On the Formation of Water Waves by Wind.

By Harold Jeffreys, M.A., D.Sc., Fellow and Lecturer of St. John's College, Cambridge.

(Communicated by Prof. G. I. Taylor, F.R.S.—Received October 24, 1924.)

[Plates 4–5.]

It is well known that in certain circumstances a type of instability may arise at the surface of separation of two fluids when there is a finite difference between the velocities on the two sides of the surface. Some disturbances of the surface, of simple harmonic type, may increase exponentially in amplitude until the customary simplifying assumption, that the terms of the second degree in the displacements from the undisturbed state can be ignored, breaks down. One would naturally expect that in the case, for instance, of a wind blowing over the surface of water, waves would be first formed when the velocity of the wind is just great enough to make one particular type of wave grow; thus the critical wind velocity and the wave-length of the waves first formed will constitute checks on any theory of wave formation. The problem for frictionless fluids has been solved by Lord Kelvin*, subject to the restriction that the disturbances considered are two-dimensional, no horizontal displacement occurring across the relative velocity of the fluids. Since, however, the possible initial deformations of a horizontal surface will not as a rule satisfy this condition, an investigation of the growth or decay of deformations of other types is desirable.

1. *Hypothesis of Irrotational Motion.*

Let the two fluids be incompressible (a legitimate approximation so long as the wave velocity is small compared with that of sound in either fluid) and of great vertical extent. Let the origin be in the undisturbed position of the surface of separation and the axis of z vertically upwards. Let ζ be the elevation of the surface, and suppose the two fluids to have initially velocities U and U′ parallel to the axis of x, accents referring to the upper fluid. Let the densities of the fluids be respectively ρ and ρ', and the velocity potentials

* 'Phil. Mag.' (4), vol. 42, pp. 368–370 (1871), or Lamb, 'Hydrodynamics,' p. 439 (1906).

in them ϕ and ϕ'. Let the operators $\partial/\partial t$, $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$ be denoted by σ, p, q, and ϑ respectively. Putting r^2 for $-(p^2+q^2)$ we see that

$$\nabla^2\phi = 0 \tag{1}$$

is equivalent to

$$(\vartheta^2 - r^2)\,\phi = 0, \tag{2}$$

whence

$$\phi = \mathrm{U}x + e^{rz}\mathrm{A}, \tag{3}$$

where A is a function of x and y, determined by the value of ϕ where z is zero.

Now the vertical velocity of a particle in the surface is $d\zeta/dt$, where d/dt denotes differentiation following a particle of the fluid, and this must be equal to the value of $\vartheta\phi$ for surface particles. Hence to the first order in the small disturbances

$$(\sigma + \mathrm{U}p)\,\zeta = r\mathrm{A}. \tag{4}$$

Thus

$$\phi = \mathrm{U}x + \frac{\sigma + \mathrm{U}p}{r}\,e^{rz}\,\zeta. \tag{5}$$

Similarly

$$\phi' = \mathrm{U}'x - \frac{\sigma + \mathrm{U}'p}{r}\,e^{-rz}\,\zeta. \tag{6}$$

If P denote the pressure and Q the resultant velocity, the pressure integral is

$$\begin{aligned}\frac{\mathrm{P}}{\rho} &= -\frac{\partial\phi}{\partial t} - \tfrac{1}{2}\mathrm{Q}^2 - gz + \text{const.}\\ &= -\sigma\phi - \mathrm{U}e^{rz}p\mathrm{A} - gz + \text{const.},\end{aligned} \tag{7}$$

to the first order. This becomes, when $z = \zeta$,

$$\frac{\mathrm{P}}{\rho} = -\left\{\frac{(\sigma + \mathrm{U}p)^2}{r} + g\right\}\zeta + \text{const.} \tag{8}$$

Similarly we have, when $z = \zeta$,

$$\frac{\mathrm{P}'}{\rho'} = \left\{\frac{(\sigma + \mathrm{U}'\,p)^2}{r} - g\right\}\zeta + \text{const.} \tag{9}$$

Also

$$\mathrm{P}' - \mathrm{P} = \mathrm{T}\,(p^2 + q^2)\,\zeta, \tag{10}$$

where T is the surface-tension. Hence ζ satisfies the differential equation

$$\{\rho\,(\sigma + \mathrm{U}p)^2 + \rho'\,(\sigma + \mathrm{U}'p)^2 + g\,(\rho - \rho')\,r + \mathrm{T}r^3\}\,\zeta = 0. \tag{11}$$

In particular, if ζ is proportional to $\cos\,(\gamma t - \kappa x)\cos\kappa' y$, where γ, κ, κ' are constants, we must have

$$\rho\,(\gamma - \mathrm{U}\kappa)^2 + \rho'\,(\gamma - \mathrm{U}'\kappa)^2 = g\,(\rho - \rho')\,r + \mathrm{T}r^3, \tag{12}$$

where

$$r^2 = \kappa^2 + \kappa'^2. \tag{13}$$

This gives

$$\gamma = \kappa \frac{\rho U + \rho' U'}{\rho + \rho'} \pm \left[\frac{g(\rho - \rho')r + Tr^3}{\rho + \rho'} - \kappa^2 \frac{\rho\rho'(U' - U)^2}{(\rho + \rho')^2}\right]^{\frac{1}{2}} \tag{14}$$

If the motion is two-dimensional, so that $\kappa' = 0$ and $r = \kappa$, the solution becomes equivalent to that of Lord Kelvin.

