Introduction and Review

COT4210 DISCRETE STRUCTURES

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PORTIONS FROM SIPSER, *INTRODUCTION TO THE THEORY OF COMPUTATION*, 3RD ED., 2013

Overview (0.1)

What is *computability*?

- What are the fundamental capabilities and limitations of computers?
- Why are some problems harder than others?
  - Sorting is pretty easy…
  - …but scheduling is very hard
- Why are some problems flat-out impossible?
  - The halting problem
  - Determining the truth or falsehood of a statement
- What are *automata*?
  - Why are they important?
  - More importantly, why are they *useful*?

Review of Mathematical Essentials

SECTION 0.2

Sets

Given elements x and y, and sets A and B:

Containment

- $x \in A$ - A contains x.
- $x \notin A$ - A doesn't contain x.
- $A = \{x, y\}$ - A contains only x and y.
- $A = \{x \mid x \in \mathrm{N}, x > 50\}$ - A contains the natural numbers higher than 50.

Operators

- $A \cup B$ – union
- $A \cap B$ – intersection
- $\overline{A}$- complement

Subsets

- $A \subseteq \mathrm{B}$ - A is a subset of B.
  - $\forall\, x \in A, x \in B$
- $A \subset \mathrm{B}$ - A is a proper subset of B.
  - $\forall\, x \in A, x \in B$ and $A \neq B$.
- The *power set* of A is the set of all subsets of A.

Common sets

- Z – the set of all integers
- N – the set of all natural numbers
- $\varnothing$ or ϕ - the empty set

Sequences and Functions

Sequences

- Like ordered sets
- Finite sequences are called *k*-tuples
- 2-tuples are also known as *ordered pairs*

Cartesian products of sets:

- $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$
- Can take it of any number of sets
- $A \times A = A^2$, $A \times A \times A = A^3$, etc.

Functions

- Map a *domain* onto a *range*
- *n*-ary functions take *n* arguments
- $f: D \rightarrow R$
  - $abs: Z \rightarrow Z$
  - $add: Z \times Z \rightarrow Z$

A function is…

- *One-to-one* (an *injection*) if it maps every element of the range from at most one element of the domain
- *Onto* (a *surjection*) if it maps every element of the range from at least one element of the domain
- A *bijection* if every element of the range is mapped by exactly one element of the domain

Relations

A *predicate* or *property* is a function with range {TRUE, FALSE}

A property with a domain of n-tuples A^n is an n-ary relation

Binary relations are common, and like binary functions, we use infix notations for them

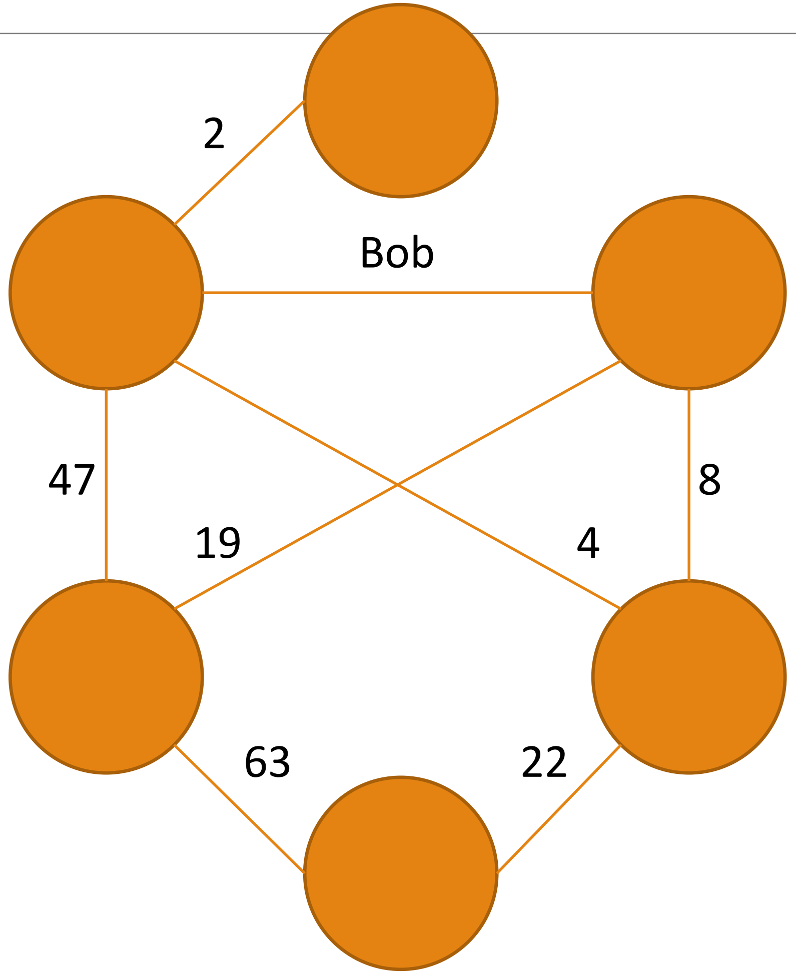
Let R be a binary relation on A^2. R is:

