

LTAT.02.004 MACHINE LEARNING II

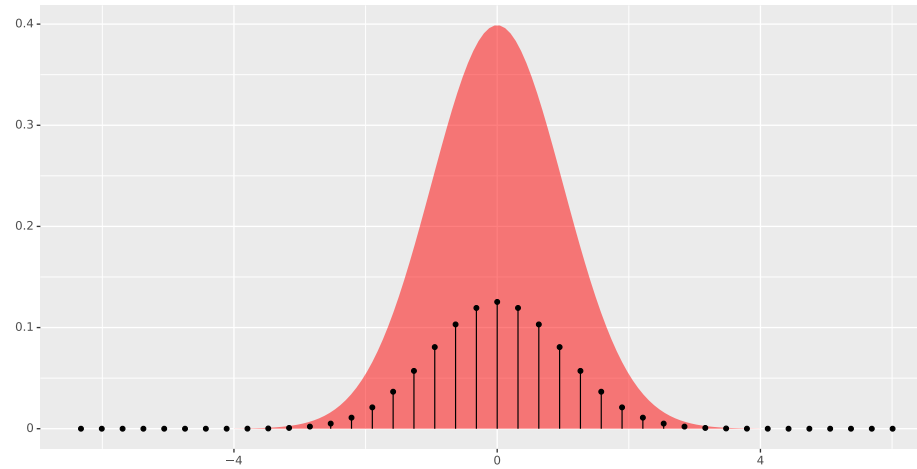
Multivariate normal distribution

Direct applications

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Univariate normal distribution

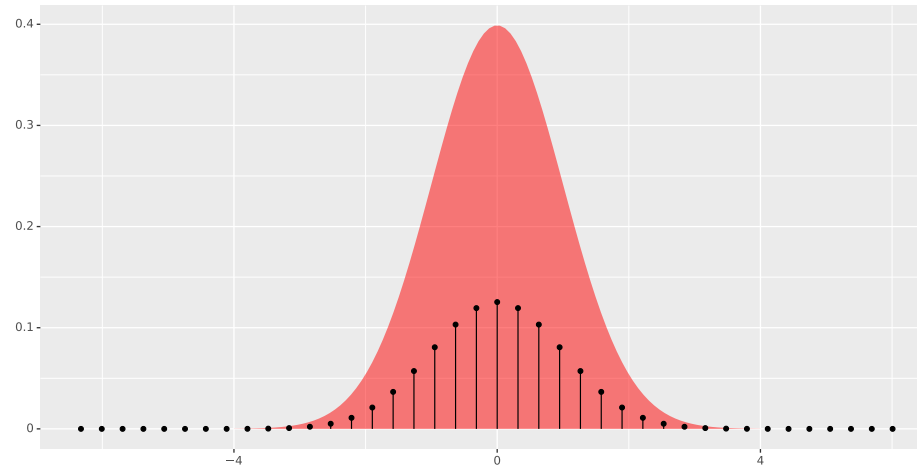
Probability density function



Definition. A real-valued random variable X comes from a continuous distribution with *a probability density function* $p : \mathbb{R} \rightarrow \mathbb{R}^+ \cup \{0\}$ if the following limit exists for any $x \in \mathbb{R}$:

$$p(x) = \lim_{\Delta x \rightarrow 0^+} \frac{\Pr [x - \Delta x \leq X \leq x + \Delta x]}{2 \cdot \Delta x} .$$