So long as γ is purely real, the given disturbance will neither increase nor decrease in amplitude. Thus the condition for a wave to develop is that γ shall be imaginary. This gives at once

$$\kappa^2 (U' - U)^2 > \frac{\rho + \rho'}{\rho\rho'} \{g(\rho - \rho')r + Tr^3\} \tag{15}$$

$$> \frac{\rho + \rho'}{\rho\rho'} \{g(\rho - \rho')\kappa + T\kappa^3\} \tag{16}$$

$$> 2\frac{\rho + \rho'}{\rho\rho'} \{Tg(\rho - \rho')\}^{\frac{1}{2}} \kappa^2 \tag{17}$$

the first sign of equality holding when the wind is just strong enough to increase the wave considered, the second when the wave is two-dimensional, and the third when the wave-length is such as to make $\frac{g(\rho - \rho')}{\kappa} + T\kappa$ a minimum. If the corresponding value of κ is κ_0, we have

$$\kappa_0^2 = \frac{g(\rho - \rho')}{T}. \tag{18}$$

For air and water we have

$$\rho = 1 \text{ gm./cm.}^3; \quad \rho' = 0 \cdot 0013 \text{ gm./cm.}^3; \quad T = 73 \text{ dyne/cm.} \tag{19}$$

Hence from (18) κ_0 is $3 \cdot 5$/cm., and the critical wave-length is $2\pi/\kappa_0$ or $1 \cdot 8$ cm. The critical value of $U' - U$, the velocity of the wind relative to the water, is given by

$$(U' - U)^2 = 2\frac{\rho + \rho'}{\rho\rho'} \{Tg(\rho - \rho')\}^{\frac{1}{2}}, \tag{20}$$

making

$$U' - U = 640 \text{ cm./sec.} \tag{21}$$

Again, if α be the rate of travel of the waves first formed, we have

$$\alpha = \gamma/\kappa \tag{22}$$

and

$$\alpha - U = \frac{\rho'}{\rho + \rho'}(U' - U) \tag{23}$$

$$= 0 \cdot 8 \text{ cm./sec.} \tag{24}$$

At each point these predictions are in disagreement with observation. The velocity of a wind just strong enough to raise waves is actually only about 110 cm./sec.; the wave-length of the waves first formed by such a wind is from 6 to 8 cm., and the rate of travel of the waves is about 30 cm./sec.

A further discrepancy is provided by the wave-length of the swell in mid-ocean. Vaughan Cornish* gives 1,150 feet as the wave-length of a typical swell. This makes κ equal to $1\cdot 8 \times 10^{-4}$/cm., and hence by (16)

$$U' - U = 6\cdot 5 \times 10^4 \text{ cm./sec.}$$

This is far beyond any actual velocity. It follows that no wind occurring at sea would be capable of raising a typical swell if the theory of irrotational motion were applicable to the formation of water waves.

2. *The Hypotheses of Sheltering and Skin Friction.*

It therefore appears that, in any theory of wave formation that is to stand the test of quantitative comparison with observation, the hypothesis of irrotational motion must be abandoned, and the effects of quasi-discontinuities and turbulence must be included. The regular form of the waves first formed suggests that discontinuities and turbulence in the water are unimportant, at least when waves first appear; so that attention will have to be given primarily to irregular motions in the air. There are two obvious mechanisms by which the existence of waves on the surface of the water may introduce rotational motions in the air, which may then react on the water in such a way as to alter the size of the waves. The first is that the air blowing over the waves may be unable to follow the deformed surface of the water. Water flowing past a sphere does not in general flow all round it; the particles that strike the front of the sphere leave it soon after they have passed the centre, and the region behind the sphere is occupied by eddying liquid with little or no systematic motion relative to the sphere. By analogy one may suggest that if waves are once formed on water, the main air current, instead of flowing steadily down into the troughs and over the crests, merely slides over each crest and impinges on the next wave at some point intermediate between the trough and the crest. The region sheltered from the main air current contains an eddy with a horizontal axis, while smaller eddies exist along the boundary between this eddy and the main current. If such a theory is correct, the pressure of the air will be greater on the slopes facing the wind than on those away from it; for the deflexion of the air upwards when it strikes the exposed slopes

* 'Waves of the Sea and Other Water Waves,' p. 97 (1910).

implies a reaction between the air and the water. By analogy with the two-dimensional problem of the thrust of a current on a plane lamina inclined to the direction of flow,* we may infer that the reaction is approximately normal to the surface and proportional to $\rho' \mathrm{U}'^2 \, \partial\zeta/\partial x$, where U' is now the velocity of the wind over the crests. Within the sheltered region the reaction will be nearly uniform. The reaction is thus not an analytic function of x, but it could be expressed as a series of harmonic functions of multiples of x. This series would evidently contain many terms; but so long as we are considering only whether the fundamental wave will increase or decrease, it will be sufficient to consider only the term of the same wave-length as the disturbance of the surface of the water. We shall, therefore, consider the reaction as equal to $s\rho' \mathrm{U}'^2 \, \partial\zeta/\partial x$, where s is a numerical constant, not necessarily small. This hypothesis will be called the hypothesis of sheltering.

