

# A3SR Math Review

## Properties of Logarithms

### Relevant Courses:

- Quantitative Methods
- Generalized Linear Models

### Notes

#### Definition

Logarithms are defined such that  $\log_b(A) = X$  is equivalent to  $b^X = A$

#### Properties

Using properties of exponents and the definition above, we can derive the following:

- The Product Rule:  $\log_b(MN) = \log_b(M) + \log_b(N)$
- The Quotient Rule:  $\log_b(\frac{M}{N}) = \log_b(M) - \log_b(N)$
- The Power Rule:  $\log_b(M^p) = p\log_b(M)$
- $\log_b(b^X) = X$
- $b^{\log_b(X)} = X$
- $\log_b b = 1$
- $\log_b 1 = 0$

#### Example 1: Expanding logarithms

$$\begin{aligned}\log_e\left(\frac{2x^3}{y}\right) &= \log_e(2x^3) - \log_e(y) \\ &= \log_e(2) + \log_e(x^3) - \log_e(y) \\ &= \log_e(2) + 3\log_e(x) - \log_e(y)\end{aligned}$$

#### Example 2: Condensing logarithms

$$\begin{aligned}2\log_3(x) + \log_3(5) - \log_3(2) &= \log_3(x^2) + \log_3(5) - \log_3(2) \\ &= \log_3(5x^2) - \log_3(2) \\ &= \log_3\left(\frac{5x^2}{2}\right)\end{aligned}$$

## Practice Problems

- Solve the following:
  - $\log_e(e^x)$
  - $\log_{10}(100)$
  - $\log_{10}(\frac{1}{10})$
  - $\log_{10}(0)$
- Expand the following:
  - $\log_{10}(\frac{5y^3}{x^2})$
  - $\log_2(\frac{4y^2}{3x})$
  - $\log_e(2x^2y^3)$
- Condense the following:
  - $4\log_3(x) - 2\log_3(y)$
  - $\log_2(x) + 5\log_2(y) - \log_2(5)$
  - $\log_{10}(5) + \log_{10}(2)$

## Answers

- Solve:
  - $x$
  - $2$
  - $-1$
  - There is no solution because there is no power of 10 that would equal 0
- Expand:
  - $\log_{10}(5) + 3\log_{10}(y) - \log_{10}(x)$
  - $2 + 2\log_2(y) - \log_2(3) - \log_2(x)$
  - $\log_e(2) + 2\log_e(x) + 3\log_e(y)$
- Condense
  - $\log_3(\frac{x^4}{y^2})$
  - $\log_2(\frac{xy^5}{5})$
  - $1$

# Matrix Algebra

## Relevant Courses:

-Quantitative Methods

## Notes

Note that these notes are a summary of relevant information from [https://www.math.psu.edu/bressan/PSPDF/M441-linalggebra\\_review.pdf](https://www.math.psu.edu/bressan/PSPDF/M441-linalggebra_review.pdf)

## Definitions

An  $m \times n$  matrix  $A$  has  $m$  rows,  $n$  columns, and can be written as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The transpose of an  $m \times n$  matrix,  $A$  is the  $n \times m$  matrix (denoted  $A^T$ ) such that every element  $a_{ij}$  in matrix  $A$  is moved to row  $j$  and column  $i$ . For example, if:

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 7 \end{bmatrix}$$

then,

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 7 \end{bmatrix}$$

The  $n \times n$  identity matrix  $I_n$  is a matrix with 1s on the diagonal and 0s everywhere else:

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

An  $n \times n$  matrix is called “diagonal” if all elements not on the diagonal are zeros. For example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

An  $n \times n$  matrix is called “upper triangular” if all elements below the diagonal are zeros. For example:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

An  $n \times n$  matrix is called “lower triangular” if all elements above the diagonal are zeros. For example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

Suppose that we have 2  $n \times n$  matrices,  $A$  and  $B$ , such that  $AB = I_n$  (note: this also implies  $BA = I_n$ ). Then we say that  $B$  is the inverse of  $A$  (and vice versa) and we can write,  $B = A^{-1}$ . A matrix  $A$  has an inverse if and only if its determinant is not equal to zero. Note that the determinant of a  $2 \times 2$  matrix can be calculated as follows (it is not important that you are able to calculate the determinant of a higher dimensional matrix by hand):

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

### Operations with matrices

- Adding matrices ( $A + B = C$ ): If we add 2  $m \times n$  matrices,  $A$  and  $B$ , we get another  $m \times n$  matrix  $C$  such that  $c_{ij} = a_{ij} + b_{ij}$
- Subtracting matrices ( $A - B = C$ ): If we subtract the  $m \times n$  matrix  $B$  from the  $m \times n$  matrix  $A$ , we get another  $m \times n$  matrix  $C$  such that  $c_{ij} = a_{ij} - b_{ij}$
- Multiplying a matrix by a scalar ( $cA = B$ ): If we multiply the  $m \times n$  matrix  $A$  by a scalar,  $c$ , then we get another  $m \times n$  matrix  $B$  such that  $b_{ij} = c * a_{ij}$
- Matrix multiplication ( $AB = C$ ): Note that it is only possible to compute  $AB$  if the number of columns in matrix  $A$  equals the number of rows in matrix  $B$ . If this is the case, then when we multiply an  $m \times n$  matrix  $A$  by an  $n \times k$  matrix  $B$ , we get an  $m \times k$  matrix,  $C$ , such that  $c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \dots + a_{in}b_{nk}$

