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2nd Sem

Goal $A\vec{x} = \vec{b}$

$\vec{x} \in \mathbb{R}^n$

$\vec{b} \in \mathbb{R}^m$

$m=n$ A is a square matrix

$$\begin{bmatrix} a_{11} & & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{n1} & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

$a_{21}x_1 + \cdots = b_2$

$\vdots$

$a_{n1}x_1 + \cdots + a_{nn}x_n = b_n$

no. of variables = no. of conditions

$$\left\langle \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right\rangle = b_1$$

$\boxed{\vec{a} \cdot \vec{x} = b}$

$\vec{a} \cdot \vec{x} = 0 \rightarrow \vec{a} \perp \vec{x}$

$\{\vec{x}\}$ would be the plane $\perp^{ar}$ to $\vec{a}$.

$S = \{\vec{x} \in \mathbb{R}^n : \vec{a} \cdot \vec{x} = b\}$

Say $\vec{x}_0 \in S$

Then $\vec{a} \cdot (\vec{x} - \vec{x}_0) = \vec{a} \cdot \vec{x} - \vec{a} \cdot \vec{x}_0 = b - b = 0$

$\therefore \vec{x} \in \vec{x}_0 + \{\vec{a}\}^{\perp}$

Translation of the linear space $\{\vec{a}\}^{\perp}$

Let $\vec{x}_1$ be the vector on S which is closest to the origin

$\left.\begin{array}{l} \\ \vec{x}_1 = c\vec{a} \end{array}\right\}$ check this!

$c\vec{a} \cdot \vec{a} = b$

$\therefore c = \dfrac{b}{\|\vec{a}\|^2}$

$\therefore \langle \vec{a} \cdot \vec{x} = b \rangle$ is a translation of $\langle \vec{a} \cdot \vec{x} = 0 \rangle$ by an amount of $\dfrac{b}{\|\vec{a}\|^2}$ in the direction of $\vec{x}$

$\dim(S) = n-1$

Case $m=n$ $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

no. of variables = no. of conditions (constraints)

$m>n$: $A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

no. of variables $<$ no. of constraints
(over deterministic)

$m<n$

no of variables $>$ no. of constraints
(under deterministic)

$\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is not invertible

$\Leftrightarrow xw - zy = 0$

Set of ~~non~~ invertible ~~sets~~ matrices is negligible in $M_2(\mathbb{R})$ $(\simeq \mathbb{R}^4)$

Special case, ~~B=0~~ $b=0$

$$A\vec{x} = 0$$

We know if

If $\vec{x}_0 \in \mathbb{R}^n$ st is a sol to $A\vec{x} = b$

$x \in \mathbb{R}^n$ " " to $A\vec{x} = 0$

then $(\vec{x} + \vec{x}_0)$ ~~is a sol to~~, $A(\vec{x} + \vec{x}_0) = b$

Proposition: Let $\vec{x}_0 \in \mathbb{R}^n$ s.t. $A\vec{x} = \vec{b}$. Then every s.l. to $A\vec{x} = \vec{b}$ is of the form $\vec{x}_0 + \vec{y}$, where $A\vec{y} = 0$

Pf: $A\vec{x} = \vec{b}$, $A\vec{x}_0 = \vec{b}$

$A \cdot (x - \vec{x}_0) = \vec{b}$

$\vec{x} = \vec{x}_0 + (x - \vec{x}_0)$

~~General~~

minimum minimum

Generally, an over deterministic system does not have a sol^n. But on that case, we rephrase the question

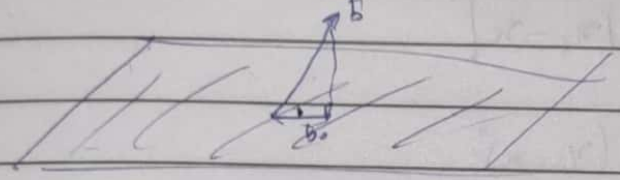
Goal: Find $\vec{x}$ s.t. $\|A\vec{x} - \vec{b}\|$ is minimised

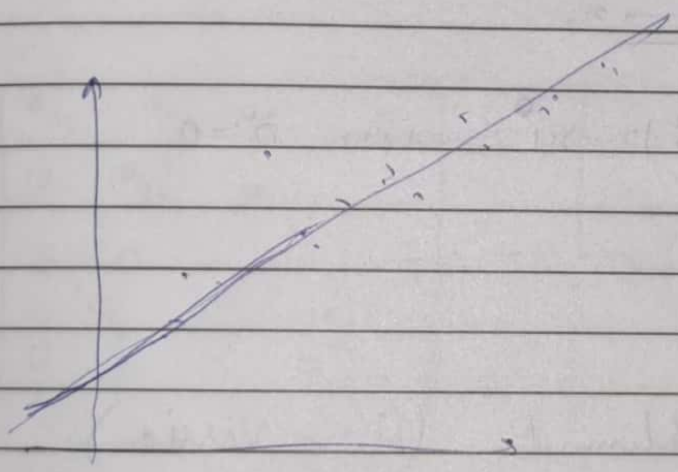
Q: Find $\vec{b_0}$ s.t. $A\vec{x} = \vec{b_0}$ has a sol^n and

$\|\vec{b} - \vec{b_0}\|$ is minimised

$\vec{b_0} \in$ col. space of A

$= \text{ran}(A)$





$(x_i, y_i) \to$ training dataset

m, c

$y = mx + c \leftarrow$ prediction model

$y_i - mx_i - c = 0$

$$\begin{bmatrix} y_1 & x_1 & -1 \\ y_2 & x_2 & -1 \\ & \vdots & \\ y_n & x_n & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -m \\ c \end{bmatrix} = 0$$

This is generally an over determined system, hence it translates to our initial question

$$\sum_{i=1}^{n} (y_i - mx_i - c)^2 \leftarrow \text{error fn.}$$

$x_1, x_2, \ldots, x_n$

$$x_{med} = \min \sum_{i=1}^{n} |x_i - x|^1$$

$$x_{mean} = \min \sum_{i=1}^{n} (x_i - x)^2$$

$$= \frac{x_1 + x_2 + \cdots + x_n}{n}$$

$$x_{mode} = \min \sum_{i=1}^{n} |x_i - x|^0 \quad \text{where } 0^0 := 0$$

$A\vec{x} = \vec{b}$

~~Geometric~~ Gaussian Elimination (Naive Version)

$$\underset{m \times (n+1)}{\left[\begin{array}{c|c} \underset{m \times n}{A} & \underset{m \times 1}{b} \end{array}\right]} \qquad A\vec{x} = \vec{b}$$

In a manner much like row eschelon elemination

$$\underbrace{C_1 C_2 \ldots C_n}_{\text{certain invertible matrices}} A \cdot \vec{x} = C_1 C_2 \ldots C_n$$

$[A_{m\times n} \mid b]_{m \times n+1} \rightarrow$ augmented matrix

$A\vec{x} = b$

$G_1 A\vec{x} = G_1\vec{b}$

$G_1 \in GL_n(\mathbb{R})$ (Invertible matrices in ...)

$G_n \cdots G_1 A\vec{x} = G_n \cdots G_1\vec{b}$

$\vec{x} = G_n \cdots G_1\vec{b}$

if $G_n \cdots G_1 A = I$

Forward elimination (variables)

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} \quad \begin{bmatrix} \vec{b} \end{bmatrix}$$

$a_{21} \rightarrow a_{21} - \left(\frac{a_{21}}{a_{11}}\right) a_{11}$

$R_2 \rightarrow R_2 - \left(\frac{a_{21}}{a_{11}}\right) R_1$

$$\begin{bmatrix} 1 & 0 \\ -\frac{a_{21}}{a_{11}} & 1 \end{bmatrix}$$

In pivoting, pick i, a_{i1} has largest magnitude

$$\left[\begin{array}{ccc|c} a_{11}' & & & b_1' \\ & a_{22}' & & \vdots \\ & & a_{nn}' & b_n' \end{array}\right]$$

converts to identity matrix

Backward substitution $x_n = \frac{b_n'}{a_{nn}}$

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$a'_{(n-1)\,n-1} x_{n-1} + a'_{n-1\,n} = b'_{n-1}$

$\Rightarrow x_{n-1} = \dfrac{b'_{n-1} - a'_{n-1\,n}}{a_{n-1\,n-1}}$

$n!$ det A

$\left(\frac{n}{e}\right)^n \sqrt{2\pi n}$ $\quad \sum_{\sigma \in S_n} \text{sign}(\sigma)$

Q Compute determinant of A.

Gaussian elimination

no. of swaps = p

Ans = $(-1)^p a'_{11} \cdots a'_{nn}$

$1 \to 2 \to 3$

$(1 \to 3)(1 \to 2)$

$$\begin{bmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ x_n - x_{n-1} \\ -x_n \end{bmatrix}$$

$\dfrac{d^2x}{dt^2} + a\dfrac{dx}{dt} + bx = 0$

$(D^2 + aD + b)x = 0$

$= 0$

A → square invertible matrix (in $M_n(\mathbb{R})$)

$[A|b]$ $\xrightarrow[\text{elimination}]{\text{Forward}}$ $\left[\begin{array}{cccc|c} a_{11} & * & * & * & \\ 0 & & \ddots & & b' \\ 0 & 0 & 0 & a_{nn} & \end{array}\right]$

↓ backward substitution

discretized version of $\frac{d}{dx}$

$$\begin{bmatrix} -1 & +1 & & 0 \\ & -1 & +1 & \\ 0 & & -1 & \ddots \\ & & & +1 \\ & & & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ \vdots \\ -x_n \end{bmatrix}$$

↳ sampling of $f : [0,1] \to \mathbb{R}$ (mesh size $\frac{1}{n}$) (grid-size)

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$x_1 = f\left(\frac{1}{n}\right) \quad x_k = f\left(\frac{k}{n}\right)^{-1}$

$x_n = f(1)$

$$\frac{x_1 + \cdots + x_n}{n} = \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$f''(x)$

$(x_{n+2} - x_{n+1}) - (x_{n+1} - x_n)$

$\frac{d^2f}{dx^2} + a\frac{df}{dx} + bf = x$

$(D^2 + aD + b)f = x$

$$\begin{pmatrix} b\left(\frac{1}{n}\right) f\left(\frac{1}{n}\right) \\ b\left(\frac{2}{n}\right) f\left(\frac{2}{n}\right) \\ \vdots \\ b(1) f(1) \end{pmatrix} = \begin{bmatrix} b\left(\frac{1}{n}\right) & & 0 \\ & \ddots & \\ 0 & & b(1) \end{bmatrix} \begin{bmatrix} f\left(\frac{1}{n}\right) \\ \vdots \\ f(1) \end{bmatrix}$$

$$\begin{bmatrix} a\left(\frac{1}{n}\right) & & 0 \\ & \ddots & \\ 0 & & a(1) \end{bmatrix} \begin{bmatrix} -1 & 1 & & 0 \\ & -1 & 1 & \\ 0 & & \ddots & \\ & & & -1 \end{bmatrix} = \begin{bmatrix} -a\left(\frac{1}{n}\right) & a\left(\frac{1}{n}\right) & & 0 \\ & -a\left(\frac{2}{n}\right) & \ddots & \\ 0 & & & \ddots \end{bmatrix}$$

~~band~~

band limited matrix

sparse matrix → most entries are zero

~~diagonal~~ mx → band → 1 ~~diagonal~~

Tridiagonal matrix

$$\begin{bmatrix} a_0 & b_1 & & 0 \\ c_0 & a_1 & b_2 & \\ & c_1 & a_2 & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix}$$

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invertible

$A \to$ Hermitian matrix

$Ax = b$ normalized

$e_1, e_2, \dots, e_n \to$ eigen vectors of A

$$\begin{bmatrix} \langle b, e_1 \rangle \\ \vdots \\ \langle b, e_n \rangle \end{bmatrix} \quad \frac{1}{\lambda_1}\langle b, e_1 \rangle e_1 + \frac{1}{\lambda_2}\langle b, e_2 \rangle e_2 + \dots + \frac{1}{\lambda_n}\langle b, e_n \rangle e_n$$

$$\frac{1}{\lambda_1}\langle b, e_1 \rangle \qquad \lambda_i \neq 0$$

$$b = \alpha_1 e_1 + \dots + \alpha_n e_n$$

$$\langle b, e_1 \rangle = \alpha_1$$

$$\begin{pmatrix} a_0 & b_0 & & & \\ c_0 & a_1 & b_1 & & 0 \\ & c_1 & a_2 & b_2 & \\ & & c_2 & a_3 & b_3 \\ 0 & & & c_3 & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad a_0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_0 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = b_0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_1 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$1\cdot 2 + 2\cdot 3 + \dots (n-1)n \qquad n(n-1) = n^2 - n$$

$$(n-1)(n-2) = (n-1)^2 - (n-1)$$

$$\vdots \qquad \vdots$$

$$\sum_{k=1.5}^{\frac{2n-1}{2}} k^2 - \frac{1}{4} \qquad - \frac{n-2}{4} \qquad 2^2 - 2$$

$$1^2 - 1$$

$$\therefore \ O(n^3).$$

$\boxed{[A|b]}$

problem: given A, find a nice form of A^{-1} (versatile)

LU decomposition $\quad L \to$ lower Δ

$U \to$ upper Δ

$a_1 \; a_2 \dots a_n$

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \dots \end{bmatrix} \begin{Bmatrix} a_1 & a_2 & \dots & a_n \end{Bmatrix}$ $a_1 \quad a_2$ Date: / /

$a_2 \dots$

Lemma: The set of invertible upper Δ (or lower Δ) matrices is a subgroup of $GL_n(\mathbb{R})$

(Inverse of an invertible upper Δ mx is upper Δ)

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix} \quad \mathbb{R}$$

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

Thm: Let $A \in GL_n(\mathbb{R})$ (A is invertible)
Then there is a permutation matrix P, an upper triangular matrix U, lower triangular matrix L, s.t. $PA = LU$

if no pivoting is required, then ...

Permutation matrix

A matrix in $M_n(\mathbb{R})$ which has exactly one non-zero entry in each row and each column and that non-zero entry is 1.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} e_1 = e_2 \qquad \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$e_1 \; e_2$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} e_2 \to e_1$$

Not a permutation mx.

$$\sigma : \langle 1, \dots, n \rangle \to \langle 1, \dots, n \rangle$$

$$P_{i, \sigma(i)} = 1 \quad i \in \langle 1, \dots, n \rangle$$

rest are zero

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$\left.\begin{array}{l} P_{i,\sigma(i)} = 1 \\ P_{j,\sigma(j)} = 1 \end{array}\right\}$ not possible unless $i = j$

as $\sigma(i) = \sigma(j) \Rightarrow i = j$

$Pe_i = i^{th}$ column of P.

$\begin{bmatrix} P_{1,i} \\ P_{2,i} \\ \vdots \\ P_{n,i} \end{bmatrix}$ $\quad e_i \to e_{\sigma(i)}$

$n = 3$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$e_1 \leftrightarrow e_2$ $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$e_3 \to e_3$ $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Row swap – Left multiplication by a permutation mx.
product of permutation matrix is a permutation mx.

$\boxed{P_{\sigma^{-1}} = (P_\sigma)^{-1}}$.

$\sigma : \langle 1, \dots, n \rangle - \langle 1, \dots, n \rangle$
$\mathrm{Sgn}(\sigma) := \det(P_\sigma)$
$\mathrm{Sgn}(\mathrm{swap}) = -1$.
(Every permutation σ can be written as a composition of swaps), $\because$ swap2 swap1) $P\begin{pmatrix} 1 & 2 & \\ & & 3 \end{pmatrix}$
$P_\sigma = (\mathrm{swap}(P))^{-1} : \mathrm{swap}(\sigma)^{-1}$.

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$A = D^{-1}AD$

$\det(P_\sigma) = \det(\text{swaps})$

$\text{sgn}(\sigma) = (-1)^{\text{no. of swaps}}$ $\rightarrow$ parity needs ev changes

$P_{ij} = \langle P_{e_i}, e_j \rangle = \langle e_{\sigma(i)}, e_j \rangle = \delta_{\sigma(i)j} = \begin{cases} 1 & \sigma(i)=j \\ 0 & j \neq \sigma(i) \end{cases}$

$\downarrow$ chronecker

$P_{i,\sigma(i)} = 1$

$P_{ij} = 0$ if $j \neq \sigma(i)$

$\begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$ $R_2 \to R_2 - \alpha R_1$

$A = PT$

$(PT)_{ij} = g_{ij}$

$(PT)_{ij} \neq 0 \Leftrightarrow P_{ji} \neq 0$ iff $i = \sigma(j) \Leftrightarrow j = \sigma^{-1}(i)$

$A^{-1} = U^{-1}L^{-1}P$

<u>QR decomposition</u>: Every invertible matrix in $M_n(\mathbb{R})$ can be decomposed as $\boxed{A = QR}$ where Q is ~~orthogonal~~ R is upper-Δ.

$A^{-1} = R^{-1}Q^{T}$

Pf is essentially Gram-Schmidt $\rightarrow$ lot of round off errors

So we use householder reflections, NF, lifting, rarely, rotations, angle.

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$P_1 \; L^{-1} U$

A

$$\begin{bmatrix} a_{11} & a_{12} & \\ a_{21} & a_{22} & \\ & & \ddots \end{bmatrix}$$

$PA = LU$

$\sigma \quad \begin{array}{l} 1 \\ 2 \\ \vdots \\ n \end{array}$

$R_k \to R_k - \frac{a_{k1}}{a_{11}} R_1$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots \\ -\frac{a_{21}}{a_{11}} & 1 & & \\ -\frac{a_{31}}{a_{11}} & & 1 & \\ \vdots & & 0 & \ddots \\ -\frac{a_{n1}}{a_{11}} & & & \end{bmatrix} \begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix} = \begin{bmatrix} a_{11}' & & \\ 0 & a_{22}' & \\ 0 & a_{32}' & \\ \vdots & \vdots & \\ 0 & a_{n2}' & \end{bmatrix}$$

A'

$\begin{bmatrix} & \end{bmatrix} A =$ upper triangular matrix U

$CU = UC = I$ (C should be upper triangular).

