Geometry through Linear Algebra

G V V Sharma*

Contents

***Abstract*—This book provides a vector approach to geometry. The content and exercises are based on NCERT textbooks from Class 6-12.**

1 Triangle Exercises

1.1. Each angle of an equilateral triangle is of 60°.
Solution: See Fig. 1.1.1
Considering A,B and C as the vertices of triangle:

$$A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} B = \begin{pmatrix} 0 \\ 0 \end{pmatrix} C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

In equilateral triangle all sides are equal.Hence,

$$\|\mathbf{A} - \mathbf{B}\| = \|\mathbf{B} - \mathbf{C}\| = \|\mathbf{A} - \mathbf{C}\| \tag{1.1.1}$$

Putting **B**= 0 in (1.1.1) we have,

$$\|\mathbf{A}\| = \|\mathbf{C}\| \tag{1.1.2}$$

*The author is with the Department of Electrical Engineering, Indian Institute of Technology, Hyderabad 502285 India e-mail: gadepall@iith.ac.in.

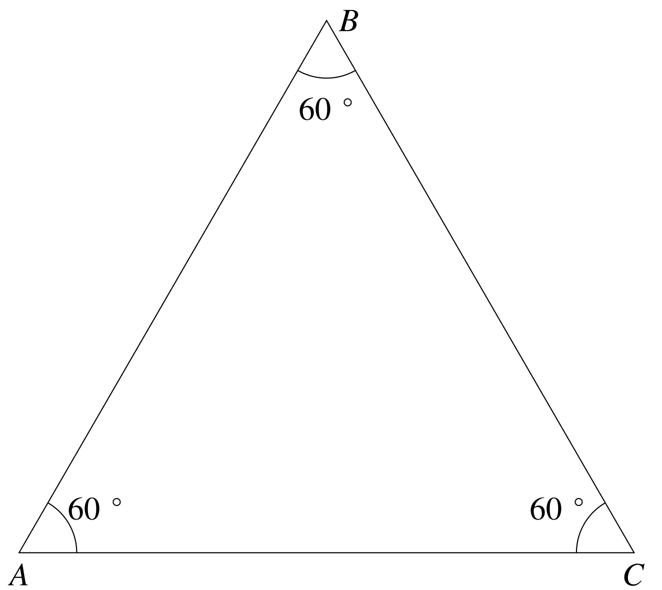


Fig. 1.1.1: Equilateral $\triangle ABC$ with A,B and C as vertices

$$\|\mathbf{A}\| = \|\mathbf{A} - \mathbf{C}\| \tag{1.1.3}$$

Squaring equation (1.1.2)

$$\|\mathbf{A}\|^2 = \|\mathbf{C}\|^2 \tag{1.1.4}$$

Squaring equation (1.1.3)

$$\|\mathbf{A}\|^2 = \|\mathbf{A}\|^2 - 2(\mathbf{A}^T)(\mathbf{C}) + \|\mathbf{C}\|^2$$
$$\implies \|\mathbf{A}\|^2 = 2(\mathbf{A}^T)(\mathbf{C}) \tag{1.1.5}$$

Taking the inner product of sides AB,BC we have:

$$(\mathbf{A}-\mathbf{B})^T(\mathbf{B}-\mathbf{C}) = \|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{B}-\mathbf{C}\|\cos ABC \tag{1.1.6}$$

The angle ABC from the above equation is:

$$\cos ABC = \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{B}-\mathbf{C})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{1.1.7}$$

Substituting value in (1.1.7) and putting we have:

$$\cos ABC = \frac{(\mathbf{A})^T(\mathbf{C})}{\|\mathbf{A}\|^2} \tag{1.1.8}$$

From (1.1.5) we have:

$$\cos ABC = \frac{(\mathbf{A})^T(\mathbf{C})}{2(\mathbf{A})^T(\mathbf{C})}$$
$$\implies \cos ABC = 1/2$$
$$\implies \angle ABC = 60^\circ \tag{1.1.9}$$

Taking the inner product of sides AB,AC we have:

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{C}) = \|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{C}\|\cos BAC \tag{1.1.10}$$

