Linear Algebra - II

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Determinant :

Let F be a field. $M_n(F) = \left\{ A_{n\times n} \middle| a_{ij} \in F \ \forall \ \begin{matrix} 1 \leq i \leq n \\ 1 \leq j \leq n \end{matrix} \right\}$.

Thus, $\det : M_n(F) \rightarrow F$, i.e. determinant is a f^n.

Let $A = [a_{ij}]_{n\times n} \in M_n(F)$.

Properties :

$\det([a]) = a$, $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$,

$$\det\Bigg(\underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}}_{A}\Bigg) = \sum_{i=1}^{n} a_{i1} \det(A_{i1}) (-1)^{i+1} \quad (*)$$

where A_{ij} denotes the sub-matrix obtained from A by deleting the i^{th}-row & the j^{th}-column.

$\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \longrightarrow \sum_{i=1}^{n} x_i y_i$$

Let $V_1, \ldots, V_m$ be m vectors ~~spaces~~ over F.

Now, $V_1 \times V_2 \times \ldots \times V_m \rightarrow F$ is called multilinear if it is linear in each coordinate.

i.e. fixing $v_2 \in V_2, \ldots, v_m \in V_m$,

The map $f\big|_{V_1 \times \{v_2\} \ldots \times \{v_m\}} (-, v_2, \ldots, v_m) \longrightarrow F$ has the property that :

$$f((\alpha v_1 + \beta \tilde{v}_1, v_2, \ldots, v_m)) = \alpha f((v_1, v_2, \ldots, v_m)) + \beta f((\tilde{v}_1, v_2, \ldots, v_m))$$

Thus, f is 'linear' in the i^{th} coordinate by fixing remaining $n-1$ coordinate $\neq i$.

Theorem ①: There is a unique f^n δ on $M_n(F)$ s.t. :-

1. $\delta(I) = 1$.
2. δ is linear in rows.
3. If two adjacent rows of A are same, then $\delta(A) = 0$.

We call this $\delta(A)$ as the determinant of A.

Theorem ②: Let $\delta : M_n(F) \to F$ satisfy properties of the theorem above,

1. If A' is obtained by adding a multiple of j^{th} row of A to the i^{th} row of A, $i \neq j$, then $\delta(A') = \delta(A)$.

2. If A' is obtained from interchanging i^{th}, j^{th} row of A, then $\delta(A') = -\delta(A)$.

3. If A' is obtained from A by multiplying i^{th} row by a constant c, then $\delta(A') = +c\,\delta(A)$.

4. If j^{th} row of A is a multiple of i^{th} row of A, then $\delta(A) = 0$.

Rolling Property : Using Thm ①,

$$\delta\begin{bmatrix} R_1 \\ \vdots \\ R_i + R_{i+1} \\ R_{i+1} + R_i \\ \vdots \\ R_n \end{bmatrix} = \delta\begin{bmatrix} R_1 \\ \vdots \\ R_i \\ R_{i+1} \\ \vdots \\ R_n \end{bmatrix} + \delta\begin{bmatrix} R_1 \\ \vdots \\ R_{i+1} \\ R_i \\ \vdots \\ R_n \end{bmatrix} + 0 + 0 = 0$$

Corollary: Let E be an elementary matrix; $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}, \ldots$

Now, $\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = -1$, $\det\left(\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}\right) = 1$, $\det\left(\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}\right) = c$.

Corollary: $\delta(EA) = \delta(E)\delta(A)$ for all $A \in M_n(F)$ & E be an elementary matrix

Proof: ① $\delta(EA)$ when E denotes interchanging of i^{th}, j^{th} rows of A,

$$\Rightarrow \delta(EA) = -\delta(A)$$
$$= \delta(E)\delta(A) \quad (\because \delta(E) = -1)$$

② if E adds cR_j to R_i, $\Rightarrow \delta(EA) = \delta(A) = \delta(E)\delta(A) \quad (\because \delta(E) = 1)$

\# $\delta(A)$ is uniquely determined due to the uniqueness of M_A.

Theorem: $\det(AB) = \det(A)\det(B)$.

Proof: If AB is not invertible, $\Rightarrow \det(AB) = 0$.

If AB is invertible, $\Rightarrow$ both A, B are invertible (Proven earlier)

$$\Rightarrow \text{if } A = E_1^{-1} \ldots\ldots E_k^{-1} \quad (A \text{ is invertible})$$

$$\Rightarrow \det(AB) = \det(E_1^{-1} \ldots E_k^{-1} B) = \det(E_1^{-1} \ldots E_k^{-1}) \det(B)$$
$$= \det(A)\det(B).$$

Theorem: Determinant defined by $(\star)$ satisfies Theorem 1 (Prove it), thereby proving its existence.

$M_n(F) \approx F^{n^2} \to n^2$ entries

$$\underbrace{F^n \times F^n \times \cdots \times F^n}_{n\text{-times}} \to F$$

Exercise: Prove that $\det(A) = \sum_{k=1}^{n} (-1)^{i+k} a_{ki} \det(A_{ki})$ $\quad \forall\ n > k, l$

$$= \sum_{l=1}^{n} (-1)^{j+l} a_{jl} \det(A_{jl})$$

Cramers Rule :

Let $A_{n\times n} X_{n\times 1} = B_{n\times 1}$ be a system of n linear eq^n's in n unknowns.
Assume $\det(A) \neq 0$.

Then, $x_j = \dfrac{\det(A_j)}{\det(A)}$ is the unique set of solutions of this system.