- *Reflexive* if $\forall\ x \in a, x\ R\ x$
- *Symmetric* if $x\ R\ y \rightarrow y\ R\ x$
- *Transitive* if $(\ x\ R\ y, y\ R\ z\ ) \rightarrow x\ R\ z$
- An *equivalence* relation if it is reflexive, symmetric and transitive

Graphs: Undirected Graphs

An undirected *graph* is a collection of *nodes* (or *vertices*) and *edges* that connect them

- The *degree* of a node is the number of edges that connect to that node
- Edges are unique – you can't have two edges between the same pair of nodes
- Nodes can have *self-loops*
- Edges can also be labeled

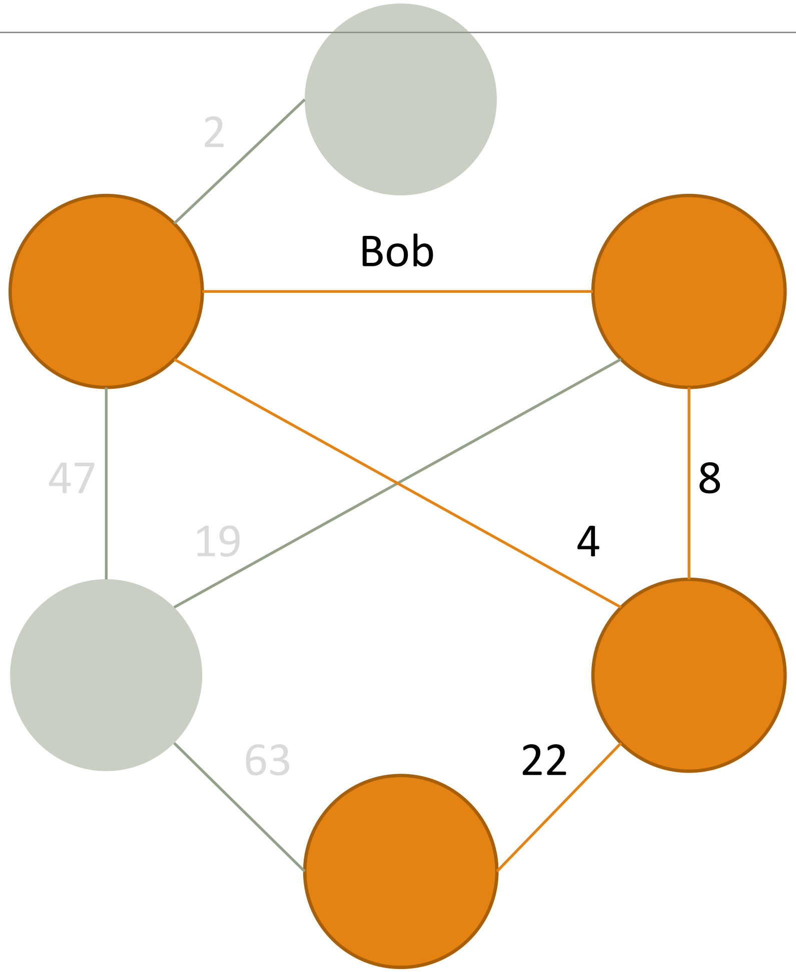


Graphs: Subgraphs

An undirected *graph* is a collection of *nodes* (or *vertices*) and *edges* that connect them

- The *degree* of a node is the number of edges that connect to that node
- Edges are unique – you can't have two edges between the same pair of nodes
- Nodes can have *self-loops*
- Edges can also be labeled

