

A3SR Math Review

Properties of Logarithms

Relevant Courses:

- Quantitative Methods
- Generalized Linear Models

Notes

Definition

Logarithms are defined such that $\log_b(A) = X$ is equivalent to $b^X = A$

Properties

Using properties of exponents and the definition above, we can derive the following:

- The Product Rule: $\log_b(MN) = \log_b(M) + \log_b(N)$
- The Quotient Rule: $\log_b(\frac{M}{N}) = \log_b(M) - \log_b(N)$
- The Power Rule: $\log_b(M^p) = p\log_b(M)$
- $\log_b(b^X) = X$
- $b^{\log_b(X)} = X$
- $\log_b b = 1$
- $\log_b 1 = 0$

Example 1: Expanding logarithms

$$\begin{aligned}\log_e\left(\frac{2x^3}{y}\right) &= \log_e(2x^3) - \log_e(y) \\ &= \log_e(2) + \log_e(x^3) - \log_e(y) \\ &= \log_e(2) + 3\log_e(x) - \log_e(y)\end{aligned}$$

Example 2: Condensing logarithms

$$\begin{aligned}2\log_3(x) + \log_3(5) - \log_3(2) &= \log_3(x^2) + \log_3(5) - \log_3(2) \\ &= \log_3(5x^2) - \log_3(2) \\ &= \log_3\left(\frac{5x^2}{2}\right)\end{aligned}$$

Practice Problems

- Solve the following:
 - $\log_e(e^x)$
 - $\log_{10}(100)$
 - $\log_{10}(\frac{1}{10})$
 - $\log_{10}(0)$
- Expand the following:
 - $\log_{10}(\frac{5y^3}{x^2})$
 - $\log_2(\frac{4y^2}{3x})$
 - $\log_e(2x^2y^3)$
- Condense the following:
 - $4\log_3(x) - 2\log_3(y)$
 - $\log_2(x) + 5\log_2(y) - \log_2(5)$
 - $\log_{10}(5) + \log_{10}(2)$

Answers

- Solve:
 - x
 - 2
 - -1
 - There is no solution because there is no power of 10 that would equal 0
- Expand:
 - $\log_{10}(5) + 3\log_{10}(y) - \log_{10}(x)$
 - $2 + 2\log_2(y) - \log_2(3) - \log_2(x)$
 - $\log_e(2) + 2\log_e(x) + 3\log_e(y)$
- Condense
 - $\log_3(\frac{x^4}{y^2})$
 - $\log_2(\frac{xy^5}{5})$
 - 1

Matrix Algebra

Relevant Courses:

-Quantitative Methods

Notes

Note that these notes are a summary of relevant information from https://www.math.psu.edu/bressan/PSPDF/M441-linalggebra_review.pdf

Definitions

An $m \times n$ matrix A has m rows, n columns, and can be written as:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The transpose of an $m \times n$ matrix, A is the $n \times m$ matrix (denoted A^T) such that every element a_{ij} in matrix A is moved to row j and column i . For example, if:

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 7 \end{bmatrix}$$

then,

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 7 \end{bmatrix}$$

The $n \times n$ identity matrix I_n is a matrix with 1s on the diagonal and 0s everywhere else:

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

An $n \times n$ matrix is called “diagonal” if all elements not on the diagonal are zeros. For example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

An $n \times n$ matrix is called “upper triangular” if all elements below the diagonal are zeros. For example:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

An $n \times n$ matrix is called “lower triangular” if all elements above the diagonal are zeros. For example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

Suppose that we have 2 $n \times n$ matrices, A and B , such that $AB = I_n$ (note: this also implies $BA = I_n$). Then we say that B is the inverse of A (and vice versa) and we can write, $B = A^{-1}$. A matrix A has an inverse if and only if its determinant is not equal to zero. Note that the determinant of a 2×2 matrix can be calculated as follows (it is not important that you are able to calculate the determinant of a higher dimensional matrix by hand):

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Operations with matrices

- Adding matrices ($A + B = C$): If we add 2 $m \times n$ matrices, A and B , we get another $m \times n$ matrix C such that $c_{ij} = a_{ij} + b_{ij}$
- Subtracting matrices ($A - B = C$): If we subtract the $m \times n$ matrix B from the $m \times n$ matrix A , we get another $m \times n$ matrix C such that $c_{ij} = a_{ij} - b_{ij}$
- Multiplying a matrix by a scalar ($cA = B$): If we multiply the $m \times n$ matrix A by a scalar, c , then we get another $m \times n$ matrix B such that $b_{ij} = c * a_{ij}$
- Matrix multiplication ($AB = C$): Note that it is only possible to compute AB if the number of columns in matrix A equals the number of rows in matrix B . If this is the case, then when we multiply an $m \times n$ matrix A by an $n \times k$ matrix B , we get an $m \times k$ matrix, C , such that $c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \dots + a_{in}b_{nk}$

