Limit along a continuous variable:

Let $h : [a, \infty) \to \mathbb{R}$ be a function.

Defⁿ :- ① The function h is said to converge to $d \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists $T_\varepsilon \in [a, \infty)$ s.t.

$$|h(t) - d| < \varepsilon \quad \text{for } t \geq T_\varepsilon \qquad ①$$

This is denoted by $\lim_{t \to \infty} h(t) = d$.

② If $\lim_{t\to\infty} h(t) = d$, for some $d \in \mathbb{R}$, then h is said to be convergent as $t \to \infty$. otherwise h is said to be divergent.

③ If for every $M \in \mathbb{R}$, there exists $T_M \in [a, \infty)$ s.t. $h(t) \geq M$, $\forall\, t \geq T_M$.
Then h is said to be diverging to $+\infty$.
It is denoted by

$$\lim_{t \to \infty} h(t) = +\infty.$$

④ If for every $M \in \mathbb{R}$, there exist $T_M \in [a, \infty)$ s.t. $h(t) \leq M$, $\forall\, t \geq T_M$.
Then h is said to be diverging to $-\infty$.
This is denoted by

$$\lim_{t \to \infty} h(t) = -\infty$$

Proposition :- Let $h : [a, \infty) \to \mathbb{R}$ be a function and $d \in \mathbb{R}$. Then $\lim_{t\to\infty} h(t) = d$, iff for every sequence $\{t_n\}_{n \geq 1}$ in $[a, \infty)$ diverging to $+\infty$, $\lim_{n\to\infty} h(t_n) =$

i.e. The following are equivalent.

① $\lim_{t \to \infty} h(t) = d$

② for every sequence $\{t_n\}$ in $[a, \infty)$ diverging to ∞, $\lim_{n \to \infty} h(t_n) = d$.

③ for every increasing sequence $\{t_n\}_{n \geq 1}$ in $[a, \infty)$, diverging to $+\infty$, $\lim_{n \to \infty} h(t_n) = d$.

<u>proof</u> $1 \Rightarrow 2$.

Let $\{t_n\}_{n \geq 1}$ be a sequence in $[a, \infty)$, diverging to $+\infty$.

As $\lim_{t \to \infty} h(t) = d$, for every $\varepsilon > 0$, $\exists\ T \in [a, \infty)$ s.t. $|h(t) - d| < \varepsilon$ for $t \geq T$.

As $\{t_n\}$ diverges to ∞, $\exists\ N \in \mathbb{N}$ s.t. $t_n > T$ for $n \geq N$

So for $n \geq N$, we have $t_n > T$ & hence $|h(t_n) - d| < \varepsilon$.

$\Rightarrow h(t_n) \longrightarrow d$. i.e. $\{h(t_n)\}_{n \geq 1}$ converges to d.

$2 \Rightarrow 3$ Trivial.

$3 \Rightarrow 1$ Suppose h doesn't converge to d as $t \to \infty$, we will arive at a contradiction.

Then there exist some $\varepsilon > 0$, s.t. ① fails for all $T \geq a$.

In particular, $\forall\ n \in \mathbb{N}$ considering ~~a+n~~ $t_{n-1} + 1$

~~there exist $t_n > a + n$, s.t. $|h(t_n) - d| \geq \varepsilon$.~~

Take $t_0 = a$

there exist $t_n > t_{n-1} + 1$

$|h(t_n) - d| \geq \varepsilon$. $\forall\ n \geq 1$ — ✡

Then $\lim_{n\to\infty} t_n = +\infty$, By ③, $\lim_{n\to\infty} h(t_n) = d$

This contradicts ☆

Thm:- Let $h : [a,\infty) \longrightarrow \mathbb{R}$ be a function.
Then the following are equivalent (TFAE)

1) h is convergent as $t \to \infty$.

2) for every sequence $\{t_n\}$ in $[a,\infty)$ with $\lim_{n\to\infty} t_n = \infty$, $\{h(t_n)\}_{n\geq 1}$ is convergent.

3) for every sequence $\{t_n\}$ in $[a,\infty)$ with $\lim_{n\to\infty} t_n = \infty$, $\{h(t_n)\}_{n\geq 1}$ is cauchy.

4) for every $\varepsilon > 0$, $\exists\, T \in [a,\infty)$ s.t.
$|h(t) - h(s)| < \varepsilon$. for $t, s \geq T$. — ☆☆

Proof $1 \Rightarrow 2$ done by the previous preposition

$2 \Rightarrow 3$ Clear as convergent sequences are Cauchy

$3 \Rightarrow 4$ Suppose this doesn't hold

There exists $\varepsilon > 0$, s.t. (☆☆) doesn't hold for any T

[Complete the proof] by yourself.

$4 \Rightarrow 1$ Consider any $\{t_n\}$ in $[a,\infty)$ s.t. $t_n \to \infty$. Then by (4), $\{h(t_n)\}_{n\geq 1}$ is cauchy.

So $\lim_{n\to\infty} h(t_n) = d$, for some $d \in \mathbb{R}$

from ④ $\exists\, T$ s.t.
$|h(t) - h(s)| < \varepsilon$ for $t, s \geq T$ — ⓐ

~~Choose n_0 s.t $t_{n_0} > T$. Then for n~~

There exists M s.t $|h(t_n) - d| < \varepsilon$ for $n \geq M$.
— ⓑ

Choose $n_0 > M$ s.t. $t_{n_0} > T$. $\Rightarrow a+b$.

Take $s = t_{n_0}$

$|h(t) - h(t_{n_0})| < \varepsilon$ & $|h(t_{n_0}) - d| < \varepsilon$

$\Rightarrow |h(t) - d| = |h(t) - h(t_{n_0})| + |h(t_{n_0}) - d|$

$< \varepsilon + \varepsilon$

$= 2\varepsilon$.

- If $\lim_{t \to \infty} h(t) = d$, Then h is said to be convergent.

- h is convergent iff it is Cauchy as $t \to \infty$, i.e. for $\varepsilon > 0$, $\exists$ $T \in [a, \infty)$ s.t $|h(t) - h(s)| < \varepsilon$ $\forall$ $s, t \geq T$.

<u>Thm</u>:- Let $h : [a, \infty) \longrightarrow \mathbb{R}$ be an increasing function. Then h is convergent as $t \to \infty$ iff h is bounded above, i.e, $\exists M \in \mathbb{R}$ s.t. $h(t) \leq M$ $\forall$ t.

In such a case $\lim_{t \to \infty} h(t) = \sup\{h(t) : t \in [a, \infty)\}$

<u>Proof</u>:- Suppose h is bounded above.

Take $d = \sup\{h(t) : t \in [a, \infty)\}$

for any $\varepsilon > 0$, as d is the supremum

$\exists$ t_0 s.t

$$d - \varepsilon \leq h(t_0) \leq d.$$

As h is increasing

$\forall$ $t \geq t_0$

$$d \geq h(t) \geq h(t_0) \geq d - \varepsilon.$$

$\Rightarrow$ $|h(t) - d| < \varepsilon$

$\therefore \lim_{t \to \infty} h(t) = d$.

$\Longrightarrow$ is an exercise.

Let $h: [a, b) \to \mathbb{R}$ be a function.

By defination, $\lim_{t \to b^-} h(t) = d$ if

for every $\varepsilon > 0$, $\exists\, T \in [a, b)$ s.t.

$|h(t) - d| < \varepsilon \quad \forall\, t \in [T, b)$.

If $\lim_{t \to b^-} h(t) = d$ for some $d \in \mathbb{R}$, then h is said to ~~...~~ be convergent as $t \to b^-$.

<u>Thm:-</u> h is convergent as $t \to b^-$ iff for ~~some~~ every $\varepsilon > 0$ there exists $T \in [a, b)$ s.t.