Here, A_j is obtained by replacing column C_j of A by B,

i.e. $A_j = (C_1 C_2 \dots C_{j-1}\ B\ C_{j+1} \dots C_n)$.

Proof: define $D_j : F^n \longrightarrow F$ by $D_j\left(\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}\right) = \det\left(\begin{bmatrix} C_1 \cdots C_{j-1} & \begin{matrix} z_1 \\ \vdots \\ z_n \end{matrix} & C_{j+1} \cdots C_n \end{bmatrix}\right)$

Thus, $D_j(C_k) = \begin{cases} 0 & ; \forall\ k \neq j \\ \det(A) & ; k = j \end{cases}$

Also, $D_j\left(\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}\right) = \sum_{i=1}^{n} C_{ij} z_j \qquad \left(C_{ij} = (-1)^{i+j} \det(A_{ij})\right)$

Now,

$$\left.\begin{aligned} C_{1j}(a_{11} x_1 + \dots + a_{1n} x_n) &= C_{1j}(b_1). \\ &\ \vdots \\ C_{nj}(a_{n1} x_1 + \dots + a_{nn} x_n) &= C_{nj}(b_n) \end{aligned}\right\} \text{Add them up}$$

Thus, we get $\det(A)\, x_j = \det(A_{ij}) \quad \forall\ 1 \leq j \leq n$.

This proves that if $AX = B$ has a sol^n, it would be this.

Def^n: $A_{n\times n} \in M_n(F)$, $Adj(A)_{n\times n}$ st.

ij^{th} entry $= (-1)^{i+j} \det(A_{ji})$.

Lemma: $A.adj(A) = adj(A).A = \det(A) I_{n\times n}$.

Proof:

Cauchy - Binet Theorem : let $A_{n \times M}$, $B_{M \times n}$ be 2 matrices st. $n < M$.

Then, $\det(BA) = 0$ & $\det(AB) = \sum_{\substack{S \subset \{1,2,\dots,M\} \\ \& \ |S| = n}} \det(A_S B_S)$

i.e. $S \in \binom{M}{n}$ choices, $A = \begin{bmatrix} C_1 & \dots & C_M \end{bmatrix}$, $B = \begin{bmatrix} R_1 \\ \vdots \\ R_M \end{bmatrix}$.

Determinant of Partitioned Matrix :

1. $\det\begin{pmatrix} I_{n\times n} & 0 \\ 0 & A_{m\times m}\end{pmatrix} = \det\left(A_{m\times m}\right) = \det\begin{bmatrix} A & 0 \\ 0 & I\end{bmatrix}$.

2. $\det\begin{pmatrix} A & B \\ 0 & D\end{pmatrix} = \det(A)\det(D)$ (A, B, D are all square matrices).

$\left(\because \begin{pmatrix} A & B \\ 0 & D\end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & D\end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I\end{pmatrix}\begin{pmatrix} I & B \\ 0 & I\end{pmatrix}\right)$

3. Assume A is invertible.

$\Rightarrow \begin{bmatrix} A & B \\ C & D\end{bmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I\end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & DA^{-1}B\end{pmatrix}\begin{pmatrix} I & A^{-1}B \\ 0 & I\end{pmatrix}$.

Thus, $\det\left(\begin{bmatrix} A & B \\ C & D\end{bmatrix}\right) = \det A \cdot \det\left(D - CA^{-1}B\right)$.

4. Idempotent $(A^2 = A)$ & Nil-potent $(A^n = 0,\ A^{n-1} \neq 0)$.

Bilinear Maps (Forms) : We stick to $\mathbb{R}$ & $\mathbb{C}$ for now.

Let V be a vector space over $\mathbb{R}$.

A Bilinear form, $f : V \times V \to \mathbb{R}$ is linear in each coordinate.

i.e. $f(\alpha v_1 + \beta v_2, v_3) = \alpha f(v_1, v_3) + \beta f(v_2, v_3)$

$\forall\, \alpha, \beta \in \mathbb{R}$ & $v_1, v_2, v_3 \in V$.

Eg: $V = \mathbb{R}^3$, $F((x_1, x_2, x_3), (y_1, y_2, y_3)) = x_1y_1 + x_2y_2 + x_3y_3$

i.e. standard inner (dot) product.

Eg: Lorentz Form :- $L((x_1, \dots, x_4), (y_1, \dots, y_4)) = x_1y_1 + x_2y_2 + x_3y_3 - x_4y_4$

Q: How many such f^ns?

sol^n: As many as $E_{n\times n}$ (elementary) matrices.

Theorem: Let $A_{n\times n}$ be a real matrix, then :-

$$f((x_1, \dots, x_n), (y_1, \dots, y_n)) = [x_1, \dots, x_n] \, A \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$ is a

bilinear map.

Conversely, for any bilinear map, $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$\exists!$ matrix $A_{n\times n}$ st. $f(x_n, y_n) = x^t A y$.

Proof: Fix $(x_1, \dots, x_n)$, look at $[x_1, \dots, x_n] \, A \begin{bmatrix} \alpha_1 x_1' + \beta_1 y_1 \\ \vdots \\ \alpha_n x_n' + \beta_n y_n \end{bmatrix}$

$$= \alpha [x_1, \dots, x_n] \, A \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} + \beta [x_1, \dots, x_n] \, A \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

similarly, we can fix $\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ & prove for the 2^{nd} coordinate as well.

