Kvitteringsskjema

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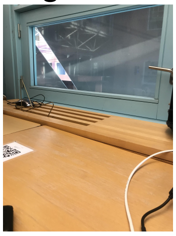

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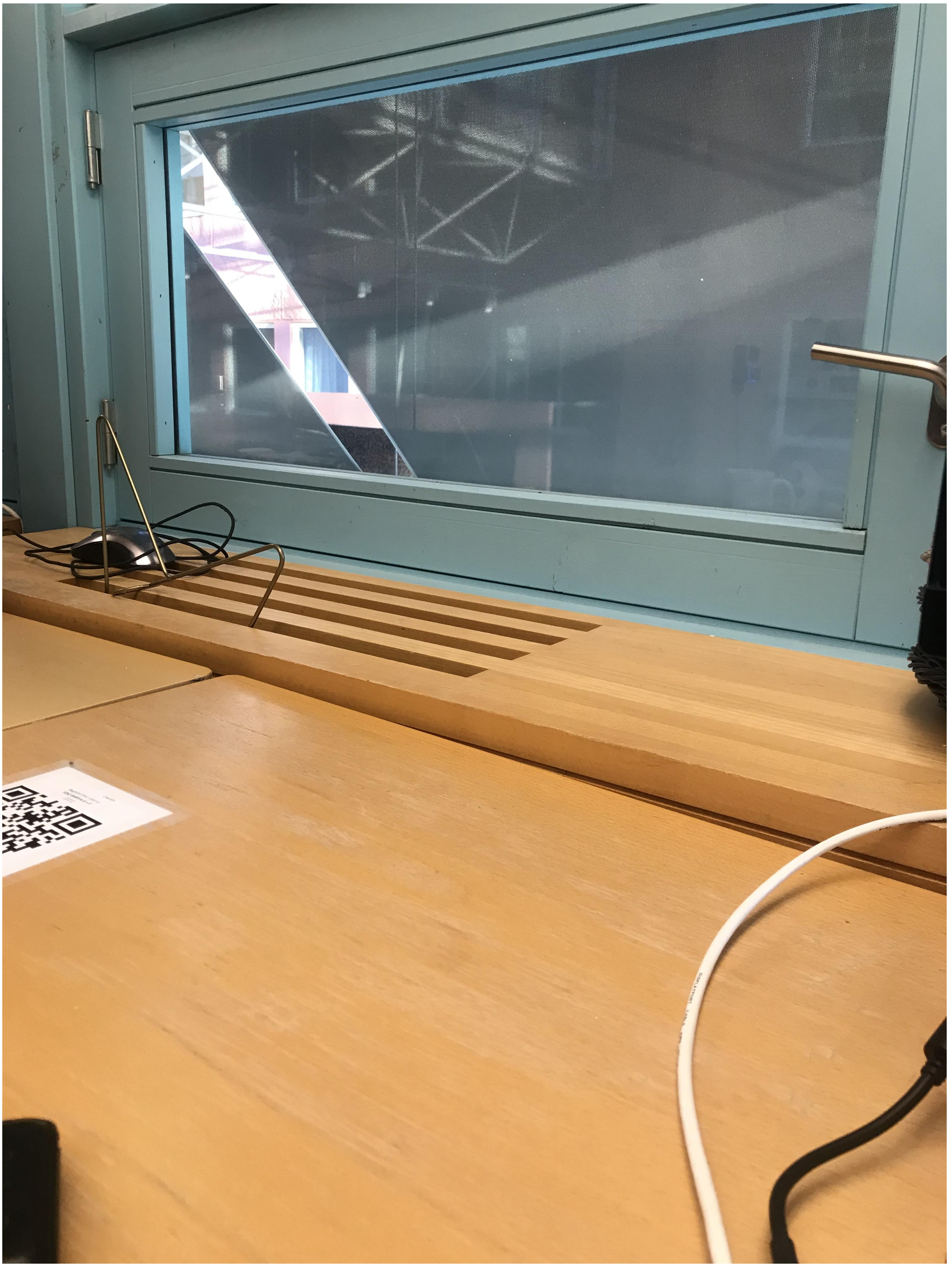

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Solution to Exercise set 3

1 **a)** Using integration by parts,

$$F(s) = \mathcal{L}[f](s) = \int_0^a e^{-st} t \, dt = -\frac{ae^{-as}}{s} + \int_0^a \frac{e^{-st}}{s} \, dt$$
$$= -\frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2}.$$

b) Substituting $x = t - \pi$,

$$\int_0^{+\infty} e^{-st} \sin t \, dt = e^{-s\pi} \int_0^{+\infty} e^{-sx} \sin(x+\pi) \, dx = -e^{-s\pi} \mathcal{L}(\sin x)$$
$$= -\frac{e^{-s\pi}}{s^2+1}.$$

c) Using that

$$\int_0^t i(\tau) \, d\tau = i(t) * 1,$$

and the Laplace transform of the convolution:

$$\mathcal{L}(f * g) = \mathcal{L}(f) \cdot \mathcal{L}(g),$$

we can transform the problem. Using the notation $\mathcal{L}(i) = I$, our ODE becomes

$$sI + 2I + \frac{I}{s} = e^{-s}.$$

Rearranging, we obtain

$$I = \frac{se^{-s}}{(s+1)^2} = \left(\frac{1}{s+1} - \frac{1}{(s+1)^2} \right) e^{-s} = e^{-s} \mathcal{L}(e^{-t} - te^{-t})$$
$$= \mathcal{L}(u(t-1)(e^{-(t-1)} - (t-1)e^{-(t-1)}).$$

