{i},t}(z_0)\eta(\phi_{\mathtt{i}}(z_0))$ is real and positive for all $\mathtt{i}$ such that $|\mathtt{i}| = n$, so we have $\eta(\phi_{\mathtt{i}}(z_0))/\xi_t(\phi_{\mathtt{i}}(z_0)) = \lambda^n e^{-n\mathscr{P}(t)}$ for all $\mathtt{i}$ such that $|\mathtt{i}| = n$. It follows that if $z_1 \in L$ denotes the fixed point of ϕ_1 then

$$\eta(z_1) = \lim_{n\to\infty} \eta(\phi_1^n(z_0)) = \lim_{n\to\infty} \left(\lambda e^{-\mathscr{P}(t)}\right)^n \xi_t(\phi_1^n(z_0))$$

but since $\lim_{n\to\infty} \xi_t(\phi_1^n(z_0)) = \xi_t(z_1) \neq 0$ this is only possible if $\lim_{n\to\infty}\left(\lambda e^{-\mathscr{P}(t)}\right)^n$ exists, which implies $\lambda e^{-\mathscr{P}(t)} = 1$. We conclude that $\lambda = e^{\mathscr{P}(t)}$ and $\eta(z) = \xi_t(z)$ for all z in the set $Z := \overline{\{\phi_{\mathtt{i}}(z_0) \colon \mathtt{i} \in \Sigma_N^*\}}$. Clearly this set Z is a closed nonempty subset of L such that $\bigcup_{i=1}^N \phi_i(Z) \subseteq Z$, and we know that such a set must equal L. We conclude that if $\mathscr{L}_t\eta = \lambda\eta$ where $|\lambda| = e^{\mathscr{P}(t)}$ and $\eta \neq 0$ then $\lambda = e^{\mathscr{P}(t)}$ and η coincides with a scalar multiple of ξ_t when restricted to L.

We have seen that $\mathscr{L}_t$ has no eigenvalues with modulus $e^{\mathscr{P}(t)}$ which are not real and positive, and since $\mathscr{L}_t$ is compact and $\rho(\mathscr{L}_t) = e^{\mathscr{P}(t)}$, by elimination $e^{\mathscr{P}(t)}$ must itself be an eigenvalue. If $\xi_t^{(1)}$, $\xi_t^{(2)}$ are eigenfunctions for this eigenvalue then both coincide on L with a scalar multiple of ξ_t, so some nontrivial linear combination of the two eigenfunctions must be identically zero on L. This linear combination is also an eigenfunction for the same eigenvalue, so by (15) this linear combination must be identically zero on Ω. The two eigenfunctions are therefore proportional to one another. We conclude that there is a one-dimensional eigenspace associated to the eigenvalue $e^{\mathscr{P}(t)}$, that no other eigenvalues of equal or greater modulus exist, and the associated eigenfunction may be chosen positive on L.

Let $\hat{\xi}_t \in \mathcal{A}^2(\Omega)$ be nonzero with $\mathscr{L}_t\hat{\xi}_t = e^{\mathscr{P}(t)}\hat{\xi}_t$ and note that $\hat{\xi}_t$ coincides on L with a nonzero scalar multiple of ξ_t. In particular $\hat{\xi}_t$ is nowhere zero on L. To complete the proof of the theorem we wish to show that the algebraic multiplicity of the eigenvalue $e^{\mathscr{P}(t)}$ is 1. If this is not the case then necessarily $\dim\ker(\mathscr{L}_t - e^{\mathscr{P}(t)}I)^2 > 1$, and in particular we may choose $\eta \in \ker(\mathscr{L}_t - e^{\mathscr{P}(t)}I)^2 \setminus \ker(\mathscr{L}_t - e^{\mathscr{P}(t)}I)$. The function $\mathscr{L}_t\eta - e^{\mathscr{P}(t)}\eta$ belongs to $\ker(\mathscr{L}_t - e^{\mathscr{P}(t)}I)$ and hence is proportional to $\hat{\xi}_t$, and is not identically zero since $\eta \notin \ker(\mathscr{L}_t - e^{\mathscr{P}(t)}I)$. Multiplying η by a suitable scalar we may assume $\mathscr{L}_t\eta - e^{\mathscr{P}(t)}\eta = e^{\mathscr{P}(t)}\hat{\xi}_t$. A simple induction shows that $\mathscr{L}_t^n\eta = e^{n\mathscr{P}(t)}(\eta + n\hat{\xi}_t)$ for all $n \geq 1$. If $z_0 \in L$ is arbitrary we obtain

$$\begin{aligned}
|\mathscr{L}_t^n\eta(z_0)| &= \left|\sum_{|\mathtt{i}|=n} \psi_{\mathtt{i},t}(z_0)\eta(\phi_{\mathtt{i}}(z_0))\right| \leq \sum_{|\mathtt{i}|=n} |\psi_{\mathtt{i},t}(z_0)\eta(\phi_{\mathtt{i}}(z_0))| \\
&\leq C_2^{\|t\|}\left(\sum_{|\mathtt{i}|=n} \left\|A_{\mathtt{i}}^{\wedge k}\right\|^{t_1} \left\|A_{\mathtt{i}}^{\wedge(k+1)}\right\|^{t_2}\right) \sup_{z\in L}|\eta(z)| \\
&\leq C_4^{\|t\|}C_2^{\|t\|}e^{n\mathscr{P}(t)} \sup_{z\in L}|\eta(z)|
\end{aligned}$$

and therefore

$$n|\hat{\xi}_t(z_0)| = \left|e^{-n\mathscr{P}(t)}\left(\mathscr{L}_t^n\eta\right)(z_0) - \eta(z_0)\right| \leq \left(C_4^{\|t\|}C_2^{\|t\|} + 1\right)\sup_{z\in L}|\eta(z)|$$

for all $n \geq 1$, which is impossible since $\hat{\xi}_t(z_0) \neq 0$. We conclude that $\dim\ker(\mathscr{L}_t - e^{\mathscr{P}(t)}I)^2 = 1$ and therefore $e^{\mathscr{P}(t)}$ is a simple eigenvalue of $\mathscr{L}_t$ as required. □

6. Proof of Theorem 3

Before starting the proof of Theorem 3 we require two preliminary lemmas, one concerning the behaviour of the leading eigenvalue of the operator $\mathscr{L}_s$ of Theorems 4 and 12 and one an abstract result concerning sequences of implicit functions in two complex variables.

Lemma 6.1. *Let $\mathsf{A} = (A_1, \ldots, A_N) \in M_d(\mathbb{R})^N$ be k- and $(k+1)$-multipositive with $N, d \geq 2$ and $0 \leq k < d$, and for each $s \in \mathbb{C}$ let $\mathscr{L}_s \colon \mathscr{H} \to \mathscr{H}$ be as given by Theorem 4. Define*

$$p(s) := \log\rho(\mathscr{L}_s) = \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{|\mathtt{i}|=n}\left\|A_{\mathtt{i}}^{\wedge k}\right\|^{k+1-s}\left\|A_{\mathtt{i}}^{\wedge(k+1)}\right\|^{s-k}\right)$$

for all $s \in \mathbb{R}$. Then p is convex. If additionally there exists a norm $|||\cdot|||$ on $\mathbb{R}^d$ with respect to which $\max_{1\leq i\leq N} |||A_i||| < 1$, then there exists $c>0$ such that

$$\frac{p(s_2)-p(s_1)}{s_2-s_1} \leq -c < 0$$

for all pairs of distinct points $s_1, s_2 \in \mathbb{R}$.

Proof. If $s_1, s_2 \in \mathbb{R}$, $\lambda \in (0,1)$ and $n \geq 1$ then

$$\begin{aligned}
&\sum_{|\mathtt{i}|=n} \left\|A_{\mathtt{i}}^{\wedge k}\right\|^{k+1-\lambda s_1-(1-\lambda)s_2} \left\|A_{\mathtt{i}}^{\wedge (k+1)}\right\|^{\lambda s_1+(1-\lambda)s_2-k}\\
&\quad= \sum_{|\mathtt{i}|=n} \left(\left\|A_{\mathtt{i}}^{\wedge k}\right\|^{k+1-s_1} \left\|A_{\mathtt{i}}^{\wedge (k+1)}\right\|^{s_1-k}\right)^{\lambda} \left(\left\|A_{\mathtt{i}}^{\wedge k}\right\|^{k+1-s_2} \left\|A_{\mathtt{i}}^{\wedge (k+1)}\right\|^{s_2-k}\right)^{1-\lambda}\\
&\quad\leq \left(\sum_{|\mathtt{i}|=n} \left\|A_{\mathtt{i}}^{\wedge k}\right\|^{k+1-s_1} \left\|A_{\mathtt{i}}^{\wedge (k+1)}\right\|^{s_1-k}\right)^{\lambda} \left(\sum_{|\mathtt{i}|=n} \left\|A_{\mathtt{i}}^{\wedge k}\right\|^{k+1-s_2} \left\|A_{\mathtt{i}}^{\wedge (k+1)}\right\|^{s_2-k}\right)^{1-\lambda}
\end{aligned}$$

using Hölder's inequality with $p=1/\lambda$ and $q=1/(1-\lambda)$. Taking n^{th} roots and letting $n\to\infty$ it follows directly that $\rho(\mathscr{L}_{\lambda s_1+(1-\lambda)s_2}) \leq \rho(\mathscr{L}_{s_1})^{\lambda}\rho(\mathscr{L}_{s_2})^{1-\lambda}$ and the convexity of p follows by taking logarithms.