The angle BAC from the above equation is:

$$\cos BAC = \frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{C}\|} \tag{1.1.11}$$

Substituting value in (1.1.11) and putting we have:

$$\cos BAC = \frac{(\mathbf{A})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{A}\|^2} \tag{1.1.12}$$
$$\implies \frac{(\mathbf{A})^T(\mathbf{A})-(\mathbf{A})^T(\mathbf{C})}{\|\mathbf{A}\|^2} \tag{1.1.13}$$

We know $(\mathbf{A})^T(\mathbf{A}) = \|\mathbf{A}\|^2$
From equation (1.1.5) we have: $(\mathbf{A})^T(\mathbf{C}) = \frac{1}{2}\|\mathbf{A}\|^2$
Substituting values in (1.1.12) we have:

$$\cos BAC = \frac{\frac{1}{2}\|\mathbf{A}\|^2}{\|\mathbf{A}\|^2}$$
$$\implies \cos BAC = 1/2$$
$$\implies \angle BAC = 60^\circ \tag{1.1.14}$$

Taking the inner product of sides AC,BC we have:

$$(\mathbf{C}-\mathbf{A})^T(\mathbf{C}-\mathbf{B}) = \|\mathbf{C}-\mathbf{A}\|\,\|\mathbf{C}-\mathbf{B}\|\cos ACB \tag{1.1.15}$$

The angle ACB from the above equation is:

$$\cos ACB = \frac{(\mathbf{C}-\mathbf{A})^T(\mathbf{C}-\mathbf{B})}{\|\mathbf{C}-\mathbf{A}\|\,\|\mathbf{C}-\mathbf{B}\|} \tag{1.1.16}$$

Substituting value in (1.1.16) and putting we have:

$$\cos ACB = \frac{(\mathbf{C}-\mathbf{A})^T(\mathbf{C})}{\|\mathbf{A}\|^2}$$
$$\implies \frac{(\mathbf{C})^T(\mathbf{C})-(\mathbf{A})^T(\mathbf{C})}{\|\mathbf{A}\|^2} \tag{1.1.17}$$

We know $(\mathbf{C})^T(\mathbf{C}) = \|\mathbf{C}\|^2$
From (1.1.4) and (1.1.5) we have:
$(\mathbf{A})^T(\mathbf{C}) = \frac{1}{2}\|\mathbf{A}\|^2 = \frac{1}{2}\|\mathbf{C}\|^2$
Substituting values in (1.1.17) we have:

$$\cos ACB = \frac{\frac{1}{2}\|\mathbf{C}\|^2}{\|\mathbf{C}\|^2}$$
$$\implies \cos ACB = 1/2$$
$$\implies \angle ACB = 60^\circ \tag{1.1.18}$$

Hence from equation (1.1.9),(1.1.14) and(1.1.16)

$$\angle ABC = \angle BAC = \angle ACB = 60^\circ \tag{1.1.19}$$

1.2. Triangles on the same base (or equal bases) and between the same parallels are equal in area.
Solution: Consider

$$\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{1.2.1}$$

$$\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{1.2.2}$$

The cross product of the 2 matrices is,

$$\mathbf{A}\times\mathbf{B} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{1.2.3}$$

Substituting $a_3 = b_3 = 0$ in (1.2.3) and simplifying,

$$\implies \mathbf{A}\times\mathbf{B} = \begin{pmatrix} 0 & a_1 \\ -a_2 & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{1.2.4}$$