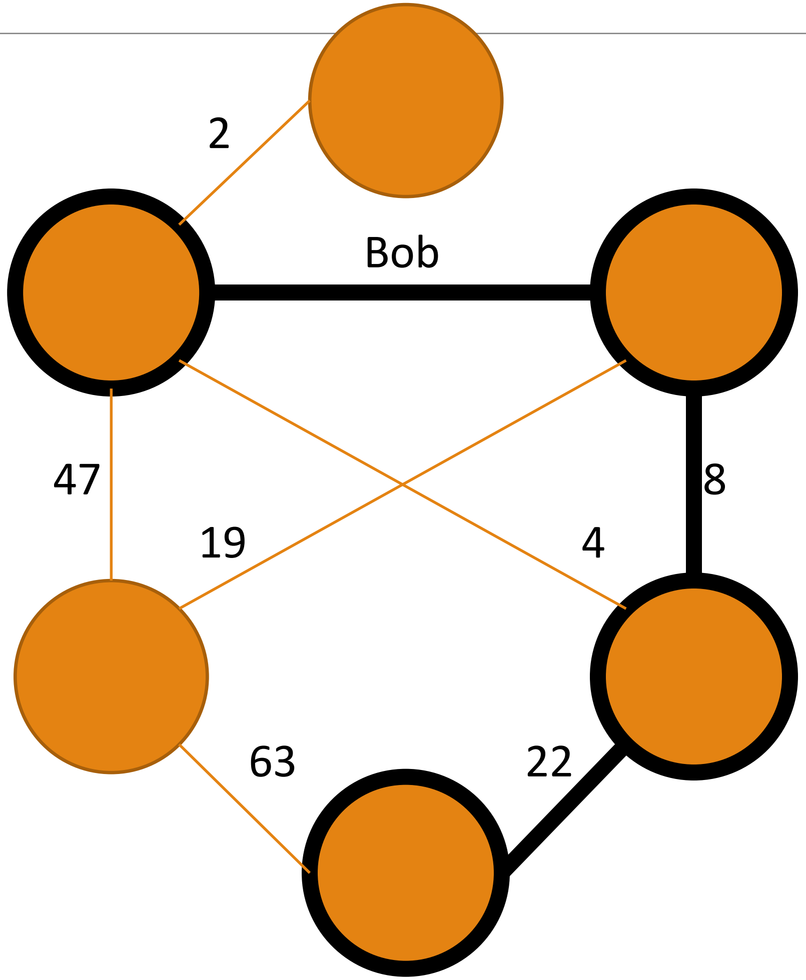
A graph *G* is a *subgraph* of graph *H* if it has a subset of *H*'s nodes and all the related edges



Graphs: Paths

A *path* is a sequence of nodes connected by edges

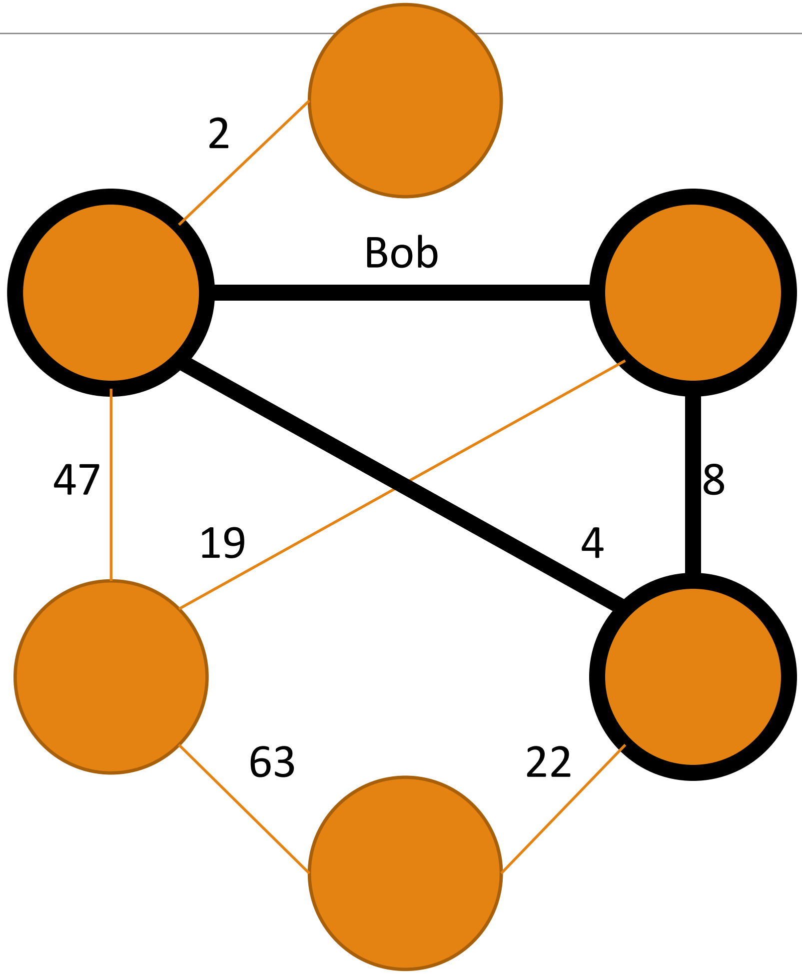
- A *simple path* doesn't repeat any nodes
- A graph is *connected* if every two nodes have a path



