Theorem: Let W be a subspace of $\mathbb{R}^n$. Let S_0 be a maximal lin. ind. subset of W. Let $|S_0| \le k$. Then any lin-ind subset of W must have cardinality $\le k$.

Proof: Left as an exercise. (Hint: Repeat the proof of the thm for $\mathbb{R}^n$

Dimension of a subspace

Def^n (Dimension). Let $W \subseteq \mathbb{R}^n$ be a subspace. The dimension of W is the cardinality of a maximal lin. ind. subset W.

Remark ① Dimension of $\{0\}$ is zero

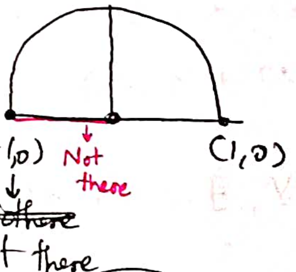
② If W contains x s.t. $x \neq 0$ then $\{x\}$ is lin. indepent.

$\Rightarrow \boxed{\dim W = 0 \Leftrightarrow W = \{0\}}$

③ If $\dim W = 1$ then $W \neq \{0\}$. Let $v \neq 0$ be any element of

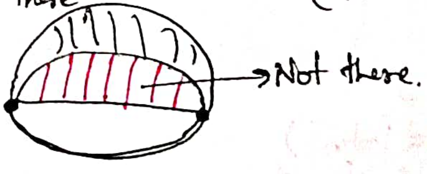
$\therefore \{v\}$ is lin. independent $\Rightarrow W = \{\alpha v \mid \alpha \in \mathbb{R}\}$

Structure for set of all dim 1 :



(In 2D)

(In 3D)



A plane in $\mathbb{R}^2$ is linear combinations of two non-zero elements v & w s.t. $w \neq \alpha v \Rightarrow \{v, w\}$ is ind.

Def^n: A linear span or span of elements $v_1, \dots, v_n$ is the subset of $\mathbb{R}^n$ given by $\{\omega \mid \omega = \sum \alpha_i v_i, \text{ for some } \alpha_i \in \mathbb{R}\}$

Note: Assume from now on that $\exists$ a finite set $S = \{e_1, \cdots, e_n\}$ s.t. $\text{span}(S) = V$ where,

$$\text{Span}(S) = \left\{ \omega \in V \;\middle|\; \begin{array}{l} \exists\, \alpha_1, \cdots, \alpha_n \in \mathbb{F} \\ s_1, s_2, \cdots s_n \in S \\ \text{s.t. } \omega = \sum_{i=1} \alpha_i s_i \end{array} \right\}$$

Theorem: Let $S \subset V$ be a subset let $\omega \in \text{span}(S)$ then $\omega = \sum_{i=1}^{n} \alpha_i s_i$ for some $a_i \in \mathbb{F}$, $s_i \in S$

If S is linearly independent then a_i, s_i are uniquely defined.

If $\omega = \sum_{j=1}^{m} B_j t_j$, $t_j \in S$ ~~$\forall j, B$~~, $B_j \in \mathbb{F}$.

"Left for more"

Thm: ① $\{W_\alpha\}_{\alpha \in \mathcal{I}} = \Gamma$ collection of subspaces of V then $\bigcap_{\gamma \in \Gamma} W_\gamma$ is also a subspace $v, \omega \in \bigcap W_\gamma$, ~~$\forall v, \omega \in \mathbb{F}$~~

then $\alpha v + \beta \omega \in W_\gamma$, $\forall \gamma$

$\therefore$ $\alpha v + \beta \omega \in \bigcap_{\alpha \in \Gamma} W_\gamma$

Note

Q1. How do you know that $\exists\, S'$ s.t. $S' \subseteq S$, $|S'| < \infty$ & $\text{span}(S') = \text{span}(S)$?

Q2. If $|S| =$ ctble, then it is possible to list them & show span(S) is a subspace. What $|S| =$ uncctble?

Q3 Do you do, $\sum_{\alpha \in \mathcal{I}} c_\alpha v_\alpha$?.

② Span(S) is the smallest ~~sp~~ subspace ~~containing v, ω~~. of V containing a subset S of V.

Reason for 2.

Clearly any subspace W of V containing S must ~~cot~~ contain span(S).

Hence, span(S) is the smallest subspace containing S.

we prove that span(s) is a subspace.

As V is a finite dimensional v-space, if $|S| = \infty$, [(uncdble or ctble) specifically uncdble, listing down the set S is not possible or do you do, $\sum_{\alpha \in \mathcal{I}} a_\alpha v_\alpha$?].

Use Solⁿ of Problem 4 in L.A. Papers.

① In ~~$\mathbb{R}^n$~~ $\mathbb{R}^n$, $\{e_1, e_2, \dots, e_n\}$ is a maximal linearly independent subset

② Any maximal lin. ind subset of $\mathbb{R}^n$ has n-elements

③ $W \subset \mathbb{R}^n$ subspace, any lin. ind subset of W is also a lin. ind subset of $\mathbb{R}^n$

④ Any lin. ind subset of $\mathbb{R}^n \subset$ Maximal lin. ind subset.

⑤ In $\mathbb{R}^n$ let S be a set s.t. $\text{span}(S) = \mathbb{R}^n$ then $\exists\, T \subset S$ s.t. T is lin. ind.

Proof: Let $v_1 \neq 0$ be an element of S. If $\text{span}(v_1) \neq \mathbb{R}^n$, $\exists\, v_2 \in S$ s.t $v_2 \notin \langle v_1 \rangle$. $\text{span}(v_1) \neq \mathbb{R}^n$, $S \subset \mathbb{R}^n$, $v_1 \in S$.

If not, then $\forall v \in S \Rightarrow S \in \text{span}(v_1)$ & $w \in \mathbb{R}^n \in \text{span}(S)$ $\forall w$. Let

$$w = \alpha_1 \underset{\beta_1 v_1}{\underset{\shortparallel}{s_1}} + \dots + \alpha_m \underset{\beta_m v_m}{\underset{\shortparallel}{s_m}}, \quad s_i \in S \qquad \beta_2 v_2 \qquad \alpha_i \in \mathbb{R}$$

$$\Rightarrow w_i = (\alpha_1\beta_1 + \dots + \alpha_m\beta_m)\, v_1$$ Contradiction.

Note: From now on we assume that a vector space V contains a finite spanning ~~space~~ set.

$\exists\, v_2 \in S$ s.t. $v_2 \notin \text{span}(v_1) \Rightarrow \{v_1, v_2\} \subseteq S$ is lin. ind.

$\Rightarrow$ If $\text{span}(\{v_1, v_2\}) \neq \mathbb{R}^n$ then we claim that $\exists\, v_3 \in S$ s.t. $v_3 \notin \text{Span}(\{v_1, v_2\})$. Prove like before (exercise). Repeat process n-times.

Thm: If $\{e_1, e_2, -, e_n\}$ is a lin. ind spanning subset of V, then any lin. ind subset S has Cordinality $\leq n$.

Pf: Remark 1 if $\omega \notin \text{span}(S)$ & S lin. ind., then $S \cup \{\omega\}$ is also lin. ind.

If not, then $S \cup \{\omega\}$ is lin. dep. $\Rightarrow \exists\ \alpha_1, \alpha_2, \dots, \alpha_n\ \beta$ s.t. $\sum \alpha_i s_i + \beta \omega = 0$. ~~Since~~ Not all zero $\Rightarrow \omega = -\frac{1}{\beta} \sum \alpha_i s_i$ contradicts the fact that $\omega \notin \text{span}(S)$.

