(Dated: January 24, 2018)

We present a comprehensible introduction to tight informationally complete measurements for arbitrarily large multipartite systems and study their allowed entanglement configurations, focusing on those requiring simplest possible quantum resources. We demonstrate that tight measurements cannot be exclusively composed neither by fully separable nor maximally entangled states, according to a selected measure of entanglement. We establish an upper bound on the maximal number of fully separable states allowed by tight measurements and we study the distinguished case in which every measurement operator has the same amount of entanglement. Furthermore, we introduce the notion of nested tight measurements, i.e. multipartite tight informationally complete measurements such that every reduction to a certain number of parties induces a lower dimensional tight measurement, proving that they exist for any number of parties and internal levels.

Keywords: Informationally complete quantum measurements, SIC-POVM, mutually unbiased bases, quantum entanglement.

I. INTRODUCTION

Information stored in physical systems can be retrieved by choosing a suitable set of measurements. In quantum mechanics this information can be obtained by considering *positive operator valued measures* (POVM). By implementing a set of measurements over an ensemble of particles it is possible to tomographically reconstruct quantum states. The information gained from some special sets of measurements is enough to reconstruct any state of a quantum system. Such kind of measurements are called *informationally complete* (IC)-POVM and the process to reconstruct the state is called *quantum state tomography.* These general measurements have a fundamental importance in quantum information theory, as they find applications in quantum processes tomography [1], quantum cryptography [2] and quantum fingerprinting [3].

Among existing schemes of generalised quantum measurements one distinguishes a class called *tight IC-POVM* [4, 5], having some remarkable properties: *(i)* they provide a simple tomographical reconstruction formula, *(ii)* they minimize the mean square error (MSE), i.e., minimal mean square Hilbert-Schmidt distance between the estimate and the true state under the presence of errors in state preparation and measurements [4]. Tight IC-POVM are closely related to *complex projective t-designs* [6] for $t \geq 2$. These measurements are optimal for quantum clonning [4] and have been widely used for theoretical [7–14] and experimental [15–18] applications in quantum information theory. They have recently been applied in a protocol for teleportation of quantum entanglement over 143 km [19].

Along this paper, we study tight IC-POVM for multipartite quantum systems and reveal several aspects concerning amount of entanglement allowed by these configurations. In particular, we shed some light onto the following basic question:
What is the minimal amount of quantum resources required to construct a tight IC-POVM?

After the introductory Section , we start Section by showing that the answer to the above question is not trivial: a tight IC-POVM cannot be composed by fully separable measurement operators. As a matter of fact, the answer depends on which measure of quantum resources do we choose to compare two given quantum measurements. We will optimize quantum resources by considering *(i)* tight measurements composed by the maximal possible number of measurement operators that can be prepared by considering local operations and classical communication (LOCC), and *(ii)* isoentangled tight measurements, in the sense that all measurement operators can be defined through a single fiducial state and local unitary operations.

This work is organized as follows: In Section , we introduce the formal definition of tight IC-POVM and study some basic properties. In Section we study the distribution of entanglement in such quantum measurements and show that they cannot be composed exclusively neither of fully separable nor k-uniform states, i.e. pure states such that every reduction to k parties is maximally mixed. We also show that Symmetric IC (SIC)-POVM cannot be composed by grouping fully separable and k-uniform states for any N qudit systems and any $k \geq 1$, establishing a remarkable difference with respect to mutually unbiased bases (MUB), where we show that this is possible for every N qudit system and $k = 1$ only. In Section we introduce the notion of nested tight IC-POVM and prove that they exist for any N-qudit system. In Section V, we present a summary of our main results and conclude the work. Appendix contains a proof of propositions, whereas Appendix shows a particular configuration of a two-qubit SIC-POVM containing five separable states, which is the maximal number we managed to find.

II. TIGHT IC-POVM

In this section we introduce the concept of *tight informationally complete-POVM* and show some basic properties and examples. Let us start by recalling the definition of *Positive Operator Valued Measure* (POVM). A POVM is a set of m positive semidefinite operators Π_j summing up to the identity. Along the work, we restrict our attention to subnormalized rank-one POVM, i.e., operators of the form

$$\Pi_j = \frac{D}{m}\,|\varphi_j\rangle\langle\varphi_j|, \quad j \in \{1,\ldots,m\}, \tag{1}$$

where $|\varphi_j\rangle \in \mathbb{C}^D$ and $m \geq D$, such that $\sum_{j=1}^{m} \Pi_j = \mathbb{I}$. Here m denotes the number of outcomes of the POVM. *Frame potentials* [20] are important to quantify efficiency of generalised quantum measurements. The weighted frame potential of order t of a given POVM $\{\Pi_j\}$ with m outcomes reads [4]:

$$F_t(\{\Pi\}) = \sum_{i,j=1}^{m} w_i w_j \big(\mathrm{Tr}(\Pi_i\Pi_j)\big)^t. \tag{2}$$

For the sake of simplicity, we are going to introduce our main figure of merit, i.e. tight IC-POVM, through the notion of frame potentials. The following adopted definition is a simplification of Corollary 14 in Ref.[4]:

DEFINITION 1. *A set of m subnormalized rank-one projectors $\{\Pi_j\}$ acting a Hilbert space of dimension D is called a tight IC-POVM if its associated t-th frame potential F_t saturates the lower bound*

$$F_t(\{\Pi\}) \geq \left(\frac{m}{D}\right)^{-t} \binom{D+t-1}{t}^{-1}, \tag{3}$$

for $t = 2$.

Eq.(3) is also known as *generalised Welch bound* [21]. A set of vectors saturating Eq.(3) for a given $t \geq 2$ defines a *weighted complex projective t-design* [4]. The factor $(m/D)^t$ appearing in Eq.(3) is due to the fact that projectors (1) are subnormalized. Note that saturation of bound (3) for $t = 1$ occurs for any rank-one POVM, equivalently, complex projective 1-design or tight frame [22]. For example, it is saturated by *von Neumann projective measurements*, i.e. orthonormal bases. As a remarkable example of tight IC-POVM we have the *Symmetric IC-POVM* [8], where $m = D^2$, the weight function is $w_i = m^{-1}$ for every $i = 1, \dots, m$, that form a 2-design in every dimension where they exist (see Ref. [23]). SIC-POVM are the unique tight IC-POVM having the minimal possible number of measurement outcomes, $m = D^2$ [4]. A set of D^2 pure quantum states $\{|\phi_i\rangle\}$ defined in dimension D forms a SIC-POVM if $|\langle\phi_i|\phi_j\rangle|^2 = 1/(D+1)$, for every $i \neq j = 1, \dots, D^2$. This special kind of measurements exists in every dimension $d \leq 21$ [23–28] and some other dimensions up to 124 [28]. Also, there are highly precise numerical solutions for every dimension $d \leq 151$ [23, 24, 28–31], $d = 323$ and $d = 844$ [28].

