A3SR Math Review

Properties of Logarithms

Relevant Courses:

-Quantitative Methods
-Generalized Linear Models

Notes

Definition

Logarithms are defined such that $log_b(A) = X$ is equivalent to $b^X = A$

Properties

Using properties of exponents and the definition above, we can derive the following:

a. The Product Rule: $log_b(MN) = log_b(M) + log_b(N)$

b. The Quotient Rule: $log_b(\frac{M}{N}) = log_b(M) - log_b(N)$

c. The Power Rule: $log_b(M^p) = plog_b(M)$

d. $log_b(b^X) = X$

e. $b^{log_b(X)} = X$

f. $log_b b = 1$

g. $log_b 1 = 0$

Example 1: Expanding logarithms

$log_e(\frac{2x^3}{y})$
$= log_e(2x^3) - log_e(y)$
$= log_e(2) + log_e(x^3) - log_e(y)$
$= log_e(2) + 3log_e(x) - log_e(y)$

Example 2: Condensing logarithms

$2log_3(x) + log_3(5) - log_3(2)$
$= log_3(x^2) + log_3(5) - log_3(2)$
$= log_3(5x^2) - log_3(2)$
$= log_3(\frac{5x^2}{2})$

Practice Problems

1. Solve the following:
   a. $log_e(e^x)$
   b. $log_{10}(100)$
   c. $log_{10}(\frac{1}{10})$
   d. $log_{10}(0)$
2. Expand the following:
   a. $log_{10}(\frac{5y^3}{x})$
   b. $log_2(\frac{4y^2}{3x})$
   c. $log_e(2x^2y^3)$
3. Condense the following:
   a. $4log_3(x) - 2log_3(y)$
   b. $log_2(x) + 5log_2(y) - log_2(5)$
   c. $log_{10}(5) + log_{10}(2)$

Answers

1. Solve:
   a. x
   b. 2
   c. -1
   d. There is no solution because there is no power of 10 that would equal 0
2. Expand:
   a. $log_{10}(5) + 3log_{10}(y) - log_{10}(x)$
   b. $2 + 2log_2(y) - log_2(3) - log_2(x)$
   c. $log_e(2) + 2log_e(x) + 3log_e(y)$
3. Condense
   a. $log_3(\frac{x^4}{y^2})$
   b. $log_2(\frac{xy^5}{5})$
   c. 1

Matrix Algebra

Relevant Courses:

-Quantitative Methods

Notes

Note that these notes are a summary of relevant information from https://www.math.psu.edu/bressan/PSPDF/M441-linalgebra_review.pdf

Definitions

An $m \times n$ matrix A has m rows, n columns, and can be written as:

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$$

The transpose of an $m \times n$ matrix, A is the $n \times m$ matrix (denoted A^T) such that every element a_{ij} in matrix A is moved to row j and column i. For example, if:

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 7 \end{bmatrix}$$

then,

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 7 \end{bmatrix}$$

The $n \times n$ identity matrix I_n is a matrix with 1s on the diagonal and 0s everywhere else:

$$I_n = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & \ddots & ... \\ 0 & 0 & ... & 1 \end{bmatrix}$$

An $n \times n$ matrix is called "diagonal" if all elements not on the diagonal are zeros. For example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

An $n \times n$ matrix is called "upper triangular" if all elements below the diagonal are zeros. For example:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

An $n \times n$ matrix is called "lower triangular" if all elements above the diagonal are zeros. For example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

Suppose that an we have 2 $n \times n$ matrices, A and B, such that $AB = I_n$ (note: this also implies $BA = I_n$). Then we say that B is the inverse of A (and vice versa) and we can write, $B = A^{-1}$. A matrix A has an inverse if and only if its determinant is not equal to zero. Note that the determinant of a 2×2 matrix can be calculated as follows (it is not important that you are able to calculate the determinant of a higher dimensional matrix by hand):

$$det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc$$

Operations with matrices

a. Adding matrices ($A + B = C$): If we add 2 $m \times n$ matrices, A and B, we get another $m \times n$ matrix C such that $c_{ij} = a_{ij} + b_{ij}$

b. Subtracting matrices ($A - B = C$): If we subtract the $m \times n$ matrix B from the $m \times n$ matrix A, we get another $m \times n$ matrix C such that $c_{ij} = a_{ij} - b_{ij}$

c. Multiplying a matrix by a scalar ($cA = B$): If we multiply the $m \times n$ matrix A by a scalar, c, then we get another $m \times n$ matrix B such that $b_{ij} = c * a_{ij}$

d. Matrix multiplication ($AB = C$): Note that it is only possible to compute AB if the number of columns in matrix A equals the number of rows in matrix B. If this is the case, then when we multiply an $m \times n$ matrix A by an $n \times k$ matrix B, we get an $m \times k$ matrix, C, such that $c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk}$