Remark 2: Any lin. ind subset can be included in a spanning lin. ind. subset.

et e_i be the first element in $B_0 = \{e_1, e_2, \dots, e_n\}$ s.t. $e_i \notin S$.

ase 1 if $e_i \notin \text{span}(S)$ then $e_i = \sum_{i=1}^{n} \alpha_i s_i$ for some $\alpha_i \in \mathbb{F}$

Since B_0 is lin. independent, $s_i \in S$ not all $s_1, s_2, \dots, s_n$ in the above eq^n are from B_0 since B_0 was lin. ind.

$\Rightarrow \exists\ s_j \in S \setminus B_0$ s.t. $\alpha_j \neq 0$. Let $S_1 = S \setminus \{s_j\} \cup \{e_i\}$

$\Rightarrow$ S_1 is lin. independent & $S_1 \cap B_0$

If $\exists\ S$ lin. ind & e_i is the first element of B_0 not in S then $\exists\ S$ lin. ind s.t. $S_1 \cap B_0 = (S \cap B_0) \cup \{e_1\}$

eat the same argument for S_1. Now to get a lin. ind subset S_2 s.t. $S_2 \cap B_0 \supset S_1 \cap B_0 \supset S \cap B_0$. Repeating this at most n-times get a lin. ind subset. T s.t. $T \supset B_0$, (& $|T| \geq |S|$) If $T \neq B_0$ then $\exists\ t \notin B_0$

T. But $t \in \text{Span}(B_0) = V$

$\hookrightarrow \{t\} \cup B_0$ is not lin. ind. contradiction

$T = B_0 \Rightarrow |T| = n \Rightarrow |S| \leq n$.

__Corollary__ : Let S_1, S_2 be the spanning lin. ind. subsets of $\mathbb{V}$ having a finite spanning lin. ind ~~subsets~~ set, then $|S_1| = |S_2|$.

__Proof__ : $|S_1| \leq |S_2|$; $|S_2| \leq |S_1|$ $\therefore$ proved.

__Exc__ : __Theorem__: If S is a finite spanning then it has a lin. ind spann set.

__Proof__: Exercise.

__Def^n__. An ordered spanning lin. ind. subset of $\mathbb{V}$ is called the basis of $\mathbb{V}$. Let $S = \{e_1, e_2, \dots, e_n\}$ spanning lin. ind. subset of $\mathbb{V}$ gives rise to n! basis.

$\therefore$ Basis given by S will be denoted by $(e_{i_1}, e_{i_2}, \dots, e_{i_n})$.

__Def^n__ : Dimension of Vector Space $\mathbb{V}$ is the cardinality of its basis, denoted by dim V or dim(V).

__Qn__ : Is there "another" vector space of dim n over $\mathbb{F}$?

__Ans__ : All polynomials of deg $< n$ with coeff $\in \mathbb{R}$

__Note__ : Any lin tran has a unique matrix.

$F[x]_{<n} / \mathbb{R}[X] = \{ \sum_{i=0}^{n-1} a_i x^i \mid a_i \in \mathbb{R} \}$ Basis is : $\{1, x, x^2, \dots, x^{n-1}\}$

__Linear Transformation(T)__ Between two func^n's vector spaces over $\mathbb{F}$ s.t

$T(a v_1 + \beta v_2) = \alpha T(v_1) + ~~\beta(T v~~ \beta T(v_2)$, $\forall v_1, v_2 \in \mathbb{V}$, $\alpha, \beta \in \mathbb{F}$.

__E.g.1__ $V = \mathcal{C}([0,1], \mathbb{R})$, $T: f \to \int_0^1 f(x)\,dx$, $f \in \mathcal{C}([0,1], \mathbb{R})$ & $\int_0^1 f(x)dx \in$

$$\Rightarrow \int_0^1 \alpha f_1 + \beta f_2 \, dx = \alpha \int_0^1 f_1 \, dx + \beta \int_0^1 f_2 \, dx$$

__E.g. 2__

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times2} \begin{bmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{bmatrix}_{2\times1} \overset{?}{=} \begin{bmatrix} (\alpha a x_1 + \beta a y_1) + (\alpha b x_2 + \beta b y_2) \\ (\alpha c x_1 + \beta c y_1) + (\alpha d x_2 + \beta d y_2) \end{bmatrix}$$

__Note__ : There is nothing (sp. 2x2 about)

$$= \alpha \begin{bmatrix} a x_1 + b x_2 \\ c x_1 + d x_2 \end{bmatrix} + \beta \begin{bmatrix} a y_1 + b y_2 \\ c y_1 + d y_2 \end{bmatrix} = \alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \beta \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y$$

Basis of vector space (finite Dimensional)

$\boxed{V}$ vector space $\exists$ a S maximal lin. ind subset

$\downarrow$

nsion = the cordinality of maximal lin. ind subset

② Any two maximal lin. ind subset V have same cordinality

Basis of V = ~~subset~~ ordered max. lin. ind subset of V.

Let $\mathcal{B}$ be a basis of V

$\mathcal{B} = (v_1, \cdots, v_n)$

Given any $\omega \in V$, $\omega = \sum_{i=1}^{n} x_i v_i$, for $x_i \in \mathbb{F}$ (be any field)

ω can be identified with $(x_1, \cdots, x_n)$ coordinates of ω w.r.t basic $\mathcal{B}$.

A digression

he that we've been working with $\mathbb{R}$'s. These are of basic 1. Consider basis R changed to $\sqrt{2}$. The no. $(300)_1 \mapsto \left(\frac{300}{\sqrt{2}}\right)_{\sqrt{2}}$ under basis $\sqrt{2}$.

ally speaking, The $\mathbb{R}$-line has moved $1 \to \sqrt{2}$ under $\sqrt{2}$ basis.

sically, $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{x}{\sqrt{2}}$.

there a formula of getting coordinates of ω w.r.t. $\mathcal{B}'$ given its ~~coordination~~ rdinates w.r.t $\mathcal{B}$.

Yes! Change of basis formula!

Change of Basic Formula

$= [v_1 \cdots v_n] \begin{bmatrix} P_{11} \\ \vdots \\ P_{n1} \end{bmatrix}$

$= [v_1 \cdots v_n] \begin{bmatrix} P_{1i} \\ \vdots \\ P_{ni} \end{bmatrix}$, $\forall\ 1 \leq i \leq n$

$\to$ each one $[v_{ij}]_{n \times 1}$ matrix, $v_{ij} \in \mathbb{F}$.

$= [P_{ij}]_{n \times n}$, then we have $[v_1 \cdots v_n]_{1 \times n} P = [\omega_1 \cdots \omega_n]_{1 \times n}$, $B_0 P = B_1$

rly let Q be an $n \times n$ matrix s.t. $B_1 Q = B_0 \Rightarrow B_1 Q P = B_1$

$\exists P'$ s.t. $B_0 P = B_0 P'$, It can be shown that this contradicts unique representation vectors w.r.t a basis.