To complete the proof suppose that there exists a norm $|||\cdot|||$ on $\mathbb{R}^d$ with respect to which $\max_{1\leq i\leq N}|||A_i|||<1$, and choose $C>0$ such that $\|B\| \leq C|||B|||$ for all $B \in M_d(\mathbb{R})$. Observe that in particular $\sigma_{k+1}(A_{\mathtt{i}}) \leq \sigma_1(A_{\mathtt{i}}) = \|A_{\mathtt{i}}\| \leq C|||A_{\mathtt{i}}|||$ for all $\mathtt{i} \in \Sigma_N^*$. If $s_1 < s_2 \in \mathbb{R}$ and $n \geq 1$ then

$$\begin{aligned}
&\sum_{|\mathtt{i}|=n} \left\|A_{\mathtt{i}}^{\wedge k}\right\|^{k+1-s_2} \left\|A_{\mathtt{i}}^{\wedge (k+1)}\right\|^{s_2-k}\\
&\quad= \sum_{|\mathtt{i}|=n} \sigma_1(A_{\mathtt{i}})\cdots\sigma_k(A_{\mathtt{i}})\sigma_{k+1}(A_{\mathtt{i}})^{s_2-k}\\
&\quad= \sum_{|\mathtt{i}|=n} \sigma_1(A_{\mathtt{i}})\cdots\sigma_k(A_{\mathtt{i}})\sigma_{k+1}(A_{\mathtt{i}})^{s_1-k}\sigma_{k+1}(A_{\mathtt{i}})^{s_2-s_1}\\
&\quad\leq \left(\max_{|\mathtt{i}|=n}\sigma_{k+1}(A_{\mathtt{i}})\right)^{s_2-s_1} \sum_{|\mathtt{i}|=n} \sigma_1(A_{\mathtt{i}})\cdots\sigma_k(A_{\mathtt{i}})\sigma_{k+1}(A_{\mathtt{i}})^{s_1-k}\\
&\quad\leq \left(\max_{|\mathtt{i}|=n} C|||A_{\mathtt{i}}|||\right)^{s_2-s_1} \sum_{|\mathtt{i}|=n} \sigma_1(A_{\mathtt{i}})\cdots\sigma_k(A_{\mathtt{i}})\sigma_{k+1}(A_{\mathtt{i}})^{s_1-k}\\
&\quad\leq C^{s_2-s_1}\left(\max_{1\leq i\leq N}|||A_i|||\right)^{n(s_2-s_1)} \sum_{|\mathtt{i}|=n} \left\|A_{\mathtt{i}}^{\wedge k}\right\|^{k+1-s_1} \left\|A_{\mathtt{i}}^{\wedge (k+1)}\right\|^{s_1-k}
\end{aligned}$$

so that by taking the n^{th} root and letting $n\to\infty$ we obtain

$$\rho(\mathscr{L}_{s_2}) \leq \left(\max_{1\leq i\leq N}|||A_i|||\right)^{s_2-s_1}\rho(\mathscr{L}_{s_1})$$

for all such s_1 and s_2. Taking logarithms and rearranging yields the claim with $c := -\log \max_{1\leq i\leq N} |||A_i||| > 0$. $\square$

As in § we shall say that X_1 is compactly contained in X_2 if the closure of X_1 is a compact subset of the interior of X_2, and express this relation with the notation $X_1 \Subset X_2$.

Lemma 6.2. *Let $D_1, D_2 \subset \mathbb{C}$ be open discs, let $f_n \colon D_1 \times D_2 \to \mathbb{C}$ be a bounded holomorphic function for each $n \geq 1$, and let $f \colon D_1 \times D_2 \to \mathbb{C}$ be bounded and holomorphic. Suppose that there exists a holomorphic function $g \colon D_1 \to D_2$ such that for all $s \in D_1$, $g(s)$ is a simple zero of the function $z \mapsto f(s,z)$ and is the unique zero of that function in D_2. Suppose also that*

$$\lim_{n\to\infty} \sup_{s\in D_1} \sup_{z\in D_2} |f_n(s,z) - f(s,z)| = 0.$$

Let D_1' be any open disc which is compactly contained in D_1. Then there exist a disc $D_2' \subseteq D_2$, which may be chosen concentric with D_2 and with radius arbitrarily close to that of D_2, an integer $n_0 \geq 1$ and holomorphic functions $g_n \colon D_1' \to D_2'$ defined for all $n \geq n_0$ such that:

(i) For all $n \geq n_0$ and $s \in D_1'$, $g_n(s)$ is a simple zero of $z \mapsto f_n(s,z)$ and is the unique zero of that function in D_2'.
(ii) For every integer $\ell \geq 0$ there exists $C_\ell > 0$ such that

$$\sup_{z\in D_1'} \left| g_n^{(\ell)}(s) - g^{(\ell)}(s) \right| \leq C_\ell \sup_{s\in D_1} \sup_{z\in D_2} |f_n(s,z) - f(s,z)|$$

for all $n \geq n_0$, where $h^{(\ell)}$ denotes the ℓ^{th} derivative of the function h.

Proof. Throughout the proof let D_3 be an open disc such that $D_1' \Subset D_3 \Subset D_1$. By compactness and continuity we have $g(\overline{D_3}) \Subset D_2$. Let $D_2' \Subset D_2$ be any disc which is concentric with D_2 and has radius large enough that $g(\overline{D_3}) \Subset D_2'$. By compactness and continuity we obtain

$$\inf_{s\in\overline{D_3}} \inf_{z\in\partial D_2'} |f(s,z)| > 0$$

and hence by uniform convergence there exists $n_1 \geq 1$ such that for all $n \geq n_1$

$$\sup_{s\in\overline{D_3}} \sup_{z\in\partial D_2'} |f(s,z) - f_n(s,z)| < \inf_{s\in\overline{D_3}} \inf_{z\in\partial D_2'} |f(s,z)|.$$

It follows by Rouché's theorem that for every $s \in \overline{D_3}$ and $n \geq n_1$ there exists a unique zero $g_n(s)$ of the function $z \mapsto f_n(s,z)$ in D_2' and this zero is simple. Since each f_n is holomorphic it follows by the holomorphic implicit function theorem (see e.g. [26, p.34]) that each $g_n \colon \overline{D_3} \to D_2'$ is holomorphic on D_3.

We claim now that

$$\lim_{n\to\infty} \sup_{s\in\overline{D_3}} |g_n(s) - g(s)| = 0.$$

Indeed, let $\varepsilon > 0$ be any number which is small enough that for every $s \in \overline{D_3}$ the closed ε-ball centred at $g(s)$ is a subset of D_2'. By compactness and the absence of zeros of $z \mapsto f(s,z)$ in $D_2 \setminus \{g(s)\}$ we have

$$\inf_{s\in\overline{D_3}} \inf_{|z-g(s)|=\varepsilon} |f(s,z)| > 0$$

so that in the same manner if n is large enough

$$\sup_{s\in\overline{D_3}}\sup_{|z-g(z)|=\varepsilon}|f(s,z)-f_n(s,z)|<\inf_{s\in\overline{D_3}}\inf_{|z-g(s)|=\varepsilon}|f(s,z)|.$$

Applying Rouché's theorem again it follows that if n is sufficiently large then for all $s\in\overline{D_3}$ there is a unique zero of the function $z\mapsto f_n(s,z)$ in the region $0\leq |z-g(s)|<\varepsilon$. This zero belongs to D_2' and hence is necessarily equal to $g_n(s)$, and we therefore have $\sup_{s\in\overline{D_3}}|g_n(s)-g(s)|\leq\varepsilon$. Since ε was arbitrary we conclude that

$$\lim_{n\to\infty}\sup_{s\in\overline{D_3}}|g_n(s)-g(s)|=0 \tag{16}$$

as claimed.

For each $s\in D_1$ the value $g(s)$ is a simple zero of the function $z\mapsto f(s,z)$, so we have $\frac{\partial f}{\partial z}(s,g(s))\neq 0$ for all $s\in D_1$. Define

$$c:=\inf_{s\in\overline{D_3}}\left|\frac{\partial f}{\partial z}(s,g(s))\right|>0.$$

Since $g(\overline{D_3})\Subset D_2'\Subset D_2$ we may choose $\tau>0$ small enough that for every $z\in\partial D_2'$ the closed ball of radius 2τ centred at z is a subset of D_2 which does not intersect $g(\overline{D_3})$. Using (16) take $n_2\geq n_1$ large enough that

$$\sup_{s\in\overline{D_3}}|g_n(s)-g(s)|<\tau$$

for all $n\geq n_2$. Observe that if $s\in\overline{D_3}$ and $n\geq n_2$ then $|g_n(s)-g(s)|<\tau$ and therefore $|g(s)-\omega|>\tau$ and $|g_n(s)-\omega|>\tau$ for all $\omega\in\partial D_2'$. Using Cauchy's integral formula, for any two distinct points $z_1,z_2\in D_2'$ we have

$$\begin{aligned}
&\frac{f(s,z_1)-f(s,z_2)}{z_1-z_2}-\frac{\partial f}{\partial z}(s,z_2)\\
&=\frac{1}{2\pi i}\int_{\partial D_2'}\frac{f(s,\omega)}{(z_1-z_2)(\omega-z_1)}-\frac{f(s,\omega)}{(z_1-z_2)(\omega-z_2)}-\frac{f(s,\omega)}{(\omega-z_2)^2}d\omega\\
&=\frac{1}{2\pi i}\int_{\partial D_2'}\frac{f(s,\omega)((\omega-z_2)^2-(\omega-z_1)(\omega-z_2)-(z_1-z_2)(\omega-z_1))}{(z_1-z_2)(\omega-z_1)(\omega-z_2)^2}d\omega\\
&=\frac{1}{2\pi i}\int_{\partial D_2'}\frac{f(s,\omega)(z_1^2-2z_1z_2+z_2^2)}{(z_1-z_2)(\omega-z_1)(\omega-z_2)^2}d\omega\\
&=\frac{1}{2\pi i}\int_{\partial D_2'}\frac{f(s,\omega)(z_1-z_2)}{(\omega-z_1)(\omega-z_2)^2}d\omega.
\end{aligned}$$

Hence if $s\in\overline{D_3}$, $n\geq n_2$ and $g_n(s)\neq g(s)$ then since $g(s),g_n(s)\in D_2'$

$$\left|\frac{f(s,g_n(s))-f(s,g(s))}{g_n(s)-g(s)}-\frac{\partial f}{\partial z}(s,g(s))\right|\leq\frac{R|g_n(s)-g(s)|}{\tau^3}\cdot\sup_{t\in D_1}\sup_{z\in D_2}|f(t,z)|$$

where R denotes the radius of D_2'. Now take $n_3\geq n_2$ large enough that additionally

$$\left(\sup_{s\in\overline{D_3}}|g_n(s)-g(s)|\right)\left(\frac{R}{\tau^3}\sup_{s\in D_1}\sup_{z\in D_2}|f(s,z)|\right)<\frac{c}{2}.$$

If $n \geq n_3$, $s \in \overline{D_3}$ and $g_n(s) \neq g(s)$ then since $f_n(s, g_n(s)) = 0 = f(s, g(s))$ we have

$$\begin{aligned}
&\left|\frac{f(s,g_n(s)) - f_n(s,g_n(s))}{g_n(s) - g(s)}\right| \\
&\quad = \left|\frac{f(s,g_n(s)) - f(s,g(s))}{g_n(s) - g(s)}\right| \\
&\quad \geq \left|\frac{\partial f}{\partial z}(s,g(s))\right| - \left|\frac{f(s,g_n(s)) - f(s,g(s))}{g_n(s) - g(s)} - \frac{\partial f}{\partial z}(s,g(s))\right| > \frac{c}{2}.
\end{aligned}$$

It follows that when $n \geq n_3$

$$\sup_{s \in \overline{D_3}} |g_n(s) - g(s)| \leq \frac{2}{c} \sup_{s \in \overline{D_3}} \sup_{z \in D_2} |f_n(s,z) - f(s,z)|.$$

