Cayley Hamilton theorem.

$T: V \to V$ is an operator. $V/\mathbb{C}$ finite dim.

Then T ~~satisfies~~ is a root of $\chi(T)$?

Endomorphism V/F of dim n.

$\text{End}(V) = \{T: V \to V \mid T \text{ is an operator}\}$

$= M_n(F)$ if we choose a basis $\mathcal{B}$.

$\text{End}(V)$
has a natural multiplication. $T_1 \circ T_2 = T_2 \circ T_1$

$T_1 \circ (T_2 + T_3) = T_1 \circ T_2 + T_1 \circ T_3$.

$\chi(T) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$. $a_i \in \mathbb{C}$.

$T: V \to V$, V over $\mathbb{C}$ of dim n.

Look at $T^n + a_{n-1}T^{n-1} + \dots + a_1 T + a_0 Id$

$\det([\lambda \ddots \lambda] - A) = \chi(A)$

Cayley Hamilton theorem says that $\chi(T)\, T \equiv 0$ in $\text{End}(V)$.

$\det(\lambda I - A)$

$$= \det\left(\begin{bmatrix} \lambda - a_{11} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \vdots \\ -a_{n1} & & \lambda - a_{nn} \end{bmatrix}\right).$$

CH Theorem> $V/\mathbb{C}$ v. space of dim. n.

$T: V \to V$ an operator.

$\mathbb{C}[\lambda] \ni P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$

be the characteristic polynomial of T.

Then $P(T) = T^n + a_{n-1}T^{n-1} + \dots + a_1 T + a_0 Id$.

is the zero operator on V.

Pf: Let $P(\lambda) = (\lambda - \alpha_1)(\lambda - \alpha_2) \dots (\lambda - \alpha_n)$.

$\Rightarrow P(T) = (T - \alpha_1 I)(T - \alpha_2 I) \dots (T - \alpha_n I)$.

This is true because

$$(T-\gamma I)(T-\delta I) = (T-\delta I)(T-\gamma I).$$

LHS = ... RHS = ...

Let $\mathcal{B} = (v_1, \dots, v_n)$ be a basis of V s.t.

$m_{\mathcal{B}}(T)$ is upper Δ^r.

Let $T(v_1) = \alpha_1 v_1 \iff (T-\alpha_1 I)(v_1) = 0$ in V.

$$T(v_2) = x_1 v_1 + \alpha_2 v_2$$

$$\Rightarrow (T-\alpha_2 I)(v_2) = x_1 v_1.$$

$$\Rightarrow (T-\alpha_2 I)\circ(T-\alpha_1 I)(v_2)$$

$$= (T-\alpha_1 I)\circ(T-\alpha_2 I)(v_2) = (T-\alpha_1 I)(x_1 v_1)$$

$$= x_1 (T-\alpha_1 I) v_1 = 0.$$

$\therefore$ By induction,

$$(T-\alpha_1 I).(T-\alpha_2 I)\dots(T-\alpha_n I)(v_i) = 0 \quad \forall i.$$

$$\Rightarrow P(T)(v_i) \equiv 0 \quad \forall i$$

$$\Rightarrow P(T) v = 0 \quad \forall v \in V$$

as $v = \sum \beta_i v_i$ (proved).

04/03

V. f.d. v. space over $\mathbb{C}$.

$T: V \longrightarrow V$ lin. operator, $\mathcal{B} = (e_1, \dots, e_n)$

let $A = m_{\mathcal{B}}(T)$.

$\chi(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$

be the characteristic polynomial of T.

CH Theorem $\rightarrow$

$\chi(T)$ is the zero endomorphism on V.

$\chi(T) = T^n + a_{n-1}T^{n-1} + \dots + a_1 T + a_0 : V \rightarrow V$.

given any polynomial $p(\lambda) \in \mathbb{C}[\lambda]$

it can be thought of an operator on V.

Defn: The minimal polynomial of T is the least degree monic polynomial in $\mathbb{C}[\lambda]$ $\ni f(T) \equiv 0$ in $\mathrm{End}(V)$.

Eg - $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\det\begin{bmatrix} t & -1 \\ 0 & t \end{bmatrix} = t^2 =$ char polynomial.

min polynomial $\equiv$ char polynomial.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\det\begin{bmatrix} t-1 & 0 \\ 0 & t-1 \end{bmatrix} = (t-1)^2 =$ char. poly.

min. poly $= t-1 \neq$ char. poly.