What does $B_1 Q P = B_1$ mean?

$$\underset{[c_{ij}]}{\underset{\downarrow}{QP}}$$

(lemma) <u>Insights</u>. If $A_{n\times n}$ matrix, then its its RREF $= I_n$ then, A ~~all~~ its and columns is madeup of vec that are lin. indp. And, every madeup of lin. ind set of vectors, rows or columns, has its RREF

$$[\omega_1 \cdots \omega_n] \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \\ c_{n1} & \vdots & \cdots & c_{nn} \end{bmatrix} = [\omega_1\ \omega_2 \cdots \omega_n]$$

$\Rightarrow$

$$\omega_1 = \omega_1 c_{11} + \omega_2 c_{21} + \cdots + \omega_n c_{n1} \Rightarrow c_{11} = 1,\ c_{21} = c_{31} = \cdots = c_{n1} = 0$$

$$\vdots$$

$$\omega_n = \omega_1 c_{n1} + \omega_2 c_{n2} + \cdots + \omega_n c_{nn} \Rightarrow c_{n1} = 0,\ \cancel{c_{n2} = c_{3n} = \cdots = c_{nn} = 0}.$$

$$c_{n2} = 0, \cdots, c_{n(n-1)} = 0,\ c_{nn} = 1.$$

(Note: $QP = I$, $Q_{n\times n}, P_{n\times n} \Rightarrow Q = P^{-1}$ or $P = Q^{-1}$). $PQ = I$.

These, $c_{11} = 1$, $c_{21} = c_{31} = \cdots = c_{n1} = 0$ and so on, are the only possible values and as $\omega_1, \omega_2, \cdots, \omega_n$ ~~are~~ is maximal. lin. ind set, Hence, their representation is unique.

Q Can any lin. ind set $\xrightarrow{\text{Transformed into}}$ Any other lin. ind set.

<u>Theorem</u>. Let V be a vector space of dim n. Let, $\mathcal{B}_0$, ~~&~~ $\mathcal{B}_1$ be two bases of

Then ∃! (there exist unique) n×n invertible matrix P s.t. $\mathcal{B}_0 P = \mathcal{B}_1$

row vector made of vectors (Cause of notation)

row vector made of vectors

Q1 What is the formula for change of coordinates?

<u>Proof</u>: It is given behind. Just the motivation proof is ex.

<u>Theorem</u>. Let V be a v-space of dim n & $\mathcal{B}_0, \mathcal{B}_1$ two bases of V. Let $\mathcal{B}_1 =$ where, $P_{n\times n}$ invertible matrix with entries from $\mathbb{F}$. Then $\forall\ \omega \in V$. If ~~$\omega \in$~~ $\omega = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ w.r.t. B_0, then $\omega = P^{-1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ w.r.t B_1. (Note: $\omega = \mathcal{B}_0 \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \Rightarrow \omega = \mathcal{B}_1 P^{-1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \Rightarrow \omega = P^{-1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ w.r.t. $\mathcal{B}_1$

(i.e. ∃ a bijection)

<u>Imp Thm</u>. The set n×n invertible matrices (n×n) with entries from $\mathbb{F}$ <u>"equals"</u> (Isomorphism) the set of all bases of $\mathbb{F}^n$.

<u>Idea of Proof/Proof is Ex.</u> $S_1 = \{P_n \mid P_{n\times n} \text{ invertible}\}$

$S_2 = \{\mathcal{B} \mid \mathcal{B} \text{ a basis of } V\}$

To define $f: S_1 \to S_2$ that is 1-1 and onto map. Fix $\mathcal{B}_0$ in S_2 & define

$f(Id) = \mathcal{B}_0$, $f(P) = \mathcal{B}_0 P$ (is a sq. matrix), $\forall\ P \in S_1$.

Proof that $\mathcal{B}_0 P$ is a basis/maximal. lin. ind ~~set~~ matrix of ve Through lemma, $\mathcal{B}_0 P$ has either I_n has its RR or 0's at the last row. If 0's are at the la row then it's not invertible. But $\mathcal{B}_0 P$ is inv. It its RREF is $I_n \Rightarrow \mathcal{B}_0 P$ has lin. ind set of vector

contradiction.

28/9/23

Recall. **Theorem** (I) If $\mathcal{B}_0, \mathcal{B}_1$ are two bases of a v-space over $\mathbb{F}$ of dim n (say).

1) Then, $\exists$ an $n \times n$ invertible matrix P with entries from $\mathbb{F}$ s.t. $\mathcal{B}_0 P = \mathcal{B}_1$
(If X, X' are coordinates of $v \in V(V)$ w.r.t. $\mathcal{B}_0$ & $\mathcal{B}_1$ respectively then $X' = P^{-1}X$).

2) If P is any invertible $n \times n$ matrix with entries from $\mathbb{F}$ then ~~$\mathcal{B}_0$, $\mathbb{F}$~~ $\mathcal{B}_0 P$ is also a basis. ((2) is the converse of (1)).

Note: Try to always ask for iff questions.

Corollary. # Basis of V = # $n \times n$ inv. matrices

xample. $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$, $V = \underbrace{(\mathbb{Z}/p\mathbb{Z}) \times (\mathbb{Z}/p\mathbb{Z}) \times \cdots \times (\mathbb{Z}/p\mathbb{Z})}_{n \text{ times}}$

$|V| = p^n$, $(v_1, v_2, \cdots, v_n)$ be a basis of V

$$\underbrace{n! \,\big|\, (p^n - 1)(p^n - p) \cdots (p^n - p^{n-1})}_{\text{prove it.}} < p^{n^2}$$

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irect Sums & Sums

m of subspaces W_1 & W_2 of V

Does there exist a subspace W_3 s.t,

① $W_1 \subset W_3$ and $W_2 \subset W_3$

② If W is any subspace containing W_1 & W_2. Then $\boxed{W \supset W_3}$ also

③ $W_1 + W_2$ is a subspace, $\boxed{W_1 + W_2 := \{x \mid x = w_1 + w_2, w_1, w_2 \in W_1, W_2 \text{ resp.}\}}$ $\boxed{W_1 + W_2}$

Span$(\mathcal{B}_1 \cup \mathcal{B}_2)$

Possible Candidates

(1) $W_1 \cup W_2$?

(2) $\langle W_1 \cup W_2 \rangle$?

(3) $\langle \mathcal{B}_1 \cup \mathcal{B}_2 \rangle$?

What is the dim of ~~W_1 & W_2~~ $W_1 + W_2$ in terms of dim W_1, dim W_2.

- $\max\{\dim W_1, \dim W_2\} \leq \dim W_1 + W_2 \leq \dim W_1 + \dim W_2$ ~~$\dim(W_1 \cap W_2)$~~

L_1 = compliment of $W_1 \cap W_2 \uparrow$ in W_1
present

L_2 = compliment of $W_1 \cap W_2 \downarrow$ in W_2
present

Claim If $\dim W_1 \cap W_2 = k$

$\dim W_1 = n+k$

$\dim W_2 = m+k$

Then, $\dim W_1 + W_2 = n+m+k$

Proof Let $\mathcal{B}_1 = (x_1, \dots, x_k)$ be a basis of $W_1 \cap W_2$

let $\mathcal{B}_1 = (x_1, \dots, x_k, y_1, \dots, y_n)$ be a basis of W_1

$\mathcal{B}_2 = (x_1, \dots, x_k, z_1, \dots, z_m)$ be a basis of W_2

Then, $|\mathcal{B}_1 \cup \mathcal{B}_2| = n+m+k$

Q. $\mathcal{B}_1 \cup \mathcal{B}_2$ lin. ind ?

$$\text{Let } \sum_{i=1}^{k} \alpha_i x_i + \sum_{j=1}^{n} \beta_j y_j + \overset{(*)}{\boxed{\sum_{l=1}^{m} \gamma_l z_l}} = 0$$