Conversely, given $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$,

$$LHS = f\left(\sum_{i=1}^{n} x_i e_i , \sum_{i=1}^{n} y_i e_i\right)$$

$$= \sum_{i=1}^{n} x_i f\left(e_i , \sum_{j=1}^{n} y_j e_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i y_j f(e_i, e_j).$$

$$RHS = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} f(e_1, e_1) & f(e_1, e_2) & \cdots & f(e_1, e_n) \\ \vdots & \vdots & \vdots & \vdots \\ f(e_n, e_1) & \cdots & \cdots & f(e_n, e_n) \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

$$= \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \sum_{j=1}^{n} f(e_1, e_j) y_j \\ \vdots \\ \sum_{j=1}^{n} f(e_n, e_j) y_j \end{bmatrix} = LHS.$$

For Uniqueness :- if f, g are bilinear forms, $f \neq g$;

$\Rightarrow f(e_i, e_j) \neq g(e_i, e_j)$ for some i, j.

But $f(e_i, e_j) = ij^{th}$ entry of A.

Hence, proved.

Symmetric Bilinear Form : $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, bilinear form is called symmetric if

$$f(x_n, y_n) = f(y_n, x_n) \ \forall \ x_n, y_n \in \mathbb{R}^n$$

Thus, its matrix is symmetric.

Positive Definite symmetric Form : $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ st.

$$f(x, x) > 0 \quad \forall \ x \neq 0.$$

Q: Prove that $f(0, y) = 0 \ \forall \ y$.

Eg: Inner Product, $f((x_1 \cdots x_n), (y_1 \cdots y_n)) = \sum_{i=1}^{n} x_i y_i$ is in the +ve definite form.

Let V be vector space over $\mathbb{R}$, $\dim(V) = n$.

Thus, Bilinear form on V is a map; $f : V \times V \to \mathbb{R}$ that is linear in each coordinate.

f is symmetric $\Rightarrow f(v, w) = f(w, v) \ \forall \ v, w \in V$.

It is +ve definite if $f(v, v) > 0 \ \forall \ v \neq 0$.

Let $\mathbb{B} = (v_1, \cdots, v_n)$ a basis of V, then any Bilinear map, $f : V \times V \to \mathbb{R}$ is uniquely determined by $f(v_i, v_j)$ by :-

$$f\left(\sum x_i v_i, \sum y_j v_j\right) = \sum \sum x_i y_j f(v_i, v_j).$$

Also, $M_{\mathbb{B}} = \left[f(v_i, v_j) \right]_{n \times n}$ will be a real matrix.

Q: Given Bilinear form $f : V \times V \to \mathbb{R}$, can one find a Basis B for V st. Matrix of f wrt. B is :-

i) identity (Sylvesters Law)

ii) $\begin{bmatrix} I_m & & 0 \\ & -I_n & \\ 0 & & O_k \end{bmatrix}_{m+n+k}$ (Gram Schmidt Law)

For a Bilinear Form, $f : V \times V \to \mathbb{R}$.

Then, if $(v_1, \dots, v_n)$ be a basis,

$$f\left(\sum \alpha_i v_i, \sum \beta_j v_j\right) = \sum \alpha_i \beta_j f(v_i, v_j)$$

i.e. $$f(w_1, w_2) = [w_1] \underbrace{\begin{bmatrix} f(v_i, v_j) \end{bmatrix}}_{\text{Matrix of } f} [w_2].$$

Q: How does Matrix change with Change of Basis ?

sol^n:- Let B' be another basis.

Then, $B' = BP_{n \times n}$.

Let $V = BX$ be vector. ($B \to$ Basis, $X \to$ coordinates) Thus, $V = B'X'$ as well.

$\Rightarrow V = BPX'$.

$\Rightarrow X = PX'$.

Now, $f(v, w) = X^t A_f Y = (X')^t A'_f Y'$

$\Rightarrow X^t A_f Y = (X')^t (P^t A_f P) Y'$ $\left(\begin{matrix}\text{By Putting} \\ X = PX', \\ Y = PY'\end{matrix}\right)$

Thus, $A'_f = P^t A_f P$.

Exercise: For Matrices A, B; if $X^t A Y = X^t B Y \ \forall\ X, Y \in \mathbb{R}^n$, Prove that $A = B$.

Theorem: $f: V \times V \to \mathbb{R}$ represents the usual inner product; wrt some basis B of V iff $\exists$ invertible matrix P st. $A_B = P^t P$.

Theorem: If $\exists$ B of $\mathbb{R}^n$ st. if $v = BX$, $w = BY$;
$\Rightarrow f(v, w) = \sum x_i y_i$ iff matrix of f wrt. standard basis is of the form $P^t P$.

\# If $f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ & $A_f = P^t P$ for some P,

if P is invertible, then $\forall\ v \in \mathbb{R}^n$, $f(v, v) = v^t P^t P v$
$= (Pv)^t P v > 0$.

Theorem: $f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ a bilinear map, then if the matrix of f wrt. some B is $I_{n \times n}$, then matrix of f wrt. standard basis is of the type $P^t P$ where $B = B_0 P$.
If this is the case, f is symmetric +ve definite.

Def^n:- Let $f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bilinear form. Let $v, w \in \mathbb{R}^n$, then if $f(v, w) = 0$, we call them orthogonal to each other wrt. f (standard basis).

Theorem (Sylvester's Law): Given a symmetric bilinear form $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\exists$ a basis $B = (x_1, \dots, x_p, y_1, \dots, y_m, z_1, \dots, z_k)$ of $\mathbb{R}^n$ st.