$I \subseteq \{1, \ldots, n\}$

$$\begin{array}{c|c} & 1 \; 2 \cdots n \\ \hline 1 & \\ 2 & \\ \vdots & \\ n & \end{array}$$

$A(I) =$

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix}$$

U

$$\begin{pmatrix} u_{11} & u_{12} & \cdots \\ & u_{22} & \\ 0 & & \ddots \end{pmatrix}$$

<u>Prop.</u> If $U \in M_n(\mathbb{R})$ is upper triangular and invertible then U^{-1} is upper triangular.

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & \\ & & \end{bmatrix} \begin{bmatrix} & \end{bmatrix} = I \quad a_{21} u_{11} = 0 \Rightarrow a_{21} = 0$$

like this get entries zero

$a_{11} = u_{11}^{-1}$, $l_{21} = a_2 \cdot a_{21}/a_1$

$a_{22} u_{22} = 1$

(2) ~~M-2~~ convert $U \cdot \Delta$ to diagonal by using $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $\Rightarrow$ you get

$$\begin{pmatrix} 1 & & & \\ -\frac{a_{21}}{a_{11}} & 1 & & \\ \vdots & & 1 & \\ -\frac{a_{n1}}{a_{11}} & & & 1 \end{pmatrix}^{-1} = \begin{bmatrix} 1 & & 0 \\ \frac{a_{21}}{a_{11}} & 1 & \\ \frac{a_{31}}{a_{11}} & & 1 \\ \vdots & & \\ \frac{a_{n1}}{a_{11}} & & \end{bmatrix}$$

[u Δ s] [Δ s]

inverse, prod = upper Δ

LU - without pivoting

Thm: $A \in M_n(\mathbb{R})$ $\exists$ permutation matrix P, lower Δ L, upper Δ U s.t. $PA = LU$.

$P_{n-1} \cdots L_2 P_2 L_1 P_1 A = U$

(Gaussian elimination with pivoting)

$P = P_{n-1} \cdots P_1$

Then during Gaussian elimination of PA, no pivoting is necessary.

$$L_1 P_1 A = \begin{bmatrix} a_{11}' & \\ 0 & \\ \vdots & \boxed{} \\ 0 & \end{bmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & P_{(n-1)\times(n-1)} \end{pmatrix} L_1 P_1 A = U$$

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diagonal

⇒ you get

prod = upper Δ

& ΔL

of PA

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$$\underbrace{L\begin{bmatrix}1 & 0\\ 0 & m\end{bmatrix}\begin{bmatrix} & \end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & p'^T\end{bmatrix}}\begin{bmatrix}1 & 0\\ 0 & p'\end{bmatrix} P_1 A$$

$$\begin{bmatrix}1 & 0 & 0\\ p'\begin{bmatrix}l_{21}\\ \vdots\\ l_{n1}\end{bmatrix} & p' & \end{bmatrix}\begin{bmatrix}1 & 0 - 0\\ 0 & p'^T\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\ 0 - 0\\ p'\begin{bmatrix}l_{21}\\ \vdots\\ l_{n1}\end{bmatrix} & \begin{matrix}1 & \\ & \ddots & 0\\ 0 & & 1\end{matrix}\end{bmatrix} P_1 A = U$$

$$L_1 P_1 A = \begin{bmatrix}a_{11}' & \cdots\\ 0 & \\ 0 & \boxed{A'}\\ 0 & \end{bmatrix}$$

$P'A' = L'U'$
(induction hyp.)

$$\begin{bmatrix}1 & 0\\ 0 & P_{n-1}'\end{bmatrix} L_1 P_1 A = \begin{bmatrix}a_{11}' & * *\\ 0 & \\ 0 & L'U'\end{bmatrix}$$

$$\begin{bmatrix}I & 0\\ 0 & P'\end{bmatrix} L_1 P A = \begin{bmatrix}a_{11}' & * \cdots\\ 0 & \\ 0 & L'U'\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & L\end{bmatrix}\begin{bmatrix}a_{11} & * * - *\\ 0 & U'\end{bmatrix}$$

$P' \rightarrow (n-1) \times (n-1)$ permutation matrix

$$\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & & &\\ 0 & & L'^{-1} & \\ 0 & & &\end{bmatrix}\begin{bmatrix}a_{11}' & * * - *\\ 0 & \\ 0 & L'U'\end{bmatrix} = \begin{bmatrix}a_{11}' & * * - *\\ 0 & \\ 0 & \\ 0 & U'\\ 0 & \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 0 \\ 0 & P' \end{bmatrix} L_1 = \begin{bmatrix} I & 0 \\ 0 & P' \end{bmatrix} L_1 \begin{bmatrix} 1 & 0 \\ 0 & P'^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & P' \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ P'\begin{pmatrix} l_{21} \\ \vdots \\ l_{n1} \end{pmatrix} & 1 & & 0 \\ & & 1 & \\ & 0 & & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & P' \end{bmatrix}$$

$$\begin{bmatrix} I & 0 \\ 0 & P' \end{bmatrix} P_1 A = L_1^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & U \end{bmatrix} \begin{bmatrix} a_{11}' & * & * \\ 0 & & \\ 0 & & 0 \end{bmatrix}$$

$A, B \to$ Hermitian matrices

$A + \varepsilon B \to \sum \mu_i P_{\mu}$

$\sum \lambda_i P_{\lambda_i} \to$ projection ~~...~~ A corresponding to λ

Defn: $A \in M_n(\mathbb{C})$ is said to be an orthogonal projection if $\underbrace{A = A^*}_{\text{Hermitian}}$ and $\underbrace{A^2 = A}_{\text{Idempotent}}$

A is the projection onto range of A

$A(Ax) = Ax$

$ran(A) = \langle Ax : x \in \mathbb{C}^n \rangle$

$y = \underbrace{Ay}_{z} + \underbrace{(y - Ay)}_{w}$ $\quad Aw = 0$, $Az = z$

$Ker(A) = \langle x : Ax = 0 \rangle$

Prop: If $A \in M_n(\mathbb{R})$ is an orthogonal projection then $ran(A) \overset{?}{=} Ker(A)^{\perp} = \langle y \in \mathbb{C}^n : x \in Ker(A) \rangle$

$\langle x, y \rangle = 0$

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Pf: $Ax \in \text{ran } A$

$y \in \ker(A) \Rightarrow Ay = 0$

$\langle Ax, y\rangle = \langle x, Ay\rangle = \langle x, 0\rangle = 0.$

$y \in \ker(A)^{\perp}$

$\langle x, y\rangle = 0$ whenever $Ax = 0$

$y \in \text{ran}(A)^{\perp} \Rightarrow \langle Ax, y\rangle = 0 \ \forall x$

$\Rightarrow \langle x, Ay\rangle = 0 \ \forall x$

$\Rightarrow Ay = 0 \Rightarrow y \in \ker(A) \Rightarrow (\ker A)^{\perp} \subseteq \text{ran}(A)$

$\mathbb{C}^n = \text{ran}(A) \oplus \ker(A)$ (A is an orthogonal projection,

$y = Ay + (y - Ay)$

range $\subseteq \ker(A)^{\perp}$

$\ker(A)^{\perp} \cap \text{ran} = \{0\}$ $\langle y, y\rangle = 0 \Rightarrow y = 0$

$V \cap V^{\perp} = \{0\}$

Thm: There is a one to one correspondence b/w subspaces of $\mathbb{C}^n$ and orthogonal projections in $M_n(\mathbb{C})$.

$P \to \text{ran}(P)$

(V) Subspaces of $\mathbb{C}^n$ $(e_1, \dots e_k)$

$\boxed{\sum_{i=1}^{k} e_i e_i^*}$ is an orthog proj. P

$n \times 1$ $\|e_1\| = 1$

$(e_1 e_1^*)(e_1 e_1^*) = e_1 e_1^*$

$(e_1 e_1^*)^* = e_1 e_1^*$ ($e_1 e_1^*$ is Hermitian)

$(e_1 e_1^*)(e_1 e_1^*) = e_1 e_1^*$ ($e_1 e_1^*$ is idempotent)

$(e_1 e_1^*) v = \langle e_1, v\rangle e_1$

rank one projection.

Qn. Is it possible $P_1 \neq P_2$ but $\text{ran}(P_1) = \text{ran}(P_2)$ & $\ker(P_1) = \ker(P_2)$

$P_1 x = 0 \Leftrightarrow P_2 x = 0$

$y \in \text{ran}(P_1) = \text{ran}(P_2)$ $\quad P_1 y = y$, $P_2 y = y$ $\quad P_1 y = P_2 y$ on V

$y \in \ker(P_1) = \ker(P_2)$ $\quad P_1 y = 0$, $P_2 y = 0$ $\quad P_1 y = P_2 y$ on V $\Rightarrow P_1 = P_2$

$x x^* = \text{rank-one projection}$ $\quad x \neq 0$

Propn: $V \to$ Subspace of $\mathbb{C}^n$ (of dim. k) and $\langle e_1, \dots, e_k \rangle$ is an ONB of V. Then $\left(\sum_{i=1}^{k} e_i e_i^*\right)$ is the orthogonal projection onto V.

Pf: $\left(\sum_{i=1}^{k} e_i e_i^*\right) x = \sum_{i=1}^{k} \langle e_i, x \rangle e_i \subseteq V$

$\left(\sum_{i=1}^{k} e_i e_i^*\right) e_j = e_j$

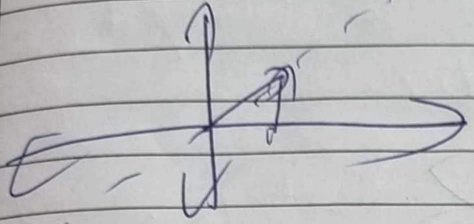
$\Rightarrow$ behaves like identity operator on V.

Cor: Let $\langle e_1, \dots, e_n \rangle$ be an ONB for $\mathbb{C}^n$

$\sum_{i=1}^{K} e_i e_i^* = I_{\mathbb{C}^n}$ (independent of choice of ONB)

Cor: $\langle e_1, \dots, e_n \rangle$ & $\langle f_1, \dots, f_n \rangle$ are two ONB of

V $\sum_{i=1}^{k} e_i e_i^* = \sum_{i=1}^{k} f_i f_i^* =$ projection onto V.



$\|u\|^2 = |\langle u, e_1\rangle|^2 + |\langle u, e_2\rangle|^2$

Assume u is a unit vector

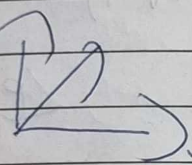
$|\langle u, e_1\rangle|^2 + |\langle u, e_2\rangle|^2 = 1$

$\Rightarrow |\langle e_1, u\rangle|^2 + |\langle e_2, u\rangle|^2 = 1$

Not a coincidence

Thm (Pythagoras Thm)

V – k-dimensional subspace of $\mathbb{C}^n$



$\langle e_1, e_2, \dots e_n\rangle \to$ ONB of $\mathbb{C}^n$

$$\sum_{j=1}^{n} \|P_V e_j\|^2 = k$$

Pf: $\langle f_1, \dots, f_k\rangle \to$ ONB of V.

$$P_V e_i = \sum_{j=1}^{k} \langle f_j, e_i\rangle e_i$$

$$\|P_V e_i\|^2 = \sum_{j=1}^{k} |\langle f_j, e_i\rangle|^2$$

$$\sum_{i=1}^{n} \|P_V e_i\|^2 = \sum_{i=1}^{n}\sum_{j=1}^{k} |\langle f_j, e_i\rangle|^2 = \sum_{j=1}^{k}\left(\sum_{i=1}^{n} |\langle f_j, e_i\rangle|^2\right)$$

$$\sum_{i=1}^{n} |\langle f_j, e_i\rangle|^2 = \|\sum_{i=1}^{n} e_i e_i^* f_j\|^2$$

$$= \langle \sum_{i=1}^{n} (e_i e_i^*) f_j, f_j\rangle$$

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$= \|e_j\|^2 = 1$

$\sum_{i=1}^{n} \|p e_i\|^2 = k$

$\|P_V e_i\|^2 = \langle P_V e_j, e_j \rangle$

$= \langle P_V e_i, P_V e_j \rangle$

$= \langle P_V^2 e_i, e_i \rangle = \langle P_V e_i, e_i \rangle = (i,i)^{th}$ entry of P_V in the basis $\langle e_1, \dots, e_n \rangle$

Cor: ~~Let P be a (orthogonal) projection matrix of rank k.~~

~~Then trace of $P = k =$ rank of P.~~

$UPU^* = \begin{bmatrix} I_k & \\ & 0 \end{bmatrix}$ rank

Q: Let $a_1, \dots, a_k$ be numbers $0 \le a_i \le 1$ s.t. $\sum a_i = k$

(Converse of Pythagoras thing) Is it always possible to find an orthogonal projection (i.e. rank $(\mathbb{C}^n, k)$) which diagonal entries are $a_1, a_2, \dots, a_n$

$UAU^* = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix} = \lambda_1 (e_1 e_1^*) + \lambda_2 (e_2 e_2^*) (?)$

$+ \dots + \lambda_n (e_n e_n^*)$

$A : \mathbb{C}^n \to \mathbb{C}^n$, Hermitian

$\lambda_1, \dots, \lambda_n$
$e_1, \dots, e_n$

$e_i e_i^* \to$ projection ~~matrix~~ on eigen vector corresponding

$A = \lambda_1 (e_1 e_1^*) + \dots + \lambda_n (e_n e_n^*)$ (Spectral Thm)

Hilbert stryles.

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$\sum \lambda_i P_{\lambda_i} = A$

$UAU^* = \sum \lambda_i (Ue_i)(Ue_i)^*$

$S_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $\quad S_1 S_1^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ & & 0 \end{bmatrix}$

$Ue_1 = S_1$ $\quad S_2 S_2^* = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ & 0 \end{bmatrix}$

$Ue_n = S_n$

$UAU^* = \lambda_1 (Ue_1)(Ue_1)^* + \dots \lambda_n (Ue_n)(Ue_n)^*$

$= \lambda_1 S_1 S_1^* + \dots + \lambda_n S_n S_n^*$

suppose A.

QR decomposition

Q is orthogonal.

$A = QR$

$A^* A = R^* R$

Cholesky decomposition

A — positive s.d matrix ($\langle Ax, x \rangle \geq 0$ $\forall x \in F^n$)

$A = LL^*$, L – lower triangular

$A = \sum \lambda_i P_{\lambda_i}$ $\quad \lambda_i \geq 0$

$\sqrt{A} = \sum \sqrt{\lambda_i} \, P_{\lambda_i}$

$\sqrt{A} = QR$

$A = (\sqrt{A})^2 = (\sqrt{A})^* \sqrt{A} = R^* R$

↳ lower A

$\boxed{\begin{array}{l} \cancel{Ax = b} \\ (A^* A) x = A^* b \end{array}}$

$x = (A^* A)^{-1} A^* b$

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Defn: $A \in M_n(\mathbb{R})$ ($A = A^T$ $A^2 = A$
Symmetric
$A \in M_n(\mathbb{C})$ ($A = A^*$ $A^2 = A$
Hermitian idempotent.

(orthogonal) projection matrix.

[$T : V \to W$ $\langle T^t x, y \rangle_V = \langle x, Ty \rangle_W$
inner product spaces. $\forall x \in W, y \in V$]

$V = \operatorname{ran}(A) \subseteq \mathbb{C}^n$ $A : \mathbb{C}^n \to \mathbb{C}^m$
$(\mathbb{R}^n \to \mathbb{R}^m)$

$P_V = \sum_{i=1}^{\dim(V)} e_i e_i^*$

$\langle e_1, e_2, \dots e_n \rangle \to$ ONB of V

[$P_V = \sum_{i=1}^{\dim(V)} e_i e_i^T$]

$V \mapsto P_V$
$V = \operatorname{ran}(A)$.

Defn: (Reflection) ~~An orthogonal (unitary), resp.~~ A symmetric (Hermitian resp.) / orthogonal (unitary resp.) A matrix is said to be a reflection if $A^2 = I$.

[A is said to be a reflection if $AA^T = A^T A = I$ and $A^2 = I$] $(T : V \to V)$ $T : V \to V$

There is a one-to-one correspondence b/w subspaces of $\mathbb{R}^n$ ($\mathbb{C}^n$, resp.) and reflection matrices.

~~Propn~~: $E \in M_n(\mathbb{R})$ is a projection if and only if $I - 2E$ is a reflection

Pf: $E = E^T$ $E^2 = E$

$(I - 2E)^T (I - 2E) = (I - 2E)^2 = I^2 - 4E + 4E^2$
$= I$.

