Probability - II

By Prof. Roy

Recall: so far, we have seen the following:

1. Countable sample space Ω.
2. $\mathcal{P}(\Omega)$ = the set of all events.
3. Probability $P : \mathcal{P}(\Omega) \to [0,1]$ satisfying the axioms.

We call the triplet $(\Omega, \mathcal{P}(\Omega), P)$ a probability space.

Now, we shall move towards uncountable sample spaces.

Motivating Example : "Choose a number (uniformly) at random strictly between 0 & 1"

Thus, $\Omega = [0,1]$.

Q: How can we make a model for this?

M_1: suppose $P(\{w\}) = a > 0 \quad \forall\, w \in (0,1)$.

This will lead to a contradiction because by Archimedian Property, $\exists\, n \in \mathbb{N}$ s.t. $na > 1$.

i.e. $P(\{w_1, \dots, w_n\}) = na > 1$, which is false.

M_2: Formalise the phrase "choosing uniformly from $(0,1)$" as follows :-

For any $(a,b) \subseteq (0,1)$, $P((a,b)) \propto (b-a)$

$$\Updownarrow$$

$$P((a,b)) = b-a.$$

Q: Does $\exists$ a set map $P : \mathcal{P}[0,1] \to \mathbb{R}$ satisfying :-

1. $P(E) \geq 0 \quad \forall\, E \in \mathcal{P}(0,1)$
2. $P((0,1)) = 1$

③ $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \quad \forall$ pairwise disjoint $A_1, A_2, \ldots \in \mathcal{P}(0,1)$

s.t. $P((a,b)) = b-a \quad \forall \ (a,b) \subseteq (0,1)$.

solⁿ:- No! This is a theorem of Ulam.

The Problem / Hindrance is that there are too many events in $\mathcal{P}(0,1)$.

Q: What else should be in the "set of all events"?

Let Ω be any (not necessarily countable) sample space.
A collection $\mathcal{A}$ of subsets of Ω is called a σ-field / σ-algebra if:

(i) $\Omega \in \mathcal{A}$

(ii) if $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

(iii) if $A_1, A_2, \ldots \in \mathcal{A} \Rightarrow \bigcup_{i=1} A_i \in \mathcal{A}$

Remark: Let us forget the fancy name σ-field / σ-algebra. Just think of $\mathcal{A}$ as a collection of events based on the sample space Ω so that we can define probability P of each event, i.e. P is a set map $P: \mathcal{A} \to \mathbb{R}$ satisfying the axioms.

① $\{\} \in \mathcal{A}$

② $A_1, A_2, \ldots \in \mathcal{A} \Rightarrow \bigcap_i A_i \in \mathcal{A}$ (Using De Morgans Law)

Two extreme examples :

1. Take any non-empty sample space Ω.
   Note that $\mathcal{A}_1 = \{ \{\}, \Omega \}$ is trivially a σ-field.

2. Take any non-empty sample space Ω.
   Note that $\mathcal{A}_2 = \mathcal{P}(\Omega)$ is also a σ-field.

Q: What should be our σ-field?

Def^n:- Let Ω be any non-empty sample space & $\mathcal{A}$ is a σ-field on it.
A set map $P : \mathcal{A} \to \mathbb{R}$ is called a probability on $(\Omega, \mathcal{A})$ if :-

① $P(E) \geq 0 \;\; \forall E \in \mathcal{A}$, ② $P(\Omega) = 1$.
③ $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ whenever $A_1, A_2, \ldots \in \mathcal{A}$ are pairwise disjoint.

Remark: ① if Ω is countable & $\mathcal{A} = \mathcal{P}(\Omega)$, then the above def^n reduces to the previous one. We have dealt with many probabilities on $(\Omega, \mathcal{P}(\Omega))$ for countable Ω.

② The triplet $(\Omega, \mathcal{A}, P)$ is called a probability space.

③ Many Properties of probability will remain valid in this more general set-up as long as the events belong to $\mathcal{A}$.

In light of the above definitions, our previous question can be rephrased as follows :

Q: Does $\exists$ a σ - field $\mathcal{A}$ on $\Omega = (0,1)$ such that we can define a probability P on $(\Omega, \mathcal{A})$ st. $P((a,b)) = b-a \; \forall \; (a,b) \subseteq (0,1)$

sol^n:- Yes !!

If such a σ - field $\mathcal{A}$ (on $\Omega = (0,1)$) exists, then it has to contain all open intervals $(a,b) \subseteq (0,1)$.

In light of Ulam's theorem, we should chose $\mathcal{A}$ "as small as possible". $\therefore$, we take $\mathcal{A} = \mathcal{B}_{(0,1)}$ = the smallest σ-field on $\Omega = (0,1)$ containing all open intervals $(a,b) \subseteq (0,1)$.

Here, $\mathcal{B}_{(0,1)}$ is called the Borel σ - field on $(0,1)$.

Theorem: $\exists$ a probability P on $((0,1), \mathcal{B}_{(0,1)})$ st. $P((a,b)) = b-a \; \forall \; (a,b) \subseteq (0,1)$. This P is the uniform probability on $(0,1)$.

This P will work as a model for choosing a number at random from $(0,1)$.

Recall: We have the uniform probability P on $(\Omega, \mathcal{A}) = (\Omega, \mathcal{B}_\Omega)$ that satisfies $P((a,b)) = b-a \; \forall \; (a,b) \subseteq \Omega = (0,1)$ here.

Now, define a "random variable" $U : \underset{\Omega}{\overset{\parallel}{(0,1)}} \to \mathbb{R}$ by

$U(\omega) = \omega$.

Clearly, Range $(U) = (0,1)$ & for any $(a,b) \subseteq (0,1)$,

$$P(a < U < b) = P\left[\{w \in (0,1) : a < U(w) < b\}\right].$$

Remark: U plays the role of a number chosen uniformly at random from $(0,1)$. In this case, we say that U follows the uniform distribution on $(0,1)$ & denoted by $U \sim \text{Unif}(0,1)$.

Theorem: $\exists$ a probability P on $((0,1), \mathcal{B}_{(0,1)})$ st. $P((a,b)) = (b-a)\ \forall (a,b) \subseteq (0,1)$ i.e. $P \stackrel{d}{=} \text{Unif}(0,1)$.

\# Go back to Discrete case i.e. Ω is countable. $\Rightarrow \mathcal{A} = \mathcal{P}(\Omega)$.

Thus, $X : \Omega \to \mathbb{R}$ st.

$$(a < X < b) = \{w \in \Omega : a < X(w) < b\} \subseteq \Omega.$$
$$= X^{-1}((a,b)).$$

Thus, $(a < X < b)$ is an event.

Hence, $P(a < X < b)$ is well-defined for discrete case.

Now, come back to General case.

Def^n: A map $X : \Omega \to \mathbb{R}$ is called a random variable if $\forall a, b \in \mathbb{R}$ with $a < b$; $(a < X < b) = \{w \in \Omega : a < X(w) < b\} \in \mathcal{A}$.

Exercise: $(\Omega, \mathcal{A}, P)$ is any probability space. $X : \Omega \to \mathbb{R}$ is a map. Then, the following are equivalent:

1. X is a random variable.
2. $X \leq x$ is an event i.e. $(X \leq x) \in \mathcal{A}\ \forall x \in \mathbb{R}$.
3. $(a \leq X \leq b) \in \mathcal{A}\ \forall a, b \in \mathbb{R}$ with $a \leq b$.
4. $(a < X \leq b) \in \mathcal{A}\ \forall a, b \in \mathbb{R}$ st. $a < b$.

5. $(a \leq X < b) \in \mathcal{A} \ \forall\ a, b \in \mathbb{R}\ (a < b)$.

6. $(X < x) \in \mathcal{A} \ \forall\ x \in \mathbb{R}$.

7. $(X \geq x) \in \mathcal{A} \ \forall\ x \in \mathbb{R}$.

8. $(X^{-1}(B)) \in \mathcal{A}$ whenever B is countable union of intervals.

Remark: ① If X is a R.V., then we can try to compute probabilities of all events mentioned in last exercise.

② If Ω is countable & $\mathcal{A} = \mathcal{P}(\Omega)$, then any map $X : \Omega \to \mathbb{R}$ is a R.V.

Example: $\Omega = (0,1)$, $\mathcal{A} = \mathcal{B}_{(0,1)}$ & P = uniform probability on $(\Omega, \mathcal{A}) = ((0,1), \mathcal{B}_{(0,1)})$.

We defined $U : \Omega \to \mathbb{R}$ st. $U(w) = w$, then, for any $a, b \in \mathbb{R}$ st. $0 \leq a < b \leq 1$, have :-

$$(a < U < b) = \{w \in \Omega : a < U(w) < b\}$$

$$= \{w \in (0,1) : a < w < b\} = (a,b) \in \mathcal{B}_{(0,1)}$$

$\Rightarrow U : \Omega \to \mathbb{R}$ is a R.V.

This R.V. U is said to follow Uniform distribution on $(0,1)$, i.e. $U \sim \text{Unif}(0,1)$.

$\Rightarrow P(a < U < b) = b - a \ \forall\ (a,b) \subseteq (0,1)$.

Properties of Probability: Suppose Ω is any sample space, $\mathcal{A}$ is any σ-field on Ω & P is any probability on $(\Omega, \mathcal{A})$. Then, the following hold :

1. $P(\{\ \}) = 0$.

2. $P(A^c) = 1 - P(A) \Rightarrow 0 \leq P(A) \leq 1 \ \forall\ A \in \mathcal{A}$.

3. If $A, B \in \mathcal{A}$ with $A \subseteq B$, $\Rightarrow P(A) \leq P(B)$.

4. P.I.E. for $A_1, A_2, \dots, A_n \in \mathcal{A}$.

5. If $A_1, A_2, \dots, A_n \in \mathcal{A}$, $\Rightarrow \max_{1 \leq i \leq n} P(A_i) \leq P\left(\bigcup_{i=1}^{n} A_i\right) \leq \min\left\{\sum_{i=1}^{n} P(A_i), 1\right\}$

6. If $A_1, A_2, \dots \in \mathcal{A}$, $\Rightarrow A_n \uparrow A$ gives $P(A_n) \uparrow P(A)$ &
$A_n \downarrow A$ gives $P(A_n) \downarrow P(A)$.

7. $\forall A_1, A_2, \dots \in \mathcal{A} \Rightarrow P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \min\left(\sum_{i=1}^{\infty} P(A_i), 1\right)$.