Example 1: Matrix Multiplication

$$\begin{bmatrix} 3 & 4 \\ 2 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 2 & 1 \\ 3 & 5 & 2 \end{bmatrix} = \begin{bmatrix} (3*7+4*3) & (3*2+4*5) & (3*1+4*2) \\ (2*7+5*3) & (2*2+5*5) & (2*1+5*2) \\ (1*7+2*3) & (1*2+2*5) & (1*1+2*2) \end{bmatrix} = \begin{bmatrix} 33 & 26 & 11 \\ 29 & 29 & 12 \\ 13 & 12 & 5 \end{bmatrix}$$

Properties of matrix operations

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $(AB)C = A(BC)$
- $(A + B)C = AC + BC$
- If A is an $m \times n$ matrix, then $I_m A = A$ and $A I_n = A$

Note that, in general $AB \neq BA$

Writing a system of equations using matrix notation

Note that, if we have a system of equations:

$$y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$y_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

\dots

$$y_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n$$

We can re-write these equations much more simply as:

$Y = AX$ where Y is a $1 \times m$ matrix, A is an $m \times n$ matrix, and X is a $1 \times n$ matrix:

$$\begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

Practice Problems

- Solve the following:
 - $\begin{bmatrix} 2 & 4 & 2 \\ 1 & 3 & 0 \\ 1 & 6 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 0 \\ -2 & -3 & 0 \\ 1 & 9 & 5 \end{bmatrix}$
 - $\begin{bmatrix} 2 & 1 \\ -2 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 5 & 2 & -1 \\ 3 & 4 & 2 \end{bmatrix}$
 - Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 4 \\ 1 & 2 \\ 7 & 0 \end{bmatrix}$ Calculate $A^T B$
 - Using the same matrices as in part c, calculate $B^T A$
- Show that A and B are inverses:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & 0.5 & 1 \\ 1 & 0 & -1 \\ 0 & -0.5 & 1 \end{bmatrix}$$
- Suppose that A is a 4×3 matrix and B is a 3×8 matrix.
 - Does AB exist? If so, what are the dimensions of AB ?
 - Does BA exist? If so, what are the dimensions of BA ?
- What is the determinant of $\begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$?

Answers

- Solve:
 - $\begin{bmatrix} 3 & 9 & 2 \\ -1 & 0 & 0 \\ 2 & 15 & 7 \end{bmatrix}$
 - $\begin{bmatrix} 13 & 8 & 0 \\ -4 & 4 & 6 \\ 26 & 16 & 0 \end{bmatrix}$
 - $\begin{bmatrix} 35 & 10 \\ 13 & 18 \end{bmatrix}$
 - $\begin{bmatrix} 35 & 13 \\ 10 & 18 \end{bmatrix}$
- Show that A and B are inverses:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0.5 & 1 \\ 1 & 0 & -1 \\ 0 & -0.5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
- Suppose that A is a 4×3 matrix and B is a 3×8 matrix.
 - AB exists and is 4×8
 - BA does not exist
- The determinant is $(1 * 3) - (-2 * 4) = (3) - (-8) = 11$

Derivatives

Relevant Courses:

- Probability
- Quantitative Methods

Notes

Definition

The derivative of a function $y = f(x)$ with respect to x is defined as a function giving the instantaneous slope of $y = f(x)$ for any value x . Notationally, a derivative can be written in any of the following ways:

$$f'(x) = y' = \frac{df}{dx} = \frac{d}{dx}(f(x)) = \frac{dy}{dx} = \frac{d}{dx}(y)$$

The second derivative of $y = f(x)$ with respect to x is a function giving the rate of change of the instantaneous slope of $f(x)$. Notationally, it can be represented in any of the following ways (note: 3rd, 4th, etc. derivatives are notated in a similar way, with increasing exponents or 's):

$$f''(x) = y'' = \frac{d^2 f}{dx^2} = \frac{d^2}{dx^2}(f(x)) = \frac{d^2 y}{dx^2} = \frac{d^2}{dx^2}(y)$$

Using derivatives to find local minima and maxima of a function

To find all local minima or maxima of a function $y = f(x)$:

- 1) Take the first derivative of $f(x)$ with respect to x ($f'(x)$).
- 2) Set this expression equal to zero and solve for x . These values of x are local maxima and minima.
- 3) Calculate the second derivative of $f(x)$ with respect to x ($f''(x)$)
- 4) Plug in the values of x calculated in part 2. If the second derivative is positive, this value of x represents a local minimum; if the second derivative is negative, it is a local maximum

Properties

Properties of derivatives:

- 1) Sum/Difference rule: $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
- 2) Constant multiple rule: $(cf(x))' = cf'(x)$ where c is a constant
- 3) Power rule: If $f(x) = x^n$, then $f'(x) = nx^{n-1}$
- 4) Product rule: $(f(x)g(x))' = f'(x)g(x) + g'(x)f(x)$
- 5) Quotient rule (given $g(x) \neq 0$): $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$
- 6) Chain rule: $(f(g(x)))' = f'(g(x)) * g'(x)$

Partial derivatives

For a function of more than one variable (i.e., $f(x, y)$), we can take partial derivatives with respect to each variable. The partial derivative of $f(x, y)$ with respect to x is often denoted by either $\frac{\partial f}{\partial x} = f_x$. The partial derivative with respect to y would be denoted by $\frac{\partial f}{\partial y} = f_y$. The partial derivative with respect to x would be calculated by treating any non- x variables as constants when applying the above properties.