Probability mass function

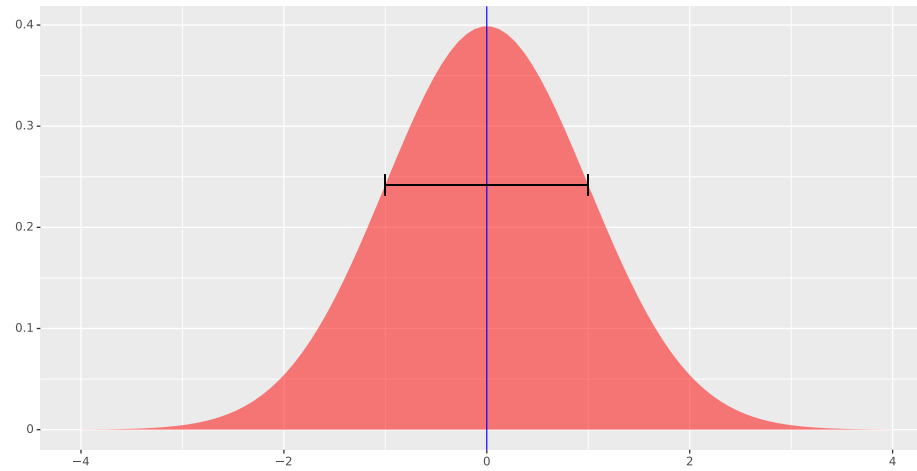


Definition. A real-valued random variable X comes from a discrete distribution with *a probability mass function* $p : \mathbb{R} \rightarrow \mathbb{R}^+ \cup \{0\}$ defined as

$$p(x) = \Pr[X = x] = \lim_{\Delta x \rightarrow 0^+} \Pr[x - \Delta x \leq X \leq x + \Delta x]$$

if there exist a sequence $(x_i)_{i=1}^{\infty}$ such that $p(x_1) + \dots + p(x_i) + \dots = 1$.

Standard normal distribution



Standard normal distribution $\mathcal{N}(\mu = 0, \sigma = 1)$ is a continuous distribution with a probability density function

$$p(x) = \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{x^2}{2}\right)$$

The mean value $\mu = 0$ and variance $\sigma^2 = 1$ for this distribution.

Univariate normal distribution

Definition. A random variable y is distributed according to a normal distribution $\mathcal{N}(\mu = a, \sigma = b)$ if it can be expressed

$$y = bx + a$$

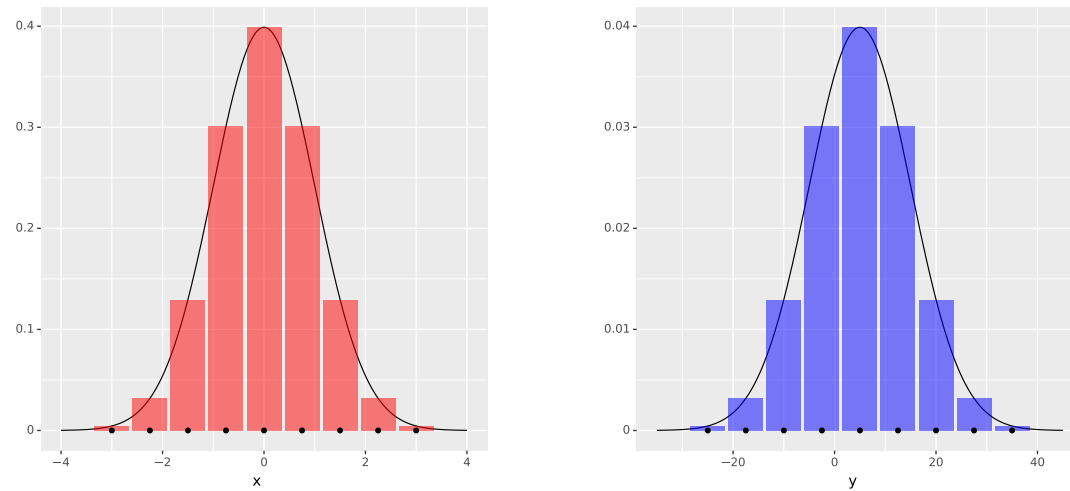
where x is distributed according to standardised normal distribution $\mathcal{N}(0, 1)$.

The corresponding probability density functions is

$$p[y|\mu, \sigma] = \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)$$

and the mean value μ and variance σ^2 for this distribution.

