Real Analysis - II

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Recap: (Defⁿ of Continuity)

ε-δ $\Rightarrow$ A fⁿ $f: A \to \mathbb{R}$ $(A \subset \mathbb{R})$ is continuous at $a \in A$ if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ st $|f(x) - f(a)| < \varepsilon \;\; \forall\, x \in A \cap (a-\delta, a+\delta)$

Sequential $\Rightarrow$ f is continuous at a if $\forall$ sequences $\{x_n\}_{n\geq 1}$ in A converging to a $\Rightarrow$ $\{f(x_n)\}_{n\geq 1}$ converges to $f(x)$.

Recap: (Defⁿ of Differentiability)

ε-δ $\Rightarrow$ $\exists\, d \in \mathbb{R}$ st. $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ satisfying $\left|\dfrac{f(x) - f(a)}{x - a} - d\right| < \varepsilon$

$\forall\, x \in A \cap (a-\delta, a+\delta) \setminus \{a\}$.

Sequential $\Rightarrow$ $\lim\limits_{\substack{(x_n)\to a \\ x_n \neq a \\ (x_n) \in A}} \dfrac{f(x_n) - f(a)}{x_n - a}$ exists, ~~converges~~ (exists) equal to d.

To denote a fⁿ, one must specify its domain & co-domain!

Example: $h(x) = \begin{cases} 0 & ;\; x \notin \mathbb{Q} \\ \frac{1}{q} & ;\; x \in \mathbb{Q} \text{ st } x = \frac{p}{q} \text{ where } p, q \in \mathbb{Z},\; q > 0 \;\&\; (p,q) = 1. \end{cases}$

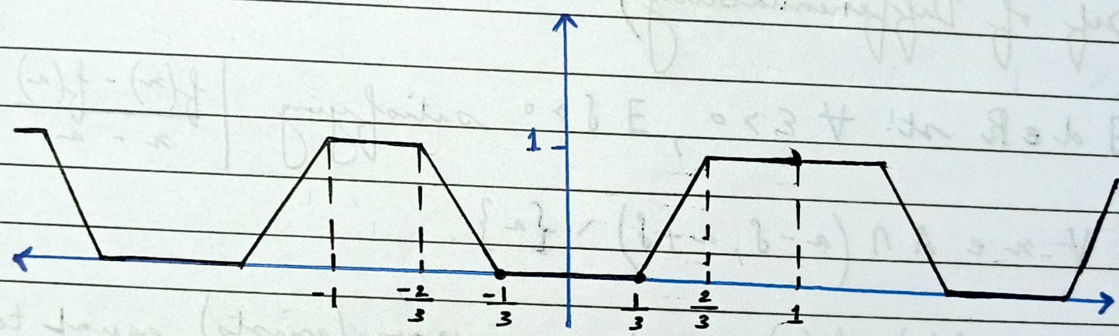
Clearly, one can prove that h is discontinuous at every rational number & continuous at every irrational number (Prove it on your own).

Space Filling Curve :

Define $\varrho : \mathbb{R} \to \mathbb{R}$ as follows :

$$\varrho(t) = \begin{cases} 0 & \text{if } t \in \left(-\frac{1}{3}, \frac{1}{3}\right) \\ 3|t|-1 & \text{if } t \in \left(-\frac{2}{3}, -\frac{1}{3}\right] \cup \left[\frac{1}{3}, \frac{2}{3}\right) \\ 1 & \text{if } t \in \left[-1, -\frac{2}{3}\right] \cup \left[\frac{2}{3}, 1\right] \end{cases}$$

Also, $\varrho(t+2) = \varrho(t)$.



Now, $\varrho : \mathbb{R} \to \mathbb{R}$ is a continuous f^n & $\varrho(t) \in [0,1] \ \forall t$.

Thus, define $P : [0,1] \to [0,1] \times [0,1]$,

st $P(t) = (f(t), g(t)) \quad \forall t \in [0,1]$ where

$$f(t) = \sum_{k=0}^{\infty} \frac{\varrho(3^{2k} t)}{2^{k+1}} \quad \& \quad g(t) = \sum_{k=0}^{\infty} \frac{\varrho(3^{2k+1} t)}{2^{k+1}}.$$

Claim : f, g are well-defined, continuous f^n on $[0,1]$ taking values on $[0,1]$.

Proof: $\sum_{k=0}^{\infty} \frac{1}{2^{k+1}} = 1$. Now, $0 \leq \frac{\varrho(3^{2k} t)}{2^{k+1}} \leq \frac{1}{2^{k+1}}$.

Thus, f, g are clearly defined $\forall t \in [0,1]$.

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Now, fix $\varepsilon > 0$. $\because \sum_{k=0}^{\infty} \frac{1}{2^{k+1}}$ converges, $\Rightarrow \exists N \in \mathbb{N}$ st.

$$\sum_{k=N}^{\infty} \frac{1}{2^{k+1}} < \frac{\varepsilon}{4}.$$

Let $f(t) = f_1(t) + f_2(t)$ where $f_1(t) = \sum_{k=0}^{N-1} \frac{\varphi(3^{2k} t)}{2^{k+1}}$.

Now, $f_2(t) = \sum_{k=N}^{\infty} \frac{\varphi(3^{2k} t)}{2^{k+1}} < \frac{\varepsilon}{4}$ as well.

Now, f_1 is a continuous f^n as it is a sum of finitely many continuous f^ns.

$\therefore$, $\exists\ \delta > 0$, st; $|f_1(s) - f_1(t)| < \frac{\varepsilon}{2}$

for $s \in [0,1]$ & $|t-s| < \delta$.

$$\text{Now, } |f(t) - f(s)| \leq |f_1(t) - f_1(s)| + |f_2(t) - f_2(s)|$$

$$< |f_1(t) - f_1(s)| + |f_2(t)| + |f_2(s)|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

Moreover, any such f^n is uniformly continuous, $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ st.

$$|f(t) - f(s)| < \varepsilon \quad ; \ s \in [0,1] \ \& \ |t-s| < \delta.$$

Hence, proved.

Now, $P(t) = (f(t), g(t))$ is just a pt. in the unit square.

Claim: Given any $(x, y) \in [0,1] \times [0,1]$, $\exists\ t \in [0,1]$ s.t.

$$(f(t), g(t)) = (x, y).$$

Proof: Consider Binary Expansion of x & y:

$$x = \frac{a_0}{2} + \frac{a_2}{2^2} + \frac{a_4}{2^3} + \cdots\cdots \; ; \quad a_i \in \{0, 1\}.$$

$$y = \frac{a_1}{2} + \frac{a_3}{2^2} + \frac{a_5}{2^3} + \cdots\cdots \; ; \quad a_i \in \{0, 1\}.$$

Take $t = \sum_{k=0}^{\infty} \frac{2a_k}{3^{k+1}}$ i.e. $t = \frac{2a_0}{3} + \frac{2a_1}{3^2} + \frac{2a_2}{3^3} + \cdots\cdots$

$$\Rightarrow 3^n t = (\text{even no.}) + \frac{2a_n}{3} + \frac{2a_{n+1}}{3^2} + \cdots\cdots$$

$$\Rightarrow \varphi(3^n t) = \varphi\Big(\underbrace{\frac{2a_n}{3} + \frac{2a_{n+1}}{3^2} + \cdots\cdots}\Big)$$

If $a_n = 0 \Rightarrow \leqslant 1/3 \Rightarrow \varphi(3^n t) = 0$.

If $a_n = 1 \Rightarrow \geqslant 2/3 \Rightarrow \varphi(3^n t) = 1$.

$$\therefore\ f(t) = \sum_{k=0}^{\infty} \frac{\varphi(3^{2k} t)}{2^{k+1}} = \sum_{k=0}^{\infty} \frac{a_{2k}}{2^{k+1}} = x,$$

$$g(t) = \sum_{k=0}^{\infty} \frac{\varphi(3^{2k+1} t)}{2^{k+1}} = \sum_{k=0}^{\infty} \frac{a_{2k+1}}{2^{k+1}} = y.$$

Hence, proved.

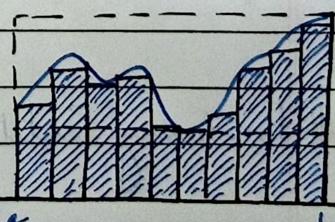
Banach Tarski Paradox : Mathematically, we can split a solid sphere of diameter 'd' into two solid spheres of diameter 'd' without changing the shapes of the pieces!!

Riemann Integration : $\int_a^b f(x)\,dx$

Set-Up :- Fix $a, b \in \mathbb{R}$ with $a < b$.
Let $f : [a, b] \to \mathbb{R}$ be a bounded f^n.
Notice that f need not be continuous.