$|h(t) - h(s)| < \varepsilon \quad \forall\, s, t \in [T, b)$.

Proof is an exercise.

<u>Thm:-</u> Let $h: [a, b) \to \mathbb{R}$ be an increasing function. Then $\lim_{t \to b^-} h(t)$ exists iff h is bounded above and in such a case,

$\lim_{t \to b^-} h(t) = \sup\{h(t) : t \in [a, b)\}$.

proof is an exercise.

~~h is convergent~~ Let $f: [a, \infty) \to \mathbb{R}$ be function s.t. for every $t \in [a, \infty)$, $f|_{[a, t]}$ is bounded & Riemann integrable. Define $h: [a, \infty) \to \mathbb{R}$ by

$$h(t) = \int_a^t f(x)\, dx.$$

Then f is said to be Riemann integrable on $[a, \infty)$ iff h is convergent as $t \to \infty$.

In such a case

$$\int_a^\infty f(x)\, dx := \lim_{t \to \infty} h(t).$$

(We may say that $\int_a^\infty f(x)\, dx$ is convergent).

The function f is said to be absolutely Riemann integrable, if $|f|$ is integrable on $[a, \infty)$.

i.e. $\lim_{t \to \infty} \int_a^t |f(x)| dx$ exist

Thm :- Let $f : [a, \infty) \to \mathbb{R}$ be absolutely Riemann integrable on $[a, \infty)$. ~~Then f is Riemann integrable on $[a, \infty)$. Converse is not true.~~

Assume that f is Riemann integrable on $[a, t]$ for every $t \geq 0$. Then f is Riemann integrable. Converse is not true.

Proof :- Define $m : [a, \infty) \to \mathbb{R}$ by $m(t) = \int_a^t |f(x)| dx$

We have $\lim_{t \to \infty} m(t)$ exists, So, $m(t)$ is Cauchy as $t \to \infty$, for $\varepsilon > 0$ $\exists$ $T \in [a, \infty)$ s.t.

$\left| \int_s^t |f(x)| dx \right| < \varepsilon$ for $s, t \geq T$.

$\left| \int_a^t f(x) dx - \int_a^s f(x) dx \right| = |m(t) - m(s)|$

Define $h : [a, \infty) \to \mathbb{R}$ by $h(t) = \int_a^t f(x) dx$.

Now for $s, t \geq T$.

$$|h(t) - h(s)| = \left| \int_s^t f(x) dx \right|$$

$$\leq \int_s^t |f(x)| dx$$

$$= \left| \int_s^t |f(x)| dx \right| < \varepsilon.$$

Example for converse being not true.

Define $f : [1, \infty) \to \mathbb{R}$ by.

$$f(t) = \begin{cases} 1 & 1 \leq x < 2 \\ -1/2 & 2 \leq x < 3 \\ +1/3 & 3 \leq x < 4 \\ -1/4 & 4 \leq x < 5 \\ \vdots & \\ (-1)^{n+1} 1/n & n \leq x < n+1 \end{cases}$$

$\int_1^\infty f(x) dx = 1 - \frac{1}{2} + \frac{1}{3} - \dots$

Show that f is R.I on but is not absolutely R.I.

Metric Spaces

Def^n Let X be a non-empty set.
A function $d : X \times X \to \mathbb{R}$ is said to be a metric on X if the following properties are satisfied

i) $d(x, y) \geq 0 \quad \forall\, x, y \in X$ (positivity)

ii) $d(x, y) = 0$ iff $x = y$ (Definiteness)

iii) $d(y, x) = d(x, y) \quad \forall\, x, y \in X$ (symmetry)

iv) $d(x, y) \leq d(x, z) + d(z, y) \quad \forall\, x, y, z \in X$
{Triangle inequality}

example (Euclidean space) $X = \mathbb{R}^n$

$x = (x_1, x_2, \ldots, x_n)$

$y = (y_1, y_2, \ldots, y_n)$.

$$d_2(x, y) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \ldots + (y_n - x_n)^2}$$

{show that this satisfy triangle inequality}
(Use Cauchy-Schwarz inequality

example $X = M_n(\mathbb{R}) = \{ A = [a_{ij}]_{1 \leq i, j \leq n} : a_{ij} \in \mathbb{R} \}$

$$d(A, B) = \left[\sum_{i=1}^{n} (b_{ij} - a_{ij})^2 \right]^{1/2} \quad A = [a_{ij}], \; B = [b_{ij}].$$

example X - nonempty set.

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Usual metric $|x - y|$

This is known as discrete metric

- Every non empty set can be made a metric space.
- On a same set, we can have several different metric.

example $X = \mathbb{R}^n$ $\quad x_1 = (x_1, \ldots, x_n)$
$y_1 = (y_1, \ldots, y_n)$

$$d_1(x, y) = \sum_{j=1}^{n} |y_j - x_j|.$$

① $d_1(x, y) \geq 0$

② $d_1(x, y) = 0 \Rightarrow \sum_{j=1}^{n} |y_j - x_j| = 0 \Rightarrow y_j = x_j \ \forall j.$

~~if $y_j = x_j$ $\forall j$ $\Rightarrow d$~~ , Check other as exercise.
it is a metric space.

$$d_p(x, y) = \left[\sum_{i=1}^{n} |x_i - y_i|^p\right]^{1/p}.$$

example $X = \mathbb{R}^n$ $\quad x = (x_1, \ldots, x_n)$
$y = (y_1, \ldots, y_n)$

$$d_\infty(x, y) = \text{Max}\{|y_j - x_j| : 1 \leq j \leq n\}$$

Proof that this is a metric space.

Defn. Let (X, d) be a metric space, Let $\{x_n\}_{n\geq 1}$ be a sequence in X. Let $a \in X$:
Then $\{x_n\}_{n\geq 1}$ is said to converge to a, if $\{d(x_n, a)\}_{n\geq 1}$ converges to zero.

Uniqueness of limit :- Let (X, d) be a metric space
Suppose a seqn $\{x_n\}_{n\geq 1}$ in X converges to $a \in X$. & to $b \in X$. Then $a = b$.

Proof :- for $\varepsilon > 0$, Choose $N_1 \in \mathbb{N}$ s.t.

$$d(x_n, a) < \varepsilon \quad \text{for } n \geq N_1$$

Choose $N_2 \in \mathbb{N}$ s.t.

$$d(x_n, b) < \varepsilon \quad \text{for } n \geq N_2$$

Take ~~$N = N_1$~~ $m = \max\{N_1, N_2\}$

Then $d(x_m, a) < \varepsilon$ & $d(x_m, b) < \varepsilon$.

We get

$$\begin{aligned} d(a,b) &\leq d(a, x_m) + d(x_m, b) \quad \{\Delta \text{ ineq} \\ &= d(x_m, a) + d(x_m, b) \quad \{\text{symm.} \\ &< \varepsilon + \varepsilon \\ &= 2\varepsilon. \end{aligned}$$

$$0 \leq d(a,b) \leq 2\varepsilon \quad \forall \varepsilon > 0 \Rightarrow d(a,b) = 0$$
$$\Rightarrow a = b \quad \{\text{definiteness}$$

Thm Let (X, d) be a metric space. Suppose $\{x_n\}_{n\geq 1}$ is a sequence in X converging to $a \in X$. Then every subsequence of $\{x_n\}_{n\geq 1}$ converges to a.

Proof :- Suppose $\{x_{n_k}\}_{k\geq 1}$ is a subsequence of $\{x_n\}_{n\geq 1}$

As $\{x_n\}$ converges to a, $\{d(x_n, a)\}_{n\geq 1}$ ~~Then $\{x_n$~~ converges to 0.