### Example 1: Matrix Multiplication

$$\begin{bmatrix} 3 & 4 \\ 2 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 2 & 1 \\ 3 & 5 & 2 \end{bmatrix} = \begin{bmatrix} (3*7+4*3) & (3*2+4*5) & (3*1+4*2) \\ (2*7+5*3) & (2*2+5*5) & (2*1+5*2) \\ (1*7+2*3) & (1*2+2*5) & (1*1+2*2) \end{bmatrix} = \begin{bmatrix} 33 & 26 & 11 \\ 29 & 29 & 12 \\ 13 & 12 & 5 \end{bmatrix}$$

### Properties of matrix operations

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $(AB)C = A(BC)$
- $(A + B)C = AC + BC$
- If  $A$  is an  $m \times n$  matrix, then  $I_m A = A$  and  $A I_n = A$

Note that, in general  $AB \neq BA$

## Writing a system of equations using matrix notation

Note that, if we have a system of equations:

$$y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$y_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

$\dots$

$$y_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n$$

We can re-write these equations much more simply as:

$Y = AX$  where  $Y$  is a  $1 \times m$  matrix,  $A$  is an  $m \times n$  matrix, and  $X$  is a  $1 \times n$  matrix:

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

## Practice Problems

- Solve the following:
  - $\begin{bmatrix} 2 & 4 & 2 \\ 1 & 3 & 0 \\ 1 & 6 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 0 \\ -2 & -3 & 0 \\ 1 & 9 & 5 \end{bmatrix}$
  - $\begin{bmatrix} 2 & 1 \\ -2 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 5 & 2 & -1 \\ 3 & 4 & 2 \end{bmatrix}$
  - Let  $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 4 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 4 & 4 \\ 1 & 2 \\ 7 & 0 \end{bmatrix}$  Calculate  $A^T B$
  - Using the same matrices as in part c, calculate  $B^T A$
- Show that  $A$  and  $B$  are inverses:
 
$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0.5 & 1 \\ 1 & 0 & -1 \\ 0 & -0.5 & 1 \end{bmatrix}$$
- Suppose that  $A$  is a  $4 \times 3$  matrix and  $B$  is a  $3 \times 8$  matrix.
  - Does  $AB$  exist? If so, what are the dimensions of  $AB$ ?
  - Does  $BA$  exist? If so, what are the dimensions of  $BA$ ?
- What is the determinant of  $\begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$ ?

## Answers

- Solve:
  - $\begin{bmatrix} 3 & 9 & 2 \\ -1 & 0 & 0 \\ 2 & 15 & 7 \end{bmatrix}$
  - $\begin{bmatrix} 13 & 8 & 0 \\ -4 & 4 & 6 \\ 26 & 16 & 0 \end{bmatrix}$
  - $\begin{bmatrix} 35 & 10 \\ 13 & 18 \end{bmatrix}$
  - $\begin{bmatrix} 35 & 13 \\ 10 & 18 \end{bmatrix}$
- Show that  $A$  and  $B$  are inverses:
 
$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0.5 & 1 \\ 1 & 0 & -1 \\ 0 & -0.5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
- Suppose that  $A$  is a  $4 \times 3$  matrix and  $B$  is a  $3 \times 8$  matrix.
  - $AB$  exists and is  $4 \times 8$
  - $BA$  does not exist
- The determinant is  $(1 * 3) - (-2 * 4) = (3) - (-8) = 11$

# Derivatives

## Relevant Courses:

- Probability
- Quantitative Methods

## Notes

### Definition

The derivative of a function  $y = f(x)$  with respect to  $x$  is defined as a function giving the instantaneous slope of  $y = f(x)$  for any value  $x$ . Notationally, a derivative can be written in any of the following ways:

$$f'(x) = y' = \frac{df}{dx} = \frac{d}{dx}(f(x)) = \frac{dy}{dx} = \frac{d}{dx}(y)$$

### Using derivatives to find local minima and maxima of a function

To find all local minima or maxima of a function  $y = f(x)$ :

- 1) Take the first derivative of  $f(x)$  with respect to  $x$  ( $f'(x)$ ).
- 2) Set this expression equal to zero and solve for  $x$ . These values of  $x$  are local maxima and minima.
- 3) Calculate the second derivative of  $f(x)$  with respect to  $x$  ( $f''(x)$ )
- 4) Plug in the values of  $x$  calculated in part 2. If the second derivative is positive, this value of  $x$  represents a local minimum; if the second derivative is negative, it is a local maximum

## Properties

Properties of derivatives:

- 1)  $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
- 2)  $(cf(x))' = cf'(x)$  where  $c$  is a constant
- 3)

## Practice Problems

1. Solve the following:
  - a.

## Answers

1. Solve:
  - a.



# Integrals

## Relevant Courses:

- Probability
- Quantitative Methods

## Notes

## Practice Problems

1. Solve the following:
  - a.

## Answers

1. Solve:
  - a.

# Summary Statistics

## Relevant Courses:

- Probability
- Quantitative Methods
- Statistical Computing

## **P-Values and T-Tests**

### **Relevant Courses:**

- Quantitative Methods
- Statistical Computing
- Causal Inference

# Correlation and Covariance

## Relevant Courses:

- Quantitative Methods
- Probability

## Notes

# Ordinary Least Squares Regression

## Relevant Courses:

-Quantitative Methods

## Probability Density/Mass Functions

### Relevant Courses:

- Quantitative Methods
- Probability
- Causal Inference

## **Expectation**

### **Relevant Courses:**

-Probability