Notation :- $m(f) = \inf\{f(x) : x \in [a, b]\}$

$M(f) = \sup\{f(x) : x \in [a, b]\}$



splitting into more & more rectangles gives a better approximation

Def^n :- A partition P of $[a, b]$ is an ordered tuple $(x_0, x_1, \ldots, x_n)$ where $a = x_0 < x_1 < \cdots < x_n = b$.

Here, $[x_{i-1}, x_i]$ is called the 'i^{th} sub-interval' of P $\forall\, 1 \leq i \leq n$.

Also, denote $m_i(f)$ & $M_i(f)$ for the i^{th} sub-interval in the same way as well.

Clearly, $m(f) \leq f(x) \leq M(f) \quad \forall\ x \in [a,b]$ &

$m_i(f) \leq f(x) \leq M_i(f) \quad \forall\ x \in [x_{i-1}, x_i]$ st $1 \leq i \leq n$

Now, equipartitioning the intervals for $n \geq 2$,

$$P = (x_0, \dots, x_n) \text{ st. } x_i = x_0 + i\left(\frac{b-a}{n}\right) = a + i\left(\frac{b-a}{n}\right).$$

Def^n :- The Lower sum of f with respect to partition P is defined as :-

$$L(P, f) = \sum_{i=1}^{n} m_i(f)(x_i - x_{i-1}).$$

Def^n :- The Upper sum of f with respect to partition P is defined as :-

$$U(P, f) = \sum_{i=1}^{n} M_i(f)(x_i - x_{i-1}).$$

For every Partition P, $m(f)(b-a) \leq L(P,f) \leq U(P,f) \leq M(f)(b-a)$ — $(*)$

Example: $[a,b] = [0,1]$ & define $f:[0,1] \to \mathbb{R}$ by $f(x) = x \ \forall x \in [0,1]$

Take $P = \left(0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}\right).$

$\Rightarrow m(f) = 0, \ M(f) = 1.$

$\Rightarrow m_i(f) = \frac{i-1}{n}, \quad M_i(f) = \frac{i}{n}.$

$$\Rightarrow L(P, f) = \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)\left(\frac{1}{n}\right) = \frac{n(n-1)}{2n^2} = \left(\frac{n-1}{2n}\right) = \frac{1}{2}\left(1 - \frac{1}{n}\right).$$

$$\Rightarrow U(P, f) = \sum_{i=1}^{n} \left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \frac{n(n+1)}{2n^2} = \left(\frac{n+1}{2n}\right) = \frac{1}{2}\left(1 + \frac{1}{n}\right).$$

$$0 \leqslant \frac{1}{2}\left(1 - \frac{1}{n}\right) \leqslant \frac{1}{2}\left(1 + \frac{1}{n}\right) \leqslant 1,$$

\# A Partition $\tilde{P} = (y_0, \dots, y_p)$ is said to be finer than (or a refinement of) partition $P = (x_0, \dots, x_n)$ if

$$\{x_0, \dots, x_n\} \subseteq \{y_0, y_1, \dots, y_p\}.$$

Theorem: Let $\tilde{P}$ be a partition finer than P. Then :

$$L(P, f) \leqslant L(\tilde{P}, f) \quad \& \quad U(P, f) \geqslant U(\tilde{P}, f).$$

Consequently, $U(\tilde{P}, f) - L(\tilde{P}, f) \leqslant U(P, f) - L(P, f)$.

Proof: Suppose $\tilde{P} = (y_0, \dots, y_p)$ & $P = (x_0, \dots, x_n)$ & $p = n+1$.

$$\Rightarrow \{y_0, \dots, y_p\} = \{x_0, \dots, x_n\} \cup \{y\} \text{ for some } y.$$

Say $x_{k-1} < y < x_k$ for some k.

$$\Rightarrow L(\tilde{P}, f) = \sum_{i=1}^{k-1} m_i(f)(x_i - x_{i-1}) + \sum_{i=k+1}^{n} m_i(f)(x_i - x_{i-1})$$

$$+ \inf\{f(x) : x \in [x_{k-1}, y]\}(y - x_{k-1})$$

$$+ \inf\{f(x) : x \in [y, x_k]\}(x_k - y)$$

$\because \inf\{f(x) : x \in [x_{k-1}, y]\}(y - x_{k-1}) \geq m_k(f)(y - x_{k-1})$

$\& \inf\{f(x) : x \in [y, x_k]\}(x_k - y) \geq m_k(f)(x_k - y)$,

Hence, we are done. $\Rightarrow L(\tilde{P}, f) \geq L(P, f)$.

Now, the general case is proven by induction.

Similarly, we can prove the Upper sum result aswell.

Aliter: Using ⊛ in $[x_{k-1}, x_k]$, we are done for $p = n+1$ & proceed similarly for the rest.

Theorem: Let $f : [a,b] \to \mathbb{R}$ be a bounded f^n. Let P, Q be any 2 partitions of $[a,b]$.

Then $L(P, f) \leq U(P, Q)$.

Proof: Let $P = \{x_0, \dots, x_n\}$ & $Q = \{y_0, \dots, y_r\}$.

Let $P \vee Q := \{z_0, \dots, z_n\}$ st. $\{z_0, \dots, z_n\} = \{x_0, \dots, x_n\} \cup \{y_0, \dots, y_r\}$

Thus, $L(P, f) \leq L(P \vee Q, f) \leq U(P \vee Q, f) \leq U(Q, f)$.

Def^n: Let $f:[a,b] \to \mathbb{R}$ be a bounded f^n. Then, the lower integral of f is defined as :-

$$\underset{a}{\overset{b}{\underline{\int}}} f := \sup \left\{ L(P,f) \; : \; P \text{ is a partition of } [a,b] \right\}$$

similarly, $\underset{a}{\overset{b}{\overline{\int}}} f := \inf \left\{ U(P,f) : P \text{ is a partition of } [a,b] \right\}.$

$\because L(P,f) \leq U(Q,f) \quad \forall \; P, Q$ of $[a,b]$,

$\Rightarrow \sup (L(P,f)) \leq \inf (U(Q,f)) \quad \forall \; P, Q$ of $[a,b]$.

$$\Rightarrow \underset{a}{\overset{b}{\underline{\int}}} f \leq \underset{a}{\overset{b}{\overline{\int}}} f .$$

Def^n: A f^n (bounded) $f:[a,b] \to \mathbb{R}$ is said to be Riemann integrable if $\underset{a}{\overset{b}{\underline{\int}}} f = \underset{a}{\overset{b}{\overline{\int}}} f$.

Also, this Riemann Integral of f '$\int_a^b f$' $= \underset{a}{\overset{b}{\underline{\int}}} f = \underset{a}{\overset{b}{\overline{\int}}} f$

Notation : $\int_a^b f = \int_a^b f(x)\,dx = \int_a^b f(y)\,dy$.

Example: Let $a, b \in \mathbb{R}$ with $a < b$.

Define $f : [a,b] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [a,b] \\ 0 & \text{if } x \notin \mathbb{Q} \ \& \ x \in [a,b] \end{cases}$$

Thus, $L(P, f) = 0 \ \forall P$, $U(P, f) = (b-a) \ \forall P$.

$$\Rightarrow \underline{\int_a^b} f = 0, \quad \overline{\int_a^b} f = b-a.$$

Hence, f is not Reimann integrable.

Example: Constant f^n :- $f(x) = c$ for some $c \in \mathbb{R}$ $\forall x \in [a,b]$.

$$\Rightarrow L(P, f) = \sum_{i=1}^{n} c(x_i - x_{i-1}) = c(b-a) \ \forall P.$$

$$\Rightarrow U(P, f) = c(b-a) \ \forall P \text{ as well.}$$

So, $\underline{\int_a^b} f = \overline{\int_a^b} f = c(b-a)$. Hence, f is Reimann Integrable.

Example: $f : [a,b] \to \mathbb{R}$ s.t. $f(x) = x \ \forall x \in [a,b]$.

Now, take $P = (x_0, \dots, x_n)$. $\Rightarrow m_i(f) = x_{i-1}$, $M_i(f) = x_i$

$$\Rightarrow L(P, f) = \sum_{i=1}^{n} x_{i-1}(x_i - x_{i-1})$$

$$\Rightarrow U(P, f) = \sum_{i=1}^{n} x_i(x_i - x_{i-1})$$

Now, fix $n \geq 1$; $x_i = a + \frac{i(b-a)}{n}$ s.t. $0 \leq i \leq n$.

$$\Rightarrow L(P,f) = \sum_{i=1}^{n}\left(a + \frac{(i-1)(b-a)}{n}\right)\left(\frac{b-a}{n}\right) = a(b-a) + \frac{(b-a)^2(n-1)}{2n}$$

Hence, $L(P,f) = a(b-a) + \frac{(b-a)^2}{2}\left(1-\frac{1}{n}\right)$.