Graphs: Cycles

A *path* is a sequence of nodes connected by edges

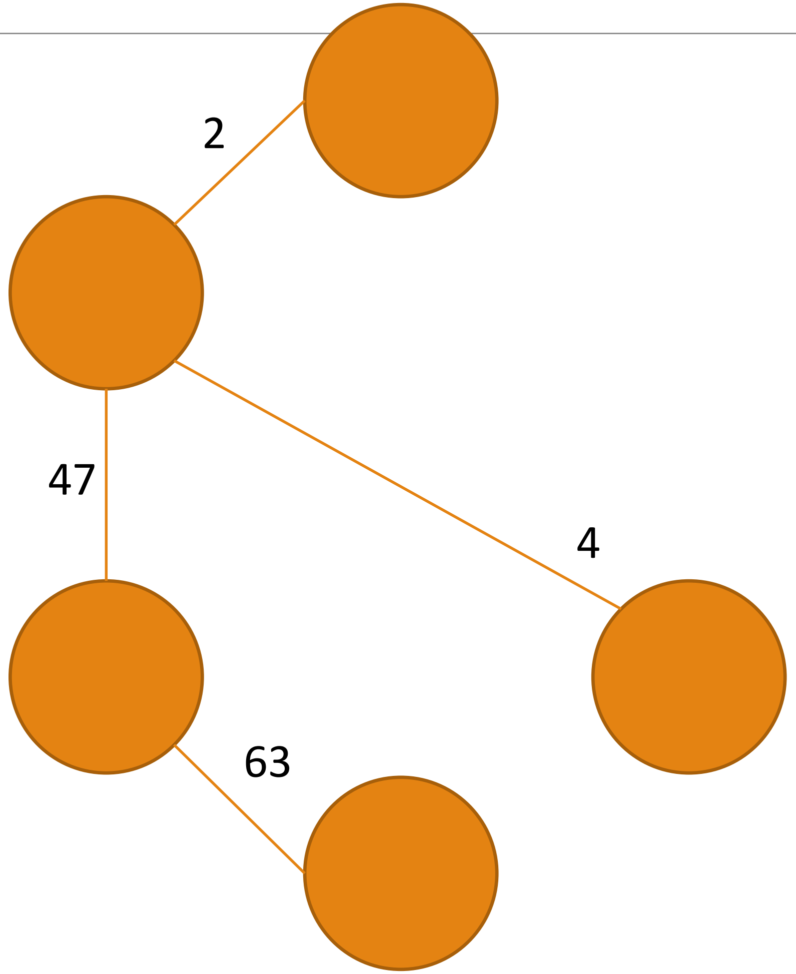
- A *simple path* doesn't repeat any nodes
- A graph is *connected* if every two nodes have a path
- A path is a *cycle* if it starts and ends on the same node
- A *simple cycle* contains at least three nodes and repeats only the first/last