Hence $\{d(x_{n_k}, a)\}$ converges to 0 $\{$by real no convergence

$\Rightarrow \{x_{n_k}\}_{k\geq 1}$ converges to a.

Thm :- Let (X,d) be a metric space. Suppose Y is a non empty subset of X. Then (Y,k) is a metric space where

$$k = d|_{Y\times Y}$$

i.e. $k(x,y) = d(x,y) \;\forall\; x,y \in Y$.

Proof :- Trivial (as a ~~work~~ exercise)

Def^n A metric space (X,d) is said to be complete if every cauchy sequence in X is convergent.

Thm :- Let (X,d) be a metric space. Suppose $\{x_n\}_{n\geq 1}$ is a sequence in X converging to $a \in X$. Then $\{x_n\}_{n\geq 1}$ is Cauchy, i.e., for $\varepsilon>0$ there exists $N \in \mathbb{N}$ s.t.

$$d(x_m, x_n) < \varepsilon \quad \forall\; m,n \geq N.$$

Proof :- As $\{x_n\}_{n\geq 1}$ converges to a.

$\{d(x_n, a)\}_{n\geq 1}$ converges to 0.

i.e. ~~for~~ $\varepsilon>0$ $\exists\, N \in \mathbb{N}$ s.t. for $n \geq N$,

$$d(x_n, a) < \varepsilon/2$$

Then $m, n \geq N$, $d(x_m, x_n) \leq d(x_m, a) + d(a, x_n)$

$$< \varepsilon/2 + \varepsilon/2$$

$$= \varepsilon.$$

eg. $[0,1]$ with usual metric is complete.

$\mathbb{Q}$ with " " is not complete.

$\mathbb{R}\setminus\mathbb{Q}$ with " " is not complete.

$\mathbb{Z}$ with " " is complete.

Consider discrete metric space (X, d_0)

$$d_0(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Suppose $\{x_n\}_{n \geq 1}$ in X is cauchy.

Take $\varepsilon = 1/2$. Then $\exists$ N s.t.

$$d(x_m, x_n) < 1/2 \text{ for } m, n \geq N.$$

As d_0 is discrete metric,

$$d_0(x_m, x_n) < \tfrac{1}{2} \Rightarrow d_0(x_m, x_n) = 0$$

$$\Rightarrow x_m = x_n \text{ for } m, n \geq N.$$

eventually constant

Take $a = x_N$. Then $\{x_n\}_{n \geq 1}$ converges to a.

$\Rightarrow$ Every discrete metric space is complete.

<u>Thm</u> :- Every discrete metric space is complete.

<u>Open balls</u>

Let (X, d) be a metric space. Then for $x \in X$ and $r > 0$, the open ball of radius r, centered at x is defined as

$$B_r(x) = \{y \in X : d(y, x) < r\}.$$

example $X = \mathbb{R}$: usual metric.

fix $x \in \mathbb{R}$, & take $r > 0$.

Then $B_r(x) = \{y \in \mathbb{R} : |y - x| < r\}$.

$$= (x - r, x + r)$$

$X = \mathbb{R}^2$: usual metric $x = (x_1, x_2)$

$$B_r(x) = \{(y_1, y_2) : d((y_1, y_2), (x_1, x_2)) < r$$

$$= \{ (y_1, y_2) : \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} < r \}$$

eg. $X = \mathbb{R}^2 \quad d_1(x, y) = |y_1 - x_1| + |y_2 - x_2|$

$x = (x_1, x_2)$

$y = (y_1 - y_2)$

$$B_1(0) = \{ (y_1, y_2) : |y_1| + |y_2| < 1 \} \qquad r = 1$$

eg. $(X, d_0) \qquad d_0 = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

Take $x \in X$

$B_r(x) = \{x\}$ if $0 < r \leq 1$.

$B_r(x) = X$ if $1 < r < \infty$

<u>Remark</u> :- A sequence $\{x_n\}_{n \geq 1}$ in (X, d) converges to $a \in X$, iff for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $x_n \in B_\varepsilon(a)$ for all $n \geq N$.

$d(x_n, a) < \varepsilon \iff x_n \in B_\varepsilon(a)$.

<u>Open set</u>

<u>def^n</u> Let A be a subset of a metric space (X, d). Then A is said to be open if for any $x \in A$, there exist $\varepsilon > 0$ such that

$$B_\varepsilon(x) \subseteq A.$$

eg. $(0, 1)$ is open in $\mathbb{R}$.

$[0, 1]$, $\{\phi\}$ is not open in $\mathbb{R}$.

singleton

eg. $\{ z : |z| < \frac{1}{2} \}$ is open in $\mathbb{C}$.

$z \in \mathbb{C}$.

Thm :- Let (X, d) is a metric space

Then

① ϕ & X are open

② Arbitrary union of open sets is open

③ finite intersections of open set is open

Proof :- ① For $x \in X$, we can take any $\varepsilon > 0$.

Then by def^n $B_\varepsilon(x) \subseteq X$.

So X is open

ϕ is open as there doesn't exists any x in ϕ. The condition is vacuously satisfied.

③ Suppose $A \subseteq X$ & $B \subseteq X$ are open. We want to show

$A \cup B$ is open.

Consider $x \in A \cup B$.

Suppose $x \in A$. Then as A is open, $\exists$ $\varepsilon > 0$ s.t.

$B_\varepsilon(x) \subseteq A$.

So, $B_\varepsilon(x) \subseteq A \cup B$.

Similarly if $x \in B$. $\exists$ $\varepsilon' > 0$. s.t

$B_{\varepsilon'}(x) \subseteq B$

So, $B_{\varepsilon'}(x) \subseteq A \cup B$.

So, we are done.

~~By induction, our claim is~~

Suppose I is a non-empty set & A_i is an open subset of X for $i \in I$.

(I is called indexing set).

Consider $B = \bigcup_{i \in I} A_i$

$x \in B$ iff $x \in A_i$ for some $i \in I$

Take $x \in B$. Then $x \in A_i$ for some $i \in I$.

Since A_i is open

$\exists\ \varepsilon > 0$ s.t.

$B_\varepsilon(x) \subseteq A_i$

$\Rightarrow B_\varepsilon(x) \subseteq B$

So, B is open.

(3) Suppose $n \in \mathbb{N}$ & $A_1, A_2, \ldots, A_n$ are open

Take $C = \bigcap_{j=1}^{n} A_j$

Consider $x \in C$, so, $x \in A_j$ $\forall j = \{1, 2, \ldots n\}$

As A_j is open, there exist $\varepsilon_j > 0$ s.t.

$B_{\varepsilon_j}(x) \subseteq A_j$.

Take $\varepsilon = \min\{\varepsilon_j \mid 1 \le j \le n\} > 0$.

Then $B_\varepsilon(x) \subseteq B_{\varepsilon_j}(x)$

$\subseteq A_j \quad \forall j$

$\Rightarrow B_\varepsilon(x) \subseteq \bigcap A_j$

Thus $\bigcap_{j=1}^{n} A_j$ is open.

Example :- Take $X = \mathbb{R}$

$A_j = (-1/j, 1/j)$.

$\bigcap_{j=1}^{\infty} A_j = \{0\} \longrightarrow$ not open

$B_j = [0, 2 + \frac{1}{j})$.

$\bigcap_{j=1}^{\infty} B_j = [0, 2]$ is not open.

Defⁿ A subset B of X is said to be closed if B^c is open.

Thm :- Let (X, d) be a metric space. Then

1. ϕ & X are closed.
2. Arbitrary intersection of closed set is closed.
3. Finite union of closed set is closed.

Proof is an exercise.