Similarly, $U(P,f) = a(b-a) + \frac{(b-a)^2}{2}\left(1+\frac{1}{n}\right)$.

$$\Rightarrow \sup(L(P,f)) \geq \sup_n\left(a(b-a) + \frac{(b-a)^2}{2}\left(1-\frac{1}{n}\right)\right) = \frac{1}{2}(b^2-a^2).$$

$$\Rightarrow \inf(U(P,f)) \leq \inf_n\left(a(b-a) + \frac{(b-a)^2}{2}\left(1+\frac{1}{n}\right)\right) = \frac{1}{2}(b^2-a^2).$$

$$\Rightarrow \overline{\int_a^b} f \leq \frac{b^2-a^2}{2} \quad \& \quad \underline{\int_a^b} f \geq \frac{b^2-a^2}{2}.$$

Hence, $\int_a^b f = \left(\frac{b^2-a^2}{2}\right)$.

Theorem: Riemanns Criteria for Integrability :-
Let $f:[a,b]\to\mathbb{R}$ be a bounded f^n, then f is a riemann integral iff $\forall\ \varepsilon>0\ \exists$ a partition P_ε st.

$$U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon$$

Proof: ($\Leftarrow$) given an $\varepsilon>0$, choose P_ε st. $U(P_\varepsilon,f) - L(P_\varepsilon,f) < \varepsilon$

$\Rightarrow U(P_\varepsilon,f) \leq L(P_\varepsilon,f) + \varepsilon$.

$\because \overline{\int_a^b} f = \inf_P(U(P,f))$, $\Rightarrow \overline{\int_a^b} f \leq L(P_\varepsilon,f) + \varepsilon$

$$\Rightarrow \overline{\int_a^b} f \leq \underline{\int_a^b} f + \varepsilon \text{ is true } \forall\, \varepsilon > 0$$

$$\Rightarrow \overline{\int_a^b} f = \underline{\int_a^b} f.$$

Hence, f is Riemann Integrable.

$(\Rightarrow)$ $\exists$ Partitions P_1, P_2 of $[a, b]$ st:

$$U(f, P_1) < \overline{\int_a^b} f + \frac{\varepsilon}{2} \quad \& \quad L(f, P_2) > \underline{\int_a^b} f - \frac{\varepsilon}{2}.$$

Let $P_\varepsilon = P_1 \vee P_2$.

$$\Rightarrow U(f, P_\varepsilon) - L(f, P_\varepsilon) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, proved.

Exercise: Fix n, Show that $f(x) = x^n$ is Riemann Integrable.

Notation: Let $f : [a, b] \to \mathbb{R}$ be a f^n, then for $A \subset [a, b]$ st. $A \neq \{\}$, $f|_A$ is the f^n $f_A : A \to \mathbb{R}$ st. $f_A(x) = f(x)\ \forall\, x \in A$.

Theorem: Let $f : [a, b] \to \mathbb{R}$ be a bounded f^n. Fix $c \in (a, b)$. Then, f is Riemann Integrable iff $f|_{[a,c]}$ & $f|_{[c,b]}$ are Riemann integrable.

Also, $\int_a^b f = \int_a^c f + \int_c^b f$ (Do it on your own)

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Proof: suppose $f|_{[a,c]}$ & $f|_{[c,b]}$ are Reimann integrable.

Thus, for $\varepsilon > 0$, $\exists$ partition P_1 of $[a,c]$ st.

$$U(P_1, f|_{[a,c]}) - L(P_1, f|_{[a,c]}) < \varepsilon.$$

similarly, $\forall \varepsilon > 0$, $\exists$ P_2 of $[c,b]$ st.

$$U(P_2, f|_{[c,b]}) - L(P_2, f|_{[c,b]}) < \varepsilon.$$

Now, say $P_1 = (x_0, \dots, x_n)$ & $P_2 = (y_0, \dots, y_m)$.

Thus, $P_1 \vee P_2 = (x_0, \dots, x_n, y_1, \dots, y_m)$ (say).

Thus, $U(P_1 \vee P_2, f) = U(P_1, f|_{[a,c]}) + U(P_2, f|_{[c,b]})$

& $L(P_1 \vee P_2, f) = L(P_1, f|_{[a,c]}) + L(P_2, f|_{[c,b]})$.

$\Rightarrow U(P_1 \vee P_2, f) - L(P_1 \vee P_2, f) < 2\varepsilon$.

Also, $\int_a^b f = \int_a^c f + \int_c^b f$.

Conversely, if f is Reimann Integrable, then, $\forall \varepsilon > 0$ $\exists P_\varepsilon$

st. $U(P_\varepsilon, f) - L(P_\varepsilon, f) < \varepsilon$.

Take $P_1 = P_\varepsilon \vee (a, c, b)$.

$\Rightarrow U(P_1, f) - L(P_1, f) < \varepsilon$ aswell.

Suppose $P_1 = (x_0, \dots, x_{k-1}, c, x_{k+1}, \dots, x_n)$ for some k.

$$\Rightarrow U(P_1, f) - L(P, f) = \sum_{i=1}^{n} (M_i(f) - m_i(f))(x_i - x_{i-1}) \leq \varepsilon.$$

$$\Rightarrow \sum_{i=1}^{k} (M_i(f) - m_i(f))(x_i - x_{i-1}) \leq \varepsilon \quad \& \sum_{i=k+1}^{n} (M_i(f) - m_i(f))(x_i - x_{i-1})$$

$\leq \varepsilon$ aswell.

Hence, $f|_{[a,c]}$ & $f|_{[c,b]}$ are Reimann integrable.

Def^n: Let $f : [a,b] \to \mathbb{R}$ be a f^n. Then, f is said to be increasing (non-decreasing) if $f(x) \leq f(y)$ $\forall$ $x, y \in [a,b]$ st. $x \leq y$.

Theorem: Let $f : [a,b] \to \mathbb{R}$ be a f^n. If f is monotonic, then it is Reimann Integrable.

Proof: Assume that f is increasing (non-decreasing) st. $f(a) \neq f(b)$. suppose $P = (x_0, \dots, x_n)$ is a partition.

$$\Rightarrow U(P, f) - L(P, f) = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1}).$$

For $\varepsilon > 0$, choose $P_\varepsilon = (x_0, \dots, x_n)$ st. $x_i - x_{i-1} \leq \dfrac{\varepsilon}{f(b) - f(a)}$

Now, $f(x_i) - f(x_{i-1}) \leq f(b) - f(a)$

Thus, $U(P, f) - L(P, f) < \varepsilon$. Hence, proved.

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Theorem: Let $f : [a,b] \to \mathbb{R}$ be a continuous (bounded) f^n. Then, f is Riemann Integrable.

Proof: Any continuous f^n on closed interval is bounded. Also, f is uniformly continuous.

Now, $\varepsilon > 0$, take $\varepsilon' = \frac{\varepsilon}{b-a}$.

By Uniform Continuity of f, $\exists \; \delta > 0$ st $|f(x) - f(y)| < \varepsilon'$

$\forall \; x, y \in [a,b]$ with $|x-y| < \delta$.

Consider a Partition $P_\varepsilon = (x_0, x_1, \ldots, x_n)$ st. $|x_i - x_{i-1}| < \delta \;\forall i$

Fix $i \in \{1, 2, \ldots, n\}$ & consider $[x_{i-1}, x_i]$.

Now, for $x, y \in [x_{i-1}, x_i]$, $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$.

Choose $u \in [x_{i-1}, x_i]$ st. $f(u) = M_i(f) = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$
similarly say v st. $f(v) = m_i(f) = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$.

Hence, $M_i(f) - m_i(f) = |f(u) - f(v)| < \frac{\varepsilon}{b-a}$.

$\therefore$, $U(P_\varepsilon, f) - L(P_\varepsilon, f) = \sum_{i=1}^{n} (M_i(f) - m_i(f))(x_i - x_{i-1}) < \varepsilon$.

Hence, f is Riemann Integrable

Set-Up: Let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded f^n. Let $P = (x_0, \dots, x_n)$ be a partition of $[a,b]$.

Def^n: A tuple $\underline{t} = (t_1, \dots, t_n)$ is said to be a tag for a partition of P if $x_{i-1} \leq t_i \leq x_i \ \forall \ i$.

Def^n: Let $\underline{t}$ be a tag for a partition P. Then, the Riemann sum is defined as (for a f^n f with partition P & tag $\underline{t}$)

$$R(P, \underline{t}, f) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}).$$

\# For any tag $\underline{t}$ of P, $L(P, f) \leq R(P, \underline{t}, f) \leq U(P, f)$.

Lemma: Under this set-up, $L(P, f) = \inf\limits_{\underline{t}} R(P, \underline{t}, f)$ &

$$U(P, f) = \sup_{\underline{t}} R(P, \underline{t}, f).$$

Proof: Here P, f are fixed.

Recall $m_i(f) = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$

For $\varepsilon > 0$, choose t_i s.t. $m_i(f) \leq f(t_i) \leq m_i(f) + \frac{\varepsilon}{b-a}$

Thus, $L(P, f) \leq \sum\limits_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \leq L(P, f) + \varepsilon$.