A_f wrt. B is of the type
$$\begin{bmatrix} I_p & 0 & 0 \\ 0 & -I_m & 0 \\ 0 & 0 & O_k \end{bmatrix}.$$

Further, p, m, k are uniquely associated to f.

Discussion: Non-Degenerate & Degenerate Bilinear Form :-

$f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a symmetric bilinear form,

Let $v \in \mathbb{R}^n$ be st. $f(v, w) = 0 \ \forall \ w \in \mathbb{R}^n$.

Let $N_f = \{ v \in \mathbb{R}^n \mid f(v, w) = 0 \ \forall \ w \in \mathbb{R}^n \}$

Clearly, it is a subspace of $\mathbb{R}^n$.

Let $k = \dim N_f$.

Def^n: A symmetric bilinear form is called Non-degenerate if $\dim(N_f) = 0$.

Otherwise, it is called Degenerate.

Claim: Let B be a basis of $\mathbb{R}^n$ & f be a symmetric bilinear form.
Let A_f^B be the matrix of f wrt. B.

$$N_f = \{ v \in \mathbb{R}^n \mid A_f^B \cdot v = 0 \}.$$

Proof: Let $v \in N_f$. if $A_f(v) \neq 0$, then say i^{th} coordinate $\neq 0$,

$$\Rightarrow e_i^t (A_f v) \neq 0. \quad \Rightarrow f(e_i, v) \neq 0. \quad \Rightarrow\Leftarrow$$

If $v \in$ Nullspace (A_f^B), then $x A_f v = 0 \ \forall \ x \in \mathbb{R}^n$.

$$\Rightarrow N_f \subseteq S_0, \quad S_0 \subseteq N_f. \quad \Rightarrow N_f = S_0.$$

Exercise: Let V be a n dimensional vector space over $\mathbb{R}$, let W_1, W_2 be m_1, m_2 dimensional subspaces of V,

$$\Rightarrow \dim(W_1 \cap W_2) \geq m_1 + m_2 - n.$$

($\Rightarrow k$ gets uniquely determined in Sylvesters theorem)

Lemma: Let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a symmetric non-zero form. Then, $\exists \ v \in \mathbb{R}^n$ st. $f(v, v) \neq 0$.

Proof: if $f(v,v) \neq 0$ for some v, then we are done.
If $f(v,v) = 0 \ \forall \ v$, then let $v, w \in \mathbb{R}^n$,

$$\Rightarrow f(v+w, v+w) = f(v,v) + 2f(v,w) + f(w,w) = 0.$$

$$\Rightarrow f(v,w) = 0 \ \forall \ v, w.$$

$$\Rightarrow\Leftarrow$$

Lemma: $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a symmetric, bilinear, non-zero form, $v \in \mathbb{R}^n$ st. $f(v,v) \neq 0$.
Then, $\langle v \rangle^{\perp} = \{ w \in \mathbb{R}^n \mid f(v,w) = 0 \}$ is an $(n-1)$ dim. subspace.

Proof:- Let $B = \{v, w_1, \ldots, w_{n-1}\}$ be a basis of $\mathbb{R}^n$.

Let $w_1' = w_1 - \dfrac{f(v,w)}{f(v,v)} v$.

Note that $B = \{v, w_1', w_2, \ldots, w_{n-1}\}$ is also a basis

Also, $f(v, w_1') = 0$.

We can repeat till the end.

Bilinear Forms : $f : V \times V \to \mathbb{R}$, V is a vector space over $\mathbb{R}$

f is bilinear if :-

i) $f(\alpha v_1 + \beta v_2, v_3) = \alpha f(v_1, v_3) + B f(v_2, v_3)$

& same for the other coordinate.

Symmetric Bilinear Form: $f(v,w) = f(w,v)$ $\forall v, w \in V$.

Non-Degenerate : $f(v,w) = 0$ $\forall w \in V \iff v = 0$.

f is bilinear in V, let $B = (v_1, \ldots, v_n)$ be a basis of V.

Then, $A = (a_{ij})_{n \times n}$ st. $a_{ij} = f(v_i, v_j)$ respectively, completely determines f.

symmetric Bilinear Form $\mapsto$ symmetric matrices

non-degenerate " " $\mapsto$ Invertible matrices

What if we choose different basis 'B'' $= BP$.

Let A, A' be matrices wrt. B, B'.

Then, $\forall v, w \in V$, let $v = BX = B'X'$

$w = BY = B'Y'$

$\Rightarrow X = PX'$, $Y = PY'$.

$f(v,w) \Rightarrow X^t A Y = (X')^t (P^t A P) Y' = (X')^t (A') Y'$.

$\Rightarrow A' = P^t A P$.

Consider inner product (standard) wrt. standard basis, then wrt to any other basis B is P^tP.

Digression: Consider the Vector space, $f \in \mathcal{C}([0,1])$ = set of all cont. f^n on $[0,1]$

define lin. transformation $g : V \to \mathbb{R}$, st. $g(f(x)) = \int f(x)\,dx$
& leads to interesting analysis of distribution.

Q: $B = \mathcal{E}_1 P$ is standard matrix of f wrt. B is Identity.