The alternative hypothesis is based on the conception of skin friction. There is a great deal of evidence to indicate that the tangential reaction of a turbulent fluid on a fixed surface is of the form $s'\rho'\mathrm{V}^2$, where s' is a numerical coefficient about equal to $0 \cdot 002$ and V is the tangential velocity in the neighbourhood of the surface. We shall overestimate this effect if we neglect the reaction of this friction on the air, which reduces the velocity of the air near the surface, and simply calculate V as if the motion of the air was irrotational.

Up to a point these two hypotheses can be treated together. The equations of viscous motion of the water are three, of the form

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial \mathrm{P}}{\partial x} + \nu\nabla^2 u, \tag{1}$$

where ν is the effective kinematic coefficient of viscosity. With our previous conventions we can write these

$$\{\sigma + \mathrm{U}p - \nu(p^2 + q^2 + \vartheta^2)\}(u, v, w) = -\frac{1}{\rho}(p, q, \vartheta)\,\mathrm{P}. \tag{2}$$

The equation of continuity is

$$pu + qv + \vartheta w = 0. \tag{3}$$

Combining (2) and (3) we have

$$(p^2 + q^2 + \vartheta^2)\,\mathrm{P} = 0, \tag{4}$$

whence

$$\frac{\mathrm{P}}{\rho} = e^{rz}\Pi \tag{5}$$

* Lamb, 'Hydrodynamics,' p. 94.

where Π is a function of x and y. Hence

$$u = U_1 e^{\lambda z} - \frac{p\Pi e^{rz}}{\sigma + Up}, \tag{6}$$

$$v = Ve^{\lambda z} - \frac{q\Pi e^{rz}}{\sigma + Up}, \tag{7}$$

$$w = We^{\lambda z} - \frac{r\Pi e^{rz}}{\sigma + Up}, \tag{8}$$

where U_1, V and W are unspecified functions of x and y, and

$$\sigma + Up - \nu(\lambda^2 - r^2) = 0. \tag{9}$$

In consequence of (3) we must have

$$pU_1 + qV + \lambda W = 0. \tag{10}$$

The surface condition is

$$\frac{d\zeta}{dt} = w \tag{11}$$

leading to

$$(\sigma + Up)\zeta = W - \frac{r\Pi}{\sigma + Up}. \tag{12}$$

The stress-components across the surface $z = 0$ are

$$p_{zz} = -P + 2\nu\rho\,\vartheta w = 2\nu\rho\lambda W - \rho\Pi\left(1 + \frac{2r^2\nu}{\sigma + Up}\right), \tag{13}$$

$$p_{zx} = \rho\nu(\vartheta u + pw) = p\nu\left(\lambda U_1 + pW - \frac{2pr\Pi}{\sigma + Up}\right), \tag{14}$$

$$p_{zy} = \qquad = p\nu\left(\lambda V + qW - \frac{2qr\Pi}{\sigma + Up}\right). \tag{15}$$

Suppose also that these stress-components are given in terms of the surface elevation by the relations

$$p_{zz} = \rho Q\zeta\,; \qquad p_{zx} = \rho pR\zeta\,; \qquad p_{zy} = \rho qR\zeta \tag{16}$$

where Q and R are linear operators. Then (14) and (15) give

$$\frac{\lambda U_1}{p} = \frac{\lambda V}{q} = \frac{R\zeta}{\nu} - W + \frac{2r\Pi}{\sigma + Up} \tag{17}$$

and on substitution in (10) we find

$$W = \frac{r^2}{\lambda^2 + r^2}\left(\frac{R\zeta}{\nu} + \frac{2r\Pi}{\sigma + Up}\right). \tag{18}$$

Eliminating W between (12) and (18) we have, using (9),

$$\Pi = rR\zeta - \frac{\nu(\lambda^2 + r^2)}{r}(\sigma + Up)\zeta, \tag{19}$$

and thence

$$W = (\sigma + Up)\,\zeta + \frac{r^2 R\zeta}{\sigma + Up} - \nu\,(\lambda^2 + r^2)\,\zeta$$

$$= -\,2\nu r^2\zeta + \frac{r^2 R\zeta}{\sigma + Up}. \tag{20}$$

Substituting in (13) we have now

$$Q\zeta = 2\nu\lambda r^2\left(\frac{R}{\sigma + Up} - 2\nu\right)\zeta$$

$$-\left(1 + \frac{2\nu r^2}{\sigma + Up}\right)\left(rR\zeta - \frac{\sigma + Up + 2\nu r^2}{r}\,(\sigma + Up)\,\zeta\right) \tag{21}$$

The amplitudes of the types of waves considered do not change by large fractions of themselves within a wave-length. Hence we shall expect that νr^2 will be small compared with $\sigma + Up$, and that $r^2 R$ will be small compared with $(\sigma + Up)^2$. If, then, we omit all terms of order higher than the first in ν and R, we have

$$\{(\sigma + Up + 2\nu r^2)^2 - Qr - Rr^2\}\,\zeta = 0 \tag{22}$$

which is the differential equation required. For further progress it is necessary to specify the forms of Q and R.