Symmetric hermitian

Def^n: (Reflection) An ~~orthogonal~~ (~~unitary~~, resp). Q matrix ~~A~~ is said to be ~~as~~ a reflection if

$\cancel{AA^T = A^TA = I}$

and $A^2 = I$

There is a one-one correspondence b/w subspaces of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$) and reflection matrices

Prop: ~~Def~~^{n}: $\mathcal{E} \in M_n(\mathbb{R})$ is a projection iff $I - 2\mathcal{E}$ is a reflection

Pf: ~~$\mathcal{E} = \mathcal{E}^T$~~ Say $\mathcal{E}$ is a projection

$\mathcal{E} = \mathcal{E}^T$, $\mathcal{E}^2 = \mathcal{E}$

$(I-2\mathcal{E})^T (I - 2\mathcal{E})$

$= (I-2\mathcal{E})^2 = I + 4\mathcal{E}^2 - 4\mathcal{E}$

$= I$

$\therefore I - 2\mathcal{E}$ is a reflection

Conversely say $I - 2\mathcal{E}$ is a reflection

$\Rightarrow I - 2\mathcal{E} = (I - 2\mathcal{E})^T$

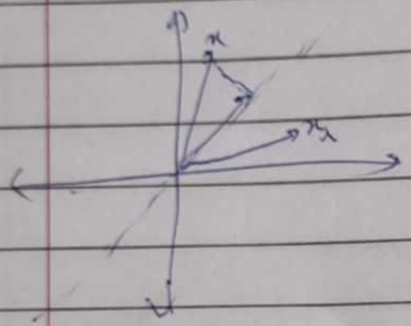
$(I-2E)^2 = I$

$\cancel{I} + 4E^2 - 4E = \cancel{I}$

$E^2 = E \longrightarrow E$ is idempotent

$(I-2E)^T = (I-2E)$

$E^T = E \longrightarrow$ Symm



$P_{span(v)} = vv^*$

$v \rightarrow$ unit vector

$(v\underbrace{v^*)(v}_{\text{inner product} \rightarrow 1}v^*)$

$= vv^*$

vv^* is a projection

$x_R = \langle x, v\rangle v + (x - \langle x, v\rangle v)$

$x_R = \langle x, v\rangle v - (x - \langle x, v\rangle v)$

$= -x + 2\langle x, v\rangle v$

$= -Ix + 2P_{sp(v)}x$

$R = 2P_{span(v)} - I$

$$\gamma(x) = \langle x, v \rangle v - (x - \langle x, v \rangle v)$$
$$= 2\langle x, v \rangle v - x$$
$$= 2P_{\langle v \rangle} x - Ix$$
$$= (2P_{\langle v \rangle} - I)x.$$

$$R = 2P_{\text{span}\{v\}} - I$$

geometrically $2P - I$ better intuition than $I - 2P$

$$P_{v^\perp} = I - P_v.$$

$P_V \to$ projection onto V

$\operatorname{eig}(P_V) = \{1, 0\}$

$$P_V = \sum_{i=1}^{\dim(V)} e_i e_i^*$$

$$\langle e_i \rangle_{i=1}^{\dim(V)} - \text{ONB for } V$$

$$\sum_{i=1}^{\dim V} 1 \cdot e_i e_i^* + \left(\sum_{i=\dim(V)+1}^{n} 0 \cdot e_i e_i^* \right)$$

$\operatorname{eig}(R_V) = \langle 1, -1 \rangle$

$$R_V = \sum_{i=1}^{\dim(V)} e_i \cdot e_i^* - \sum_{i=\dim(V)+1}^{n} e_i \cdot e_i^* \qquad \{e_i\} \to \text{ONB for } \mathbb{F}^n$$

$$\langle e_1, \dots, e_{\dim(V)} \rangle \to \text{ONB for } V$$

<u>Thm</u>: (QR decomposition)

$A \in M_n(\mathbb{F})$. Then $\exists$ a unitary Q and an upper-triangular R s.t. $A = QR$.

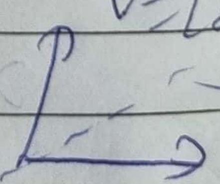
Pf:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & & \\ \vdots & & \\ a_{n1} & & \end{bmatrix}$$

$(Q_n \cdots Q_1) A = R$

Assume $v \neq 0$

$v = \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix}$



span$(1, 0)$

Householder Reflection

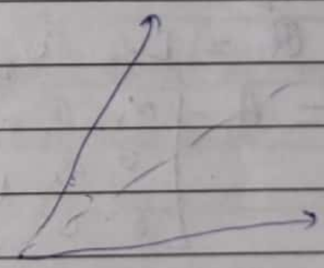
$Z\ v \mapsto V$

Thm: ~~Q~~ QR decomposition

$A \in M_n(\mathbb{C})$. Then $\exists$ a unitary Q and upper Δ R

s.t. $A = QR$

Pf: $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & & \end{bmatrix}$



span$(1, 0, 0, 0, \dots, 0)$

Assume $v \neq 0$

$v \to \frac{v}{\|v\|}$

normalised

$$w = \text{nor}\left(\frac{1}{\sqrt{a_{11}^2 + a_{21}^2 + \dots + a_{n1}^2}} \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right)$$

$$Q_1 = 2ww^* - I =$$

$$Q_1 \begin{bmatrix} a_{11} \\ \vdots \\ \vdots \\ a_{n1} \end{bmatrix} = \begin{bmatrix} \sqrt{a_{11}^2 \dots a_{n1}^2} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Inductively, we get to $\cancel{Q = Q_1}\ Q_2, \dots, Q_n$

$$Q_1 Q_2 \cdots Q_n A = R$$

$$A = \cancel{R}\,(\cancel{Q}\;\underbrace{(Q_1 Q_2 \cdots Q_n)}_{Q})^{-1} R$$

Pf 2: if Assume A is invertible

$$A = [v_1 \mid v_2 \mid \cdots \mid v_n]$$

$$\text{span}(v_1, v_2, \ldots, v_n)$$

$$w_1 = \frac{v_1}{\|v_1\|}$$

$$Q_1 = [w_1 \mid v_2 \cdots \mid v_n]$$

$$= A \begin{bmatrix} \frac{1}{\|v_1\|} & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & \end{bmatrix}$$

α_1

$$w_2 = \text{normal}(v_2 - \langle v_2, w_1 \rangle w_1) \qquad w_2 \perp w_1$$

$$Q_2 = [w_1 \; w_2 \; v_3 \; \cdots \; v_n]$$

$$= A \begin{bmatrix} \alpha_1 & \beta_1 & & 0 \\ 0 & \alpha_2 & 1 & \\ 0 & 0 & & \ddots \end{bmatrix} \to R_2$$

etc

Using Gram-~~Smith~~ Smith method, we get

$$Q_n = [w_1 \; w_2 \; \cdots \; w_n]$$

$$= A \underset{R}{R_n} = \begin{bmatrix} \alpha_1 & \beta_1 & & \\ & \alpha_2 & \beta_2 & \\ 0 & & \ddots & \ddots \\ & & & \beta_{n-1} \\ & & & \alpha_n \end{bmatrix}$$

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n \neq 0$$

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$Q_1 = 2ww^* - I$

$$Q_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} = \begin{bmatrix} \sqrt{a_{11}^2 + \dots + a_{n1}^2} \\ \vdots \\ 0 \end{bmatrix}$$

Inductively find $Q_1, \dots, Q_n$

(Assume A is invertible)

$$A = [v_1 | v_2 | \dots | v_n] \quad \text{Span} \langle v_1, v_2, \dots, v_n \rangle = \mathbb{C}^n$$

$$A \begin{bmatrix} \frac{1}{\|v_1\|} & 0 \\ 0 & \ddots \end{bmatrix} = \left[\frac{v_1}{\|v_1\|} \Big| v_2 \Big| \dots \Big| v_n \right]$$

$$w_1 = \frac{v_1}{\|v_1\|}$$

$$w_2 = \underset{\text{normal}}{} (v_2 - \langle v_2, w_1 \rangle w_1) \quad (w_2 \perp v_1)$$

If span$\langle v_1 \rangle$, span$\langle v_2 \rangle$ have a small angle, then, this step is prone to round off errors.

$$w_3 = \text{normalized } (v_3 - \langle v_3, w_2 \rangle w_2 - \langle v_3, w_1 \rangle w_1)$$

$$C_2 \to \alpha (C_2 - \beta C_1)$$

$$[C_1 | C_2 | \dots] \begin{bmatrix} 1 & -\alpha\beta & 0 \\ 0 & \alpha & \\ 0 & & 1 \end{bmatrix} R_2$$

$$[C_1 | \alpha (C_2 - \beta C_1) | \dots]$$

$$= [w_1 | w_2 | w_3 | \dots]$$

$$C_3 \to \alpha (C_3 - \beta C_2 - \gamma C_1)$$

$$[C_1 | C_2 | C_3 | \dots] \begin{bmatrix} 1 & 0 & -\alpha\gamma & 0 & 0 \\ 0 & 1 & -\alpha\beta & 0 & 0 \\ 0 & 0 & \alpha & & \\ & & & 1 & \\ & & & & \ddots \end{bmatrix} R_3$$

A R_1R_2 q_n $= [w_1|w_2| \dots |w_n]$ is a unitary matrix $= Q$

Gram-Schmidth $\Rightarrow$ round off problems

If A is invertible the QR decomposition is essentially unique $Q_1R_1 = Q_2R_2$

$$Q_2^{-1}Q_1 = R_2R_1^{-1}$$

unitary upper Δ

$$\Rightarrow \begin{pmatrix} a_{11} & \geq 0 & \\ 0 & a_{22} & \\ 0 & \geq & a_{nn} \end{pmatrix}$$

(diagonal unitary)

$$\therefore Q_2^{-1}Q_1 = D$$

$$\Rightarrow Q_1 = Q_2D$$

$$R_2 = DR_1$$

(A is invertible) $\exists!$ Q unitary and R upper triangular with positive diagonal entries s.t $\boxed{A = QR}$

$Q_1R_1 = [\ ]$

$Q_1[\ ][\ ]$

Qn: Give a method to uniformly sample from the sphere $S^2 = \{(x,y,z) : x^2+y^2+z^2=1\} \subseteq \mathbb{R}^3$

$\frac{(X,Y,Z)}{\sqrt{X^2+Y^2+Z^2}} \in S^2$

$X, Y, Z \sim \text{Unif}(0,1)$

However, when plotting the above, it will exhibit a bias

It turns out if
$X, Y, Z \sim \text{Nor}(0,1)$, this gives us a truly uniform sample.

Qn: $U_n \longrightarrow$ Unitary matrices in $M_n(F)$
How to uniform randomly sample from U_n?
We know that
$A(A^*A)^{-1/2}$ is unitary
but $\hookrightarrow$ not uniform
but using $A = QR$ we can get uniform dist.

Thm: (Cholesky decomp).
$A \in M_n(\mathbb{C}) \longrightarrow$ +ve semidefinite matrix
$\exists$ a lower Δ mx s.t
$A = LL^*$.
$A = \sum_{i=1}^{n} \lambda_i (e_i e_i^*)$ $\leftarrow$ Spectral Thm
$\lambda_i \leftarrow$ eigenvalues

let $\sqrt{A} = \sum_{i=1}^{n} \sqrt{\lambda_i} (e_i e_i^*)$

$A = (\sqrt{A})^2 = (\sqrt{A})(\sqrt{A})^*$

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$\sqrt{A} = QR$

$\sqrt{A}^* = R^* Q^* = \sqrt{A}$

$A = (\sqrt{A})^2 = R^* Q^* Q R$

$= R^* (R^*)^*$

$= LL^*$

Thm: Singular Value decomp.

$A \in M_{m \times n}(\mathbb{C})$

Then there are unitary matrices $U \in M_m(\mathbb{C})$,

$V \in M_n(\mathbb{C})$,

and a "diagonal" matrix $\Sigma \in M_{m \times n}(\mathbb{C})$

s.t. $A = U \Sigma V^*$

$\hookrightarrow \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \end{bmatrix}$ Like this

Pf. $A^* A = \sum \lambda_i (e_i e_i^*) \in M_n(\mathbb{C})$

$\hookrightarrow$ +ve ~~sym~~ semidef.

$(\langle A^*Ax, x\rangle = \|Ax\|^2 \geq 0)$

$(AA^*)^2 = AA^*AA^*$

$= \sum \lambda_i (Ae_i)(Ae_i)^*$

$\lambda_i = \sigma_i^2$

let $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_p > 0$

$p \leq n$ [$p = \text{rank}(A)$]

let $f_i = \frac{1}{\sigma_i}(Ae_i)$
$(1 \leq i \leq p)$

$(AA^*) f_i = \frac{1}{\sigma_i} A(A^*Ae_i) = \frac{1}{\sigma_i} A(\sigma_i^2 e_i)$

$= \sigma_i A e_i = \sigma_i^2 f_i$

$\lambda_i = \sigma_i^2 = \|Ae_i\|^2 \geq 0$ $\quad f_i = \frac{1}{\sigma_i}(Ae_i)$ $\quad (AA^*)f_i = \frac{1}{\sigma_i} A(A^*Ae_i) = \sigma_i(Ae_i)$

$= \sigma_i^2 f_i$

$\langle f_i, f_j \rangle = \frac{1}{\sigma_i \sigma_j} \langle A^*A e_i, e_j \rangle = \frac{\sigma_i}{\sigma_j} \langle e_i, e_j \rangle$

$= \delta_{ij} \frac{\sigma_i}{\sigma_j}$

$\{f_i\} \to$ orthonormal vectors in $\mathbb{C}^m$

$\|f_i\|^2 = \frac{1}{\sigma_i^2} \langle A^*A e_i, e_i \rangle = I$

$\{f_i\}_{i=1}^{n} \Rightarrow$ orthonormal set in $\mathbb{C}^m$

$\lambda_1 \geq \cdots \geq \lambda_p > 0$

$p \leq n$

$p = \text{rank}(A)$ $\quad A = \boxed{\sum_{i=1}^{n} \lambda_i e_i e_i^*}$ $\quad Ae_i = \lambda_i e_i$

$A^*A = \sum_{i=1}^{n} \lambda_i (e_i e_i^*) \in M_n(\mathbb{C})$

$\lambda_1 \geq \cdots \geq \lambda_p > 0$ $\quad p = \text{rank}(A)$

(claim $\text{rank}(A) = \text{rank}(A^*A)$)

$f_i = \frac{1}{\sigma_i} Ae_i \quad (i = 1, \ldots, p)$

$\|f_i\|^2 = \frac{\langle Ae_i, Ae_i \rangle}{\sigma_i^2} = \frac{\langle A^*A e_i, e_i \rangle}{\sigma_i^2} = 1$

$\langle f_i, f_j \rangle = \delta_{ij}$

$\{f_i\}_{i=1}^{p}$ maybe extended to an ONB of $\mathbb{F}^m$

$A: e_i \to \sigma_i f_i$

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$\mathbb{C}^n \to \mathbb{C}^n$

V $\beta = [e_1 | \dots | e_n] \in M_n(\mathbb{F})$ $\quad$ U $\beta = [f_1 | \dots | f_n] \in M_n(\mathbb{F})$

$A [e_1 | \dots | e_n] = [f_1 | \dots | f_m] \begin{bmatrix} \sigma_1 & & \Sigma \\ & \sigma_m & \\ & & 0 \end{bmatrix}$

$\quad V \qquad\qquad U$

$A = U \Sigma V^*$, $\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \sigma_p & \\ & & 0 \end{bmatrix}$ $\quad \sigma_i = \lambda_i ((A^*A)^{1/2})$

ith largest eigenvalue of $(A^*A)^{1/2}$.

A sends the unit ball to an ellipsoid whose axes are of size $\sigma_1, \sigma_2, \dots \sigma_p$.

$A : \mathbb{R}^2 \to \mathbb{R}^2$

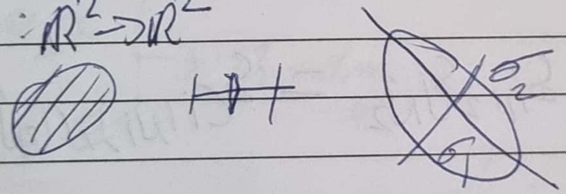

Image compression

$A = \sum \sigma_i f_i e_i^*$

rank 1 matrix

$Tr(A^*A) = \sum_{ij} |a_{ij}|^2$

$A, B \in M_n(\mathbb{F})$

$\langle A, B \rangle = Tr(B^*A)$

$A = \sum_{i=1}^{\text{rank } A} \sigma_i \cancel{e_i f_i^*}\ f_i e_i^* \to$ ~~rank one operator~~.

$\langle e_i \rangle \to$ ONB of eigen vectors of A^*A

$\langle f_i \rangle \Rightarrow$ ONS of eigen vectors of AA^*

$(e_1 \dots e_n)$ $(f_1 \dots f_m)$

$A : \mathbb{C}^n \to \mathbb{C}^m$ $\quad x = \sum_{i=1}^{n} \lambda_i e_i \in V$, $\lambda = \langle e_i, x \rangle$

$\dim(V) = n$.