Graphs: Trees

A *path* is a sequence of nodes connected by edges

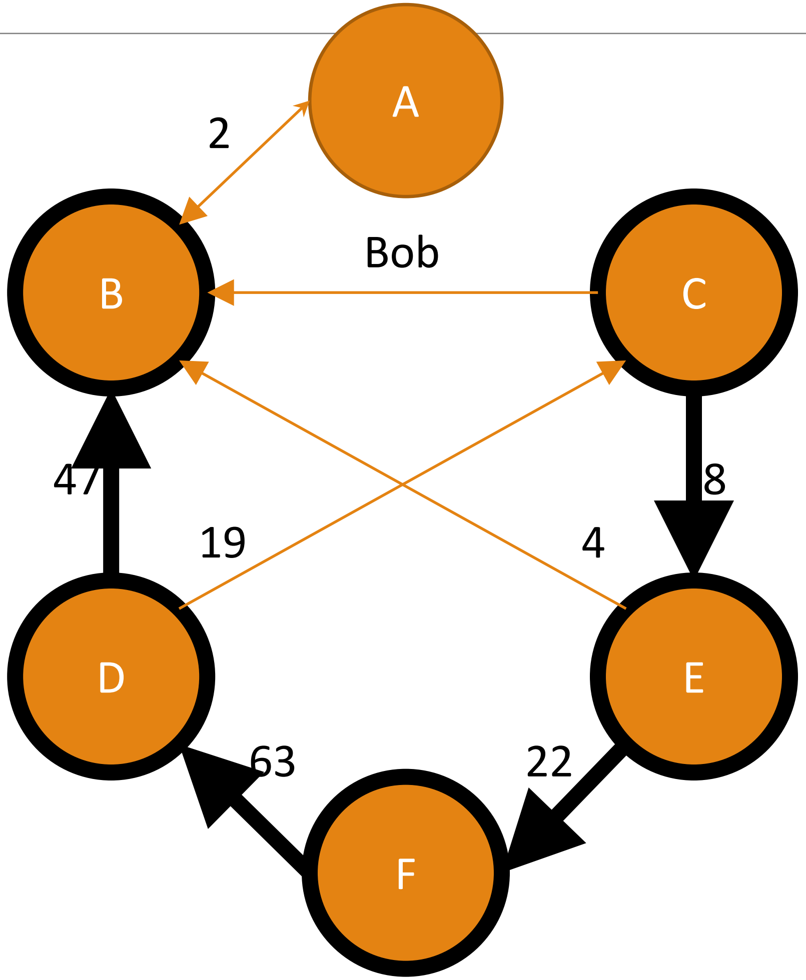
- A *simple path* doesn't repeat any nodes
- A graph is *connected* if every two nodes have a path
- A path is a *cycle* if it starts and ends on the same node
- A *simple cycle* contains at least three nodes and repeats only the first/last
- A graph is a *tree* if it is connected and has no simple cycles



Graphs: Directed Graphs

A *directed graph* is a graph with arrows instead of lines

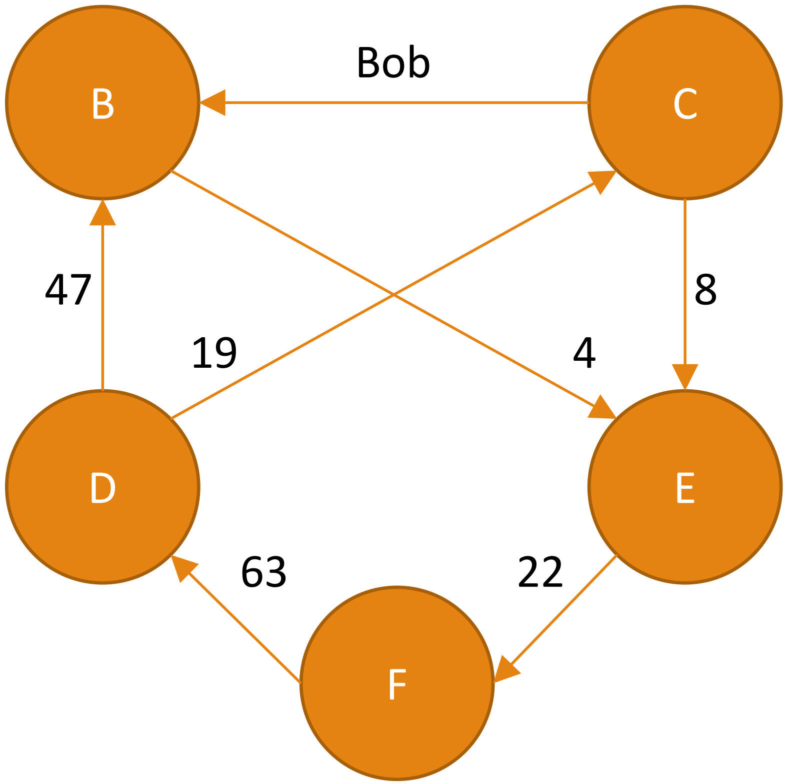
- Edges between nodes *i* and *j* are *ordered* pairs (*i*, *j*)
- *Directed paths* are paths that follow the direction of the edges