3. *Hypothesis of Sheltering.*

On the hypothesis of discontinuous motion of the air there is no tangentia traction, so that R is zero. The vertical pressure across a given horizontal surface will depart from its value in the undisturbed state for three reasons. First, there is a depth ζ of water above the surface instead of an equal depth of air; hence the pressure is increased by $g\,(\rho - \rho')\,\zeta$, or, since in (13) tensions are reckoned positive, p_{zz} must contain a term $-g\,(\rho - \rho')\,\zeta$. Second, as in the irrotational case, the surface tension introduces a discontinuity of amount $T(p^2 + q^2)\,\zeta$. Third, the pressure of the air on the side of the wave facing the wind introduces a term $s\rho' U'^2 p\zeta$, where such a velocity has been supposed superposed on the whole system that the crests of the waves have no horizontal motion. The exposed side of the wave is that facing in the direction of x increasing, if U′ is negative, which will be the case considered. Thus p_{zz} is positive when $p\zeta$ is positive, and this term must therefore be associated with a positive sign. Thus in all

$$\rho Q = -\,g\,(\rho - \rho') - Tr^2 + s\rho' U'^2 p \tag{1}$$

and the differential equation satisfied by ζ becomes

$$\left\{ (\sigma + \mathrm{U}p + 2\nu r^2)^2 + c^2 r^2 - \frac{s\rho'}{\rho} \mathrm{U}'^2 pr \right\} \zeta = 0, \tag{2}$$

where $$\rho c^2 = \frac{g}{r}(\rho - \rho') + \mathrm{T}r. \tag{3}$$

In accordance with our conventions the wave is a stationary one, with gradually varying amplitude. Thus we can take ζ as proportional to $e^{\gamma t} \cos \kappa x \cos \kappa' y$, where γ, κ and κ' are constants. Then

$$\sigma\zeta = \gamma\zeta\,; \quad p^2\zeta = -\kappa^2\zeta\,; \quad f(r^2)\,\zeta = f(\kappa^2 + \kappa'^2)\,\zeta. \tag{4}$$

Then (2) gives

$$\{(\gamma + 2\nu r^2)^2 + c^2 r^2 - \mathrm{U}^2\kappa^2\}\,\zeta + \left\{ 2\mathrm{U}\,(\gamma + 2\nu r^2) - \frac{s\rho'}{\rho}\mathrm{U}'^2 r \right\} p\zeta = 0 \tag{5}$$

Neither ζ nor $p\zeta$ is in general zero, nor is their ratio a constant. Hence the coefficients must both vanish, giving

$$\gamma = -2\nu r^2 + \frac{s\rho'}{2\rho}\frac{\mathrm{U}'^2 r}{\mathrm{U}}, \tag{6}$$

$$\kappa^2\mathrm{U}^2 = c^2 r^2 + \text{a second-order quantity}. \tag{7}$$

We notice from (6) that the wave will necessarily die down if U is negative. Thus when U′ is negative U must be positive. Remembering that these velocities were both referred to the crests of the waves, we see that this implies that the velocity of the waves is intermediate between those of the air and the water. Again, (3) and (7) give

$$\mathrm{U}^2 = \frac{gr}{\kappa^2}\frac{\rho - \rho'}{\rho} + \frac{\mathrm{T}}{\rho}\frac{r^3}{\kappa^2} \tag{8}$$

and if γ is positive we must also have

$$\mathrm{U}'^2 \geqslant \frac{4\nu\rho}{s\rho'}\mathrm{U}r. \tag{9}$$

Let us denote the coefficient $4\nu\rho/s\rho'$ by C. For every value of κ, r increases steadily with κ'. Hence U and $|\mathrm{U}'|$ both increase with κ'. But the observed speed of the wind relative to the water is $\mathrm{U}' - \mathrm{U}$, and its absolute value is $|\mathrm{U}'| + \mathrm{U}$. Hence, for any given wave-length in the x-direction, the wind velocity relative to the water needed to excite the disturbance is least when the wave-length in the y-direction is infinite; that is, when the disturbance is two-dimensional. Thus the waves produced by the gentlest wind that can ruffle the water at all should be two-dimensional.

Let us assume that the value of κ corresponding to actual waves is such as to ake $T\kappa^2/g(\rho - \rho')$ small; that is, that wind-raised waves are gravity waves d not ripples. Also let the velocity of the wind relative to the water be V. hen (9) gives

$$r \leqslant \frac{(V - U)^2}{CU}, \tag{10}$$

e sign of equality holding when the wave can just be maintained. Again, e have

$$r \geqslant \kappa \tag{11}$$

e sign of equality holding when the disturbance is two-dimensional. Then, bject to the neglect of T, we have from (8)

$$\kappa U^2 \geqslant g \frac{\rho - \rho'}{\rho}, \tag{12}$$

d, therefore,

$$r \geqslant \frac{g}{U^2} \frac{\rho - \rho'}{\rho}. \tag{13}$$

om (10) and (13) then

$$U(V - U)^2 \geqslant Cg \frac{\rho - \rho'}{\rho}. \tag{14}$$

r a given value of V, the value of U between O and V that makes the ntity on the left greatest is

$$U = \tfrac{1}{3}V \tag{15}$$

that the condition for a wind to be able to raise a wave at all is

$$V^3 \geqslant \frac{27}{4} Cg \frac{\rho - \rho'}{\rho} \tag{16}$$

nce, with $\nu = 0{\cdot}018$ cm.^{2}/sec.

$$V \geqslant 73 s^{-\frac{1}{3}} \text{ cm./sec.} \tag{17}$$

d from (12)

$$\kappa^3 = 5s^2 \tag{18}$$

the critical case. Thus, so long as s is less than unity, as it must be, the sumption made in the neglect of the surface tension is justified.