$A \in M_{m \times n}(\mathbb{F}) \to$ is an inner product space with the Frobenius inner product

$\langle A, B \rangle_F := Tr(A^*B)$ $\quad M_{m \times n}(\mathbb{F}) \times M_{m \times n}(\mathbb{F}) \to \mathbb{F}$

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$$A = \sum_{i=1}^{rk(A)} \sigma_i^{2} e_i f_i^{*}$$

$\{e_i\}$ → o.n basis of eigenvectors of A^*A

$\{f_i\}$ → o.n system of eigenvectors of AA^*

$A : \mathbb{C}^n \to \mathbb{C}^m$

$A \in M_{m \times n}(\mathbb{C})$ is an inner product space with the Frobenius inner product

$$\langle A, B \rangle_F = tr(A^* B)$$

the above is a bilinear map

$$\|A\|_F = tr(A^* A)$$

$$tr\left(\begin{bmatrix} \bar{a}_{11} & \bar{a}_{21} & \cdots \\ \vdots & \bar{a}_{22} & \\ & & \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \\ & \vdots & \end{bmatrix} \right)$$

$$= tr \begin{bmatrix} \|A_{c_1}\|^2 & * & * \\ * & \|A_{c_2}\| & * \\ & * & \|A_{c_3}\| \end{bmatrix}$$

$$= \sum_{\substack{1 \le i \le n \\ 1 \le j \le n}} \|a_{ij}\|$$

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$$\{ b_i e_j^* \}_{\substack{1\le i\le n \\ 1\le j\le n}}$$

$$\langle b_{i_1} e_{j_1}^*, b_{i_2} e_{j_2}^* \rangle = tr(e_{j_1} b_{i_1}^* b_{i_2} e_{j_2}^*)$$

$$= tr(b_{i_1}^* b_{i_2} e_{j_1} e_{j_2}^*)$$

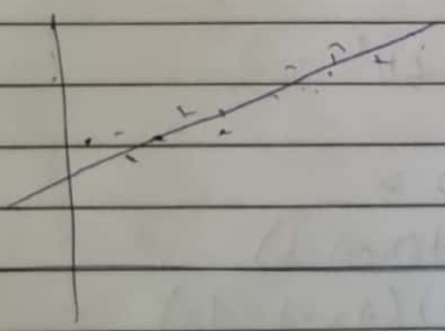
$$\delta_{(i_1,i_2)} \times \delta_{(j_1,j_2)}$$

$\therefore \{ b_i e_j^* \}_{1\le i\le n, 1\le j\le n}$ is an ONB of $M_{n\times n}(\mathbb{C})$

$A = U \Sigma V^*$ $\quad (A = \sum \sigma_i^2 b_i e_j^*$

$Ax = b$, if A is square and invertible, we know how to solve

Problem: Find x st $\|Ax - b\|_2$ is minimal
standard



y have approx linear
$y_i \approx m x_i + c$

So our solution will help us in linear regression: $\rightarrow$ find the "best" linear fit to the given data

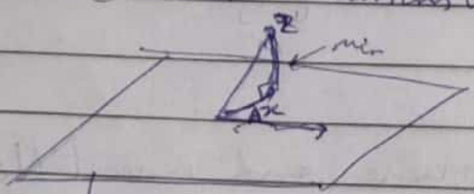
$$\arg\min_{m,c} \sum_{i=1}^{n} (y_i - m x_i - c)^2$$

$$\underbrace{\begin{bmatrix} m & c \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ 1 & 1 & & 1 \end{bmatrix}}_{x'} \simeq \underbrace{\begin{bmatrix} y_1 & \cdots & y_n \end{bmatrix}}_{y}$$

$$\underbrace{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \\ \vdots & \\ x_n & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ c \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_n \end{bmatrix}}_{y}$$

The least squared error $= \|Ax - y\|^2$

We want to minimise the error



plane of A

Let x be such a sol^n

$$\|A(x+y) - b\|_2^2 \geq \|Ax - b\|^2$$

$$\langle A(x+y) - b, A(x+y) - b \rangle$$

$$= ((x+y)^* A^* - b^*)(A(x+y) - b)$$

$$= (x^*A^* - b^* + y^*A^*)(Ax - b + Ay)$$

$$= (x^*A^*Ax - x^*A^*b + x^*A^*Ay$$

$$- b^*Ax + b^*b - b^*Ay +$$

$$y^*A^*Ax - y^*A^*b + y^*A^*Ay)$$

$\|Ax-b\|^2$

$= (x^*A^* - b^*)(Ax-b) +$
$y^*A^*Ay + (x^*A^* - b^*)Ay + y^*A^*(Ax-b)$

$\Rightarrow 2\,\mathrm{Re}\langle Ax-b, Ay\rangle + \|Ay\|_2^2 \geq 0 \quad \forall y$

$\Rightarrow 2\,\mathrm{Re}\langle Ax-b, Ay\varepsilon\rangle + \|Ay\varepsilon\|^2 \geq 0$

$\Rightarrow 2\,\mathrm{Re}\langle Ax-b, Ay\rangle + \varepsilon\|Ay\|^2 \geq 0 \quad \forall \varepsilon$

$\Rightarrow \underbrace{2\,\mathrm{Re}\langle Ax-b, Ay\rangle}_{S} \geq 0 \quad \forall y$

~~let $y = iy$~~

if $y \to -y$

$2\,\mathrm{Re}\langle Ax-b, -Ay\rangle \geq 0$

$S \geq 0$

$-S \geq 0$

$\Rightarrow S = 0$

$\therefore \mathrm{Re}\langle Ax-b, Ay\rangle = 0$

$y \to ye^{i\theta}$, where $\theta = \arg(\langle Ax-b, Ay\rangle)$

$z - \left[\begin{array}{l} \because z = re^{i\theta} \\ \Rightarrow r = ze^{-i\theta} \end{array}\right]$

$\Rightarrow \mathrm{Re}\left(e^{-i\theta}\langle Ax-b, Ay\rangle\right) = 0$

$\Rightarrow |\langle Ax-b, Ay\rangle| = 0 \quad \forall y$

$\Rightarrow \langle A^*(Ax-b), y\rangle = 0 \quad \forall y \in \mathbb{C}^n$

$[\langle A^T(Ax-b), y\rangle = 0 \quad \forall y \in \mathbb{R}^n]$

$A^*(Ax-b) \perp$ every vector

$\therefore A^*(Ax-b) = 0$

$\Rightarrow A^*Ax = A^*b$

[$A^TAx = A^Tb$ in the real case]

Thm: x minimizes $\|Ax-b\|$ iff $A^*Ax = A^*b$

Prop^n: $A^*Ax = A^*b$ always has a soln for any choice of A, b.

Pf:

Range(A) = $\ker(A^*)^\perp$ ← Check (personal)

$x \in \ker(A^*A)$

$\Leftrightarrow A^*Ax = 0$

$\Leftrightarrow \langle A^*Ax, x\rangle = 0$

$\Leftrightarrow \langle Ax, Ax\rangle = 0$

$\Leftrightarrow \|Ax\|^2 = 0$

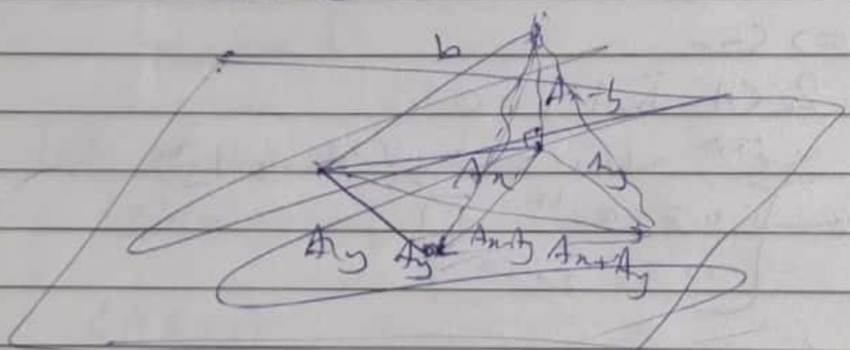
$\Leftrightarrow Ax = 0$

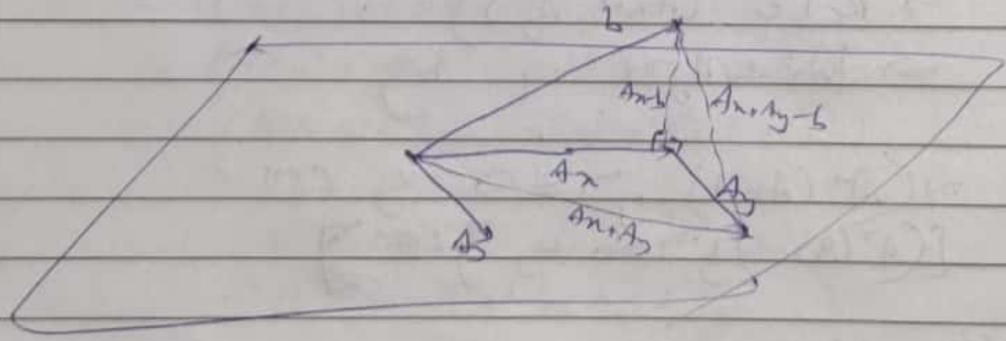
$\therefore x \in \ker(A) \rightarrow \ker(A^*A) = \ker(A)$

Range(A^*) = $\ker(A)^\perp$ = $\ker(A^*A)^\perp$ = Range(A^*A)

$\therefore A^*b$ is in range of A^*A

$\Rightarrow$ Done





$\therefore \|Ax+Ay-b\|^2 = \|Ay\|^2 + \|Ax-b\|^2$

$$A^*Ax = A^*b \quad \leftarrow \text{Pseudo-inverse}$$

$$x = (A^*A)^{+}A^*b$$

$$A^*A = \sum_{i}^{n} \lambda_i P_{\lambda_i} \qquad P_{\lambda_i} = e_i e_i^*$$

$$(A^*A)^{+} = \sum (\lambda_i)^{+} P_{\lambda_i}$$

$$= \lambda_i^{+} = \begin{cases} 1/\lambda_i & \text{if } \lambda_i \neq 0 \\ 0 & \text{if } \lambda_i = 0 \end{cases}$$

Moore-Penrose inverse

$A \in M_{m\times n}(\mathbb{C})$

Then $\exists!\ A^{+} \in M_{n\times m}(\mathbb{C})$ s.t.

i) $AA^{+}A = A$

ii) $A^{+}AA^{+} = A^{+}$

iii) $A^{+}A$ is a hermetian matrix in $M_n(\mathbb{C})$

(iv) AA^{+} " " " " $M_m(\mathbb{C})$

Pf: $A = U\Sigma V^*$, $\Sigma^{+} = \begin{bmatrix} \lambda_1^{+} & & & 0 \\ & \ddots & & \\ & & \lambda_p^{+} & \\ 0 & & & \cancel{\lambda}\,0 \end{bmatrix}$

$A^{+} = V\Sigma^{+}U^*$

$A^{+}A = V(\Sigma^{+}\Sigma)V^*$

$\Sigma^{+}\Sigma = \begin{bmatrix} \lambda_1\lambda_1^{+} & & & 0 \\ & \ddots & & \\ & & \lambda_p\lambda_p^{+} & \\ & & & 0_0 \end{bmatrix}$

$= \begin{bmatrix} 1 & & & 0 \\ 0 & 1 & & \\ & & \ddots\ 1 & \\ 0 & & & 0 \end{bmatrix}$

A^{+} ~~$\Sigma^{+}\Sigma$~~

$A^{+}A$ = Projection onto range $(A^*) = \operatorname{ran}(A^T)$

AA^{+} = " " " (A)

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(2) $A^+ A A^+ = A^+$

(3) A^+A is a Hermitian matrix in $M_n(\mathbb{C})$

(4) AA^+ // // in $M_m(\mathbb{C})$

Pf $A = U\Sigma V^*$, $\Sigma^+ = \begin{bmatrix} \lambda_1^+ & & 0 \\ & \ddots & \\ 0 & & \lambda_p^+ \end{bmatrix}$

$A^+ = V\Sigma^+U^*$

$A^+A = V(\Sigma^+\Sigma)V^* = V\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 0 \end{bmatrix}V^*$

Projection onto Range of (A^+)

$AA^+ \to$ projection onto ran(A)

$A^*Ax = A^*b$

claim: $x = (A^*A)^+ A^*b$ is the least square error solution to $Ax = b$.

Pf: want to show x satisfies the normal eqn

$(A^*A)x = (A^*A)(A^*A)^+A^*b$

$= P_{ran(A^*A)}A^*b = P_{ran(A^*)}A^*b$

$= A^*b$

Mod Results

$A/B = A \cdot \text{pinv}(B)$

$A = \sum_{i=1}^{rk(A)} \sigma_i^2 e_i f_i^*$

Proj onto span f_i

$A^+ = \sum \frac{1}{\sigma_i^2} f_i e_i^*$

$A^+A = \sum P_{span f_i} = P_{span\{f_1 \dots f_r\}}$

range$(A^+A) = $ span$\langle f_1 \dots f_r\rangle = $ ran(A^*)

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$Ax=b$. Find x s.t $\|Ax-b\|_2$ is minimized.

$$(A^*A)x = A^*b$$

$$x = (A^*A)^+ A^*b$$

$Hx = d$

$H \to$ psd $\qquad x+y,\ y \in \ker(A)$

$$x \in \operatorname{ran}(A^*A) = \ker(A)^\perp \qquad \|Ax+Ay-b\|_2^2$$

$$= \|Ax-b\|_2^2 + \|Ay\|_2^2$$

$(A^*A)^+ \to$ ~~projection onto~~ $\operatorname{ran}(A^*)$

$x \in \operatorname{ran}(A^*A) = \operatorname{ran}(A)$

$y \in \ker(A)$

$\Rightarrow x \perp y.\quad \|x+y\|_2^2 = \|x\|_2^2 + \|y\|_2^2$

$x = (A^*A)^+A^*$ is the soln to

Normal eqn of minimal norm.

$$y_i \approx mx_i + c \qquad \sum_{i=1}^{n}(y_i - mx_i - c)^2 = \text{Sum of Square errors.}$$

$$\underbrace{\begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}}_{X} \begin{bmatrix} m \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}}_{y}$$

$$\begin{bmatrix} m \\ c \end{bmatrix} = (X^TX)^+X^Ty$$

$X_1 - X_k$

$$y_i = \sum_{j=1}^{k} a_j x_{ij} + a_{k+1} \quad 1 \le i \le n.$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \underbrace{\begin{array}{c} \begin{array}{cccc} X_1 & \cdots & X_k & \end{array} \\ \begin{bmatrix} x_{11} & \cdots & x_{1k} & 1 \\ \vdots & & & \vdots \\ x_{n1} & \cdots & x_{nk} & 1 \end{bmatrix} \end{array}}_{A} \underbrace{\begin{bmatrix} a_1 \\ \vdots \\ a_{k+1} \end{bmatrix}}_{x}$$

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Find $a_1, \ldots, a_k$ s.t $\sum_{i=1}^{n} \left(y_i - \sum_{j=1}^{k} a_j x_{ij} + a_{k+1}\right)$ is minimized.

Multiple linear Regression

$y = a x_i^2 + b x_i + c$

Eigenvalue algorithm:

Schur upper-Δ form of a mx.

~~Given $A \in M_n(\mathbb{C})$~~

$\langle f, g \rangle = \int_0^1 f \bar{g}\, dx$

$\|f\|_2^2 = \int_0^1 |f|^2 dx$

QR Algorithm A - invertible and $|\lambda_1| > \ldots > |\lambda_n| > 0$

$A = A_1 = Q_1 R_1$ $R_1 \to$ upper-Δ with positive entries

$A_2 := R_1 Q_1 = Q_2 R_2$

$A_3 = R_2 Q_2 = Q_3 R_3$

Thm: The sequence $P_n^* A P_n$

$A_n = (Q_1 \ldots Q_n)^* A (Q_1 \ldots Q_n)$ $A_2 = Q_1^* (Q_1 R_1) Q_1$

Converges to an upper-triangular matrix.

$= Q_1^* A_1 Q_1$

Pf. $P_j = Q_1 \ldots Q_j$.

$A = X D X^{-1}$ D - diagonal $\mathrm{diag}(\lambda_1 \ldots \lambda_n)$ A_2, A_1 are unitarily equivalent

$U_j = R_j R_{j-1} \ldots R_1$ (upper-Δ)

$P_j U_j = (Q_1 \ldots Q_j)(R_j \ldots R_1) = Q_1 \ldots Q_{j-1} A_j R_{j-1} \ldots R_1$

$= P_{j-1} A_j U_{j-1}$

$\boxed{P_j U_j = P_{j-1} A_j U_{j-1}}$

$X = QR$ $A = QRDR^{-1}Q^{-1}$

$\Rightarrow Q^* A Q = RDR^{-1}$ (upper Δ)

Multiple Linear Regression

$y_i = ax_i^2 + bx_i + c$

Now if the x_i's have very small angles b/w them, a

Quick digression,

— Eigenvalue Algorithm

Schur upper triangular form of a matrix

~~Given~~ $A \in M_n(\mathbb{C})$ ↳ Done in LA

QR Algorithm:

$A = A_1 = Q_1 R_1$ [QR decomp]

Assumption: A — Invertible and $|\lambda_1| > |\lambda_2| > \dots > |\lambda_n| > 0$

$A_2 := R_1 Q_1 = Q_2 R_2$

$A_3 := R_2 Q_2 = Q_3 R_3$ ...

$A_2 = Q_1^{-1} Q_1 R_1 Q_1$

$= Q_1^*(Q_1 R_1) Q_1$

$= Q_1^* A_1 Q_1$

$\therefore$ A_1 and A_2 are unitarily equivalent

⇒ Their eigenvalues do not change

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Thm: The sequence
$A_n = (Q_1 Q_2 \cdots Q_n)^* A (Q_1 \cdots Q_n)$ converges to an upper diagonal matrix

Pf: Define $P_i = Q_1 \cdots Q_i$
$A = XDX^{-1}$ $\quad D \to$ diagonal
$\quad\quad \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$
$U_i := R_i R_{i-1} \cdots R_1$ $\quad R_i \to$ upper Δ

$P_i U_i = Q_1 \cdots Q_i R_i \cdots R_1$
$= Q_1 \cdots Q_{i-1} A_{i-1} R_{i-1} \cdots R_1$
$P_i U_i = P_{i-1} A_i U_{i-1}$
$X = QR$
$A = QRDR^{-1}Q^{-1}$
$\Rightarrow Q^* A Q = RDR^{-1}$
To show:
$\lim_{i \to \infty} P_i = Q$

$P_{i-1} A = A_i P_{i-1} \Rightarrow P_i U_i = A_1 P_{i-1} U_{i-1}$
$= (A_1)^2 P_{i-2} U_{i-2}$
$\vdots$
$= (A_1)^{i-1} P_1 U_1$
$= (A_1)^i$
$= X D^i X^{-1}$

$A^i = QRD^i LUP = QR(D^i L D^{-i}) D^i UP$
$\lim D^n L D^{-n} = \begin{bmatrix} l_{11} & & 0 \\ & \ddots & \\ 0 & & l_{nn} \end{bmatrix}$

Just search it up and finish it yourself

Gauss-Seidel method

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$
$$a_{12}, a_{21} \neq 0$$
$$x_2 = \frac{b_1 - a_{11}x_1}{a_{12}}$$
$$x_1 = \frac{b_2 - a_{22}x_2}{a_{21}}$$

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$T_{A,b}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longrightarrow \begin{bmatrix} \dfrac{b_2 - a_{22}x_2}{a_{21}} \\ \dfrac{b_1 - a_{11}x_1}{a_{12}} \end{bmatrix}$$

Find ~~the~~ fixed points of T_{Ab} using fixed pt iteration method

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Polynomial Interpolation (Curve fitting)

Def^n: $p : [a,b] \longrightarrow \mathbb{R}$ (or $\mathbb{C}$ / any field)
$p(x) = a_0 + a_1x_1 + \cdots + a_nx^n$, $a_n \neq 0$ → polynomial.

