A3SR Math Review

Properties of Logarithms

Relevant Courses:

-Quantitative Methods
-Generalized Linear Models

Notes

Definition

Logarithms are defined such that $log_b(A) = X$ is equivalent to $b^X = A$

Properties

Using properties of exponents and the definition above, we can derive the following:

a. The Product Rule: $log_b(MN) = log_b(M) + log_b(N)$

b. The Quotient Rule: $log_b(\frac{M}{N}) = log_b(M) - log_b(N)$

c. The Power Rule: $log_b(M^p) = plog_b(M)$

d. $log_b(b^X) = X$

e. $b^{log_b(X)} = X$

f. $log_b b = 1$

g. $log_b 1 = 0$

Example 1: Expanding logarithms

$log_e(\frac{2x^3}{y})$
$= log_e(2x^3) - log_e(y)$
$= log_e(2) + log_e(x^3) - log_e(y)$
$= log_e(2) + 3log_e(x) - log_e(y)$

Example 2: Condensing logarithms

$2log_3(x) + log_3(5) - log_3(2)$
$= log_3(x^2) + log_3(5) - log_3(2)$
$= log_3(5x^2) - log_3(2)$
$= log_3(\frac{5x^2}{2})$

Practice Problems

1. Solve the following:
   a. $log_e(e^x)$
   b. $log_{10}(100)$
   c. $log_{10}(\frac{1}{10})$
   d. $log_{10}(0)$
2. Expand the following:
   a. $log_{10}(\frac{5y^3}{x})$
   b. $log_2(\frac{4y^2}{3x})$
   c. $log_e(2x^2y^3)$
3. Condense the following:
   a. $4log_3(x) - 2log_3(y)$
   b. $log_2(x) + 5log_2(y) - log_2(5)$
   c. $log_{10}(5) + log_{10}(2)$

Answers

1. Solve:
   a. x
   b. 2
   c. -1
   d. There is no solution because there is no power of 10 that would equal 0
2. Expand:
   a. $log_{10}(5) + 3log_{10}(y) - log_{10}(x)$
   b. $2 + 2log_2(y) - log_2(3) - log_2(x)$
   c. $log_e(2) + 2log_e(x) + 3log_e(y)$
3. Condense
   a. $log_3(\frac{x^4}{y^2})$
   b. $log_2(\frac{xy^5}{5})$
   c. 1

Matrix Algebra

Relevant Courses:

-Quantitative Methods

Notes

Note that these notes are a summary of relevant information from https://www.math.psu.edu/bressan/PSPDF/M441-linalgebra_review.pdf

Definitions

An $m \times n$ matrix A has m rows, n columns, and can be written as:

$$A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$$

The transpose of an $m \times n$ matrix, A is the $n \times m$ matrix (denoted A^T) such that every element a_{ij} in matrix A is moved to row j and column i. For example, if:

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 7 \end{bmatrix}$$

then,

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 7 \end{bmatrix}$$

The $n \times n$ identity matrix I_n is a matrix with 1s on the diagonal and 0s everywhere else:

$$I_n = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & \ddots & ... \\ 0 & 0 & ... & 1 \end{bmatrix}$$

An $n \times n$ matrix is called "diagonal" if all elements not on the diagonal are zeros. For example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

An $n \times n$ matrix is called "upper triangular" if all elements below the diagonal are zeros. For example:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

An $n \times n$ matrix is called "lower triangular" if all elements above the diagonal are zeros. For example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

Suppose that an we have 2 $n \times n$ matrices, A and B, such that $AB = I_n$ (note: this also implies $BA = I_n$). Then we say that B is the inverse of A (and vice versa) and we can write, $B = A^{-1}$. A matrix A has an inverse if and only if its determinant is not equal to zero. Note that the determinant of a 2×2 matrix can be calculated as follows (it is not important that you are able to calculate the determinant of a higher dimensional matrix by hand):

$$det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc$$

Operations with matrices

a. Adding matrices ($A + B = C$): If we add 2 $m \times n$ matrices, A and B, we get another $m \times n$ matrix C such that $c_{ij} = a_{ij} + b_{ij}$