Another highly interesting class of tight IC-POVM is represented by maximal sets of mutually unbiased bases (MUB) [32], where $m = D(D+1)$, the weight function is $w_i = m^{-1}$ for every $i = 1, \dots, m$. These configurations of m states form a 2-design in every dimension in which they exist [14]. Two orthonormal bases $|\phi_i\rangle$ and $|\psi_j\rangle$ defined in dimension D are *unbiased* if $|\langle\phi_i|\psi_j\rangle|^2 = 1/D$, for every $i, j = 1, \dots, D$. A set of more that two orthonormal bases form mutually unbiased bases (MUB) if they are pairwise unbiased. It is known that at most $D + 1$ MUB exist in every dimension D, and it is saturated for any dimension D which can be factored as a power of a prime d, $D = d^N$ [33]. Some more general classes of tight IC-POVM have been constructed in Ref. [5].

The existence and construction of tight IC-POVM is a complicated open problem. Indeed, the existence of MUB in arbitrary dimensions remain open and SIC-POVM are known to exist in a finite set of dimensions only, so far. A remarkable property of tight IC-POVM $\{\Pi_j\}$ is the fact that any D dimensional quantum state ρ can be decomposed as follows:

$$\rho = \frac{m(D+1)}{D} \sum_{j=1}^{m} p_j \Pi_j - \mathbb{I}_D, \tag{4}$$

where $p_j = \mathrm{Tr}(\rho \Pi_j)$ is a normalized probability distribution that can be measured in the laboratory. Even more, tight IC-POVM have high fidelity of reconstruction under the presence of errors in state preparation and quantum measurements [4].

Let us show that probability distribution $\{p_j\}$ from Eq.(4) satisfies a special constraint.

PROPOSITION 1. *Let $\{\Pi_j\}$ be a tight IC-POVM acting on dimension D and having m outcomes. Then, the following relation holds,*

$$\sum_{j=1}^{m} p_j^2 = \frac{D(\mathrm{Tr}(\rho^2)+1)}{m(D+1)}. \tag{5}$$

Proof. From multiplying Eq.(4) by ρ and taking trace at both hand sides we have the desired result. □

In particular, Proposition 1 holds for any maximal set of MUB and SIC-POVM. This proposition induces a sort of Bloch representation for any tight IC-POVM. In particular, for a single-qubit system and maximal set of MUB measurements, three out of six probabilities are independent. By applying a suitable redefinition of these independent coordinates we obtain the standard Bloch representation of a one-qubit state.

III. ENTANGLEMENT IN TIGHT IC-POVM

In this section we study entanglement properties of multipartite tight IC-POVM. We derive some general results that do not depend on global unitary transformations applied to the entire set of measurement operators. Let us start by showing a simple fact.

OBSERVATION 1. *An IC-POVM can be exclusively formed by local measurements.*

For instance, in the case of N qudit systems the fully separable eigenbases of all tensor product Pauli operators allows us to determine the full set of Stock parameters, that univocally determine any density matrix. This method to tomographically reconstruct quantum states is known as *standard quantum tomography*, and it is highly costly because the number of projective measurements scales as D^3, where $D = d^N$ is the total dimension of the Hilbert space [34]. On the other hand, let us show that a tight IC-POVM does not satisfy the above property.

PROPOSITION 2. *A tight IC-POVM cannot be exclusively formed by local measurements.*

Proof. Suppose that $\{\Pi_j\}$ is a bipartite tight IC-POVM such that $\Pi_j = \Pi_j^A \otimes \Pi_j^B$, where $\{\Pi_j^A\}$ and $\{\Pi_j^B\}$ are tight IC-POVM having m_A and m_b outcomes and acting on dimensions D_A and D_B, respectively. From Eq.(4) we have

$$\rho = (D_A D_B + 1)\frac{m_A m_B}{D_A D_B}\sum_{j=1}^{m} p_j \Pi_j^A \otimes \Pi_j^B - \mathbb{I}_{D_A D_B}. \tag{6}$$

Let us consider the separable state $\rho = \rho_A \otimes \mathbb{I}_{D_B}$ and the identities

$$p_j = \mathrm{Tr}[((\rho_A) \otimes \mathbb{I}_{D_B})(\Pi_j^A \otimes \Pi_j^B)] = \frac{p_j^A}{m_B}, \tag{7}$$

where $p_j^A = \mathrm{Tr}(\rho_A \Pi_j^A)$ and

$$\sum_{j=1}^{m} p_j \Pi_j^A = m_B \sum_{j=1}^{m_A} p_j \Pi_j^A. \tag{8}$$

Identity (8) holds because of the $m = m_A m_B$ operators $\Pi_j^A \otimes \Pi_k^B$ cover all possible combination of indices $j = 1, \ldots, m_A$ and $k = 1, \ldots, m_N$. Thus, from taking partial trace over the subsystem B in Eq.(6) we obtain

$$\rho_A = \frac{m_A(D_A D_B + 1)}{D_A}\sum_{j=1}^{m_A} p_j \Pi_j^A - D_B\, \mathbb{I}_{D_A}. \tag{9}$$

From here we find a contradiction, as the reduced measurement $\{\Pi_A\}$ would not be a tight IC-POVM for any dimension $D_B \geq 2$. That is, Eq.(4) does not hold for any $D_B \geq 2$, which concludes the proof. For multipartite systems the proof follows in the same way by considering every possible bipartition. $\square$

Tensor product of tight IC-POVM defines an informationally complete set of measurements, but not a tight one. For instance, the set of N partite measurements composed by product of monopartite SIC-POVM is informationally complete and optimal among all product measurements [35]. However, the fact that the resulting multipartite measurement is not tight has important consequences: *robustness of fidelity reconstruction under the presence of measurement errors decreases exponentially with the number of parties, with respect to the tight SIC-POVM* [35].