This is possible $\forall \ \varepsilon > 0$, Hence, $L(P, f) = \inf\limits_{\underline{t}} R(P, \underline{t}, f)$.

similarly, $U(P, f) = \sup\limits_{\underline{t}} R(P, \underline{t}, f)$.

Theorem: Let $f : [a,b] \to \mathbb{R}$ be a bounded f^n. Then f is Riemann integrable iff $\forall \ \varepsilon > 0$, $\exists \ P_\varepsilon$ s.t. $|R(P_\varepsilon, s, f) - R(P_\varepsilon, t, f)| <$ for any 2 tags s, t of P_ε.

Claim: Suppose $|R(P_\varepsilon, \underline{s}, f) - R(P_\varepsilon, \underline{t}, f)| < \varepsilon \ \forall$ tags $\underline{s}, \underline{t}$ of P_ε, Then for any refinement P of P_ε :-

$$|R(P, \underline{u}, f) - R(P, \underline{v}, f)| < \varepsilon \ \forall \text{ tags } \underline{u}, \underline{v} \text{ of } P.$$

Theorem: Let $f : [a, b] \to \mathbb{R}$ be a bounded f^n. Then f is Riemann integrable iff $\exists$ a real number I st. $\forall \varepsilon > 0$, $\exists P_\varepsilon$:

$$|I - R(P_\varepsilon, \underline{t}, f)| < \varepsilon \ \forall \text{ tags } \underline{t} \text{ of } P_\varepsilon.$$

Thus, $\int_a^b f = I$.

Proof: ($\Leftarrow$) i.e. $\forall \varepsilon > 0$, $\exists P_\varepsilon$ st. $|I - R(P_\varepsilon, \underline{t}, f)| < \varepsilon \ \forall$ tags $\underline{t}$ of P_ε.

Now, $I - \varepsilon < R(P_\varepsilon, \underline{t}, f) < I + \varepsilon$.

$\Rightarrow I - \varepsilon < U(P_\varepsilon, f) \leq I + \varepsilon$ & $I - \varepsilon \leq L(P_\varepsilon, f) < I + \varepsilon$.

Thus, $U(P_\varepsilon, f) - L(P_\varepsilon, f) \leq 2\varepsilon$.

Hence, f is Riemann Integrable.

Aliter: Using Δ inequality:

Now, $U(P_\varepsilon, f) \leq I + \varepsilon$. $\Rightarrow \int_a^b f = \overline{\int_a^b} f \leq I + \varepsilon$; $\forall \varepsilon > 0$

similarly, $\int_a^b f = \underline{\int_a^b} f \geq I - \varepsilon$; $\forall \varepsilon > 0$.

Thus, $\int_a^b f = I$.

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$(\Rightarrow)$ Take $I = \int_a^b f$.

$\forall \varepsilon > 0, \exists P_1$ st. $U(P_1, f) \leq I + \frac{\varepsilon}{2}$.

similarly, $\exists P_2$ st. $L(P_2, f) \geq I - \frac{\varepsilon}{2}$.

Take $P_\varepsilon = P_1 \vee P_2$. Thus, $I - \frac{\varepsilon}{2} \leq L(P_\varepsilon, f) \leq R(P_\varepsilon, \underline{t}, f) \leq$

$U(P_\varepsilon, f) \leq I + \frac{\varepsilon}{2}$ $\forall$ tags $\underline{t}$ of P_ε.

Thus, $\left| I - R(P_\varepsilon, \underline{t}, f) \right| < \varepsilon$ $\forall$ tags $\underline{t}$ of P_ε

Theorem: Let $f, g : [a, b] \to \mathbb{R}$ be bounded f^ns. Suppose f, g are Riemann integrable. Then, $f+g$ is Riemann Integrable &

$$\int_a^b f+g = \int_a^b f + \int_a^b g.$$

Proof: Take $P = (x_0, \dots, x_n)$.

Now, $M_i(f+g) = \sup \{ (f+g)(x) : x \in [x_{i-1}, x_i] \}$

$\Rightarrow M_i(f+g) \leq M_i(f) + M_i(g)$.

similarly, $m_i(f+g) \geq m_i(f) + m_i(g)$.

Thus, $U(P, f+g) - L(P, f+g) \leq \left[U(P,f) - L(P,f) \right] + \left[U(P,g) - L(P,g) \right]$

Now, take P_1 for f, P_2 for g for some $\frac{\varepsilon}{2}$ to get $P_\varepsilon = P_1 \vee P_2$

st. $U(P_\varepsilon, f+g) - L(P_\varepsilon, f+g) < \varepsilon$.

Hence, $f+g$ is Riemann Integrable.

Aliter: $\forall\ \varepsilon > 0,\ \exists\ P_1$ st. $\left|\int_a^b f - R(P_1, t, f)\right| < \frac{\varepsilon}{2}\ \forall\ t$ of P_1

& $\exists\ P_2$ st. $\left|\int_a^b g - R(P_2, s, g)\right| < \frac{\varepsilon}{2}\ \forall\ s$ of P_2.

$\Rightarrow$ for $P_\varepsilon = P_1 \vee P_2$, $\left|\int_a^b f + \int_a^b g - R(P_\varepsilon, t, f+g)\right| < \varepsilon$.

Hence, $f+g$ is Riemann Integrable & $\int_a^b f+g = \int_a^b f + \int_a^b g$.

Theorem: Let $f : [a, b] \to \mathbb{R}$ be a bounded f^n. Suppose f is Riemann integrable, then for any $r \in \mathbb{R}$, rf is Riemann Integrable & $\int_a^b (rf) = r \int_a^b f$.

Theorem: Let $f, g : [a, b] \to \mathbb{R}$ be bounded f^ns. Suppose f, g are Riemann Integrable. Then, $f.g$ is Riemann Integrable.

Proof: Now, $|f(x)| \leq M$ & $|g(x)| \leq N\ \ \forall\ x \in [a, b]$ for some $M, N \in \mathbb{R}$.
Thus, $|fg(x)| \leq MN$.

Hence, $f.g$ is bounded.

Let $P = (x_0, \dots, x_n)$ be a partition of $[a, b]$.

For $x, y \in [x_{i-1}, x_i]$, $(fg)(x) - (fg)(y) = fg(x) - fg(y) \pm f(x)g(y)$

$$\Rightarrow |fg(x) - fg(y)| \leq |f(x)||g(x)-g(y)| + |g(y)||f(x)-f(y)|$$

$$\leq \underbrace{M\left(M_i(g) - m_i(g)\right) + N\left(M_i(f) - m_i(f)\right)}_{\text{Independant over } x,y}$$

$$\Rightarrow M_i(fg) - m_i(fg) \leq M\left(M_i(g) - m_i(g)\right) + N\left(M_i(f) - m_i(f)\right)$$

$$\Rightarrow U(P, fg) - L(P, fg) \leq M\left(U(P,g) - L(P,g)\right) + N\left(U(P,f) - L(P,f)\right)$$

Now, we are done.

Hence, $f \cdot g$ is Riemann Integrable as well.

Theorem: Let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded f^n. Suppose f is Riemann Integrable & $\exists\ \delta > 0$ st. $|f(x)| \geq \delta \ \forall\ x \in [a,b]$.
Then $\dfrac{1}{f(x)}$ is Riemann Integrable.

Proof: Thus, $\forall\ x \in [a,b]$, $\left|\dfrac{1}{f(x)}\right| \leq \dfrac{1}{\delta}$. Hence, it is bounded as well.

Let $P = (x_0, \dots, x_n)$ be a partition of $[a,b]$.
Now, for $x, y \in [x_{i-1}, x_i]$,

$$\left|\frac{1}{f(x)} - \frac{1}{f(y)}\right| = \frac{|f(x) - f(y)|}{|f(x)\, f(y)|} \leq \frac{M_i(f) - m_i(f)}{\delta^2}$$

$$\Rightarrow U\left(P, \frac{1}{f}\right) - L\left(P, \frac{1}{f}\right) \leq \left(\frac{U(P,f) - L(P,f)}{\delta^2}\right)$$

Hence, $\dfrac{1}{f(x)}$ is Riemann Integrable.

Lemma: Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded f^n. Assume that f is Riemann Integrable. Suppose $g : [a,b]$ is a f^n st. $g(x) = f(x)$ $\forall$ $x \in (a, b]$. Then, g is Riemann Integrable.

Proof: say $h(x)$ st $h = g - f$. Thus, $h(x) = \begin{cases} 0 & ; x \neq a \\ g(a) - f(a) & ; x = a \end{cases}$

It is enough to show that h is Riemann Integrable.

Clearly, $\int_a^b h(x)\, dx = 0$.

$$\Rightarrow \int_a^b f = \int_a^b g.$$

similarly, if $g(x) = f(x)$ $\forall$ $x \notin S$ where $S \subseteq [a, b]$ is a finite set, then the lemma is true aswell.