As we are ~~mapping~~ fitting points to a curve exactly for n pts $(x_1,y_1) \cdots (x_n,y_n)$
If $x_i = x_j \Rightarrow y_i = y_j$.

Lemma: If α is a root of p, then $\exists\ q \in \mathbb{R}[x]$
w/ $\deg(q) = \deg(p) - 1$
s.t. $p(x) = (x-\alpha)q(x)$

Pf: Follows from division algorithm

Cor: $p \in \mathbb{R}[x]$ cannot have more than $\deg(p)$ roots

Cor 2: If $n = \max\{\deg(p), \deg(q)\}$
and $\alpha_1, \alpha_2, \ldots, \alpha_{n+1} \in \mathbb{R}$ s.t. $p(\alpha_1) = q(\alpha_1)$
$p(\alpha_2) = q(\alpha_2)$
$\vdots$
$p(\alpha_{n+1}) = q(\alpha_{n+1})$

Then $p \equiv q$

Pf: Look at $p-q$.
No. of roots > deg
$\therefore$ This only happens if $\deg(p-q) = 0$
$\Rightarrow p \equiv q$

Cor: $p, q : [a,b] \to \mathbb{R}$
Then $p \equiv q$ iff $p(x) = q(x)$ in $\mathbb{R}[x]$

Cor. $(x_1, y_1), \ldots, (x_{n+1}, y_{n+1})$
There is at most one polynomial of degree ($\leq n$) that plot the given data points

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$p(x) = a_0 + a_1 x + \cdots + a_n x^n$
$p(x) = a_0' + a_1'(x-c) + \cdots + a_n'(x-c)^n$
(Shifted power form)
↳ centering reduces correlation.
Correlation b/w terms can ultimately may lead to round-off errors.

$p(c) = a_0'$
$p'(c) = a_1'$
$a_0' = a_0 + a_1 c + \cdots + a_n c^n$
$\therefore$
$$a_k' = \frac{p^{(k)}(c)}{k!} = a_k + (k+1)a_{k+1} + \cdots + \frac{n!}{(n-k)!} a_n c^{n-k}$$

Ex: $p \rightarrow$ linear polyn.
$p(6000) = \frac{1}{3}$
$p(6001) = \frac{-2}{3}$

$p(x) = 6000.3 - x$
$p(6000) = 0.3$
$p(6001) = -0.7$
→ Smith Smith errors, closeness etc.
$c = 6000$
$\dot{p}(0) = \frac{1}{3}$
$\dot{p}^*(1) = \frac{-2}{3}$
$p^*(x)$
$\cancel{a_0 + = \frac{1}{3} + c(x-6000)}$

$\hat{x} = 1$
$p^*(x) = \frac{1}{3} + cx^*$

$p(0) = \frac{1}{3}$

$p(1) = \frac{1}{3} + c$

$-\frac{2}{3} = \frac{1}{3} + c \longrightarrow c = -1$

$p^{*}(x^{*}) = \frac{1}{3} - x_{*}$

$p(x) = \frac{1}{3} - (x - 6000)$

$= 0.333333\ldots - (x - 6000)$

Newton form

$$p(x) = a_0' + a_1'(x - c_1) + a_2'(x - c_1)(x - c_2) + \cdots$$
$$+ a_n'(x - c_1)(x - c_2)\cdots(x - c_n)$$

$$= a_0' + (x - c_1)[a_1' + (x - c_2)[a_2' + (x - c_3)[\cdots]]]$$

$$a_k' = \frac{p^{(k)}(c)}{k!} = \cancel{a_k + k a_{k+1}} \; a_k + (k+1)a_{k+1}c + \cdots + n(n-1)\cdots a_n c^{n-k}$$

Ex: $p \to$ lineal polynomial

$$p(6000) = 1/3, \quad p(6001) = -\frac{2}{3}$$

$$p(x) = 6000 \cdot 3 - x$$

$$c = 6000, \quad p'(0) = 1/3, \quad p'(1) = -\frac{2}{3}$$

$$p(x) = 0.333333 - (x - 6000.0)$$

Newton form:

$$p(x) = a_0' + a_1'(x - c_1) + a_2'(x - c_1)(x - c_2) + \cdots + a_n'(x - c_1)(x - c_2)\cdots(x - c_n)$$

$$\cancel{= a_0' + (x - c_1)[a_1' + (x - c_2)[a_2' \cdots}$$

$$= a_0' + (x - c_1)\{a_1' + (x - c_2)\{a_2' + \cdots + a_{n-1}' + (x - c_n)(a_n')\}\}$$

$$a_0' = p(c_1), \quad a_1' = \frac{p(c_2) - a_0'}{c_2 - c_1}$$

Thm: $p, q \in \mathbb{R}[x]$, $n = \max\{\deg(p), \deg(q)\}$

If $p(x_i) = q(x_i)$ for $(n+1)$-many points $x_i, i = 1, \ldots$

then $p \equiv q$.

$$p_0 + p_1 x + \cdots + p_n x^n, \quad q_0 + q_1 x + \cdots + q_n x^n$$

$$p(x) = a_0 + a_1(x - c) + \cdots + a_n(x - c)^n$$

$$a_k = \frac{p^{(k)}(c)}{k!} \qquad p(x) = p_0 + p_1 x + \cdots + p_n x^n$$

$$= p_0 + p_1(x + p_2(x + p \cdots (p_n x)))$$

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$P_0 + x(P_1 + x(P_2 + x \cdots x(P_n)))$.

$\deg(P) = n$ we can evaluat $P(x_0)$ using $(n-1)$ multiplication and $(n-1)$ addition.

$$P(x) = a_0 + a_1(x-c_1) + a_2(x-c_1)(x-c_2) + \cdots + a_n(x-c_1)\cdots(x-c_n).$$

$a_0 = P(c_1)$, $a_1 = ??$ $a_2 = ??$

$$P(c_2) = a_0 + (c_2 - c_1)a_1$$

$$\Rightarrow a_1 = \frac{P(c_2) - P(c_1)}{c_2 - c_1} \text{ (first divided difference)}$$

Problems: (x_i, y_i), $i = 1, \dots, n$ $(y_i \neq y_j, x_i \neq x_j)$. Find a polynomial p s.t $p(x_i) = y_i$

$$P(c_3) = P(c_1) + \frac{P(c_2) - P(c_1)}{c_2 - c_1}(c_3 - c_1) + a_2(c_3 - c_1)(c_3 - c_2)$$

$$\frac{P(c_3) - P(c_1)}{(c_3 - c_1)(c_3 - c_2)} - \frac{P(c_2) - P(c_1)}{(c_2 - c_1)(c_3 - c_2)} = a_2$$

$$\frac{1}{(c_3 - c_2)}\left[\frac{P(c_3) - P(c_1)}{c_3 - c_1} - \frac{P(c_2) - P(c_1)}{c_2 - c_1}\right] = a_2$$

Second divided difference.

Newton form

$$P(x) = P[x_0] + P[x_0, x_1](x - x_0) + P[x_0, x_1, x_2](x - x_0)(x - x_1) \cdots + P[x_0 \cdots x_n](x - x_0)\cdots(x - x_{n-1}).$$

$P[x_0] = P(x_0)$.

$P[x_0, x_1] = \dfrac{P(x_1) - P(x_0)}{x_1 - x_0}$ (1st divided difference).

$$P[x_0,\dots,x_k] = \frac{P[x_1,\dots,x_k] - P[x_0,\dots,x_{k-1}]}{x_k - x_0}$$

$$P(c_3) - P(c_2) = P(c_1) - P(c_2) + \frac{P(c_2)-P(c_1)}{c_2-c_1}(c_3-c_1) + a_2(c_3-c_1)(c_3-c_2)$$

$$P(c_3) - P(c_2) = P(c_1) - P(c_2) + \frac{P(c_2)-P(c_1)}{c_2-c_1}(c_3-c_1) + a_2(c_3-c_1)(c_3-c_2)$$

$$= \frac{(c_2-c_3)P(c_1) - P(c_2)c_2}{c_2-c_1}$$

$$= \frac{P(c_1)(c_2-c_3) - P(c_2)(c_3-c_2)}{c_2-c_1} + a_2(c_3-c_1)(c_3-c_2)$$

$$\frac{P(c_3)-P(c_2)}{(c_3-c_2)(c_3-c_1)} + \frac{P(c_1)(c_2-c_3)+P(c_2)(c_2-c_3)}{(c_2-c_1)(c_3-c_1)}$$

<u>Thm:</u> P is a polynomial of degree n, $x_0, x_1, \dots, x_n$ are distinct points in $\mathbb{R}$.
Then $P = P[x_0] + P[x_0,x_1](x-x_0) + P[x_0,x_1,x_2](x-x_0)(x-x_1) + \dots + P[x_0,\dots,x_n](x-x_0)\cdots(x-x_{n-1})$

<u>Pf:</u> $P_i(x)$ $(\deg(P_i) \le i)$

$P_i(x) = P[x_0] + \dots + P[x_0,\dots,x_i](x-x_0)\cdots(x-x_{i-1})$

$(P_n(x) = RHS)$.

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$p_i(x_0) = p(x_0)$ (Induction Hypothesis)

$\vdots$

$p_i(x_i) = p(x_i)$

$p_1(x_0) = p(x_0)$

$p_1(x_1) = p(x_1)$

$p_1(x) = p(x_0) + p[x_0, x_1](x - x_0)$ (Base case)

$$\frac{x - x_0}{x_{i+1} - x_0} p_i(x) + \frac{x_{i+1} - x}{x_{i+1} - x_0} (\quad)$$

$q_i(x)$ $x_1, \ldots, x_{i+1}$

$= p[x_1] + p[x_1, x_2](x - x_1) + \cdots + p[x_1, \ldots, x_{i+1}](x - x_1) \cdots (x - x_i)$

$\deg(q_i) \le i$ $\quad q_i(x_1) = p(x_1)$

$q_i(x_2) = p(x_2)$

$\vdots$

$q_i(x_{i+1}) = p(x_{i+1})$

Claim: $p_{i+1}(x) = \dfrac{x - x_0}{x_{i+1} - x_0} q_i(x) + \dfrac{x_{i+1} - x}{x_{i+1} - x_0} p_i(x)$.

$p_i(x_j) = q_i(x_j)$ for $j = 1, \ldots, i$ $\equiv p$ on $x_0, x_1, \ldots, x_{i+1}$

$\deg(p_{i+1}) \le i+1$.

$p[x_0, \ldots, x_{i+1}]$ = leading coefficient of p_{i+1}

$= \dfrac{p[x_1, \ldots, x_{i+1}] - p[x_0, \ldots, x_i]}{x_{i+1} - x_0}$

~~Leading coefficient of RHS~~ $= \dfrac{1}{x_{i+1} - x_0}$ l.c of q_i $\pm \dfrac{\text{l.c of } p_i}{x_{i+1} - x_0}$

leading coeff

$\deg(p) \le n$

Thm: $p \to$ polynomial, $x_0, \ldots, x_n$ distinct points in $\mathbb{R}$.

There is a unique choice of coeffs. $A_0, A_1, \ldots, A_n$

s.t. $p = A_0 + A_1(x - x_0) + A_2(x - x_0)(x - x_1) + \cdots + A_n(x - x_0) \cdots (x - x_{n-1})$

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Pf: $\dfrac{p - p(x_0)}{x - x_0} = B_0 + B_1(x - x_1) + \dots + B_{n-1}(x - x_{n-1})$

$$P_{i+1}(x) = P_i(x) + \left(\frac{x - x_0}{x_{i+1} - x_0}\right)\left(q_i(x) - P_i(x)\right)$$

Thm: (Newton interpolation)

$y_i = f(x_i)$ $(x_0, y_0) \dots (x_n, y_n)$

$$p = f[x_0] + f[x_0, x_1](x - x_0) + \dots + f[x_0 \dots x_n](x - x_0) \dots (x - x_{n-1})$$

is the unique polynomial of degree $\le n$

$f[x_0, x_1]$ Divided difference table

x_0 $f(x_0)$ $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$ $\dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$

x_1 $f(x_1)$ $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = f[x_1, x_2]$

x_n $f(x_n)$

(x_i, y_i) find p of deg $\le n$ s.t. $p(x_i) = y_i$
$i = 1, \dots, n$

real-valued.

Space of functions on $\langle x_1, \dots x_n \rangle$
(vector space of dim n)

x_1 x_2 x_3 $\dots$ x_n

$(y_1, \dots, y_n)$

$y_1(1, 0, \dots 0) + y_2(0, 1, 0 \dots 0)$

$\dots + y_n(0, \dots, 0, 1)$

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$p(x_1) = 1$; $p(x_j) = 0$, $2 \le j \le n$

$p(x) = (x - x_2) \cdots (x - x_n)\, \alpha$ constant

$p(x_1) = (x_1 - x_2) \cdots (x_1 - x_n)\, \alpha_1 = 1$

$\Rightarrow \alpha_1 = \dfrac{1}{(x_1 - x_2) \cdots (x_1 - x_n)}$

$$P_j = \frac{(x - x_1) \cdots (x - x_{j-1}) (x / x_j) (x - x_{j+1}) \cdots (x - x_n)}{(x_j - x_1) \cdots (x_j - x_{j-1}) \cdots (x_j - x_n)}$$

The interpolating polynomial

$$\sum_{j=1}^{n} y_j P_j = \sum_{j=1}^{n} y_j \frac{\prod_{i=1}^{n} (x - x_i) / (x - x_j)}{(x_j - x_1) \cdots (x_j - x_{j-1}) (x_j - x_{n})}$$

$\hookrightarrow \deg (n-1)$

Lagrange Interpolation

* With new training data, Lagrange interpolation poly has to be computed from scratch whereas Newton interpolation polynomial can use previous information.

$\begin{pmatrix} \lambda_1 & \cdots & 0 \\ 0 & \cdots & \lambda_n \end{pmatrix}$ $(A - \lambda_1 I) \cdots (A - \lambda_n I) = 0$

$\underbrace{(A - \lambda_1 I) \cdots (A - \lambda_{n-1} I)}_{}$

$\underbrace{(\lambda_n - \lambda_1) \cdots (\lambda_n - \lambda_{n-1})}_{}$

is an idempotent mx.

$E_1 + \cdots + E_k = I$

$E_i E_j = \delta_{ij}$

Commutatively commutative idempotents

$A = \sum \lambda_i E_i$

$E_i = \dfrac{(A - \lambda_1 I) \cdots (A - \lambda_{i-1} I) \cdots (A - \lambda_k I)}{(\lambda_i - \lambda_1) \cdots (\lambda_i - \lambda_{i-1}) \cdots (\lambda_i - \lambda_k)}$

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$f(x_i) = x_i \quad x_1 - x_n$

$\boxed{A = \Sigma \lambda_i E_i}$ $\quad x = \Sigma c_i \dfrac{x - x_1 \ldots (x - x_i) \ldots (x - x_n)}{(x_i - x_1) \ldots (x_i - x_n)}$

$(x_0, y_0) - (x_1, y_1)$.

Find the unique polynomial P of ~~degree~~ $\leq n$ s.t

$$P(x_i) = y_i \quad (i = 0, 1, \ldots n).$$

$$f(x_i) = y_i$$

~~New~~

Newton form: $P(x) = y_0 + [y_0 y_1](x - x_0) + [y_0, y_1, \ldots y_n](x - x_0) \ldots (x - x_n)$

$f[x_0 \ldots x_n]$

Lagrange form: $P(x) = \sum_{i=0}^{n} y_i \dfrac{(x - x_1) \ldots (x - x_i) \ldots (x - x_n)}{(x_i - x_1) \ldots (x_i - x_i) \ldots (x_i - x_n)}$

$P_0 + P_1 x + \ldots + P_n x^n$.