$\Rightarrow$ {roots of min poly} $\subseteq$ {roots of char. poly.}

$\therefore$ Min polynomial $\equiv$ char. polynomial.

$\Leftarrow$ all roots of char. poly. are different.

($\because$ Different v_i's in $m_{\mathcal{B}}(T)$ will give different results).

Alternate proof:

$p(\lambda) = \alpha_n \lambda^n + \dots + \alpha_1 \lambda + \alpha_0$

$p(\lambda): V \to V$

$v \to \alpha_n T^n(v) + \dots + \alpha_0 v$

" V becomes a $\mathbb{C}[\lambda]$ module "

Let $V^k = \underbrace{V \times V \times \dots \times V}_{k \text{ times}}$.

given any $p(\lambda) \in \mathbb{C}[\lambda]$

$p(\lambda): V^k \to V^k$

$(v_1, \dots, v_k) \mapsto (p(\lambda)(v_1), \dots, p(\lambda)(v_k))$

$A: V \to V$ be an operator

Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \ddots & \\ \vdots & & \\ a_{n1} & & a_{nn} \end{bmatrix} \Rightarrow A^t = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ a_{12} & \ddots & \vdots \\ \vdots & & a_{nn} \\ a_{1n} & & \end{bmatrix}$

$$B = \begin{bmatrix} \lambda - a_{11} & \cdots & -a_{n1} \\ -a_{12} & \lambda - a_{22} & \vdots \\ \vdots & & \lambda - a_{nn} \\ -a_{12} & & \end{bmatrix} = \lambda I - A^t.$$

$B: V^n \to V^n$ an operator.

$(v_1, \dots, v_n) \longrightarrow B \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$$B \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} \lambda - a_{11} & \cdots & -a_{n1} \\ -a_{12} & \ddots & \vdots \\ -a_{1n} & \cdots & \lambda - a_{nn} \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix}$$

$$= \begin{bmatrix} (\lambda - a_{11}) e_1 + (-a_{21}) e_2 + \dots + (-a_{n1}) e_n \\ \vdots \end{bmatrix}$$

Switching with T in ~~as~~ place of λ

$$= \begin{bmatrix} T(e_1) - (a_{11} e_1 + a_{21} e_2 + \dots + a_{n1} e_n) \\ (-a_{12} e_1) + T(e_2) + (-a_{22} e_2) + \dots + (-a_{n2} e_n) \\ \vdots \end{bmatrix}$$

(Nakayema Lemma)

$= 0.$

But $\quad Adj(M) \cdot M = M \cdot Adj(M) = \det(M) \cdot I$.

$$\Rightarrow \quad (Adj(B) \cdot B) \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} = 0.$$

$$\Rightarrow \begin{bmatrix} \chi(A^t) & & 0 \\ & \chi(A^t) & \\ 0 & & \ddots \chi(A^t) \end{bmatrix} \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

$$\Rightarrow \chi(A^t) = 0.$$

$B = \lambda I - A^t \quad \Rightarrow \quad \det(B) = \chi(A^t)$.

$$\chi(A^t) = \det(\lambda I - A^t) = \det(\lambda I - A) = \chi(A).$$

$$\Rightarrow \chi(A) \equiv 0.$$

V. vector space over F.

$T: V \to V$. $\quad\quad B = (e_1, \ldots, e_n)$ basis of V.

$A = m_B(T)$.

$(M_k)_{k \in \mathbb{N}} \longrightarrow M_0$ iff $m_k^{ij} \longrightarrow m_0^{ij} \quad \forall\, 1 \leq i,j \leq n$.

$P_0, P_k \in \mathbb{C}[\lambda] \qquad P_k \to P_0 = ?$

<u>Proposition</u>> Let $P_k \in \mathbb{C}[\lambda]$ be a sequence of polynomials of degree 'n'. Let $P_0 \in \mathbb{C}[\lambda]$.

Then $P_k \to P_0$ iff we can renumber the roots of P_k say $\alpha_1^{(k)}, \ldots, \alpha_n^{(k)}$ such that

$\alpha_i^{(k)} \longrightarrow \alpha_i$ - th root of P_0.

Let $\alpha_1^{(k)}, \ldots, \alpha_n^{(k)}$ be roots of P_k and $\alpha_1, \ldots, \alpha_n$ are roots of P_0.