Density derivation



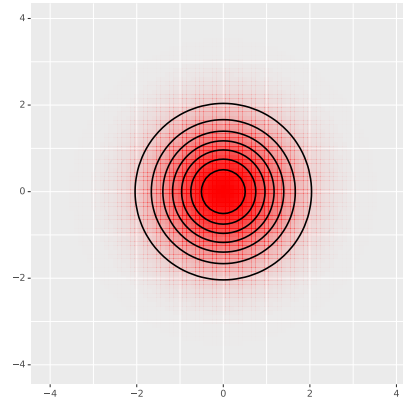
Let $y = ax + b$ the the relation between densities

$$p_x(x) = \sigma \cdot p_y(y)$$

follows form the fact that areas of red and blue columns must be the same.

Multivariate normal distribution

White Gaussian noise



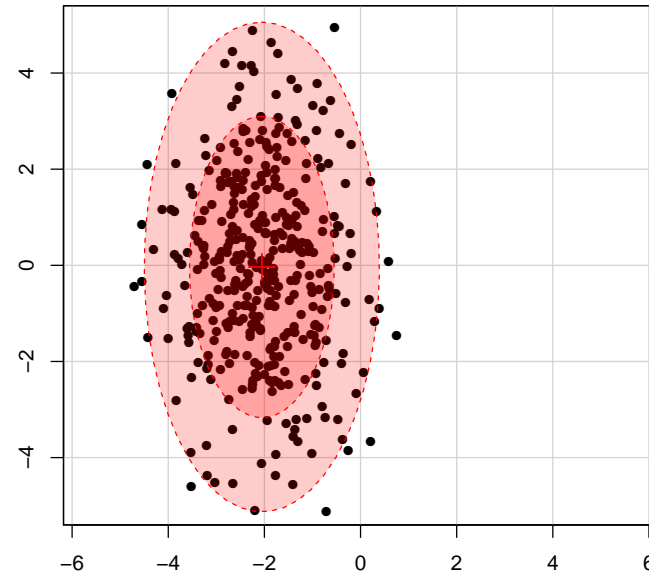
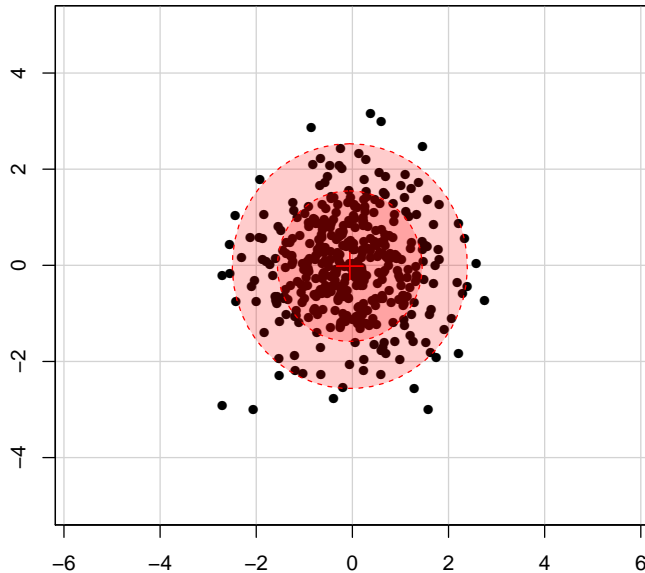
Definition. A random vector X_1, \dots, X_n is a standard normal random vector if all of its components are independent and $X_i \sim \mathcal{N}(0, 1)$.

▷ The density can be computed based on independence:

$$p(x_1, \dots, x_n) = p(x_1) \cdots p(x_n) = \frac{1}{(2\pi)^{n/2}} \cdot \exp\left(-\frac{x_1^2 + \cdots + x_n^2}{2}\right) .$$