Defⁿ :- Let (X, d), (Y, k) be metric space. Let $f : X \to Y$ be a function. Suppose $a \in X$. Then f is said to be continuous at a, if for every $\varepsilon > 0$, $\exists$ $\delta > 0$ s.t. ~~$x \in B_\delta(a)$~~ ~~implies~~ $f(x) \in \underbrace{B_\varepsilon(f(a))}_{\subseteq Y}$ $\forall x \in \underbrace{B_\delta(a)}_{\subseteq X}$.

Thm A subset A of a metric space (X, d) is open iff it is a union of open ball.
~~If A is non-empt~~

Proof :- Let $A \subseteq X$ be open. If $A = \phi$, then it is an empty union of open balls.
If A is non empty, for $x \in A$, choose $\varepsilon_x >$ s.t. $B_{\varepsilon_x} \subseteq A$. This can be done as A is open. Now

$$A = \bigcup_{x \in A} B_{\varepsilon_x}(x)$$

Any $x \in A$, is in $B_{\varepsilon_x}(x)$. Hence

$$A \subseteq \bigcup_{x \in A} B_{\varepsilon_x}(x)$$

Also $B_{\varepsilon_x}(x) \subseteq A$, $\forall x$

Hence $\bigcup_{x \in A} B_{\varepsilon_x}(x) \subseteq A$.

Conversely, if A is a union of open balls then it is open, as every open ball is open (exercise), and abritrary union of open set are open.

<u>Thm</u>:- Let $\{x_n\}_{n \geq 1}$ be a sequence in a metric space (X, d), converging to a point $a \in X$. Suppose A is an open set containing a.

Then there exist $N \in \mathbb{N}$ s.t. $x_n \in A$ for all $n \geq N$.

<u>Proof</u>:- We have $a \in A$ and A is open.

So, there exist $\varepsilon > 0$ such that $B_\varepsilon(a) \subseteq A$.

As $\lim_{n \to \infty} x_n = a$, we have $\lim_{n \to \infty} d(x_n, a) = 0$.

So, there exist $N \in \mathbb{N}$ s.t.

$$d(x_n, a) < \varepsilon \text{ for } n \geq N.$$

$$\Rightarrow x_n \in B_\varepsilon(a)$$

$$\& \; B_\varepsilon(a) \subseteq A$$

$$\Rightarrow x_n \in A, \quad \text{for all } n \geq N.$$

<u>Thm</u>:- Let $B \subseteq X$, be a non-empty closed subset of a metric space (X, d). Suppose $\{y_n\}_{n \geq 1}$ is a sequence of elements in B converging to a point $b \in X$.

Then $b \in B$.

<u>Proof</u>:- Take $A = B^c$. Suppose $b \in A$

A is open. Then by previous thm, there exist $N \in \mathbb{N}$ s.t. $y_n \in A$ for $n \geq N$.

This is a contradiction, as we have assumed $y_n \in B = A^c$, $\forall n$.

Def. (X, d), (Y, k) be metric space.

Let $f: X \longrightarrow Y$ be a function.

Fix $a \in X$.

Then f is said to be continuous to a if for every $\varepsilon > 0$, there exist $\delta > 0$ s.t. $f(x) \in \underset{\subseteq Y}{B_\varepsilon(f(a))}$ for all $x \in \underset{\subseteq X}{B_\delta(a)}$

or $d(x, a) < \delta \Rightarrow k(f(x), f(a)) < \varepsilon$

Def:- If f is continuous at every $a \in X$, then it is said to be a continuous function.

Thm:- Let (X, d), (Y, k) be metric spaces. Let $f: X \to Y$ be a function. Then f is continuous iff $f^{-1}(C)$ is open for every open subset C of Y.

$$f^{-1}(C) = \{a \in X : f(a) \in C\}$$

Proof:- We have to proof that f is continuous iff inverse images of open sets are open.

Suppose f is continuous & $C \subseteq Y$ is open. If $C = \phi$ then $f^{-1}(C) = f^{-1}(\phi) = \phi$ which is open. So consider $C \neq \phi$.

We want to show that $A = f^{-1}(C)$ is open.

Take $a \in A$. Then as $A = f^{-1}(C)$, $f(a) \in C$.

As C is open, there exist $\varepsilon > 0$ s.t.

$B_\varepsilon(f(a)) \subseteq C$.

Then there exists $\delta > 0$, s.t. $x \in B_\delta(a)$

$\Rightarrow f(x) \in B_\varepsilon f(a) \subseteq C$.

$x \in B_\delta(a) \Rightarrow f(x) \in B_\varepsilon(f(a)) \subseteq C$.

i.e. $x \in B_\delta(a) \Rightarrow f(x) \in C$.

$x \in B_\delta(a) \Rightarrow x \in f^{-1}(U)$

$\Rightarrow$ in other words $B_\delta(a) \subseteq f^{-1}(U) = A$.

for $a \in A$, there exist $\delta > 0$ s.t.

$B_\delta(a) \subseteq A$

So, A is open.

Converse is an exercise

→ **Pointwise Convergence**

Def:- Let X be a non-empty set and let $\{f_n\}_{n\geq 1}$ be a sequence of real valued function on X. Let $f: X \to \mathbb{R}$ be a function. Then $\{f_n\}_{n\geq 1}$ is said to converge to f pointwise if $\{f_n(x)\}_{n\geq 1}$ converges to $f(x)$ $\forall\, x \in X$.

example-1 $X = \{1, 2\}$

$$f_n(j) = \begin{cases} 1 + \frac{1}{n} & \text{if } j = 1 \\ 1/n^2 & \text{if } j = 2 \end{cases}$$

Define f by $f(1) = 1$ & $f(2) = 0$.

Then $\{f_n\}_{n\geq 1}$ is converging to f pointwise.

example-2 :- Take $X = \mathbb{R}$. Define $g_n: \mathbb{R} \to \mathbb{R}$ by $g_n(x) = \frac{x}{n}$, $\forall\, x \in \mathbb{R}$.

Define g by $g(x) = 0$ $\forall\, x \in \mathbb{R}$

Then $\{g_n\}_{n\geq 1}$ converges pointwise to g.

Each g_n is unbounded. But the limit g is a bounded function

Example-3 Take $X = [0, 1]$. Define $f_n : [0,1] \to \mathbb{R}$ by $f_n(x) = x^n$

for $x = 0$ $\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} 0 = 0$,

for $x = 1$ $\lim_{n \to \infty} f_n(x) = 1$.

for $0 < x < 1$, Claim $\lim_{n \to \infty} x^n = 0$

$\frac{1}{x} = 1 + \beta$ for some $\beta > 0$

$\left(\frac{1}{x}\right)^n = (1+\beta)^n = (1 + n\beta + \ldots) > 1 + n\beta$

$x^n < \frac{1}{1+n\beta} = \frac{1}{n(1+\frac{\beta}{n})} = \frac{1}{n}\left(\frac{1}{1+\frac{\beta}{n}}\right)$.
$\downarrow 0 \quad \downarrow 1$

$\Rightarrow x^n \to 0$.

---

for $\varepsilon > 0$ $N \in \mathbb{N}$ $N \in \mathbb{N}$

$|x^n| < \varepsilon$. Take N ~~$\log_x \varepsilon$~~ $\frac{1}{N} < \log_x$

$n \log x < \log \varepsilon$ ~~$n > N$~~ $\frac{1}{N} < \varepsilon$.

$n < \frac{\log \varepsilon}{\log x}$ $n > N > \frac{1}{\log_x \varepsilon}$ $n > N$ $\frac{1}{n} < \frac{1}{N} < \varepsilon$

$n < \log_x \varepsilon$ $n > -\log_\varepsilon x$

$\frac{1}{n} > \log_\varepsilon x$.