<u>Proof</u>: ① (If) $\alpha_i^{(k)} \longrightarrow \alpha_i$ then $P_k \to P_0$.

Since coeff. are "elementary symmetric functions" of roots.

$(a_n) \to a_0$ $\quad|\quad$ $(a_n b_n) \to ab$

$(b_n) \to b_0$ $\quad|\quad$ $(a_n + b_n) \to a+b$.

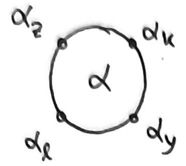
② Conversely, assume $P_k \to P_0$.

amongst roots of P_k

let $\alpha_1^{(k)}$ be (a) root closest to α_1

$$|\alpha_1^{(k)} - \alpha_1| < |\alpha_j^{(k)} - \alpha_j|$$

$$\Rightarrow |\alpha_1^{(k)} - \alpha_1|^n \leq |\alpha_1 - \alpha_1^{(k)}|\,|\alpha_1 - \alpha_2^{(k)}| \ldots$$
$$\ldots |\alpha_1 - \alpha_n^{(k)}|$$



$$\begin{tikzcd} |P_k(\alpha_1)| \arrow[r] \arrow[dr] & 0 \\ & |P_0(\alpha_1)| \arrow[u] \end{tikzcd}$$

Write,

$$P_k(\lambda) = (\lambda - \alpha_1^{(k)})\, \boxed{q_k(\lambda)}$$
$$\downarrow \qquad\qquad \downarrow$$
$$P_0(\lambda) = (\lambda - \alpha_1)\, \boxed{q_0(\lambda)} \quad \text{deg } (n-1)$$

Use induction on degree of p, q

Proposition) $M_{n\times n}$ any matrix

then $\exists$ a sequence M_k of matrices of order n such that

1) $M_k \longrightarrow M_0$.

2) M_k has n distinct eigen values.

$$A_n = \begin{bmatrix} 1/n & 1 \\ 0 & 1/n^2 \end{bmatrix}$$

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Proof: $\exists$ a matrix P_0 s.t.

$P_0 M_0 P_0^{-1}$ = Upper Δ^r with eigen values of A on diagonal.

$$\det(\lambda I - P_0 M_0 P_0^{-1})$$
$$= \det(P_0(\lambda I - M_0)P_0^{-1})$$
$$= \det(\lambda I - M_0)$$

~~Proposition~~ ~~$M_{n\times n}$ any matrix~~

~~then~~

$$P_0^{-1}\begin{bmatrix} a_{11}+\frac{1}{n} & \cancel{\times} & \cdots & \cancel{\times} \\ 0 & a_{22}+\frac{1}{n^2} & & \vdots \\ \vdots & & \ddots & \times \\ 0 & \cdots & & a_{nn}+\frac{1}{n^n}\end{bmatrix} P_0 \longrightarrow P_0^{-1}(P_0 M P_0^{-1}) P_0 .$$

$$= M^0.$$

Add terms to diagonal in such a way so that all the elements are distinct. Since its upper Δ^r, the diagonal forms the eigen values (distinct).

<u>Lemma</u> $M_k \to M_0$ seq. of matrices

P an invertible matrix.

~~Pf.~~ $\sum m_{ij}^k P^{-1} M_k P$ is convergent and converges to $M_0 P$.

<u>Proposition</u> C-H Hamiton theorem is true for diagonal matrices.

<u>Proof</u>: If $A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$

$$\det(\lambda I - A) = \delta\begin{bmatrix} \lambda - d_1 & & \\ & \ddots & \\ & & \lambda - d_n \end{bmatrix} = \prod(\lambda - d_i).$$

Now, $(\lambda - d_i)(e_i) = 0 \Rightarrow \prod$ is a zero map.

CH <u>Thm</u>

<u>Proof</u> 3) $M_k \longrightarrow M_0 \Rightarrow \chi(M_k) \longrightarrow \underbrace{\chi(M_0)}_{\text{has to be } 0 \text{ endomorphism}}$

M_k: diagonal matrices

M_0: might not be diagonal

$\chi(M_k)$: 0 endomorphism

Hadamard Inequality

$[m_{ij}] = M_{n\times n}$ complex matrix

$$\Delta(M) = \sum_{\sigma \in \text{Perm}\{1,2,..,n\}} \text{sign}(\sigma) \prod_{i=1}^{n} m_{i\sigma(i)}$$

$\rightarrow$ Show that these satisfy axioms of dets.

$$\delta \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = m_{11}m_{22} - m_{12}m_{21}.$$

$$\{1,2,\ldots,n\} \xrightarrow[\text{bijection}]{\sigma} \{1,2,\ldots,n\}$$

σ: a permutation.