In Section , we show that some products of tight IC-POVM can be complemented with POVM composed of maximally entangled states in such a way that the entire set forms a tight IC-POVM.

Let us now consider higher classes of multipartite entangled states. A multipartite quantum pure state is called k-uniform if every reduction to k parties is maximally mixed [36, 37]. Therefore, we have the following result.

PROPOSITION 3. *Tight IC-POVM cannot be exclusively composed of k-uniform states for any $k \geq 1$.*

Proof. Suppose there is a tight IC-POVM such that every measurement operator is a k-uniform state. Therefore, reconstruction formula (4) does not work for states ρ having non-maximally mixed reductions. □

Propositions 2 and 3 reveal that for a tight IC-POVM only intermediate degrees of entanglement are allowed on average. There is a simple argument to estimate the mean amount of entanglement characterising the quantum states which lead to a tight IC-POVM. Consider a quantum system composed by N qudits having d levels each, where the total dimension of the Hilbert space is $D = d^N$. Also consider reductions $\rho_{X_i} = Tr_{\overline{X}_i}(\rho)$, where X_i denotes the i-th subset of k out of N parties, $\overline{X}_i$ is the complementary set and $i = 1, \ldots, \binom{N}{k}$. For instance, for $N = 4$ and $k = 2$ we have $X_1 = \{1,2\}$, $X_2 = \{1,3\}$, $X_3 = \{1,4\}$, $X_4 = \overline{X}_3 = \{2,3\}$, $X_5 = \overline{X}_2 = \{2,4\}$, $X_6 = \overline{X}_1 = \{3,4\}$. A tight IC-POVM is a 2-design, which implies that average purity of reductions over a tight IC-POVM coincides with average purity over the entire set of quantum states, according to the Haar measure distribution. That is,

$$\frac{1}{m}\sum_{j=1}^{m} \mathrm{Tr}({\sigma_j}^2_{X_i}) = \langle \mathrm{Tr}(\rho^2_{X_i})\rangle_{\mathrm{Haar}}, \tag{10}$$

where $\sigma_j = \frac{m}{D}\Pi_j = |\varphi_j\rangle\langle\varphi_j|$ are normalised projectors and ρ_{X_i} is a k-qudit reduction of the state ρ, with respect to the subset of parties X_i. Using the average values derived by Lubkin [38], we have

$$\frac{1}{m}\sum_{j=1}^{m} \mathrm{Tr}({\sigma^2_j}_{X_i}) = \frac{d^k + d^{N-k}}{d^N + 1}, \tag{11}$$

for every reduction to k parties X_i. In the particular case of maximal sets of MUB for bipartite systems this expression reduces to Eqs.(3-5) in Ref. [39]. From Eq.(11), we see that average purity of reductions of a tight IC-POVM behaves asymptotically as

$$\frac{1}{m}\sum_{j=1}^{m}\mathrm{Tr}(\sigma_{j\,X_i}^2) \to \frac{1}{d^k}. \tag{12}$$

This statement holds if either the number of levels d or the number of parties N is large, for any $k \leq N/2$. Therefore, projectors σ_j forming a tight IC-POVM asymptotically become close to *absolutely maximally entangled* (AME) states in large Hilbert spaces. We recall that AME states for N qudit systems are k-uniform states for the maximal possible value $k = \lfloor N/2 \rfloor$ [41]. For instance, the two-qudit generalized Bell states and three-qubit GHZ states belong to this class.

Despite the above described asymptotic behavior, it is always possible to apply a rigid rotation to a tight IC-POVM in such a way that some measurement operators become fully separable, i.e form local measurements. An interesting question concerns establishing how many of the measurement operators can be chosen to be fully separable. From Eq.(11) we can estimate this maximal possible number m_{sep} of fully separable operators, i.e. those satisfying the constraint $\mathrm{Tr}(\rho_{j\,X_i}^2) = 1$, $j = 1, \ldots, m_{sep}$, for every subset X_i. To this end, we impose the extremal condition that the remaining $m - m_{sep}$ operators are k-uniform states, i.e. $\mathrm{Tr}(\sigma_{j\,X_i}^2) = 1/d^k$, for every subset X_i and every $j = m_{sep} + 1, \ldots, m$. As a consequence, we arrive at the following statement.

PROPOSITION 4. *Suppose that a tight IC-POVM for N qudit systems contains m_{sep} fully separable operators, out of the total number m of operators. Then, the following relation holds*

$$m_{sep}\,(d^N + 1) \leq m\,(d^k + 1). \tag{13}$$

This inequality is saturated if and only if the remaining $m - m_{sep}$ states are k-uniform, for a given $k \geq 1$.

Proof. Consider Eq.(11), where m_{sep} reductions are pure states and $m - m_{sep}$ reductions are maximally mixed for every possible subset X_i consisting of k out of N parties. Therefore we have

$$\frac{1}{m}[m_{sep} + d^{-k}(m - m_{sep})] = \frac{d^k + d^{N-k}}{d^N + 1}. \tag{14}$$

From here we obtain Eq.(13) as an equality. The inequality occurs because the possible number of separable vectors m_{sep} is strictly lower when not all of the $m - m_{sep}$ entangled states are k-uniform. □

In particular, Proposition 4 implies that a maximal set of $d^N + 1$ mutually unbiased bases in dimension d^N cannot contain more than $d^k + 1$ fully separable bases, where $k \leq \lfloor N/2 \rfloor$. Moreover, at most $d + 1$ fully separable bases exist for N qudit systems [40]. Thus, we have the following result.

COROLLARY 1. *A maximal set of MUB for N qudit systems exclusively composed of fully separable and k-uniform states is only possible for $k = 1$.*

For instance, Corollary 1 forbids existence of a maximal set of 5-qubit MUB exclusively composed by fully separable and absolutely maximally entangled states [41], which are $k = 2$ uniform. There is another interesting consequence of Proposition 4. For a fixed number of parties N and reductions $k \leq N/2$, left hand side of Eq.(13) increases faster than the right hand side, as a function of local dimension d. As a consequence, there is an upper bound on d for the existence of a tight IC-POVM exclusively composed by the union set of fully separable and k-uniform states.