Q) $a_n > 0$; $a_{n+1} > a_n$; for every natural no. $p \in \mathbb{N}$ $\exists\, a_k$ s.t $p \mid a_k$; $a_n \mid a_{n+1}$; $a_n \in \mathbb{N}$

Given $b_n = \frac{1}{a_n}$

1) Proof that $\sum_{n=1}^{\infty} b_n$ converges.

2) ~~Proof~~ Check whether $\sum_{n=1}^{\infty} b_n$ is rational or irrational.

Q) If f is R.I, then f should have atleast one point continuity

→ pointwise Convergence

for every $x \in X$ & $\varepsilon > 0$, there exists $N = N_{x,\varepsilon}$ s.t.

$$|f_n(x) - f(x)| < \varepsilon \text{ for all } n \geq N.$$

eg. $X = [0,1]$; $f_n(x) = x^n \quad \forall x \in [0,1]$.

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & 1 \end{cases}$$

Uniform Convergence

Def. :- Let X be a non-empty set, Let $\{f_n\}_{n\geq 1}$, f be real valued functions on X. Then $\{f_n\}_{n\geq 1}$ is said to be converge to f uniformly on X if for every $\varepsilon > 0$, there exist $N = N_\varepsilon$ s.t.

$$|f_n(x) - f(x)| < \varepsilon \quad \forall\ n \geq N \ \& \ x \in X.$$

eg. $X = [0,1]$ $\quad g_n(x) = x/n \quad \forall\ 0 \leq x \leq 1$

$$g(x) = 0 \quad \forall\ 0 \leq x \leq 1$$

$\{g_n\}_{n\geq 1}$ converges to g uniformly

Thm :- Uniform Convergence implies pointwise.

Suppose $\{f_n\}_{n\geq 1}$ converges to f uniformly on X.

Take $\varepsilon = 1$. Then $\exists\ N_0 \in \mathbb{N}$ s.t.

$$|f_n(x) - f(x)| < 1 \quad \forall\ n \geq N_0 \ \& \ x \in X$$

for $n \geq N_0$, take

$$\alpha_n = \sup\{|f_n(x) - f(x)| : x \in X\}.$$

$$\alpha_n \leq 1 \quad \forall\ n \geq N_0$$

for $n < N_0$ $\sup\{f_n(x) - f(x)\}$ could be infinity

Theorem:- If $\{f_n\}_{n\geq 1}$ converges to f uniformly then $\{\alpha_n\}_{n\geq N_0}$ converges to 0.

Proof:- for $\varepsilon>0$, there exist $N=N_\varepsilon$, s.t.

$$|f_n(x)-f(x)|<\varepsilon \text{ for } n\geq N \text{ \& } x\in X.$$

Hence $\sup\{|f_n(x)-f(x)| : x\in X\}\leq\varepsilon$ for $n\geq N$

Therefore $\alpha_n\leq\varepsilon$ for $n\geq \cancel{N} \max\{N, N_0\}$

Thm:- Suppose $\{f_n\}_{n\geq 1}$, f are real valued function on a set X. Suppose there exists $N_0\in\mathbb{N}$ s.t.

$$\alpha_n=\sup\{|f_n(x)-f(x)| : x\in X\} \text{ is}$$

finite for $n\geq N_0$ & $\lim_{n\to\infty}\alpha_n=0$,

Then $\{f_n\}_{n\geq 1}$ converges to f uniformly

Proof:- for $\varepsilon>0$, there exists $N\geq N_0$, s.t.

$|\alpha_n|<\varepsilon$ for $n\geq N$

$\Rightarrow 0\leq\alpha_n<\varepsilon$ for $n\geq N$. {by def of α_n}

Hence for $n\geq N$

$$\sup\{|f_n(x)-f(x)| : x\in X\}<\varepsilon.$$

Therefore $|f_n(x)-f(x)|<\varepsilon$ $\forall x\in X$ & $n\geq N$.

Def^n:- Let $\{f_n\}_{n\geq 1}$ be a sequence of real valued functions on a non-empty set X. Then $\{f_n\}_{n\geq 1}$ is said to be uniformly cauchy if for every $\varepsilon>0$, there exist $N=N_\varepsilon\in\mathbb{N}$ s.t.

$$|f_m(x)-f_n(x)|<\varepsilon \ \forall m,n\geq N$$
$$\forall x\in X.$$

Thm :- A sequence $\{f_n\}_{n\geq 1}$ of real valued function on a non-empty set X is uniformly convergent iff it is uniformly Cauchy.

Pf :- Suppose $\{f_n\}_{n\geq 1}$ converges to f uniformly. Then for $\varepsilon>0$, there exists $N = N_\varepsilon \in \mathbb{N}$ s.t.

$$|f_n(x) - f(x)| < \varepsilon/2 \text{ for all } n \geq N \text{ \& } x \in X.$$

Hence for $x \in X$, $m,n \geq N$

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f_n(x) - f(x)|$$

$$< \varepsilon/2 + \varepsilon/2$$

$$= \varepsilon. \quad \forall m,n \geq N$$

So, $\{f_n\}_{n\geq 1}$ is uniformly cauchy.

$\Leftarrow$ Now suppose $\{f_n\}_{n\geq 1}$ is uniformly cauchy for $x \in X$, we see that $\{f_n(x)\}_{n\geq 1}$ is cauchy. Hence it is convergent. Define $f : X \to \mathbb{R}$ by

$$f(x) = \lim_{n\to\infty} f_n(x)$$

Exercise :- Show that $\{f_n\}$ converges to f uniformly.

Def^n :- Let X be a non-empty set. A function $f : X \to \mathbb{R}$ is said to be bounded if $\exists$ $M \in \mathbb{R}$ s.t

$$|f(x)| \leq M \quad \forall x \in X.$$

A sequence $\{f_n\}_{n\geq 1}$ of real valued function on X is said to be uniformly bounded if there exists $k \in \mathbb{R}$ s.t.

$$|f_n(x)| \leq k \quad \forall x \in X \text{ \& } n \in \mathbb{N}$$

Thm:- Suppose $\{f_n\}_{n\geq 1}$ converges to f uniformly on X. Suppose f is bounded. Then there exist $N_0 \in \mathbb{N}$ s.t

$\{f_n\}_{n\geq N_0}$ is uniformly bounded.

Proof:- As f is bounded, there exist $M \in \mathbb{R}$ s.t

$|f(x)| \leq M \quad \forall x \in X.$

Take $\varepsilon = 1$. As $\{f_n\}_{n\geq 1}$ converges to f uniformly, $\exists N_0 \in \mathbb{N}$ s.t

$|f_n(x) - f(x)| < 1 \quad \forall x \in X \ \& \ n \geq N_0$

Then for $n \geq N_0$ & $x \in X$,

$$|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)|$$
$$\leq 1 + M$$

$\Rightarrow \{f_n\}_{n\geq N_0}$ is uniformly bounded.

Thm:- Let $X \subseteq \mathbb{R}$ be a non-empty set. Let $\{f_n\}_{n\geq 1}$ and f be real valued functions on X. Suppose $\{f_n\}_{n\geq 1}$ converges to f uniformly. Fix $c \in X$.
Suppose f_n is continuous at c, $\forall n \in \mathbb{N}$.
Then f is continuous at c.

Proof:- Consider any $\varepsilon > 0$. As $\{f_n\}_{n\geq 1}$ converges to f uniformly, there exists $N \in \mathbb{N}$ s.t.

$|f_n(x) - f(x)| < \frac{\varepsilon}{3}$, $\forall$ ~~n ∈ N~~ $n \geq N$ & $x \in X$.

Consider f_N. It is continuous at c.
So, $\exists \ \delta > 0$ s.t.

$|f_N(x) - f_N(c)| < \varepsilon/3$

for x satisfying $|x - c| < \delta$ & $x \in X$.