If $\gamma_1 \neq 0$ $z_1 = (\sum a_i x_i + \sum_{\cancel{j=1}} b_j y_j) + \sum_{\cancel{l}} c_l z_l$

$$v = \underset{\substack{\cap \\ W_2}}{\sum \gamma_l z_l} = \underset{\substack{\cap \\ W_1}}{\sum \alpha_i x_i + \sum \beta_j y_j} \Rightarrow v \in W_1 \cap W_2 \Rightarrow v = \sum \phi_i x_i \cancel{\Rightarrow \sum}$$

$\Rightarrow \cancel{v =} \sum \phi_i x_i + \sum \gamma_l z_l = 0 \Rightarrow \phi_i = 0, \forall i \ \& \ \gamma_l = 0 \ \forall l.$

$\Rightarrow \sum \alpha_i x_i + \sum \beta_i y_i = 0$ by $(*)$ $\Rightarrow \alpha_i = 0 \ \forall i, \ \beta_j = 0 \ \forall j$

$\Rightarrow x_i, y_i, z_i$ form a lin. ind set whose span is $W_1 + W_2$

$\Rightarrow \dim W_1 + W_2 = n+m+k.$

$\boxed{W_1 + W_2 \text{ is written as } W_1 \oplus W_2 \text{ if } W_1 \cap W_2 = \{0\}.}$

$$L_1 := \underbrace{(W_1 + W_2) + W_3 \Rightarrow (W_1+W_2) \cap W_3}_{\dim(W_1+W_2+W_3)}$$

Back to lin. Transformations

$T: V_1 \to V_2$ be a map b/w two vector spaces $/\overset{(\text{over } F)}{\mathbb{F}}$. T is called a lin. transformation if, $\forall \alpha, \beta \in \mathbb{F}$ & $v_1, v_2 \in V_1$ we have,

$$T(\underset{\substack{\cap \\ V_1}}{\alpha v_1 + \beta v_2}) = \underset{\substack{\cap \\ V_2}}{\alpha T(v_1) + \beta T(v_2)}$$

By Induction, $T(\sum_i \alpha_i v_i) = \sum_i \alpha_i T(v_i)$

Hence, we can say that $\forall v \in V_2$ (T

$v = \sum_i \alpha_i T(u_i)$, $u_i \in V_1$ as,

$v = T(\underbrace{\sum_i \alpha_i u_i}_{\text{Input}})$. We can write v

lin. combn of T(u

$: V_1 \to V_2$ be a map b/w two v-spaces over $\mathbb{F}$. T is called linear formation if $\forall\ \alpha, \beta \in F$ & $v_1, v_2 \in V_1$ we have $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$

$\left(\sum_{i=1}^{n} \alpha_i v_i\right) = \sum_i \alpha_i T(v_i)$. If $\mathcal{B} = (v_1, \dots, v_n)$ is a basis of V_1 &

$: V_1 \to V_2$ is a lin. ind transformation then $\forall\ \omega \in V_1$. We know that $\omega)$ in terms of $T(v_i)$ $1 \le i \le n$ why? because let $\omega = \sum \alpha_i v_i$, Then,

$$T(\omega) = \sum_i \alpha_i T(v_i)$$

① $V_1 = V = V_2$, $T(v) = v$, $\forall\ v \in \mathbb{R}$. The "Identity" map.

② $T: V_1 \to V_2$, $T(v) = 0$, $\forall\ v$

③ i^{th} projection map π_i, $\pi_i : \mathbb{R}^n \mapsto \mathbb{R}$, $(x_1, x_2, \dots, x_n) \mapsto x_i$

④ let $(a_1, \dots, a_n)$ be a fixed vector in $\mathbb{R}^n$. $T_{(a_1, \dots a_n)} : \mathbb{R}^n \to \mathbb{R}$

$$\cancel{(x_1,}\ (x_1 .. x_n) \mapsto (a_1, \dots a_n)\cdot(x_1, \dots, x_n)$$

$$\| $$

$$\sum_{i=1}^{n} a_i x_i$$

<u>Theorem</u>. let $T: \mathbb{R}^n \to \mathbb{R}$ be a lin. Transformation. Then, $\exists!$ vector $v_0 \in \mathbb{R}^n$ s.t. $T(\omega) = v_0 \cdot \omega$ $\forall\ \omega \in \mathbb{R}^n$ i.e. $T = T_{v_0}$.

<u>Note</u>: What is special about $\mathbb{R}^n \to \mathbb{R}$? Dot product's extension is matrix product.

<u>Proof</u>. Exc, Proof. As, $\omega \in \mathbb{R}^n$, $\omega = \sum_i \alpha_i e_i$, $\{e_i\}_{i=1}^{n}$ is a $\mathcal{B}$ of $\mathbb{R}^n$.
$T(\omega) = \sum_i \alpha_i T(e_i)$, let $v_0 = (T(e_1), T(e_2), \dots, T(e_n))$
For uniqueness, let $v_0 \cdot \omega = u_0 \cdot \omega$. let $\omega = e_i$, $v_0 e_i = u_0 \cdot e_i$ $\boxed{x_{i v} = x_{u_0 i}}$, $\forall i \Rightarrow \boxed{v_0 = u_0}$. ∎

---

s a vector space over $\mathbb{R}$. Let $\mathcal{B} := (v_1, v_2, \dots, v_n)$ be a basis, then, $\forall\ x \in V \xrightarrow{} \mathbb{R}^n$

$= \sum_{i=1}^{n} a_i v_i \rightsquigarrow \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$. Basically, given just the coeffs, one can determine the vector. $\begin{bmatrix} a_1 v_1 \\ \vdots \\ a_n v_n \end{bmatrix} \mapsto \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ is a Lin. Transformation.

are showing that $(a_1, a_2, \dots, a_n)$ is a linear transformation. (is isomorphic to V)

$$x + \beta y) \iff \begin{cases} T(x+y) = T(x) + T(y) & \forall\ x, y \in V \\ T(\alpha x) = \alpha T(x) & \forall\ x \in V;\ \alpha \in \mathbb{R} \end{cases}$$

$\in \mathbb{R}, x, y \in V$

$+y) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$, if $x+y = \sum c_i v_i$. Let $x = \sum a_i v_i$, $y = \sum b_i v_i$

$= \sum a_i v_i + \sum b_i v_i = \sum (a_i + b_i) v_i \Rightarrow T(x+y) = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}$ — since $x+y$ is uniquely expressed as $\sum (a_i + b_i) v_i$

$\Big] + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = T(x) + T(y)$. ∎

Theorem Let V be a n-dim vectorspace over $\mathbb{F}$. Then V is isomorphic to $\mathbb{F}^n$.

Defn. The vector space V and W over $\mathbb{F}$ are said to be isomorphic (to each other) if $\exists$ a lin. transformation $T : V \to W$ s.t.

(For Thm! Proof is given behind. Notice that we have just used the fact that $\dim V = n$.)