Orthogonalisation Process:

given $(e_1, \ldots, e_n)$; find $B(v_1, \ldots, v_n)$ st.

$$f(v_i, v_j) = 0 \quad \forall\, i \neq j \quad \text{i.e. diagonal matrix.}$$

Proof: if $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is zero f^n, matrix of $f = 0_{n\times n}$.

if $f \not\equiv 0$ (not identically equal to 0), then $\exists\ v \in \mathbb{R}^n$ st.

$$f(v, v) \neq 0,$$

because $2f(v,w) = \underbrace{f(v+w, v+w) - f(v,v) - f(w,w)}_{\text{Thus, atleast one of them is non-zero.}} \neq 0$

Let $(x_1, y_2 \cdots y_n)$ be the basis.

Let $x_2, \ldots, x_n$ be defined by $\quad x_i = y_i - \dfrac{f(y_i, x_i)}{f(x_i, x_i)}\, x_1$.

Note that $f(x_1, x_i) = 0 \quad \forall\, i \in \{2, \ldots, n\}$.

Now, consider W = subspace generated by $\{x_2 \ldots x_n\}$.

$$f|_{W\times W}\ \ W \times W \longrightarrow \mathbb{R}.$$

If $f|_{W \times W} \equiv 0$, matrix of f wrt $(x_1 \cdots x_n)$

is $\begin{bmatrix} f(x_1, x_1) & 0 \\ 0 & 0 \end{bmatrix}$ then, we are done.

If $f|_{W \times W} \not\equiv 0$, then $\exists\, y_2 \in W$ st $f(y_2, y_2) \neq 0$

& repeat the process.

This process must end in atmost $(n-2)$ steps & we get a diagonal matrix.

Now, substitute $w_i = \dfrac{z_i}{\sqrt{f(z_i, z_i)}}$ if $f(z_i, z_i) > 0$

or $w_i = \dfrac{z_i}{\sqrt{-f(z_i, z_i)}}$ if $f(z_i, z_i) < 0$.

So, $f(w_i) = 1$ or -1 depending on $f(z_i, z_i)$ being +ve or -ve.

Theorem: If f is a symmetric, Bilinear form on $\mathbb{R}^n$, then $\exists$ a basis $\mathcal{B}$ of $\mathbb{R}^n$ st. Matrix of f wrt $\mathcal{B}$ is of the form

$\begin{bmatrix} I_p & 0 & 0 \\ 0 & -I_m & 0 \\ 0 & 0 & O_k \end{bmatrix}$ with $p + m + k = n$.

Also, the numbers p, m, k are unique for f.

Proof: $k = \dim(\text{Null space}(A))$ if A is the matrix of f wrt B. (any B)

this is because if A is the matrix of f wrt. B,

$$k = n - rk(A)$$

For any basis $A' = P^t A P$ (P invertible),

$$\Rightarrow rk(A') = rk(A). \quad \text{Hence, true.}$$

Let B, B' be 2 bases of $\mathbb{R}^n$ st. Matrix of f wrt B, B' are :-

$$\begin{bmatrix} I_p & 0 & 0 \\ 0 & -I_m & 0 \\ 0 & 0 & O_k \end{bmatrix} \quad \& \quad \begin{bmatrix} I_{p'} & 0 & 0 \\ 0 & -I_{m'} & 0 \\ 0 & 0 & O_k \end{bmatrix}.$$

$B = (v_1 \dots v_n)$ & $B' = (w_1 \dots w_n)$.

Consider the set / ordered-pair $(v_1, \dots, v_p, w_{p'+1}, \dots, w_n)$

Claim : This set is lin. ind.

Take their linear combination ~~This is true as any vector 'x' is denoted by~~

(say) $x = \sum_{i=1}^{p} \alpha_i v_i = -\sum_{j=p'+1}^{n} \beta_j w_j$

take $f(x, x) = \sum_{i=1}^{p} \alpha_i^2 f(v_i, v_i) = \sum_{i=1}^{p} \alpha_i^2$.

Also, $f(x, x) = \sum_{j=p'+1}^{n} \beta_j^2 f(w_j, w_j) = -\sum_{j=p'+1}^{n} \beta_j^2$.

$$\Rightarrow \sum \alpha_i^2 + \sum \beta_j^2 = 0 \quad \Rightarrow \alpha_i = \beta_j = 0 \;\; \forall i, j.$$

Thus, $p + (n - p') \leq n$ i.e. $p \leq p'$.

similarly, $p' + (n - p) \leq n$ i.e. $p' \leq p$. $\quad \Rightarrow p = p'$.

Q: Can we relate m', p' to some subspace of $\mathbb{R}^n$?

sol^n:- Sadly, No.

$\left(\because \right.$ We cannot define a subspace where all elements end up being +ve or -ve (inclusive or) $\left. \right)$

Corollary: If f is +ve symmetric, bilinear form, $\exists$ Basis of $\mathbb{R}^n$ st. Matrix of f wrt. $\mathcal{B}$ is $I_{n\times n}$

OR

$f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a +ve definite symmetric form, then matrix of f wrt. $\mathcal{E}_\ell$ is of the form $P^t P$ for some invertible P.

Note: We used the fact that, if $v_0 \in \mathbb{R}^n$ st. $f(v_0, v_0) \neq 0$, then, $\exists\, W \subset \mathbb{R}^n$ st. $\dim(W) = n-1$ & $w \in W \Leftrightarrow f(v, w) = 0$.

Generalisation : If $W_0 \subset \mathbb{R}^n$ be a subspace st. $f/_{W_0}$ is non-degenerate

① $\exists\, W' \subset \mathbb{R}^n$ st. $\Leftrightarrow f(x, w') = 0 \;\forall\, w \in W_0$

② $\dim(W') = n - \dim(W_0)$.