Proposition: Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded f^n. Say, $C \leq f(x) \leq D$ $\forall$ $x \in [a, b]$ & f is Riemann Integrable. Then :-

$$C(b-a) \leq \int_a^b f(x)\, dx \leq D(b-a).$$

Lemma: Let $f, g : [a, b] \rightarrow \mathbb{R}$, be bounded & Riemann Integrable f^ns. If $f(x) \leq g(x)$ $\forall$ $x \in [a, b]$, then $\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$.

Theorem: Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded f^n which is Riemann Integrable as well. Then, $|f|$ is Riemann Integrable &

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx$$

Proof: Let $P = (x_0, \dots, x_n)$ be a partition of $[a, b]$. Consider $[x_{i-1}, x_i]$

Now, $|f(x)| - |f(y)| \leq |f(x) - f(y)| \; \forall \; x, y \in [a,b]$.

Thus, $\underbrace{\sup_{x,y} |f(x)| - |f(y)|}_{M_i(|f|) - m_i(|f|)} \leq \underbrace{\sup_{x,y} |f(x) - f(y)|}_{M_i(f) - m_i(f)}$

$\Rightarrow U(P, |f|) - L(P, |f|) \leq U(P, f) - L(P, f)$.

Hence, $|f|$ is Riemann Integrable.

Now, we are done.

Theorem: Let $f : [a,b] \to \mathbb{R}$ be a f^n. Then, f decomposes uniquely as $f = f_+ - f_-$ where $f_+ : [a,b] \to \mathbb{R}$ & $f_- : [a,b] \to \mathbb{R}$ are non-negative & $f_+ \cdot f_- = 0$.

Proof: Only Possible f^ns (Check on your own) :-

$$f_+ = \begin{cases} f(x) & ; f(x) \geq 0 \\ 0 & ; \text{otherwise.} \end{cases} \quad \& \quad f_- = \begin{cases} -f(x) & ; f(x) \leq 0 \\ 0 & ; \text{otherwise.} \end{cases}$$

Lemma: $\int_a^b f(x)\,dx = \int_a^b f_+(x)\,dx - \int_a^b f_-(x)\,dx$.

\# Let $f : [a,b] \to \mathbb{R}$ be a bounded, Riemann Integrable f^n.
For $c \in (a,b]$,

$$\int_a^c f := \int_a^c f|_{[a,c]}.$$

By Convention, $\int_a^a f := 0$.

For $a \leq d \leq c \leq b$, $\int_c^d f = -\int_d^c f$.

Let $f : [a,b] \to \mathbb{R}$ be a bounded, Riemann Integrable f^n.
Define f^n $F : [a,b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\,dt = \int_a^x f.$$

Note that $F(a) = \int_a^a f = 0$.

In good structures, F is differentiable & $F'(x) = f(x)\ \forall\, x \in [a,b]$.

Theorem: Lipschitz Continuity : Let $f : [a,b] \to \mathbb{R}$ be a bounded, Riemann integrable f^n. Define $F : [a,b] \to \mathbb{R}$ by $F(x) = \int_a^x f(t)\,dt$.
Then, f is Lipschitz continuous.

Proof: Choose $M > 0$ st. $|f(x)| \leq M\ \forall\, x \in [a,b]$.
For $x, y \in [a,b]$, we want to estimate $|F(x) - F(y)|$.
If $x = y$, $|F(x) - F(y)| = 0$.

If $x > y$, $|F(x) - F(y)| = \left| \int_a^x f - \int_a^y f \right| = \left| \int_y^x f(t)\,dt \right|$

$$\leq \int_y^x |f(t)|\,dt$$

$$\Rightarrow |F(x) - F(y)| \leq M|x-y| \quad \forall\, x, y \in [a,b].$$

Def^n: Let $f : [a,b] \to \mathbb{R}$ be a f^n. Suppose $G : [a,b] \to \mathbb{R}$ is a differentiable f^n & $G'(x) = f(x)\ \forall\, x \in [a,b]$.
Then, G is said to be the antiderivative of f.

Remark: If G_1, G_2 are antiderivatives of f, Then, $G_1 - G_2$ is differentiable & $(G_1 - G_2)'(x) = 0 \ \forall x \in [a,b]$.

$\therefore$, By Rolles Theorem, $(G_1 - G_2)(x) = c$ (constant).

Fundamental Theorem of Calculus (Part - I):

Let $f : [a,b] \to \mathbb{R}$ be a bounded, Riemann Integrable f^n. Suppose f admits anti-derivatives & $F : [a,b] \to \mathbb{R}$ is an anti-derivative of f.

Then, $$\int_a^x f(t)\,dt = F(x) - F(a) \quad \forall x \in [a,b].$$

Proof: For $x = a$, there is nothing to prove.

Now, take $x = b$. We want to show $\int_a^b f = F(b) - F(a)$.

For $\varepsilon > 0$, choose $P = (x_0, \dots, x_n)$ s.t. $U(P,f) - L(P,f) < \varepsilon$.

Consider $[x_{i-1}, x_i] \ \forall i \in \{1, 2, \dots, n\}$.

Consider F on $[x_{i-1}, x_i]$.

By Mean Value Theorem, $\exists \ t_i \in [x_{i-1}, x_i]$,

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = f(t_i).$$

Thus, $$F(b) - F(a) = \sum_{i=1}^{n} F(x_i) - F(x_{i-1})$$

$$= \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$$

$$= R(P, \underline{t}, f).$$

Now, $L(P,f) \leq R(P, \underline{t}, f) \leq U(P,f)$.

Also,
$$L(P,f) \leq \int_a^b f \leq U(P,f).$$

Thus, $\left| \int_a^b f - R(P, \underline{t}, f) \right| < \varepsilon$ as well.

Thus, $\left|\int_a^b f - (F(b) - F(a))\right| < \varepsilon \quad \forall\ \varepsilon > 0$.

Hence, $\int_a^b f(t)\,dt = F(b) - F(a)$.

Now, we are done.

Fundamental Theorem of Calculus (Part-II):

Let $f : [a,b] \to \mathbb{R}$ be a bounded, Riemann Integrable f^n. Define $F : [a,b] \to \mathbb{R}$ by $F(x) = \int_a^x f(t)\,dt$. Suppose, f is continuous at $c \in [a,b]$. Then F is differentiable at c & $F'(c) = f(c)$.

In particular, if f is continuous, F is differentiable & $F'(x) = f(x)\ \forall\ x \in [a,b]$.

Proof: Assume that f is continuous at $c \in [a,b]$.

For $\varepsilon > 0$, choose $\delta > 0$ s.t. $|f(x) - f(c)| < \varepsilon \quad \forall\ x \in (c-\delta, c+\delta) \cap [a,b]$.

For $x \in (c-\delta, c+\delta) \cap [a,b]$ & $x \neq c$,

$$\frac{F(x) - F(c)}{x - c} = \frac{1}{x-c} \int_c^x f(t)\,dt.$$

$$\Rightarrow \frac{F(x) - F(c)}{x - c} - f(c) = \frac{1}{x-c} \int_c^x f(t)\,dt - f(c).$$

$$\Rightarrow \left|\frac{F(x) - F(c)}{x - c} - f(c)\right| = \left|\frac{1}{x-c} \int_c^x (f(t) - f(c))\,dt\right|$$

$$\Rightarrow \left|\frac{F(x) - F(c)}{x - c} - f(c)\right| \leq \frac{1}{|x-c|} \int_c^x |f(t) - f(c)|\,dt.$$

$$\leq \frac{1}{|x-c|} \int_c^x \varepsilon\,dt = \varepsilon.$$

Now, we are done.

Lemma: A f^n $F:[a,b] \longrightarrow \mathbb{R}$ is continuously differentiable iff $\exists$ a continuous f^n $f:[a,b] \longrightarrow \mathbb{R}$ st. $F(x) = F(a) + \int_a^x f(t)\,dt$.
In such a case, $F'(x) = f(x)$ $\forall$ $x \in [a,b]$.

Proof: suppose F is continuously differentiable.
Define $f:[a,b] \longrightarrow \mathbb{R}$ by $f(x) = F'(x)$ $\forall$ $x \in [a,b]$.
Thus, $F(x) = F(a) + \int_a^b f(t)\,dt$ (by FToC part - I)
The Converse follows by FToC part - II.

Notation: $F(x)\Big|_{x=a}^{x=b} := F(b) - F(a)$.

Theorem: Let $f, g : [a,b] \longrightarrow \mathbb{R}$ be f^ns. Assume that :-
1. f is differentiable & f' is integrable.
2. g is integrable with an anti-derivative G.