Now, for $x \in X$, with $|x-c|<\delta$

$$|f(x) - f(c)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)|.$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3$$

$$= \varepsilon.$$

Hence f is continuous at c.

<u>Corollary</u>:- Let X be a non empty subset of $\mathbb{R}$ & let $\{f_n\}_{n\geq 1}$, f be real valued functions on X. Suppose $\{f_n\}_{n\geq 1}$ converges to f uniformly & f_n is continuous for every n. Then f is continuous.

eg. $X : [0,1]$ $\quad f_n(x) = x^n, \; x \in [0,1]$

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

$f_n(x)$ converges pointwise & f_n is continuous also f is not continuous.

$\Rightarrow f_n$ doesn't converge uniformly.

eg. $X = [0,1)$ $\quad f_n(x) = x^n, \; x \in [0,1)$

$f(x) = 0 \quad x \in [0,1)$.

then $\{f_n\}$ converges to f pointwise, f is continuous, but the convergence is not uniform.

eg. $X = [0,1]$

$$f_n(x) = \begin{cases} (n+1)x, & 0 \leq x \leq \frac{1}{n+1} \\ * & \frac{1}{n+1} \leq x < \frac{1}{n} \\ 0 & \frac{1}{n} \leq x \leq 1 \end{cases}$$

$f_n(x) = 0$ for $n > \frac{1}{x}$

$\Rightarrow f_n$ converges pointwise to 0.

f_n & f_0 are continuous but f_n convergence is not uniform.

Take $\varepsilon < 1$, then $\forall n$ $f_n\left(\frac{1}{n+1}\right) = 1 > \varepsilon$

i.e. convergence is not uniform.

Fix $a, b \in \mathbb{R}$ with $a < b$.

Take $C_{\mathbb{R}}([a,b])$ = The set of real valued continuous function on $[a,b]$.

If $f \in C_{\mathbb{R}}[a,b]$ then f is bounded & uniformly continuous.

$C_{\mathbb{R}}[a,b]$ is a vector space over $\mathbb{R}$.

for $f, g \in C_{\mathbb{R}}[a,b]$, take

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in [a,b]\}$$

Claim :- d is metric on $C_{\mathbb{R}}[a,b]$.

① As f, g is continuous, it is bounded. Hence $d(f,g) \in \mathbb{R}$.

② Clearly $d(f,g) \geq 0$ $\forall f, g$.

③ $d(f,g) = 0$ then $|f(x) - g(x)| = 0$ $\forall x \in$ [

hence $f(x) - g(x) = 0$ or $f = g$

conversely if $f = g \Rightarrow d(f,g) = 0$.

④ Symmetry, $d(g,f) = d(f,g)$ {clea

(5) $\triangle$ inequality

Suppose $f, g, h \in C_{\mathbb{R}}[a,b]$.

for any $x \in [a,b]$

$$|f(x) - g(x)| \le |f(x) - h(x)| + |h(x) - g(x)|$$

$$|f(x) - g(x)| \le \sup(|f(x) - h(x)| + |h(x) - g(x)|)$$

$$\sup|f(x) - g(x)| \le \sup|f(x) - h(x)| + \sup|h(x) - g(x)|$$

$$\le \sup|f(x) - h(x)| \quad \forall x.$$

$$\Rightarrow d(f,g) \le d(f,h) + d(h,g).$$

Or $\quad |f(x) - g(x)| \le d(f,h) + d(h,g) \quad \forall x \in [a,b].$

$$\Rightarrow \sup_{x \in X} |f(x) - g(x)| \le d(f,h) + d(h,g)$$

$$\Rightarrow d(f,g) \le d(f,h) + d(h,g).$$

Thm:- $C_{\mathbb{R}}[a,b]$ is a complete metric space.

Proof:- ~~is exercise~~

Suppose $\{f_n\}_{n \ge 1}$ is cauchy sequence

w.r.t this metric.

for $\varepsilon > 0$, $\exists$ $N \in \mathbb{N}$ s.t.

$$d(f_m, f_n) < \varepsilon \quad \forall\, m,n \ge N.$$

That is $\sup_{x \in X} |f_m(x) - f_n(x)| < \varepsilon \quad \forall\, n,m \ge N.$

So, $\{f_n\}$ is uniformly Cauchy, So

$\{f_n\}_{n \ge 1}$ converge uniformly

to some function f.

Uniform convergence of continuous

function implies that the limit function

is continuous.

So, $f \in C_{\mathbb{R}}[a,b]$.

Note $\{f_n\}$ converges to f iff $d(f_n, f) \to 0$

$d(f_n, f) = \sup |f_n(x) - f(x)| = \alpha_n$

So, $\{f_n\}_{n\geq 1}$ converges to f in this metric iff $\{f_n\}$ converge to f uniformly.

Dini's theorem :- Let $\{f_n\}_{n\geq 1}$ be a sequence (monotonic) of function in $C_{\mathbb{R}}[a,b]$ converging to a function $f \in C_{\mathbb{R}}[a,b]$ pointwise. Then the ~~fu~~ convergence is monotonic.

Proof :- Suppose it is a decreasing sequence.

$$f_1 \geq f_2 \geq \cdots \geq f_n \geq f_{n+1} \geq \cdots$$

$\{f_n\}$ converging to f pointwise, with f continuous.

Take $g_n = f_n - f$

Then g_n is continuous for every n &

$$g_1 \geq g_2 \geq \cdots \geq g_n \geq g_{n+1}.$$

$$g_n \geq 0 \quad \forall n$$

$$\lim_{n\to\infty} g_n(x) = 0 \quad \forall x \in [a,b]$$

$$d(g_n, 0) = \sup\{|g_n(x)| ; x \in [a,b]\}$$

Take $M_n = d(g_n, 0)$

We want to show that

$$\lim_{n\to\infty} M_n = 0.$$

Take $\delta = \lim_{n\to\infty} M_n$, assume that $\delta > 0$.

We want to arrive at a contradiction.

We have
$$\sup\{g_n(x) : x \in [a,b]\} \geq \delta$$
for all n.

As it attains its supremum, we get $x_n \in [a,b]$

s.t. $\underset{m_n}{\underset{\shortparallel}{g_n(x_n)}} \geq \delta$.

As $x_n \in [a,b]$, $\forall n$ $\{x_n\}_{n\geq 1}$ has a convergent subsequence, say $\{x_{n_k}\}_{k\geq 1}$.

Take $y = \lim_{x\to\infty} x_{n_k}$

$g_{n_1}(x_{n_1}) \geq \delta$ & $g_{n_2}(x_{n_2}) \geq \delta$

$\ldots g_{n_k}(x_{n_k}) \geq \delta$ $\forall k$

We have
$\lim_{n\to\infty} g_n(x) = 0$ $\forall x \in [a,b]$

In particular

$\lim_{n\to\infty} g_n(y) = 0$ — ①

$\lim_{k\to\infty} x_{n_k} = y$ — ②

$g_{n_k}(x_{n_k}) \geq \delta$ $\forall k$ — ③.

from ①, $\exists$ $N_0 \in \mathbb{N}$ s.t. $g_n(n) < \delta/2$ for $n \geq N$.

This explains that $g_{n_k}(y) < \delta/2$ for $k \geq N$

$\Rightarrow \underbrace{g_{n_k}(y) \leq g_k(y)}_{\text{as } n_k \geq k \text{ \& } g_n \text{ is decreasing}} < \delta/2$ for $k \geq N$.

① $g_{n_k}(y) < \delta/2$ for $k \geq N$.

② but $g_{n_k}(x_{n_k}) \geq \delta$ for all k.

③ & $\lim_{k\to\infty} x_{n_k} = y$.