To complete the proof of the lemma let $\delta > 0$ be small enough that for every $s \in D_1'$ the closed δ-ball centred at s is a subset of D_3. By the Cauchy integral formula we have for each integer $\ell \geq 0$ and every $n \geq n_3$

$$\begin{aligned}
\sup_{s \in D_1'} \left|g_n^{(\ell)}(s) - g^{(\ell)}(s)\right| &\leq \sup_{s \in D_1'} \left|\frac{\ell!}{2\pi i} \int_{|s-t|=\delta} \frac{g_n(t) - g(t)}{(t-s)^{\ell+1}} dt\right| \\
&\leq \delta^{-\ell} \ell! \sup_{s \in \overline{D_3}} |g_n(s) - g(s)| \\
&\leq \frac{2\ell!}{c\delta^\ell} \sup_{s \in D_1} \sup_{z \in D_2} |f_n(s,z) - f(s,z)|
\end{aligned}$$

as required. The proof is complete. $\square$

Proof of Theorem 3. Let $\mathsf{A} = (A_1, \ldots, A_N) \in M_d(\mathbb{R})^N$ be k- and $(k+1)$-multipositive where $N, d \geq 2$ and $0 \leq k < d$. For all $s \in \mathbb{C}$ let $\mathscr{L}_s \colon \mathscr{H} \to \mathscr{H}$ be as given by Theorem 4. Define

$$t_n(s) := \sum_{|\mathtt{i}|=n} \frac{\lambda_1\left(A_{\mathtt{i}}^{\wedge k}\right)^{\binom{d}{k}-1} \lambda_1\left(A_{\mathtt{i}}^{\wedge(k+1)}\right)^{\binom{d}{k+1}-1} \rho\left(A_{\mathtt{i}}^{\wedge k}\right)^{k+1-s} \rho\left(A_{\mathtt{i}}^{\wedge(k+1)}\right)^{s-k}}{p'_{A_{\mathtt{i}}^{\wedge k}}\left(\lambda_1\left(A_{\mathtt{i}}^{\wedge k}\right)\right) p'_{A_{\mathtt{i}}^{\wedge(k+1)}}\left(\lambda_1\left(A_{\mathtt{i}}^{\wedge(k+1)}\right)\right)}$$

and

$$a_n(s) := \frac{(-1)^n}{n!} \det \begin{pmatrix}
t_1(s) & n-1 & 0 & \cdots & 0 & 0 \\
t_2(s) & t_1(s) & n-2 & \cdots & 0 & 0 \\
t_3(s) & t_2(s) & t_1(s) & \ddots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
t_{n-1}(s) & t_{n-2}(s) & t_{n-3}(s) & \cdots & t_1(s) & 1 \\
t_n(s) & t_{n-1}(s) & t_{n-2}(s) & \cdots & t_2(s) & t_1(s)
\end{pmatrix}$$

for every $s \in \mathbb{C}$ and $n \geq 1$. Define also $a_0(s) := 1$. We claim that there exist $\tilde{K}, \tilde{\gamma}, \kappa > 0$ such that

$$|a_n(s)| \leq \tilde{K}^n e^{n\kappa|s|} \exp\left(-\tilde{\gamma} n^\alpha\right) \tag{17}$$

for all $n \geq 1$ and $s \in \mathbb{C}$, where $\tilde{K}, \tilde{\gamma}$ and κ do not depend on s or n and where

$$\alpha := \frac{\binom{d+1}{k+1} - 1}{\binom{d+1}{k+1} - 2}.$$

By Theorem 4 there exist constants $C, \gamma, \kappa > 0$ such that

$$\mathfrak{s}_n(\mathscr{L}_s) \leq C \exp\left(\kappa|s| - \gamma n^{\beta}\right) \tag{18}$$

for all $n \geq 1$ and $s \in \mathbb{C}$ where $\beta := (\binom{d+1}{k+1} - 2)^{-1} = \alpha - 1$, and $\mathscr{L}_s$ is trace-class with $\operatorname{tr} \mathscr{L}_s^n = t_n(s)$ for all $s \in \mathbb{C}$. By Theorem 7 we have

$$|a_n(s)| \leq \sum_{i_1 < \cdots < i_n} \mathfrak{s}_{i_1}(\mathscr{L}_s) \cdots \mathfrak{s}_{i_n}(\mathscr{L}_s) \tag{19}$$

for all $n \geq 1$. In order to proceed further we require two elementary inequalities. We first note that for every integer $m \geq 1$

$$\begin{aligned} \sum_{\ell=m}^{\infty} e^{-\gamma \ell^{\beta}} \leq \int_{m-\frac{1}{2}}^{\infty} e^{-\gamma t^{\beta}} dt = \frac{1}{\beta} \int_{\left(m-\frac{1}{2}\right)^{\beta}}^{\infty} u^{\frac{1}{\beta}-1} e^{-\gamma u} du \\ \leq \frac{K}{\beta} \int_{\left(m-\frac{1}{2}\right)^{\beta}}^{\infty} e^{-\frac{\gamma}{2} u} du \\ \leq \frac{2K}{\beta\gamma} e^{-\frac{\gamma}{2}\left(m-\frac{1}{2}\right)^{\beta}} \leq \frac{2K}{\beta\gamma} e^{-\frac{\gamma}{2^{1+\beta}} m^{\beta}} \end{aligned} \tag{20}$$

where $K := \sup\{x^{\frac{1}{\beta}-1} e^{-\gamma x/2} \colon x \geq \frac{1}{2}\} > 0$ depends only on β and γ, and where $e^{-\gamma \ell^{\beta}} \leq \int_{\ell-\frac{1}{2}}^{\ell+\frac{1}{2}} e^{-\gamma t^{\beta}} dt$ follows from Jensen's inequality. Secondly we notice that

$$\sum_{\ell=1}^{m} \ell^{\beta} \geq \int_0^m t^{\beta} dt = \frac{m^{1+\beta}}{1+\beta} \tag{21}$$

for all integers $m \geq 1$ since the series is an upper Riemann sum for the integral. Combining (18), (19), (20) and (21) we may now obtain

$$\begin{aligned}
|a_n(s)| &\leq \sum_{i_1<\cdots<i_n} \prod_{\ell=1}^{n} C\exp(\kappa|s| - \gamma i_\ell^\beta) \\
&= \left(Ce^{\kappa|s|}\right)^n \sum_{i_1<\cdots<i_n} \exp\left(-\gamma\left(i_1^\beta + \cdots + i_n^\beta\right)\right) \\
&\leq \left(Ce^{\kappa|s|}\right)^n \sum_{i_1=1}^{\infty}\sum_{i_2=2}^{\infty}\cdots\sum_{i_n=n}^{\infty} \exp\left(-\gamma\left(i_1^\beta + \cdots + i_n^\beta\right)\right) \\
&= \left(Ce^{\kappa|s|}\right)^n \prod_{m=1}^{n}\sum_{\ell=m}^{\infty} \exp\left(-\gamma \ell^\beta\right) \\
&\leq \left(\frac{2KCe^{\kappa|s|}}{\beta\gamma}\right)^n \prod_{m=1}^{n} \exp\left(-\frac{\gamma}{2^{1+\beta}} m^\beta\right) \\
&\leq \left(\frac{2KCe^{\kappa|s|}}{\beta\gamma}\right)^n \exp\left(-\frac{\gamma}{(1+\beta)2^{1+\beta}} n^{1+\beta}\right)
\end{aligned}$$

which establishes the claimed inequality (17) with $\tilde{\gamma} := \gamma/(2^{1+\beta}(1+\beta))$ and $\tilde{K} := 2KC/\beta\gamma$.

Now define a function $d_n \colon \mathbb{C}^2 \to \mathbb{C}$ for each $n \geq 1$ by $d_n(s,z) := \sum_{m=0}^{n} a_m(s) z^m$, and define also $d_\infty(s,z) := \sum_{m=0}^{\infty} a_m(s) z^m$, the convergence of the series being guaranteed by (17). As a consequence of (17) it is clear that

$$(22) \qquad |d_n(s,z) - d_\infty(s,z)| = \left|\sum_{m=n+1}^{\infty} a_m(s) z^m\right| = O\left(\exp\left(-\frac{\tilde{\gamma}}{2} n^\alpha\right)\right)$$

uniformly on compact subsets of $\mathbb{C}^2$. It is clear by inspection that each d_n is holomorphic, and using the convergence of d_n to d_∞ uniformly on compact sets together with Cauchy's theorem and Morera's theorem it follows easily that $d_\infty \colon \mathbb{C}^2 \to \mathbb{C}$ is holomorphic. By Theorem 7 we have $d_\infty(s,z) = \det(I - z\mathscr{L}_s)$ for every $(s,z) \in \mathbb{C}^2$. In particular for every $s \in \mathbb{C}$ the zeros of $z \mapsto d_\infty(s,z)$ are precisely the reciprocals of the nonzero eigenvalues of $\mathscr{L}_s$, with the degree of each zero being equal to the algebraic multiplicity of the corresponding eigenvalue.

For each $s \in \mathbb{R}$ define $r_\infty(s) := \rho(\mathscr{L}_s)^{-1} \in (0, +\infty)$. We observe that $p(s) = -\log r_\infty(s)$ is a continuous function of s by Lemma 6.1 since it is a convex function of $s \in \mathbb{R}$, so $r_\infty(s) \colon \mathbb{R} \to (0,+\infty)$ is continuous. By the combination of Theorem 4 and Theorem 7, for each $s \in \mathbb{R}$ the function $z \mapsto d_\infty(s,z)$ has a simple zero at $r_\infty(s)$ and has no zeroes with equal or smaller absolute value. We claim that there exist $n_0 \geq 1$, an open set $U \subset \mathbb{C}$ containing $[k, k+1]$, a holomorphic extension of $r_\infty|_{[k,k+1]}$ to U and a sequence of holomorphic functions $r_n \colon U \to \mathbb{C}$ defined for all $n \geq n_0$ such that

$$(23) \qquad \sup_{s \in U}\left|r_n^{(\ell)}(s) - r_\infty^{(\ell)}(s)\right| = O\left(\exp\left(-\frac{\tilde{\gamma}}{2} n^\alpha\right)\right)$$

for all integers $\ell \geq 0$ and such that for all $n \geq n_0$ and $s \in [k, k+1]$, $r_n(s)$ is the smallest positive real number x such that $d_n(s, x) = 0$.

To prove the claim it is clearly sufficient, by the compactness of $[k, k+1]$, to show that every $s_0 \in [k, k+1]$ admits an open neighbourhood $U(s_0)$ such that r_∞ extends holomorphically from $U(s_0) \cap [k, k+1]$ to all of $U(s_0)$, such that there exists a sequence of functions $r_n \colon U(s_0) \to \mathbb{C}$ defined for all large enough n such that for all $s \in [k, k+1] \cap U(s_0)$, $r_n(s)$ is the smallest positive real number x such that $d_n(s, x) = 0$, and such that

$$\sup_{s \in U(s_0)} \left| r_n^{(\ell)}(s) - r_\infty^{(\ell)}(s) \right| = O\left(\exp\left(-\frac{\tilde{\gamma}}{2} n^\alpha \right) \right)$$

for all integers $\ell \geq 0$. The open set U can then be taken equal to the union of a finite cover of $[k, k+1]$ by different sets $U(s)$, and the characterisation of $r_n(s)$ as the smallest positive root of $d_n(s, x) = 0$ ensures that for each n the local functions $r_n \colon U(s) \to \mathbb{C}$ extend consistently to a single well-defined function $r_n \colon U \to \mathbb{C}$.