Determining $\text{sign}(\sigma)$:

Take $1, \sigma(1), \sigma(\sigma(1)), \ldots, \sigma^k(1)$

Let k_1 be the smallest $\ni \sigma^{k_1}(1) = 1$.

Then $2, \sigma(2), \ldots$

Let k_2 " " $\ni \sigma^{k_2}(2) = 2$.

$\vdots$

$\sigma^{k_n}(n) = n$.

$\Rightarrow \text{sign}(\sigma) = (-1)$ [Min. no. of switches required]

For $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$, $\sigma = \underset{\downarrow}{(1,2)}$ or $\underset{\downarrow}{(2,1)} \Rightarrow m_{11}m_{22} - m_{12}m_{21}$

$\qquad 1 \qquad -1$

Corollary> If each entry $|m_{ij}| \leq N$

$$\Rightarrow |\delta(M)| \leq n!\,(N)^n.$$

* Thm] Let $M_{n\times n}$ be a matrix, with columns $v_1, \ldots, v_n$.

Then $|\delta(M)| \leq \prod_{i=1}^{n} \|v_i\|$.

Further equality holds $\Leftrightarrow (v_1, \ldots, v_n)$ orthogonal

Proof : (I) $\delta(M) = 0$, trivially proved.

(II) $\delta(M) \neq 0 \Rightarrow M$ is Invertible.

Let $\omega_i = \dfrac{v_i}{\|v_i\|}$ & $N = [\omega_1 \dots \omega_n]$

If $|\delta(N)| \leq 1 \Rightarrow |\delta(M)| = |\delta([\|v_1\| \omega_1 \dots \|v_n\| \omega_n])|$

$= \|v_1\| \dots \|v_n\| \, |\delta(N)| \leq \prod_{i=1}^{n} \|v_i\|$

if $|\delta(N)| \leq 1$

* We need to prove that $|\delta(N)| \leq 1$ if columns of N are unit vectors and equality holds $\Leftrightarrow$

$\Leftrightarrow$ columns are orthogonal, i.e. column form orthogonal basis of $\mathbb{C}^n$.

Notice from the formula of $\delta(M)$ that $\overline{\delta(M)} = \delta(\bar{M})$

Let $P = N^* N \Rightarrow \delta(P) = \delta(N^*) \delta(N)$

$= \delta(\bar{N}) \delta(N) = \overline{\delta(N)} \delta(N) = |\delta(N)|^2$.

& $P^* = N^* N = P$ (Hermitian)

$\therefore$ P is +ve definite hermitian form

Let $\lambda_1, \dots, \lambda_n$ be eigen values of P.

$\exists$ unitary $Q \ni QPQ^* = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

by Spectral theorem.

~~$\delta(QPQ) = \delta(Q$~~

$\delta(QPQ^*) = \delta(Q) \delta(P) \delta(Q^*) = \delta(P) = \prod_{i=1}^{n} \lambda_i$

$\text{Trace}(P) = \sum_{i=1}^{n} \lambda_i$

Claim : $\sum_{i=1}^{n} \lambda_i = n$,

Proof : $\text{Tr}(P) = \sum_{i=1}^{n} p_{ii} = \sum_{i=1}^{n} (nn^*)_{ii} = \sum_{i=1}^{n} \|n\|^2$

$= \sum_{i=1}^{n} 1 = n$

normalized

Now, $AM \geq GM$

$$\Rightarrow \frac{\sum_{i=1}^{n} \lambda_i}{n} \geq \left(\prod_{i=1}^{n} \lambda_i\right)^{1/n}$$

$$\Rightarrow 1 \geq \delta(P)^{1/n}$$

$$\Rightarrow |\delta(N)|^2 \leq 1 \Rightarrow |\delta(N)| \leq 1.$$

$AM = GM \Leftrightarrow \lambda_i = \lambda_j \;\forall\; 1 \leq i,j \leq n$

$\Leftrightarrow \lambda_i = 1 \;\forall\; 1 \leq i \leq n$ ~~(because normalized)~~

$(\because \sum \lambda_i = n \Rightarrow n\lambda_i = n \Rightarrow \lambda_i = 1)$

$\Leftrightarrow QPQ^* = Id.$

$\Leftrightarrow P = Id \Leftrightarrow N^*N = Id.$

$\Leftrightarrow N^* = N^{-1} \Leftrightarrow N$ is unitary.