Considering three points $\mathbf{A}, \mathbf{B}, \mathbf{C}$ on a triangle

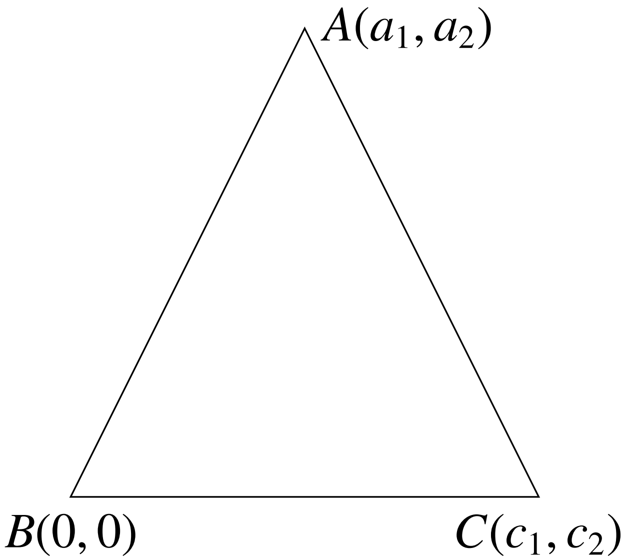


Fig. 1.2.1: $\triangle ABC$ with B at origin

and $\mathbf{B}$ at origin,

$$\mathbf{A} - \mathbf{B} = \mathbf{A} \tag{1.2.5}$$
$$\mathbf{C} - \mathbf{B} = \mathbf{C} \tag{1.2.6}$$

Area of triangle is,

$$Area(\triangle ABC) = \frac{1}{2} \left\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{C} - \mathbf{B})\right\| \tag{1.2.7}$$

Substituting (1.2.5), (1.2.6) in (1.2.7),

$$\implies Area(\triangle ABC) = \frac{1}{2} \left\|\mathbf{A} \times \mathbf{C}\right\| \tag{1.2.8}$$

Constructing another triangle DBC with base as BC,
Since AD || BC,

$$\mathbf{A} - \mathbf{D} = k(\mathbf{B} - \mathbf{C}) \tag{1.2.9}$$

Now calculating the area of $\triangle DBC$,

$$Area(\triangle DBC) = \frac{1}{2} \left\|(\mathbf{D} - \mathbf{B}) \times (\mathbf{C} - \mathbf{B})\right\| \tag{1.2.10}$$

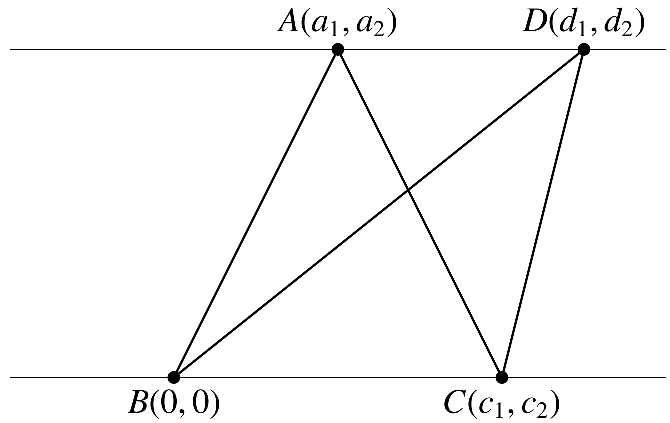


Fig. 1.2.2: $\triangle ABC$ and $\triangle DBC$ with BC as common base

Substituting (1.2.9) in (1.2.10),

$$\begin{aligned} \implies Area(\triangle DBC) &= \frac{1}{2} \left\|(\mathbf{A} - k(\mathbf{B} - \mathbf{C})) \times (\mathbf{C} - \mathbf{B})\right\| \\ \implies Area(\triangle DBC) &= \frac{1}{2} \left\|(\mathbf{A} + k\mathbf{C}) \times \mathbf{C}\right\| \\ &(\because \mathbf{C} - \mathbf{B} = \mathbf{C}) \\ \implies Area(\triangle DBC) &= \frac{1}{2} \left\|\mathbf{A} \times \mathbf{C}\right\| (\because \mathbf{A} \times \mathbf{A} = 0) \end{aligned} \tag{1.2.11}$$

It can be observed that (1.2.11) is same as (1.2.8) Hence, triangles on the same base(or equal bases) and between the same parallels are equal in area.