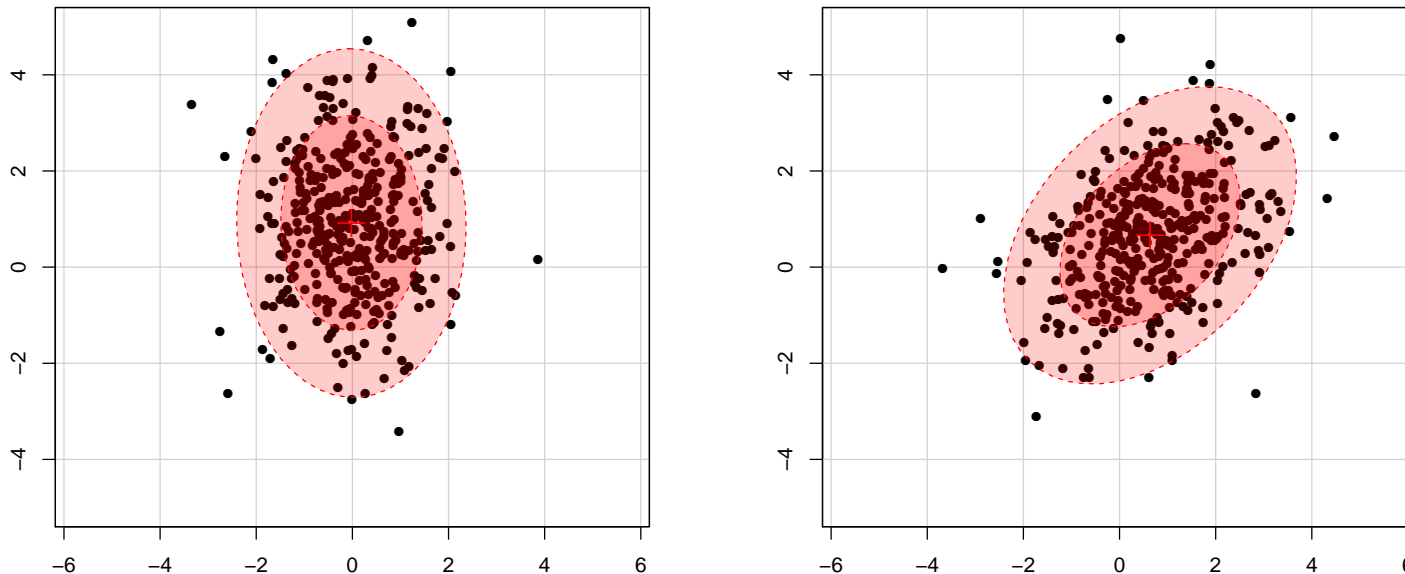
Scaling and shifting

By shifting and scaling the source distribution $\mathcal{N}(\mathbf{0}, I)$ we can obtain some other instances of multivariate normal distribution.



Necessity of rotations

As the choice of coordinate axis is sometimes arbitrary, there must be other ways to form a normal distribution – rotations of coordinate axis.



Any affine transformation can be expressed as scaling, rotating and shifting.

Affine transformations

Let \mathbf{x} be standard normal random vector and let \mathbf{y} be obtained the scaling, translation and rotation of the coordinate plane.

Then we can express \mathbf{x} and \mathbf{y} in terms of an affine transformation

$$\begin{aligned}\mathbf{y} &= A\mathbf{x} + \boldsymbol{\mu} \ , \\ \mathbf{x} &= A^{-1}(\mathbf{y} - \boldsymbol{\mu}) \ .\end{aligned}$$

Observation. Affine transformations are closed with respect to composition, i.e., applying two affine transformations yields a new affine transformation.

Remark. Not all affine transformations are invertible.

What is density in 2D?

Recall that density assigns probability to small enough regions \mathcal{R} :

$$\Pr \left[\begin{array}{l} x_1^* \leftarrow \mathcal{N}(0, 1) : x_1 \leq x_1^* \leq x_1 + \Delta x_1 \\ x_2^* \leftarrow \mathcal{N}(0, 1) : x_2 \leq x_2^* \leq x_2 + \Delta x_2 \end{array} \right] = p(x_1, x_2) \cdot \underbrace{\Delta x_1 \Delta x_2}_S + \varepsilon$$

where $\varepsilon = o(\Delta x_1 \cdot \Delta x_2)$ in the process $\Delta x_1 \rightarrow 0$ and $\Delta x_2 \rightarrow 0$.