COROLLARY 2. *A tight IC-POVM composed by m rank-one projectors, where $m_{sep} \leq m$ are fully separable, can only exist for local dimensions $d \leq d_{max}$, where d_{max} is implicitly defined as*

$$m_{sep}\,(d_{max}^N + 1) = m\,(d_{max}^{N/2} + 1). \tag{15}$$

Proof. Consider Eq.(13) from Proposition 4 and the maximal possible value for $k = N/2$. In general, this assumption is an overestimation (e.g. for $N = 4$ and $d = 2$ the value $k = 2$ is not possible [42]). □

The only examples saturating the value $d = d_{max}$ established by Corollary 2 is achieved by the two-qudit MUB for prime power values of d, as far as we know.

Let us now consider an implication for SIC-POVM.

COROLLARY 3. *For any $k \geq 1$, an N qudit SIC-POVM cannot be composed by the union set of fully separable and k-uniform states.*

Proof. Inequality $m_{sep} \leq \frac{(1+d^k)d^{2N}}{1+d^N}$ comes from considering $m = d^2$ in Eq.(13). According to Proposition 4 we require saturation of the inequality in order to have a POVM exclusively composed by m_{sep} fully separable and $m - m_{sep}$ k-uniform elements. However, the inequality cannot be saturated as the upper bound is not an integer number for any value of the involved parameters. □

Corollary 3 implies that at most nine separable vectors are allowed by a two-qubit SIC-POVM. However, by considering exhaustive numerical optimization of 2-nd frame potential (2), we have found no more than *five* separable vectors (see Appendix). Corollaries 1 and 3 reveal fundamental differences existing between MUB and SIC-POVM from the point of view of quantum entanglement.

Before ending this section, let us emphasise that isoentangled tight IC-POVM exist for multipartite systems. The most remarkable example is provided by the so-called *Hoggar lines* [43], a special kind of SIC-POVM existing for three-qubit systems. They are given by the following 64 states:

$$|\phi_{ijk}\rangle = \sigma_i \otimes \sigma_j \otimes \sigma_k |\phi_{000}\rangle, \tag{16}$$

where $\sigma_0 = \mathbb{I}_2$, while σ_1, σ_2 and σ_3 are the Pauli matrices. Hoggar lines are isoentangled, as all its 64 elements $|\phi_{ijk}\rangle$ are related by local unitary operations through Eq.(16). Among the entire set of 240 Hoggar lines [44], there are only two inequivalent fiducial states under local unitary operators, which can be written in the following symmetric form [**?**]:

$$\begin{aligned} |\phi^{(1)}_{000}\rangle &= \frac{1}{\sqrt{6}}\big(|000\rangle + i(|011\rangle + |101\rangle + |110\rangle) - (1-i)|011\rangle\big), \\ |\phi^{(2)}_{000}\rangle &= \frac{1}{\sqrt{6}}\big(|000\rangle + i(|011\rangle + |101\rangle + |110\rangle) - (1+i)|011\rangle\big). \end{aligned} \tag{17}$$

A symmetric fiducial state equivalent to $|\phi^{(2)}_{000}\rangle$ was already noted by Jedwab and Wiebe [45]. Symmetric fiducial state for Hoggar lines is unique up to unitary or anti-unitary transformations. However, from the point of view of quantum information theory, fiducial states $|\phi^{(1)}_{000}\rangle$ and $|\phi^{(2)}_{000}\rangle$ are essentially different, as they have different values of the three-tangle τ_{ABC}, a three-qubit entanglement invariant [46]. These values are given by $\tau^{(1)}_{ABC} = 2/3$ and $\tau^{(2)}_{ABC} = 2/9$, respectively. Even thought these states are inequivalent under local unitary operations, the state $|\phi^{(1)}_{000}\rangle$ can be obtained from $|\phi^{(2)}_{000}\rangle$ by considering stochastic local operations, as both states belong to the GHZ class of states characterized by the restriction $\tau_{ABC} > 0$.

For bipartite systems, any existing tight IC-POVM has an average amount of entanglement - quantified by the purity of reductions- that only depends on the total dimension of the Hilbert space and the dimension of the reduced space, as we have seen in Eq. (10). We observe that the same result *cannot be generalized* to multipartite systems.

PROPOSITION 5. *For three-partite systems, the average purity of any single particle reductions of a tight IC-POVM is not univocally determined by the dimension of the Hilbert space D and the dimension of reduction space k.*

In other words, the law of Lubkin formulated in [38] for bipartite systems (see Eq. (11)) cannot be generalized in the above sense to three-partite systems. However, it is worth to mention that the Lubkin's law applies to any bipartition of a multipartite system.

One can pose the question whether similar isoentangled sets can be defined for MUB acting on two-qudit systems, i.e $D = d^2$. As we have seen in Eq.(11), the average entanglement among all the states of a maximal set of MUB is fixed by the size of the system and the reductions. However, by applying global unitary operations one may obtain different entanglement distributions having the same average entanglement. In particular, isoentangled sets are remarkably interesting, as they can be prepared in the laboratory by considering a single *fiducial* state and local unitary operations. In the simplest case of two qubit systems the answer has recently been shown to be positive [49].

IV. NESTED TIGHT IC-POVM

In Section , we have shown that tight IC-POVM cannot be exclusively composed neither of fully separable nor of 1-uniform states. Here, we study the possibility to construct tight IC-POVM composed by grouping fully separable and k-uniform states. Let us start by showing the following result.

PROPOSITION 6. *Let $\{\Pi_j\}$ be a tight IC-POVM defined for N qudit systems and composed by m rank-one projectors, where m_{sep} of them are fully separable and $m - m_{sep}$ are k-uniform states, for a given $k \geq 1$. Every reduction to k parties of the entire set of projectors induce a tight IC-POVM for k qudit systems if and only if*

$$m_{sep}(d^N + 1) = m(d^k + 1). \tag{18}$$

A proof of Proposition 6 is given in Appendix .

From Eq.(18) we note that if $(d^N + 1)$ is not divisible by $(d^k + 1)$ then the minimal value for m_{sep} is $d^k + 1$, which represents the same number of fully separable vectors required by a maximal set of MUB for N qudit systems. Note that Proposition 6 generalizes the result found in Proposition 4.