$\Leftrightarrow (w_1, \ldots, w_n)$ is orthonormal basis of $\mathbb{C}^n$.

$(v_1, \ldots, v_n)$ is orthogonal basis of $\mathbb{C}^n$.

Thm: P has real eigen values (or $\lambda_i \in \mathbb{R} \;\forall i$)

$P(v) = \lambda v \Rightarrow \langle P(v), v\rangle = \langle \lambda v, v\rangle = \bar{\lambda}\langle v, v\rangle$

$$\begin{aligned} &\quad\;\, \| \\ &\langle v, P^*(v)\rangle = \langle v, P(v)\rangle = \langle v, \lambda v\rangle \\ &\qquad\qquad = \lambda \langle v, v\rangle. \end{aligned}$$

Corollary: If $|m_{ij}| < 1 \;\forall\; 1 \leq i,j \leq n$, $\|v_i\| \leq \sqrt{n}$.

$$\Rightarrow |\delta(M)| \leq n^{n/2}.$$

Proof: $P = N^*N \Rightarrow |\delta(P)| = |\delta(N)|^2 \leq \prod_{i=1}^{n} \|v_i\|^2$

$$\Rightarrow |\delta(P)| \leq \prod_{i=1}^{n} P_{ii}$$

Unsolved Problem: Does $\exists$ a Hadamard matrix of order $4k$ $\forall k \geq 1$?

* **Defn.:** A Hadamard matrix is a $n \times n$ matrix whose entries are ± 1 and $|\delta(H)| = n^{n/2}$

Open Q: For which n, do Hadamard matrixes exist?

Eg - $[1]$, $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$ $H_n \rightarrow$ to get H_{2n} $\forall n$.
Sylvester construction (Wall's notation).

Proof: Can't exist for odd $n > 1$.

We are dealing with the matrix, where ~~et~~ multiplication with ± 1 doesn't matter

$\Rightarrow$ rearrangement of row / column doesn't matter.

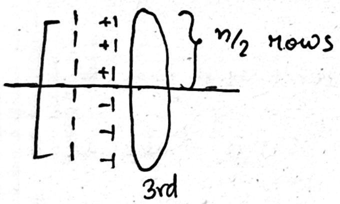
WLOG assume that 1st column is all 1.

2nd column must contain equal number of -1 and $+1$ $\because \langle c_1, c_2 \rangle = 0$.

$\Rightarrow n$ must be even.

Claim: $n = 4k$

After odd rearrangement,



Let $\exists\, k$ '-1's in first $n/2$ rows

for $\langle c_2, c_3 \rangle = 0$,

$$\{H_{i2} = H_{i3}\} \equiv \{H_{j2} = -H_{j3}\}$$

$$\Rightarrow |\{H_{i2} \neq H_{i3}\}| = \frac{n}{2}.$$

$\therefore$ $\exists$ $\frac{n}{2} - k$ '$+1$'s in second $n/2$ rows

Similarly, $\exists\ \frac{n}{2} - k$ '$+1$'s in first $n/2$ rows
and k '-1's in ~~first~~ second $n/2$ rows

But $|\{\text{'}+1\text{'s}\}| \equiv |\{\text{'}-1\text{'s}\}|$ $[\because \langle c_3, c_1 \rangle = 0]$

$\longrightarrow$ $\Rightarrow 2k = 2\left(\frac{n}{2} - k\right) \Rightarrow \boxed{n = 4k}$.

Singular Value Decomposition

11/03

Statement: Let $M_{m\times n}$ be a complex matrix. The singular value decomposition is the factorisation of M as:

$$M_{m\times n} = U_{m\times m} \Sigma_{m\times n} V^*_{n\times n}$$

U: rotates back on std. basis + scales back

Σ: rescales

V^*: rotates on eigenvectors.

where U, V are unitary ($U^* = U^{-1}$, $V^* = V^{-1}$) and Σ is diagonal (rectangular).

Proof:

Let $P_{n\times n} = M^*_{n\times m} M_{m\times n}$.

$$P^* = (M^*M)^* = M^*(M^*)^* = M^*M = P$$

$\Rightarrow P$ is Hermitian

Also, $X^*PX = X^*M^*MX = (MX)^*(MX)$

$= \langle MX, MX\rangle \geq 0$.