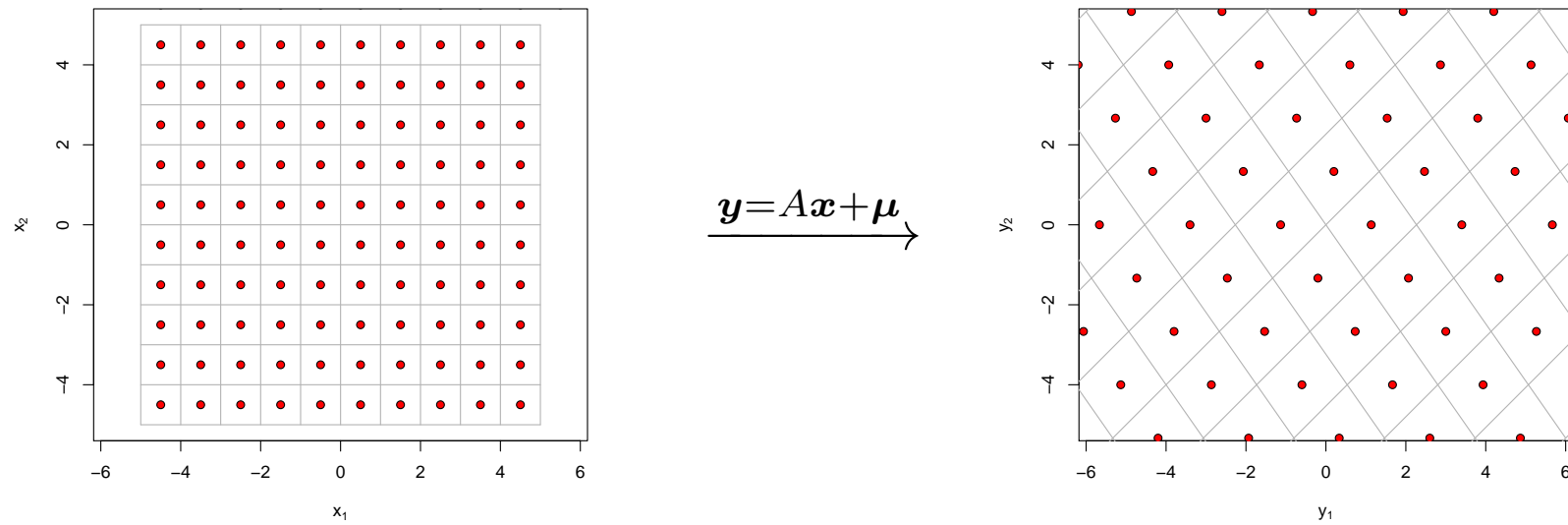
Remark. Regions \mathcal{R} do not have to be rectangular as long as:

- ▷ The area $S(\mathcal{R})$ of a region can be computed.
- ▷ Probability can be assigned to the region \mathcal{R} and its scalings.

Then $\varepsilon = o(S)$ when we rescale the region \mathcal{R} around the point (x_1, x_2) .

Density recalibration

Any affine transformation changes a square grid into parallelograms.



As a result, the area of the regions is different on the left and on the right:

$$p(x_1, x_2) \cdot S_1 \approx q(y_1, y_2) \cdot S_2 \quad \implies \quad q(y_1, y_2) = \frac{S_1}{S_2} \cdot p(x_1, x_2)$$

Fortunately, the ratio between areas are constant over the entire plane!

Density of two-variate normal distribution

The density of (x_1, x_2) pairs can be computed based on independence:

$$p(x_1, x_2) = p(x_1) \cdot p(x_2) = \frac{1}{2\pi} \cdot \exp\left(-\frac{x_1^2 + x_2^2}{2}\right) .$$

