Lecture 3: Taylor's theorem and extreme values

1. Higher order derivatives

Since the partial derivatives of a function are themselves functions, we can take the partial derivatives of them, giving second order partial derivatives.

Definition: The **second order partial derivative functions** of $f(x, y)$ are defined as follows.

$$\frac{\partial^2 f}{\partial x^2} = f_{xx} = (f_x)_x, \quad \frac{\partial^2 f}{\partial x \partial y} = f_{yx} = (f_y)_x$$
$$\frac{\partial^2 f}{\partial y \partial x} = f_{xy} = (f_x)_y, \quad \frac{\partial^2 f}{\partial y^2} = f_{yx} = (f_y)_y$$

We can also define third order partial derivatives such as

$$\frac{\partial^3 f}{\partial x^3} = f_{xxx} = ((f_x)_x)_x, \quad \frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx} = ((f_y)_y)_x$$

and so on. In this course, most of time, we only need partial derivatives up to second order.

The functions $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are called mixed partial derivatives, and they are usually **different** functions. However we have the following theorem.

Theorem: If f_{xy} and f_{yx} are continuous functions, we have

$$f_{xy} = f_{yx}.$$

For most functions f we encounter, their higher order derivatives are continuous, so the mixed partial derivatives are equal.

Example

$$
\begin{aligned}
f(x,y) &= xy^2 + 3x^2e^y \\
f_x &= y^2 + 6xe^y \\
f_y &= 2xy + 3x^2e^y \\
f_{xx} &= 6e^y \\
f_{xy} &= 2y + 6xe^y \\
f_{yx} &= 2y + 6xe^y \\
f_{yy} &= 2x + 3x^2e^y
\end{aligned}
$$

2. Taylor's theorem of degree 2

Recall that for a single variable function, the local linearity tells us that the best linear approximation is the degree 1 Taylor polynomial, which is the equation of the tangent line.

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) \text{ for } x \text{ near } x_0.$$

A better approximation to $f(x)$ is given by the degree 2 Taylor polynomial

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 \text{ for } x \text{ near } x_0.$$

We can continue to consider higher degree Taylor polynomial, and the Taylor series, which is the infinite degree Taylor polynomial, gives us the best approximation.

For a function of two variables, the local linearization for $f(x,y)$ near (x_0, y_0) is given by its tangent plane, as we learned last week.

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0) \text{ for } (x,y) \text{ near } (x_0,y_0).$$

A better approximation to $f(x,y)$ is a quadratic polynomial, which we describe as follows.

Taylor's theroem of degree two: Suppose the second order derivatives of $f(x,y)$ around (x_0,y_0) exist and are continuous functions. Then we have

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &+ \frac{f_{xx}(x_0,y_0)}{2}(x-x_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) \\ &+ \frac{f_{yy}(x_0,y_0)}{2}(y-y_0)^2 \\ &+ o((x-x_0)^2+(y-y_0)^2), \end{aligned}$$

where $o((x-x_0)^2+(y-y_0)^2)$ describes a function which, after being divided by $(x-x_0)^2+(y-y_0)^2$, tends to zero when (x,y) tends to (x_0,y_0).
Another way to write it is as follows.

$$\begin{aligned} f(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &+ \begin{bmatrix} x-x_0 & y-y_0 \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} \\ &+ o((x-x_0)^2+(y-y_0)^2). \end{aligned}$$

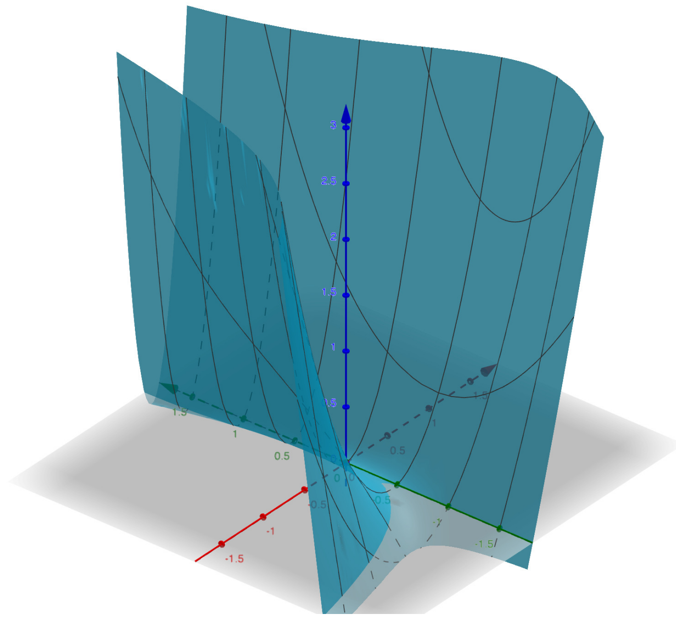
The sketch of the proof will be given in the lecture.