Idea of Proof: Let $(x_1 \dots x_k)$ be a orthogonal basis of W_0 & $(x_1, \dots, x_k, y_{k+1}, \dots, y_n)$ be a basis of $\mathbb{R}^n$.

Now, define $x_i = y_i - \sum_{j=1}^{k} \dfrac{f(y_i, x_j)}{f(x_j, x_j)} x_j$.

classmate

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Theorem: Let V be a vector space of $\mathbb{R}^n$, f is a +ve definite, symmetric, bilinear form. $W \subset V$ be a subspace.
Then $\exists !$ subspace called orthogonal complement of W, denoted by $W^\perp$ s.t. $V = W \oplus W^\perp$.
In fact, $W^\perp = \{ v \in V \,/\, f(v, w) = 0 \; \forall \, w \in W \}$.

Proof: We need to show that $W \cap W^\perp = \{0\}$ & an $v \in V$ can be written as $v = w_1 + w_2$ ($w_1 \in W$ & $w_2 \in W^\perp$).

1. Let $w_0 \in W \cap W^\perp$ ($w_0 \neq 0$)
$\Leftrightarrow f(w_0, v) = 0 \; \forall \, v \in W$.
$\Rightarrow f(w_0, w_0) = 0 \quad \Rightarrow\Leftarrow$

2. Extend the orthonormal basis of W to basis of V.
i.e. Let $\mathcal{B}_1 = (v_1 \cdots v_k)$ be orthonormal basis of W.

Thus, $\tilde{\mathcal{B}} = (v_1, \ldots, v_k, w_{k+1}, \ldots, w_n)$ be a basis of V.
Now, define :-
$$v_{k+i} = w_{k+i} - \sum_{j=1}^{k} f(w_{k+i}, v_j)\, v_j \quad \forall \, 1 \leq i \leq n-k$$

Note that $\text{span}(\{v_{k+i}, W\}) = \text{span}(\{w_{k+i}, W\})$.

Thus, $\mathcal{B} = (v_1, \ldots, v_k, v_{k+1}, \ldots, v_n)$ is a basis of V & $v_{k+i} \in V^\perp$ $\forall \, 1 \leq i \leq n-k$

Aim: To extend this discussion on Vector spaces over $\mathbb{R}$ to vector spaces over $\mathbb{C}$.

Hermitian Form is the generalisation of Bilinear form $/ \mathbb{R}$.

Def^n: Let V be a vector space $/\mathbb{C}$.

$f: V\times V \to \mathbb{C}$ is called hermitian if :-

$$f(\alpha z_1 + \beta z_2, w) = \bar{\alpha} f(z_1, w) + \bar{\beta} f(z_2, w),$$

$$f(z, r w_1 + \delta w_2) = r f(z, w_1) + \delta f(z, w_2),$$

$$f(v, w) = \overline{f(w, v)} \qquad \forall\, \alpha, \beta, r, \delta \in \mathbb{C}.$$

Eg: $f: \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$

$$(v_1, \dots, v_n), (w_1, \dots, w_n) \longmapsto \sum_{i=1}^{n} \overline{v_i}\, w_i$$

\# For hermitian f, in general, $f(v, w) \in \mathbb{C}$.
But $f(v, v) \in \mathbb{R} \subset \mathbb{C}$ $\forall\, v \in V$.

Def^n: +ve definite hermitian form is an hermitian form st.
$f(v, v) > 0$ $\forall\, v \neq 0$.

Lemma: f is a non-zero Hermitian form st. $\exists\, v \in V$ which gives $f(v, v) \neq 0$.

Proof: If $\underset{c}{\underset{\shortparallel}{f(v, w)}} \neq 0$ in $\mathbb{C}$. $\Rightarrow f(cv, w) = \bar{c} f(v, w) = c\bar{c} = |c|^2$.

Thus, $f(cv, w) \in \mathbb{R} \setminus \{0\}$.

$\Rightarrow f(cv + w, cv + w) = f(cv, cv) + f(w, w) + 2 f(cv, w)$.

Thus, $\exists$ some v_0 st $f(v_0, v_0) \neq 0$.

Theorem: $f : V \times V \to \mathbb{C}$ is a Hermitian Form

Then, f is uniquely defined by an $n \times n$ matrix A where $A = [a_{ij}]_{n\times n}$ st. $a_{ij} = f(v_i, v_j)$ where $\mathcal{B} = (v_1 \ldots v_n)$ is basis of V.

By The Formula, $f((x_1 \ldots x_n), (y_1 \cdots y_n))$

$$\| $$

$$(\bar{x}_1 \ldots \bar{x}_n) A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = X^* A Y.$$

Linear Operators:

Let V be a vector space $(\mathbb{R}, \mathbb{C})$.

$f : V \to V$ be a linear transformation.

Then, f is said to be Linear Operator on V.

Let B be a basis of V. Then, matrix of f wrt. B makes sense.

Theorem: Let f be an Operator on V. Let A be its matrix wrt. B.