Let us therefore prove this local version of the preceding claim. Fix $s_0 \in [k, k+1]$. Since $z \mapsto d_\infty(s_0, z)$ has a unique zero in the closed disc with centre 0 and radius $r_\infty(s_0)$, and all of its zeros are isolated, we may choose an open disc $D_2(s_0)$ with centre $z_0 \in \mathbb{R}$ and radius $R > 0$ such that $[0, r_\infty(s_0)] \subset D_2(s_0)$ and such that $\overline{D_2(s_0)}$ contains no other zeros of $z \mapsto d_\infty(s_0, z)$. A simple argument using compactness shows that we may choose a small open disc $D_1(s_0)$ centred at s_0 such that

$$\sup_{s \in D_1(s_0)} \sup_{|z - z_0| = R} |d_\infty(s, z) - d_\infty(s_0, z)| < \inf_{|z - z_0| = R} |d_\infty(s_0, z)|$$

and by shrinking the neighbourhood $D_1(s_0)$ further if necessary we may assume using continuity that additionally $r_\infty(s) \in D_2(s_0)$ for all $s \in D_1(s_0) \cap [k, k+1]$.

By Rouché's theorem, for all $s \in D_1(s_0)$ the function $z \mapsto d_\infty(s, z)$ has a unique zero in $D_2(s_0)$ and this zero is simple. When $s \in D_1(s_0) \cap [k, k+1]$ this zero must be equal to $r_\infty(s) \in D_2(s_0)$ by uniqueness. Extend $r_\infty \colon D_1(s_0) \cap [k, k+1] \to \mathbb{R}$ to a function $D_1(s_0) \to \mathbb{C}$ by defining $r_\infty(s)$ to be the unique zero of $z \mapsto d_\infty(s, z)$ in $D_2(s_0)$ for each $s \in D_1(s_0)$. By the holomorphic implicit function theorem and the simplicity of the zero $r_\infty \colon D_1(s_0) \to D_2(s_0)$ is holomorphic. Applying Lemma 6.2 we find, shrinking $D_1(s_0)$ and $D_2(s_0)$ if necessary, that there exist constants $C_\ell > 0$, an integer $n_1 \geq 1$ and holomorphic functions $r_n \colon D_1(s_0) \to D_2(s_0)$ defined for all $n \geq n_1$ such that

$$\begin{aligned} \sup_{s \in D_1(s_0)} \left| r_n^{(\ell)}(s) - r_\infty^{(\ell)}(s) \right| &\leq C_\ell \sup_{s \in D_1(s_0)} \sup_{z \in D_2(s_0)} |d_n(s, z) - d_\infty(s, z)| \\ &= O\left(\exp\left(-\frac{\tilde{\gamma}}{2} n^\alpha \right) \right) \end{aligned}$$

for every integer $\ell \geq 0$, such that $r_n(s)$ is the unique zero of $z \mapsto d_n(s, z)$ in $D_2(s_0)$ for all $s \in D_1(s_0)$ and $n \geq n_1$ and is a simple zero for all such s and n, such that $[0, r_\infty(s_0)] \subseteq D_2(s_0)$, and such that $D_2(s_0)$ is an open disc centred on the real axis. For all $s \in D_1(s_0) \cap [k, k+1]$ and $n \geq n_0$ the numbers $r_n(s)$ and $r_n(s)^*$ both lie in $D_2(s_0)$ and are both zeros of the polynomial $d_n(s, z) = \sum_{m=0}^n a_n(s) z^m$ since the coefficients of that polynomial are real and since $D_2(s_0)$, being a disc centred on the real axis, is symmetric with respect to complex conjugation. By the uniqueness of the zero $r_n(s)$ in $D_2(s_0)$ this is possible only if $r_n(s) = r_n(s)^*$, which is to say if $r_n(s)$ is real. Since $D_2(s_0)$ contains the interval from 0 to $r_n(s)$, it follows that if $r_n(s)$ is positive then it is the smallest positive real root of $\sum_{m=0}^n a_n(s)x^m$ for all $s \in D_1(s_0) \cap [k, k+1]$. To complete the proof of the claim it therefore suffices to show that if n is sufficiently large then $r_n(s) > 0$ for all $s \in D_1(s_0)$. To see this choose $\delta \in (0, r_\infty(s_0))$ small enough that the open δ-ball centred at $r_\infty(s_0)$ is contained in $D_2(s_0)$, and observe that by shrinking $D_1(s_0)$ further if necessary we may obtain

$$\inf_{s \in D_1(s_0)} \inf_{|z - r_\infty(s_0)| = \delta} |d_\infty(s,z)| > 0$$

and hence for all large enough n

$$\sup_{s \in D_1(s_0)} \sup_{|z - r_\infty(s_0)| = \delta} |d_n(s,z) - d_\infty(s,z)| < \inf_{s \in D_1(s_0)} \inf_{|z - r_\infty(s_0)| = \delta} |d_\infty(s,z)|.$$

By Rouché's theorem this implies that there exists $n_0 \geq n_1$ such that for all $n \geq n_0$ and all $s \in D_1(s_0)$ there is a unique zero of $z \mapsto d_n(s,z)$ inside the circle of radius δ and centre $r_\infty(s_0)$, and since this region is a subset of $D_2(s_0)$ this root must equal $r_n(s)$ by the uniqueness of that root in $D_2(s_0)$. In particular for all $n \geq n_0$ and $s \in D_1(s_0) \cap [k, k+1]$ we have $r_n(s) > r_\infty(s_0) - \delta > 0$ and no other root lies in $(0, r_n(s)) \subset D_2(s_0)$. Hence $r_n(s)$ is the smallest positive real root of $\sum_{m=0}^n a_n(s)x^m$ for all $s \in D_1(s_0) \cap [k, k+1]$ as required to prove the local version of the claim with $U(s_0) := D_1(s_0)$. The full statement of the claim follows.

We may now complete the proof of the theorem. Define $P_n(s) := r_n(s)^{-1} > 0$ for all $s \in [k, k+1]$ and $n \geq n_0$, and $P(s) := r_\infty(s)^{-1} > 0$ for all $s \in \mathbb{R}$. Observe that by Theorem 4 we have $e^{P(A_1, \ldots, A_N; s)} = P(s)$ for all $s \in [k, k+1]$. Since $r_\infty \colon U \to \mathbb{C}$ is holomorphic, P is real-analytic at least on a neighbourhood of $[k, k+1]$. Since $r_\infty(s)$ is positive for all real s and $[k, k+1]$ is compact it follows that

$$\inf_{s \in [k, k+1]} r_\infty(s) > 0 \tag{24}$$

and by the case $\ell = 0$ of (23) we deduce that

$$\lim_{n \to \infty} \inf_{s \in [k, k+1]} r_n(s) > 0. \tag{25}$$

Using (23), (24), (25) and the expressions

$$|P_n(s) - P(s)| = \left| \frac{1}{r_n(s)} - \frac{1}{r_\infty(s)} \right|,$$

$$|P_n'(s) - P'(s)| = \left| \frac{r_n'(s)}{r_n(s)^2} - \frac{r_\infty'(s)}{r_\infty(s)^2} \right|$$

and

$$|P_n''(s) - P''(s)| = \left| \frac{r_n''(s) r_n(s) - r_n'(s)^2}{r_n(s)^4} - \frac{r_\infty''(s) r_\infty(s) - r_\infty'(s)^2}{r_\infty(s)^4} \right|$$

it follows by elementary manipulations that

$$\sup_{s \in [k, k+1]} |P_n(s) - P(s)| = O\left(\exp\left(-\frac{\tilde{\gamma}}{2} n^\alpha \right) \right), \tag{26}$$

$$\sup_{s\in[k,k+1]} |P_n'(s) - P'(s)| = O\left(\exp\left(-\frac{\tilde{\gamma}}{2}n^\alpha\right)\right) \tag{27}$$

and

$$\sup_{s\in[k,k+1]} |P_n''(s) - P''(s)| = O\left(\exp\left(-\frac{\tilde{\gamma}}{2}n^\alpha\right)\right). \tag{28}$$

In the case where we do not assume that $\max_{1\leq i\leq N} |||A_i||| < 1$ for some norm on $\mathbb{R}^d$ the estimate (26) already completes the proof of Theorem 3. Otherwise, we claim that $\inf_{s\in[k,k+1]} P''(s) > 0$ and $\sup_{s\in[k,k+1]} P'(s) < 0$. Let $p(s) := \log P(s)$ for $s \in \mathbb{R}$ so that $P'(s) = p'(s)P(s)$ and $P''(s) = p''(s)P(s)+p'(s)^2P(s)$. Obviously p is real-analytic on $[k, k+1]$ since P is positive and real-analytic there, and p is convex by Lemma 6.1, so necessarily $p''(s) \geq 0$ for all $s \in [k, k+1]$. By Lemma 6.1 we have $p'(s) < 0$ for all $s \in [k, k+1]$ and therefore

$$\sup_{s\in[k,k+1]} P'(s) = \sup_{s\in[k,k+1]} p'(s)P(s) < 0. \tag{29}$$

Similarly we observe that $\inf_{s\in[k,k+1]} |p'(s)| > 0$, and since $P''(s) = p''(s)P(s) + p'(s)^2P(s) \geq p'(s)^2P(s)$ we likewise deduce that $\inf_{s\in[k,k+1]} P''(s) > 0$ as claimed.

Combining the previous claim with (28) we find in particular that $\inf_{s\in[k,k+1]} P_n''(s) > 0$ for all large enough n, which proves that each such function $P_n\colon [k, k+1] \to \mathbb{R}$ is convex. By the hypothesis $\dim_{\mathsf{aff}}(A_1, \ldots, A_N) \in (k, k+1)$ of Theorem 3 there exists a solution $s \in (k, k+1)$ to $P(s) = 1$, and since P has negative derivative on $[k, k+1]$ this implies that $P(k) > 1 > P(k+1)$. Combining this observation with (26) we find that $P_n(k) > 1 > P_n(k+1)$ for all large enough n, and by the combination of (29) and (27) we find that $\sup_{s\in[k,k+1]} P_n'(s) \leq -c < 0$ for all large enough n where $c > 0$ is some positive constant. It follows that for all large enough n there exists a unique $s_n \in [k, k+1]$ such that $P_n(s_n) = 1$. Let $s_\infty := \dim_{\mathsf{aff}}(A_1, \ldots, A_N) \in [k, k+1]$ be the unique solution to $P(s_\infty) = 1$. If $s_n \neq s_\infty$ then by the Mean Value Theorem there exists t strictly between s_n and s_∞ such that

$$P'(t) = \frac{P(s_n) - P(s_\infty)}{s_n - s_\infty}$$

and therefore since $P_n(s_n) = 1 = P(s_\infty)$ we obtain

$$|s_n - s_\infty| = \frac{|P(s_n) - P(s_\infty)|}{|P'(t)|} = \frac{|P(s_n) - P_n(s_n)|}{|P'(t)|} \leq c^{-1}|P(s_n) - P_n(s_n)|.$$