Practice Problems

- Find $f'(x)$ for each of the following. Then compute $f''(x)$ for a-d:
 - $f(x) = 5x^2 + 3x + 1$
 - $f(x) = \frac{5}{x^3} + 2x^4$
 - $f(x) = (3x + 1)^5$
 - $f(x) = 2x^3(x^2 + 1)$
 - $f(x) = \frac{2x+1}{x^2-5}$
- Find the partial derivative with respect to x for each of the following:
 - $f(x, y) = 3xy^2 + 2x$
 - $f(x, y) = (xy^4 + 2y)^3$
 - $f(x, y) = 4x^3 + xy + x^2y^2 + 4x + 2$
- Find all local minima and maxima for the following function (and note whether they are minima or maxima): $\frac{2}{3}x^3 - x^2 - 12x$

Solutions

- $f'(x)$ and $f''(x)$
 - $f'(x) = 10x + 3$ and $f''(x) = 10$
 - $f'(x) = \frac{-15}{x^4} + 8x^3$ and $f''(x) = \frac{-60}{x^5} + 24x^2$
 - $f'(x) = 15(3x + 1)^4$ and $f''(x) = 180(3x + 1)^3$
 - $f'(x) = 10x^4 + 6x^2$ and $f''(x) = 40x^3 + 12x$
 - $f'(x) = \frac{2(x^2-5)-2x(2x+1)}{(x^2-5)^2}$ and $f''(x)$
- $f_x(x, y) =$
 - $3y^2 + 2$
 - $3y^4(xy^4 + 2y)^2$
 - $12x^2 + y + 2xy^2 + 4$
- Local minima and maxima
 - $f'(x) = 2x^2 - 2x - 12 = (2x + 4)(x - 3)$, so setting the first derivative equal to 0, we get $0 = (2x + 4)(x - 3)$, with solutions $x = 3$ and $x = -2$. $f''(x) = 4x - 2$. $f''(3) = 10$ and $f''(-2) = -10$. So, $x = 3$ is a local minimum and $x = -2$ is a local maximum. $f(3) = 18 - 9 - 36 = -27$ and $f(-2) = \frac{-16}{3} - 4 + 24 = \frac{44}{3} \approx 14.67$

Integrals

Relevant Courses:

- Probability
- Quantitative Methods

Notes

Definition

Indefinite integrals

In the previous section, we calculated the derivative of a function, $f(x)$. Finding an indefinite integral (also called an anti-derivative) involves a simple reversal of this process. The indefinite integral, $F(x)$, of a function, $f(x)$, is defined such that $F'(x) = f(x)$ and is written as follows:

$$\int f(x)dx = F(x) + C$$

where C is a constant.

Definite integrals

The definite integral of $f(x)$ from a to b gives the area under the curve of $f(x)$ on the interval between a and b and is calculated/notated as follows:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Basic properties

1. $\int cf(x)dx = c \int f(x)dx$ where c is a constant
2. $\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$
3. $\int \frac{1}{x}dx = \ln|x| + C$

Practice Problems

1. Evaluate the following indefinite integrals:

- a. $\int 5x^2 dx$
- b. $\int 2x^3 - \frac{3}{x^2} dx$
- c. $\int (2x - 2)(x + 3) dx$
- d. $\int \frac{2}{x} dx$

2. Evaluate the following definite integrals:

- a. $\int_0^1 4x^3 dx$
- b. $\int_{-2}^2 8x^3 + 3x^2 - 5x + 2 dx$
- c. $\int_1^\infty \frac{2}{x^2} dx$
- d. $\int_1^2 \int_0^2 2x^3 + 3y^4 x^2 \, dx dy$

Answers

1. Solve:

- a.

Summary Statistics

Relevant Courses:

- Probability
- Quantitative Methods
- Statistical Computing

P-Values and T-Tests

Relevant Courses:

- Quantitative Methods
- Statistical Computing
- Causal Inference

Correlation and Covariance

Relevant Courses:

- Quantitative Methods
- Probability

Notes

Ordinary Least Squares Regression

Relevant Courses:

-Quantitative Methods

Probability Density/Mass Functions

Relevant Courses:

- Quantitative Methods
- Probability
- Causal Inference

Expectation

Relevant Courses:

-Probability