4. *Hypothesis of Skin Friction.*

In discussing this hypothesis the motion of the air will be treated as if nmodified by friction. In this way the velocity of the air in contact with the vater surface will be exaggerated, and therefore the effects of skin friction will be over-estimated. The pressure in the air is given by

$$\begin{aligned} \frac{P'}{\rho'} &= -gz - \sigma\phi' - \tfrac{1}{2}Q'^2 + \text{const.} \\ &= -(\sigma + U'p)\phi' - gz + \text{const.}, \end{aligned} \tag{1}$$

and the value over the boundary is $-g\zeta + \dfrac{(\sigma + U'p)^2}{r}\zeta +$ constant. Then

$$\rho Q\zeta = \left\{-g(\rho - \rho') + \frac{(\sigma + U'p)^2}{r}\rho' - Tr^2\right\}\zeta. \quad ($$

The tangential velocity over the surface is, to the first order, $p\phi'$, and the sk friction is $s'\rho'(p\phi')^2$. The components p_{zx} and p_{zy} cannot as a rule be put the form used in 3 (16), except when the motion is two-dimensional. Restricti the discussion to this case, we have for the first order part of p_{zx},

$$\begin{aligned} p_{zx} &= -2\rho' s' U' \frac{\sigma + U'p}{r} p\zeta \\ &= \rho p R\zeta, \end{aligned} \quad ($$

and from 3 (22)

$$\left\{(\sigma + Up + 2\nu r^2)^2 + g\frac{(\rho - \rho')}{\rho} r + \frac{T}{\rho} r^3 - (\sigma + U'p)^2 \frac{\rho'}{\rho} \right.$$
$$\left. + \frac{2\rho' s' U'}{\rho}(\sigma + U'p)\, r\right\}\zeta = 0. \quad ($$

If now we suppose, as before, that ζ is proportional to $e^{\lambda t}\cos\kappa x$, and introdu c^2 as before, we have

$$(\gamma + 2\nu\kappa^2)^2 + (c^2 - U^2)\kappa^2 - (\gamma^2 - U'^2\kappa^2 - 2s'\kappa\gamma)\frac{\rho'}{\rho} = 0. \quad ($$

$$(\gamma + 2\nu\kappa^2)\,U - \gamma U'\frac{\rho'}{\rho} + \frac{\rho' s'\kappa U'^2}{\rho}. \quad ($$

First approximations to these are

$$\gamma = -2\nu\kappa^2 - \frac{\rho'}{\rho}\frac{s'\kappa U'^2}{U} \quad (7$$

$$U^2 = c^2. \quad (8$$

In deriving the skin friction we have tacitly supposed U' to be positive Hence (7) shows that for a wave to grow U must be negative, and the wave velocity must, as before, be between those of the water and the air. But is known to be about 0·002. Hence, for a wave to grow we must have

$$U'^2 > 14400\,\kappa c$$

and thence

$$U' - U > 120\,(g\kappa)^{\frac{1}{4}} + \left(\frac{g}{\kappa}\right)^{\frac{1}{2}}.$$

The right side of this inequality has its least value when κ is 1/23 cm., making the wave-length 140 cm. The corresponding value of $U' - U$ is 480 cm./sec

5. *Comparison with Observation.*

bservations on the initial stages of wave formation have been described J. Scott Russell.* He states that a wind with a velocity between half a e and two miles an hour produces small capillary waves, and that regular vity waves, with a wave-length of about 2 inches, first develop when the d reaches about two miles an hour.

ese observations seem to have been non-instrumental, and I considered it able to check them against a modern instrument. A Negretti and Zambra Meter was kindly lent to me by the Meteorological Office, to the Director hich I am indebted, and observations were made on the River Cam, where ters St. John's College grounds, and just above Jesus Lock, and also on ge pond at Barnwell produced by the flooding of a clay pit. It was found velocities up to 3·4 ft./sec. produced no change in the appearance of the ce of the water, which always showed a slight unevenness however gentle wind seemed, but that strong regular waves were produced by winds of and 4 ft./sec.; the latter should be reduced by a few inches per second low for the speed of the water. On the pond ripples were once observed velocity of 3·6 ft./sec.

ese velocities are very much lower than would be consistent with either rrotational theory or the skin-friction theory. They agree well, however, the theory of sheltering. Using the relations found under this theory, nd the following values for s and for the initial wave-length for various es of the critical wind velocity—

V ft./sec.	3·4	3·6	3·8
V cm./sec.	104	110	116
s	0·318	0·269	0·229
κ (1/cm.)	0·79	0·71	0·64
$2\pi/\kappa$ (cm.)	8·0	8·8	9·8

quantity s is a pure number, and since it expresses the amount of a wave at is exposed to the action of the wind, it will be called the exposure -efficient. There are few *a priori* conditions to guide us as to its value, and ch as there are indicate that the values just found are plausible.