The inequality $|s_n - s_\infty| \leq c^{-1}|P(s_n) - P_n(s_n)|$ obviously also holds when $s_n = s_\infty$, so

$$|s_n - s_\infty| = O\left(\exp\left(-\frac{\tilde{\gamma}}{2}n^\alpha\right)\right)$$

as $n \to \infty$ using (26). The proof of the theorem is complete. $\square$

n	Approximation to affinity dimension	CPU time
2	1.14341 79598 76019 95000 60486 91827 85789 60135	0.043s
3	1.11827 23247 08006 28499 89060 66409 13091 47143	0.044s
4	1.11538 89736 67461 99644 51849 00512 18003 54788	0.053s
5	1.11560 42107 66261 56209 11669 09958 04069 77087	0.075s
6	1.11560 31850 39305 08475 98379 83168 80085 68510	0.11s
7	1.11560 32522 24751 03699 38823 87724 66623 37012	0.16s
8	1.11560 32579 27402 64806 11546 27227 11083 45893	0.30s
9	1.11560 32577 86505 71154 77556 50836 85812 53178	0.39s
10	1.11560 32577 87028 88533 65835 00045 83936 61000	0.67s
11	1.11560 32577 87030 91898 36777 33249 49956 17495	1.2s
12	1.11560 32577 87030 89197 97928 71446 51257 73313	2.0s
13	1.11560 32577 87030 89218 88050 96492 48585 23429	4.3s
14	1.11560 32577 87030 89218 84942 17623 75680 33697	8.8s
15	1.11560 32577 87030 89218 84937 14660 75123 27001	20s
16	1.11560 32577 87030 89218 84937 14840 85419 85122	44s
17	1.11560 32577 87030 89218 84937 14840 24544 08248	100s
18	1.11560 32577 87030 89218 84937 14840 24574 24137	210s
19	1.11560 32577 87030 89218 84937 14840 24574 25551	440s
20	1.11560 32577 87030 89218 84937 14840 24574 25551	990s

Table 1. Approximations to the affinity dimension of Example 1 calculated using Theorem 3 and the secant method as described in §, implemented in Wolfram Mathematica. The CPU time used in each approximation is as reported by Mathematica's `Timing` function. For $n = 1$ the approximation to the pressure function has no root in $(1, 2)$ and this line is therefore omitted from the table. Digits which are empirically observed to have converged to a stable value are underlined.

7. Examples

There are two intuitively natural mechanisms by which to make the approximations given in Theorem 3 yield an approximation to the affinity dimension. On the one hand since $e^{P(A_1,\ldots,A_n;s)}$ is decreasing in s and since the affinity dimension is the unique $s \in [k, k+1]$ such that 1 is the leading eigenvalue of $\mathscr{L}_s$, the affinity dimension corresponds to the smallest $s \in [k, k+1]$ such that $\det(I - \mathscr{L}_s) = 0$, which is to say the smallest $s \in [k, k+1]$ such that $\sum_{m=0}^{\infty} a_m(s) = 0$. It is therefore natural to attempt to approximate the affinity dimension by looking for the smallest solution s to the equation $\sum_{m=0}^{n} a_m(s) = 0$ for each fixed n. In practice however this is problematic since $\mathscr{L}_s$ may in general have infinitely many positive real eigenvalues and the number of solutions to $\sum_{m=0}^{n} a_m(s) = 0$ may therefore be extremely large and the function itself highly oscillatory.

In practice, therefore, we adopt the following alternative approach. For large n the smallest positive real root $x = r_n(s)$ of $\sum_{m=0}^{n} a_m(s)x^m$ approximates the reciprocal of the leading eigenvalue of $\mathscr{L}_s$. Moreover, for large n the function

$s \mapsto r_n(s)^{-1}$ is convex and strictly decreasing with a unique root in $[k, k+1]$ by virtue of Theorem 3. Computing the unique root of a convex decreasing function is a far more tractable enterprise than finding the smallest root of an oscillating function, and for this reason our application of Theorem 3 follows the approach of solving $r_n(s) = 1$. For this problem we use the *secant method*: for fixed n we evaluate $r_n(s)^{-1}$ firstly at $s_1 := k+1$ and secondly at $s_2 := k$. Given two approximations s_m, s_{m-1} we then define s_{m+1} by extrapolating the location of the root from s_m, s_{m-1} as if the function $r_n(s)^{-1}$ were affine:

$$s_{m+1} := \frac{s_{m-1} - s_m - r_n(s_m)^{-1} s_{m-1} + r_n(s_{m-1})^{-1} s_m}{r_n(s_{m-1})^{-1} - r_n(s_m)^{-1}}.$$

When r_n^{-1} is convex and decreasing the convergence of the sequence (s_m) is guaranteed with super-exponential rate $O(\theta^{m^{(1+\sqrt{5})/2}})$ for some $\theta \in (0,1)$. In practical instances we found that the sequence (s_m) consistently converged empirically to 40 decimal places by around $m \simeq 12$ independently of n. The results of this procedure applied to some examples of two- and three-dimensional affine iterated function systems are presented in this section.

For large n one may show that the trace $t_n(s)$ appearing in Theorem 3 approximates the value $e^{nP(A_1,\ldots,A_N;s)}$ whereas the coefficients $a_n(s)$ are shown in Theorem 3 to decrease to zero with super-exponential speed. The small size of $a_n(s)$ is thus attributable to additive cancellation between potentially very large summands. It is therefore important in implementation to record the traces $t_n(s)$ to much higher accuracy than is desired for the ultimate approximation. In the computations which follow the traces $t_n(s)$ were calculated in arbitrary precision, reducing to finite precision only for the outcome of the calculation of the coefficients $a_n(s)$.

Define

$$A_1 := \begin{pmatrix} -\frac{4}{7} & \frac{5}{7} \\ 0 & \frac{1}{7} \end{pmatrix}, \qquad A_2 := \begin{pmatrix} \frac{1}{7} & 0 \\ -\frac{5}{7} & -\frac{4}{7} \end{pmatrix}.$$

We claim that the pair (A_1, A_2) is 1-dominated. Indeed, define

$$\mathcal{C}_1 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \colon |x| \geq 2|y| \right\},$$

$$\mathcal{C}_2 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \colon |y| \geq 2|x| \right\}.$$

If $(x,y)^\top \in \mathcal{C}_1$ then

$$\left| \frac{5}{7}y - \frac{4}{7}x \right| \geq \frac{4}{7}|x| - \frac{5}{7}|y| \geq \frac{3}{7}|y| \geq \left| \frac{2}{7}y \right|$$

and equality of the first and last terms is only possible if $y = 0$ and consequently $x = 0$. In particular if $(x,y)^\top \in \mathcal{C}_1$ is nonzero we obtain $A_1(x,y)^\top \in \operatorname{Int} \mathcal{C}_1$. Moreover for $(x,y)^\top \in \mathcal{C}_1$ we also have

$$\left| \frac{4}{7}y + \frac{5}{7}x \right| \geq \frac{5}{7}|x| - \frac{4}{7}|y| \geq \frac{3}{7}|x| \geq \left| \frac{2}{7}x \right|$$

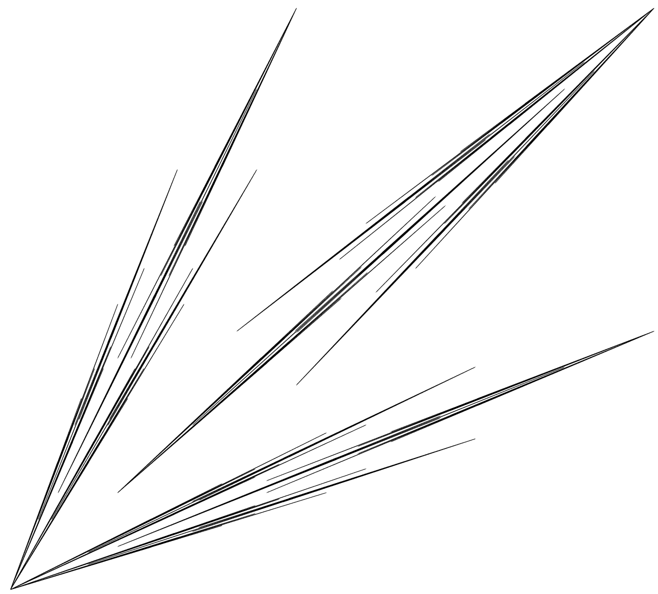

FIGURE 1. The attractor of the iterated function system (T_1, T_2, T_3) defined in Example 2. It has Hausdorff dimension equal to the affinity dimension of its defining iterated function system, which we compute in Table 2 to be approximately 1.44676377386238429715620108279097222804 36...

which yields $A_2(x,y)^\top \in \operatorname{Int}\mathcal{C}_2$ when $(x,y)^\top$ is nonzero. In a similar manner, if $(x,y)^\top \in \mathcal{C}_2$ then

$$\left|\frac{5}{7}y - \frac{4}{7}x\right| \geq \frac{5}{7}|y| - \frac{4}{7}|x| \geq \frac{3}{7}|y| \geq \left|\frac{2}{7}y\right|$$

and

$$\left|\frac{4}{7}y + \frac{5}{7}x\right| \geq \frac{4}{7}|y| - \frac{5}{7}|x| \geq \frac{3}{7}|x| \geq \left|\frac{2}{7}x\right|$$

which respectively give $A_1(x,y)^\top \in \operatorname{Int}\mathcal{C}_1$ and $A_2(x,y)^\top \in \operatorname{Int}\mathcal{C}_2$ when $(x,y)^\top$ is nonzero.

If we now let $w = (1,1)^\top$ then $\langle u, w\rangle$ is never zero for any nonzero $u \in \mathcal{C}_1 \cup \mathcal{C}_2$, so defining

$$\mathcal{K}_i := \{u \in \mathcal{C}_i \colon \langle u, w\rangle > 0\}$$

for $i = 1, 2$ it is not difficult to see that $(\mathcal{K}_1, \mathcal{K}_2)$ is a multicone for (A_1, A_2). In particular Theorem 3 may be applied to estimate the affinity dimension of the pair

n	Approximation to affinity dimension	CPU time
1	1.57850 39107 24303 42569 39013 22778 88907 20542	0.046s
2	1.43428 20777 82633 21247 87188 76730 31996 86014	0.044s
3	1.44698 63740 68855 64166 13462 60397 02738 95013	0.064s
4	1.44676 23250 25528 19736 40628 61933 67159 40086	0.11s
5	1.44676 37772 54098 43296 70430 41085 33834 29566	0.20s
6	1.44676 37738 59463 32542 61749 94490 38856 75805	0.47s
7	1.44676 37738 62385 23207 08694 66251 21939 16812	1.1s
8	1.44676 37738 62384 29704 44057 28444 21373 26314	3.3s
9	1.44676 37738 62384 29715 62060 91538 64348 53245	9.7s
10	1.44676 37738 62384 29715 62010 82706 74910 92449	36s
11	1.44676 37738 62384 29715 62010 82790 97276 01619	114s
12	1.44676 37738 62384 29715 62010 82790 97222 80423	360s
13	1.44676 37738 62384 29715 62010 82790 97222 80436	1300s
14	1.44676 37738 62384 29715 62010 82790 97222 80436	4400s

TABLE 2. Approximations to the affinity dimension of Example 2 calculated using Theorem 3 and the secant method as described in §, implemented in Wolfram Mathematica. The CPU time used in each approximation is as reported by Mathematica's `Timing` function. Digits which are empirically observed to have converged to a stable value are underlined.