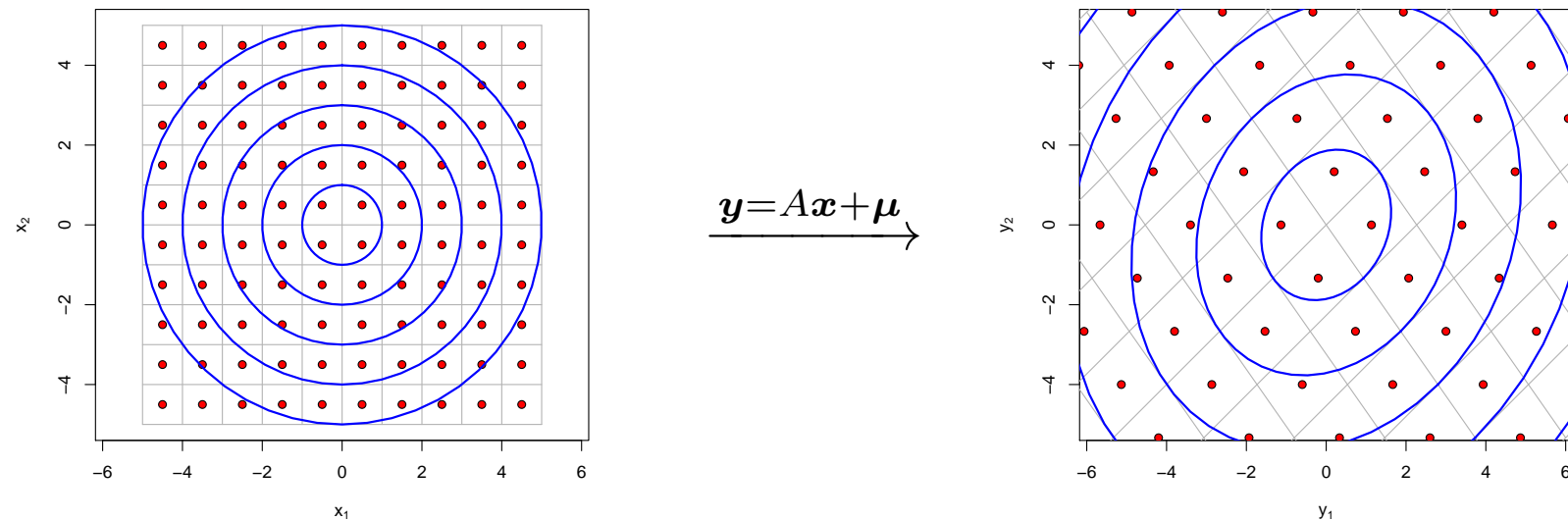
To estimate density $q(y_1, y_2)$, we must find the corresponding (x_1, x_2) :

$$\mathbf{y} = A\mathbf{x} + \boldsymbol{\mu} \quad \Leftrightarrow \quad \mathbf{x} = A^{-1}(\mathbf{y} - \boldsymbol{\mu}) .$$

Thus we get

$$\begin{aligned} q(y_1, y_2) &= \frac{S_1}{S_2} \cdot \frac{1}{2\pi} \cdot \exp\left(-\frac{(\mathbf{y} - \boldsymbol{\mu})^T A^{-T} A^{-1}(\mathbf{y} - \boldsymbol{\mu})}{2}\right) \\ &= \frac{1}{\sqrt{\det(\Sigma)}} \cdot \frac{1}{2\pi} \cdot \exp\left(-\frac{(\mathbf{y} - \boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{y} - \boldsymbol{\mu})}{2}\right) . \end{aligned}$$

Illustrative example



- ▷ Affine transformation changes the square grid into parallelograms.
- ▷ Affine transformation changes circular equiprobability lines into ellipses.
- ▷ The axes of the ellipses may intersect with the sides of parallelograms.

Generalisation to multivariate case

If observed quantities \mathbf{y} are generated by applying the affine transformation

$$\mathbf{y} = A\mathbf{x} + \boldsymbol{\mu} \quad \Leftrightarrow \quad \mathbf{x} = A^{-1}(\mathbf{y} - \boldsymbol{\mu})$$

to the *independent source signals* $x_1, \dots, x_n \sim \mathcal{N}(0, 1)$, then the resulting distribution is *a multivariate normal distribution* with the density:

$$p(\mathbf{y}) = \frac{1}{(2\pi)^{n/2}} \cdot \frac{1}{\sqrt{\det(\Sigma)}} \cdot \exp\left(-\frac{(\mathbf{y} - \boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{y} - \boldsymbol{\mu})}{2}\right)$$

where $\Sigma^{-1} = A^{-T}A^{-1}$ is *a positively definite symmetric matrix*.