Def^n: Suppose $X : \Omega \to \mathbb{R}$ is a R.V. Then, the cummulative distribution f^n (cdf) of X is defined as $F_X : \mathbb{R} \to [0,1]$ st.

$$F_X(u) = P[X \leq u] \quad \forall u \in \mathbb{R}.$$

Exercise: Show that $F = F_X$ satisfies the following properties :-

i) F is non - decreasing i.e. $\forall u \leq v$, $F(u) \leq F(v)$.

ii) F is right - continuous i.e. $\forall v \in \mathbb{R}$, $\lim_{u \to v+} F(u) = F(v)$.

iii) $\lim_{u \to \infty} F(u) = 1$ & $\lim_{u \to -\infty} F(u) = 0$.

Let us compute the cdf of $U \sim \text{Unif}(0,1)$.

Claim: $\forall u \in \mathbb{R}$, $P(U = u) = 0$.

Now, if $u \notin (0,1)$, $(U = u) = \{\} \Rightarrow P(U = u) = 0$.

If $u \in (0,1)$: get N large enough so that : $0 < u - \frac{1}{N} < u < u + \frac{1}{N} < 1$

Then, $\forall n \geq N$, $\left(u - \frac{1}{n}, u + \frac{1}{n}\right) \subseteq (0,1)$.

Define $A_m = \left(u - \frac{1}{N+m} < U < u + \frac{1}{N+m}\right) \ \forall\, m \in \mathbb{N}$.

Clearly, $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ Also, $\bigcap_{m=1}^{\infty} A_m = (U = u)$.

$\Rightarrow A_m \downarrow (U=u)$. $\Rightarrow P(A_m) \downarrow P(U=u)$.

Now, $P(A_m) = P\left(u - \frac{1}{N+m} < U < u + \frac{1}{N+m}\right) = \frac{2}{N+m}$.

$\Rightarrow P(A_m) \downarrow 0$.

$\Rightarrow P(U=u) = 0$.

For any $v \in \mathbb{R}$, we need to compute $F_U(v) = P[U \leq v]$.

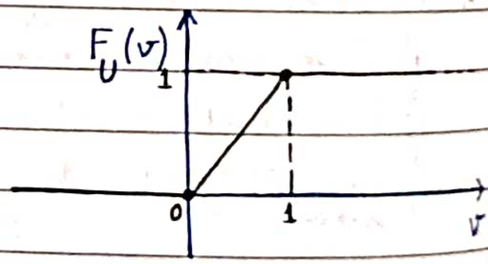
Clearly, if $v < 0 \Rightarrow F_U(v) = P(U \leq 0) = 0$.

Also, if $v \geq 1 \Rightarrow P(U \leq v) = 1$.

Also, $v = 0 \Rightarrow P(U \leq v) = 0$.

Take $v \in (0,1)$. $\Rightarrow F_U(v) = P(U \leq v) = P(U < v) + P(U = v)$
$= P(0 < U < v) = v - 0 = v$.

Thus, $F_U(v) = \begin{cases} 0 & ; v < 0 \\ v & ; 0 \leq v < 1 \\ 1 & ; v \geq 1 \end{cases}$



Remark: We shall prove later in the semester that the converse of last exercise holds i.e. if $F : \mathbb{R} \to [0,1]$ satisfying (i), (ii) & (iii), then $\exists$ a R.V. X st. $F(u) = P[X \leq u] \ \forall\, u \in \mathbb{R}$.

$U \sim Unif(0,1)$ basically means that U is a R.V. that takes value in $(0,1)$ with "no preference of one value to another". This means that all the numbers in $(0,1)$ get equal weightage. This can be finalized by the following observation :

$\forall\, a, b \in \mathbb{R}$ with $a < b$, $P(a < U < b) = \int_a^b f(u)\, du$, where

$$f(u) = \begin{cases} 1 & ; u \in (0,1) \\ 0 & ; u \notin (0,1) \end{cases}$$

This example motivates the following question.

Q: In general, are there R.V.s X s.t. $\forall\, a, b \in \mathbb{R}$ with $a < b$,

$$P(a < X < b) = \int_a^b f(x)\, dx \text{ for some } f^n\ f(x) \geq 0 ?$$

sol^n:- Yes, Many such R.V.s exist.

Def^n: We say that a R.V. X is an absolutely continuous R.V. (or a continuous R.V.) if $\exists$ an integrable f^n $f_X : \mathbb{R} \to [0, \infty)$ s.t. $\forall\, a, b \in \mathbb{R}$ with $a < b$,

$$P(a < X < b) = \int_a^b f_X(x)\, dx.$$

In this case, $f_X(x)$ is called the Probability Density f^n (pdf) of the R.V. X.

$$f_U(u) = \begin{cases} 1 & \text{if} \\ 0 & \text{otherwise} \end{cases}$$

\# Suppose X is a continuous R.V. with pdf $f_X(x)$.

Q: How can we express the cdf of X in terms of its pdf?

sol^n:- $F_X(u) = P(X \leq u) = \int_{-\infty}^{u} f_X(x)\, dx \quad \forall\, u \in \mathbb{R}$.

In all the examples, f_X will have finitely many discontinuities.

Remark: ① The integral $\int_{-\infty}^{u} f_X(x)\, dx = \lim_{y \to -\infty} \int_y^u f_X(x)\, dx$.

② We shall also specify / state the meaning of integrability of f_X on $\mathbb{R}$.

Fix $b \in \mathbb{R}$. If f is integrable on $[a, b]$ $\forall a \leq b$ & the limit $\lim_{a \to -\infty} \int_a^b f(x)\,dx = I_{-\infty, b}$ exists & is finite,

We say that f is integrable on $(-\infty, b]$ & $\int_{-\infty}^{b} f(x)\,dx = I_{-\infty, b}$

Similarly, fixing $a \in \mathbb{R}$, one can define the integrability of f on $[a, \infty)$, we can define $\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$

If for some $c \in \mathbb{R}$, f is integrable both on $(-\infty, c]$ & $[c, \infty)$, then we say that f is integrable on $\mathbb{R}$ &

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx.$$

If f is integrable on $\mathbb{R}$, then $\forall a \in \mathbb{R}$:-

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx.$$

If f is integrable on $\mathbb{R}$, $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_{-b}^{b} f(x)\,dx.$

However, the limit of RHS may exist even if f is not integrable. for example, $f(x) = x, x^3, \ldots$ etc.

Theorem: X is a continuous R.V. with p.d.f. $f(x)$ iff $\forall u \in \mathbb{R}$,

$$F_X(u) = P[X \leq u] = \int_{-\infty}^{u} f(x)\,dx.$$

Proof: $(\Rightarrow)$ We know that $\forall\, a, u \in \mathbb{R}\ (a < u)$, $P(a < X < u) = \int_a^u f(x)\,dx$

Consider a sequence $\{a_n\}$ s.t. $a_n < u\ \forall\, n \in \mathbb{N}$ & $a_n \downarrow -\infty$.

Thus, $\displaystyle\int_{a_n}^{u} f(x)\,dx = P(a_n < X < u)$.

$$\Rightarrow \lim_{n\to\infty} \int_{a_n}^{u} f(x)\,dx = \lim_{n\to\infty} P(a_n < X < u).$$

Note that $(a_n < X < u) \uparrow (-\infty < X < u)$

$$\Rightarrow \lim_{n\to\infty} P(a_n < X < u) = P(X < u).$$

On the other hand, $\displaystyle\lim_{n\to\infty} \int_{a_n}^{u} f(x)\,dx = \int_{-\infty}^{u} f(x)\,dx$ since f is integrable

$$\therefore,\ P[X < u] = \int_{-\infty}^{u} f(x)\,dx.$$

\# Claim : $P(X = u) = 0$.

Now, We shall use a result of Analysis :-

namely, The First Fundamental theorem of Integral Calculus :

Suppose f is integrable on $[a, b]$. Define $F : [a, b] \to \mathbb{R}$ by $F(u) = \int_a^u f(x)\,dx$ $\ (F(a) := 0)$.

Then, we have :

i) F is continuous f^n on $[a, b]$.

ii) $\forall$ continuity point $u_0 \in (a, b)$ of f, F is differentiable at u_0 & $F'(u_0) = f(u_0)$.

classmate

Date

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Proof of claim: $\forall n \geq 1$, $P\left(u - \frac{1}{n} < X < u + \frac{1}{n}\right) = \int_{u-\frac{1}{n}}^{u+\frac{1}{n}} f(x)\,dx$

$$= \int_{u-\frac{1}{n}}^{u} f(x)\,dx + \int_{u}^{u+\frac{1}{n}} f(x)\,dx$$

Now, $\left(u - \frac{1}{n} < X < u + \frac{1}{n}\right) \downarrow (X = u)$.

$$\Rightarrow P\left(u - \frac{1}{n} < X < u + \frac{1}{n}\right) \downarrow P(X = u).$$

Note that $\lim_{n \to \infty} \int_{u}^{u+\frac{1}{n}} f(x)\,dx = 0$.

Define $F : [u, u+1] \to \mathbb{R}$ by $F(y) = \begin{cases} \int_u^y f(x)\,dx & \text{if } y \in (u, u+1] \\ 0 & \text{if } y = u \end{cases}$

Now, F is continuous f^n by Fundamental theorem of Calculus.

In part, $\lim_{n \to \infty} F\left(u + \frac{1}{n}\right) = F(u) = 0$.

$$\Rightarrow \lim_{n \to \infty} \int_{u}^{u+\frac{1}{n}} f(x)\,dx = 0.$$

Also, $\int_{u-\frac{1}{n}}^{u} f(x)\,dx = \int_{-u}^{-u+\frac{1}{n}} f(-y)\,dy \quad [x \longrightarrow -x]$

$$\Rightarrow \lim_{n \to \infty} \int_{u-\frac{1}{n}}^{u} f(x)\,dx = 0 \text{ as well.}$$

Thus, $P(X = u) = 0$.

Proof: $(\Leftarrow)$ suppose for all $u \in \mathbb{R}$, $P[X \leq u] = F_X(u) = \int_{-\infty}^{u} f(x)\,d$

To show : X is a continuous R.V. with pdf $f(x)$.
i.e. $\forall\, a, b \in \mathbb{R}\ (a < b)$:-
$$P(a < X < b) = \int_a^b f(x)\,dx.$$

Thus, take $a, b \in \mathbb{R}\ (a < b)$.

$$\Rightarrow P(X \leq b) - P(X \leq a) = \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx$$

$$\Rightarrow P(a < X \leq b) = \int_a^b f(x)\,dx.$$

$$\Rightarrow P(a < X < b) = \int_a^b f(x)\,dx.$$

Hence, Proved.

As a byproduct of the above proofs, we obtain the following properties of continuous R.V.s :

Theorem: Suppose X is a continuous R.V with pdf f_X & cdf F_X.
Then, we have :

a) $\forall\, c \in \mathbb{R}$, $P(X = c) = 0$.

b) $\forall\, a, b \in \mathbb{R}$ with $a < b$,
$P(a < X < b) = P(a \leq X < b) = P(a < X \leq b)$
$= P(a \leq X \leq b)$

$$= F_X(b) - F_X(a) = \int_a^b f_X(x)\,dx.$$

c) F_X is a continuous f^n.