In order to illustrate nested tight IC-POVM let us consider a maximal set of 5 MUB for two-qubit systems, given by the columns of the following unitary matrices [47]:

$$\mathcal{B}_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \\ 1 & 1 & \bar{1} & \bar{1} \end{pmatrix} \quad \mathcal{B}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ i & \bar{i} & i & \bar{i} \\ i & \bar{i} & \bar{i} & i \\ \bar{1} & \bar{1} & 1 & 1 \end{pmatrix} \quad \mathcal{B}_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \bar{1} & 1 & \bar{1} \\ \bar{i} & i & i & \bar{i} \\ i & i & \bar{i} & \bar{i} \end{pmatrix} \quad \mathcal{B}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \bar{i} & i & i & \bar{i} \\ 1 & \bar{1} & 1 & \bar{1} \\ i & i & \bar{i} & \bar{i} \end{pmatrix},$$

where the upper bar denotes negative sign and a normalization factor 1/2 has to be applied to every vector. The remaining unbiased basis is the computational basis $\mathcal{B}_0 = \mathbb{I}_4$. Note that $\mathcal{B}_0, \mathcal{B}_1$ and $\mathcal{B}_2$ are separable bases, whereas $\mathcal{B}_3$ and $\mathcal{B}_4$ are maximally entangled bases. Both Alice and Bob reductions of the three separable bases yield a maximal set of 3 MUB for the single qubit system, whereas the remaining reductions are maximally mixed.

Following the main result of Ref. [40], we have the following observation.

OBSERVATION 2. *Nested tight IC-POVM exist for N qudit systems.*

This observation relies on the fact that there is a maximal set of $d^N + 1$ MUB in prime power dimension $D = d^N$ such that every single particle reduction determines a maximal set of $d+1$ MUB in every prime dimension d [40]. On the other hand, we have the following consequence of Proposition 6.

OBSERVATION 3. *Nested SIC-POVM do not exist for N qudit systems.*

In other words, Eq.(18) cannot be satisfied for a tight IC-POVM having the minimal possible number of measurement outcomes $m = d^{2N}$, for any number of parties N, single particle levels d and reductions k.

To conclude the section, let us mention that Figure 1 summarizes the most important results of the paper.

V. CONCLUSIONS

We studied entanglement configurations allowed by tight informationally complete quantum measurements having any number of outcomes for arbitrary large multipartite quantum systems. These informationally complete measurements, MUB and SIC-POVM in particular, have the advantage to maximize the robustness of fidelity reconstruction under the presence of errors in both state preparation and measurement stages. We focused our study on the existence of tight measurements having not only the highest possible robustness but also allowing the simplest possible implementation, in the sense of minimizing the physical resources required to implement the measurements in the laboratory. We have shown that tight quantum measurements cannot be exclusively composed neither of fully separable nor of k-uniform states, for any $k \geq 1$ (see Propositions 2 and 3). In particular, the result holds from multipartite GHZ qudit states ($k = 1$) to absolutely maximally entangled states ($k = \lfloor N/2 \rfloor$). On the other hand, it is worth to mention that non-tight measurements can be actually composed by fully separable states (see Observation 1). Furthermore, we established an upper bound for the maximal number of fully separable states that can be a part of a tight measurement. We also shown that a tight measurement composed by the union set of fully separable and k-uniform states exist only for $k = 1$ (see Corollary 1). Here, a remarkable example is provided by some special maximal sets of mutually unbiased bases existing in a prime power dimension. The same conclusion cannot be achieved by

SIC-POVM for any number of particles and any local dimension, establishing a fundamental difference with respect to mutually unbiased bases (see Corollary 3). Also, we showed that the well-known result by Lubkin for the average entanglement of reductions for bipartite systems, cf. Eq.(10), cannot be extended beyond bipartitions. Specifically, the average amount of entropy of reductions of a tripartite tight measurement, equivalently a complex projective 2-design, is not solely determined by the number of outcomes, number of parties, local and global dimensional Hilbert spaces (see Proposition 5). We introduced the notion of *nested tight IC-POVM*, i.e. tight IC-POVM such that every subset of reductions to a certain number k of parties induces a k-dimensional tight IC-POVM (see Section). These nested measurements exist for N qudit system for any number of parties $N \geq 2$ and any number of internal levels $d \geq 2$ (see Observation 2). Finally, we provided necessary and sufficient conditions to have a tight measurement for N qudit systems composed by both fully separable and k-uniform states (see Proposition 6).

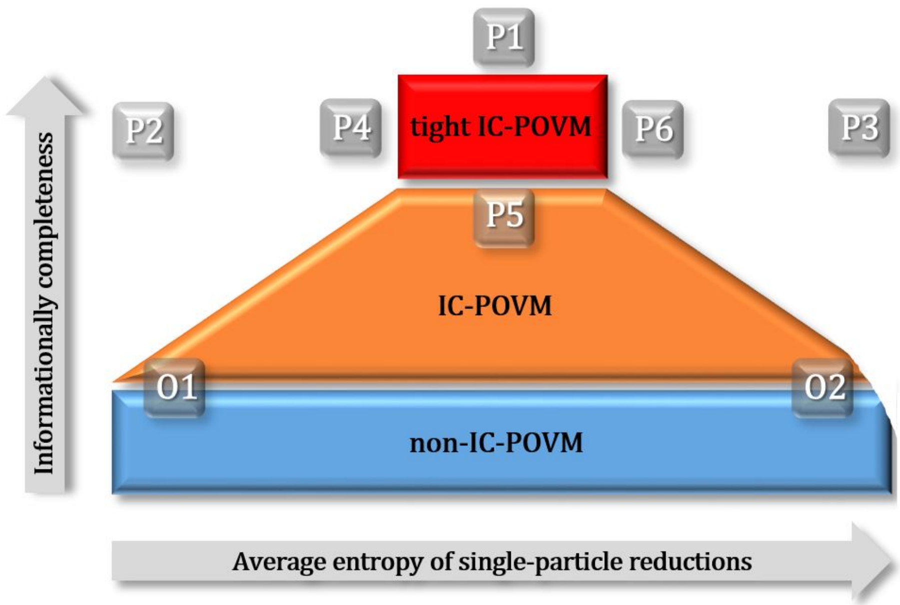

FIG. 1: Informationally completeness of generalised quantum measurement as a function of average entropy of single-particle reductions. Strength of informationally completeness is characterized by its robustness of fidelity reconstruction under noisy state preparation and imperfect measurements. Observations (O) and Propositions (P) derived along the work are illustrated (see details in Conclusions).

Acknowledgements

We are specially thankful to Marcus Appleby, Ingemar Bengtsson, Markus Grassl and Zbigniew Puchała for fruitful discussions about average moments of reduced density matrices, Hoggar lines and informationally complete measurements in general. Financial support by Narodowe Centrum Nauki under the grant number DEC-2015/18/A/ST2/00274 is gratefully acknowledged.