$P_0 + P_1 x_0 + \ldots + P_n x_0^n = y_0$

$P_0 + P_1 x_1 + \ldots + P_n x_1^n = y_1$

$\vdots$

$P_0 + P_1 x_n + \ldots + P_n x_n^2 = y_n$

$$\begin{bmatrix} 1 & x_0 & \ldots & x_0^n \\ 1 & x_1 & \ldots & x_1^n \\ \vdots & & & \\ 1 & x_n & \ldots & x_n^n \end{bmatrix} \begin{bmatrix} P_0 \\ \vdots \\ P_n \end{bmatrix} = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}$$

$x_i \neq x_j$ for $i \neq j$

Find the unique poly p of degree $\leq n$ s.t.
$p(x_i) = y_i \quad i = 0, 1, \dots, n$

Newton form:

$f(x_0, x_1)$

$$p(x) = y_0 + [y_0, y_1](x - x_0) + \dots + [y_0, y_1, \dots, y_n](x - x_0)(x - x_1)\cdots(x - x_n)$$

Lagrange form

$$p(x) = \sum_{i=0}^{n} \frac{y_i (x - x_1) \cdots (x - x_n)}{(x_i - x_1) \cdots (x_i - x_i) \cdots (x_i - x_n)}$$

$$p_0 + p_1 x_0 + \cdots + p_n x_0^n = y_0$$
$$p_0 + p_1 x_1 + \cdots + p_n x_1^n = y_1$$
$$\vdots$$

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & \vdots & & \vdots \\ 1 & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} y_0 \\ \vdots \\ \vdots \\ y_n \end{bmatrix}$$

This is invertible

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Pf.

$$\det \begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ 1 & x_2 & \cdots & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^n \end{bmatrix}$$

$C_n \to C_n - x_0 C_{n-1}$

$$= \det \begin{bmatrix} 1 & \cdots & x_0^{n-1} & 0 \\ 1 & \cdots & x_1^{n-1} & (x_1 - x_0) x_1^{n-1} \\ \vdots & & \vdots & \vdots \\ 1 & \cdots & & (x_n - x_0) x_n^{n-1} \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & (x_1 - x_0) & x_1(x_1 - x_0) & \cdots & x_1^{n-1}(x_1 - x_0) \\ 1 & & \ddots & & \\ \vdots & & & & \\ 1 & & & & \end{bmatrix}$$

$$= (x_1 - x_0)(x_2 - x_0) \cdots (x_n - x_0) \det \begin{bmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & & \ddots & \\ \vdots & & & \\ 1 & & & x_n^{n-1} \end{bmatrix}$$

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Now for $n=2$

$$\det\begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix} = (x_1 - x_0)$$

$n=3$: $$\det\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \end{bmatrix}$$

$= \det$

$$(x_1-x_0)(x_2-x_0)\det\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}$$

$$= (x_1-x_0)(x_2-x_0)(x_2-x_1)$$

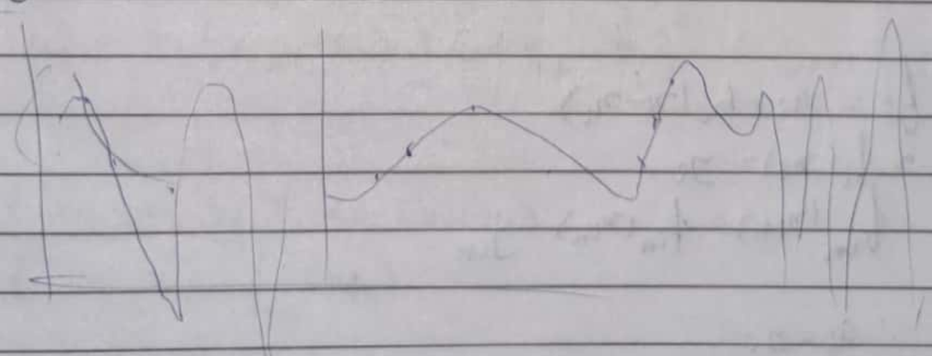
$\therefore$ 1. Hi

$$\det\begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & & & \vdots \\ \vdots & & & \\ 1 & \cdots & \cdots & x_n^n \end{bmatrix} = \prod_{i>j\geq 0} (x_i - x_j)$$

$$= \cancel{x_1 - x_0} \prod_{i=1}^{n} (x_i - x_0) \prod_{1\leq j<i} (x_i - x_j)$$

$$= \prod_{0\leq j<i} (x_i - x_j)$$

Spline



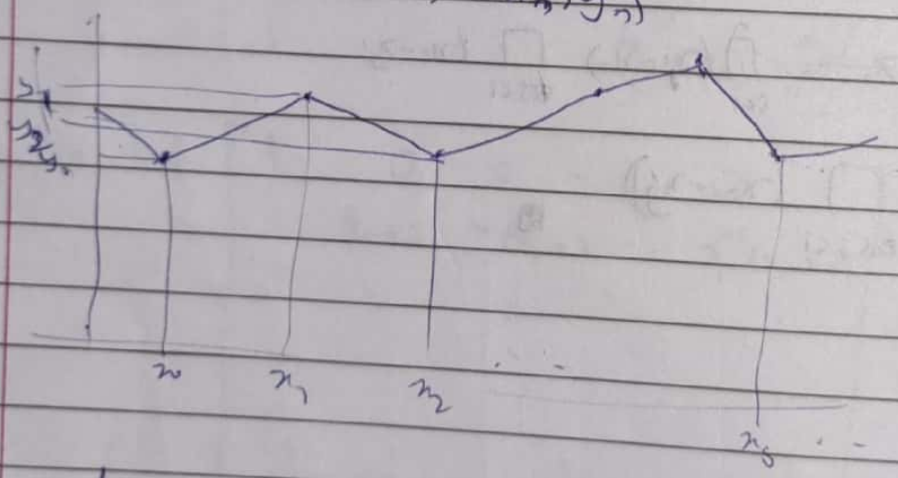
As n increases, the polynomial can get very oscillatory

$\Rightarrow$ We want to find f smooth function w/ control over some order of derivative

[Insert quick de boor here about Bezier curves]

Linear splice

$(x_0, y_0), \ldots, (x_n, y_n)$



$f_i = a_i + b_i(x - x_i)$

$f_i(x_i) = y_i$

$f_{i+1}(x_{i+1}) = f_{i+1}(x_{i+1}) = y_{i+1}$

$\therefore a_i = y_i$

$y_i + b_i(x_{i+1} - x_i) = y_{i+1}$

$\therefore b_i = \dfrac{y_{i+1} - y_i}{x_{i+1} - x_i}$

$$f_i(x) = y_i + \left(\frac{y_{i+1}-y_i}{x_{i+1}-x_i}\right)(x-x_i)$$

~~piecewise function~~

for $[x_i, x_{i+1}]$

Quadratic splice

$$f_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 \text{ on } [x_i, x_{i+1}]$$

~~Be~~ Problem: for $(x_0, y_0), (x_1, y_1) \ldots, (x_n, y_n)$
find $a_0, \ldots, a_{n-1}$
$b_0, \ldots, b_{n-1}$
$c_0, \ldots, c_{n-1}$
($3n$ values)

Constraint 1:

$f_i(x_i) = y_i$

$a_i = y_i$

2: $f_i(x_{i+1}) = f_{i+1}(x_{i+1}) = y_{i+1}$

$$y_i + b_i(x_{i+1}-x_i) + c_i(x_{i+1}-x_i)^2 = y_{i+1} \quad -①$$

3: $f_i'(x) = b_i + 2c_i(x-x_i)$

$f_i'(x_{i+1}) = f_{i+1}'(x_{i+1})$

$$b_i + 2c_i(x_{i+1}-x_i) = b_{i+1} \quad -②$$

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some arbitrary

~~If we fix b_0, then we can get c_0~~

c_0 using ① we can get c_0

using ② " " " " b_1

and so on

up to b_n c_{n-1}

~~To get c_n we need one final~~

$$c_i = \frac{\frac{y_{i+1} - y_i}{x_{i+1} - x_i} - b}{x_{i+1} - x_i}$$

$$= \frac{y_{i+1} - y_i}{(x_{i+1} - x_i)^2} - \frac{b}{(x_{i+1} - x_i)}$$

$$b_i + 2\left(\frac{y_{i+1} - y_i}{x_{i+1} - x_i}\right) - 2b_i = b_{i+1}$$

$$b_{i+1} + b_i = 2\left(\frac{y_{i+1} - y_i}{x_{i+1} - x_i}\right)$$

$$\begin{bmatrix} 1 & 1 & & & 0 \\ & 1 & 1 & & \\ & & 1 & 1 & \\ 0 & & & \ddots & \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} * \\ * \\ * \\ \vdots \\ *\alpha \end{bmatrix}$$

choose arbitrarily $= b_n$

Cubic spline

$$f_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3$$

Constraints:

1) $f_i(x_i) = y_i$

$a_i = y_i$ — ①

2) ~~$f_{i+1}(x_{i+1}$~~

$f_i(x_{i+1}) = f_{i+1}(x_{i+1}) = y_{i+1}$

~~$y_i + b_i(x$~~ Let ~~x_{i+1}~~ $(x_{i+1} - x_i) = h_i$

$y_i + b_i h_i + c_i h_i^2 + d_i h_i^3 = y_{i+1}$ — ②

3) $f_i'(x_{i+1}) = f_{i+1}'(x_{i+1})$

$b_i + 2c_i h_i + 3d_i h_i^2 = b_{i+1}$ — ③

4) $f_i''(x_{i+1}) = f_{i+1}''(x_{i+1})$

$2c_i + 6d_i h_i = 2c_{i+1}$

$c_i + 3d_i h_i = c_{i+1}$ — ④

Solving ①, ②, ③, ④, we'll get values for

$d_i = \frac{c_{i+1} - c_i}{3h_i}$

$b_i + 2c_i h_i + 3(c_{i+1} - c_i)h_i = b_{i+1}$

~~$b_i\ c_i h_i + c_{i+1} h_i = b_{i+1} - b_i$~~

$b_{i+1} = b_i + h_i(c_i + c_{i+1})$

$$b_i + c_i h_i + d_i h_i^2 = \frac{y_{i+1} - y_i}{h_i}$$

$$b_i + c_i h_i + \frac{(c_{i+1} - c_i)}{3} h_i^2 = \frac{y_{i+1} - y_i}{h_i}$$

$\cancel{h_{i-1} c_{i-1} + 2(h_{i-1} - h_i}$

$$3b_{i+1} - 3b_{i-1} = 3h_i(c_{i+1} + c_{i-1})$$

$$= 3\left(\frac{y_{i+1} - y_i}{h_i}\right) - 3\left(\frac{y_i - y_{i-1}}{h_{i-1}}\right)$$

$3c_i h_i + c_{i+1} h_i - c_i h_i \quad - \cancel{c_i h_{i-1} + c_{i-1} h_{i-1}}$

$2c_i h_i + c_{i+1} h_i \quad - 2c_i h_i + 2c_{i-1} h_{i-1} - c_{i+1} h_i + c_i h_{i-1}$

$3h_i c_i + 3h_i c_{i-1} + 2.$

Long
tedious
calculation

$\cancel{c_i h_i + d_i h_i^2}$

$$-c_i h_i - d_i h_i^2 + \frac{y_{i+1} - y_i}{h_i} + 3 d_i h_i^2$$
$$= -c_{i+1} h_{i+1} - d_{i+1} h_{i+1}^2 + \frac{y_{i+2} - y_{i+1}}{h_{i+1}}$$

$$d_i = \frac{c_{i+1} - c_i}{3 h_i}$$

$$b_i + c_i h_i + \frac{c_{i+1} - c_i}{3} h_i = \frac{y_{i+1} - y_i}{h_i}$$

$$b_i + 2 c_i h_i + (c_{i+1} - c_i) h_i = b_{i+1}$$

$$\Rightarrow b_{i+1} = b_i + h_i (c_{i+1} + c_i)$$

$$h_{i-1} c_{i-1} + 2(h_{i-1} + h_i) c_i + h_i c_{i+1} = 3\frac{(y_{i+1} - y_i)}{h_i} - 3\frac{(y_i - y_{i-1})}{h_{i-1}}$$

$\cancel{a_i c_{i-1} + b_i}$

$$\gamma_i c_{i-1} + s_i c_i + t_i c_{i+1} = \alpha_i$$

$i = 1, \dots, n-2$

$$\begin{bmatrix} 1 & & & \\ \gamma_i & s_i & t_i & \\ & \ddots & \ddots & \ddots \\ & & & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} B_1 \\ \alpha_0 \\ \vdots \\ B_2 \end{bmatrix}$$

Clamped End Condition

Specify first derivative at the first and last nodes.

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"Not-a-Knot" Condition:

Force continuity of third derivative at the second and last but one nodes.

~~Bernstein polynomials~~

$$B_{n_i}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

(polynomial of degree n)

0 ... 1

At level n, there are $(n+1)$ ~~continuous~~ Bernstein Polynomials

$B_{n_i}(t)$ are linearly independent

$$\sum_{i=0}^{n} \alpha_i t^i (1-t)^{n-i} = 0 \Rightarrow \sum_{i=0}^{n} \alpha_i \left(\frac{1-t}{t}\right)^{n-i} = 0$$

$$\sum_{i=0}^{n} \alpha_i x^{n-i} = 0 \quad \forall x \in B_0$$

$Sp < B_{n_i}(t) > =$ polys of deg $\leq n$

$$\text{argmax } B_{n,i}(t) = 1 - \frac{i}{n}$$

$t = \frac{5}{6} \dots$ midpoint over $\frac{?}{2}$

$$\sum_{i=0}^{n} B_{n,i}(t) = 1 \cdot \left(\text{i.e. } (t+1-t)^n\right)$$

$B_n(t) \geq 0$ $i = n$, on $[0,1]$.

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$\sum_{k=0}^{n} f(\frac{k}{n}) B_{n,n}(t) \approx f$

$c_0, \dots, c_n \Rightarrow$ control points $\in \mathbb{R}^n$

$\sum_{k=0}^{n} c_k B_{n,k}(t)$ $t \in [0,1]$

is a Bizier curve.

"passing through" $c_0, \dots, c_n$

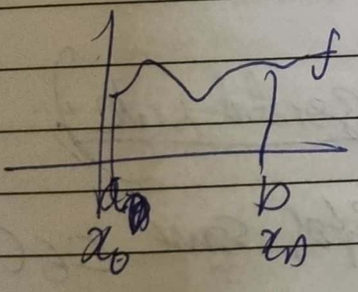
3.

2, 4 for cubic

1 similarly for s^{th} degree poly, $s+1$ points

Numerical Integration:

Problem: Given f on $I = [a, b]$ - Evaluate $\int_a^b f\,dx$ (area under the curve).



$P = \langle x_0 < x_1 < \dots < x_n = b \rangle$

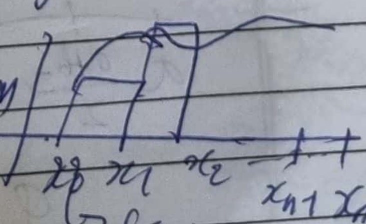
partition of $[a, b]$.

$\left[\mu(P) = \max_{0 \le i \le n-1} |x_{i+1} - x_i|\right]$.

$R(P, f) = \sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i) \to$ Sum of areas of rectangles

If f is ~~said to be~~ integrable if then

$\lim_{\mu(P) \to 0} R(P, f)$ exists



$\to f(x_0)(x_1 - x_0)$

$\int_a^b f\,dx = \lim_{\mu(P) \to 0} L(P, f)$.

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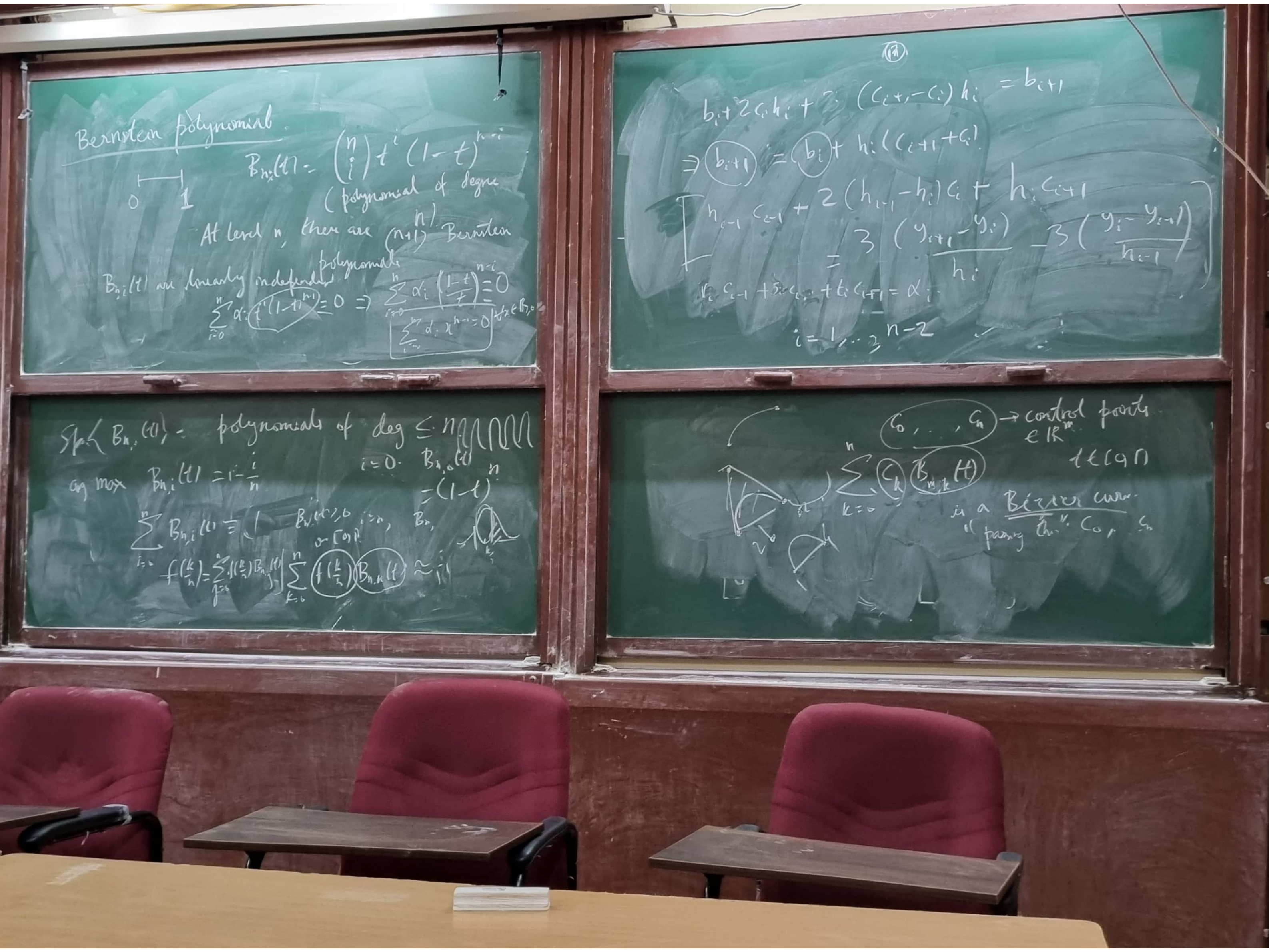
Bernstein polynomials
At level n, there are (n+1) Bernstein polynomials
$B_{n,i}(t)$ are linearly independent
polynomials of deg ≤ n
is a Bézier curve
control points

Bernstein polynomial

$$B_{n_i}(t) = \binom{n}{i} t^i (1-t)^{n-t} \quad \text{(poly of degree } n\text{)}$$