Proof of (c): Fix $u \in \mathbb{R}$. $\lim_{y \to u^+} F_X(y) = F_X(u)$ (same proof as in discrete case)

Also, $\lim_{y \to u^-} F_X(y) = P(X < u) = P(X \leq u) = F_X(u)$.

Using the fundamental theorem of Calculus, we can show the following: Assume that X is a continuous R.V. with pdf f_X.

Suppose that f_X is continuous at $u_0 \in \mathbb{R}$.

Then, F_X is differentiable at u_0 & $F'(u_0) = f(u_0)$.

This means that $f_X(u_0) = F_X'(u_0) = \lim_{h \to 0} \frac{F_X(u_0+h) - F_X(u_0)}{h}$

$$= \lim_{h \to 0} \frac{P(u_0 < X < u_0+h)}{h}$$

This is why we call f_X the probability density f^n.

In particular, pdf can take any non-negative value.

Suppose $X \sim \text{Unif}(0,1)$. It is a continuous R.V. with pdf $f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

It can be verified that $\forall\, a, b \in \mathbb{R}\ (a < b)$, $P(a < X < b) = \int_a^b f(x)\,dx$.

Suppose, we take $g(x) = \begin{cases} 1 & ; x \in [0,1] \\ 0 & ; \text{otherwise} \end{cases}$

Hence, $\forall a, b$, $P(a < X < b) = \int_a^b g(x)\,dx$.

$\therefore$, the pdf is not unique.

Q: Suppose we know cdf of a continuous R.V 'X', how to compute pdf of X ?

sol^n:- Roughly speaking,

$$\begin{tikzcd} \boxed{\text{pdf}} \arrow[r, bend left, "\text{integration}"] & \boxed{\text{cdf}} \arrow[l, bend left, "\text{differentiation}"] \end{tikzcd}$$

Take $u \in \mathbb{R}$. Fix $a \in \mathbb{R}$ s.t. $a < u$.

Then, $F_X(u) = \int_{-\infty}^{u} f_X(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^u f(x)\,dx$.

If u is a continuity point of $f_X(x)$,

then, $\frac{d}{du}\left(F_X(u)\right) = \frac{d}{du}\left(\int_a^u f_X(x)\,dx\right) = f_X(u)$.

However, if u is not a continuity point of $f_X(x)$, We cannot compute $f_X(u)$ from $F_X(u)$.

Furthermore, we wont even know whether u is a point of continuity of f_X or not.

Recipe : Define $f(x) = \begin{cases} F_X'(x) & ; \text{if } F_X \text{ is diff. at } x \\ 0 & ; \text{otherwise} \end{cases}$

A deep result from Real Analysis will ensure that this recipe will always work. However, we are not in a position to use that result.

We shall use this recipe to make a guess of a pdf of X & then establish that it is indeed ~~that~~ the case by verifying that $F_X(u) = \int_{-\infty}^{u} f(x)\,dx$.

Suppose you know the cdf $F_X(x)$ of a R.V. X.

Q: How to find whether X is a continuous R.V.?
If yes, then how to find a pdf of X?

Recipe: Define $f(x) = \begin{cases} F_X'(x) & ; \text{ if } F \text{ is diff at } x \\ 0 & ; \text{ otherwise} \end{cases}$

If X is indeed a continuous R.V, then $f(x)$ would actually be a pdf of $f(x)$.

We shall verify whether $\forall\, u \in \mathbb{R}$,
$$F_X(u) = \int_{-\infty}^{u} f(x)\,dx \text{ or not.}$$

Eg: Suppose X is a R.V. with cdf :-
$$F_X(u) = \begin{cases} 0 & ; u < 0 \\ u^2 & ; u \in [0,1) \\ 1 & ; u \geq 1 \end{cases}$$
Find whether X is a continuous R.V.

solⁿ:- Our guess for a pdf of X is the following :-

$$f(x) = \begin{cases} F_X'(x) & ; x \neq 1 \\ 0 & ; x = 1 \end{cases}$$

Thus, $$f(x) = \begin{cases} 2x & ; x \in (0,1) \\ 0 & ; \text{otherwise} \end{cases}$$

Claim : X is a cont. R.V. with pdf as above.

Proof : We shall show that $\forall\, u \in \mathbb{R}$,

$$F_X(u) = \int_{-\infty}^{u} f(x)\,dx.$$

$C_1 : u < 0 \Rightarrow \int_{-\infty}^{u} f(x)\,dx = 0 = F_X(u) \;\forall\, u.$

$C_2 : u \geq 1 \Rightarrow \int_{-\infty}^{u} f(x)\,dx = \int_{0}^{1} f(x)\,dx = 1 = F_X(u) \;\forall\, u.$

$C_3 : 0 \leq u < 1 \Rightarrow \int_{-\infty}^{u} f(x)\,dx = \int_{0}^{u} f(x)\,dx = u^2 = F_X(u) \;\forall\, u.$

Note that from the def^n of a pdf $f(x)$ of a cont. R.V. X, it is clear that f_X is integrable on $\mathbb{R}$ & $f_X(x) \geq 0 \;\forall\, x \in \mathbb{R}$.

Also, $P[-\infty < X \leq n] = \int_{-\infty}^{n} f_X(x)\,dx \;\forall\, n \geq 1.$

$$\Rightarrow \lim_{n\to\infty} P(-\infty < X \leq n) = \lim_{n\to\infty} \int_{-\infty}^{n} f_X(x)\,dx.$$

$$\Rightarrow \lim_{n\to\infty} P(-\infty < X \leq n) = \lim_{n\to\infty} \left(\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{n} f(x)\,dx \right)$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

Also, note that $(-\infty < X \leq n) \uparrow (-\infty < X < \infty)$.

$$\Rightarrow P(-\infty < X \leq n) \uparrow 1.$$

$$\Rightarrow 1 = \int_{-\infty}^{\infty} f(x)\, dx.$$

Theorem: If $f(x)$ is a Pdf of a cont. R.V. X, then the following properties hold:

1. $f(x)$ is integrable on $\mathbb{R}$
2. $f(x) \geq 0 \ \forall x \in \mathbb{R}$
3. $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

The converse of this theorem also holds.

Remark: We shall introduce many families of cont. RVs with help of converse of the above Theorem. We shall just verify that (I), (II) & (III) hold for a f^n $f(x)$ & conclude that it is indeed a pdf of some cont. R.V. X.

Continuous Probability Distributions:

1. Uniform Distribution on (a, b):

Fix $a, b \in \mathbb{R}$ $(a < b)$. A cont. R.V. X is said to follow Uniform Distribution on the interval (a, b) if it has a pdf

$$f_X(x) = \begin{cases} \frac{1}{b-a} & ; \text{ if } a < x < b \\ 0 & ; \text{ otherwise} \end{cases}$$

Notated by: $X \sim \text{Unif}(a, b)$.

Also, the cdf of X is :-

$$F_X(u) = \begin{cases} 0 & ; u < a \\ \frac{u-a}{b-a} & ; u \in [a, b) \\ 1 & ; u \geq b \end{cases}$$

2. ~~Exponential~~ Distribution : A cont. R.V. X is said to follow exponential distribution if it has the
Exponential
pdf of the form :-

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & ; x > 0 \\ 0 & ; x \leq 0 \end{cases}$$

Notated by : $X \sim Exp(\lambda)$.

Exponential distribution is typically used to model life-times, waiting times, inter-arrival times b/w two successive phone calls in a telephone etc.

Properties of Exponential Distribution :

1. $X \sim exp(\lambda) \Rightarrow$ cdf of X is :

$$F_X(u) = \begin{cases} 0 & ; u < 0 \\ 1 - e^{-\lambda u} & ; u \geq 0 \end{cases}$$

Proof: Clearly, for $u < 0$, $F_X(u) = P(X \leq u) = 0$.

For $u \geq 0$, $F_X(u) = P(X \leq u) = \int_{-\infty}^{u} f(x)\,dx = \int_{-\infty}^{u} \lambda e^{-\lambda x} dx$.

$$= 1 - e^{-\lambda u}.$$

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2. $X \sim Exp(\lambda) \Rightarrow \overline{F}_X(x) := P(X > x) = 1 - F_X(x)$

i.e. $\overline{F}_X(x) = \begin{cases} 1 & \text{if } x < 0 \\ e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$ (called the survival f^n of x)

3. Lack of Memory Property : Exponential distribution is memory-less in the sense that $\forall\, s > 0$ & $t > 0$, we have
$$P(X > t+s \mid X > s) = P(X > t).$$

This means the following :- Given that something has survived upto time 's', the conditional probability that it will survive t more amount of time in the future is equal to the unconditional probability that it survives upto time 't'.

Proof: $\forall\, s > 0$ & $t > 0$, $P(X > t+s / X > s) = \dfrac{P(X > t+s, X > s)}{P(X > s)}$

$$= \frac{P(X > t+s)}{P(X > s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda(s)}} = e^{-\lambda t} = P(X > t).$$

Eg: suppose the waiting time (in minutes) for the first telephone call in office follows an exponential distribution with $\lambda = 0.2$ & the office opens at 8 am.

(a) What is the chance that there is no telephone call till 9:00 am?

(b) Given that there has no telephone call till 9:00 am, What is the condititional probability that there is no call till 9:30 am

3. Gamma Distribution :

$$\forall \alpha > 0, \int_0^\infty y^{\alpha-1} e^{-y}\, dy = \underbrace{\Gamma(\alpha)}_{\text{Gamma F}^n}$$

Properties :-

1. $\Gamma(1) = 1$.
2. $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha) \quad \forall \alpha > 0 \qquad \Rightarrow \Gamma(n+1) = n! \quad \forall n > 0 \; (n \in \mathbb{N})$

Def^n: A cont. R.V. X is said to follow Gamma-distribution with shape parameter $\alpha > 0$ & scale parameter $\lambda > 0$ if it has a pdf :-

$$f_X(x) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & ; x > 0 \\ 0 & ; x \leq 0 \end{cases}$$

This is a valid pdf because $f_X \geq 0$, $\int_{-\infty}^{\infty} f_X(x)\, dx$

$$= \int_0^\infty f_X(x)\, dx = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\, dx.$$

Put $y = \lambda x$. $\Rightarrow \int_0^\infty \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)} \left(\frac{y}{\lambda}\right)^{\alpha-1} e^{-y}\, dy = \frac{1}{\Gamma(\alpha)} \Gamma(\alpha) = 1.$

We shall show later that if $X_1, \dots, X_n$ are independent R.Vs & they all follow exponential distribution with parameter λ, then $X_1 + \dots + X_n \sim \text{Gamma}(n, \lambda)$.