Then,
$$\int_a^b f(t)\,g(t)\,dt = f(b)\,G(b) - f(a)\,G(a) - \int_a^b f'(t)\,G(t)\,dt$$

Proof: Define $H:[a,b] \longrightarrow \mathbb{R}$ by $H(x) = f(x)\,G(x)$ $\forall$ $x \in [a,b]$.
As f, G are differentiable $\Rightarrow$ H is differentiable.

$$\text{Thus, } H'(x) = f(x)\,G'(x) + f'(x)\,G(x)$$
$$= f(x)\,g(x) + f'(x)\,G(x)$$

$\because$ f is differentiable, f is continuous $\Rightarrow$ f is integrable.
similarly, G is integrable as well.
Thus, $H'(x)$ is integrable.

Now, H' has an anti-derivative, namely H.
Hence, FtoC is applicable.
$\therefore$, $\int_a^b H'(t)\,dt = H(b) - H(a)$.

Thus, $$\int_a^b f(t)g(t)\,dt = f(b)G(b) - f(a)G(a) - \int_a^b f'(t)G(t)\,dt.$$

Theorem: Let $\varphi : [\alpha, \beta] \to \mathbb{R}$ be a differentiable f^n s.t. φ' is integrable on $[\alpha, \beta]$. Suppose $\varphi([\alpha, \beta]) = [a, b]$. If $f : [a, b] \to \mathbb{R}$ is continuous, then the f^n $(f(\varphi(x))\varphi'(x)$ is Riemann Integrable &
$$\int_\alpha^\beta (f \circ \varphi) \cdot \varphi'(t)\,dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(t)\,dt ;$$

$a = \{\varphi(t) : t \in [\alpha, \beta]\}$ & $b = \sup\{\varphi(t) : t \in [\alpha, \beta]\}$.

Proof: Consider $f : [a, b] \to \mathbb{R}$, defined by $F(x) = \int_a^x f(t)\,dt$.
Then, by FtoC (Part-II), F is differentiable &
$$F'(x) = f(x) \;\forall\; x \in [a, b].$$

Define $H : [\alpha, \beta] \to \mathbb{R}$ by $H = F \circ \varphi$.
Thus, $H(x) = F(\varphi(x))$.
Now, $H'(x) = F'(\varphi(x))\varphi'(x) = f(\varphi(x))\varphi'(x) \;\forall\; x \in [\alpha, \beta]$.

Thus, H' is Riemann Integrable.
Note that H is an anti-derivative for H'.
Thus, by FtoC (Part-I),
$$\int_\alpha^\beta H'(t)\,dt = H(\beta) - H(\alpha).$$

$$\Rightarrow \int_{\alpha}^{\beta} f(\varphi(t)) \cdot \varphi'(t)\,dt = F(\varphi(\beta)) - F(\varphi(\alpha)).$$

$$= \int_{a}^{\varphi(\beta)} f(t)\,dt - \int_{a}^{\varphi(\alpha)} f(t)\,dt.$$

If $\varphi(\alpha) \leq \varphi(\beta) \Rightarrow \displaystyle\int_{\varphi(\alpha)}^{\varphi(\beta)} f(t)\,dt = \int_{\alpha}^{\beta} f(\varphi(t))\,\varphi'(t)\,dt.$

otherwise, $F(\varphi(\beta)) - F(\varphi(\alpha)) = \displaystyle\int_{a}^{\varphi(\beta)} f(t)\,dt - \left(\int_{a}^{\varphi(\beta)} f(t)\,dt + \int_{\varphi(\beta)}^{\varphi(\alpha)} f(t)\,dt \right)$

$$= \int_{\varphi(\alpha)}^{\varphi(\beta)} f(t)\,dt.$$

Hence, proved.

\# In Ghorpade, there is a proof of a better statement where f is not continuous.

Lemma :

Integrals of polynomials of low degree:

Let $f : [a,b] \to \mathbb{R}$ be a polynomial:-

1. If $deg(f) \leq 1$, then $\int_a^b f(t)\,dt = \left(\frac{b-a}{2}\right)\left(f(a) + f(b)\right)$.
2. If $deg(f) \leq 2$, then $\int_a^b f(t)\,dt = \left(\frac{b-a}{2}\right)\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$.

Consider polynomials, $f_0(x) = 1 \;\forall x$, $f_1(x) = x \;\forall x$, $f_2(x) = x \;\forall x$.

It is enough to prove the formula for these f^n's $\because$ they form a basis of the vector space of polynomials of degree 2 or less.

Problems:

1. $f : [a, \infty)$ is twice differentiable & $M_i = \sup\left\{ |f^i(x)| : x \in (a, \infty) \right\}$

$$i = 0, 1, 2$$

Then $M_1^2 \leq 4 M_0 M_2$.

2. $f : [0,1] \to [0,1]$ is continuous $\Rightarrow$ it has fixed point

$f : [0, \infty) \to [0, \infty)$ is continuous & bounded $\Rightarrow$ it has fixed point.

3. $f : [0,1] \to [0, \infty)$ is continuous & satisfies $\int_0^x f(t)\,dt \geq f(x) \;\forall x$,

$\Rightarrow f$?

4. $g(x) := \int_0^x f(t)\,dt$, $g'(x) = f(x)$, $F(x) = e^{-x} g(x)$, comment on sign of $F(x)$ ($f : [0,1] \to [0, \infty)$ is as follows here)

5. $\alpha \in \mathbb{Q}^c$, $a_n = \frac{[n\alpha]}{n} \Rightarrow a_n \to \alpha$.

6. $\mathcal{P}_0 \subseteq \mathcal{P}(\mathbb{N})$ st. $\phi \neq A, B \in \mathcal{P}_0 \Rightarrow A \cap B = \phi$. Give example of $\mathcal{P}_0$:- (finite, countably infinite, $\underbrace{\text{Uncountable}}_{\text{Not possible}}$)

Disconnected subset of $\mathbb{R}$:-

$A \subseteq \mathbb{R}$ is said to be disconnected if $\exists$ 2 open subsets of $\mathbb{R}$; U, V st :-

1. $A \subseteq U \cup V$
2. $U \cap V = \{ \}$
3. $U, V \neq \{ \}$
4. $A \cap V \neq \{ \}$, $A \cap U \neq \{ \}$

$[a, b]$ is connected.

Proof: Suppose $[a, b]$ is disconnected.

$\Rightarrow [a, b] \subseteq U \cup V$, $U \cap V = \{ \}$, $U, V \neq \{ \}$.

Without loss of generality, say $a \in U$.

Consider $S = \{ x \in [a, b] : [a, x) \subseteq U \} \neq \{ \}$ $\because a \in U$ & U is an open set.

Note that $b \notin S$, $y \in [a, b] \cap V \Rightarrow x \leq y \ \forall x \in S$.

Thus, S has an lub (say L).

Also, $a \leq L \leq b$.

$(a, L) \subseteq U \cap [a, b]$. if $z \in U \cap [a, b]$, $z \in (a, L)$.

$\Rightarrow z \in V \cap [a, b]$.

But this contradicts that L is the lub of S.

Q: $L \in U \cap [a, b]$?

$\Rightarrow \exists \delta$ st $L \in (L - \delta, L + \delta) \subseteq U \cap [a, b]$.

$\Rightarrow (a, L) \subseteq U \cap [a, b]$.

$\Rightarrow [a, L + \delta) \subseteq U \cap [a, b]$.

But L is the lub of S.

Thus, $L \notin U \cap [a, b]$.

suppose $L \in V \cap [a,b]$.

$\Rightarrow \exists\ \delta>0$ st $(L-\delta, L+\delta) \subseteq V \cap [a,b]$.

But, this implies $\forall\ x \in (L-\delta, L)$, x is upper bound of S & $x \leq L$,

which is a contradiction.

Hence, $L \notin V \cap [a,b]$.

Thus, $[a,b]$ is connected.

Def^n: Let $P = (x_0, \dots, x_n)$ be a partition of $[a,b]$.
The mesh of P is defined as :- $\mu(P) = \max\limits_i (x_i - x_{i-1})$.

Lemma: Let $f : [a,b] \to \mathbb{R}$ be a bounded f^n. Then for $\varepsilon > 0$, $\exists\ \delta > 0$ st.
if P is any partition with $\mu(P) < \delta$, then $U(P,f) < \overline{\int_a^b} f + \varepsilon$

Proof: Consider $M \in \mathbb{R}^{+ve}$ st. $|f(x)| \leq M\ \forall\ x \in [a,b]$.

$\forall\ \varepsilon > 0$, choose $P_0 = (y_0, \dots, y_{n_0})$ st. $\overline{\int_a^b} f + \frac{\varepsilon}{2} > U(P_0, f)$

Take $\delta = \frac{\varepsilon}{4Mn_0}$.

Let $P = (x_0, \dots, x_n)$ be a partition st $\mu(P) < \delta$.

Take $P' = P_0 \vee P$. $\because$ P' is finer than P_0,

$\Rightarrow U(P', f) \leq U(P_0, f) < \overline{\int_a^b} f + \frac{\varepsilon}{2}$

Fix $i \in \{1, 2, \dots, n\}$ & consider $[x_{i-1}, x_i]$.