(i.e. $v \in V = BX$; then, $f(v) = AX$) If $B' = BP$ is another basis, then :-

$$v = B'X' = BPX' = BX. \quad \Rightarrow X = PX'.$$

If A' is the matrix of f wrt B', then $f(v) = B'A'X' = BAX$

$$= BAPX'$$

$$= BPA'X'$$

$$\Rightarrow A' = P^{-1}AP.$$

Q: Can one find B st. matrix of f wrt. matrix of f wrt. B is Diagonal Matrix?

sol^n:- In general, we cant! Example, rotation by $\theta \neq 0$ in $\mathbb{R}^2$.

i.e. $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

If $(v_1, v_2) = \mathcal{B}$ is a basis of $\mathbb{R}^2$ st. matrix of f wrt. $\mathcal{B}$ is diagonal, then $f(v_1) = \alpha_1 v_1 + \cancel{\alpha_2 v_2}^{\,0}$

Eigen Vectors : Let $f: V \to V$ be an operator on V. $v \in V$ is called an eigenvector for f if :-

① $v \neq 0$

② $f(v) = \alpha v$ for some α.

$T: V \to F$ a lin. transformation st. $T \not\equiv 0 \Rightarrow T$ is onto.

Def^n: $T: V \to V$ a linear transformation. $W \subset V$ a subspace. If $T(W) \subset W$, then W is called T-invariant subspace of V.

Def^n: $T: V \to V$ a linear transformation. A vector $v \neq 0$ is called a characteristic vector (eigen vector) of T if $T(v) = \alpha v$ for some α.

Eg: any element in the Kernel (T) is an eigenvector.

Def^n: $T: V \to V$ a linear transformation, $v \neq 0$ & $v \in V$ st. $T(v) = \alpha v$ for some α, then α is called an eigen value of T.

Q: How can one find eigen vectors / eigen values of T ?

Consequence of existence of an eigen value / vector :

$\exists\ v \neq 0$ & $\alpha \in F$ s.t. $T(v) = \alpha v$.

$\Rightarrow T(v) - \alpha v = 0$.

$\Rightarrow (T - \alpha Id) v = 0 = 0 \cdot v$.

$\Leftrightarrow (\alpha I - A)$ is a singular matrix.

$\Leftrightarrow \det(M) = 0$ where M is the matrix of $\alpha I - T$ w.r.t. some basis $\mathcal{B}$ of V.

Thus, matrix of $\alpha I - T$ (w.r.t. $\mathcal{B}$) is $\alpha I - A$ where $\mathcal{B} = (v_1 \dots v_n)$, A is the matrix of T w.r.t. $\mathcal{B}$.

$\because \alpha v_1 - Tv_1 = \alpha v_1 - \sum_{j=1}^{n} a_{j,1} v_j, \dots\dots$ & so on.

$\Rightarrow \det(\alpha I - A) = 0$ iff α is an eigen value (do it on your own).

$\therefore$, if we look at : $\det\begin{bmatrix} t-a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & t-a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & t-a_{nn} \end{bmatrix} = 0$.

$\Rightarrow$ It is a polynomial of degree 'n' in t with coefficients in $\mathbb{C}$.

Theorem: $T: V \to V$ a linear operator, $\dim(V) = n$, then $\exists$ exactly n eigenvalues (some may be repeated) of T.

Note that if A & A' are 2 matrices of T w.r.t. two bases $\mathcal{B}$ & $\mathcal{B}'$ respectively & if :- $\mathcal{B}' = \mathcal{B}P$

$\Rightarrow A' = P^{-1}AP$.

$\Rightarrow \det(tI - A') = \det(P^{-1}Pt - P^{-1}AP)$

$= \det(P^{-1}(tI - A)P) = \det(tI - A) = 0$

Theorem: $T: V \to V$ be a linear operator ($\dim(V) = n$). Then $\exists$ a Basis of V (say $\mathcal{B}$) w.r.t. that basis, the matrix of T is diagonal iff $\exists$ n linearly independent eigen vectors of T. In fact, $\mathcal{B}$ consists of those eigen vectors.

Proof: $m(T) = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$ then $\left(\begin{bmatrix} d_1 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ d_2 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ d_n \end{bmatrix} \right)$ is a Basis & vice-versa.

$d_1, d_2, \ldots, d_n$

\# $T: V \to V$ be a linear operator, f be a +ve definite Hermitian form on V, $\mathcal{B}$ be the orthonormal basis. Let A be the matrix of T w.r.t. $\mathcal{B}$, $A^* = \overline{A_{ji}}$.

$\Rightarrow T^*: V \to V$ is a linear operator whose matrix w.r.t. $\mathcal{B}$ is A^*.

Notation: $f(v, w) =: \langle v, w \rangle$.

Theorem: For any $v, w \in V$, $\langle T(v), w \rangle = \langle v, T^*(w) \rangle$.

Proof: $v = \sum a_j v_j$, $w = \sum b_k v_k$ (say)

$$\Rightarrow \langle T(v), w \rangle = \left\langle \sum a_j v_j, \sum b_k v_k \right\rangle$$

$$\Rightarrow \langle v, w \rangle = \sum a_j b_k \langle v_j, v_k \rangle.$$

Now, $T(v) = \mathcal{B}\left(A \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \right)$.

Thus, $\langle T(v), w \rangle = \left\langle A\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\rangle$

$$= \left\langle \begin{bmatrix} \sum a_{1j} a_j \\ \vdots \\ \sum a_{nj} a_j \end{bmatrix}, \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \right\rangle$$

$$= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} \overline{a_{kj} a_j} \right) b_k .$$

Also, $\langle v, T^*(w) \rangle = \left\langle \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}, \begin{bmatrix} \sum \bar{a}_{l1} b_l \\ \vdots \\ \sum \bar{a}_{ln} b_l \end{bmatrix} \right\rangle$

$$=$$

\# Hermitian Operator : let V be a hermitian space.
$T : V \to V$ be a linear operator.