b. Subtracting matrices ($A - B = C$): If we subtract the $m \times n$ matrix B from the $m \times n$ matrix A, we get another $m \times n$ matrix C such that $c_{ij} = a_{ij} - b_{ij}$

c. Multiplying a matrix by a scalar ($cA = B$): If we multiply the $m \times n$ matrix A by a scalar, c, then we get another $m \times n$ matrix B such that $b_{ij} = c * a_{ij}$

d. Matrix multiplication ($AB = C$): Note that it is only possible to compute AB if the number of columns in matrix A equals the number of rows in matrix B. If this is the case, then when we multiply an $m \times n$ matrix A by an $n \times k$ matrix B, we get an $m \times k$ matrix, C, such that $c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk}$

Example 1: Matrix Multiplication

$$\begin{bmatrix} 3 & 4 \\ 2 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 2 & 1 \\ 3 & 5 & 2 \end{bmatrix} = \begin{bmatrix} (3*7+4*3) & (3*2+4*5) & (3*1+4*2) \\ (2*7+5*3) & (2*2+5*5) & (2*1+5*2) \\ (1*7+2*3) & (1*2+2*5) & (1*1+2*2) \end{bmatrix} = \begin{bmatrix} 33 & 26 & 11 \\ 29 & 29 & 12 \\ 13 & 12 & 5 \end{bmatrix}$$

Properties of matrix operations

a. $A + B = B + A$

b. $(A + B) + C = A + (B + C)$

c. $(AB)C = A(BC)$

d. $(A + B)C = AC + BC$

e. If A is an $m \times n$ matrix, then $I_m A = A$ and $AI_n = A$

Note that, in general $AB \neq BA$

Writing a system of equations using matrix notation

Note that, if we have a system of equations:
$y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$
$y_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$
$\cdots$
$y_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n$

We can re-write these equations much more simply as:
$Y = AX$ where Y is a $1 \times m$ matrix, A is an $m \times n$ matrix, and X is a $1 \times n$ matrix:

$$\begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$$

Practice Problems

1. Solve the following:

   a. $\begin{bmatrix} 2 & 4 & 2 \\ 1 & 3 & 0 \\ 1 & 6 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 0 \\ -2 & -3 & 0 \\ 1 & 9 & 5 \end{bmatrix}$

   b. $\begin{bmatrix} 2 & 1 \\ -2 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 5 & 2 & -1 \\ 3 & 4 & 2 \end{bmatrix}$

   c. Let A$= \begin{bmatrix} 1 & 2 \\ 3 & 5 \\ 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 4 \\ 1 & 2 \\ 7 & 0 \end{bmatrix}$ Calculate $A^T B$

   d. Using the same matrices as in part c, calculate $B^T A$

2. Show that A and B are inverses:

   $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

   $B = \begin{bmatrix} -1 & 0.5 & 1 \\ 1 & 0 & -1 \\ 0 & -0.5 & 1 \end{bmatrix}$

3. Suppose that A is a 4x3 matrix and B is a $3x8$ matrix.

   a. Does AB exist? If so, what are the dimensions of AB?

   b. Does BA exist? If so, what are the dimensions of BA?

4. What is the determinant of $\begin{bmatrix} 1 & -2 \\ 4 & 3 \end{bmatrix}$?

Answers

1. Solve:

   a. $\begin{bmatrix} 3 & 9 & 2 \\ -1 & 0 & 0 \\ 2 & 15 & 7 \end{bmatrix}$

   b. $\begin{bmatrix} 13 & 8 & 0 \\ -4 & 4 & 6 \\ 26 & 16 & 0 \end{bmatrix}$

   c. $\begin{bmatrix} 35 & 10 \\ 13 & 18 \end{bmatrix}$

   d. $\begin{bmatrix} 35 & 13 \\ 10 & 18 \end{bmatrix}$

2. Show that A and B are inverses:

   $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0.5 & 1 \\ 1 & 0 & -1 \\ 0 & -0.5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3. Suppose that A is a 4x3 matrix and B is a $3x8$ matrix.

   a. AB exists and is 4x8

   b. BA does not exist

4. The determinant is $(1 * 3) - (-2 * 4) = (3) - (-8) = 11$

Derivatives

Relevant Courses:

-Probability
-Quantitative Methods

Notes

Definition

The derivative of a function $y = f(x)$ with respect to x is defined as a function giving the instantaneous slope of $y = f(x)$ for any value x. Notationally, a derivative can be written in any of the following ways:

$$f'(x) = y' = \frac{df}{dx} = \frac{d}{dx}(f(x)) = \frac{dy}{dx} = \frac{d}{dx}(y)$$

The second derivative of $y = f(x)$ with respect to x is a function giving the rate of change of the instantaneous slope of $f(x)$. Notationally, it can be represented in any of the following ways (note: 3rd, 4th, etc. derivatives are notated in a similar way, with increasing exponents or $'$s):

$$f''(x) = y'' = \frac{d^2 f}{dx^2} = \frac{d^2}{dx^2}(f(x)) = \frac{d^2 y}{dx^2} = \frac{d^2}{dx^2}(y)$$

Using derivatives to find local minima and maxima of a function

To find all local minima or maxima of a function $y = f(x)$:
1) Take the first derivative of $f(x)$ with respect to x ($f'(x)$).
2) Set this expression equal to zero and solve for x. These values of x are local maxima and minima.
3) Calculate the second derivative of $f(x)$ with respect to x ($f''(x)$)
4) Plug in the values of x calculated in part 2. If the second derivative is positive, this value of x represents a local minimum; if the second derivative is negative, it is a local maximum

Properties

Properties of derivatives:
1) Sum/Difference rule: $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
2) Constant multiple rule: $(cf(x))' = cf'(x)$ where c is a constant
3) Power rule: If $f(x) = x^n$, then $f'(x) = nx^{n-1}$
4) Product rule: $(f(x)g(x))' = f'(x)g(x) + g'(x)f(x)$
5) Quotient rule (given g(x) != 0): $(\frac{f(x)}{g(x)})' = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$
6) Chain rule: $(f(g(x)))' = f'(g(x)) * g'(x)$

Partial derivatives

For a function of more than one variable (i.e., $f(x, y)$), we can take partial derivatives with respect to each variable. The partial derivative of $f(x, y)$ with respect to x is often denoted by either $\frac{\partial f}{\partial x} = f_x$. The partial derivative with respect to y would be denoted by $\frac{\partial f}{\partial y} = f_y$. The partial derivative with respect to x would be calculated by treating any non-x variables as constants when applying the above properties.

Practice Problems

1. Find $f'(x)$ for each of the following. Then compute $f''(x)$ for a-d:
   a. $f(x) = 5x^2 + 3x + 1$
   b. $f(x) = \frac{5}{x^3} + 2x^4$
   c. $f(x) = (3x+1)^5$
   d. $f(x) = 2x^3(x^2+1)$
   e. $f(x) = \frac{2x+1}{x^2-5}$

2. Find the partial derivative with respect to x for each of the following:
   a. $f(x, y) = 3xy^2 + 2x$
   b. $f(x, y) = (xy^4 + 2y)^3$
   c. $f(x, y) = 4x^3 + xy + x^2y^2 + 4x + 2$

3. Find all local minima and maxima for the following function (and note whether they are minima or maxima): $\frac{2}{3}x^3 - x^2 - 12x$

Solutions

1. $f'(x)$ and $f''(x)$
   a. $f'(x) = 10x + 3$ and $f''(x) = 10$
   b. $f'(x) = \frac{-15}{x^4} + 8x^3$ and $f''(x) = \frac{-60}{x^5} + 24x^2$
   c. $f'(x) = 15(3x+1)^4$ and $f''(x) = 180(3x+1)^3$
   d. $f'(x) = 10x^4 + 6x^2$ and $f''(x) = 40x^3 + 12x$
   e. $f'(x) = \frac{2(x^2-5)-2x(2x+1)}{(x^2-5)^2}$ and $f''(x)$

2. $f_x(x, y) =$
   a. $3y^2 + 2$
   b. $3y^4(xy^4 + 2y)^2$
   c. $12x^2 + y + 2xy^2 + 4$