Thus

$$i(t) = u(t-1)(e^{-(t-1)} - (t-1)e^{-(t-1)}).$$

2 Here we use the notation $\mathcal{L}(y) = Y$. The Laplace transform of the convolution gives

$$Y - \frac{Y}{s^2} = \frac{1}{s},$$

and rearranging we find

$$Y = \frac{s}{s^2-1}$$

Hence, taking the inverse Laplace transform, we obtain

$$y = \cosh(t).$$

$\boxed{3}$ We use the notation $\mathcal{L}(y) = Y$ and $\mathcal{L}(x) = X$. Transforming the system we obtain

$$\begin{cases} sX = 2X - Y, \\ sY = 3X - 2Y + 1. \end{cases}$$

Rearranging the first equation gives $Y = (2-s)X$, and substituting this into the second equation, we get

$$X = -\frac{1}{s^2-1} \implies Y = \frac{s-2}{s^2-1} = \frac{s}{s^2-1} - \frac{2}{s^2-1}.$$

Finally, taking the inverse Laplace transform we obtain

$$\begin{cases} x = -\sinh(t), \\ y = -2\sinh(t) + \cosh(t). \end{cases}$$

$\boxed{4}$ **a)** Substituting x into our formula for c_n, and integrating by parts, we get

$$\begin{aligned} c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\,dx &= \frac{1}{2\pi}\left[\Big[-\frac{xe^{-inx}}{in}\Big]_{-\pi}^{\pi} - \frac{i}{n}\underbrace{\int_{-\pi}^{\pi} e^{-inx}\,dx}_{=0}\right] \\ &= \frac{1}{2\pi}\left[\frac{2i\pi(-1)^n}{n}\right] \\ &= \frac{i(-1)^n}{n}, \qquad n \neq 0. \end{aligned}$$

Where we have used the fact that

$$e^{-in\pi} = \cos(n\pi) - i\sin(n\pi) = (-1)^n - 0.$$

Note, we have only calculated c_n for $n \neq 0$. If $n = 0$, we obtain

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0$$

Substituting our c_n into the Fourier series identity, we get

$$x = \sum_{n\in\mathbb{Z}} c_n e^{inx} = \sum_{n\neq 0} \frac{i(-1)^n}{n} e^{inx}, \qquad \text{when } -\pi < x < \pi.$$

b) Substituting $x(2\pi - x)$ into our formula for c_n, we get

$$c_n = \frac{2\pi}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\,dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}\,dx.$$

We first integral is just the c_n from **(a)**, multiplied by 2π, so we focus on the second integral.
We first note

$$\int_{-\pi}^{\pi} x^2 e^{-inx}\,dx = \int_{-\pi}^{\pi} x^2\cos(nx)\,dx + i\int_{-\pi}^{\pi} x^2\sin(x)\,dx.$$

The integral on the imaginary part vanishes, as $x^2\sin(x)$ is an odd function (as it is the product of an odd and even function), and the integral of an odd function over a symmetric integral is 0.
Hence we only need to calculate the first integral. This is the integral of an even function, leading to the first equality in what follows:

$$\begin{aligned}\int_{-\pi}^{\pi} x^2\cos(nx)\ dx &= 2\int_0^{\pi} x^2\cos(nx)\ dx\\ &= 2\Bigg[\underbrace{\Big[\frac{x^2\sin(x)}{n}\Big]_0^{\pi}}_{=0} - \frac{2}{n}\int_0^{\pi} x\sin(nx)\ dx\Bigg]\\ &= 2\Bigg[-\frac{2}{n}\Big[-\frac{x\cos(nx)}{n}\Big]_0^{\pi} - \frac{2}{n^2}\underbrace{\int_0^{\pi}\cos(nx)\ dx}_{=0}\Bigg]\\ &= \frac{4\pi(-1)^n}{n^2}, \qquad n\neq 0.\end{aligned}$$

We need to calculate what this integral is when $n=0$,

$$\int_{-\pi}^{\pi} x^2\ dx = \frac{2\pi^3}{3}.$$

Putting everything together we obtain that

$$c_n = \begin{cases} \frac{2\pi i(-1)^n}{n} - \frac{2(-1)^n}{n^2}, & n\neq 0,\\ -\frac{\pi^2}{3}, & n=0.\end{cases}$$

Substitution into the Fourier series identity, and using $-(-1)^n = (-1)^{n+1}$ leads to the desired result.