Suppose $(v_0, \dots, v_r)$ are the points of P' in $[x_{i-1}, x_i]$.

Thus, $M_i(f)(x_i - x_{i-1}) - \sum\limits_{i=1}^{r} M_i'(f)(v_i - v_{i-1}) = U(P,f) - U(P', f)$ in $[x_{i-1}, x_i]$

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$$= \sum_{i=1}^{l} \left(M_i(f) - M_i'(f)\right)(v_i - v_{i-1}) \leq \sum_{i=1}^{l} (2M)(v_i - v_{i-1})$$

$$= 2M(x_i - x_{i-1})$$

$$< 2M\delta$$

$$\Rightarrow \underset{\text{in } [x_{i-1}, x_i]}{U(P, f) - U(P', f)} \leq \frac{\varepsilon}{2n_0}.$$

Now, the no. of terms in $P' = P \vee P_0$, is atmost n_0 more than number of terms in P.

$$\Rightarrow U(P, f) - U(P', f) \leq n_0 \frac{\varepsilon}{2n_0} = \frac{\varepsilon}{2}.$$

$$\Rightarrow U(P, f) \leq U(P', f) + \frac{\varepsilon}{2} \leq \overline{\int_a^b} f + \varepsilon$$

Now, we are done.

Lemma: Let $f : [a, b] \to \mathbb{R}$ be a bounded f^n. Then, $\forall \varepsilon > 0$, $\exists \delta > 0$ st $\forall$ partitions P with $\mu(P) < \delta$;

$$U(P, f) < \overline{\int_a^b} f + \varepsilon \quad \& \quad L(P, f) > \underline{\int_a^b} f - \varepsilon$$

Proof: Choose $\delta_1 > 0$ st. if $\mu(P) < \delta_1$,

$$U(P, f) < \overline{\int_a^b} f + \varepsilon$$

Choose $\delta_2 > 0$ st. if $\mu(P) < \delta_2$,

$$L(P, f) < \underline{\int_a^b} f - \varepsilon.$$

Take $\delta = \min(\delta_1, \delta_2)$.
Now, we are done.

Corollary: Let $f : [a,b] \to \mathbb{R}$ be a bounded f^n. Then, $\forall \varepsilon > 0$, $\exists \delta > 0$ st. for any partition P with $\mu(P) < \delta$ & any tag $\underline{t}$ of P :-

$$\underline{\int_a^b} f - \varepsilon < R(P, \underline{t}, f) < \overline{\int_a^b} f + \varepsilon .$$

Theorem: Let $f : [a,b] \to \mathbb{R}$ be a bounded f^n. Say f is Riemann Integrable. Then, $\forall \varepsilon > 0$ $\exists \delta > 0$ st. for any partition P with $\mu(P) < \delta$ & tag $\underline{t}$ of P,

$$\left| \int_a^b f - R(P, \underline{t}, f) \right| < \varepsilon .$$

Conversely, if $\exists I \in \mathbb{R}$ st. $\forall \varepsilon > 0$, $\exists \delta > 0$ satisfying :- for any partition P with $\mu(P) < \delta$ & tag $\underline{t}$,

$$\text{If } \left| \underbrace{\int_a^b}_{I \text{ (replace)}} f - R(P, \underline{t}, f) \right| < \varepsilon \Rightarrow f \text{ is Riemann integrable } \& \int_a^b f = I.$$

Corollary: Let $f : [a,b] \to \mathbb{R}$ be a bounded f^n.
Assume that f is Riemann integrable.
Let $\{P_n\}$ be a sequence of partitions in $[a,b]$; $n \geq 1$
st. $\lim_{n \to \infty} \mu(P_n) = 0$,

$$\text{then } \lim_{n \to \infty} R(P_n, t_n, f) = \int_a^b f .$$

Riemann - Stieltjes Integration :

Given an increasing f^n $\alpha : [a,b] \to \mathbb{R}$

Note: ① α need not be strictly increasing.
② α need not be continuous.
③ α is bounded as $\alpha(a) \leq \alpha(x) \leq \alpha(b)$ $\forall x \in [a,b]$.

Let $f : [a,b] \to \mathbb{R}$ be a bounded f^n.
Let $P = (x_0, \dots, x_n)$ be a partition of $[a,b]$.
Let $m_i(f)$, $M_i(f)$ be as usual.
Let $m(f) = \inf\{f(x) : x \in [a,b]\}$, $M(f) = \sup\{f(x) : x \in [a,b]\}$

Def^n: $L(P, f, \alpha) = \sum\limits_{i=1}^{n} m_i(f)(\alpha(x_i) - \alpha(x_{i-1}))$.

$U(P, f, \alpha) = \sum\limits_{i=1}^{n} M_i(f)(\alpha(x_i) - \alpha(x_i))$.

Proposition :

① For any partition P, $m(f)(\alpha(b) - \alpha(a)) \leq L(P, f, \alpha) \leq U(P, f, \alpha) \leq M(f)(\alpha(b) - \alpha(a))$

② If Q is a partition finer than P, then :
$L(P, f, \alpha) \leq L(Q, P, \alpha)$ & $U(Q, f, \alpha) \leq U(P, f, \alpha)$.

③ If P, Q are any 2 partitions of $[a,b]$, $L(P, f, \alpha) \leq U(Q, f, \alpha)$

Def^n: Let $f : [a,b] \to \mathbb{R}$ be a bounded f^n. Let $\alpha : [a,b] \to \mathbb{R}$ be an increasing f^n. Then, the lower R.S.I. of f wrt. α is :- $\underline{\int_a^b} f \, d\alpha = \sup\{L(P, f, \alpha) : P \text{ is a partition of } [a,b]\}$

Similarly, $\overline{\int_a^b} f \, d\alpha = \inf \{ U(P, f, \alpha) : P \text{ is a partition of } [a,b] \}$

Further, f is said to be R.S.I. if $\overline{\int_a^b} f \, d\alpha = \underline{\int_a^b} f \, d\alpha$.

Theorem: Let $f : [a, b] \to \mathbb{R}$ be a bounded f^n. Then f is R.I. wrt α iff $\forall \, \varepsilon > 0$, $\exists$ a $P_\varepsilon > 0$ st.

$$U(P_\varepsilon, f, \alpha) - L(P_\varepsilon, f, \alpha) < \varepsilon.$$

Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is R.S.I. wrt α.

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Theorem: Let $\alpha : [a,b] \to \mathbb{R}$ be increasing & continuous. Suppose $f : [a,b] \to \mathbb{R}$ is monotonic. Then, f is R.S.I. wrt α.

Proof: Assume that f is increasing & $f(a) \neq f(b)$.
$\forall\, \varepsilon > 0$, choose $\delta > 0$ st.

$$|\alpha(x) - \alpha(y)| < \frac{\varepsilon}{(f(b) - f(a))} \quad \text{for } |x-y| < \delta$$

Now, take any partition $P = (x_0, \dots, x_n)$ st. $\mu(P) < \delta$.

Then, $\sum_{i=1}^{n} (M_i(f) - m_i(f))(\alpha(x_i) - \alpha(x_{i-1}))$

$$= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(\alpha(x_i) - \alpha(x_{i-1}))$$

$$\leq \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\left(\frac{\varepsilon}{f(b) - f(a)}\right) = \varepsilon.$$

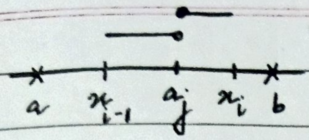
Now, we are done.

\# Suppose $a < a_1 < a_2 < \dots < a_k < b$ & X is a R.V., taking values in $\{a_1, \dots, a_k\}$ st.
$P(x = a_i) = p_i$ & $\sum_{j=1}^{k} p_j = 1$.

\# Then, the distribution of X is given by $G : \mathbb{R} \to [0,1]$;
$G(x) = P(X \leq x)$

\# Let $f : [a,b] \to \mathbb{R}$ be a continuous f^n. Then f is R.S.I wrt. G

Q: $\int_a^b f \, dG = ?$

\# 

Claim : $\int f\, dG = \sum_{i=1}^{k} f(a_i)\, p_i = \mathbb{E}(f(x))$

$\forall \varepsilon > 0$, choose $\delta > 0$ st. $|f(x) - f(y)| < \frac{\varepsilon}{\text{[redacted]}}$ if $|x-y| < \delta$.

Now, choose a partition P of $[a,b]$ st $\mu(P) < \delta$, & for every j, $\exists$ unique $[x_{i-1}, x_i]$ st. $a_j \in (x_{i-1}, x_i)$, $a_k \notin (x_{i-1}, x_i)$ $\forall\, k \neq j$.