$h_k(y) < \delta/2 \quad \forall\, k \geq N$, $h_k = g_{n_k}$

$h_k(y_k) \geq \delta$, $y_k = x_{n_k}$.

$y_k \to y$. i.e. $\lim_{k\to\infty} y_k = y$.

$h_k(y_k) \geq \delta \quad \forall\, k$

for $l \geq k$

$h_k(y_l) \geq h_l(y_l) \geq \delta$

$h_k(y_l) \geq \delta$ for $k \leq l$

$\lim_{l\to\infty} h_k(y_l) \geq \delta \Rightarrow h_k(y) \geq \delta \quad \forall\, k$

$\Rightarrow \lim_{k\to\infty} h_k(y) \geq \delta$.

Weistrass Approximation theorem

Thm :- Let $a, b \in \mathbb{R}$ with $a<b$. Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then there exists a sequence of polynomials converging to f uniformly in $[a,b]$.

Proof :- first we consider $[0,1]$.

Suppose $f : [0,1] \to \mathbb{R}$ is a continuous function.

for $n \geq 0$, the bunstien polynomial of f is defined as

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

Claim :- $\{B_n(f)\}_{n \geq 0}$ converges to f uniformly as $n \to \infty$.

Suppose $X \sim Bin(n, x)$

X is a random variable taking values in $\{0, 1, 2, \ldots, n\}$

$$P(X=k) = \binom{n}{k} x^k (1-x)^{n-k}$$

$0 \leq k \leq n$.

① $E(X) = nx$ & $V(X) = nx(1-x)$

$E(X^2) = V(X) + [E(X)]^2$, $V(X) = E(X^2) - E(X)^2$

$= nx(1-x) + n^2 x^2$

② Chebyshev's inequality.

for $\delta > 0$

$$P(|X - E(X)| \geq \delta) \leq \frac{V(X)}{\delta^2}$$

$$P(|X - nx| \geq \delta) \leq \frac{nx(1-x)}{\delta^2}.$$

Take $q_0(x) = 1$, $q_1(x) = x$, $q_2(x) = x^2$

$0 \leq x \leq 1$

$$B_n(q_0)(x) = \sum_{k=0}^{n} 1 \binom{n}{k} x^k (1-x)^{n-k} = 1.$$

$\{B_n(q_0)\}$ converges to q_0

$$B_n(q_1)(x) = \sum_{k=0}^{n} \frac{k}{n} \binom{n}{k} x^k (1-x)^{n-k}$$

$$= \frac{1}{n} E(x) = \frac{1}{n} \cdot nx = x$$

$$B_n(q_1) = q_1$$

$$B_n(q_2)(x) = \sum_{k=0}^{n} \left(\frac{k}{n}\right)^2 \binom{n}{k} x^k (1-x)^{n-k}$$

$$= \frac{1}{n^2} E(x^2)$$

$$= \frac{1}{n^2}\left[nx(1-x) + n^2x^2\right]$$

$$= \frac{1}{n}\left[x - x^2 + nx^2\right]$$

$$B_n(q_2)(x) = \frac{1}{n}x + \left(1 - \frac{1}{n}\right)x^2$$

Clearly $B_n(q_2)$ converges to q_2.

Proof: ~~[illegible]~~ of claim

Choose $M > 0$ s.t. $|f(x)| \leq M$ $\forall x \in [0,1]$

As f is uniformly continuous, for $\epsilon > 0$, choose $\delta > 0$ s.t.

$$|f(x) - f(y)| < \frac{\epsilon}{2} \text{ for } x, y \in [0,1]$$
with $|x - y| < \delta$

Now, for $x \in [0,1]$

$$f(x) - B_n(f)(x)$$

$$B_n(f)(x) - f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$
$$- f(x) \cdot \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}$$

$$|B_n(f)(x) - f(x)| = \left|\sum_{k=0}^{n} \left(f\left(\frac{k}{n}\right) - f(x)\right) \binom{n}{k} x^k (1-x)^{n-k}\right|$$

$$\leq \sum_{k=0}^{n} \left|f\left(\frac{k}{n}\right) - f(x)\right| \binom{n}{k} x^k (1-x)^{n-k}$$

$$= \sum_{\left|\frac{k}{n} - x\right| < \delta} \left|f\left(\frac{x}{n}\right) - f(x)\right| \binom{n}{k} x^k (1-x)^{n-k} + \sum_{\left|\frac{k}{n} - x\right| \geq \delta} \left|f\left(\frac{k}{n}\right) - f(x)\right| \binom{n}{k} x^k (1-x)^{n-k}$$

$$T_1 \sum_{k:\left|\frac{k}{n} - x\right| < \delta} \left|f\left(\frac{k}{n}\right) - f(x)\right| \binom{n}{k} x^k (1-x)^{n-k} + \sum_{k:\left|\frac{k}{n} - x\right| \geq \delta} \left|f\left(\frac{k}{n}\right) - f(x)\right| \binom{n}{k} x^k (1-x)^{n-k}$$

$$\sum_{k:\left|\frac{k}{n} - x\right| < \delta} \left|f\left(\frac{k}{n}\right) - f(x)\right| \binom{n}{k} x^k (1-x)^{n-k} \leq \sum \left(\frac{\varepsilon}{2}\right)(\;) \leq \varepsilon/2$$

$$T_2 \sum_{k:\left|\frac{k}{n} - x\right| \geq \delta} \left|f\left(\frac{x}{n}\right) - f(x)\right| \binom{n}{k} x^k (1-x)^{n-k}$$

$$\leq 2M \sum_{k:\left|\frac{k}{n} - x\right| \geq \delta} \binom{n}{k} x^k (1-x)^{n-k}$$

$$= 2M \sum_{k:|k - nx| \geq n\delta} \binom{n}{k} x^k (1-x)^{n-k}$$

$$= 2M\, P(|X - E(x)| \geq n\delta)$$

$$\leq 2M \frac{V(x)}{n^2\delta^2} = 2M \frac{nx(1-x)}{n^2\delta^2}$$

$$= \frac{2Mx(1-x)}{n\delta^2}$$

Choose N s.t. ~~large~~ ~~enough~~ $\frac{M}{2N\delta^2} < \varepsilon/2$

Then ~~Choose~~ for $n \geq N$, $T_2 \leq \varepsilon/2$

So for $n \geq N$

$$|B_n(f)(x) - f(x)| \leq \varepsilon \quad \forall x \in [0,1]$$

This proves the claim

Power Series

~~Suppose a_0, a_1~~
A series of the form $\sum_{k=0}^{\infty} a_k x^k$, where $a_k \in \mathbb{R}$ $\forall k$ and x is a real variable is called power series.

Domain of convergence.

$$\mathcal{D}(\{a_k\}) = \{x \in \mathbb{R} : \sum_{k=0}^{\infty} a_k x^k \text{ converges}\}$$

$\mathcal{D}(\{a_k\})$ is always non empty as $0 \in \mathcal{D}(\{a_k\})$.

example 1

$a_k = 1$ $\forall k$. We have the series $\sum_{k=0}^{\infty} x^k$. If $|x| < 1$, then $\sum_{k=0}^{\infty} x^k$ converges.

$x = 1$, $\mathcal{D}(\{1\}) = (-1, 1)$

example 2

$a_k = 0$ for $k \geq 10$. Then $\sum_{k=0}^{\infty} a_k x^k$ is convergent for all x

So, $\mathcal{D} = \mathbb{R}$.

example-3

$a_k = k^k$

for $x \neq 0$, consider $\sum_{k=0}^{\infty} a_k x^k = 1 + \sum_{k=1}^{\infty} k^k x^k$

Take $x > 0$,
Consider $(kx)^k$. Then $(kx)^k > 1$ for $k > \frac{1}{x}$.