(A_1, A_2). Let $(B_1, B_2) := (A_1, -A_2)$. Since

$$\begin{aligned} e^{P(A_1,A_2;1)} = e^{P(B_1,B_2;1)} &= \lim_{n\to\infty} \left(\sum_{|\mathtt{i}|=n} \|B_{\mathtt{i}}\| \right)^{\frac{1}{n}} \\ &\geq \lim_{n\to\infty} \left\| \sum_{|\mathtt{i}|=n} B_{\mathtt{i}} \right\|^{\frac{1}{n}} \\ &= \lim_{n\to\infty} \|(B_1+B_2)^n\|^{\frac{1}{n}} = \rho(B_1+B_2) = \frac{\sqrt{50}}{7} > 1 \end{aligned}$$

and

$$e^{P(A_1,A_2;2)} = |\det A_1| + |\det A_2| = \frac{8}{49} < 1$$

we infer that $\dim_{\mathsf{aff}}(A_1, A_2) \in (1, 2)$. The first 20 approximations to the affinity dimension of (A_1, A_2) are tabulated in Table 1.

FIGURE 2. A projection of the attractor of the iterated function system defined by Example 3. Approximations to the affinity dimension computed using Theorem 3 are listed in Table 3. It is known from work of Falconer [21, §5] that the upper box dimension $\overline{\dim}_B X$ is bounded above by $\dim_{\mathsf{aff}}(A_1, A_2)$, but unlike the planar example in Figure 1 current techniques are not powerful enough to determine whether or not $\dim_H X = \dim_{\mathsf{aff}}(A_1, A_2)$.

Define

$$\begin{aligned} T_1\begin{pmatrix} x \\ y \end{pmatrix} &:= \begin{pmatrix} \frac{1}{3} & \frac{1}{9} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \\ T_2\begin{pmatrix} x \\ y \end{pmatrix} &:= \begin{pmatrix} -\frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\ T_3\begin{pmatrix} x \\ y \end{pmatrix} &:= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{9} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \end{aligned}$$

and let $A_1, A_2, A_3 \in M_2(\mathbb{R})$ denote the linear parts of T_1, T_2, T_3 respectively. Let $X \subset \mathbb{R}^2$ denote the attractor of (T_1, T_2, T_3). One may show that the strong open set condition is satisfied by (T_1, T_2, T_3) and using [5] one may show that $\dim_H X = \dim_B X = \dim_{\mathsf{aff}}(A_1, A_2, A_3)$. It is easily verified that $\rho(A_1 - A_2 + A_3) > 1$ and $|\det A_1| + |\det A_2| + |\det A_3| < 1$ and in a similar manner to the previous example we deduce that $\dim_{\mathsf{aff}}(A_1, -A_2, A_3) = \dim_{\mathsf{aff}}(A_1, A_2, A_3) \in (1, 2)$. The Hausdorff dimension $\dim_H X = \dim_{\mathsf{aff}}(A_1, A_2, A_3)$ may thus be approximated using Theorem 3; the first 14 approximations are presented in Table 2.

n	Approximation to affinity dimension	CPU time
3	1.74010 38961 34544 64381 66016 57752 82592 79145	0.067s
4	1.53612 13489 34570 18769 13237 56458 61628 45041	0.10s
5	1.58779 31446 44939 17928 98900 28708 16065 92496	0.15s
6	1.58459 23810 06597 43285 21249 54866 32813 68839	0.22s
7	1.58477 97771 44149 34557 48903 92413 22985 52229	0.33s
8	1.58477 17757 07488 53767 71488 42424 52891 52003	0.63s
9	1.58477 20386 65944 76377 72361 85895 44529 09738	0.80s
10	1.58477 20318 53062 52952 58955 36166 25319 46959	1.4s
11	1.58477 20319 95110 47059 43620 26740 31575 13317	2.4s
12	1.58477 20319 92686 60697 00747 19778 01115 41015	5.4s
13	1.58477 20319 92720 93370 05697 62846 36869 58071	12s
14	1.58477 20319 92720 52545 02878 00445 78535 74528	27s
15	1.58477 20319 92720 52956 88351 89418 63989 50927	59s
16	1.58477 20319 92720 52953 32862 81715 84179 24019	130s
17	1.58477 20319 92720 52953 35507 79078 84111 41677	270s
18	1.58477 20319 92720 52953 35490 71502 87276 30757	560s
19	1.58477 20319 92720 52953 35490 81124 12318 84553	1200s
20	1.58477 20319 92720 52953 35490 81076 56294 07542	2800s
21	1.58477 20319 92720 52953 35490 81076 77018 06325	5900s

TABLE 3. Approximations to the affinity dimension of Example 3 calculated using Theorem 3 and the secant method as described in §, implemented in Wolfram Mathematica. The CPU time used in each approximation is as reported by Mathematica's `Timing` function. Digits which are empirically observed to have converged to a stable value are underlined. Convergence is noticeably slower than for two-dimensional examples: in this context our bound for the error in the n^{th} approximation is $O(\exp(-\gamma n^{5/4}))$ as opposed to $O(\exp(-\gamma n^2))$ in the other examples. For $n = 1, 2$ the approximation to the pressure function has no root in $(1, 2)$ and these lines are therefore omitted.

Consider (A_1, A_2) where

$$A_1 := \frac{1}{12}\begin{pmatrix} 5 & 4 & 1 \\ 5 & 5 & 4 \\ 0 & 1 & 5 \end{pmatrix}, \qquad A_2 := \frac{1}{12}\begin{pmatrix} 5 & 5 & 0 \\ 4 & 5 & 1 \\ 1 & 4 & 5 \end{pmatrix} = A_1^\top$$

and note that A_1 and A_2 are contractions in the Euclidean norm. It is easily checked that $(A_1A_1, A_1A_2, A_2A_1, A_2A_2)$ is a tuple of positive invertible matrices and is therefore 1-dominated. By consideration of Theorem 2 it follows that (1) holds for $(A_1A_1, A_1A_2, A_2A_1, A_2A_2)$ and obviously (1) therefore also holds for (A_1, A_2). We conclude that (A_1, A_2) is likewise 1-dominated.

We identify each A_i with the corresponding linear map $\mathbb{R}^3 \to \mathbb{R}^3$ defined by A_i with respect to the standard basis e_1, e_2, e_3 of $\mathbb{R}^3$. With respect to the basis

$e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3$ for $\wedge^2\mathbb{R}^3$ we have

$$A_1^{\wedge 2} = \frac{1}{144}\begin{pmatrix} 5 & 15 & 11 \\ 5 & 25 & 19 \\ 5 & 25 & 21 \end{pmatrix}, \qquad A_2^{\wedge 2} = \frac{1}{144}\begin{pmatrix} 5 & 5 & 5 \\ 15 & 25 & 25 \\ 11 & 19 & 21 \end{pmatrix}.$$

Since $(A_1^{\wedge 2}, A_2^{\wedge 2})$ is thus representable by a pair of positive matrices we see that (A_1, A_2) is both 1-and 2-dominated. Using non-negativity it follows by a theorem of Yu. V. Protasov ([54]) that

$$\lim_{n\to\infty}\left(\sum_{|\mathtt{i}|=n} \|A_{\mathtt{i}}\|\right)^{\frac{1}{n}} = \rho(A_1 + A_2) > 1$$

and

$$\lim_{n\to\infty}\left(\sum_{|\mathtt{i}|=n} \left\|A_{\mathtt{i}}^{\wedge 2}\right\|\right)^{\frac{1}{n}} = \rho\left(A_1^{\wedge 2} + A_2^{\wedge 2}\right) < 1.$$

Thus $P(A_1, A_2; 1) > 0 > P(A_1, A_2; 2)$ and consequently $\dim_{\mathsf{aff}}(A_1, A_2) \in (1, 2)$, and we conclude that Theorem 3 is applicable to the computation of $\dim_{\mathsf{aff}}(A_1, A_2)$. The first 21 approximations to $\dim_{\mathsf{aff}}(A_1, A_2)$ are presented in Table 3. An illustration of the attractor of the iterated function system

$$T_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} := \frac{1}{12}\begin{pmatrix} 5 & 4 & 1 \\ 5 & 5 & 4 \\ 0 & 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$T_2\begin{pmatrix} x \\ y \\ z \end{pmatrix} := \frac{1}{12}\begin{pmatrix} 5 & 5 & 0 \\ 4 & 5 & 1 \\ 1 & 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is given in Figure 2.

8. Non-dominated matrices

If $(A_1, \ldots, A_N) \in M_2(\mathbb{R})^N$ is a tuple of invertible matrices which is not 1-dominated then by a line of reasoning due to A. Avila [60] there exist tuples $(A_1', \ldots, A_N')$ arbitrarily close to $(A_1, \ldots, A_N)$ with the property that some product $A_{i_1}' \cdots A_{i_n}'$ has complex eigenvalues. For such matrices the formula for $t_n(s)$ in Theorem 3 has no clear meaning, and also for such matrices no open subset of $\mathbb{RP}^1$ may be found which is mapped strictly inside itself by the action of the matrices A_i', preventing the construction of a trace-class transfer operator in direct mimicry of Theorem 3. For such matrices it is therefore difficult to see how any reasonable adaptation of Theorem 3 could be made. In this sense we believe that 1-domination, or multipositivity, is the weakest open condition on the matrices $A_1, \ldots, A_N$ which permits a version of Theorem 3 to be proved.

However, for non-dominated matrices it is still possible to obtain non-rigorous estimates of the affinity dimension by other techniques. Given $A_1, \ldots, A_N \in GL_2(\mathbb{R})$ and $s \in [0, 1]$ we may define an operator $\mathscr{L}_s \colon C^\alpha(\mathbb{RP}^1) \to C^\alpha(\mathbb{RP}^1)$ by

$$(\mathscr{L}_s f)(\overline{u}) := \sum_{i=1}^{N}\left(\frac{\|A_i u\|}{\|u\|}\right)^s f\left(\overline{A_i u}\right),$$

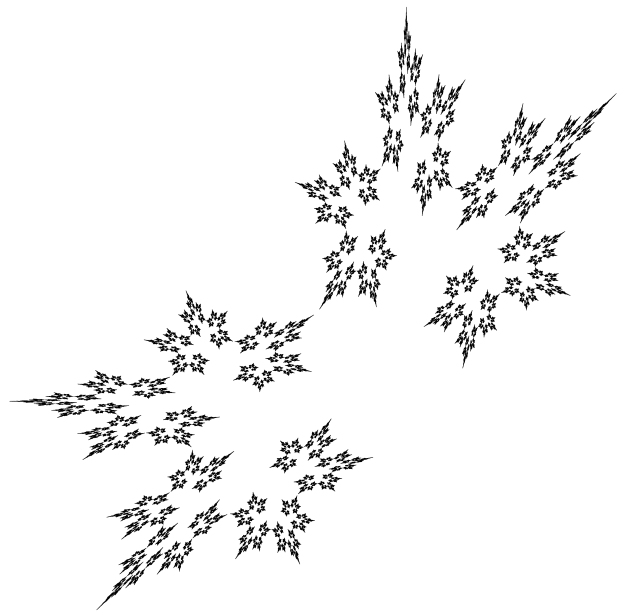

FIGURE 3. This self-affine set was shown in [46, §6.6] to have Hausdorff dimension equal to the affinity dimension of the defining iterated function system. However, the linear parts of the defining affine transformations have non-real eigenvalues and Theorem 3 is not applicable. Non-rigorous estimates using the discretisation method described in § as tabulated in Table 5 suggest that the affinity dimension is equal to approximately 1.522688.