Example

$$\begin{aligned} f(x,y) = xy^2 + 3x^2e^y &\quad f(0,0) = 0 \\ f_x = y^2 + 6xe^y &\quad f_x(0,0) = 0 \\ f_y = 2xy + 3x^2e^y &\quad f_y(0,0) = 0 \\ f_{xx} = 6e^y, &\quad f_{xx}(0,0) = 6 \\ f_{xy} = 2y + 6xe^y &\quad f_{xy}(0,0) = 0 \\ f_{yy} = 2x + 3x^2e^y &\quad f_{yy}(0,0) = 0 \end{aligned}$$

Therefore around $(0,0)$

$$f(x,y) = 3x^2 + o(x^2+y^2).$$



3. Extreme values

In this part, we study how to find the maximum and minimum of a 2-variable function.

> Definition:
>
> - $f(x,y)$ has a **local maximum** at the point (x_0,y_0) if $f(x,y) \leq f(x_0,y_0)$ in a near neighborhoold of (x_0,y_0).
> - $f(x,y)$ has a **local minimum** at the point (x_0,y_0) if $f(x,y) \geq f(x_0,y_0)$ in a near neighborhoold of (x_0,y_0).
> - $f(x,y)$ has a **absolute maximum** at the point (x_0,y_0) if $f(x,y) \leq f(x_0,y_0)$ for all the points (x,y) in the domain of f.
> - $f(x,y)$ has a **absolute minimum** at the point (x_0,y_0) if $f(x,y) \geq f(x_0,y_0)$ for all the points (x,y) in the domain of f.

The most important candidates of local maximum/minimum points are:

Definition: A point (x, y) is called a **critical point** of $f(x, y)$ if either at least one partial derivative does not exist, or $f_x(x, y) = f_y(x, y) = 0$.

Here is a simple observation.

Lemma: Suppose $f(x, y)$ has partial derivatives at (x_0, y_0) and (x_0, y_0) is a local maximum/minimum point, then (x_0, y_0) is a critical point of f.

The proof of the lemma will be given in the lecture.

Therefore to find all the local maximum/minimum points of a given function f, we need to

1. Find all the points where the partial derivatives are not defined.
2. Find all the points where the partial derivatives vanish.

Having found all the candidates, we use the second order partial derivatives to classify them.

Definition: Suppose all the second order derivatives exist. Then

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$$

is called the **Hessian matrix** of $f(x, y)$.

The determinant of the Hessian matrix is

$$D = f_{xx} f_{yy} - f_{xy}^2.$$

Here is an important test.

Second derivative test: Suppose all the second order derivatives exist and are continuous function. Suppose the partial derivatives vanish at (x_0, y_0).

1. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).
2. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).
3. If $D < 0$, then f has a saddle point at (x_0, y_0).
4. If $D = 0$, anything can happen: f can have a local maximum, or a local minimum, or a saddle point, or none of the above, at (x_0, y_0).

The idea of the proof will be given in the lecture. Applications will be given as well.

Summary:
Suppose $f(x, y)$ is a function defined in a domain D (e.g. It can be $\mathbb{R}^2$, or a domain bounded by a closed curve.) To find the local maxima/minima, we need to

1. Find all the points where the partial derivatives are not defined.
2. Find all the points where the partial derivatives vanish.
3. Use second derivative test to classify points found at step 2.
4. If $D = 0$, try another method.

The absolute maximum, if exists, is the maximum of

1. Local maxima found by using second derivative test.
2. Values of interior points of D where partial derivatives are not defined.
3. Values of interior points of D where the partial derivatives vanish and $D = 0$.

4. Values of boundary points of D.

The absolute minimum, if exists, can be found in the same vein.

4. Homework

Q. 1 For each of the following conditions, consider what types of functions can satisfy it.

$$(1) \quad f_{xx} = f_{yy} = 0 \qquad (2) \quad f_{xy} = 0$$

Q. 2 Calculate the second order partial derivatives.

$$(1) \quad x^3 + xy^2 + \frac{y}{x} \qquad (2) \quad x^y \qquad (3) \quad \sin(x \cos y)$$

Q. 3 Find the critical points and classify them as local maxima, local minima, saddle points, or none of these.

1. $x^2 - 2xy + 3y^3 - 8y$
2. $(x + y)(xy + 1)$
3. $e^{-x^2-y^2}(x + y)$
4. $2x^2 - 3x^2y + y^2$