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Remarks :

① $X \sim \text{Gamma}(1, \lambda) \iff X \sim \exp(\lambda)$.

② If the inter-arrival times of phone-calls (i.e. the time b/w 2 successive calls) follow $\text{Exp}(\lambda)$ distribution, & they are independent, the arrival time of n^{th} phone call follows $\text{Gamma}(n, \lambda)$ distribution.

4. Beta Distribution :

① $\forall\, a, b > 0, \quad \underbrace{\int_0^1 u^{a-1}(1-u)^{b-1}\,du}_{= B(a,b)} \in (0, \infty)$. (Beta f^n at $(a,b) \in (0,\infty) \times (0,\infty)$).

② $B(a,b) = \dfrac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}$.

Def^n: A cont. R.V. X is said to follow beta distribution (parameters $a, b > 0$) if it has a pdf :

$$f_X(x) = \begin{cases} \dfrac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} & ; \; x \in (0,1) \\ 0 & ; \; \text{otherwise} \end{cases}$$

Notated by $X \sim \text{Beta}(a,b)$.

Note : $B(1,1) = \dfrac{\Gamma(1)\Gamma(1)}{\Gamma(2)} = 1$. Thus, $X \sim \text{Beta}(1,1) \iff X \sim \text{Unif}(0,1)$

Remarks :

① For any continuous R.V. Y with pdf f_Y,

$\text{Range}(Y) := \{ y \in \mathbb{R} \mid f_Y(y) > 0 \}$

(2) In particular, this means that Range $(X) = (0,1)$ whenever $X \sim \text{Beta}(a,b)$ for some $a>0$, $b>0$.
$\therefore$, Beta distribution is used to model R.V.s which take values b/w 0 & 1.

5. Normal Distribution :

Def^n: A continuous R.V. X is said to follow Normal (or Gaussian) distribution with parameters $\mu \in \mathbb{R}$ & $\sigma \in (0,\infty)$ if its pdf is :

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\} \quad \forall\, x \in \mathbb{R}.$$

Notated by $X \sim N(\mu, \sigma^2)$.

Remarks:

(1) If $\mu = 0$ & $\sigma = 1$, (i.e. if $X \sim N(0,1)$) we say that X follows standard normal distribution. Here, X has pdf :

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \forall\, x \in \mathbb{R}.$$

(2) It can be checked that $g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $x \in \mathbb{R}$ is an integrable f^n using results from Real Analysis. However, computing

$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx$ is harder. We shall show later that this value is $\sqrt{2\pi}$.

Lets assume this from now on.

Distributions of f^{ns} of Random Variables:

Let us start with an example.

Eg: Say $X \sim N(0,1)$ & $Y := X^2$.

Q: What is distribution of Y?

sol^n:- $\text{Range}(X) = \mathbb{R} \Rightarrow \text{Range}(Y) = (0, \infty)$.

We shall find the cdf of Y i.e. $F_Y(u)$.

If $u \leq 0$, then $F_Y(u) = 0$.

If $u > 0$, then we have $F_Y(u) = P(Y \leq u) = P(X^2 \leq u)$
$= P(-\sqrt{u} \leq X \leq \sqrt{u})$.

$$\text{Thus, } F_Y(u) = \int_{-\sqrt{u}}^{\sqrt{u}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx.$$

$$= 2 \int_{0}^{\sqrt{u}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx.$$

Thus,

$$F_Y(u) = \begin{cases} 2 \int_0^{\sqrt{u}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx & ; u > 0 \\ 0 & ; u \leq 0 \end{cases}$$

Now, we shall make a guess of pdf of Y using our recipe.

Our guess of pdf of Y is:

$$h(y) = \begin{cases} \frac{d}{dy} F_Y(y) & ; F_Y \text{ is diff}^{ble} \text{ at } y \\ 0 & ; \text{otherwise} \end{cases}$$

Clearly, $h(y) = 0$ whenever $y < 0$.

Take $y \in (0, \infty)$. Then, $H(y) = \frac{d}{dy}\left(F_Y(y)\right) = \frac{d}{dy}\left[2\int_0^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx\right]$

$$= \frac{d}{dy}\left[2\int_0^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx\right] \quad (z = \sqrt{y})$$

$$= \left(\frac{dz}{dy}\right)\left[\frac{2}{\sqrt{2\pi}} e^{-z^2/2}\right] \quad (\text{By FtoC})$$

$$= \frac{1}{2\sqrt{y}}\left[\frac{2}{\sqrt{2\pi}} e^{-y/2}\right] = \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2}.$$

\# We guess that Y is a continuous R.V. with pdf:

$$h(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2} & \text{if } y > 0 \\ 0 & \text{if } y \leq 0 \end{cases}$$

Exercise: Check that Y is indeed a continuous R.V. with the pdf $h(y)$ given above.

Remarks:

① Recall that Gamma $\left(\frac{1}{2}, \frac{1}{2}\right)$ distribution has a pdf :-

$$g(y) = \begin{cases} \frac{\left(\frac{1}{2}\right)^{1/2}}{\Gamma\left(\frac{1}{2}\right)} y^{-1/2} e^{-\frac{1}{2}y} & ; y > 0 \\ 0 & ; \text{otherwise} \end{cases}$$

② If one pdf is a constant multiple of another pdf, then the two pdfs are equally ($\because \int$ pdf on $\mathbb{R}$ is 1 for both)

Hence, $Y \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$.
We also get that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

$X \sim N(0,1) \Rightarrow X^2 \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$.

Exercise: Using a similar method, show that $U \sim \text{Unif}(0,1) \Rightarrow -\log U \sim \text{Exp}(1)$.

Theorem: Change of Density Formula :-
Suppose $I, J \subseteq \mathbb{R}$ be two open sets (intervals) & $g: I \to J$ is a differentiable, bijective f^n st. either $g' > 0$ on I or $g' < 0$ on I. If X is a continuous R.V. with pdf f_X that vanishes outside I (this means Range (X) is contained inside I). Then $Y = g(X)$ is also a continuous R.V, with a pdf :-

$$f_Y(y) = \begin{cases} 0 & ; y \in J^c \\ f_X(g^{-1}(y)) \left| \dfrac{d(g^{-1}(y))}{dy} \right| & ; y \in J. \end{cases}$$

Remark: The conditions on g ensure that $g^{-1}(y)$ is differentiable on J & $\dfrac{d(g^{-1}(y))}{dy} \neq 0 \quad \forall y \in J$.

If $y = g(x)$. $\Rightarrow x = g^{-1}(y)$. Hence, $\dfrac{d(g^{-1}(y))}{dy} = \dfrac{dx}{dy}$

$$= \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{g'(x)} \quad \forall x \in J.$$

Range $(Y) \subseteq J$. When Range $(X) = I \Rightarrow$ Range $(Y) = J$.

Application: $U \sim \text{Unif}(0,1) \Rightarrow V := -\log U \sim \text{Exp}(1)$.

Proof: Note that $\text{Range}(U) = (0,1) =: I$.

$\Rightarrow \text{Range}(V) = (0,\infty) =: J$.

Clearly, $g(x) = -\log x$.

Q: Check that g is bijective, differentiable on I with $g'(u) = \frac{-1}{u} < 0$ $\forall\, u \in I$.

Solving the eq^n $g(u) = j$ for each $j \in J$, we get that the inverse map $g^{-1}: J \to I$ is of the form $g^{-1}(v) = e^{-v}$ $\forall\, v \in J$.

We have verified the assumptions of change of density formula.

Hence, we got that $V = g(U) = -\log U$ is also a continuous R.V. with pdf :-

$$f_V(v) = \begin{cases} f_U(g^{-1}(v)) \left| \dfrac{d\, g^{-1}(v)}{dv} \right| & ; v \in J \\ 0 & ; v \in J^c \end{cases}$$

Thus, $f_V(v) = \begin{cases} e^{-v} & ; v \in J \\ 0 & ; v \in J^c \end{cases}$

$\therefore$, $V \sim \text{Exp}(1)$.

Remarks:

1. We have shown $U \sim \text{Unif}(0,1) \Rightarrow V := -\log U \sim \text{Exp}(1)$. Thus, we can simulate a R.V. $V \sim \text{Exp}(1)$.
2. If $U \sim \text{Unif}(0,1) \Rightarrow (1-U) \sim \text{Unif}(0,1)$ can be verified by change of density formula.
3. $\Rightarrow -\log(1-U) \sim \text{Exp}(1)$.

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(4) There can be examples where Range $(X) \subsetneq I$.

Exercise: Suppose $X \sim Unif\left((0,1) \cup (2,3)\right)$, i.e. X has pdf :-

$$f_X(x) = \begin{cases} \frac{1}{2} & ; x \in (0,1) \cup (2,3) \\ 0 & ; \text{otherwise} \end{cases}$$

Show that $Y = X^2$ is also a continuous R.V. & find a pdf of Y.

Exercise: Suppose $U \sim Unif(-1,1)$. Show that $V = U^3$ is also a continuous R.V. & find a pdf of V (Use the cdf method).

Proof of Change of Density Formula :

given : $g : I \to J$ is differentiable, bijective f^n with either $g'(x) > 0$ or $g'(x) < 0 \ \forall x \in I$.

$\Rightarrow g^{-1}$ is differentiable on J, also $\forall y \in J$;

$$g(g^{-1}(y)) = y. \Rightarrow g'(g^{-1}(y)) \frac{d(g^{-1}(y))}{dy} = 1.$$

Thus, $\frac{d(g^{-1}(y))}{dy} = \begin{cases} +ve & \forall y \\ -ve & \forall y \end{cases}$ & not equal to zero.

We know that,

$$\text{Range}(X) \subseteq I = (a,b) \quad (\text{say}) \quad \&$$
$$\text{Range}(Y) \subseteq J = (c,d) \quad (\text{say})$$

Here; a, b, c, d are not necessarily all finite.

Case I: $g'(x) > 0 \ \forall x \in I$.

This means that both g & g^{-1} are strictly increasing f^ns.
The proof of change of density formula will be complete if we establish that $\forall u \in \mathbb{R}$,

$$P(Y \leq u) = \int_{-\infty}^{u} h(y)\,dy \;;$$

$$h(y) = \begin{cases} f_X(g^{-1}(y)) \dfrac{d(g^{-1}(y))}{dy} & ; \; y \in (c,d) \\ 0 & ; \; y \notin (c,d) \end{cases} \quad \circledast$$

$\because$ Range $(Y) \subseteq (c,d)$, $\circledast$ holds trivially if $u \leq c$

$\because u \leq c \Rightarrow P(Y \leq u) = 0$.