and for $s \in [1,2]$ by

$$(\mathscr{L}_s f)(\overline{u}) := \sum_{i=1}^{N} \left(\frac{\|A_i u\|}{\|u\|}\right)^{2-s} |\det A_i|^{s-1} f\left(\overline{A_i u}\right),$$

in such a manner that

$$\rho(\mathscr{L}_s) = \lim_{n\to\infty} \left(\sum_{i_1,\ldots,i_n=1}^{N} \varphi^s\left(A_{i_1}\cdots A_{i_n}\right)\right)^{\frac{1}{n}}$$

and such that $\rho(\mathscr{L}_s)$ is a simple eigenvalue of $\mathscr{L}_s$, as long as $\alpha \in (0,1)$ is chosen suitably small (in a manner which in general will depend on s) and mild algebraic non-degeneracy conditions on $(A_1,\ldots,A_N)$ are met. (These spectral properties are guaranteed by, for example, [29, Théorème 8.8].) We could then hope to estimate the spectral radius of $\mathscr{L}_s$ for different values of s by discretising the phase space $\mathbb{RP}^1$, constructing a large matrix representing a discretised action of $\mathscr{L}_s$, and working on the supposition that the spectral radius of the matrix is a good approximation to $\rho(\mathscr{L}_s)$ and hence to $e^{P(A_1,\ldots,A_N;s)}$. In practical experiments we were able to obtain around five decimal places of accuracy for the affinity dimension by discretising

Mesh size	Approximation to affinity dimension	CPU time
2	1.45960943	0.0048s
2^2	$\underline{1}.\underline{4}3685698$	0.0061s
2^3	$\underline{1}.\underline{4}3981279$	0.0096s
2^4	$\underline{1}.\underline{44}686353$	0.047s
2^5	$\underline{1}.\underline{44}707711$	0.070s
2^6	$\underline{1}.\underline{44}682259$	0.11s
2^7	$\underline{1}.\underline{446}69990$	0.21s
2^8	$\underline{1}.\underline{446}78054$	0.42s
2^9	$\underline{1}.\underline{4467}7211$	0.83s
2^{10}	$\underline{1}.\underline{4467}6722$	1.5s
2^{11}	$\underline{1}.\underline{44676}500$	3.2s
2^{12}	$\underline{1}.\underline{44676}360$	6.0s
2^{13}	$\underline{1}.\underline{446763}39$	12s
2^{14}	$\underline{1}.\underline{446763}78$	31s
2^{15}	$\underline{1}.\underline{446763}85$	96s
2^{16}	$\underline{1}.\underline{446763}76$	460s
2^{17}	$\underline{1}.\underline{4467637}5$	2400s

TABLE 4. Estimates of the affinity dimension of Example 2 calculated using the non-rigorous discretisation method described in §. At mesh sizes above around 2^{10} the result shows good agreement with Table 2 but convergence is slow thereafter. Digits which are empirically observed to have converged to a stable value are underlined.

$\mathbb{RP}^1$ into approximately 10^5 evenly-spaced mesh points: see Tables 4 and 5. We observe in particular that the results obtained in Table 4 show good agreement with Theorem 3 when tested on the multipositive matrix set described in Example 2. However, we have not been able to make this method of estimation rigorous. This approach could also be applied to higher-dimensional affine iterated function systems but we have not investigated the matter of finding suitable discretisations of the more complicated phase spaces required in this context.

9. Acknowledgements

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Appendix A. On the equivalence of domination and multipositivity

Proposition A.1. *Let* $\mathsf{A} \subset M_d(\mathbb{R})$ *be compact, and suppose that every* $A \in \mathsf{A}$ *is invertible. Then* A *is* 1*-dominated if and only if it is multipositive.*

Mesh size	Approximation to affinity dimension	CPU time
2	1.50000000	0.0028s
2^2	$\underline{1}.\underline{5}1578683$	0.0025s
2^3	$\underline{1}.\underline{5}1254065$	0.0047s
2^4	$\underline{1}.\underline{52}070716$	0.033s
2^5	$\underline{1}.\underline{52}415711$	0.059s
2^6	$\underline{1}.\underline{52}305542$	0.079s
2^7	$\underline{1}.\underline{522}90806$	0.13s
2^8	$\underline{1}.\underline{522}62668$	0.26s
2^9	$\underline{1}.\underline{522}69395$	0.61s
2^{10}	$\underline{1}.\underline{522}70408$	1.1s
2^{11}	$\underline{1}.\underline{522}69152$	2.2s
2^{12}	$\underline{1}.\underline{5226}8717$	4.5s
2^{13}	$\underline{1}.\underline{52268}810$	7.7s
2^{14}	$\underline{1}.\underline{52268}795$	18s
2^{15}	$\underline{1}.\underline{522687}80$	55s
2^{16}	$\underline{1}.\underline{5226878}0$	220s
2^{17}	$\underline{1}.\underline{5226878}2$	1400s

Table 5. Estimates of the affinity dimension of the iterated function system defined in [46, §6.6] and illustrated in Figure 3, calculated using the non-rigorous discretisation method described in §. Digits which are empirically observed to have converged to a stable value are underlined. No rigorous estimate of the affinity dimension of this IFS is currently available.

The proof of Proposition A.1 is complicated to express in full technical detail but follows a simple idea. Using Theorem 2 we may find an "almost multicone" which satisfies all the criteria of Definition 1.1 except that the sets $\mathcal{K}_j \setminus \{0\}$ are connected but perhaps non-convex. If we replace each $\mathcal{K}_j$ with its convex hull then we establish Definition 1.1(i) in full whilst retaining Definition 1.1(ii)–(iii), but potentially lose Definition 1.1(iv), which stipulates that the sets $\mathcal{K}_j \setminus \{0\}$ should be disjoint. If Definition 1.1(iv) has not been lost then we have constructed the desired multicone. Otherwise, we can counteract the loss of disjointness by replacing any newly-overlapping sets $\mathcal{K}_j \setminus \{0\}$ with connected, nonoverlapping unions of those sets; but by doing so we risk losing convexity again. Crucially however, by taking unions in this manner the overall number of sets $\mathcal{K}_j$ has been reduced. Since the number of cones $\mathcal{K}_j$ is a positive integer, by repeating these two successive processes sufficiently many times we must eventually reach a position where the cardinality of the set of cones cannot be reduced any further and the only possibility is that the process terminates with a true multicone in the sense of Definition 1.1.

The rigorous presentation of this informal argument is substantially simplified by the following formal definition:

Definition A.2. *Let* $\mathsf{A} \subset M_d(\mathbb{R})$ *be nonempty and let* $\mathcal{K}_1, \ldots, \mathcal{K}_m \subset \mathbb{R}^d$. *We say that* $(\mathcal{K}_1, \ldots, \mathcal{K}_m)$ *is an* almost multicone of the first kind with cardinality m *for* A *if all of the criteria of Definition 1.1 are satisfied except that for each* $j = 1, \ldots, m$ *the closed set* $\mathcal{K}_j$ *is not necessarily convex, but the set* $\mathcal{K}_j \setminus \{0\}$ *is connected. We say that* $(\mathcal{K}_1, \ldots, \mathcal{K}_m)$ *is an* almost multicone of the second kind with cardinality m *for* A *if all of the criteria of Definition 1.1 are satisfied except possibly for Definition 1.1(iv).*

The following two lemmas formalise the processes described informally at the beginning of the section:

Lemma A.3. *Let* $\mathsf{A} \subset M_d(\mathbb{R})$ *be nonempty. If* A *has an almost multicone of the first kind with cardinality* m*, then it has an almost multicone of the second kind with cardinality* m*.*

Proof. Let $(\mathcal{K}_1, \ldots, \mathcal{K}_m)$ be an almost multicone of the first kind for A with cardinality m, and for each $j = 1, \ldots, m$ let $\hat{\mathcal{K}}_j$ be the closed convex hull of $\mathcal{K}_j$. By Carathéodory's theorem on convex hulls every $u \in \hat{\mathcal{K}}_j$ is equal to a convex combination of at most $d+1$ elements of $\mathcal{K}_j$. We claim that $(\hat{\mathcal{K}}_1, \ldots, \hat{\mathcal{K}}_m)$ is an almost multicone of the second kind for A.

It is obvious that each $\hat{\mathcal{K}}_j$ satisfies (i). If $u \in \hat{\mathcal{K}}_j$ is nonzero then it is equal to a convex combination of at most $d+1$ elements of $\mathcal{K}_j$ at least one of which must be nonzero, and (ii) follows. To see (iii), let $u \in \hat{\mathcal{K}}_j \setminus \{0\}$ be written as $u = \sum_{i=1}^{d+1} a_i u_i$ where each $u_i \in \mathcal{K}_j \setminus \{0\}$ and where each a_i is a non-negative real number. Since $\mathcal{K}_j \setminus \{0\}$ is connected and A is continuous it follows from (ii) that the sign of $\langle Au_i, w\rangle$ is independent of i. Using the fact that $(\mathcal{K}_1, \ldots, \mathcal{K}_m)$ satisfies Definition 1.1(iii) it follows that there exists ℓ such that either $Au_i \in \operatorname{Int} \mathcal{K}_\ell$ for every i, or $Au_i \in -\operatorname{Int} \mathcal{K}_\ell$ for every i. In the former case we see that u belongs to the convex hull of the interior of $\mathcal{K}_\ell$, which (since the convex hull of an open set is open) is an open subset of the convex hull of $\mathcal{K}_\ell$, hence a subset of $\operatorname{Int} \hat{\mathcal{K}}_\ell$. Thus $u \in \operatorname{Int} \hat{\mathcal{K}}_\ell$. In the latter case we similarly have $u \in -\operatorname{Int} \hat{\mathcal{K}}_\ell$. It follows that $A(\hat{\mathcal{K}}_j \setminus \{0\}) \subseteq (\operatorname{Int} \hat{\mathcal{K}}_\ell) \cup (-\operatorname{Int} \hat{\mathcal{K}}_\ell)$ as required. □

Lemma A.4. *Let* $\mathsf{A} \subset M_d(\mathbb{R})$ *be nonempty, and suppose that* A *has an almost multicone of the second kind with cardinality* m *which is* not *a multicone for* A*. Then* A *has an almost multicone of the first kind with cardinality strictly smaller than* m*.*

Proof. Let $(\mathcal{K}_1, \ldots, \mathcal{K}_m)$ be an almost multicone of the second kind for A with cardinality m. Define a relation $\sim$ on $\{\mathcal{K}_1, \ldots, \mathcal{K}_m)$ by $\mathcal{K}_{j_1} \sim \mathcal{K}_{j_2}$ if and only if $\mathcal{K}_{j_1} \cap \mathcal{K}_{j_2} \neq \{0\}$. Define an equivalence relation $\sim'$ on $\{\mathcal{K}_1, \ldots, \mathcal{K}_m\}$ by $\mathcal{K}_{j_1} \sim' \mathcal{K}_{j_2}$ if and only if there exist $k_1, \ldots, k_r \in \{1, \ldots, m\}$ such that $\mathcal{K}_{j_1} \sim \mathcal{K}_{k_1} \sim \cdots \sim \mathcal{K}_{k_r} \sim \mathcal{K}_{j_2}$. (Note that we do not require any of the indices k_i, j_i to be distinct.) Since $(\mathcal{K}_1, \ldots, \mathcal{K}_m)$ is an almost multicone of the second kind which is not a multicone, there exists at least one pair of distinct integers j_1, j_2 such that $\mathcal{K}_{j_1} \cap \mathcal{K}_{j_2} \neq \{0\}$

and therefore $\mathcal{K}_{j_1} \sim \mathcal{K}_{j_2}$. Let m' be the number of equivalence classes of $\sim'$ and notice that $m' < m$.