Example 1: Matrix Multiplication

$$\begin{bmatrix} 3 & 4 \\ 2 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 2 & 1 \\ 3 & 5 & 2 \end{bmatrix} = \begin{bmatrix} (3*7+4*3) & (3*2+4*5) & (3*1+4*2) \\ (2*7+5*3) & (2*2+5*5) & (2*1+5*2) \\ (1*7+2*3) & (1*2+2*5) & (1*1+2*2) \end{bmatrix} = \begin{bmatrix} 33 & 26 & 11 \\ 29 & 29 & 12 \\ 13 & 12 & 5 \end{bmatrix}$$

Properties of matrix operations

a. $A + B = B + A$

b. $(A + B) + C = A + (B + C)$

c. $(AB)C = A(BC)$

d. $(A + B)C = AC + BC$

e. If A is an $m \times n$ matrix, then $I_m A = A$ and $AI_n = A$

Note that, in general $AB \neq BA$

Writing a system of equations using matrix notation

Note that, if we have a system of equations:
$y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$
$y_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$
$\ldots$
$y_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n$

We can re-write these equations much more simply as:
$Y = AX$ where Y is a $1 \times m$ matrix, A is an $m \times n$ matrix, and X is a $1 \times n$ matrix:

$$\begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$$

Practice Problems

1. Solve the following:
   a. $\begin{bmatrix} 2 & 4 & 2 \\ 1 & 3 & 0 \\ 1 & 6 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 0 \\ -2 & -3 & 0 \\ 1 & 9 & 5 \end{bmatrix}$
   b. $\begin{bmatrix} 2 & 1 \\ -2 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 5 & 2 & -1 \\ 3 & 4 & 2 \end{bmatrix}$
   c. Let A= $\begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 4 \\ 1 & 2 \\ 7 & 0 \end{bmatrix}$ Calculate $A^T B$
   d. Using the same matrices as in part c, calculate $B^T A$

2. Show that A and B are inverses:
   $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$
   $B = \begin{bmatrix} -1 & 0.5 & 1 \\ 1 & 0 & -1 \\ 0 & -0.5 & 1 \end{bmatrix}$

3. Suppose that A is a 4x3 matrix and B is a $3x8$ matrix.
   a. Does AB exist? If so, what are the dimensions of AB?
   b. Does BA exist? If so, what are the dimensions of BA?

4. What is the determinant of $\begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$?

Answers

1. Solve:
   a. $\begin{bmatrix} 3 & 9 & 2 \\ -1 & 0 & 0 \\ 2 & 15 & 7 \end{bmatrix}$
   b. $\begin{bmatrix} 13 & 8 & 0 \\ -4 & 4 & 6 \\ 26 & 16 & 0 \end{bmatrix}$
   c. $\begin{bmatrix} 35 & 10 \\ 13 & 18 \end{bmatrix}$
   d. $\begin{bmatrix} 35 & 13 \\ 10 & 18 \end{bmatrix}$

2. Show that A and B are inverses:
   $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0.5 & 1 \\ 1 & 0 & -1 \\ 0 & -0.5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. Suppose that A is a 4x3 matrix and B is a $3x8$ matrix.
   a. AB exists and is 4x8
   b. BA does not exist

4. The determinant is $(1 * 3) - (-2 * 4) = (3) - (-8) = 11$

Derivatives

Relevant Courses:

-Probability
-Quantitative Methods

Notes

Definition

The derivative of a function $y = f(x)$ with respect to x is defined as a function giving the instantaneous slope of $y = f(x)$ for any value x. Notationally, a derivative can be written in any of the following ways:

$$f'(x) = y' = \frac{df}{dx} = \frac{d}{dx}(f(x)) = \frac{dy}{dx} = \frac{d}{dx}(y)$$

The second derivative of $y = f(x)$ with respect to x is a function giving the rate of change of the instantaneous slope of $f(x)$. Notationally, it can be represented in any of the following ways (note: 3rd, 4th, etc. derivatives are notated in a similar way, with increasing exponents or $'$s):

$$f''(x) = y'' = \frac{d^2 f}{dx^2} = \frac{d^2}{dx^2}(f(x)) = \frac{d^2 y}{dx^2} = \frac{d^2}{dx^2}(y)$$