Now, $\sum_{i=1}^{k} M_i(f)\left(G(x_i) - G(x_{i-1})\right) - \sum_{j=1}^{k} f(a_j)\, p_j$ —— ⊛

If $[x_{i-1}, x_i]$ does not contain any a_j, $G(x_i) = G(x_{i-1})$.

⊛ $= \sum_{j=1}^{k} \left(f(y_j) - f(x_j)\right) p_j$; where $y_j \in [x_{i-1}, x_i]$ st. $M_i(f) = f(y_j)$.

$\Rightarrow$ ⊛ $\leq \sum_{j=1}^{k} \frac{\varepsilon}{\text{[redacted]}}\, p_j = \varepsilon$.

Now, we are done.

Theorem: Let $\alpha : [a,b] \to \mathbb{R}$ be a differentiable, increasing f^n st. α' is integrable, then a bounded f^n $f : [a,b] \to \mathbb{R}$ is R.S.I. wrt. α iff $f \cdot \alpha'$ is Riemann Integrable on $[a,b]$. In such a case,

$$\int_a^b f(x)\, d(\alpha(x)) = \int_a^b f(x)\, \alpha'(x)\, dx.$$

Proof: Fix $M \geq 0$ st. $|f(x)| \leq M$ $\forall x$.

$\forall \varepsilon > 0$, choose a partition P st. $P = (x_0, \dots, x_n)$,

$U(P, \alpha') - L(P, \alpha') < \frac{\varepsilon}{M}$.

We will show that; $\int_a^b f\,d\alpha = \int_a^b f\alpha'$.

Now, by LMVT, $\forall\, i$ st. $1 \leq i \leq n$, $\exists\, t_i \in [x_{i-1}, x_i]$ st.

$$\alpha(x_i) - \alpha(x_{i-1}) = \alpha'(t_i)(x_i - x_{i-1}).$$

Consider $t = (t_1 \dots t_n)$.

Now, take any tag $s = (s_1 \dots s_n)$ of P.

$$\Rightarrow \sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)|(x_i - x_{i-1}) \leq \sum_{i=1}^{n} (M_i(\alpha') - m_i(\alpha'))(x_i - x_{i-1})$$

$$\leq \frac{\varepsilon}{M}.$$

Now, $\sum_{i=1}^{n} f(s_i)(\alpha(x_i) - \alpha(x_{i-1})) = \sum_{i=1}^{n} f(s_i)\alpha'(t_i)(x_i - x_{i-1})$

$$\Rightarrow \left| \sum_{i=1}^{n} f(s_i)(\alpha(x_i) - \alpha(x_{i-1})) - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)(x_i - x_{i-1}) \right|$$

$$= \left| \sum_{i=1}^{n} f(s_i)\alpha'(t_i)(x_i - x_{i-1}) - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)(x_i - x_{i-1}) \right|$$

$$\leq M \sum_{i=1}^{m} |\alpha'(t_i) - \alpha'(s_i)|(x_i - x_{i-1}) \leq \varepsilon.$$

$$\Rightarrow \sum_{i=1}^{n} f(s_i)(\alpha(x_i) - \alpha(x_{i-1})) \leq \sum_{i=1}^{n} f(s_i)\alpha'(s_i)(x_i - x_{i-1}) + \varepsilon.$$

Thus, taking inf on LHS & RHS respectively one after other, we get that :-

$$U(P, f, \alpha) \leq U(P, f\alpha') + \varepsilon.$$

similarly, $U(P, f\alpha') \leq U(P, f, \alpha) + \varepsilon$.

$$\Rightarrow |U(P, f\alpha') - U(P, f, \alpha)| < \varepsilon.$$

$\because U(P, \alpha') - L(P, \alpha') < \frac{\varepsilon}{M}$ is true for any finer partition,

$|U(Q, f\alpha') - U(Q, f, \alpha)| < \varepsilon; \; \forall Q$ finer than P.

$\exists$ a partition R_1 st. $\left|U(R_1, f\alpha') - \underline{\int_a^b} f\alpha'\right| < \varepsilon,$

$\exists$ a partition R_2 st. $\left|U(R_2, f, \alpha) - \underline{\int_a^b} f\, d\alpha\right| < \varepsilon.$

Consider $Q \vee R_1 \vee R_2$,

$$\Rightarrow \left| \underline{\int_a^b} f\, d\alpha - \underline{\int_a^b} f\alpha' \right| < 3\varepsilon.$$

Note: Domain additivity holds for RSI too.
Also, if α_1, α_2 are increasing, $\int f\, d(\alpha_1 + \alpha_2) = \int f\, d\alpha_1 + \int f\, d\alpha_2$.

Improper Riemann Integrals:

Fix $a \in \mathbb{R}$.

Let $f : [a, \infty) \to \mathbb{R}$ be a f^n.

Def^n: Suppose $f|_{[a,x]}$ is bounded & Riemann integrable $\forall x > a$.

Then,

$$\int_{[a,\infty)} f \text{ exists if } \lim_{x \to \infty} \int_a^x f \text{ exists.}$$

Then, $\int_a^\infty f$ is defined as $\lim_{x \to \infty} \int_a^x f$.

Remark: Say $F(x) = \int_a^x f = \int_a^x f(t)\,dt$.

Then, $\lim_{x\to\infty} F(x)$ exists iff $\exists\, I \in \mathbb{R}$ st. $\forall \varepsilon > 0$, $\exists N > 0$ st $|F(x) - I| < \varepsilon \quad \forall\, x \geq N$.

Remark: The f^n f on $[a, \infty)$ need not be bounded for it to be Riemann Integrable.

Lemma: Let $f : [a, \infty) \to \mathbb{R}$ be a f^n st. $f|_{[a,x]}$ is bounded & Riemann Integrable. Then $\int_a^\infty f$ exists iff $\int_{a'}^\infty f$ exists $\forall\, a' > a$,
$$\int_a^\infty f = \int_a^{a'} f + \int_{a'}^\infty f.$$

Proof: $\forall\, x \geq a$, let $\int_a^x f = F(x)$.

Then, $\forall\, x \geq a'$, $F(x) = \int_a^x f = \int_a^{a'} f + \int_{a'}^x f$.

$\Rightarrow \lim_{x\to\infty} F(x) = \int_a^{a'} f + \lim_{x\to\infty} \int_{a'}^x f$. Now, we are done.

Remark: Instead of saying $\int_a^\infty f$ exists, we may say that $\int_a^\infty f$ is convergent.

Also, $\int_a^\infty f$ converges absolutely if $\int_a^\infty |f|$ exists.

Exercise: If $\int_a^\infty f$ is absolutely convergent, it is convergent as well.

Exercise: Give an example where f converges but not absolutely.

Cauchy Criterion:

Let $f:[a,b]\to\mathbb{R}$ be a f^n st. $f|_{[a,x]}$ is Riemann Integrable $\forall\, x>a$. Then, f is Riemann Integrable on $[a,\infty)$ iff $\forall\, \varepsilon>0$, $\exists\, M>a$ st. $\left|\int_x^y f\right|<\varepsilon \;\; \forall\, M\leq x<y<\infty$.

Proof: Say $\lim_{x\to\infty} F(x)=I$.

$\Rightarrow \forall\, \varepsilon>0,\ \exists\, M>a$ st. $|I-F(x)|<\frac{\varepsilon}{2}\;\; \forall\, x>M$.

Thus, $|F(y)-F(x)|\leq |F(y)-I|+|I-F(x)|<\varepsilon$.

Say $\int_a^\infty f$ exists. Let $\{a_n\}_{n\geq 1}$ be a sequence st. $a=a_1<a_2<a_3\ldots$ with $\lim_{n\to\infty} a_n=\infty$.

Then, $\int_a^\infty f=\sum_{k=1}^{\infty}\left(\int_{a_k}^{a_{k+1}} f\right)$.

Similarly, $f:(-\infty,b]\to\mathbb{R}$ is a f^n st. $f|_{[x,b]}$ is Riemann Integrable $\forall\, x\leq b$.

Then, $\int_{-\infty}^b f$ exists if $\lim_{x\to-\infty}\int_x^b f$ exists.

Def^n: Let $f:\mathbb{R}\to\mathbb{R}$ be a f^n st. $f|_{[x,y]}$ is a Riemann integrable f^n $\forall\, -\infty<x<y<\infty$.

Note: $\lim_{x\to\infty}\int_{-\infty}^{\infty} f$ exists does not mean that $\int_{-\infty}^{\infty} f$ exists.

eg. $f(t)=t$.

Thus, $\int_{-\infty}^{\infty} f$ exists if $\int_{-\infty}^{0} f+\int_0^{\infty} f$ exists & $\int_{-\infty}^{\infty} f=\int_{-\infty}^{0} f+\int_0^{\infty} f$.

If $\int_{-\infty}^{\infty} f$ exists $\Rightarrow \int_{-\infty}^{\infty} f = \int_{-\infty}^{a} f + \int_{a}^{\infty} f \quad \forall\ a \in \mathbb{R}$.