Note S_1, S_2 are sets. Then $S_1 \times S_2$ (Generalisation of Cartesian Product) $= \{ f \mid f : \{1,2\} \to S_1 \cup S_2,\ f(1) \in S_1,\ f(2) \in S_2 \}$

$\overset{?}{\parallel}$

$\{ (s_1, s_2) : s_1 \in S_1, s_2 \in S_2 \}$

- $T : V \to W$ lin. transformation then, we can define two subspaces (one each in V and W)

$T(\alpha x + \beta y) =$ (Null Space) Kernel of $T = \{ v \in V \mid T(v) = 0_W \}$

Image of $T = \{ \omega \in W \mid \exists\ v \in V \text{ s.t. } T(v) = \omega \}$ (Column Space)

Let $\omega_1 = T(v_1)$ & $\omega_2 = T(v_2)$, $T(\alpha v_1 + \beta v_2) = \cancel{\alpha v_1 + \beta v_2}$ $\alpha T(v_1) + \beta T(v_2)$

Theorem $T : V \to W$ lin. transformation. Then, $\dim V = \dim(\ker(T)) + \dim(\operatorname{Im}(T$

Proof. Let $(x_1, x_2, \ldots, x_r)$ be a basis of $\ker(T)$

Let $(x_1, x_2, \ldots, x_r, y_1, \ldots, y_t)$ be a basis of V

- **Claim.** $\overbrace{(T(y_1), T(y_2), \ldots, T(y_t))}^{S}$ is a basis of $\operatorname{Im}(T)$.

① Proof that S is lin. ind.

~~We need~~ Let $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{F}$ be s.t. $\sum_{i=1}^{s} \alpha_i T(y_i) = 0 \Rightarrow$ $T\left(\sum_{i=1}^{s} \alpha_i y_i\right) = 0$

$\Rightarrow \sum \alpha_i y_i \in \ker(T) \Rightarrow \sum_{i=1}^{s} \alpha_i y_i = \sum_{i=1}^{r} \beta_j x_j$

$\Rightarrow \sum \alpha_i y_i - \sum \beta_j x_j = 0 \Rightarrow \alpha_i = 0, \forall i.$

② Proof that S is maximal. $\{T(y_1), T(y_2), \ldots, T(y_t)\}$ or spans $\operatorname{Im}(T)$.

Take $\omega \in \operatorname{Im}(T) \Rightarrow \exists\, v \in V$ s.t $T(v) = \omega \Rightarrow v = \sum_{i=1} \alpha_i x_i + \sum_{j=1}^{t} \beta_j$

$\Rightarrow \omega = T(\underbrace{\sum \alpha_i x_i}_{0}) + T(\sum \beta_j y_j)$ (As $\Rightarrow \omega = \sum_{j=1}^{s} \beta_j T(y_j)$)

[As x_i are basis of $\ker(T)$]

Matrix of a linear Transformation

Let $T: V \to W$ be a lin. Transf.

Let $\dim V = n$, $\dim W = m$.

Let $\mathcal{B}_1 := (x_1, x_2, \ldots, x_n)$ & $\mathcal{B}_2 := (y_1, y_2, \ldots, y_m)$ be bases of V & W resp.

$$T(x_j) = \sum_{i=1}^{m} a_{ij} y_i, \quad \forall\ 1 \le j \le n, \text{ i.e.,}$$

$$T(x_1) = a_{11} y_1 + a_{21} y_2 + \cdots + a_{m1} y_m$$

$$T(x_2) = a_{12} y_1 + a_{22} y_2 + \cdots + a_{m2} y_m$$

$$\vdots$$

$$T(x_n) = a_{1n} y_1 + a_{2n} y_2 + \cdots + a_{mn} y_m \qquad \text{i.e.,}$$

Q. What if $T(x_i) = y_i$, $\forall i \in \{1, \ldots, n\}$?

Ans. This actually a special case, of, where some $T(x_i)$'s equal 0. i.e. some of the x_i's $\in \ker(T)$. The case above dealt w/ no $x_i \in \ker(T)$.

$$[y_1\ y_2\ \cdots\ y_m] \overbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}^{\leftarrow \text{matrix of the lin. Transformation} \rightarrow \text{ w.r.t } \mathcal{B}_1 \,\&\, \mathcal{B}_2.} = [T(x_1) \cdots T(x_n)]_{1\times n}$$

$M(T) := [a_{ij}]_{m\times n}, \ \substack{1 \le i \le m \\ 1 \le j \le n} = [\underset{\underset{W}{\cap}}{T(x_1)}\ \underset{\underset{W}{\cap}}{T(x_2)} \cdots \underset{\underset{W}{\cap}}{T(x_n)}]$ is called the matrix of the lin. transformation T w.r.t the bases $\mathcal{B}_1$ & $\mathcal{B}_2$ of V & W.

Can one chose bases $\mathcal{B}_1$ & $\mathcal{B}_2$ s.t., $\underset{\underset{M(T).}{\|}}{m(T)} = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]_{m\times n}$?

. Yes!

orem. Let $T: V \to W$ be a lin. transformation. Then, we can choose bases $\mathcal{B}_1$ of V & $\mathcal{B}_2$ of W s.t. the matrix of T w.r.t $\mathcal{B}_1$ & $\mathcal{B}_2$ is of the type,

$$\left[\begin{array}{c|c} I_r & \overset{\leftarrow n-r \rightarrow}{O} \\ \hline O & O \end{array}\right]_{m\times n} \updownarrow m-r$$

. Let $(x_1, \ldots, x_{n-r})$ be a basis of $\ker(T)$. (Implies, $T(x_1) = T(x_2) = \cdots = T(x_{n-r}) = 0$).

Let $\mathcal{B}_1 = (y_1, \ldots, y_r, x_1, \ldots, x_{n-r})$ be a basis of V.

Let $\mathcal{B}_2 = (T(y_1), \ldots, T(y_r), z_{r+1}, \ldots, z_m)$ be a basis of W

ince, $\{T(y_1), T(y_2), \ldots, T(y_r)\}$ is lin. ind, $m(T)$ w.r.t this choice of $\mathcal{B}_1$ & $\mathcal{B}_2$ oks like, Since, $T(\mathcal{B}_1) = (\underbrace{T(y_1), \dots, T(y_r)}, 0, 0, \dots, 0)$

$\hookrightarrow$ rank of the lin. transformation (r) i.e. the dim of

$\Rightarrow T(y_1) = 1 \cdot T(y_1) + 0 \cdot T(y_2) + \cdots$

$\Rightarrow T(y_2) = 0 \cdot T(y_1) + 1 \cdot T(y_2) + \cdots$

$\vdots$

$\Big\}$ $T(y_i) = 1 \cdot T(y_i)$, $\forall i$. (Due to lin. indepen

$T(x_j) = 0$, $\forall j$.

$\rightarrow$ This is unique!

So, $$m(T) = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]_{m \times n}$$ ($\because$ It's the dimension of $\operatorname{Im}(T)$).

⊛ Special Cases.

$V = \mathbb{F}^n$, $\quad \mathcal{B}_1 = (e_1^{(n)}, \dots, e_n^{(n)})$

$W = \mathbb{F}^m$, $\quad \mathcal{B}_2 = (e_1^{(m)}, \dots, e_m^{(m)})$

A Lin. Transformation $T: \mathbb{F}^n \to \mathbb{F}^m$ is nothing but a $m \times n$ matrix w.r.t $\mathcal{B}_1$ & $\mathcal{B}$

We denote it by $M(T)$.

Q. What is the matrix of T if we change $\mathcal{B}_1$ to $(v_1, \dots, v_n)$?

$M(T)$ changes &c. (may not be $m(T)$).