3. Local minima and maxima
   a. $f'(x) = 2x^2 - 2x - 12 = (2x+4)(x-3)$, so setting the first derivative equal to 0, we get $0 = (2x+4)(x-3)$, with solutions $x = 3$ and $x = -2$. $f''(x) = 4x - 2$. $f''(3) = 10$ and $f''(x) = -10$. So, $x = 3$ is a local minimum and $x = -2$ is a local maximum. $f(3) = 18 - 9 - 36 = -27$ and $f(-2) = \frac{-16}{3} - 4 + 24 = \frac{44}{3} \approx 14.67$

Integrals

Relevant Courses:

-Probability
-Quantitative Methods

Notes

Definition

Indefinite integrals

In the previous section, we calculated the derivative of a function, $f(x)$. Finding an indefinite integral (also called an anti-derivative) involves a simple reversal of this process. The indefinite integral, $F(x)$, of a function, $f(x)$, is defined such that $F'(x) = f(x)$ and is written as follows:

$$\int f(x)dx = F(x) + C$$

where C is a constant.

Definite integrals

The definite integral of $f(x)$ from a to b gives the area under the curve of $f(x)$ on the interval between a and b and is calculated/notated as follows:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Basic properties

Note: There are many other properties of integrals, which may be useful once or twice throughout this program. The following are what you will see most commonly, but you should feel comfortable looking up and using additional properties as needed.
1. $\int cf(x)dx = c\int f(x)dx$ where c is a constant
2. $\int f(x) \pm g(x)dx = \int f(x)dx \pm \int g(x)dx$
3. $\int \frac{1}{x}dx = ln|x| + C$ 4. $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$

Practice Problems

1. Evaluate the following indefinite integrals:
   a. $\int 5x^2 dx$
   b. $\int 2x^3 - \frac{3}{x^2} dx$
   c. $\int (2x-2)(x+3) dx$
   d. $\int \frac{2}{x} dx$
2. Evaluate the following definite integrals:
   a. $\int_0^1 4x^3 dx$
   b. $\int_{-2}^2 8x^3 + 3x^2 - 5x + 2dx$
   c. $\int_1^\infty \frac{2}{x^2} dx$
   d. $\int_1^2 \int_0^2 2x^3 + 3y^3x^2 \ dxdy$

Answers

1. Indefinite integrals:
   a. $\int 5x^2 dx = \frac{5}{3}x^3$
   b. $\int 2x^3 - \frac{3}{x^2} dx = \int 2x^3 - 3x^{-2} dx = \frac{1}{2}x^4 + 3x^{-1} + C$
   c. $\int (2x-2)(x+3) dx = \int 2x^2 + 4x - 6dx = \frac{2}{3}x^3 + 2x^2 - 6x + C$
   d. $\int \frac{2}{x} dx = 2 \int \frac{1}{x} dx = 2ln|x| + C$
2. Definite integrals:
   a. $\int_0^1 4x^3 dx = x^4|_0^1 = 1^4 - 0^4 = 1$
   b. $\int_{-2}^2 8x^3 + 3x^2 - 5x + 2dx = (2x^4 + x^3 - \frac{5}{2}x^2 + 2x)|_{-2}^2 = (32 + 8 - 10 + 4) - (32 - 8 - 10 - 4) = 24$
   c. $\int_1^\infty \frac{2}{x^2} dx = \int_1^\infty 2x^{-2} dx = -2x^{-1}|_1^\infty = 0 - -2 = 2$
   d. $\int_1^2 \int_0^2 2x^3 + 3y^3x^2 \ dxdy = \int_1^2 (\frac{1}{2}x^4 + y^3x^3)|_0^2 dy = \int_1^2 8 + 8y^3 dy = (8y + 2y^4)|_1^2 = (16+32) - (8+2) = 38$

Summary Statistics

Relevant Courses:

-Probability
-Quantitative Methods
-Statistical Computing

P-Values and T-Tests

Relevant Courses:

-Quantitative Methods
-Statistical Computing
-Causal Inference

Correlation and Covariance

Relevant Courses:

-Quantitative Methods
-Probability

Notes

Ordinary Least Squares Regression

Relevant Courses:

-Quantitative Methods

Probability Density/Mass Functions

Relevant Courses:

-Quantitative Methods
-Probability
-Causal Inference

Expectation

Relevant Courses:

-Probability