Important properties of normal distributions

Closeness under marginalisation

Let $\mathbf{x}_{\mathcal{I}} = (x_i)_{i \in \mathcal{I}}$ be a subvector determined by the coordinate set \mathcal{I} . Then $\mathbf{x}_{\mathcal{I}}$ is distributed according to a multivariate normal distribution as long as the vector \mathbf{x} comes from a multivariate normal distribution $\mathcal{N}(\boldsymbol{\mu}, \Sigma)$.

▷ Moment matching gives the parameters of the resulting distribution

$$\begin{aligned}\mathbf{E}(\mathbf{x}_{\mathcal{I}}) &= \mathbf{E}(\mathbf{x})_{\mathcal{I}} = \boldsymbol{\mu}_{\mathcal{I}} \\ \mathbf{Cov}(\mathbf{x}_{\mathcal{I}}) &= \mathbf{Cov}(\mathbf{x})_{\mathcal{I} \times \mathcal{I}} = \Sigma[\mathcal{I}, \mathcal{I}]\end{aligned}$$

Closeness under linear combinations

Linear combination $y = \alpha_1^T x_1 + \alpha_2^T x_2$ of independent multivariate normal distributions $x_1 \sim \mathcal{N}(\mu_1, \Sigma_1)$ and $x_2 \sim \mathcal{N}(\mu_2, \Sigma_2)$ is also a multivariate normal distribution.

▷ Moment matching gives the parameters of the resulting distribution

$$\begin{aligned}\mathbf{E}(y) &= \alpha_1^T \mathbf{E}(x_1) + \alpha_2^T \mathbf{E}(x_2) = \alpha_1^T \mu_1 + \alpha_2^T \mu_2 \\ \mathbf{Var}(y) &= \mathbf{Cov}(\alpha_1^T x_1) + \mathbf{Cov}(\alpha_2^T x_2) \\ &= \alpha_1^T \mathbf{Cov}(x_1) \alpha_1 + \alpha_2^T \mathbf{Cov}(x_2) \alpha_2 \\ &= \alpha_1^T \Sigma_1 \alpha_1 + \alpha_2^T \Sigma_2 \alpha_2\end{aligned}$$

▷ Closeness under linear combinations holds also for matrix combinations.

Closeness under conditioning

Let \mathbf{x} and \mathbf{y} be related random variables. Let $\mathbf{x}|\mathbf{y}_*$ denote the conditional distribution of \mathbf{x} given that a random variable \mathbf{y} has a fixed value \mathbf{y}_* . Then $\mathbf{x}|\mathbf{y}_*$ is distributed according to a multivariate normal distribution provided that (\mathbf{x}, \mathbf{y}) comes from a multivariate normal distribution $\mathcal{N}((\boldsymbol{\mu}_i), (\Sigma_{ij}))$

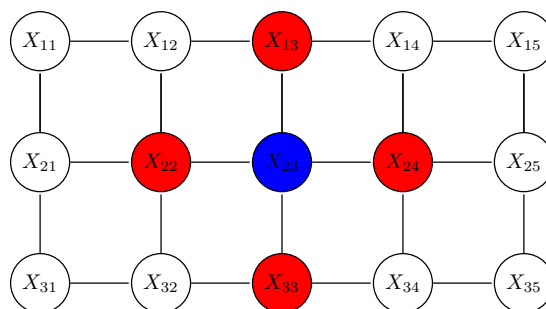
▷ Moment matching gives the parameters of the resulting distribution

$$\mathbf{E}(\mathbf{x}|\mathbf{y}_*) = \boldsymbol{\mu}_1 + \Sigma_{1,2}\Sigma_{2,2}^{-1}(\mathbf{y} - \boldsymbol{\mu}_2)$$

$$\mathbf{Cov}(\mathbf{x}|\mathbf{y}_*) = \Sigma_{1,1} - \Sigma_{1,2}\Sigma_{2,2}^{-1}\Sigma_{2,1}$$

Markov Random Fields

Background model for digital images



In most images intensity of pixel is influenced only by its neighbours:

- ▷ For simple textures the neighbourhood consist of four adjacent pixels.
- ▷ For complex textures the the neighbourhood contains much more pixels.
- ▷ For homogenous textures the conditional probabilities are universal.
 - ◇ Generative repetitive patterns for textile and grass
- ▷ For complex patterns conditional probabilities can be location dependent.
 - ◇ Generative patterns for human faces and fashion accessories

General form of the likelihood function

The celebrated Hammersley-Clifford theorem fixes the format in which the corresponding probability distribution must be sought:

$$p[\mathbf{x}|\omega] = \frac{1}{Z(\omega)} \cdot \exp \left(- \sum_{c \in \text{MaxClique}} \Psi_c(\mathbf{x}_c, \omega) \right)$$

where

- ▷ ω is a set of model parameters
- ▷ $Z(\omega)$ is a normalising constant
- ▷ MaxClique is the set of maximal cliques in the Markov random field
- ▷ Ψ_c is defined on the variables x_i in the clique c .

Multivariate normal distribution as likelihood

If individual sub-potentials $\Psi_c(\mathbf{x}_c, \omega)$ are quadratic forms then the energy

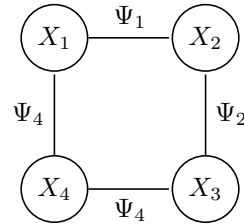
$$\Psi(\mathbf{x}) = \sum_{c \in \text{MaxClique}} \Psi_c(\mathbf{x}_c, \omega)$$

is also a quadratic form and thus $p[\mathbf{x}|\omega]$ is a multivariate normal distribution.

Sub-potentials are often fixed directly based on smoothness constraints

- ▷ Intensities have bounded variance: $\Psi_e = \delta^2 x_{ij}^2$.
- ▷ Intensity changes smoothly vertically: $\Psi_e = \beta(x_{i,j} - x_{i+1,j})^2$.
- ▷ Intensity changes smoothly horizontally: $\Psi_e = \alpha(x_{i,j} - x_{i,j+1})^2$.

Toy example



Sub-potentials corresponding four edges are:

$$\Psi_1(x_1, x_2) = \alpha_1(x_1 - x_2)^2 = \alpha_1 x_1^2 - 2\alpha_1 x_1 x_2 + \alpha_1 x_2^2$$

$$\Psi_2(x_2, x_3) = \alpha_2(x_2 - x_3)^2 = \alpha_2 x_2^2 - 2\alpha_2 x_2 x_3 + \alpha_2 x_3^2$$

$$\Psi_3(x_3, x_4) = \alpha_3(x_3 - x_4)^2 = \alpha_3 x_3^2 - 2\alpha_3 x_3 x_4 + \alpha_3 x_4^2$$

$$\Psi_4(x_4, x_1) = \alpha_4(x_4 - x_1)^2 = \alpha_4 x_4^2 - 2\alpha_4 x_4 x_1 + \alpha_4 x_1^2$$

Sub-potentials corresponding to four vertices are $\Psi_i^*(x_i) = \delta_i^2 x_i$

Resulting potential function

$$\Psi(\mathbf{x}) = \mathbf{x}^T \begin{pmatrix} \alpha_1 + \alpha_4 + \delta_1^2 & -\alpha_1 & 0 & -\alpha_4 \\ -\alpha_1 & \alpha_1 + \alpha_2 + \delta_2^2 & -\alpha_2 & 0 \\ 0 & -\alpha_2 & \alpha_2 + \alpha_3 + \delta_3^2 & -\alpha_3 \\ -\alpha_4 & 0 & -\alpha_3 & \alpha_3 + \alpha_4 + \delta_4^2 \end{pmatrix} \mathbf{x}$$

and thus the covariance matrix Σ and mean $\boldsymbol{\mu}$ can be computed by matching the shape of the multivariate normal density

$$p[\mathbf{x}|\boldsymbol{\mu}, \Sigma] \propto \frac{1}{\det \Sigma} \cdot \exp \left(-\frac{1}{2} \cdot (\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right)$$

Kalman filter

To be added next year