So what does it change to?

* Theorem Let $A_{m \times n}$ be a matrix with entries from $\mathbb{F}$. Then,

Crucial.

$\exists!$ invertible matrices $Q_{m \times m}$ & $P_{n \times n}$ s.t. $Q_{m \times m} A_{m \times n} P_{n \times n} = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$

(my claim of uniqueness)

Note: This is nothing but applying row & column operations simultaneously. This is the most reduced form a matrix can become.

This is what is used to create a bijection: $\mathbb{F}^{n \times n} \to$ Basis of $\mathbb{F}^n$.

(similar idea) (invertible)

Exercise. $(A^t)^{-1} = (A^{-1})^t$ or $(A^T)^{-1} = (A^{-1})^T$

$\boxed{\text{Def}^n \quad \text{Rank of } A \overset{?}{:=} \operatorname{Im}(T) = r.}$

Prove $rk(A) = rk(A^T)$.

Proof Using Theorem*, $QAP = m(T) \Rightarrow P^T A^T Q^T = (m(T))^T$

(using the fact, $(AB)^T = B^T A^T$). $\quad P^T A^T Q^T = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]_{n \times m}$

$\leftarrow$ prove this $\longrightarrow$

(Try Lin. Trans.)

$\downarrow$ Elementary matrices $\quad \downarrow$ Elementary matrices

Note: S.T. $E^T \in \{$elementary matrices$\}$.

Note: Given a bases $\mathcal{B}_1$ of V_1 & $\mathcal{B}_2$ of V_2, notice that $\mathcal{B}_1$ & $\mathcal{B}_2$ together determine the lin. transformation i.e. the matrix of lin. transf. is determined!.

of Converse. Given a $\underset{(n)}{T : V} \overset{(\text{or } \mathbb{R}^n \to \mathbb{R}^m)}{\longrightarrow} \underset{(m)}{W}$, $\exists !\, M(T)$ viz. the matrix of the lin. transf.

$M(T) = [a_{ij}]_{m \times n}$,

$$T(x_1, x_2, \ldots, x_n) = M(T) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = [a_{ij}]_{m \times n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$$

$$\Rightarrow T(x_1, \ldots, x_n) = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix} \begin{matrix} \to \ell_1 \\ \vdots \\ \to \ell_m \end{matrix} \Bigg\}$$ linear polynomials. (w.r.t lin. transf.)

So, we get homogeneous polynomials.

Claim (Not Proved yet) Let $\mathcal{B}_1$ be a basis of $\underset{(\dim V = n)}{V}$ defined as, $\mathcal{B}_1 := \{v_1, \ldots, v_n\}$.

Then, $T(\mathcal{B}_1) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ has some v_i s.t. $T(v_i) = 0 \Rightarrow v_i \in \ker(T)$.

let (WLOG), $v_{r+1}, v_{r+2}, \ldots, v_n \in \ker(T)$. Show that, the set $\{T(v_1), \ldots, T(v_r)\}$ is lin. ind. & $\{v_{r+1}, \ldots, v_n\}$ is a basis of $\ker(T)$.

Special case when $r = n$, $\boxed{rk(T_A) = n}$. Note: $\underset{(n)}{T : V} \to \underset{(m)}{W}$.

To associate an $m \times n$ matrix to a lin. transf. $T : \mathbb{F}^n \to \mathbb{F}^m$, we choose bases $\mathcal{B}_1 = (e_1^{(n)}, \ldots, e_n^{(n)})$ of $\mathbb{F}^n$. We choose basis $\mathcal{B}_2 = (e_1^{(m)}, \ldots, e_m^{(m)})$ of $\mathbb{F}^m$.

Where, $e_k^{(\ell)} := \overbrace{(0, 0, \ldots, \underset{k^{th}\text{ entry}}{1}, 0, \ldots, 0)}^{\ell \text{ entries}}$

Let $T(e_j^{(n)}) = \sum_{i=1}^{m} a_{ij} e_i^{(m)}$, $m(T) := [\,\underset{\text{expand into column vectors}}{T(e_1^{(n)})}\ T(e_2^{(n)}) \ldots T(e_n^{(n)})\,]$

$\downarrow$

$\{T(e_1^{(n)}), \ldots, T(e_n^{(n)})\}$

$m(T) = [a_{ij}]_{m \times n}$

Claim. $Im(T) = \text{span}\{T(e_j^{(n)})\}_{1 \le j \le n}$

Note: Replace V by $\mathbb{F}^n$ & W by $\mathbb{F}^m$.

Proof. $Im(T) := \{\omega \in \overset{\mathbb{F}^m}{W} \mid \exists\, v_0 \in \overset{\mathbb{F}^n}{V} \text{ s.t. } T(v_0) = \omega\}$

As $v_0 \in V$ s.t. $T(v_0) = \omega$, $v_0 = \sum_{i=1}^{n} b_{ij} e_i^{(n)}$ (As, $\{e_1^{(n)}, \ldots, e_n^{(n)}\}$ is basis for V)

$\Rightarrow T(v_0) = \sum_{i=1}^{n} b_{ij} T(e_i^{(n)})$ viz. $\boxed{\omega = \sum_{i=1}^{n} b_{ij} T(e_i^{(n)})}$.

Now consider, $v \in V \Rightarrow T(v) \in Im(T)$ (By def^n or...)

As $T(v) \in Im(T) \Rightarrow v = \sum_{i=1}^{n} b'_{ij} e_i^{(n)} \Rightarrow T(v) = \sum_{i=1}^{n} b'_{ij} T(e_i^{(n)}) \Rightarrow$

$Im(T) = \text{span}\{T(e_j^{(n)})\}_{1 \le j \le n}$. ■

Proof (1) $Im(T) := \{\omega \in \mathbb{F}^m \mid \exists\, v_0 \in \mathbb{F}^n \text{ s.t. } T(v_0) = \omega\}$ But, $v_0 = \sum_{i=1}^{n} b_{ij} e_i^{(n)} \Rightarrow T(v_0) = \sum_{i=1}^{n} b_{ij} T(e_i^{(n)})$

$\Rightarrow Im(T) \subseteq \text{span}\{T(e_j^{(n)})\}_{1 \le j \le n}$.

(2) let $v = \sum_{i=1}^{n} b'_{ij} T(e_i^{(n)})$ or $T(e_i^{(n)})$ $\forall\, 1 \le i \le n$, $\boxed{T(e_i^{(n)}) \in Im(T)}$ & as $Im(T)$ is a subspace, $\text{span}(T(e_i^{(n)})) \subseteq Im(T) \Rightarrow \boxed{Im(T) = \text{span}(T(e_i^{(n)}))}$

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(16/10/2

* Recap

(1) Fix a field $\mathbb{F}$

(2) V & W are two v-spaces /$\mathbb{F}$ (eq. vectors spaces over $\mathbb{F}$).