1.3. Triangles on the same base (or equal bases) and having equal areas lie between the same parallels.
Solution: Consider $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are three points of a triangle $\triangle ABC$, $\mathbf{D}, \mathbf{B}, \mathbf{C}$ are three points of another triangle $\triangle DBC$, both triangles having same base BC and $\mathbf{B}$ at origin, then

$$\mathbf{A} - \mathbf{B} = \mathbf{A} \tag{1.3.1}$$
$$\mathbf{C} - \mathbf{B} = \mathbf{C} \tag{1.3.2}$$
$$\mathbf{D} - \mathbf{B} = \mathbf{D} \tag{1.3.3}$$

The area of the triangle $\triangle ABC$ is,

$$Area\,(\triangle ABC) = \frac{1}{2} \left\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{C} - \mathbf{B})\right\| \tag{1.3.4}$$

Substituting (1.3.1) and (1.3.2) in (1.3.4),

$$\implies Area\,(\triangle ABC) = \frac{1}{2} \left\|(\mathbf{A} \times \mathbf{C})\right\| \tag{1.3.5}$$

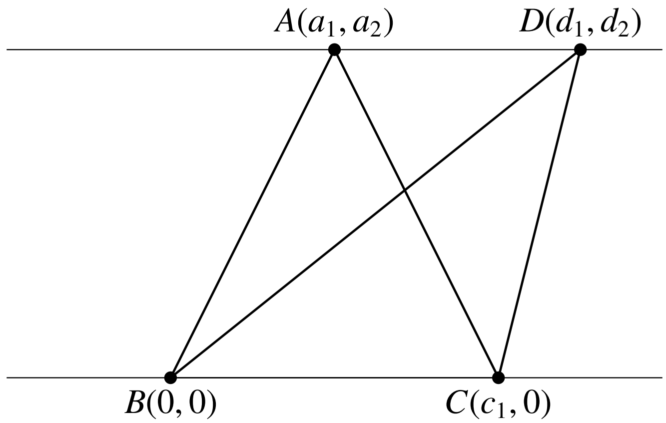


Fig. 1.3.1: $\triangle ABC$ and $\triangle DBC$ with BC as common base

The area of the triangle $\triangle DBC$ is,

$$Area\left(\triangle ABC\right) = \frac{1}{2}\left\|\left(\mathbf{D}-\mathbf{B}\right)\times\left(\mathbf{C}-\mathbf{B}\right)\right\| \tag{1.3.6}$$

Substituting (1.3.2) and (1.3.3) in (1.3.6),

$$\implies Area\left(\triangle DBC\right) = \frac{1}{2}\left\|\left(\mathbf{D}\times\mathbf{C}\right)\right\| \tag{1.3.7}$$

Given the area of $\triangle ABC$ is equal to the area $\triangle DBC$, from equations (1.3.5) and (1.3.7)

$$\frac{1}{2}\left\|\left(\mathbf{A}\times\mathbf{C}\right)\right\| = \frac{1}{2}\left\|\left(\mathbf{D}\times\mathbf{C}\right)\right\| \tag{1.3.8}$$

Squaring on both sides of the equation (1.3.8), we get

$$\left\|\left(\mathbf{A}\times\mathbf{C}\right)\right\|^2 = \left\|\left(\mathbf{D}\times\mathbf{C}\right)\right\|^2 \tag{1.3.9}$$

$$\implies \left(\mathbf{A}\times\mathbf{C}\right)^T\left(\mathbf{A}\times\mathbf{C}\right) = \left(\mathbf{D}\times\mathbf{C}\right)^T\left(\mathbf{D}\times\mathbf{C}\right) \tag{1.3.10}$$

$$\implies \left(\mathbf{A}^T\mathbf{A}\right)\left(\mathbf{C}^T\mathbf{C}\right)-\left(\mathbf{A}^T\mathbf{C}\right)\left(\mathbf{C}^T\mathbf{A}\right) = \tag{1.3.11}$$

$$\left(\mathbf{D}^T\mathbf{D}\right)\left(\mathbf{C}^T\mathbf{C}\right)-\left(\mathbf{D}^T\mathbf{C}\right)\left(\mathbf{C}^T\mathbf{D}\right) \tag{1.3.12}$$

$$\implies \|\mathbf{A}\|^2\|\mathbf{C}\|^2-\left(\mathbf{A}^T\mathbf{C}\right)^2 = \|\mathbf{D}\|^2\left\|\mathbf{C}^2\right\|-\left(\mathbf{D}^T\mathbf{C}\right)^2 \tag{1.3.13}$$