Let $\mathcal{B}$ be the basis of V, A be matrix of T wrt. $\mathcal{B}$.
Define T^* to be that linear operator whose matrix wrt $\mathcal{B}$. is A^*.

T^* is called Adjoint of T.

\# Let $\mathcal{B}_1$ be orthonormal basis of V (V Hermitian).

$\Rightarrow \mathcal{B}_1 = \mathcal{B} P.$

$\langle w_i, w_j \rangle = \delta_{ij}$

Lemma: Let $\mathcal{B}$ be orthonormal basis of V. $P_{n\times n}$, an invertible matrix s.t. $\mathcal{B}_1 = \mathcal{B}P$ is orthonormal iff $P^*P = I$.

Def^n: An $n\times n$ complex matrix P is called <u>Unitary</u> iff $P^*P = I$.

Let $\mathcal{B}, \mathcal{B}_1$ be orthonormal basis of V.

$\mathcal{B}_1 = \mathcal{B}P \Leftrightarrow P^*P = I$.

$T: V \to V$ be an operator.

Let A, A_1 be matrix of T wrt $\mathcal{B}, \mathcal{B}_1$.

$\Rightarrow A_1 = P^{-1}AP = P^*AP$.

Let T_1, T_2 be linear transformations whose matrix wrt. $\mathcal{B}, \mathcal{B}_1$ is A^*, A_1^*.

Q: Is $T_1(v_0) = T_2(v_0)\ \forall v_0 \in V$? Yes

Theorem: V be a hermitian space, $T: V \to V$ be an operator, T^* is its adjoint. Then $\langle Tv, w\rangle = \langle v, T^*w\rangle\ \forall v, w \in V$.

Def^n: T is called Hermitian Operator if $T = T^*$.
T is called Normal Operator if $TT^* = T^*T$.

Lemma: T is normal iff $\langle Tv, Tw\rangle = \langle T^*v, T^*w\rangle$.

Lemma: T is Unitary $\Rightarrow T^* = T^{-1}$, $\langle v, w\rangle = \langle Tv, Tw\rangle$.

Lemma: T is hermitian iff $\langle Tv, w\rangle = \langle v, Tw\rangle$.

$(TT^* = T^*T)$

Theorem: V be a Hermitian Space. Let $T: V \to V$ be a Normal Operator. Then $\exists$ an orthonormal basis of V consisting of eigen vectors of T.

$\mathbb{C}^n$ (wrt. standard Hermitian form). A be a normal matrix $(AA^* = A^*A)$, Then $\exists$ a Unitary matrix P st. P^*AP is a diagonal matrix.

Proof: We know that T has an eigen vector $v_1 \neq 0$ in V with eigen value α_1.

(Eigen vector $\Leftrightarrow$ Eigen Value $\Leftrightarrow$ roots of a polynomial over $\mathbb{C}$)

Without Loss of generality, assume $\langle v_1, v_1 \rangle = 1$.

Lemma: Let T be normal on Hermitian Space V. $W \subset V$ subspace st. $T(W) \subset W$. Then $T(W^\perp) \subset W^\perp$, $\dim(W) = 1$.

Proof: Let $w \in W^\perp$. $\Leftrightarrow \langle v, w \rangle = 0 \ \forall v \in W$.

TPT: $T(w) \in W^\perp$.

i.e. $\langle v, T(w) \rangle = 0 \ \forall v \in W$.

$\Rightarrow \langle T^*(v), w \rangle = 0 \ \forall v \in W$.

Q: If v is eigen vector for T, is it eigen vector for T^* if T is normal? i.e. if $T(v) = \lambda v$, $\Rightarrow T^*(v) = \bar{\lambda} v$?

solⁿ:- If $\lambda = 0 \Rightarrow Tv = 0$. $\Leftrightarrow \langle Tv, Tv \rangle = 0$.

$$\langle Tv, Tv \rangle = \langle T^*v, T^*v \rangle$$

$\Rightarrow T^*v = 0$.

If $\lambda \neq 0$, let $S = \lambda I - T$. $\Rightarrow T(v) = \lambda v$ gives $S(v) = 0$.

Thus, $S^* = \bar{\lambda} I - T^*$. $\Rightarrow SS^* = (\lambda I - T)(\bar{\lambda} I - T^*)$

$$= |\lambda|^2 I - \lambda T^* - \bar{\lambda} T + TT^*.$$

Also, $S^* S = |\lambda|^2 I - \lambda T^* - \bar{\lambda} T + T^* T$.

$$\Rightarrow SS^* = S^* S \quad (\because TT^* = T^* T).$$

Thus, S is Normal as well.

$\because S(v) = 0 \cdot v$, we must have $S^*(v) = 0 \cdot v$ as well.

Thus, $T^*(v) = \bar{\lambda} v$.

For $w \in \langle v_1 \rangle^{\perp}$, $\Rightarrow \langle v_1, T(w) \rangle = \langle T^* v_1, w \rangle = \langle \bar{d_1} v_1, w \rangle$

$$= d_1 \langle v_1, w \rangle = 0; \quad \forall w \in \langle v_1 \rangle^{\perp}.$$

$\Rightarrow \langle v_1 \rangle^{\perp}$ is T-invariant.

$\Rightarrow$ Matrix of T wrt. $(v_1, \underbrace{w_2, \ldots, w_n}_{\text{Basis of } \langle v_1 \rangle^{\perp}})$

is of the type $\left[\begin{array}{c|c} d_1 & 0 \\ \hline 0 & A_1 \end{array}\right]$