Appendix A: Proof of Propositions

In this appendix, we provide a proof of Proposition 6.

PROPOSITION 6. *Let $\{\Pi_j\}$ be a tight IC-POVM defined for N qudit systems and composed by m rank-one projectors, where m_{sep} of them are fully separable and $m - m_{sep}$ are k-uniform states. Every reduction to k parties of the fully separable projectors forms a tight IC-POVM for k qudit systems if and only if*

$$m_{sep}(d^N + 1) = m(d^k + 1). \tag{A1}$$

Proof. Let $\{\Pi_j\}_{j=1}^m = \{\Pi_j^s\}_{j=1}^{m_{sep}} \cup \{\Pi_j^e\}_{j=m_{sep}+1}^m$ be the tight IC-POVM composed by m_{sep} fully separable and $m-m_{sep}$ entangled (k-uniform) subnormalized rank-one projectors. The proof basically consists in taking partial trace to Eq.(4) over a given set of k parties denoted X_i, for $i \in \{1, \ldots, \binom{N}{k}\}$. Calculation of these reductions involve some fine details to be one-by-one considered below. Let us start applying partial trace to the separable POVM projectors

$$\begin{aligned}
\mathrm{Tr}_{X_i}(\Pi_j^s) &= \frac{d^N}{m}\mathrm{Tr}_{X_i}(|\varphi_j^s\rangle\langle\varphi_j^s|) = \frac{d^N}{m}|\tilde{\varphi}_j^{(i)}\rangle\langle\tilde{\varphi}_j^{(i)}| \\
&= \frac{d^N}{m}\frac{m_{sep}}{d^k}\left(\frac{d^k}{m_{sep}}|\tilde{\varphi}_j^{(i)}\rangle\langle\tilde{\varphi}_j^{(i)}|\right) = \frac{m_{sep}d^N}{md^k}\tilde{\Pi}_j^{(i)},
\end{aligned} \tag{A2}$$

where $\tilde{\Pi}_j^{(i)} = \frac{d^k}{m_{sep}}|\tilde{\varphi}_j^{(i)}\rangle\langle\tilde{\varphi}_j^{(i)}|$ are the suitable subnormalized rank-one POVM projectors according to definition of a tight IC-POVM composed by m_{sep} operators for the subset of k parties X_i (see Section). For k-uniform projectors Π_j^e we have uniform reductions to k parties, that is

$$\mathrm{Tr}_{X_i}(\Pi_j^e) \;=\; \frac{d^N}{m}\mathrm{Tr}_{X_i}(|\varphi_j^e\rangle\langle\varphi_j^e|) = \frac{d^N}{md^k}\,\mathbb{I}_{d^k}. \tag{A3}$$

Also, for the identity operator we have

$$\mathrm{Tr}_{X_i}(\mathbb{I}_{d^N}) = d^{N-k}\mathbb{I}_{d^k}. \tag{A4}$$

Another important detail to be taken into account is the suitable expression for probabilites. In order to simplify expressions we assume that $\rho = \tilde{\rho}\otimes\mathbb{I}/d^{N-k}$. This assumption can be done without loosing of generality, as $\tilde{\rho}$ is an arbitrary density matrix acting on the subset of k parties X_i. Therefore, probabilities associated to separable projectors Π_j^s can be written as:

$$\begin{aligned} p_j \;&=\; \mathrm{Tr}(\rho\Pi_j^s) = \frac{d^N}{m}\frac{d^k}{d^N}\mathrm{Tr}[(\rho_{X_i}\otimes\mathbb{I}_{d^{N-k}})(\Pi_j^{(i)}\otimes\Pi_{j\,\bar{X}_i})] \\ &=\; \frac{d^N}{m}\frac{d^k}{d^N}\frac{m_{sep}}{d^k}\frac{d^k}{m_{sep}}\mathrm{Tr}(\rho_{X_i}\Pi_j^{(i)}) = \frac{m_{sep}}{m}\mathrm{Tr}(\rho_{X_i}\tilde{\Pi}_j^{(i)}) = \frac{m_{sep}}{m}\tilde{p}_j^{(i)}, \end{aligned} \tag{A5}$$

where we assumed the decomposition $\Pi_j^s = (d^N/m)\,\Pi_{j\,X_i}\otimes\Pi_{j\,\bar{X}_i}$, for $j\in\{1,\ldots,m_{sep}\}$. Here, $\bar{X}_i$ denotes the subset of $N-k$ parties complementary to X_i. The remaining ingredient concerns to the simplified expression for a partial sum of probabilities. That is,

$$\sum_{j=m_{sep}+1}^{m} p_j = 1 - \sum_{j=1}^{m_{sep}} p_j^{(i)} = 1 - \frac{m_{sep}}{m}\sum_{j=1}^{m_{sep}}\tilde{p}_j^{(i)} = 1 - \frac{m_{sep}}{m}, \tag{A6}$$

where $\tilde{p}_j^{(i)} = \mathrm{Tr}(\tilde{\rho}\,\tilde{\Pi}_j^{(i)})$ and condition $\sum_{j=1}^{m_{sep}}\tilde{p}_j^{(i)} = 1$ is imposed because we require that the set of operators $\{\tilde{\Pi}_j^{(i)}\}$, $j\in\{1,\ldots,m_{sep}\}$ form a tight IC-POVM for every subset of k parties X_i, $i\in\{1,\ldots,\binom{N}{k}\}$.