Hence P +ve semi definite Hermitian form.

$\exists$ o.n. basis (unitary) $\mathcal{B}$ of $\mathbb{C}^n$. [spectral thm].

$$\ni \mathcal{B}P\mathcal{B}^* = \begin{bmatrix} D_4 & 0 \\ 0 & 0 \end{bmatrix}$$

D is diagonal $d_{ij} > 0$. $\hookrightarrow$ eigen values of P.

Let $\mathcal{B} = [\mathcal{B}_1 \ \ \mathcal{B}_2]$

$\mathcal{B}_2$: basis of $\ker(P)$.

If $w \in \ker(P) \Rightarrow P(w) = 0 \Rightarrow M^*M(w) = 0$.

$\Rightarrow \langle M^*M(w), w\rangle = 0 \Rightarrow \langle M(w), M(w)\rangle = 0$.

$\Rightarrow M(w) = 0 \Rightarrow w \in \ker(M)$.

If $w \in \ker(M) \Rightarrow M(w) = 0 \Rightarrow M^*M(w) = 0$.

$\Rightarrow P(w) = 0 \Rightarrow w \in \ker(P)$.

$\therefore\ \ker(P) = \ker(M) \Rightarrow \text{rank}(P) = \text{rank}(M) = r.$

$\Rightarrow r \leq \min(m,n).$

$B_1 = [c_1 \dots c_r]_{n\times r}$ $\quad Im(P)$

Let

$U_{1\,m\times r} = M_{m\times n}\, \underbrace{B_{1\,n\times r}}_{\text{for rotating back}}\, D^{-1/2}_{r\times r} \rightarrow$ for scaling back, where $D^{-1/2}_{r\times r} = [d_{ij}^{-1/2}]_{r\times r}$

<u>Claim</u>: $U_{1\,m\times r} D^{1/2}_{r\times r} B^*_{1\,r\times n} = M_{m\times n}.$

$U_1 D^{1/2} B_1^* = M B_1 D^{-1/2} D^{1/2} V_1^* = M \underbrace{B_1 B_1^*}_{} \overset{?}{=} M.$

Since columns of B_1 are o.n. by choice $B_1 B_1^* = I$.

As $B_1^* = B_1^{-1}$

B_1 and B_1^* rotates the vector with equal 'θ' in opposite directions.

$U_1 = [w_1 \dots w_r] = M B_1 D^{-1/2}$

$B_1 = [c_1 \dots c_r].$

$\Rightarrow B_1 D^{-1/2} = [c_1 \dots c_r]_{n\times r} \left[D^{-1/2} \right]_{r\times r}$

$= \left[\frac{c_1}{\sqrt{d_{11}}} \cdots \frac{c_r}{\sqrt{d_{rr}}} \right]$

$\left\langle \frac{Mc_i}{\sqrt{d_{ii}}}, \frac{Mc_i}{\sqrt{d_{ii}}} \right\rangle = \frac{1}{|d_{ii}|} \langle Mc_i, Mc_i \rangle$

$= \frac{1}{|d_{ii}|} \langle M^*M c_i, c_i \rangle = \frac{1}{|d_{ii}|} \langle Pc_i, c_i \rangle$

$\hookrightarrow$ eigen value d_{ii} vector c_i, basis B o.n.

$= \frac{1}{|d_{ii}|} \langle d_{ii} c_i, c_i \rangle$

$= \langle c_i, c_i \rangle \frac{d_{ii}}{|d_{ii}|} = 1.$

Similarly,

$$\left\langle \frac{Mc_i}{\sqrt{d_{ii}}}, \frac{Mc_j}{\sqrt{d_{jj}}} \right\rangle = 0 \quad \forall\ i \neq j.$$

$\Rightarrow [w_1 \dots w_r]$ is also o.n.

Let $[U_{1\,m\times r}\ U_{2\,m\times(m-r)}] = U_{m\times m}$ be an o.n. basis of $\mathbb{C}^n$

Define $\Sigma = \left[\begin{matrix} \begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix}_{n\times n} \\ 0 \end{matrix}\right]_{m\times n}$ or $\begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix}_{m\times n}$

$\hookrightarrow m\times n$ $\quad$ $m<n$.