Suppose $x < 0$, $\lim_{k \to \infty} (kx)^k \neq 0$

So, $\sum_{k=0}^{\infty} k^k x^k$ is not convergent for $x \neq 0$.

$\mathcal{D}(\{k^k\}) = \{0\}$.

Abel's lemma

Consider a power series $\sum_{k=0}^{\infty} a_k x^k$.

Suppose $t \neq 0$ and $\{a_k t^k : k \geq 0\}$ is bounded

Then $\sum_{k=0}^{\infty} a_k x^k$ is absolutely convergent for $|x| < |t|$.

Proof:- Consider any $x \in \mathbb{R}$ with $|x| < |t|$

Then $x = ct$ for some $c \in (-1, 1)$

Suppose $|a_k t^k| \leq M \; \forall k$

Now $\sum_{k=0}^{\infty} |a_k x^k| = \sum_{k=0}^{\infty} |a_k (ct)^k|$

$= \sum_{k=0}^{\infty} |a_k t^k| |c|^k$

$\leq \sum_{k=0}^{\infty} M |c|^k = M \sum_{k=0}^{\infty} |c|^k < \infty$

So, $\sum_{k=0}^{\infty} a_k x^k$ is absolutely convergent.

Thm :- Let $\sum_{k=0}^{\infty} a_k x^k$ be a power series. Then either $\sum_{k=0}^{\infty} a_k x^k$ is absotely convergogent for all $x \in \mathbb{R}$ or there exist $R \geq 0$ s.t. $\sum_{k=0}^{\infty} a_k x^k$ is absolutely convergent for $|x| < R$ & $\sum_{k=0}^{\infty} a_k x^k$ is not convergent for $|x| > R$.

Proof :- Suppose $\sum_{k=0}^{\infty} a_k t^k$ is convergent

Then $\lim_{k \to \infty} a_k t^k = 0$. In particular $\{a_k t^k\}_{k \geq 0}$ is bounded.

Then by Abel's lemma $\sum_{k=0}^{\infty} a_k x^k$ is absolutely convergent for $|x| < t$.

Take $R = \sup \{ t : \sum_k a_k t^k$ is convergent $\}$.

R is known as the radius of convergence.

$R \in [0, \infty]$.

examples $a_k = \frac{1}{k^2}$ $R = 1$

$\mathcal{D} = [-1, 1]$.

example 2 $a_k = \frac{1}{k}$. $R = 1$

$\mathcal{D} = [-1, 1)$.

<u>examples</u> $a_n = \frac{(-1)^k}{k}$ $R = 1$

$\mathcal{D} = (-1, 1]$.

formula for radius of convergence:

$$\frac{1}{R} = \limsup_{k \to \infty} |a_k|^{1/k}.$$

[If limsup is zero, R is taken to be ∞].

$R = \infty$ if $\limsup_{k\to\infty} |a_k|^{1/k} = 0$ and.

$R = 0$ if $\limsup_{k\to\infty} |a_k|^{1/k} = 0$.

Take $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $x \in (-R, R)$

$$f_n(x) = \sum_{k=0}^{n} a_k x^k$$

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Do we have uniform convergence?

$f(x) = 1 + x + x^2 + \dots = \frac{1}{1-x}$, $x \in (-1, 1)$

$f_n(x) = 1 + x^2 + \dots + x^n$

$|f(x) - f_n(x)| < \varepsilon \to$ can't hold for any ε.

No uniform convergence in $(-1, 1)$.

Fix r, $0 < r < 1$. Consider $|x| < r$.

$$|f(x) - f_n(x)| = \left| \frac{1}{1-x} - \frac{1-x^{n+1}}{1-x} \right|$$

$$= \left| \frac{x^{n+1}}{1-x} \right|$$

$$= \frac{|x|^{n+1}}{|1-x|} \le \frac{r^{n+1}}{1-r} \to 0$$

as $n \to \infty$

So, $\{f_n\}$ converges to f uniformly on $[-r, r]$.

General Case

Assume $R > 0$.

Consider $0 < r < R$

Consider $\{f_n\}_{n \ge 1}$ and f on $[-r, r]$.

Use the proof of Abel's lemma to see that $\{f_n\}$ converges uniformly to f on $[-r, r]$.

Integrate & differentiate term wise

$$g_n(x) = \sum_{k=0}^{n} \frac{a_k x^{k+1}}{k+1}, \quad h_n(x) = \sum_{k=0}^{n} k a_k x^{k-1}$$

$$\left\{0, \frac{a_0}{1}, \frac{a_1}{2}, \ldots\right\}, \quad \{a_1, 2a_2, 3a_3, \ldots\}$$

$$\quad\;\; 0 \quad\; 1 \quad\; 2 \qquad\qquad\quad 0 \quad\; 1 \quad\; 2$$

Power series $\displaystyle\sum_{k=1}^{\infty} \frac{a_{k-1}}{k} x^k$ $\qquad \displaystyle\sum_{k=0}^{\infty} (k+1) a_{k+1} x^k$

$$\limsup_{k \to \infty} \left(\left|\frac{a_{k-1}}{k}\right|\right)^{1/k} = \frac{1}{R}$$

Fix any r such that $0 < r < R$

Here $\{f_n\}$ converges uniformly to f

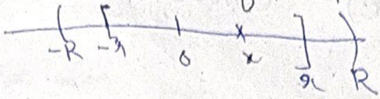
$\{g_n\}$ " " to g

$\{h_n\}$ " " to h.

By theorems proved before g is the integral of f, $g' = f$.

... termwise differentiation or integration is permitted & we have power series converging uniformly, for both.

...ume $R \neq 0$. Define, $f : (-R, R) \to \mathbb{R}$, by $f(x) = \sum_{k=0}^{\infty} a_k x^k$



for $x \in (-R, R)$, Choose r s.t.

$$0 \le |x| < r < R$$

On $[-r, r]$, the series is uniformly convergent.

On $[-r, r]$, we can do term-wise differen.

So, f is differentiable at every $x \in (-R, R)$

In fact f is infinitely differentiable on $(-R, R)$ by the same argument.

$a_0 = f(0)$

$a_1 = f'(0)$

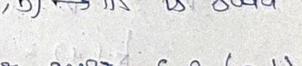
$\frac{f^k(0)}{k!} = a_k$ {by induction

So, $f(x) = \sum_{k=0}^{\infty} \frac{f^k(0)}{k!} x^k$.

Def :- A function $f : (a, b) \to \mathbb{R}$ is said to be real analytic if for every $c \in (a, b)$, there exists $\varepsilon > 0$ s.t. $f|_{(c-\varepsilon, c+\varepsilon)}$ has a power series expansion.

example :- $f : (-1, 1) \to \mathbb{R}$, $f(x) = \frac{1}{1-x}$

$0 < c < 1$



Theorem 7 If f and g are integrable, then product fg ~~belong~~ is also integrable.

Theorem 8 Integration by parts

Let F, G be differentiable on $[a,b]$ and let $f := F'$ and $g = G'$. g and f are integrable. Then,

$$\int_a^b fG = G\int_a^b f - \int_a^b g\left(\int_a^b f\right)$$

$$= GF\Big|_a^b - \int_a^b Fg.$$

Theorem 9 Taylor's theorem with remainder

Suppose that $f', \dots, f^n, f^{n+1}$ exist on $[a,b]$ and that f^{n+1} is ~~also~~ integrable then we have $t \in [a,b]$ s.t

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \dots + \frac{f^n(a)}{n!}(b-a)^n + R_n$$

when R_n is the remainder given by

$$R_n = \frac{1}{n!}\int_a^b f^{n+1}(t)\cdot(b-t)^n\, dt.$$