Taking into account all the above described ingredients we are now in position to apply partial trace to Eq.(4) with respect to the subset of k parties X_i, thus obtaining

$$\begin{aligned}
\mathrm{Tr}_{X_i}(\rho) &= \frac{m(d^N+1)}{d^N}\left(\sum_{j=1}^{m_{sep}} p_j^s \mathrm{Tr}_{X_i}(\Pi_j) + \sum_{j=m_{sep}+1}^{m} p_j \mathrm{Tr}_{X_i}(\Pi_j^k)\right) - \mathrm{Tr}_{X_i}(\mathbb{I}_{d^N}) \\
&= \frac{m(d^N+1)}{d^N}\left(\frac{m_{sep}d^N}{md^k}\sum_{j=1}^{m_{sep}} p_j\tilde{\Pi}_j + \frac{d^N}{md^k}\sum_{j=m_{sep}+1}^{m} p_j \mathbb{I}_{d^k}\right) - d^{N-k}\mathbb{I}_{d^k} \\
&= \frac{m(d^N+1)}{d^N}\left(\frac{m_{sep}d^N}{md^k}\frac{m_{sep}}{m}\sum_{j=1}^{m_{sep}} \tilde{p}_j\tilde{\Pi}_j + \frac{d^N}{md^k}(1-\frac{m_{sep}}{m})\mathbb{I}_{d^k}\right) - d^{N-k}\mathbb{I}_{d^k} \\
&= \frac{m_{sep}^2(d^N+1)}{md^k}\sum_{j=1}^{m_{sep}} \tilde{p}_j\tilde{\Pi}_j + \frac{m-(d^N+1)m_{sep}}{md^k}\mathbb{I}_{d^k}. \qquad \text{(A7)}
\end{aligned}$$

From comparing Eqs. (4) and (A7) we obtain the following restrictions on the parameters:

$$\frac{m_{sep}^2(d^N+1)}{md^k} = \frac{m_{sep}(d^k+1)}{d^k}, \qquad \text{(A8)}$$

and

$$\frac{m-(d^N+1)m_{sep}}{md^k} = 1. \qquad \text{(A9)}$$

These two equations are equivalent and reduced to $m_{sep}(d^N+1) = m(d^k+1)$, which concludes the proof. □

Appendix B: Two-qubit tight IC-POVM composed by four separable vectors

In this section we provide a numerical study of two-qubit tight IC-POVM. Below we present a list of a two-qubit SIC-POVM, i.e. tight IC-POVM having $m = 16$ outcomes, such that 5 out of its 16 rank-one projectors are separable.

$$
\begin{aligned}
v_1^{sep} &= (1,0)\otimes(1,0)\\
v_2^{sep} &= (0.829450, -0.071288+0.554012i)\otimes(0.538325, -0.738169+0.406584i)\\
v_3^{sep} &= (0.558493, 0.656070+0.507598i)\otimes(0.801919, 0.432837+0.411797i)\\
v_4^{sep} &= (0.822550, -0.502837+0.265641i)\otimes(0.543781, 0.835135+0.082763i)\\
v_5^{sep} &= (0.543031, 0.153431+0.825576i)\otimes(0.823280, -0.032621-0.566696i)
\end{aligned}
$$

$$
\begin{aligned}
v_6 &= (0.447615, -0.389125-0.446939i, -0.454783+0.230108i, -0.276499+0.335022i)\\
v_7 &= (0.446765, 0.604844-0.274049i, -0.291296-0.259105i, 0.439704-0.118884i)\\
v_8 &= (0.447075, 0.00858814-0.303724i, 0.241987+0.435012i, -0.638674-0.228265i)\\
v_9 &= (0.446722, 0.334228+0.585485i, 0.0484938+0.315171i, 0.385974-0.308672i)\\
v_{10} &= (0.446515, -0.00980095-0.028985i, 0.0311724-0.417279i, -0.0431798-0.789133i)\\
v_{11} &= (0.447508, 0.340026+0.0366212i, 0.422419-0.587076i, -0.394988+0.060542i)\\
v_{12} &= (0.447241, -0.200413-0.735024i, 0.340286-0.312953i, 0.012735-0.0751886i)\\
v_{13} &= (0.448257, -0.213794+0.454123i, -0.24267-0.231547i, -0.354814+0.555638i)\\
v_{14} &= (0.447515, -0.205972-0.235433i, -0.0725021-0.207019i, 0.614807+0.525144i)\\
v_{15} &= (0.44708, -0.45748+0.428185i, 0.470043-0.350614i, 0.237987+0.0835461i)\\
v_{16} &= (0.447179, -0.109299+0.192073i, -0.833937-0.142906i, 0.0951736-0.162052i)
\end{aligned}
$$

The smallest frame potential F_2 achieved by the above solution is 25.600034, where 25.6 is the value of the Welch bound (3). As a further numerical study, in Table I we show that tight IC-POVM seem to exist for any number of vectors $m \geq 16$, where $m = 16$ correspond to the minimal possible number of outcomes (SIC-POVM).

m	Welch bound	Frame potencial (F_2)
16	25.600	25.600
17	28.900	28.914
18	32.400	32.414
19	36.100	36.101
20	40.000	40.000

TABLE I: Numerical optimization of frame potential F_2 for generalised measurements of two-qubit systems ($D = 2^2$), having $16 \leq m \leq 20$ measurement outcomes (see Eq.(2)). Lower bound for F_2, i.e. Welch bound (3), can be achieved if and only if the quantum measurement is tight. Our study suggests that tight IC-POVM may exist for any number of measurement outcomes $m \geq D^2$.

[1] A. Scott, Optimizing quantum process tomography with unitary 2-designs, J. Phys. A 41, 055308 (2008).

[2] J. Renes, Equiangular spherical codes in quantum cryptography, Quantum Inf. Comput. 5, 81 (2005).

[3] A. J. Scott, J. Walgate, B. Sanders, Quantum Inf. Comput. 7, 243 (2007). quant-ph/0507048.

[4] A. J. Scott, Tight informationally complete quantum measurements, J. Phys. A 39, 13507 (2006).

[5] A. Roy, A. Scott, J. Math. Phys. 48, 072110 (2007).

[6] A. Neumaier, *Combinatorial configurations in terms of distances*, Dept. of Mathematics Memorandum 81-09 (Eindhoven University of Technology, 1981).

[7] M. Graydon, D. Appleby, Quantum conical designs, J. Phys. A: Math. Theor. 49, 085301 (2016).

[8] G. Zauner, Ph.D. thesis, University of Vienna, 1999.

[9] C. Dankert, R. Cleve, J. Emerson, E. Livine, Exact and approximate unitary 2-designs: constructions and applications, Phys. Rev. A 80, 012304 (2009).

[10] D. Gross, K. Audenaert, J. Eisert, Evenly distributed unitaries: on the structure of unitary designs, J. Math. Phys. 48, 052104 (2007).

[11] B. Chen, T. Li, S. Ming Fei, General SIC-Measurement Based Entanglement Detection, Quan-

tum Information Processing 14, 2281–2290 (2015).