Also, if $u \geq d$, RHS of $\circledast$ $= \underbrace{\int_{-\infty}^{u} h(y)\,dy = \int_{c}^{d} f_X(g^{-1}(y)) \frac{d(g^{-1}(y))}{dy}\,dy}_{\text{(This step may require us to take limits due to improper integrals)}}$

$= \int_{c}^{d} f_X(x)\,dx$.

(This step may require us to take limits due to improper integrals)

$= 1$ (since Range $(X) \subseteq (a,b)$)

$= P(Y \leq u)$ (since Range $(Y) \subseteq (c,d)$ & $u \geq d$)

$=$ LHS of $\circledast$.

Case-II: $g'(x) < 0 \;\forall\; x \in I$ is left as an exercise.

finally, if $c < u < d$, RHS of $\circledast$ $= \int_{-\infty}^{u} h(y)\,dy = \int_{c}^{u} h(y)\,dy$

$$\Rightarrow \int_{c}^{u} f_X(g^{-1}(y))\, d(g^{-1}(y)) = \int_{a}^{g^{-1}(u)} f_X(x)\,dx = P(X \leq g^{-1}(u))$$

$$= P(g(X) \leq u)$$

$$= P(Y \leq u)$$

($\because g$ is strictly increasing)

Exercise: Suppose $X \sim Exp(1)$, show that $Y = \frac{1}{X}$ is a continuous R.V. & find its pdf.

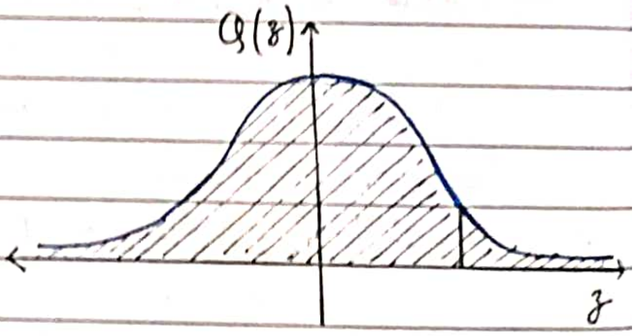
Exercise: $X \sim Normal(\mu, \sigma^2) \Rightarrow aX + b \sim Normal(a\mu + b, a^2\sigma^2)$ $\forall a \neq 0$, $a, b \in \mathbb{R}$. In particular :-

1. $Z \sim Normal(0,1) \Rightarrow \mu + \sigma Z \sim Normal(\mu, \sigma^2)$.
2. $Z \sim Normal(\mu, \sigma^2) \Rightarrow \frac{X - \mu}{\sigma} \sim Normal(0,1)$.

A few Quick Observations :-

Suppose $Z \sim Normal(0,1)$.
This means that Z has a pdf :-

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad \forall z \in \mathbb{R}$$



Area under the curve is cdf of the random variable.

Note :- cdf of $z = \Phi(u)$ (say).

① $\Rightarrow \Phi(0) = \frac{1}{2}$ (By symmetry)

② Also, $1 - \Phi(a) = P(Z > a) = P(Z \leq -a) = \Phi(-a)$.

$$\Rightarrow \Phi(a) + \Phi(-a) = 1 \quad \forall a \in \mathbb{R}.$$

Exercise: Suppose $U \sim Unif(0,1)$. Find the cdf of $X = \sqrt{u}$.
(You'll find that cdf of X is :- $F_X(x) = \begin{cases} 0 & ; x \leq 0 \\ x^2 & ; x \in (0,1) \\ 1 & ; x \geq 1 \end{cases}$)

Recall: We have already proven that: $U \sim Unif(0,1) \Rightarrow X = -\log(1-U) \sim Exp(1)$

i.e. $F_X(x) = \begin{cases} 1-e^{-x} & ; x>0 \\ 0 & ; x \leq 0 \end{cases}$

Note:

① $g(u) = \sqrt{u}$, $u \in (0,1)$ is the inverse f^n of $F(x) = x^2$, $x \in (0,1)$.

② $g(u) = -\log(1-u)$; $u \in (0,1)$ is the inverse f^n of $F(x) = 1-e^{-x}$; $x \in (0,\infty)$.

These two examples motivate the following result.

Theorem: Suppose $U \sim Unif(0,1)$ & $F(x)$ is a cdf which is continuous & strictly increasing on $R = \{x \in \mathbb{R} : F(x) \in (0,1)\}$. Then, the f^n F^{-1} is well-defined on $(0,1)$ & $X = F^{-1}(u)$ has cdf F.

Remarks:

① Using this theorem, we simulate an R.V. X with this cdf, provided that F satisfies the above said conditions & F^{-1} can be computed explicitly.

② If the condition of the theorem is satisfied, then R becomes the range of the R.V. i.e. $R = Range(X)$.

Proof: Take $u \in (0,1)$.

We have to show that $\exists$ unique $x \in \mathbb{R}$ st. $f(x) = u$.

$\because \lim_{y \to -\infty} F(y) = 0 \Rightarrow \exists M$ st. $y \leq M \Rightarrow F(y) \leq a$.

$\because \lim_{y \to \infty} F(y) = 1 \Rightarrow \exists M' > M$ st. $y \geq M' \Rightarrow F(y) \geq b$.

$\because f$ is continuous & $F(M) < u < F(M')$,

Thus, by IVT, $\exists\, x$ s.t. $F(x) = u$.
Also, it is unique $\because F(x)$ is strictly increasing f^n on $\mathbb{R}$.

Thus, F^{-1} is a well-defined map; $F^{-1} : [0,1] \to \mathbb{R}$. Now, we take $U \sim \text{Unif}(0,1)$ & define $X = F^{-1}(U)$.

$\Rightarrow$ Range$(X) \subseteq \mathbb{R}$. It is enough to show that $\forall\, x \in \mathbb{R}$,

$P(X \leq x) = F(x)$ because after end-point of $\mathbb{R}$, it is trivially 1 & before start-point, it is 0.

Take $x \in \mathbb{R}$. Then, $P[X \leq x] = P[F^{-1}(U) \leq x] = P[U \leq F(x)]$
$\left(\because f^n\ F \text{ is } \uparrow \text{ increasing on } (0,1) \quad \mathbb{R} \underset{F^{-1}}{\overset{F}{\rightleftarrows}} (0,1)\right)$

$\Rightarrow F(x) = P(U \leq F(x))$. $\quad \Rightarrow F(x) = P(U \leq u) \quad \left(\because u = F(x) \in (0,1)\right)$

Theorem: If $F : \mathbb{R} \to \mathbb{R}$ is a f^n s.t.
1. F is non-decreasing
2. F is right continuous
3. $\lim_{x \to \infty} F(x) = 1$.
4. $\lim_{x \to -\infty} F(x) = 0$,

then $\exists$ R.V. X s.t. $F(x) = P(X \leq x)\ \forall\, x \in \mathbb{R}$.

Observation:

① In the proof, all we needed was ③ & ④.
$\Rightarrow$ we have proved special case of previous theorem where $F(x)$ is continuous instead of just right-continuous & $\uparrow$ increasing instead of non-decreasing respectively.

② Note that the only property of F^{-1} that has been used in the proof is the following :-

$\forall\, u \in (0,1)$, $F^{-1}(u) \leq x$ iff $F(x) \geq u$ — ⊗

∴, we can assume a fⁿ, say $F^{\leftarrow}$ st. $\forall\, u \in (0,1)$ $F^{\leftarrow}$ satisfies ⊗, then if we define $X = F^{\leftarrow}(U)$, (here $U \sim \text{Unif}(0,1)$) will have cdf F.

Q: How to find such $F^{\leftarrow}$?

solⁿ:- If F satisfies the conditions of previous theorem, then $F^{\leftarrow} = F^{-1}$ & we are done.

In general, F may have jump discontinuities &/or flat regions.

The Problems are as follows:

1. $Y = u$, may not intersect $y = F(x)$.
2. $Y = u$ may intersect $y = F(x)$ in the entire flat region.
   $\Rightarrow$ too many x's.

Moral: Do not try to look for an intersection of $y = u$, $y = F(x)$. Rather, look at the first time $y = F(x)$ crosses or touches the line $y = u$. More precisely, first times means, define

$$F^{\leftarrow}(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\} \quad ; u \in (0,1)$$

This is called Generalised inverse of F.

Proof of Theorem:

Defⁿ: The generalised of such an F is defined as :-

$$F^{\leftarrow}(u) = \inf \underbrace{\{x \in \mathbb{R} : F(x) \geq u\}}_{A_u} \quad ; u \in (0,1)$$

Now, the set A_u is well-defined because $\forall\, u \in (0,1)$, $A_u \neq \{\}$ & A_u is a bounded f^n.

Proof of $A_u \neq \{\}$

To show $A_u = \{x \in \mathbb{R} : F(x) \geq u\} \neq \{\}$,

(iii) $\Leftrightarrow \lim\limits_{x \to \infty} f(x) = 1 \Rightarrow \exists M \in \mathbb{R}$ st. $x \geq M$

$\Rightarrow F(x) > u$

i.e. $f(M) > u$.

$\Rightarrow A_u \neq \{\}$.

Proof that A_u is bounded below

suppose A_u is not bounded below for some $u \in (0,1)$,

Thus, $\exists\, u \in (0,1)$ st. $A_u = \{x \in \mathbb{R} : f(x) \geq u\}$ is not bounded below.

$\therefore$, we can extract a sequence $x_n \in A_u$ st. $x_n \downarrow -\infty$.

However, by (ii) $\lim\limits_{x \to -\infty} F(x) = 0 \Rightarrow \lim\limits_{n \to \infty} F(x_n) = 0 \Rightarrow\Leftarrow$

$\because x_n \in A_u$ & $F(x_n) \geq u \Rightarrow \lim\limits_{n \to \infty} F(x_n) \geq u > 0 \Rightarrow\Leftarrow$

Jump discontinuities in F give rise to flat regions in $F^{\leftarrow}$.

Flat region in F gives rise to a jump discontinuity from the right in $F^{\leftarrow}$.

Exercise: Show that $F^{\leftarrow}$ is left continuous. Show that $\forall\, u \in (0,1)$, $F(F^{\leftarrow}(u)) \geq u$ & give example to show that strict inequality may hold.

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Lemma: $\forall u \in (0,1)$, $F^{\leftarrow}(u) \geqslant x$ iff $u \leqslant F(x)$.

Proof: Recall :- $\forall u \in (0,1)$, $A_u = \{x \in \mathbb{R} : F(x) \geqslant u\}$, $F^{\leftarrow}(u) = \inf A_u$

Fix $u \in (0,1)$, to show that : $u \leqslant F(x) \Rightarrow F^{\leftarrow}(u) \leqslant x$.

We shall show that $F^{\leftarrow}(u) > x \Rightarrow u > F(x)$.