At level n, there are $(n+1)$ Bernstein polynomials

$B_{n_i}(t)$ are lin ind

$$\sum_{i=1}^{n} \alpha_i t^i (1-t)^{n-i} = 0$$

$$\Leftrightarrow \sum_{i=}^{n} \alpha_i \left(\frac{1-t}{t}\right)^{n-i} = 0$$

$$\Leftrightarrow \sum_{i=1}^{n} \alpha_i x^{n-i} = 0$$

$$\Leftrightarrow \alpha_i = 0 \ \forall \alpha_i$$

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Numerical Integration

Problem: Given f on $[a,b]$

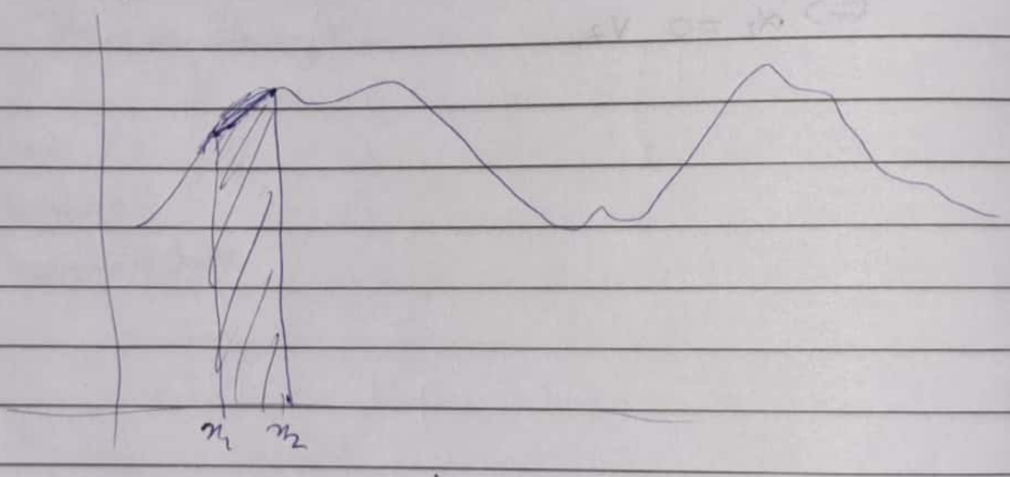
evaluate $\int_a^b f$

$P = \{x_0 < x_1 < \cdots < x_n\}$ partition of $[a,b]$ $\qquad \mu(P) = \max(|x_{i+1} - x_i|)$

$L(P,f) =$

$\int f = \lim_{\mu(P)\to 0} L(P,f)$ if f is integrable

Trapezoidal rule



$$\int_{x_1}^{x_2} f \approx (x_2 - x_1)\left(\frac{f(x_2) + f(x_1)}{2}\right)$$

let P be $(a, a+h, a+2h, \ldots, a+(n-1)h, b)$

$h = \frac{b-a}{n}$ $\qquad b = a+nh$

$$\Rightarrow \int_a^b f \approx \sum_{i=0}^{n-1} (x_{i+1} - x_i) \cdot \left(\frac{f(x_{i+1}) + f(x_i)}{2} \right)$$

$$= \sum_{i=0}^{n-1} h \left(\frac{f(a+ih) + f(a+(i+1)h)}{2} \right)$$

$$= \left(\frac{b-a}{2n}\right) \left(f(a) + 2f(a+h) + 2f(a+2h) + \cdots + 2f(a+(n-1)h) + f(b) \right)$$

(trapezoidal rule)

Error term on trapezoidal rule:

Thm: $\mathcal{E}_T = \frac{-1}{12n^2} f''(c)(b-a)^3$ for some $c \in (a,b)$

[assume f is twice cont diff.

$$\mathcal{E}_T = \frac{-1}{12} \left(\frac{b-a}{n}\right)^2 f''(c)\,(b-a)$$

quadratic convergence

Pf: $a \qquad \frac{a+b}{2} = x \qquad b$

$$g(t) := \int_{x-t}^{x+t} f(y)\,dy - t\left(f(x-t) + f(x+t)\right)$$

$g \to$ error in trapezoidal rule as one expands the interval symmetrically around $x\left(\frac{a+b}{2}\right)$

$$g'(t) = (f(x+t) + f(x-t)) - (f(x-t) + f(x+t))$$
$$- t(-f'(x-t) + f'(x+t))$$
$$= t(f'(x-t) - f'(x+t))$$
$$\frac{g'(t)}{t} = f'(x-t) - f'(x+t)$$

As $t \to 0$, $\frac{g'(t)}{t} \to 0$

$\therefore$ let $\frac{g'(0)}{0} := 0$

removable discontinuity

$$\frac{\frac{g'(t)}{t} - \frac{g'(0)}{0}}{t} = -2f''(c) \quad c \in [a,b]$$

$-f''(x-t) - f''(x+t)$

$$\cancel{g'(t) = -} \cancel{(f''(x-\alpha) + f''(x+\alpha))} = -2\left(\frac{f''(x-\alpha) + f''(x+\alpha)}{2}\right)$$
$$= -2f''(c) \text{ for some } c \in [a,b]$$
[IVP]

$$g'(t) = -2t^2 f''(c_t)$$

$$g'(h) = -2h^2 f''(c_h)$$

Let $s(t) = g(t) - \left(\frac{t}{h}\right)^3 g(h)$

$s(0) = 0$

$s(h) = g(h) - g(h) = 0$

$\therefore \exists\, t_0 \in (0,h)$ st $s'(t_0) = 0$

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$$s'(t) = g'(t) - 3\frac{t^2}{h^2}g(h)$$

$$\cancel{s'(t_0)} = -\cancel{2h^2} - 2t^2 f''(c_0) - 3\frac{t^2}{h^2}g(h)$$

$$= -t^2\left(2f''(c_0) - \frac{3}{h^2}g(h)\right)$$

$$\Rightarrow -t_0^2\left(2f''(c_0) - \frac{3}{h^2}g(h)\right) = 0$$

$$g(h) = \frac{h^2}{3}\left(2f''(c_{t_0})\right)$$

$$g(h) = \frac{2h^2}{3}f''(c_{t_0})$$

$$\int_a^b f = \sum_{k=0}^{n-1}\int_{a+kh}^{a+(k+1)h} f$$

$$\cancel{= \sum\left[\left(\frac{b-a}{n}\right) - \frac{(b-a)^3}{12n^3}\right]}$$

$$= \sum\left[\left(\frac{b-a}{n}\right)\left(\frac{f(a+kh) + f(a+(k+1)h)}{2}\right) - \frac{(b-a)^3}{12n^3}f''(c)\right]$$

$$= \sum\left(\frac{b-a}{n}\right)\left(\frac{f(a)+2f(a+h)+\dots+f(b)}{2}\right)$$

$$- \sum\frac{(b-a)^3}{12n^2}\left(\frac{1}{n}\sum f''(c_k)\right)$$

$\to = f''(c)$ for some $c \in [a,b]$ by IVP

Cor: $f \in [a,b] \to \mathbb{R}$ is twice cont diff.

$$|f''(t)| \leq M \quad \forall t \in [a,b]$$

$$\Rightarrow |E_1| \leq \frac{1}{12n^2} M(b-a)^3$$

2) Another potential strategy is to take the function, take several points (say, n points), interpolate to them and then take the integration b/w each point

$$f(x) = p_k(x) + f[x_0, \ldots, x_k, x]\, \psi_k(x)$$

$$\psi_k(x) = (x-x_0)(\cdots)(x-x_k)$$

Thm: Let f be n times cont$^{\underline{n}}$ diff.

$y_0, \ldots, y_n \in [a,b]$ (May not be distinct)

Then

(1) $\exists\ \xi \in [\min y_i, \max y_i]$ s.t.

$$f[y_0, y_1, \ldots, y_n] = \frac{f^{(n)}(\xi)}{n!}$$

(2) If for each $\lambda \in \mathbb{N}$, $x_0^{(\lambda)}, \ldots, x_n^{(\lambda)}$ $(n+1)$ points on $[a,b]$ and $\lim_{\lambda \to \infty} x_i^{(\lambda)} = y_i$, $i = 0, \ldots, n$, then

$$\lim_{\lambda \to \infty} f[x_0^{\lambda}, \ldots, x_n^{\lambda}] = f[y_0, \ldots, y_n]$$

Pf of (2)

Assume $y_0, \dots, y_n$ are not all same ~~distinct~~

WLOG, $y_0 \leq y_1 \leq \dots \leq y_n$ and $y_0 < y_n$

(i.e. $y_0 = y_1 = \dots = y_n$ is not true)

$x_0^{(\lambda)} < x_n^{(\lambda)}$ is eventually always true

$$\lim_{\lambda \to \infty} f[x_0^{(\lambda)}, x_1^{(\lambda)}, \dots, x_n^{(\lambda)}]$$

$$= \lim_{\lambda \to \infty} f[x$$

→ we'll prevsit this next class lmao

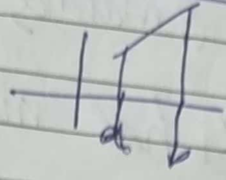
$x_0 \quad x_1, \dots \quad x_k \quad y$

$f(x_0) \quad f(x_1) \dots \quad f(x_k) \quad f(y)$

$$f(y) = p_k(y) + f[x_0, x_1, \dots, x_n, y]\,\psi_k(y)$$

$f[x_0, \ldots, x_n, x]$ is a function of x.

Numerical Integration:



$(x_0,y_0) \ldots (x_n,y_n)$

$f \to$ function on $[a,b]$

$p(x) = A_0 + A_1(x-x_0) + \cdots + A_n(x-x_0)\cdots(x-x_{n-1})$ $\quad (b-a)\left(\frac{f(a)+f(b)}{2}\right)$

$0 = A_0 + A_1(x-x_0) + \cdots + A_n(x-x_0)\cdots(x-x_{n-1})$ $\quad \|f''\|_\infty \le M$ and bounded.

Does it imply $A_0 = A_1 = \cdots = A_n = 0$?

$E_T \le \frac{M(b-a)^3}{12}$

& Newton form unique. Ans. yes

$P_n(x) = f(x_0) + f[x_0,x_1](x-x_0) + \cdots + \overbrace{f[x_0,x_1,\ldots,x_n]}^{A_n}(x-x_0)\cdots(x-x_{n-1})$

$f[x_0 \ldots x_n] = \frac{f[x_1 \ldots x_n] - f[x_0 \ldots x_{n-1}]}{x_n - x_0}$

$P_n(x) = \left(\frac{x-x_0}{x_n-x_0}\right) q_{n-1}(x) + \left(\frac{x_n-x}{x_n-x_0}\right) p_{n-1}(x)$ $\quad p_{n-1}(x)$

$q_{n-1}(x_i) = f(x_i)$, $i = 1, \ldots, n$. $\quad p_{n-1}(x_i) = f(x_i)$, $i = 0, \ldots, n-1$

Let f be n times continuously diff $\quad f[x_0, \ldots, x_n] = f^{n}[\xi]$

Ex: $f[x_0, \ldots, x_n]$ is a function on $[a,b] \times \cdots \times [a,b]$

prove that f is symmetric

(need not be distinct) $x_0, x_1, \ldots, x_n$ $\quad f \to m$ times continuously ~~differentiable~~ (no point is repeated more than $(m+1)$ times) then, there is a unique polynomial

Multiplicity and [illegible]

q degree $\le n$.
s.t. f matches p on $x_0, \ldots, x_n$ with multiplicity.

If x_i has multiplicity k, $f(x_i) = p(x_i)$
$p'(x_i) = p'(x_i)$
$f^{k}(x_i) = p^{(k)}(x_i)$.

$\frac{1}{x_0}, \frac{1}{x_1}, \ldots, \frac{1}{x_n}$ find p s.t $p(x_i) = f(x_i)$ and $\deg(p) \le n$.

$(p-q)(x_i) = 0 \cdot (p-q)^{(k)}(x_i) = 0$ $\Rightarrow$ $p-q$ is divisible by $\prod (x-x_i)^{m_i}$

$$P(x) = a_0 + a_1(x-x_0) + \cdots + a_n(x-x_0)^n$$

$$a_0 = f(x_0),\ a_1 = f'(x_0), \ldots, a_n = \frac{f^n(x_0)}{n!}$$

Thm: $x_0^{(r)} \to y_0, \ldots, x_n^{(r)} \to y_n$ $f \to n$ times continuously differentiable

Then $f[x_0^{(r)}, \ldots, x_n^{(r)}] \to f[y_0, \ldots, y_n]$
($f[x_0, \ldots, x_n]$ is cont on $[a,b]^{n+1}$)

$$f[x_{\sigma(0)}, \ldots, x_{\sigma(n)}] = \frac{f[x_{\sigma(1)}, \ldots, x_{\sigma(n)}] - f[x_{\sigma(0)}, \ldots, x_{\sigma(n-1)}]}{x_{\sigma(n)} - x_{\sigma(0)}}$$

Pf: $y_0, \ldots, y_n \to$ not all of them are same
$y_0 \le y_1 \le \cdots \le y_n$ (wlog)

$$\lim_{r\to\infty} f[x_0^{(r)}, \ldots, x_n^{(r)}] = \lim_{r\to\infty} \frac{f[x_1^{(r)}, \ldots, x_n^{(r)}] - f[x_0^{(r)}, \ldots, x_{n-1}^{(r)}]}{x_n^{(r)} - x_0^{(r)}}$$

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$= f[y_0, \ldots, y_n]$

Claim: $f[y_0, \ldots, y_n] = \frac{f^{(n)}(\xi)}{n!}$ for some $\xi \in [\min_i y_i, \max_i y_i]$

$p(y_i) = f(y_i) \Rightarrow (p-f)(y_i) = 0$

If y_i has multiplicity k, $f^{(j)}(y_i) = p^{(j)}(y_i)$

$(p-f)^{(j)}(y_i) = 0$

$g = p - f$

$y_0 \quad y_1 \quad y_n$ $g^{(n)}$ has atleast one zero.

(By Rolle's thm) $(p^{(n)} - f^{(n)})(\xi) = 0$ in $[y_0, y_n]$

Rolle's Thm: $[a,b]$ $f : [a,b] \to \mathbb{R}$ is n times continuously differentiable.

$x_0 \in (a,b)$ is said to be a zero of ~~multiple~~ multiplicity K if $f(x_0) = \cdots = f^{(k-1)}(x_0) = 0$.

$x_0 \quad x_1$

If f has 'm' zeroes on (a,b) (counting multiplicity)

$f'(\xi) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = 0$ then f' has atleast '$m-1$' zeroes in (a,b) (counting multiplicity)

$k_0 \quad k_1$

$x_0 \quad x_1$ $(k_0 - 1) + (k_1 - 1) + 1 = k_0 + k_1 - 1$

$k_0 \quad k_n$

$x_0 \quad x_n$

$k_0 - 1 \quad k_1 - 1 \quad k_n - 1$ $\Rightarrow \boxed{k_0 + \cdots + k_n - (n+1) + n} = m - 1$

$f[y_0, \ldots, y_0] = \frac{f^{(n)}(y_0)}{n!}$

$f[y_0^{(l)}, \ldots, y_n^{(l)}] = \frac{f^{(n)}(\xi^{(l)})}{n!}$ $y_n^{(l)} = y_0$

f-smm. f is cont. $\underline{\text{Imp}}$

$$\Rightarrow \lim_{n\to\infty} \frac{f^{(n)}(\xi^{(r)})}{n!} = f[y_0, \dots, y_0] = \frac{f^{(n)}(y_0)}{n!}$$

$y_0 = y_1 = \dots = y_n$

$x_0^{(r)} \to y_0, \dots, x_n^{(r)} \to y_n$

$$f[x_0^{(r)}, x_1^{(r)}] = \frac{f(x_1^{(r)}) - f(x_0^{(r)})}{x_1^{(r)} - x_0^{(r)}}$$

Imp stuff

$$f[x_0^{(r)}, \dots, x_n^{(r)}] = \frac{f^{(n)}(\xi^{(r)})}{n!}, \quad \xi^{(r)} \in [\min x_i^{(r)}, \max x_i^{(r)}]$$

$\lim_{n\to\infty} \xi^{(r)} = y_0$

$$\frac{f^{(n)}(y_0)}{n!} = f[y_0, \dots, y_0] \quad \left(\lim_{r\to\infty} f[x_0^{(r)}, \dots, x_n^{(r)}] = f[y_0, \dots, y_n]\right)$$

$$I(f) = \int_a^b f\,dx$$

$$f(x) = P_k(x) + f[x_0, \dots, x_k, x](x - x_0) \dots (x - x_k)$$

$$P_k(x) = f(x_0) + f[x_0, x_1](x - x_0) \dots f[x_0, \dots, x_k](x - x_0) \dots (x - x_{k-1})$$

$P_{k+1}(x_k) = f(x_k)$ $\quad P_{k+1}(x_{k+1}) = f(x_{k+1})$

$P_{k+1}'(x_k) = f'(x_k)$

Strategy: Approximate $I(f)$ by $I(P_k)$

$$E(f) = I(f) - I(P_k)$$
$$= \int_a^b f[x_0, \dots, x_k, x]\,\psi_k(x)\,dx$$

$$\psi_k(x) := (x - x_0) \dots (x - x_k)$$

Simplifications of the error term:

Assumption 1 on ψ_k: Either $\psi_k \geq 0$ on $[a,b]$

or $\psi_k \leq 0$ on $[a,b]$

$\varphi_k \neq 0$

$$\int_a^b f(x)\,\varphi_k(x)\,dx = g(\xi)\int_a^b \varphi_k(x)\,dx \text{ for some } \xi \in [a,b]$$

$M \geq g \geq m$

$$M\varphi_k \geq g\varphi_k \geq m\varphi_k \Rightarrow M \geq \frac{\int_a^b g\varphi_k\,dx}{\int_a^b \varphi_k(x)\,dx} \geq m.$$

$$E(f) = f[x_0, \ldots, x_k, \xi]\left(\int_a^b \varphi_k(x)\,dx\right) \text{ for some } \xi \in [a,b]$$

Assume $f \in C^{k+1}([a,b])$.