Using derivatives to find local minima and maxima of a function

To find all local minima or maxima of a function $y = f(x)$:
1) Take the first derivative of $f(x)$ with respect to x ($f'(x)$).
2) Set this expression equal to zero and solve for x. These values of x are local maxima and minima.
3) Calculate the second derivative of $f(x)$ with respect to x ($f''(x)$)
4) Plug in the values of x calculated in part 2. If the second derivative is positive, this value of x represents a local minimum; if the second derivative is negative, it is a local maximum

Properties

Properties of derivatives:
1) Sum/Difference rule: $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
2) Constant multiple rule: $(cf(x))' = cf'(x)$ where c is a constant
3) Power rule: If $f(x) = x^n$, then $f'(x) = nx^{n-1}$
4) Product rule: $(f(x)g(x))' = f'(x)g(x) + g'(x)f(x)$
5) Quotient rule (given g(x) != 0): $(\frac{f(x)}{g(x)})' = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$
6) Chain rule: $(f(g(x)))' = f'(g(x)) * g'(x)$

Partial derivatives

For a function of more than one variable (i.e., $f(x, y)$), we can take partial derivatives with respect to each variable. The partial derivative of $f(x, y)$ with respect to x is often denoted by either $\frac{\partial f}{\partial x} = f_x$. The partial derivative with respect to y would be denoted by $\frac{\partial f}{\partial y} = f_y$. The partial derivative with respect to x would be calculated by treating any non-x variables as constants when applying the above properties.

Practice Problems

1. Find $f'(x)$ for each of the following. Then compute $f''(x)$ for a-d:
   a. $f(x) = 5x^2 + 3x + 1$
   b. $f(x) = \frac{5}{x^3} + 2x^4$
   c. $f(x) = (3x + 1)^5$
   d. $f(x) = 2x^3(x^2 + 1)$
   e. $f(x) = \frac{2x+1}{x^2-5}$

2. Find the partial derivative with respect to x for each of the following:
   a. $f(x, y) = 3xy^2 + 2x$
   b. $f(x, y) = (xy^4 + 2y)^3$
   c. $f(x, y) = 4x^3 + xy + x^2y^2 + 4x + 2$

3. Find all local minima and maxima for the following function (and note whether they are minima or maxima): $\frac{2}{3}x^3 - x^2 - 12x$

Solutions

1. $f'(x)$ and $f''(x)$
   a. $f'(x) = 10x + 3$ and $f''(x) = 10$
   b. $f'(x) = \frac{-15}{x^4} + 8x^3$ and $f''(x) = \frac{-60}{x^5} + 24x^2$
   c. $f'(x) = 15(3x + 1)^4$ and $f''(x) = 180(3x + 1)^3$
   d. $f'(x) = 10x^4 + 6x^2$ and $f''(x) = 40x^3 + 12x$
   e. $f'(x) = \frac{2(x^2-5)-2x(2x+1)}{(x^2-5)^2}$ and $f''(x)$

2. $f_x(x, y) =$
   a. $3y^2 + 2$
   b. $3y^4(xy^4 + 2y)^2$
   c. $12x^2 + y + 2xy^2 + 4$

3. Local minima and maxima
   a. $f'(x) = 2x^2 - 2x - 12 = (2x + 4)(x - 3)$, so setting the first derivative equal to 0, we get $0 = (2x + 4)(x - 3)$, with solutions $x = 3$ and $x = -2$. $f''(x) = 4x - 2$. $f''(3) = 10$ and $f''(x) = -10$. So, $x = 3$ is a local minimum and $x = -2$ is a local maximum. $f(3) = 18 - 9 - 36 = -27$ and $f(-2) = \frac{-16}{3} - 4 + 24 = \frac{44}{3} \approx 14.67$

Integrals

Relevant Courses:

-Probability
-Quantitative Methods

Notes

Definition

Indefinite integrals

In the previous section, we calculated the derivative of a function, $f(x)$. Finding an indefinite integral (also called an anti-derivative) involves a simple reversal of this process. The indefinite integral, $F(x)$, of a function, $f(x)$, is defined such that $F'(x) = f(x)$ and is written as follows:

$$\int f(x)dx = F(x) + c$$

where c is a constant.

Definite integrals

Properties

Practice Problems

1. Solve the following:
   a.

Answers

1. Solve:
   a.

Summary Statistics

Relevant Courses:

-Probability
-Quantitative Methods
-Statistical Computing

P-Values and T-Tests

Relevant Courses:

-Quantitative Methods
-Statistical Computing
-Causal Inference

Correlation and Covariance

Relevant Courses:

-Quantitative Methods
-Probability

Notes

Ordinary Least Squares Regression

Relevant Courses:

-Quantitative Methods

Probability Density/Mass Functions

Relevant Courses:

-Quantitative Methods
-Probability
-Causal Inference

Expectation

Relevant Courses:

-Probability