Suppose $\inf A_u = F^{\leftarrow}(u) > x \Rightarrow x \notin A_u = \{x \in \mathbb{R} : F(x) \geqslant u\}$

Now, to show that : $F^{\leftarrow}(u) \leqslant x \Rightarrow u \leqslant F(x)$.

Note that $F^{\leftarrow}(u) = \inf A_u$.

$\Rightarrow \exists$ a sequence $\{x_n\}_{n \geq 1} \subseteq A_u$ s.t. $x_n \downarrow F^{\leftarrow}(u)$.

$\because$ Each $x_n \in A_u \Leftrightarrow F(x_n) \geqslant u$ for each n. Also $x_n \downarrow F^{\leftarrow}(u)$

$\xrightarrow[\text{continuity of } F]{\text{by right}} \lim_{n \to \infty} F(x_n) = F(F^{\leftarrow}(u)) \leqslant F(x)$ ($\because$ F is non-decreasing)

$\Rightarrow u \leqslant \lim_{n \to \infty} F(x_n) \leqslant F(x)$. This completes the proof.

Proof of Theorem: Take $U \sim \text{Unif}(0,1)$. Define $X = F^{\leftarrow}(U)$.

Following the proof of previous theorem, we get,

$$P(X \leqslant x) = P(F^{\leftarrow}(U) \leqslant x) = P(U \leqslant F(x)) = F(x).$$

This proves the theorem.

Remarks:

1. By virtue of the theorem, we can characterize all cdfs.
2. The theorem also gives us an algorithm to simulate the given cdf, provided $F^{\leftarrow}(u)$ can be computed explicitly $\forall u \in (0,1)$.

Exercise: Give an algorithm to simulate the following random variable :-

(i) $X \sim \text{Poi}(1)$ (ii) $X \sim \text{Unif}\ (0,1) \cup (2,3)$.

Corollary: If $f : \mathbb{R} \to \mathbb{R}$ is an integrable f^n s.t :-

1. $f(x) \geq 0 \ \forall x \in \mathbb{R}$ 2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Then, $\exists$ a continuous R.V. whose pdf is f.

Proof: Define $F(t) = \int_{-\infty}^{t} f(x)\, dx$, $t \in \mathbb{R}$

Then, F satisfies the following properties :-

1. F is non-decreasing.
2. F is continuous (by FtoC)
3. $\lim_{t \to \infty} F(t) = \lim_{t \to \infty} \int_{-\infty}^{t} f(x)\, dx = 1$.
4. $\lim_{t \to -\infty} F(t) = 0$.

For ④ :- $1 = \int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{t} f(x)\, dx + \int_{t}^{\infty} f(x)\, dx$.

$\Rightarrow F(t) = 1 - \int_{t}^{\infty} f(x)\, dx$.

Thus, $\lim_{t \to -\infty} F(t) = 1 - 1 = 0$.

Thus, because of the Theorem, $\exists$ an R.V. X st. F is the cdf of X.

i.e. $\forall t \in \mathbb{R}$, $P(X \leq t) = F(t) = \int_{-\infty}^{t} f(x)\, dx$.

Hence, it follows that X is a cont. R.V. with pdf f.

Remark: Just as in the discrete case, we do not need the sample space & the exact map to define a cont. R.V. X. It is enough to verify that X has a valid pdf by checking the conditions of the corollary.

Observation: If $X \sim Exp(\lambda)$, then we know that its survival f^n,
$P(X > x) = e^{-\lambda x}$, $x > 0$.
If u is very large, $P(X > u) = e^{-\lambda u}$ is very low.

Def^n: Suppose $h_1 : \mathbb{R} \to (0, \infty)$, $h_2 : \mathbb{R} \to (0, \infty)$ are 2 f^ns.
We write $h_1(u) \sim h_2(u)$ as $u \to \infty$ (They have same behaviour) if $\lim_{u \to \infty} \frac{h_1(u)}{h_2(u)} = 1$.
We say that assymptotically, h_1 & h_2 are the same as $u \to \infty$.

Exercise: Suppose $X \sim N(0,1)$, show that, $P(X > u) = 1 - \Phi \sim \frac{\varphi(u)}{u}$ as $u \to \infty$.

Here, $\varphi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$, $u \in \mathbb{R}$ and Φ is the cdf of standard normal distribution.

\# $\therefore$, the tails of standard normal distribution decay to 0, exponentially fast.

\# If the right / left tail of X decays significantly slower than exponential, then these random variables will take large values with slightly higher probability than exponential distribution.

For eg. if $P(X > u) \sim Cu^{-\alpha}$ as $u \to \infty$ $(C, \alpha > 0)$, then this will be the case.
Such R.Vs are called Heavy tailed R.Vs & are useful in economics, insurance etc.

Examples of Heavy Tailed Random Variables :-

1. Pareto Distribution :
A R.V. X is said to follow Pareto distribution (Pareto introduced this to model income distributions) with parameters, $A > 0$ (called scale parameter) & $\alpha > 0$ (called tail parameter) iff X has a pdf :-

$$f_X(x) = \begin{cases} \alpha A^{\alpha} x^{-\alpha-1} & ; x > A \\ 0 & ; \text{otherwise} \end{cases}$$

Notated by : $X \sim \text{Pareto}(A, \alpha)$.
If $A = 1$, $X \sim \text{Pareto}(\alpha)$.

Exercise: Show that F_X is a valid pdf.

A is called the scale parameter because,
$$X \sim \text{Pareto}(A, \alpha) \Rightarrow \frac{X}{A} \sim \text{Pareto}(\alpha).$$

Exercise: $X \sim \text{Gamma}(\alpha, \lambda) \Rightarrow \lambda X \sim \text{Gamma}(\alpha, 1)$.
(This is why λ is called the scale parameter in Gamma Distribution)

On the other hand, α is called the tail parameter, because
$$\forall u > A, \; P(X > u) = \int_u^{\infty} \alpha A^{\alpha} x^{-\alpha-1}\, dx = A^{\alpha} u^{-\alpha}.$$

$\therefore$, $\forall\ u > A$, the right tail,
$P(X > u)$ if $X \sim \text{Pareto}(A, \alpha)$ decays polynomially fast.

$\Rightarrow X$ is heavy tailed.

2. Cauchy Distribution:
A R.V. X is said to be following Cauchy Distribution with scale parameter $\sigma > 0$ if it has a pdf:-

$$f_X(x) = \frac{\sigma}{\pi(x^2+\sigma^2)} \quad ; \quad x \in \mathbb{R}.$$

Notated by $X \sim \text{Cauchy}(\sigma)$.

Exercise: Check that $f_X(x)$ is a valid pdf.

If $\sigma = 1$, we say $X \sim \text{Cauchy}$.

Exercise: If $X \sim \text{Cauchy}(\sigma)$ & $\alpha \in (0, \infty)$, find the distribution of αX.

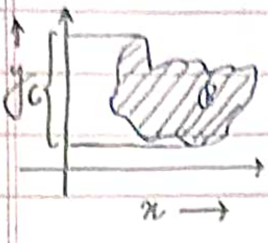
Exercise: If $X \sim \text{Cauchy}(\sigma)$, show that $\exists\ c \in (0, \infty)$ s.t. $P(X > u) \sim c u^{-1}$ as $u \to \infty$.

Exercise: Give an algorithm to simulate the following:
1. $X \sim \text{Pareto}(A, \alpha)$ 2. $X \sim \text{Cauchy}(\sigma)$

Double Integrals:
Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ is a f^n & $B \subseteq \mathbb{R}^2$ is a "nice" set.

Q: What do we understand by $\iint_B f(x,y)\, dx\, dy$?

We shall understand that this integral is a repeated integral.



Let $C = \{ y \in \mathbb{R} : (x, y) \in B \text{ for some } x \}$, be the projection of B to the vertical axis. We will deal with B nice enough so that C is countable union of intervals.

Fix $y \in C$. Look at $B^y = \{ x \in \mathbb{R} : (x, y) \in B \}$

Hence, B^y is a section of B formed by a horizontal line through $(0, y)$.

We will assume that B is nice enough so that B^y is a countable union of intervals.

\# The Integral I will be understood as follows :- first integrate over x, each B^y keeping y constant & then integrate over y on C.

Again, we will be dealing with f^n's whose integral exists & is finite.

In other words, we shall first compute $g(y) = \int_{B^y} f(x, y)\, dx \in (-\infty, \infty)$ $\forall\, y \in C$.

This gives us a f^n $g : C \to \mathbb{R}$. We shall now compute the double integral I by,

$$I = \int g(y)\, dy \in (-\infty, \infty).$$

\# This means that, $I = \iint_B f(x, y)\, dx\, dy = \int_C \left(\int_{B^y} f(x, y)\, dx \right) dy.$

Hence, we integrated w.r.t. B, x first & then integrated w.r.t. y.

What if we integrated w.r.t. y first & then w.r.t. x ?

There is a version of Fubini's Theorem that says that all nice f^n's (the ones we'll work with), the order of integration does not matter.

Eg. if $f(x,y) \geq 0 \ \forall (x,y) \in \mathbb{R}^2$, then order of integrals never matters.

Examples:

① If $\mathcal{B} = \mathbb{R}^2$, $\Rightarrow C = \mathbb{R}$, $\forall y \in \mathbb{R}$, $\mathcal{B}^y = \mathbb{R}$.

$$\Rightarrow \iint_{\mathcal{B}} f(x,y)\,dx\,dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,dx \right) dy.$$

② If $\mathcal{B} = (-\infty, u] \times (-\infty, y]$, $C = (-\infty, y]$ & $\mathcal{B}^y = (-\infty, u]$.

$$\Rightarrow \iint_{(-\infty,u]\times(-\infty,y]} f(x,y)\,dx\,dy = \int_{-\infty}^{y} \left(\int_{-\infty}^{u} f(x,y)\,dx \right) dy.$$

Bivariate Continuous Random Vector:

Def^n: A bivariate R.V (X, Y) (i.e. X, Y are R.Vs on the same sample space) is called absolutely continuous if $\exists$ a f^n $f_{X,Y} : \mathbb{R}^2 \to [0,\infty)$ st. $\forall (u,v) \in \mathbb{R}^2$,

$$F_{X,Y}(u,v) = P(X \leq u, Y \leq v) = \int_{-\infty}^{v} \left(\int_{-\infty}^{u} f_{X,Y}(x,y)\,dx \right) dy$$

In this case, $f_{X,Y}(x,y)$ is called the joint probability distribution f^n (joint pdf) or joint density f^n of (X,Y).
We shall say that X, Y are jointly continuous.