A crucial test of the theory should be provided, however, if the critical ve-length could be found by observation and compared with the theoretical imates just made from the critical wind velocity. Unfortunately the critical

* 'Brit. Assoc. Report,' pp. 317–8 (1844).

wave-length is rather difficult to measure. When waves are formed the wi must be systematically somewhat above the true critical value; thus s w be systematically under-estimated, and so will κ. Thus the wave-leng calculated from the velocity of the wind will, as a rule, be too great. On t other hand, a wind rather above the critical velocity will raise at first wa whose lengths differ over a finite range. The longest of these move faste and thus the shortest tend to remain at the rear of the group. If, then, t waves at the rear of the group are taken as the standard, the wave-length w be systematically under-estimated. By eye estimation I find the length waves raised when the wind is under 4 ft./sec. to be from 6 to 8 cm., which consistent with the above considerations; attempts were made to find t length indirectly from the number of waves passing a given fixed object in known time, but they proved unsuccessful on account of the difficulty counting disturbances with such a short period.

The observational evidence, therefore, appears consistent with the theo that discontinuous motion in the air plays a fundamental part in the formati of water waves.

6. Other Aspects of the Sheltering Theory.

The account of the sheltering theory given in 3 describes only the init stages of waves produced by winds. Winds with velocities greater than t critical velocity there found will be capable of raising waves with any length either the x or the y direction within certain finite ranges. In general, all su waves will be formed; but the longer ones travel more rapidly than the shor ones, and therefore the shortest waves are at the rear of the train produc by a given puff of wind. This agrees with observation. Again, the values U that make 3 (14) an equality are the extreme possible values of U; and has been seen that the equality can hold only if κ' is zero. Thus both t fastest and the slowest waves, relative to the water, that can be generated by given wind are two-dimensional. Three-dimensional waves will be forme but will travel with intermediate velocities. This, again, appears consiste with observation. The swiftest wave in a storm is the swell, which is a tw dimensional wave. (See Plate 4, fig. 1.)

The most striking feature about waves at sea, however, is their irregularit There is never any definite wave pattern, except in so far as the swell can b traced through the shorter and slower waves. (See fig. 2.) As a rule, th distance between consecutive maxima of elevation is of the same order o magnitude in whatever direction it is measured. Such wave-systems will b lled "short-crested," as distinguished from waves whose crests have lengths ny times the distance between consecutive crests; the latter type, luding as particular cases the swell and the first waves raised by a wind t strong enough to raise waves at all, will be called "long-crested."

n the case of a sudden gust with a velocity considerably more than the tical velocity (say, 6 ft./sec.) the equation (14) shows easily that U may have value from about 1/26 of the critical wind velocity up to almost the actual l velocity. The wave-length corresponding to the former velocity is about centimetres. This is short enough to show that surface tension is not gible in this case. However, it is clear that such a wind will be able to e very short waves, and they will include short-crested waves. Now it can bserved that when such a gust occurs the first effect is to produce an gular pattern of short-crested waves, often short enough to be considered llary waves. This is to be expected, since it is for intermediate types of e that the rate of growth will be most rapid. After an interval of 10 to econds the long-crested waves may be seen emerging from the catspaw, ng meanwhile had time to grow. (See Plate 5, fig. 3.)

hen the wind continues steadily for a long time, several other factors become vant. The swifter waves continually gain on the slower, so that at any rmediate point of the wave train many different types of waves might be ected to be superposed; as indeed is the case. But a new feature appears: shorter waves are wholly obliterated. The theory of sheltering offers two lanations of this fact, which probably co-operate in producing the change. e development of the theory all terms depending on the squares and products he deviations from steady motion have been ignored. It has, however, been d that when a wave is actually formed its amplitude will increase exponen-y with the time, so that a stage must be reached when the neglect is no er justifiable. The nature of the change is well known. The form of the es ceases to be purely sinusoidal; the crests become sharper and the troughs ter than in a simple sine curve. (See fig. 4.) This is to be expected in es of considerable amplitude on water unaffected by air; the wind only rovides a mechanism that causes this stage to be attained.

There is a limit to the height that such waves can attain; in a gravity wave is fixed by the fact that the crests become definitely angular, the angle eing 120°. Thus a wave of given length can never grow beyond a certain eight, this height being probably proportional to the wave-length. Further, ind acting on a wave in this critical state will only cause water to be projected rom the crests, leaving the height unaltered. This is what actually happens;

the crests do become sharp ridges, and the tops curl over, casting white fo
or spray on to the sheltered side of the wave. This effect is presumal
prominent in producing turbulence in the water.

When waves of different lengths are superposed, it will not in general
possible to discuss them separately as if the equations of the problems w
linear. The linearity of equations 2 (16) is due to the selection from
actual stresses across the boundary between the fluids of the harmonic c
stituents with wave-lengths equal to those of the wave whose history we
considering. Actually the complete expression of the stresses would requ
the overtones of these constituents. When two non-commensurable wa
lengths are combined the discussion becomes still more difficult.

If, however, we return to the physical description of the sheltering eff
it becomes possible to make some headway. A short wave superposed o
long one will be of smaller height than the latter when both are fully develop
Hence the short one will be sheltered from the wind for the whole time exc
when it is near the crest of the long one and on the exposed side. Its opp
tunities for growth are therefore very much less than if it were alone, w
the damping effect of viscosity will be at least as great. Further, when i
on the sheltered side of the crest of the long wave, the splashing of the wa
from the crest of the long wave will tend to fill up the trough of the sh
one. On both grounds, therefore, it may be expected that the short w
will die out; its motion also will then make a contribution to the turbule
of the water.

The waves produced by a long-continued wind would therefore be expec
to consist mainly of large short-crested waves, combined with a fast-travell
swell of long wave-length, which has not had time to develop a height gr
enough to overshadow the larger short-crested waves. This is in qualitat
agreement with what is observed in mid-ocean.