$T: V \to W$ be a lin. Transf

if, $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$, $\forall \alpha, \beta \in \mathbb{F}$, $v_1, v_2 \in V$

(i) Null space of T or kernel of T = $\ker(T) \subset V$

- It is a subspace of V, $\ker(T) := \{v \in V \mid T(v) = 0_W\}$

(ii) Image of T = $\mathrm{Im}(T) \subset W$,

$$\mathrm{Im}(T) := \{\omega \in W \mid \underset{\text{may not be unique}}{\exists\, v_0 \in V} \text{ s.t. } T(v_0) = \omega\}$$

Note

If $T(v) = v$, ∀

then $\ker(T) = \{0$

$\Rightarrow \dim V = \dim($

Notice, $\dim(\mathrm{Im}(T) + \ker(T)$

$= \dim(\mathrm{Im}(T)) + \dim(\ker(T)$

$- \dim(\mathrm{Im}(T) \cap \ker(T)$

Theorem $\dim V = \dim(\mathrm{Im}(T)) + \dim(\ker(T))$

Proof (done last time)

Case 1 If V = W, then does this mean that Im(T) & ker(T) are complement subspaces of each other? i.e. is this $\mathrm{Im}(T) \oplus \ker(T)$?

Note: Let $\ker_{\mathcal{B}} := \{x_1, x_2, \ldots, x_{n-r}\}$ be a basis of ker(T)

(Lin. Trans. $T: V \to V$ (As V = W)), $\dim V = n$.

Extend $\ker_{\mathcal{B}}$ to get basis of $V \Rightarrow \mathcal{B}_1 = \{x_1, \ldots, x_{n-r}, y_{n-r+1}, \ldots, y_n\}$

$T(\mathcal{B}_1) = \{T(y_{n-r+1}), \ldots, T(y_n)\}$, As, $T(x_i) = 0$, $x_i \in \ker(T)$

(We ~~will~~ showed that $T(\mathcal{B}_1)$ is a lin. ind set & span $\mathrm{Im}(T) \Rightarrow$ $T(\mathcal{B}_1)$ is a basis for Im(T))

Viz. ~~$I(\mathcal{B}_1)$~~ $\underbrace{\mathrm{span}(T(\mathcal{B}_1))}_{\mathrm{Im}(T)} \subseteq W = V$. Then, if a lin. combn of elts of $T(\mathcal{B}_1) \in \ker(T)$ (except 0) $\Rightarrow \ker(T) \cap \mathrm{Im}(T) \neq \{0\}$ the minimal req. for (or one) non-complemency.

Note: There's a reason why we take basis from kernel & extend it to basis of V. Let $\mathcal{B}_1$ be a basis of V. $\mathcal{B}_1 := \{v_1, \ldots, v_n)$ $T(\mathcal{B}_1) = (T(v_1), \ldots, T(v_n))$ some of $T(v_i)$ could be zero, Implying, $v_i \in \ker(T)$ & for v_j s.t. $T(v_j) \neq 0$, $v_j \notin \ker(T)$. Hence, Start w/ ker(T) basis & extend it V.

Counter e.g. $T: \mathbb{R}^2 \to \mathbb{R}^2$

$(x, y) \mapsto (y, 0)$. Check that this is a lin. transf

$\ker(T) = X$-axis

$\mathrm{Im}(T) = X$-axis

e.g. $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$

In general, $(x_1, \ldots, x_n) \mapsto (L_1, \ldots, L_m)$, where each L_i is a linear polynomial (w/o constant) in $x_1, x_2, \ldots, x_n$, i.e., each L_i is a homogeneous degree 1 polynomial in $x_1, \ldots, x_n$.

Check that this a lin. Transf

So, if all the L_i's are homogeneous degree 1 polynomials in $x_1, \ldots, x_n$, then T is indeed a lin transformation.

Q Is the converse true? Yes! (It has to be b/c lin. trans are "linear" in the sense, eqn $mx + b$ are not linear!)

A

Theorem: Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a lin. Transf. Let $m(T)$ be the matrix of T w.r.t bases $\mathcal{B}_1$ & $\mathcal{B}_2$ of $\mathbb{F}^n$ & $\mathbb{F}^m$ respectively.

Then, $\dim \operatorname{Im}(T)$ = column rank of $m(T)$ where column rank of an $m\times n$ matrix is the size of the maximal linearly ind. subset of its columns.

Defn: Rank of an $m \times n$ matrix is its column rank = $\dim(\operatorname{Im}(T))$ where,

$$T\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = A_{m\times n}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Figure

$$\begin{tikzcd} A \arrow[r, squiggly] & T_A : \mathbb{F}^n \to \mathbb{F}^m \\ \dim(\operatorname{Im}(T_A)) & \operatorname{Im}(T_A) \subset \mathbb{F}^m \arrow[l, squiggly] \end{tikzcd}$$

equality ← || rank of A

Note

Every dual set of Basis of V & W i.e. $\{\mathcal{B}_1, \mathcal{B}_2\}$ corresponds to a matrix. Converse need not be true. (Work on it.)

Theorem $rk(A) = rk(A^T)$

Proof Case 1. If $A = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]_{m\times n}$, $A^T = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]_{n\times m}$

Case 2. In general, we can choose basis $\mathcal{B}_1$ & $\mathcal{B}_2$ of $\mathbb{F}^n$ & $\mathbb{F}^m$ respectively s.t.

$$m(T_A) = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]_{m\times n} \quad \text{w.r.t } \mathcal{B}_1 \text{ \& } \mathcal{B}_2$$

↳ is written w.r.t. $(e_1^{(n)}, \dots, e_n^{(n)})$ & $(e_1^{(m)}, \dots, e_n^{(m)})$

$A \rightsquigarrow T_A$

$\operatorname{Im}(T_A) \subset \mathbb{F}^m$

$$\underbrace{\omega_1, \omega_2, \dots, \omega_r}_{\text{Basis of } \operatorname{Im}(T_A)}, \omega_{r+1}, \dots, \omega_m$$

→ Basis of dim m.

Further, if $\mathcal{B}_0^{(n)}$ & $\mathcal{B}_0^{(m)}$ denote ? (left).

↳ Basis of dim n

Note

Commutative diagrams are extremely useful in mathematics and mathematicians love it! Look at Artin for fur. reading

Commutative Diagram

$$\begin{tikzcd} \mathbb{F}^n \arrow[r, "A"] \arrow[d, shift right, "P"'] \arrow[r, bend left] & \mathbb{F}^m \arrow[d, shift left, "Q"] \\ \mathbb{F}^n \arrow[r, "A'"] \arrow[u, shift right, "P^{-1}"'] \arrow[ur, phantom, "\circlearrowleft"] & \mathbb{F}^m \arrow[u, shift left, "Q^{-1}"] \end{tikzcd}$$

$A' := \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]_{m\times n}$, for $v \in \mathbb{F}^n$, $Av = Q^{-1}A'Pv$ — $(*)$

$(*)$ is eq. to saying, for $v \in \mathbb{F}^n$,

$$\begin{tikzcd} v \arrow[r] & Pv \in \mathbb{F}^n \arrow[r, "A'"] & A'Pv \in \mathbb{F}^m \arrow[d, "Q^{-1}"] \\ & & Q^{-1}A'Pv \in \mathbb{F}^m \end{tikzcd}$$

$rk(A) = rk(A')$ ∵ Image doesn't change.

$Q^{-1}A'P = A \Rightarrow P^T(A')^T(Q^{-1})^T = A^T$
$\Rightarrow A^T = P^T(A')^T(Q^T)^{-1}$, $rk((A')^T) = rk(A^T) = rk(A')$

A commutative diagram is a diagram ~~wherever~~ whichever path taken doesn't matter i.e. operations from A to B = operations from A to C to B. (Hess's law for energy addition is an e.g.)

$\text{rk } A = \text{rk } A'$. BA'.