Recall: If (X,Y) is a discrete R.V, then for any $\mathbb{B} \subseteq \mathbb{R}^2$,

$$P[(X,Y) \in \mathbb{B}] = \sum\sum_{(x,y)\in\mathbb{B}} P_{X,Y}(x,y), \quad P_X(x) = \sum_{y\in\mathbb{R}} P_{X,Y}(x,y) \quad \forall x \in \mathbb{R}.$$

Similarly, it can be shown $\forall$ nice $\mathbb{B} \subseteq \mathbb{R}^2$,

$$P((X,Y)\in\mathbb{B}) = \iint_{\mathbb{B}} f_{X,Y}(x,y)\,dx\,dy \quad \text{——} \quad (*)$$

Proposition: If X & Y are jointly continuous with a joint pdf $f_{X,Y}$, then marginally, both X & Y are continuous random variables & their marginal pdfs are given by :-

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \quad \forall x \in \mathbb{R}$$ & similarly for $f_Y(y)$ as well.

(The converse, unlike in discrete case, is not true. $\Rightarrow$ 2 continuous R.V.s may not be jointly continuous)

Proof: We will only show that X is a continuous R.V. with a marginal pdf f_X as above.

We need to show that $\forall u \in \mathbb{R}$, $P(X \leq u) = \int_{-\infty}^{u} f_X(x)\,dx$.

Note that $\forall u \in \mathbb{R}$, $P(X \leq u) = P(X \leq u, -\infty < Y < \infty)$

$$= P(X,Y \in (-\infty,u] \times (-\infty,\infty))$$

$$= \iint_{(-\infty,u]\times(-\infty,\infty)} f_{X,Y}(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{u} f_{X,Y}(x,y)\,dx\,dy = \int_{-\infty}^{u}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_{-\infty}^{u} f_X(x)\, dx.$$

Remark: The above fact shows that if X, Y are jointly continuous, then they are both marginally continuous however the converse is not true as shown in the exercise below.

Exercise: Take $X \sim \text{Unif}(0,1)$. Define $Y = X$ & Show that X & Y are not jointly continuous, even though marginally, each is a continuous R.V.

(Define $B = \{(x,y) \in (0,1)\times(0,1) : x = y\}$. Using ⊛, show that $P[(x,y) \in B] = 0$ if X, Y have a joint pdf.)

In parallel to the univariate case, if (u,v) is a continuity point of $f_{X,Y}$ (this means that $f_{X,Y}(u_n, v_n) \to f_{X,Y}(u,v)$ whenever $u_n \to u$, $v_n \to v$),

then we can show that,

$$f_{X,Y}(u,v) = \frac{\partial}{\partial u}\frac{\partial}{\partial v} F_{X,Y}(x,y).$$

Here, $\frac{\partial}{\partial v}$ refers to the partial derivative of $F_{X,Y}(u,v)$ w.r.t v

i.e. taking the derivative of F treating u as a constant.

$$\Rightarrow \frac{\partial}{\partial v}\left(F_{X,Y}(u,v)\right) = \lim_{h\to 0}\left(\frac{F_{X,Y}(u, v+h) - F_{X,Y}(u,v)}{h}\right).$$

Also, the order of the partial derivative will not matter.

∴, we shall use the following recipe to guess a joint pdf of a continuous r.v. (X,Y) from its cdf.

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$$h(x,y) = \begin{cases} \frac{\partial}{\partial x}\frac{\partial}{\partial y} F_{X,Y}(x,y) & ; \text{ if the partial derivative exists} \\ 0 & ; \text{ otherwise} \end{cases}$$

In order to show that h is indeed a joint pdf of a continuous R.V. (X,Y), we have to verify :-

$$F_{X,Y}(u,v) = \int_{-\infty}^{v}\int_{-\infty}^{u} h(x,y)\,dx\,dy .$$

(This will always work, & whenever it doesn't, there is a Theorem to show that X,Y would not be continuous jointly)

Whenever (u,v) is a continuity point of $f_{X,Y}$, we can show that :-

$$\lim_{\Delta u \to 0^+}\lim_{\Delta v \to 0^+} \frac{P(u \leq X \leq u+\Delta u,\ v \leq Y \leq v+\Delta v)}{\Delta u\,\Delta v} = f_{X,Y}(u,v)$$

This somewhat explains $f_{X,Y}$ is called a joint probability density f^n.

$$\Rightarrow P\left[(X,Y) \in [u,u+\Delta u]\times[v,v+\Delta v]\right] \approx f_{X,Y}(u,v)\,du\,dv$$

Any Joint density f^n $f_{X,Y}$ satisfies :-

1. $f_{X,Y}(x,y) \geq 0 \quad \forall (x,y) \in \mathbb{R}^2$
2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y) = 1.$

(This is going to be a characterizing property of a joint pdf)

Recall : We say that (X,Y) is a continuous R.V. X & Y are jointly continuous if (X,Y) has a joint pdf ie.
$\exists\ f_{X,Y} : \mathbb{R}^2 \to \mathbb{R}$ s.t.

$$P(X \leq u, Y \leq v) = \int_{-\infty}^{v}\int_{-\infty}^{u} f_{X,Y}(x,y)\,dx\,dy \quad \forall (u,v) \in \mathbb{R}^2$$

Joint pdf is not unique.

Eg: Consider a continuous R.V. (X, Y) with a joint pdf :-

$$f(x,y) = \begin{cases} 1 & ; x \in (0,1) \\ 0 & ; x \in \mathbb{R} \setminus (0,1) \end{cases}$$

For this R.V, another joint pdf is :-

$$g(x,y) = \begin{cases} 1 & ; x \in (0,1] \\ 0 & ; x \in \mathbb{R} \setminus (0,1] \end{cases}$$

Exercise : Suppose (X, Y) is a continuous R.V. with joint pdf :

$$f_{X,Y}(x,y) = \begin{cases} c(x+y)^2 & ; x>0, y>0, x+y<1 \\ 0 & ; \text{otherwise} \end{cases}$$

where $c > 0$ is a constant.

(a) Find c.
(b) Compute $P(X<Y)$ & $P(Y<X)$
(c) Compute marginal pdfs of X & Y.

sol^n :- Range$(X,Y) = \{(x,y) \in \mathbb{R}^2 : x>0, y>0, x<1-y\}$

Now, $1 = \iint_{\mathbb{R}^2} c(x+y)^2 \, dx\, dy = \int_0^1 \int_0^{1-x} c(x+y)^2 \, dy\, dx.$

Now, consider x as constant & put $z = x+y$.

$$\Rightarrow 1 = \int_0^1 \left(\int_x^1 c z^2 \, dz \right) dx = c \int_0^1 \left[\frac{z^3}{3} \right]_x^1 dx = \frac{c}{3} \int_0^1 (1-x^3) \, dx.$$

$$\Rightarrow \frac{c}{3}\left(\frac{3}{4}\right) = 1. \qquad \Rightarrow c = 4.$$

Note that $P(X<Y) + P(X>Y) + P(X=Y) = 1.$

Also, $P(X=Y) = \iint\limits_{\substack{x>0,\, y>0,\\ x+y<1,\, x=y}} 4(x+y)^2\, dx\, dy = 0$

Now, $P(X<Y) = 1 - P(X>Y)$

(We may notice that $4(x+y)^2$ is symmetric, so, $P(X<Y) = P(X>Y) = \frac{1}{2}$.)

$$\Rightarrow P(X>Y) = \iint\limits_{\substack{x>0,\, y>0\\ x+y<1,\, x>y}} 4(x+y)^2\, dx\, dy.$$

$\{(x,y) \in \mathbb{R}^2 : x, y > 0,\ x+y<1,\ x>y\} = \{(x,y) \in \mathbb{R}^2 : 0 < y < \frac{1}{2},\ y < x < 1-y\}$

$$\Rightarrow P(X>Y) = \int_0^{1/2} \int_y^{1-y} 4(x+y)^2\, dx\, dy = \frac{1}{2} \quad \text{(Check on your own)}$$

Hence, joint pdf of (X,Y) is :-

$$f_{X,Y}(u,v) = \begin{cases} 4(u+v)^2 & ; u, v > 0,\ u+v<1 \\ 0 & ; \text{otherwise} \end{cases}$$

$\therefore$, the random vectors (X,Y) & (Y,X) have same joint pdf.

Define $B = \{(u,v) \in \mathbb{R}^2 : u < v\}$.

$$\Rightarrow P(X<Y) = P[(X,Y) \in B] \overset{*}{=} \iint\limits_B f_{X,Y}(u,v)\, du\, dv$$

$$= \iint\limits_B f_{Y,X}(u,v)\, du\, dv = P(Y<X) \qquad \Rightarrow P(X<Y) = \frac{1}{2}.$$

Remark: Whenever $f_{X,Y}(u,v) = f_{Y,X}(u,v)$, we can conclude,
$P(X<Y) = P(Y<X) = \frac{1}{2}$.

A marginal pdf of X is $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$, $x \in \mathbb{R}$.

Also, range$(X) = (0,1)$.

$$\Rightarrow f_X(x) = 0 \quad \forall\ x \notin (0,1).$$

$$\Rightarrow f_X(x) = \int_0^{1-x} f_{X,Y}(x,y)\,dy = \frac{4}{3}(1-x^3)$$

Thus, $f_X(x) = \begin{cases} \frac{4}{3}(1-x^3) & ;\ x \in (0,1) \\ 0 & ;\ x \notin (0,1) \end{cases}$

Independence of two jointly distributed random variables:

Recall: We say that two R.V.s X & Y are independent ($X \perp\!\!\!\perp Y$) if
$\forall (u,v) \in \mathbb{R}^2$, $P(X \leq u, Y \leq v) = P(X \leq u)\,P(Y \leq v)$.

In other words, $X \perp\!\!\!\perp Y \Leftrightarrow F_{X,Y}(u,v) = F_X(u)\,F_Y(v)\ \forall (u,v) \in \mathbb{R}^2$

Theorem: Two discrete R.Vs are independant iff $P_{X,Y}(x,y) = P_X(x)\,P_Y(y)$; $\forall (x,y) \in \mathbb{R}^2$.

Corollary: For a discrete R.V. (X,Y),
$X \perp\!\!\!\perp Y$ iff $P[X \in C, Y \in D] = P[X \in C]\,P[Y \in D]$
$\forall C, D \subseteq \mathbb{R}$.

Theorem: Suppose X, Y are continuous R.Vs with pdfs $f_X(x)$, $f_Y(y)$ respectively, then $X \perp\!\!\!\perp Y$ iff (X, Y) is a continuous R.V. with joint pdf $g(x,y) = f_X(x) f_Y(y) \ \forall (x,y) \in \mathbb{R}^2$.