For each equivalence class under $\sim'$ define a set $\mathcal{K}'_j \subset \mathbb{R}^d$ by taking $\mathcal{K}'_j$ to be the union of all of the sets $\mathcal{K}_j$ which belong to the equivalence class. Let $\mathcal{K}'_1, \ldots, \mathcal{K}'_{m'}$ be a complete list of the sets which may be formed in this manner from the m' equivalence classes. We claim that $(\mathcal{K}'_1, \ldots, \mathcal{K}'_{m'})$ is an almost multicone of the first kind for A, which suffices to prove the lemma. Let $w \in \mathbb{R}^d$ be a vector such that Definition 1.1(ii) is satisfied for $(\mathcal{K}_1, \ldots, \mathcal{K}_m)$.

To see that the required parts of Definition 1.1(i) are satisfied for $(\mathcal{K}'_1, \ldots, \mathcal{K}'_m)$ we notice that if $\mathcal{K}_{j_1} \sim \mathcal{K}_{j_2}$ then each $\mathcal{K}_{j_i} \setminus \{0\}$ is connected (since convex) and $(\mathcal{K}_{j_1} \cup \mathcal{K}_{j_2}) \setminus \{0\} = (\mathcal{K}_{j_1} \setminus \{0\}) \cup (\mathcal{K}_{j_2} \setminus \{0\})$ contains a point since $\mathcal{K}_{j_1} \sim \mathcal{K}_{j_2}$. It follows that $(\mathcal{K}_{j_1} \cup \mathcal{K}_{j_2}) \setminus \{0\}$ is equal to the union of two overlapping connected sets, and hence is connected. If $\mathcal{K}_{j_3} \sim \mathcal{K}_{j_1}$ or $\mathcal{K}_{j_3} \sim \mathcal{K}_{j_2}$ then it is clear that $(\mathcal{K}_{j_1} \cup \mathcal{K}_{j_2} \cup \mathcal{K}_{j_3}) \setminus \{0\}$ is connected by the same reasoning. Proceeding inductively it is clear that we may add further elements of the equivalence class to this union one by one in until the equivalence class is exhausted, retaining connectedness at each step. This proves that each $\mathcal{K}'_j \setminus \{0\}$ is connected. It is obvious that each $\mathcal{K}'_j$ is closed and has the positive homogeneity property required by Definition 1.1(i). Since $\bigcup_{j=1}^m \mathcal{K}_j = \bigcup_{j=1}^{m'} \mathcal{K}'_j$ by construction, Definition 1.1(ii) obviously holds for $(\mathcal{K}'_1, \ldots, \mathcal{K}'_{m'})$.

Let us now establish Definition 1.1(iii). Let $A \in \mathsf{A}$ and let $\mathcal{K}_{j_1}, \ldots, \mathcal{K}_{j_r}$ form an equivalence class whose union is equal to $\mathcal{K}'_j$, say. For each $i = 1, \ldots, r$ choose ℓ_i such that $A(\mathcal{K}_{j_i} \setminus \{0\}) \subseteq (\operatorname{Int} \mathcal{K}_{\ell_i}) \cup (-\operatorname{Int} \mathcal{K}_{\ell_i})$, which is possible since $(\mathcal{K}_1, \ldots, \mathcal{K}_m)$ satisfies Definition 1.1(iii). We claim that in fact ℓ_i is constant with respect to i. Indeed, if $\mathcal{K}_{j_{i_1}} \sim \mathcal{K}_{j_{i_2}}$ let $v \in (\mathcal{K}_{j_{i_1}} \cap \mathcal{K}_{j_{i_2}}) \setminus \{0\}$, where we notice that the existence of such a vector v is guaranteed by the definition of $\sim$. We then have $Av \in ((\operatorname{Int} \mathcal{K}_{\ell_{i_1}}) \cup (-\operatorname{Int} \mathcal{K}_{\ell_{i_1}})) \cap ((\operatorname{Int} \mathcal{K}_{\ell_{i_2}}) \cup (-\operatorname{Int} \mathcal{K}_{\ell_{i_2}}))$, but it follows from Definition 1.1(ii) and (iv) that this set is empty unless $\ell_{i_1} = \ell_{i_2}$. It follows easily that ℓ_i is constant on the entire equivalence class, being equal to ℓ, say. Choose $\mathcal{K}'_{\ell'}$ such that $\mathcal{K}_\ell \subseteq \mathcal{K}'_{\ell'}$: we have $A(\mathcal{K}_{j_i} \setminus \{0\}) \subseteq (\operatorname{Int} \mathcal{K}_\ell) \cup (-\operatorname{Int} \mathcal{K}_\ell) \subseteq (\operatorname{Int} \mathcal{K}'_{\ell'}) \cup (-\operatorname{Int} \mathcal{K}'_{\ell'})$ for each $i = 1, \ldots, r$ and therefore $A(\mathcal{K}'_j \setminus \{0\}) \subseteq (\operatorname{Int} \mathcal{K}'_{\ell'}) \cup (-\operatorname{Int} \mathcal{K}'_{\ell'})$ as required to prove (iii).

Finally, suppose that $v \in (\mathcal{K}'_{j_1} \cap \mathcal{K}'_{j_2}) \setminus \{0\}$ where j_1 and j_2 are distinct. Choose distinct k_1 and k_2 such that $v \in \mathcal{K}_{k_1} \cap \mathcal{K}_{k_2}$, which is necessarily possible by definition of $\mathcal{K}'_{j_1}$ and $\mathcal{K}'_{j_2}$. We have $v \in (\mathcal{K}_{k_1} \cap \mathcal{K}_{k_2}) \setminus \{0\}$ with $k_1 \neq k_2$, which contradicts Definition 1.1(iv) for $(\mathcal{K}_1, \ldots, \mathcal{K}_m)$ and hence contradicts the hypothesis of the lemma. We conclude that Definition 1.1(iv) holds for $(\mathcal{K}'_1, \ldots, \mathcal{K}'_{m'})$. The proof is complete. $\square$

A third lemma demonstrates that the process described at the start of the section can begin in the first place.

Lemma A.5. *Let $\mathsf{A} \subset M_d(\mathbb{R})$ be compact, and suppose that every $A \in \mathsf{A}$ is invertible. If A is 1-dominated then there exists an almost multicone of the first kind for A.*

Proof. For every nonzero $v \in \mathbb{R}^d$ let us write $\overline{v}$ for the one-dimensional subspace of $\mathbb{R}^d$ spanned by v. By Theorem 2(iii) there exist a closed nonempty set $\mathcal{C} \subset \mathbb{RP}^{d-1}$ with finitely many connected components $\mathcal{C}_1, \ldots, \mathcal{C}_m$ and a $(d-1)$-dimensional subspace V of $\mathbb{R}^d$ such that for all $\overline{v} \in \mathcal{C}$ we have $v \notin V$, and for all $A \in \mathsf{A}$ the closure of $A\mathcal{C}$ is a subset of $\operatorname{Int}\mathcal{C}$. It is clear that if any $\mathcal{C}_j$ has empty interior then we may discard it and consider only the remaining sets $\mathcal{C}_k$ without affecting any of the previous properties; we therefore assume without loss of generality that each $\mathcal{C}_j$ has nonempty interior.

Let $w \in \mathbb{R}^d$ be a unit normal vector to V, and for each $j = 1, \ldots, m$ define

$$\mathcal{K}_j := \left\{u \in \mathbb{R}^d \colon \overline{u} \in \mathcal{C}_j \text{ and } \langle u, w\rangle > 0\right\}.$$

It is clear that each $\mathcal{K}_j$ is closed and satisfies $\lambda\mathcal{K}_j = \mathcal{K}_j$ for all $\lambda > 0$, and each $\mathcal{K}_j \setminus \{0\}$ is connected and has nonempty interior since the corresponding properties hold for $\mathcal{C}_j$. In particular Definition 1.1(i) and (ii) are satisfied with the exception that each $\mathcal{K}_j \setminus \{0\}$ is connected but some sets $\mathcal{K}_j$ may fail to be convex. It is clear that for each $A \in \mathsf{A}$ and $j = 1, \ldots, m$ we have $A\mathcal{C}_j \subseteq \operatorname{Int}\mathcal{C}_\ell$ for some $\ell = \ell(j, A)$ by connectedness, and this clearly implies $A(\mathcal{K}_j \setminus \{0\}) \subseteq (\operatorname{Int}\mathcal{K}_\ell) \cup (-\operatorname{Int}\mathcal{K}_j)$ which is Definition 1.1(iii). If $v \in \mathcal{K}_{j_1} \cap \mathcal{K}_{j_2}$ with v nonzero and $j_1 \neq j_2$ then $\overline{v} \in \mathcal{C}_{j_1} \cap \mathcal{C}_{j_2}$ which contradicts the fact that distinct connected components do not overlap. This proves Definition 1.1(iv). The proof is complete. $\square$

Proof of Proposition A.1. Suppose first that A is 1-dominated. Consider the set of all integers $m \geq 1$ such that there exists an almost multicone of the first kind for A with cardinality m. By Lemma A.5 this set is nonempty, so it has a smallest element m_0, say. Consider an almost multicone of the first kind for A which has cardinality m_0. If this almost multicone is a multicone then the proof is complete. Otherwise, by Lemma A.3 we may construct an almost multicone of the second kind for A which has the same cardinality m_0. If this is also not a multicone for A then by Lemma A.4 we may construct an almost multicone of the first kind for A which has a cardinality smaller than m_0, but this contradicts the definition of m_0. We conclude that A admits a multicone, which in fact must have cardinality m_0.

If conversely A has a multicone $(\mathcal{K}_1, \ldots, \mathcal{K}_m)$, it is clear that the set

$$\mathcal{C} := \left\{\overline{u} \in \mathbb{RP}^{d-1} \colon u \in \bigcup_{j=1}^{m} \mathcal{K}_j \setminus \{0\}\right\}$$

satisfies the criteria of Theorem 2(iii) with $k = 1$ and $F := \{u \in \mathbb{R}^d \colon \langle u, w\rangle = 0\}$. The proof is complete. $\square$

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