Let $\mathbf{A} = \begin{pmatrix} a1 \\ a2 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ .Then equation (1.3.13) can be written as,

$$\left(a_1^2+a_2^2\right)\left(c_1^2\right)-\left(a_1c_1\right)^2 = \left(d_1^2+d_2^2\right)\left(c_1^2\right)-\left(d_1c_1\right)^2 \tag{1.3.14}$$

$$\implies a_2 = d_2 \tag{1.3.15}$$

Now we have,

$$\left(\mathbf{D}-\mathbf{A}\right) = \begin{pmatrix} d_1-a_1 \\ d_2-a_2 \end{pmatrix} = \begin{pmatrix} d_1-a_1 \\ 0 \end{pmatrix} \tag{1.3.16}$$

$$\left(\mathbf{C}-\mathbf{B}\right) = \begin{pmatrix} c_1-0 \\ 0-0 \end{pmatrix} = \begin{pmatrix} c_1 \\ 0 \end{pmatrix} \tag{1.3.17}$$

From equations (1.3.16) and (1.3.17), we can say

$$\left(\mathbf{D}-\mathbf{A}\right) = \frac{d_1-a_1}{c_1}\left(\mathbf{C}-\mathbf{B}\right) \tag{1.3.18}$$

$$\implies \left(\mathbf{D}-\mathbf{A}\right) = k\left(\mathbf{C}-\mathbf{B}\right) \tag{1.3.19}$$

where k is a constant. From the equation (1.3.19), we can say that $AD \parallel BC$. Hence the two triangles $\triangle ABC$ and $\triangle DBC$ lie between the same parallels AD and BC

1.4. In $\triangle ABC, D, E$ and F are respectively the mid-points of sides AB, BC and CA. Show that $\triangle ABC$ is divided into four congruent triangles by joining D, E and F.

Solution: Given : $\triangle ABC$, **D**,**E** and **F** are the midpoints of AB,BC and CA respectively

$$\mathbf{D} = \frac{A+B}{2} \tag{1.4.1}$$

$$\mathbf{E} = \frac{B+C}{2} \tag{1.4.2}$$

$$\mathbf{F} = \frac{A+C}{2} \tag{1.4.3}$$

The direction vector $\mathbf{m}_{DF}$ is given by ,

$$\mathbf{m}_{DF} = \mathbf{D}-\mathbf{F} \tag{1.4.4}$$

Since **D** is the mid-point of AB and **F** is the mid-point of AC,from equation 1.4.4,

$$\mathbf{m}_{DF} = \frac{\mathbf{A}+\mathbf{B}}{2}-\frac{\mathbf{A}+\mathbf{C}}{2} \tag{1.4.5}$$