$[\because r \leq \min(n,m)]$

$$[U_1\ U_2]_{m\times m}\, \Sigma_{m\times n}\, [B_1\ B_2]^*_{n\times n}$$

$$= [U_1\ U_2]_{m\times m} \begin{bmatrix} \sqrt{D}\, B_1^* \\ 0 \end{bmatrix}_{m\times n}.$$

$$= U_1 \sqrt{D}\, B^* = M$$

~~Exercise~~

Exercises

1) M is Hermitian, SVD = Spectral Thm Decomposition.

$$VMV^* D \Rightarrow M = V^{-1} D (V^*)^{-1} = \underset{U}{\underset{\parallel}{V^{-1}}} D \underset{U^*}{\underset{\parallel}{V}}.$$

2) $rk(M) = 1. \Rightarrow$ Say G be 1 dim column span of M

$= y,\ \|y\| = 1.$

Soln: Let $x \in \mathbb{R}^n \ni \|x\| = 1$ and $M(x) = y$.

$\langle x, \underbrace{k_1, \dots, k_{n-1}}_{\ker(M)} \rangle$ $\quad$ $rk(M) = rk(P) = 1$

$\hookrightarrow$ only 1 nonzero eigen value.

Scale it so $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{m\times n}.$

~~3~~ 1st column of V → eigen vector of P.

$U\Sigma V^T = M$.

3) Find SVD of $M = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$

Soln.:

$P = M^* M = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}$.

$P \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 45 \\ 45 \end{bmatrix}, \quad d_{11} = 45.$

$P \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}, \quad d_{22} = 5.$

$V_1 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \qquad \langle (1,1), (1,-1) \rangle = 0. \qquad V_1' = \frac{V_1}{\sqrt{2}}$

$U_1 = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{45} & 0 \\ 0 & 1/\sqrt{5} \end{bmatrix}$

$= \begin{bmatrix} 1/\sqrt{5} & 3/\sqrt{5} \\ -1/\sqrt{45} & 9/\sqrt{5} \end{bmatrix}$

$\left\langle \left(\frac{1}{\sqrt{5}}, \frac{-1}{\sqrt{45}}\right), \left(\frac{3}{\sqrt{5}}, \frac{9}{\sqrt{5}}\right) \right\rangle = \frac{3}{5} - \frac{9}{3 \cdot 5} = 0.$

$\therefore \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -1/3\sqrt{10} & 9/\sqrt{10} \end{bmatrix} \begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{45} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

(M) $\qquad$ (U_1') $\qquad$ (D) $\qquad$ (V_1^*)

14/03

A is of order n

$\chi(A) = \delta(\lambda I - A)$

[Monic polynomial with degree 'n' in variable 'λ' whose roots are eigen values of A]

If A is a complex matrix.

$\chi(A) = (\lambda - \alpha_1)^{a_1}(\lambda - \alpha_2)^{a_2} \cdots (\lambda - \alpha_k)^{a_k}$

where $a_1 + a_2 + \dots + a_k = n$.

<u>Defn</u>: a_i is the multiplicity of α_i (algebraic)

In general, if α_i occurs with multiplicity a_i in $\chi(A)$ it doesn't mean $\exists\, a_i$ linearly independent eigen vectors of A with eigen values α_i.

<u>Eg</u>: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\chi(A) = \delta\begin{pmatrix} t & -1 \\ 0 & t \end{pmatrix} = t^2$

$\hookrightarrow (y,0) = \lambda(x,y)$.

$\Rightarrow y = 0, \quad x \cdot 0 = y = 0$

$\therefore (x,0)$ are eigen vectors.

→ (1,0) is an eigen vector (Only 1)

But no. of repeated roots $= 2 \neq 1$.

⇒ <u>Defn</u>: If $\exists\, n_i$ lin. ind. eigen vectors of eigen value α_i, then n_i is called the geometric multiplicity of α_i.

Always, Geo. multiplicity $\leq$ algebraic multiplicity.

Thm]. Equality holds $\Leftrightarrow$ A is diagonalisable.

If $A_{m\times n}, B_{n\times m}$ be two matrices with $\mathbb{R}$ entries.

Then $\chi(AB_{m\times m})$ has degree m.

$\chi(BA_{n\times n})$ has degree n.

Thm $\rightarrow$ $\lambda^n(\chi(AB)) = \lambda^m(\chi(BA))$

Note : Not true for minimal polynomial.