Then $E(f) = \dfrac{f^{(k+1)}(\xi)}{(k+1)!}\left(\int_a^b \varphi_k(x)\,dx\right)$ for some $\xi \in [a,b]$.

Assumption 2 on φ_k (i) $\int_a^b \varphi_k(x)\,dx = 0$

$x - \frac{a+b}{2}$

[a,b]

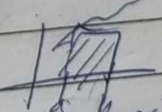
(ii) we can choose $x_{k+1} \in [a,b]$ for $\varphi_{k+1}(x) = (x - x_{k+1})(\varphi_k(x))$

does not change sign on (a,b).

$$f[x_0, \ldots, x_k, x] = f[x_0, \ldots, x_k, x_{k+1}] + f[x_0, \ldots, x_{k+1}, x](x - x_{k+1})$$

$$\Rightarrow E(f) = \int_a^b f[x_0, \ldots, x_k, x_{k+1}]\,\varphi_k(x)\,dx + \int_a^b f[x_0, \ldots, x_{k+1}, x]\,\varphi_{k+1}(x)\,dx.$$

$$= \frac{f^{(k+2)}(\xi)}{(k+2)!}\left(\int_a^b \varphi_{k+1}(x)\,dx\right) \text{ for some } \xi \in [a,b]$$



Rectangle Rule

$k=0$ $\quad f(x) = f(x_0) + f[x_0, x](x - x_0)$

$I(f_k) = (b-a)\,f(x_0)$

If $x_0 = a$ $\quad I(f) \approx (b-a)\,f(a)$

$\varphi_0(x) = x - x_0 = x - a$

$E(f) = f''(\xi)\left(\int_a^b (x-a)\,dx\right) = f''(\xi)\frac{(b-a)^2}{2}$ for some $\xi \in (a,b)$

If $\|f''\|_\infty \le M$ then $|E(f)| \le M\frac{(b-a)^2}{2}$

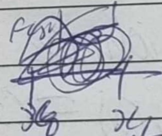
Midpoint Rule: $k=0$ $P_k = f\left(\frac{a+b}{2}\right)$, $x_0 = \frac{a+b}{2}$

$\psi_0(x) = x - \left(\frac{a+b}{2}\right)$ $(x-x_{k+1})\psi_0(x) \ge 0 \; \forall x \in [a,b]$, $x_1 = \frac{a+b}{2}$

$$E(f) = \frac{f''(\xi)}{2!}\left(\int_a^b \left(x - \frac{a+b}{2}\right)^2 dx\right)$$

$$= \frac{f''(\xi)}{2!}\, 2\, \frac{\left(\frac{b-a}{2}\right)^3}{3} = \frac{f''(\xi)(b-a)^3}{24}$$ for some $\xi \in [a,b]$

Trapezoidal Rule:-



$K=1$ $f(x) = f(x_0) + f[x_0,x_1](x-x_0) + f[x_0,x_1,x](x-x_0)(x-x_1)$

$x_0 = a$, $x_1 = b$

$$I(f) \approx f(a)(b-a) + \frac{f(b)-f(a)}{b-a}\int_a^b (x-a)\,dx$$

$$= f(a)(b-a) + \frac{(f(b)-f(a))(b-a)}{2}$$

$$= (b-a)\left(\frac{f(b)+f(a)}{2}\right)$$ $\psi_1(x) = (x-a)(x-b)$

$E(f) = \frac{f''(\xi)}{2}\left(\int_a^b (x-a)(x-b)\,dx\right)$, for some $\xi \in [a,b]$

$$\int_0^{b-a} x(x+a-b)\,dx$$

$$\frac{2(b-a)^3}{3} + \frac{(a-b)(b-a)^2}{2} = -\frac{(b-a)^3}{6}$$

$$\boxed{E(f) = -\frac{f''(\xi)(b-a)^3}{12} \text{ for some } \xi \in (a,b)}$$

<u>Simpson's rule</u>

$a \quad b$

$k=2$ $\quad f(x) = P_2(x) + f[x_0, x_1, x_2, x](x-x_0)(x-x_1)(x-x_2)$

$x_0 = x_1 = a,\ x_2 = b \quad \psi_2(x) \geq 0$ on $[a,b]$

$x_0 = a,\ x_1 = \frac{a+b}{2},\ x_2 = b$

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$f = p_n + f[x_0 \ldots x_n, x] \psi_n(x)$

Strategy: Approximate $I(f) = \int_a^b f \, dx$ by

$I(p_n) = \int_a^b p_n \, dx$

$\cancel{I(p_n) - I(f)}$ $I(f) - I(p_n) = \varepsilon(f)$ → error

$= \int_a^b f[x_0, \ldots, x_n, x] \, \psi_n(x)$

$= \dfrac{f^{(n+1)}(\eta)}{(n+1)!} \int_a^b \psi_n(x)$ for some $\eta \in [a,b]$

Assumption 1:

$\psi_n(x) \geq 0 \quad \forall x \in [a,b]$

(or $\psi_n(x) \leq 0 \quad \forall x \in [a,b]$)

$\varepsilon(f) = I(f) - I(p_n) = \dfrac{f^{(n+1)}(\eta)}{(n+1)!} \int_a^b \psi_n(x) \, dx$

Assumption 2: $\int_a^b \psi_n(x) \, dx = 0$. if $f \in C^{n+1}((a,b))$

for some $\eta \in (a,b)$

$f[x_0, \ldots, x_n, x] = f[x_0, \ldots, x_n, x_{n+1}] + f[x_0, \ldots, x_{n+1}, x](x - x_{n+1})$

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$$E(f) = \frac{f^{(k+2)}(\eta)}{(k+2)!}\int_a^b \psi_{n+1}(x)\,dx$$

$$\psi_{n+1}(x) = (x-x_0)(x-x_1)\cdots(x-x_{n+1})$$

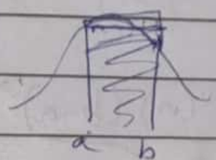
for some $\eta \in (a,b)$

(1) Rectangle Rule

$k=0$, $\psi_k(x) = (x-a)$ $\rightarrow f(x) = f(x_0) + f[x.$

$x_0 = a$

$I(f) \approx f(a)(b-a)$



$$E(f) = f'(\eta)\int_a^b (x-a)\,dx$$

$$E(f) = \;= f'(\eta)\frac{(b-a)^2}{2}$$

$|f'| \leq M$ on $[a,b]$

~~Then~~ $\Rightarrow \frac{M}{2}\left(\frac{b-a}{n}\right)^2 n$

$$= \frac{M(b-a)^2}{2n}$$

(2) Midpoint Rule

$k=0$

$f(x) = f(x_0) + f[x_0, x](x-x_0)$

$x_0 = \frac{a+b}{2}$

$\psi_0(x) = x - \left(\frac{a+b}{2}\right)$

$x_0 = \frac{a+b}{2}$

$I(f) \approx f\left(\frac{a+b}{2}\right)(b-a)$

$x_1 = \frac{a+b}{2}$

$\psi_1(x) = (x-x_0)(x-x_1)$
$= \left(x - \frac{a+b}{2}\right)^2$

$$E(f) = \frac{f''(\eta)}{2} \int_a^b \left(x - \left(\frac{a+b}{2}\right)\right)^2 dx$$

$$= \frac{f''(\eta)}{2} \int_{-\left(\frac{b-a}{2}\right)}^{\left(\frac{b-a}{2}\right)} x^2 dx$$

$$= \frac{f''(\eta)}{2} \frac{(b-a)^3}{3\times 4}$$

$$= \frac{f''(\eta)(b-a)^3}{24}$$

$$= \frac{f''(\eta)}{24} \frac{(b-a)^3}{n^3} \cdot n$$ looks good as we have $\frac{1}{n^2}$, but we may not have control over $f''(\eta)$

(3) Trapezoidal Rule

$k=1$

$f(x) = f[x_0] + \cancel{f[x_0, x_1](x}$ $+ f[x_0, x_1](x-x_0)$
$+ f[x_0, x_1, x_2]\,\psi_1(x)$

$x_0 = a, \quad x_1 = b$

$$I(f) \approx f(a)(b-a) + \frac{f(b) - f(a)}{b-a} \int_a^b (x-a)\,dx$$

$$= f(a)(b-a) + \frac{f(b) - f(a)}{2}(b-a)$$

$$= (b-a)\left(\frac{f(a)+f(b)}{2}\right)$$

$$\mathcal{E}(f) = \frac{f''(\eta)}{2!}\int_a^b (x-a)(x-b)\,dx \quad \text{for some } \eta \in (a,b)$$

$\vdots$

$$= -\frac{f''(\eta)(b-a)^3}{12} \quad \text{for some } \eta$$

4) Simpson's Rule

$k=2$

$x_0 = a, \quad x_1 = \frac{a+b}{2}, \quad x_2 = b \qquad \left(x_3 = \frac{a+b}{2}\right)$ → so that Assumption 2 is satisfied

$$f(x) = f(x_0) + f[x_0, x](x-x_0) + f[x_0, x_1, x](x-x_0)(x-x_1) + f[x_0, x_1, x_2, x]\,\psi_2(x)$$

→ $(x-x_0)\cdots(x-x_2)$

For ~~const~~ computation's sake, we will swap values of x_1 and x_2 → This will not change the answer

$$I(f) \approx \int_a^b \left[f(a) + f[a,b](x-a) + f\left[a, b, \tfrac{a+b}{2}\right](x-a)(x-b)\right] dx$$

$$= f(a)(b-a) + \left(\frac{f(b)-f(a)}{b-a}\right)\frac{(b-a)^2}{2}$$
$$+ f\left[a, b, \tfrac{a+b}{2}\right]\left(-\frac{1}{6}(b-a)^3\right)$$

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$$f\left[a, \frac{a+b}{2}, b\right] = \frac{f\left[\frac{a+b}{2}, b\right] - f\left[\frac{a+b}{2}, a\right]}{b-a}$$

$$= \frac{\left(f(b) - f\left(\frac{a+b}{2}\right)\right) - \left(f\left(\frac{a+b}{2}\right) - f(a)\right)}{(b-a)^2/2}$$

$$= \frac{2\left(f(b) + f(a) - 2f\left(\frac{a+b}{2}\right)\right)}{(b-a)^2}$$

$$I(f) \approx f(a)(b-a) + \frac{f(b) - f(a)}{2}(b-a) + \frac{-1}{3}\left(f(b) + f(a) - 2f\left(\frac{a+b}{2}\right)\right)(b-a)$$

$$= \frac{b-a}{6}\left(6f(a) + 3f(b) - 3f(a) - 2f(b) - 2f(a) + 4f\left(\frac{a+b}{2}\right)\right)$$

$$= \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$

$$\mathcal{E}_{sa}(f) = \frac{f^{(4)}(\eta)}{24} \underbrace{\int_a^b (x-a)\left(x - \frac{a+b}{2}\right)^2 (x-b)\, dx}$$

$$= \int_{-\alpha}^{\alpha} (x-\alpha) x^2 (x+\alpha)\, dx$$

$$\alpha = \frac{b-a}{2}$$

$$= \int_{-\alpha}^{\alpha} x^4 - \alpha^2 x^2 \, dx$$

$$= \left[\frac{x^5}{5} - \frac{\alpha^2 x^3}{3}\right]_{-\alpha}^{\alpha}$$

$$= \frac{\alpha^5}{5} - \frac{\alpha^5}{3} - \frac{-\alpha^5}{5} + \frac{-\alpha^5}{3}$$

$$= \frac{2\alpha^5}{5} - \frac{2\alpha^5}{3}$$

$$= 2\alpha^5\left(\frac{1}{5} - \frac{1}{3}\right)$$

$$= \frac{\cancel{2}\cdot(b-a)^5}{3\cancel{2}_8}\left(\frac{-\cancel{2}}{15}\right)$$

$\frac{3-5}{15}$

$$= \frac{-1}{15 \times 8}(b-a)^5$$

$$= \frac{-1}{120}(b-a)^5$$

$$\therefore \varepsilon(f) = \frac{f^{(4)}(n)\,(b-a)^5}{24 \times 120}$$

(5) Corrected : ~~$h=x_3$~~ l

trapezoidal rule

$h = 3$

$$f(x) = p_3(x) + f[x_0, x_1, x_2, x_3, x]\,\psi_3(x)$$

$$x_0 = a \quad x_1 = a, \quad x_2 = x_3 = b$$

$$I(f) \approx \int_a^b \Big(f(a) + f[a,a](x-a) + f[a,a,b](x-a)^2 + f[a,a,b,b](x-a)^2(x-b) \Big)\, dx$$

$$f[a,a] = f'(a)$$

$$f[a,a,b] = \frac{f[a,b] - f'(a)}{b-a}$$

$$f[a,a,b,b] = \frac{f'(b) - 2f[a,b] + f'(a)}{(b-a)^2}$$

$$I(f) \approx \int_a^b f(a)(b-a) + f'(a)\frac{(b-a)^2}{2} + \frac{f[a,b] - f'(a)}{(b-a)}\,\frac{(b-a)^3}{3} + \frac{f'(b) - 2f[a,b] + f'(a)}{(b-a)^2}\left(\frac{(b-a)^4}{4} - \frac{(b-a)^4}{3}\right)$$

$\dots$

$$= \frac{b-a}{2}\big(f(a) + f(b)\big) + \frac{(b-a)^2}{12}\big(f'(a) - f'(b)\big)$$

$$E(f) = \frac{f^{(4)}(\eta)}{720}(b-a)^5$$

(calculated)

In general, most of our integration rules look like

$$I(f) \approx A_0 f(x_0) + \dots + A_n f(x_n)$$

In essence, we approximate f by polynomial of degree $\leq k$.
(If we are free to choose $x_0, \dots, x_n$, we can make the rule exact for polynomials of degree $\leq 2n-1$)

Orthogonal Polynomials

$\omega : [-1, 1] \longrightarrow \mathbb{R}_{>0}$
$\omega \rightarrow$ weight function
(Riemann integrable)

eg

$\omega = 1$
$\omega = e^{-x^2}$
$\omega = x^{a} e^{-x}$

~~defined on~~ defines an inner product on the space of real valued polynomials fns on $[-1, 1]$

$\langle p, q \rangle = \int_{-1}^{1} p(x)\, q(x)\, \omega(x)\, dx$

$\langle x, 1 \rangle = \int_{-1}^{1} x\, \omega(x)\, dx$

$\|p\|^2 = \int_{-1}^{1} p^2(x)\, \omega(x)\, dx$

$\|p\| = \sqrt{\int_{-1}^{1} p^2(x)\, \omega(x)\, dx}$

We say $P_0(x), P_1(x), \dots$ is a sequence of orthogonal polynomials (w.r.t ω) if

(1) $\langle P_i, P_j \rangle = 0$ if $i \neq j$

(2) deg $P_i = i$

$P_0 = 1 \quad P_0(x) = 1 \quad \neq 0$

$P_1(x) = \beta_1 x + \alpha$

$\int_{-1}^{1} (\beta x + \alpha) \omega(x) = 0$

Properties of the sequence

Property 1:

If $p(x)$ has degree $\leq k$, then $p(x)$

$p_k(x) \in \text{Span}\{P_0, P_1, \dots, P_k\}$

2) If $p(x)$ has degree $\leq k-1$, then

$p_k(x) \perp P_k(x)$

3) $P_k(x)$ has k simple zeroes in $[-1, 1]$

Pf: Take $k > 0$

$\xi_{1k}, \dots, \xi_{k}$, points on $[-1, 1]$ where

P_k changes sign

Could it be empty?

If the list is empty, then

$$\langle \beta P_k, 1 \rangle = \int_{-1}^{1} \underbrace{P_k(x)}_{\text{wlog } \geq 0} \underbrace{\omega(x)}_{>0} = 0$$

This is only possible if $P_k(x) = 0$

→ not possible

Assume $0 < r < k$

$$p(x) = P_k(x)(x - \xi_{1,k})(x - \xi_{2,k}) \cdots (x - \xi_{r,k})$$

$p(x) \geq 0$ ~~or~~ $p(x) \leq 0$ $\forall x$

Since the sign changes in βP_k and resp. $(x - \xi_{i,k})$ will cancel each other out

~~$\langle p(x), 1 \rangle$~~

$$\langle P_k(x), \underbrace{(x - \xi_{1,k}) \cdots (x - \xi_{r,k})}_{r\text{-deg}} \rangle = 0 \quad (\text{by Prop-2})$$

$$\int_{-1}^{1} p(x)\, \omega(x)\, dx = 0$$

$\Leftrightarrow p(x) = 0$

$\Leftrightarrow P_k(x) = 0 \quad \forall x \in (-1, 1)$

This is impossible

$\therefore r = k$

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$(b-a)$

$(b-a)\left(f(a)+f(b) \right) - \frac{1}{}\left(f(a)+f(b)\right) + \frac{2}{} f\left(\frac{a+b}{}\right)$

Property 4:

$$\boxed{x P_k(x) = \alpha_{k+1} P_{k+1}(x) + \alpha_k P_k(x) + \alpha_{k-1} P_{k-1}(x)} + \dots + \alpha_0 \quad = 0.$$

$$\langle x P_k(x), 1 \rangle = \langle \alpha_0, 1 \rangle$$

$$\| $$

$$\langle P_k(x), x \rangle = 0 \text{ if } k \geq 2.$$

$$\langle x P_k(x), x^m \rangle = \langle P_k(x), x^{m+1} \rangle = 0 \text{ if } m+1 \leq k-1 \iff m \leq k-2$$

$I(f) = A_0 f(x_0) + \dots + A_k f(x_k)$

f – poly of deg $\leq n$