Proof:

($\Leftarrow$) Take any $(u, v) \in \mathbb{R}^2$. Then,

$$F_{X,Y}(u,v) = P(X \leq u, Y \leq v) = \int_{-\infty}^{v} \int_{-\infty}^{u} g(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{v} \int_{-\infty}^{u} f_X(x) f_Y(y)\, dx\, dy = \int_{-\infty}^{v} f_Y(y)\, F_X(u)\, dy$$

$$= F_X(u) F_Y(v).$$

($\Rightarrow$) Say X, Y are cont. R.Vs with pdfs f_X, f_Y respectively & $X \perp\!\!\!\perp Y$.

Using independance of X, Y; we get,

$$F_X(u) F_Y(v) = \left(\int_{-\infty}^{u} f_X(x)\, dx\right)\left(\int_{-\infty}^{v} f_Y(y)\, dy\right) = F_{X,Y}(u,v).$$

$$= \int_{-\infty}^{v} \int_{-\infty}^{u} f_X(x) f_Y(y)\, dx\, dy.$$

Remark:

1. We know that X & Y may not in-general be jointly continuous, even ~~if~~ when they are marginally continuous. Under the assumption of independence, Marginal continuity of R.Vs X & Y $\Rightarrow$ Joint Continuity.

2. $\because$ joint & marginal pdfs are not unique, the above theorem should not be used to justify that $X \not\perp\!\!\!\perp Y$, in which case, it is better to prove from the defn.

(5) However, for establishing independence of 2 continuous R.Vs, this theorem is very useful.

Corollary: Suppose X, Y are marginally continuous R.Vs, then $X \perp\!\!\!\perp Y$ iff,

$$P[X \in C, Y \in D] = P[X \in C]\,P[Y \in D] \quad \forall\, C, D \subseteq \mathbb{R}$$

s.t. both C, D are countable, disjoint union of intervals.
(We can use this too to show that $X \not\perp\!\!\!\perp Y$).

Example: Suppose $X \sim Exp(\lambda)$, $Y \sim Exp(\mu)$ & $X \perp\!\!\!\perp Y$.
a) Calculate $P(X < Y)$
b) Assuming $\lambda = \mu$, compute $P(X \leq aY)$ $\forall\, a > 0$.
c) Using b), compute the cdf & pdf of $\frac{X}{Y}$.

a) In this example,
marginal pdf of X ; $f_X(x) = \lambda e^{-\lambda x}$; $x > 0$ &
marginal pdf of Y ; $f_Y(y) = \mu e^{-\mu y}$; $y > 0$.

$\because X \perp\!\!\!\perp Y$, $\Rightarrow$ Joint pdf of (X, Y) is :-

$$f_{X,Y}(x,y) = \lambda \mu e^{-\lambda x - \mu y} \quad ; \quad x, y > 0.$$

Now, we have $P[X < Y] = P[(X,Y) \in B] = \iint_B f_{X,Y}(x,y)\,dx\,dy$.

$$= \int_{0}^{\infty}\int_0^y \lambda\mu e^{-\lambda x - \mu y}\,dx\,dy = \mu\int_0^\infty e^{-\mu y}\left(\int_0^y \lambda e^{-\lambda x}\,dx\right)dy$$

$$= \mu\int_0^\infty e^{-\mu y} F_X(y)\,dy = \mu\int_0^\infty e^{-\mu y}\left(1 - e^{-\lambda y}\right)dy = \frac{\lambda}{\mu + \lambda}.$$

b) In particular, if $\lambda = \mu$, then $P(X < Y) = \frac{1}{2}$, which also follows from the symmetry of R.Vs.

From now, we assume $\lambda = \mu$, then, (X,Y) has a joint pdf,

$$f_{X,Y}(x,y) = \lambda^2 e^{-\lambda(x+y)} \quad ; \quad x,y > 0.$$

Fix $a > 0$, then $P(X \leq aY) = \int_0^\infty \int_0^{ay} \lambda^2 e^{-\lambda(x+y)} \, dx \, dy$

$$= \int_0^\infty \lambda^1 e^{-\lambda y} \left(\int_0^{ay} \lambda e^{-\lambda x} \, dx \right) dy = \int_0^\infty \lambda e^{-\lambda y} F_X(ay) \, dy$$

$$= \int_0^\infty \lambda e^{-\lambda y} \left(1 - e^{-\lambda a y}\right) dy = \frac{a}{1+a}.$$

Exercise: solve b) by finding the distn of aY.

c) Define $Z = \frac{X}{Y}$. Clearly, $\text{Range}(X) = \text{Range}(Y) \neq \text{Range}(Z)$
$= (0,\infty)$.

$\Rightarrow \text{Range}(Z) \subseteq (0,\infty)$.

$\therefore$, if $a \leq 0$, $\Rightarrow P[Z \leq a] = 0$.

Take $a > 0$. Then, $F_Z(a) = P[Z \leq a] = P\left[\frac{X}{Y} \leq a\right] = P[X \leq aY]$
$= \frac{a}{1+a}$.

$\Rightarrow$ we can guess that Z has a pdf,

$$h(z) = \begin{cases} \frac{1}{(1+z)^2} & ; z > 0 \\ 0 & ; z \leq 0 \end{cases}$$

It is easy to check that $h(z)$ is differentiable $\forall a \in \mathbb{R} \setminus \{0\}$

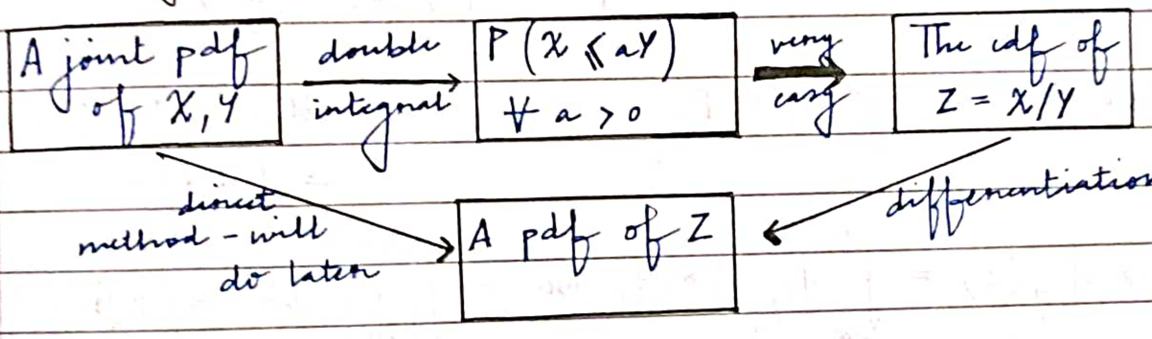
Thus, $F_Z'(a) = \frac{d}{da}\left(\frac{a}{1+a}\right) = \frac{1}{(1+a)^2} \quad \forall a > 0$,

$F_Z'(a) = 0 \quad \forall a < 0$.

Now, $\int_{-\infty}^{a} h(z)\,dz = F_Z(a) = \begin{cases} 0 & ; a \leq 0 \\ \frac{a}{1+a} & ; a > 0 \end{cases}$ (check this on your own)

On the other hand, if $a > 0$, then $\int_{-\infty}^{a} h(z)\,dz = \frac{a}{1+a}$ which shows that above given claim holds.

Summary of What we did :



Example: Suppose (X, Y) is uniformly distributed on the disc $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.

a) Write down a joint pdf of (X, Y).

b) Find marginal pdfs of X, Y.

c) Suppose Δ denotes the distance of the Random point (X, Y) from $(0, 0)$. Find cdf of Δ.

a) The phrase "(X, Y) is uniformally distributed on D" means that (X, Y) has a joint pdf :-

$$f_{X,Y}(x, y) = \begin{cases} c & ; (x, y) \in D \\ 0 & ; (x, y) \notin D \end{cases}$$

Now, $\iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy = 1 = c\iint_D dx\,dy = c\,\text{Area}(D)$
$= c\pi$.
$\Rightarrow c = \frac{1}{\pi}$.

$$\Rightarrow f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi} & ; (x,y) \in D \\ 0 & ; (x,y) \notin D \end{cases}$$

b) We shall find a marginal pdf $f_Y(y)$ of Y.

Note that Range(Y) = Projection of D on the vertical axis
$= (-1,1)$.

$$\Rightarrow f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(\frac{1}{\pi}\right) dx = \left(\frac{2\sqrt{1-y^2}}{\pi}\right).$$

Thus, $$f_Y(y) = \begin{cases} \left(\frac{2\sqrt{1-y^2}}{\pi}\right) & ; y \in (-1,1) \\ 0 & ; y \notin (-1,1) \end{cases}$$

c) Clearly, $\Delta = \sqrt{x^2+y^2}$. Also, Range$(\Delta) = (0,1)$.

$$\Rightarrow F_\Delta(a) = P(\Delta \leq a) = \begin{cases} 0 & ; a \leq 0 \\ 1 & ; a \geq 1 \end{cases}$$

Take $a \in (0,1)$. $\Rightarrow F_\Delta(a) = P\left(\sqrt{X^2+Y^2} \leq a\right)$

$\Rightarrow F_\Delta(a) = P\left(X^2+Y^2 \leq a^2\right)$

$$= \iint_{x^2+y^2 \leq a^2} \left(\frac{1}{\pi}\right) dx\,dy = a^2.$$

Thus, cdf of Δ is,

$$F_\Delta(a) = \begin{cases} 0 & ; a \leq 0 \\ a^2 & ; a \in (0,1) \\ 1 & ; a \geq 1 \end{cases}$$

Using the recipe, we guess :-

$$\text{The pdf '}f_\Delta(x)\text{'} = \begin{cases} F'_\Delta(x) & ; F_\Delta \text{ is differentiable at } x \\ 0 & ; \text{otherwise} \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} 2x & ; x \in (0,1) \\ 0 & ; x \notin (0,1) \end{cases}$$

Exercise: Show that Δ is a continuous R.V. with pdf $f(x)$.

\# If X & Y would have been independant, Range of (X,Y) would be cartesian product of Range of X & Y.

This is an informal way to deal with independance.

\# Here, $T : \text{Range}(X,Y) \to \mathbb{R}$,

$T(x,y) = \frac{x}{y}$, $(x,y) \in (0,\infty)^2$ in the first example.

In the second example, $T(x,y) = \sqrt{x^2+y^2}$, $(x,y) \in D$

\# If $T(x,y) = x+y$ or $x-y$, xy or $\frac{x}{y}$,

then we have direct method of computing a pdf of $T(x,y)$ for a joint pdf of (X,Y).

These are called examples of real-valued f^ns of R.Vs.

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Exercise: Suppose (X,Y) is uniformly distributed on the space,

$$S = \{(x,y) \in \mathbb{R}^2 : |x|+|y| < 1\}.$$

a) Find joint pdf of (X,Y).

b) find marginal pdf of X & Y.

c) find the cdf of $U = |x|+|y|$

d) show that U is a continuous R.V. & find pdf of U.