There are, however, difficulties in the way of a quantitative comparis
though quantitative tests may be sought in several directions. In the fi
place, we may consider the total thrust of the wind in a horizontal directi
on the water. This should be equal to the skin friction as determined
other means. Now the horizontal component in the x-direction of the press
of the wind on the surface is $p_{zz}\dfrac{\partial\zeta}{\partial x}$ approximately, and hence the thr
within a wave-length is $\displaystyle\int_0^{2\pi/\kappa} s\mathrm{U}'^2\rho'\left(\frac{\partial\zeta}{\partial x}\right)^2 dx$. This can only be evaluat
when we have some knowledge of the relation of the height of the wave

its length; but an estimate of this may be made from the fact that the angle at the crest of a fully developed wave must be 120°. As an approximation let us suppose that the section of the wave is an arc of a circle, so that we can write

$$\zeta = a(1 - \cos\theta); \tag{1}$$

$$x = a \sin\theta, \tag{2}$$

and the limits for θ are 0 and $\frac{1}{6}\pi$. Then the mean value of $(\partial\zeta/\partial x)^2$ is

$$2\int_0^{\frac{1}{6}\pi} \sin\theta \tan\theta \, d\theta = \log_e 3 - 1$$
$$= 0\cdot0986. \tag{3}$$

If then s' be the coefficient of skin friction, the horizontal thrust of the wind on the water surface will be $s'\rho'\mathrm{V}^2$ per unit area, and if we take s to be $0\cdot3$ we have the relation

$$s'\mathrm{V}^2 = 0\cdot029\mathrm{U}'^2. \tag{4}$$

Now when waves are just on the verge of formation we have

$$\mathrm{U}' = \tfrac{2}{3}\mathrm{V}. \tag{5}$$

Hence

$$s' = 0\cdot013. \tag{6}$$

This value is much larger than has been found for the coefficient of skin friction in fluids flowing over smooth solids, or even rough solids where the roughnesses are irregular. Experiments have usually given numbers in the neighbourhood of $0\cdot002$. Perhaps, however, it would be premature to say that the above estimate cannot be right for the conditions it refers to, namely, a wind just strong enough to raise waves at all, blowing over water for a time long enough for them to develop fully. If the ordinary estimate for a smooth surface is correct until waves start, as would be expected, it would not be surprising if a sudden increase in the friction accompanied the formation of the first waves. The point should be capable of experimental test in a wind channel; the stress to be sought is of the order of a gram weight per square metre.

When the velocity is greater than has just been considered, the prevalent waves travel with velocities greater than $\frac{1}{3}\mathrm{V}$; if there is no scale effect tending to alter s, as is probable, the anticipated value of s' would therefore be smaller. Observations of the wind at sea have shown no reason to believe that it follows laws very different from those applicable on land, so that the true value of s' is probably not very different from 0·002. Substitution in (4) now gives

$$U' = 0 \cdot 26V \tag{7}$$

so that the waves that produce most of the resistance (that is, the largest waves) should travel with about three-quarters of the velocity of the wind. Cornish* makes the velocity of the waves, when well developed, about 80 to 90 per cent. of that of the wind; but as the wind was logged on the Beaufort Scale an error of 7 per cent. in the wind could escape being logged, merely on account of the coarseness of the scale. The agreement is therefore as good as could be expected.

If now we return to 3 (9) we have for the condition that a wave of given type may be able to grow at all

$$U'^2 \geqslant \frac{4\nu\rho}{s\rho'} Ur, \tag{8}$$

and if we apply 3 (8), taking

$$r = \kappa\sqrt{2}, \tag{9}$$

as is roughly true, we have

$$\nu \leqslant \frac{s\rho'}{8\rho} \frac{U'^2 U}{g}, \tag{10}$$

giving with the value of U' just found

$$\nu \leqslant 6 \cdot 7 \times 10^{-9} V^3 \tag{11}$$

in C.G.S. units. If the sign of equality holds, the wave will be the longest that can grow.

Now the damping of waves in mid-ocean must arise mainly from turbulence rather than true viscosity, and independent determinations of turbulence in water are available. From a comparison of the velocities of wind and ocean currents I have found† values of the eddy-viscosity in mid-ocean ranging from 4 to 460 cm.^{2}/sec. Durst,‡ by a similar method, but with much more numerous and accurately comparable data, has recently carried out an elaborate investigation, and finds that the eddy-viscosity in mid-ocean is well represented by a formula which, when corrected for an arithmetical error, is