Since, $\text{rk } A' = \text{rk}((A')^t)$ we must have

Also, is $\text{rk}(AB) = \text{rk}(A)$.
What are the properties of rk

13/10/23
(16/10/23). **Remark/Recall.** (Note: All v-spaces are finite dimensional).

(1) Let V, W be vector spaces/$\mathbb{F}$

(2) Let $V_0 \subset V$ be a subspace.

(3) Let $T: V \to W$ be a lin. Transf.

Q1. Is $T(V_0) \subset W$, a subspace?

Q2. If so, what can be said about $\dim(T(V_0))$?

Ans. Q1 Yes!

Pf. If $w_0, w_1 \in T(V_0) \Rightarrow \exists\, v_0, v_1 \in V_0$ s.t. $T(v_0) = w_0$, $T(v_1) = w_1$.

$\alpha w_0 = \alpha T(v_0) = T(\alpha v_0)$ & $\beta w_1 = \beta T(v_1) = T(\beta v_1)$

$\Rightarrow \alpha w_0 + \beta w_1 = T(\underset{\underset{V_0}{\cap}}{\alpha v_0 + \beta v_1}) \Rightarrow \alpha w_0 + \beta w_1 \in T(V_0)$. ■

Hence, $T(V_0)$ is a subspace.

Q2. $\dim(T(V_0)) = ?$

~~Guess~~ 1. $0 \le \dim(T(V_0)) \le \min\{\dim(T(V_0)), \dim W\}$ — observation.

Proof. Let $\mathcal{B}_0 = (v_1, \cdots, v_K)$ be a basis of V_0. Then,

- $S := T(\mathcal{B}_0) = \{T(v_1), \cdots, T(v_K)\}$ spans $T(V_0)$.

$\Rightarrow \exists$ lin. ind subset S_1 of S s.t. $\text{span}(S_1) = \text{span}(S)$, $\kappa |S| \geqslant |S_1| = \dim($
(if not inf. card.)

Guess 2. Look at $T: V_0 \to W$, $T/_{V_0}$ or $T|_{V_0}$ i.e. T restricted to V_0. (domain)

Using dimension formula,

$$\dim V_0 = \dim(\ker(T|_{V_0})) + \dim(T(V_0))$$

(from, $\dim V = \dim(\ker(T)) + \dim(\text{Im}(T))$)

$$\Rightarrow \dim(T(V_0)) = -\dim(\ker(T|_{V_0})) + \dim V_0$$

$$~~= -\dim(\ker(T) \cap V_0)~~ = \dim(T(V_1)).$$

How?

Note: $\dim(\ker(T|_{V_0}) + V_0) = \dim(\ker(T|_{V_0})) + \dim V_0 - \dim(\ker(T|_{V_0}) \cap V_0)$

$\Rightarrow \dim V_0 = \dim V_0 + \dim(\ker(T|_{V_0})) - \dim(\ker(T|_{V_0}) \cap V_0)$ ($\dim(U+V) = \dim U + \dim V - \dim(U \cap V)$)

$\Rightarrow \dim(\ker(T|_{V_0}) \cap V_0) = \dim(\ker(T|_{V_0}))$

$\Rightarrow \dim(T(V_0)) = \dim V_0 - \dim(\ker(T|_{V_0}) \cap V_0)$. □

orem let V, W be v-spaces$|\mathbb{F}$. & $T: V \to W$ be a 1-1 map. Then,
$\dim(T(V)) = \dim V$.

Claim We claim that $T: V \to W$ is a 1-1 map iff $\ker(T) = \{0\}$

Proof of Claim ① let $\ker(T) = \{0\}$

To prove, $T(v_1) = T(v_2)$ iff $v_1 = v_2$. (Defⁿ of one-one).

$\Rightarrow T(v_1) - T(v_2) = T(v_1 - v_2)$ [As T is a lin. Transf.]

$\Rightarrow T(v_1 - v_2) = 0_W \Rightarrow v_1 - v_2 \in \ker(T) \Rightarrow v_1 - v_2 = 0_V$

$\Rightarrow \boxed{v_1 = v_2}$ ∎

Ⓘ Assume T is 1-1. & let $x \in \ker(T)$. $\Leftrightarrow T(x) = 0_W = T(0_V)$

← clever part ⟶

$\Rightarrow x = 0_V$ as T is 1-1!

So, $\ker(T) = \{0_V\}$.

Hence, $T: V \to W$ is 1-1 iff $\ker(T) = \{0_V\}$. ∎

T is 1-1, then, $\dim(T(V)) = \dim(\mathrm{Im}(T))$

$$= \dim(V) - \underbrace{\dim(\ker(T))}_{0}$$

$$= \dim V$$

orollary. Let $T: V \to W$ be an isomorphism $W \subset V$ any subspace of V.
Then, $\dim(T(W)) = \dim W$.

back to the Thm in the end of previous class i.e. $\mathrm{rk}(A) = \mathrm{rk}(A^T)$.

T be a lin. transf. from $\mathbb{F}^n \to \mathbb{F}^m$.

A = m(T) be its matrix w.r.t standard bases.

$\mathcal{B}_0 = (e_1^{(n)}, \dots, e_n^{(n)})$

$\mathcal{B}_1 = (e_1^{(m)}, \dots, e_m^{(m)})$

, we choose another bases $\mathcal{C}_0, \mathcal{C}_1$ of $\mathbb{F}^n$ & $\mathbb{F}^m$ respectively s.t. $A' = m(T)$ w.r.t.
$\mathcal{C}_1$ is $\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]_{m\times n}$.

Claim $A' = Q^{-1}AP$

$$\boxed{\begin{aligned} \mathcal{C}_0 &= \mathcal{B}_0 P_{n\times m} \\ \mathcal{C}_1 &= \mathcal{B}_1 P_{m\times n} \end{aligned}}$$

$$\begin{tikzcd}
\mathbb{F}^n \arrow[r, "A'"] \arrow[d, "P"] & \mathbb{F}^m \\
\mathbb{F}^n \arrow[r, "A"] & \mathbb{F}^m \arrow[u, "Q^{-1}"'] \arrow[l, shift left, "A^T"]
\end{tikzcd}$$

ther notion of matrix equality!

matrices are said to be equal as isomorphisms b/w $\mathbb{F}^n \to \mathbb{F}^m$. i.e.,

$[a_{ij}]_{m\times n} X_{n\times 1} = Y_{m\times 1}$, $\forall X_{n\times 1} \in \mathbb{F}^n$ } (iff) Then, $[a_{ij}]_{m\times n} = [b_{ij}]_{m\times n}$.

&

$[b_{ij}]_{m\times n} X'_{n\times 1} = Y'_{m\times 1}$, $\forall X'_{n\times 1} \in \mathbb{F}^n$ } (just take $\begin{bmatrix} x_1 \\ x_n \end{bmatrix} = x$ as e_i's to complete the proof.)

⊗ **Def**

⊗ **Def^n.** Linear Operator is a linear Transformation from V to V.
We always write the matrix of an operator with same basis for domain & Co-domain.

If $\mathcal{B}_1$ & $\mathcal{B}_2$ are 2 bases of V & $T: V \to V$, operator $\boxed{m(T)_{\mathcal{B}_2} = P^{-1} m(T)_{\mathcal{B}_1}}$

where $\mathcal{B}_2 = \mathcal{B}_1 P$

Conjugate of $m(T)_{\mathcal{B}_1}$ w.r.t P.

e.g.