[12] Y. Xi, Z Jun Zheng, Entanglement detection via general SIC-POVMs, Quant. Inform. Process. 15, 5119 (2016)

[13] A. Ambainis, J. Emerson, Quantum t-designs: t-wise independence in the quantum world, in the proceedings of the Twenty-Second Annual IEEE Conference on Computational Complexity, June 13-16, Washington, U.S.A. (2007).

[14] A. Klappenecker, M. Roetteler, Mutually Unbiased Bases are Complex Projective 2-Designs, Proceedings of the IEEE International Symposium on Information Theory, Adelaide, Australia, September 2005, p. 1740. Available online: arXiv:quant-ph/0502031

[15] G. Lima, L. Neves, R. Guzmán, E. Gómez, W. A. T. Nogueira, A. Delgado, A. Vargas, C. Saavedra, Experimental quantum tomography of photonic qudits via mutually unbiased basis, Opt. Express,19, 3542 (2011).

[16] S. Etcheverry, G. Cañas, E. Gómez, W. A. T. Nogueira, C. Saavedra, G. B. Xavier, G. Lima, Quantum key distribution session with 16-dimensional photonic states, Sci. Rep. 3, 2316 (2013).

[17] N. Bent *et. al.*, Experimental Realization of Quantum Tomography of Photonic Qudits via Symmetric Informationally Complete Positive Operator-Valued Measures, Phys. Rev. X, 5, 041006 (2015).

[18] M. Dall'Arno, S. Brandsen, F. Buscemi, V. Vedral, Device-independent tests of quantum measurements, Phys. Rev. Lett. 118, 250501 (2017).

[19] T. Herbst, T. Scheidl, M. Fink, J. Handsteiner, B. Wittmann, R. Ursin, A. Zeilinger, Teleportation of entanglement over 143 km, PNAS 112(46), 14202 (2015).

[20] J. Benedetto, M. Fickus, Finite normalized tight frames, Adv. Comput. Math. 18, 357 (2003).

[21] A. Belovs, Welch bounds and quantum state tomography. MSc thesis, University of Waterloo, Canada (2008). Available at `https://uwspace.uwaterloo.ca/bitstream/handle/10012/4159/ABelovs.pdf`

[22] S. Waldron, Generalized Welch bound equality sequences are tight frames, IEEE Trans. Inform. Theory, 49 (2003), 2307.

[23] J. M. Renes, R. Blume-Kohout, A. J. Scott, C. Caves. Symmetric informationally complete quantum measurements, J. Math. Phys 45, 2171 (2004).

[24] G. Zauner: *Quantendesigns. Grundzüge einer nichtkommutativen Designtheorie*, PhD thesis,

Univ. Wien 1999. Available in English translation in Int. J. Quant. Inf. **9** (2011) 445.

[25] M. Appleby, T.-Y. Chien, S. Flammia, and S. Waldron, Constructing exact symmetric informationally complete measurements from numerical solutions, arXiv:1703.05981.

[26] D. M. Appleby, SIC-POVMs and the extended Clifford group, J. Math. Phys. 46, 052107 (2005).

[27] M. Grassl, On SIC-POVMs and MUBs in dimension 6, quant-ph/0406175.

[28] M. Grassl, A. Scott, Fibonacci-Lucas SIC-POVMs, J. Math. Phys. 58, 12, 122201 (2017).

[29] A. J. Scott and M. Grassl, SIC-POVMs: A new computer study, J. Math. Phys. 51, 042203 (2010).

[30] A. J. Scott, SICs: Extending the list of solutions, arXiv:1703.03993.

[31] C. A. Fuchs, M. C. Hoan, and B. C. Stacey, The SIC question: History and state of play, Axioms 6, 3, 21 (2017).

[32] I.D. Ivanovic, Geometrical description of quantal state determination, J. Phys. A, 14, 3241 (1981).

[33] W. Wootters, B. Fields, Optimal State-Determination by Mutually Unbiased Measurements, Ann. Phys. 191, 363 (1989).

[34] U. Leonhardt, *Measuring the Quantum State of Light*, Measurement Science and Technology, **11**, 12, Cambridge University Press, Cambridge, UK (1997)

[35] H. Zhu, B Englert, Quantum state tomography with fully symmetric measurements and product measurements, Phys. Rev. A 84, 022327 (2011).

[36] A. Scott, Multipartite entanglement, quantum-error-correcting codes, and entangling power of quantum evolutions, Phys. Rev. A 69, 052330 (2004).

[37] L. Arnaud, N. Cerf, Exploring pure quantum states with maximally mixed reductions, Phys. Rev. A 87, 012319 (2013).

[38] E. Lubkin, Entropy of an n-system from its correlation with a k-reservoir, J. Math Phys. 19 1028 (1978).

[39] M. Wiesniak, T. Paterek, A. Zeilinger, Entanglement in mutually unbiased bases, New J. Phys. 13, 053047 (2011).

[40] J. Lawrence, Entanglement patterns in mutually unbiased basis sets, Phys. Rev. A 84, 022338 (2011).

[41] W. Helwig, W. Cui, A. Riera, J. I. Latorre, H. Lo, Absolute Maximal Entanglement and

Quantum Secret Sharing, Phys. Rev. A 86, 052335 (2012).

[42] A. Higuchi, A. Sudbery, How entangled can two couples get? Phys. Lett. A 272, 213 (2000).

[43] S. G. Hoggar, 64 lines from a quaternionic polytope, Geometriae Dedicata 69, 287–289 (1998).

[44] H. Zhu, Ph.D. thesis, National University of Singapore (2012).

[45] J. Jedwab and A. Wiebe, A simple construction of complex equiangular lines, in Algebraic Design Theory and Hadamard Matrices, C. J. Colbourn, ed. Springer, (2015) – available online: arXiv:1408.2492 [math.CO]

[46] V. Coffman, J. Kundu, W. Wootters, Distributed Entanglement, Phys. Rev. A 61, 052306 (2000).

[47] S. Bandyopadhyay, P. Boykin, V. Roychowdhury, F. Vatan, A new proof for the existence of mutually unbiased bases, Algorithmica 34, 4, 512–528 (2002).

[48] S. Hill, W. K. Wootters, Entanglement of a Pair of Quantum Bits, Phys. Rev. Lett. 78, 5022–5025 (1997).

[49] J. Czartowski, D. Goyeneche, M. Grassl, K. Życzkowski, to be published (2018).

[50] M. Grass, private communication, 26-10-2017.