$$\nu = 8 \times 10^{-4} V^2. \tag{12}$$

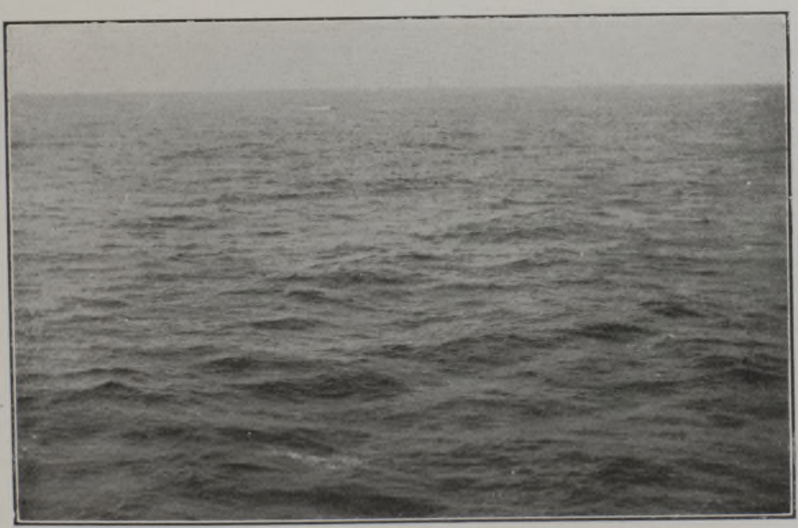
His work amounts to a verification of this formula for values of V ranging from 200 to 1,200 cm./sec. For a wind of 10 m/s. this makes ν equal to 800 cm.^{2}/sec.; whereas formula (11) makes it less than 6·7 cm.^{2}/sec.

* *Loc. cit.*, pp. 111–113.

† 'Phil. Mag.,' vol. 39, pp. 578–586 (1920).

‡ 'Q. J. R. Met. Soc.,' vol. 50, pp. 113–119 (1924).

1.



2.

[Facing p. 204.]

3.

4.

We have here, therefore, a definite inconsistency, which becomes more serious as the wind velocity is reduced.

A satisfactory explanation is not yet available. It might be suggested that variation of turbulence with depth would provide a possible explanation, but numerical comparison does not support this suggestion. If, for instance, ν is equal to 400 cm.^{2}/sec., the current is reduced to e^{-1} of its surface value at a depth of 25 metres; the motion due to a wave of length 200 metres is reduced in a similar ratio at a depth of 32 metres. Thus there is no reason to expect waves and currents to be very differently affected by any variation of turbulence with depth.

Summary.

Kelvin's theory of the formation of water waves by wind, which supposed the motion in both air and water to be irrotational, has been found to lead to several quantitative predictions that disagree with observation. An alternative theory is developed in the present paper. According to this theory, the wind presses more strongly on the slopes of the waves facing it than on the sheltered slopes, and it is when the resulting tendency of the waves to grow is just able to overcome viscosity that waves are first formed. A numerical constant in the theory can be adjusted to make the wind velocity required to produce waves agree with observation; and when this is done the predicted wave-length of the waves first formed agrees with observation without further assumption. Several other facts of wave-formation are readily explained by the theory.

An attempt has been made to account for the skin friction of the wind over the seas as the resultant drag due to the horizontal thrust of the wind on the exposed sides of the waves. Agreement with the ordinarily accepted value of the skin friction can be attained if the wave velocity is about three-quarters of the wind velocity, which is in accordance with observation. The formation of waves with such a velocity, however, appears to require values of the eddy-viscosity much smaller than are indicated by observations of ocean currents; thus there is an outstanding discrepancy.

DESCRIPTION OF PLATES 4 AND 5.

PLATE 4.

FIG. 1.—Combination of two wind-formed swells in Newnham Millpond, Cambridge, June 15, 1924. Notice the remarkable straightness of the crests of the longer swell, in spite of the fact that the photograph covers in the middle distance approximately $\frac{1}{3}$ the diameter of the pond. (See p. 200.)

FIG. 2.—Short-crested waves in North Atlantic, July 31, 1924 (see p. 200). Swell is almost imperceptible.

PLATE 5.

FIG. 3.—Long-crested waves emerging from the front of a catspaw. (Jesus L
Cambridge, June 29, 1924.) (See p. 201.)

FIG. 4.—Effect of an increase in the wind-velocity on a swell already established.
crests have become sharp and are curling over; the formation of new th
dimensional disturbances is shown by the running together of the crests. (Newnl
Millpond, June 15, 1924.) (See p. 201.)

The Thermal and Electrical Conductivities of some Pure Meta

By F. H. SCHOFIELD, B.A., B.Sc.

(Communicated by Sir Joseph Petavel, F.R.S.—Received October 22, 1924.

(Physics Department, the National Physical Laboratory.)

I.—INTRODUCTION.

The relation between the thermal and electrical conductivities of me has for a long time engaged the attention of physicists. As far back as 1 Wiedemann and Franz* propounded the law to the effect that the ratio of two conductivities was the same for all metals. In 1872 Lorenz†, both on th retical and experimental grounds, sought to establish that the above-mentio ratio was proportional to the absolute temperature. On the development the electron theory Drude, H. A. Lorentz, J. J. Thomson and others‡ ha on the basis of various assumptions, arrived at the same conclusion as Lore Up to 1900, however, the experimental values were too uncertain to allow definite confirmation of the theory. In that year Jaeger and Diesselho published the result of their investigation, which gave directly the ratio of conductivities for a number of metals and alloys over the range 18° to 100 Lees|| has since, by an independent method, confirmed the values of Jaeger Diesselhorst for a number of metals at 18° C. and has carried the investigat

* 'Ann. der Phys.,' vol. 89, p. 497 (1853).

† 'Ann. der Phys.,' vol. 147, p. 429 (1872).

‡ For a critical review of the most recent theories, see Meissner, 'Jahrbuch der Rad aktivität und Elektronik,' vol. 17, p. 260 (1920).

§ 'Abh. der Phys.-Tech. Reichsanstalt,' vol. 3, p. 282 (1900).

|| 'Phil. Trans.,' A, vol. 208, p. 381 (1908).