Eg: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $AB = A \rightarrow t^2$, $BA = [0] \rightarrow t$.

Proof :

$$X = \begin{bmatrix} AB_{m\times m} & O_{m\times n} \\ B_{n\times m} & O_{n\times n} \end{bmatrix}_{(m+n)\times(m+n)}$$

and
$$Y = \begin{bmatrix} O_{n\times n} & O_{m\times n} \\ B_{n\times m} & BA_{m\times m} \end{bmatrix}_{(m+n)\times(m+n)}.$$

Let
$$C = \begin{bmatrix} I_m & -A_{m\times n} \\ O_{n\times m} & I_n \end{bmatrix}$$

$$CX = \begin{bmatrix} O_m & O_{m\times n} \\ B_{n\times m} & O_{n\times n} \end{bmatrix} = YC.$$

$\Rightarrow CX = YC$.

$\delta(C) = 1 \Rightarrow$ Invertible.

$\therefore CXC^{-1} = Y \Rightarrow \delta(\lambda I - CXC^{-1}) = \delta(C(\lambda I - X)C^{-1})$

$= \delta(\lambda I - X) = \delta(\lambda I - Y)$

$\Rightarrow \chi(Y) = \chi(X)$.

$$\Rightarrow \delta\begin{bmatrix} \lambda I - AB & 0 \\ -B & \lambda I_n - O_n \end{bmatrix} = \delta\begin{bmatrix} \lambda I_m - O_m & 0 \\ -B & \lambda I - BA \end{bmatrix}$$

$\Rightarrow \delta(\lambda I - AB)\,\delta(\lambda I - O_n) = \delta(\lambda I - O_m)\,\delta(\lambda I - BA)$

$\Rightarrow \lambda^n \delta(\lambda I - AB) = \lambda^m \delta(\lambda I - BA)$

$\Rightarrow \lambda^n \chi(AB) = \lambda^m \chi(BA)$

Simultaneous Orthogonal diagonalisation of Commutative Symmetric Matrices.

$\Leftrightarrow$ ~~Simultaneous~~ finding orthogonal basis of eigen vectors for $A \neq B$.

Let A, B be symmetric.

$AB = BA$ & $V_\lambda = \{ v \in V \mid A(v) = \lambda v \}$.

Then $BV_\lambda \subset V_\lambda$

$V \xrightarrow{B} V$ (basis of eigenvectors)

Now, B is diagonalisable.

Does this mean $B|_W$ is diagonalisable too?

Where $W \subset V$ and $B(W) \subset W$.

$$\begin{array}{l} V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \dots \oplus V_{\lambda_n} \\ \cup \\ W \end{array}$$

(proof later).

Let's assume $B|_W$ is diagonalisable

In that case, if $W = V_{\lambda_i}$, basis consists of eigen vectors of B.

$\because$ A is diagonalisable,

$\mathbb{R}^n = V_{\lambda_1} \oplus \dots \oplus V_{\lambda_k}$

where V_{λ_i} are eigen v. spaces for A.

$\therefore$ V_{λ_i} consists of eigen vectors for both A, B.

Repeat this $\forall i$ and since distinct eigen values eigen vectors are lin. ind.

$\Rightarrow$ We get an o.n. basis of V that consists of eigen vectors of both A, B.

Thm] Let A, M be real symmetric matrices of order n.
M is +ve definite.

Then $\exists$ invertible C $\ni$ $C^t M C = Id$

$C^{-1} A C$ = Diagonal.

$$\boxed{e_i^t M e_j = \delta_{ij}}$$

<u>Proof</u>: $M \to$ +ve definite symmetric

$\Rightarrow$ o.n. basis of M, i.e., $\exists R \ni M = R^T R$.

Now, $(R^{-1})^t A R^{-1}$ symmetric

$\to$ base change of A $\to$

By spectral,

$\exists B_{n\times n}$ o.n. matrix $\to$ unitary

$$\ni B^{-1} (R^{-1})^t A R^{-1} B = \text{diagonal}.$$

let $C = R^{-1} B$. $\Rightarrow C^T = B^T (R^{-1})^T = B^{-1} (R^T)^{-1}$

$$C^T M C = B^{-1} (R^T)^{-1} (R^T R) R^{-1} B = Id.$$

$B^T = B^{-1}$ ($\because$ real)

$$C^T A C = B^{-1} (R^{-1})^T A R^{-1} B = \text{diagonal}.$$