Graphs: Directed Graphs

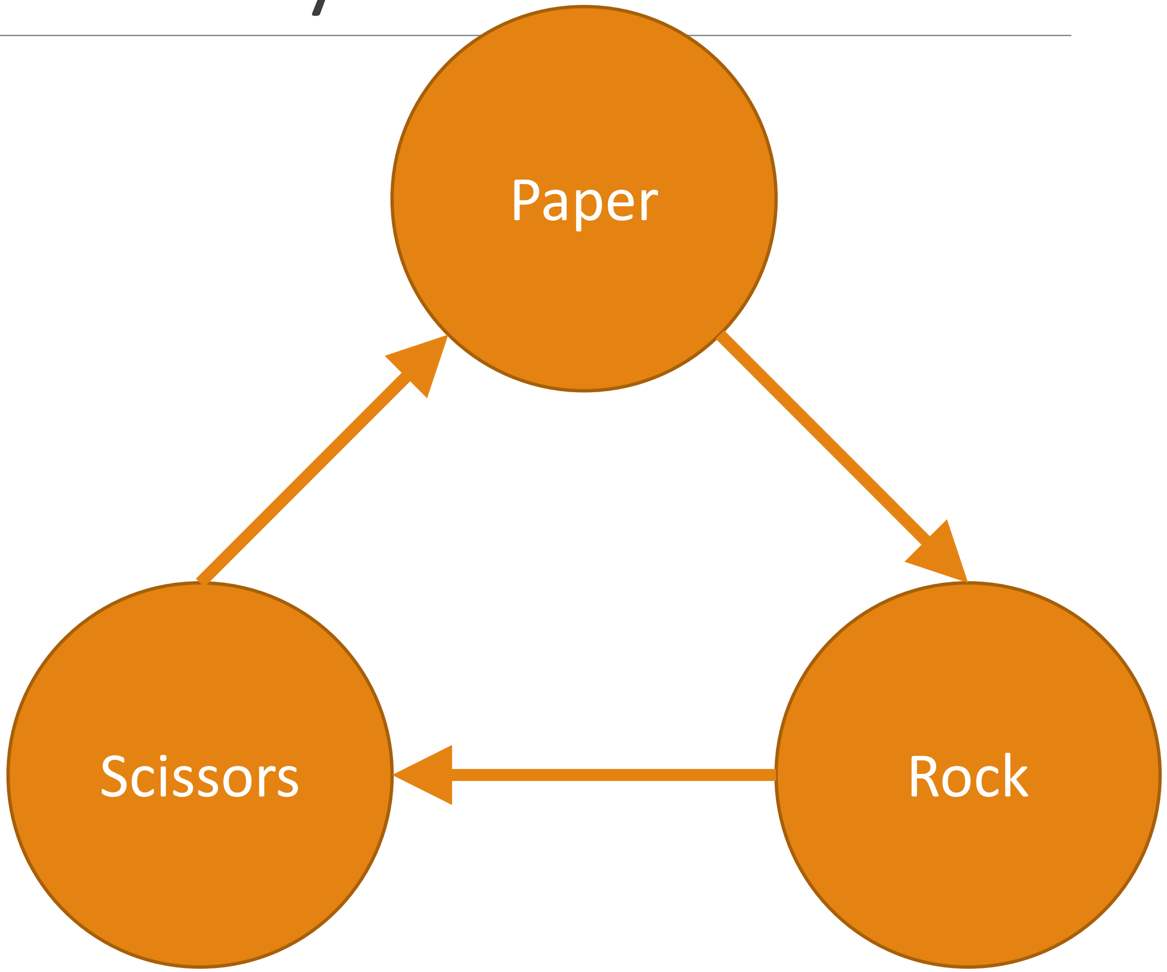
A *directed graph* is a graph with arrows instead of lines

- Edges between nodes *i* and *j* are *ordered* pairs (*i*, *j*)
- *Directed paths* are paths that follow the direction of the edges
- A directed graph is *strongly connected* if every pair of nodes has a directed path



Directed Graphs and Binary Relations

Consider the relation "beats"



Strings

- An *alphabet* is a non-empty, finite set of *symbols*
- A *string* over an alphabet is a finite sequence of symbols from that alphabet
- Strings have *length*, like any sequence; the empty string ε is the string with length 0
- A *language* is a set of strings over a given alphabet
- ***Do not confound the empty language with the empty string***

Given strings S, T, U and V, we write:

- S_i to denote the i^{th} symbol in S
- ST to denote the *concatenation* of S and T
- S^R to denote the *reverse* of S

…and we say:

- S is a *substring* of V if $\exists\ T, U \ni TSU = V$
  - …and a *proper substring* if $S \neq V$
- S is a *prefix* of V if $\exists\ T \ni ST = V$
  - …and a *proper prefix* if $S \neq V$
- S is a *suffix* of V if $\exists\ T \ni TS = V$
  - …and a *proper suffix* if $S \neq V$

Proofs

SECTIONS 0.3-0.4

Proofs and Friends

All of these *should* be clear and concise; they *must* be precise

- **Definitions** describe the mathematical objects and ideas we want to work with
- **Statements** or **assertions** are things we say about mathematics; they can be true or false
- **Proofs** are unassailable logical demonstrations that statements are true
- **Theorems** are statements that have been proven true
- **Lemmas** are theorems that are only any good for proving other theorems
- **Corollaries** are follow-on theorems that are easy to prove once you prove their parent theorems

How To Prove Something

1. Understand the statement
2. Convince *yourself* of whether it is true or false
3. Work out its implications until you have a general sense of *why* it is true or false
   - “Warm fuzzy feelings” don’t prove anything – but they can help you get *ready* to prove something
4. Break down any sub-cases you will need to prove
   - After this you may need to cycle back to step 2
5. Get started

Formats of Proofs

- The book uses a highly narrative proof format
- There are several other valid ones
- Let’s look at two

Quasi-Narrative Format

Prove $\overline{A \cup B} = \overline{A} \cap \overline{B}$

We can show this by showing $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

Suppose $x \in \overline{A \cup B}$.
Then by definition of complement, $x \notin A \cup B$.
Then by definition of union, $x \notin A$ and $x \notin B$.
Then by def. of complement, $x \in \overline{A}$ and $x \in \overline{B}$.
Then by definition of intersection, $x \in \overline{A} \cap \overline{B}$.
We have shown that if $x \in \overline{A \cup B}$, $x \in \overline{A} \cap \overline{B}$.
Hence by definition of subset, $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

Now suppose $x \in \overline{A} \cap \overline{B}$.
Then by def. of intersection, $x \in \overline{A}$ and $x \in \overline{B}$.
Then by def. of complement, $x \notin A$ and $x \notin B$.
Then by definition of union, $x \notin A \cup B$.
Then by definition of complement, $x \in \overline{A \cup B}$.
We have shown that if $x \in \overline{A} \cap \overline{B}$, $x \in \overline{A \cup B}$.
Hence by definition of subset, $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

We have shown that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

Hence by set equality, $\overline{A \cup B} = \overline{A} \cap \overline{B}$, QED.

Two-Column Format

Prove $\overline{A \cup B} = \overline{A} \cap \overline{B}$

1	STS $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}, \overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$	set equality
2	Let $x \in \overline{A \cup B}$	
3	$\therefore x \notin A \cup B$	complement
4	$\therefore x \in \overline{A}, x \in \overline{B}$	union
5	$\therefore x \in \overline{A} \cap \overline{B}$	intersection
6	$x \in \overline{A \cup B} \Rightarrow x \in \overline{A} \cap \overline{B}$	2-5
7	$\therefore \overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$	subset
8	Let $x \in \overline{A} \cap \overline{B}$	
9	$\therefore x \in \overline{A}, x \in \overline{B}$	intersection
10	$\therefore x \notin A, x \notin B$	complement
11	$\therefore x \notin A \cup B$	union
12	$\therefore x \in \overline{A \cup B}$	complement
13	$x \in \overline{A} \cap \overline{B} \Rightarrow x \in \overline{A \cup B}$	9-13
14	$\therefore \overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$	subset
15	$\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}, \overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$	7, 14
16	$\therefore \overline{A \cup B} = \overline{A} \cap \overline{B}$	set equality
	$\square$	

Types of Proofs

Direct Argument
- What we just did

Construction
- Prove something exists by showing how to make it

Contradiction
- Prove something is true by showing it can't be false

Weak Induction
- Show that a statement is true for the case of 0
- Show that *if* it's true for the case of i, *then* it's true for the case of $i + 1$

Strong Induction
- Show that a statement is true for the case of 0
- Show that *if* it's true for all of the cases $< i$, *then* it's true for the case of i

Next Time:
Finite Automata