$$\mathbf{m}_{DF} = \frac{\mathbf{B}-\mathbf{C}}{2} \tag{1.4.6}$$

$$\mathbf{m}_{DF} = \frac{\mathbf{m}_{BC}}{2} \tag{1.4.7}$$

where $\mathbf{B}-\mathbf{C}$ is direction vector of line segment

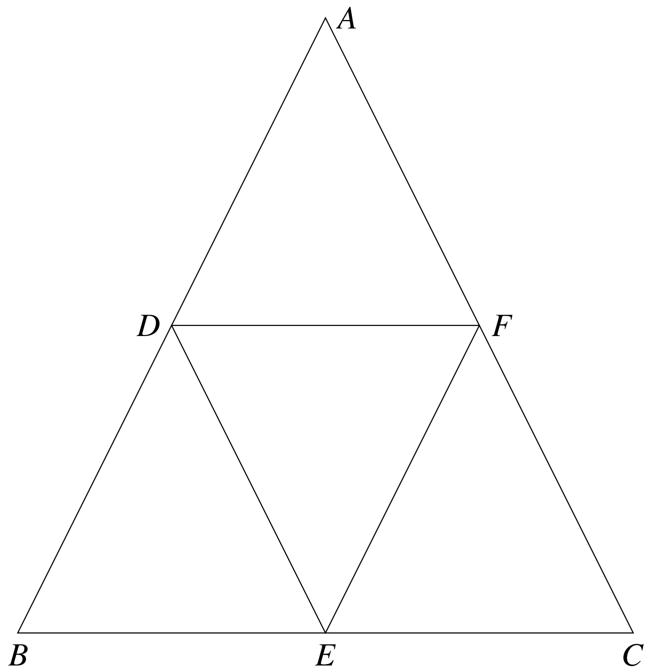


Fig. 1.4.1: Right Angled Triangle by Latex-Tikz

BC From equation 1.4.6 we could say that,

$$DF \parallel BC \tag{1.4.8}$$

Similarly we could show that ,

$$DE \parallel AC \tag{1.4.9}$$

$$EF \parallel AB \tag{1.4.10}$$

Since given **E** is the mid-point of BC,

$$\mathbf{m}_{BE} = \mathbf{m}_{EC} = \frac{\mathbf{m}_{BC}}{2} \tag{1.4.11}$$

$$\mathbf{m}_{BC} = 2\mathbf{m}_{BE} \tag{1.4.12}$$

Substituting equation 1.4.12 in equation 1.4.7,we get,

$$\mathbf{m}_{DF} = \mathbf{m}_{BE} \tag{1.4.13}$$

From equation 1.4.13,since the opposite sides are equal and parallel (DF=BE and DF$\parallel$ BE),we could say that BDFE is parallelogram From Fig.1.4.1, Consider parallelogram BDFE, where DE is the diagonal of the parallelogram BDFE as shown in Fig.1.4.2
Since BDFE is a parallelogram,

$$EF = BD \tag{1.4.14}$$

From the above equations 1.4.13,1.4.14 and

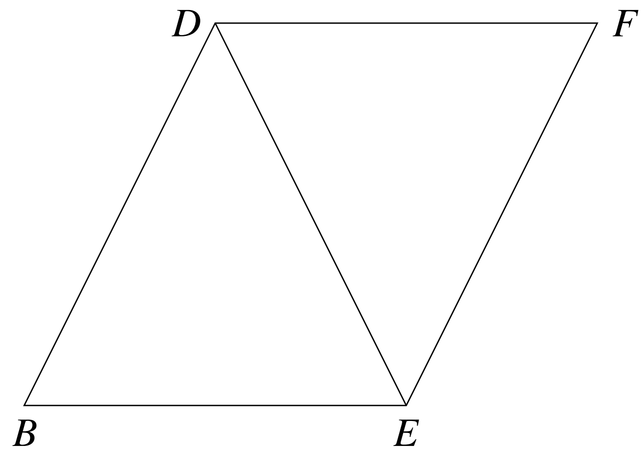


Fig. 1.4.2: Parallelogram by Latex-Tikz

DE is common side to both the $\triangle BDE$ and $\triangle DEF$, by Side-Side-Side (SSS) rule, if all the three sides of one triangle are equivalent to the corresponding three sides of the second triangle, then the two triangles are said to be congruent.

$$\triangle DBE \cong \triangle DEF \tag{1.4.15}$$

Similarly,

$$\triangle ADF \cong \triangle DEF \tag{1.4.16}$$

$$\triangle CEF \cong \triangle DEF \tag{1.4.17}$$

From equations 1.4.15,1.4.16 and 1.4.17,we could conclude that,

$$\triangle DBE \cong \triangle ADF \cong \triangle CEF \cong \triangle DEF \tag{1.4.18}$$

From equation 1.4.18 we could say that all four triangles are congruent which is obtained by joining the midpoints of the $\triangle ABC$.

1.5. The line-segment joining the mid-points of any two sides of a triangle is parallel to the third side and is half of it.
Solution: Let us consider a $\triangle ABC$, and let **D**,**E** and **F** be the mid- points of sides AB,BC and CA respectively.
Let us consider a line-segment joining the points **D** and **F** which are midpoints of line AB and CA.

As **D** is midpoint of line AB, **E** is midpoint of line BC and **F** is midpoint of line CA, they

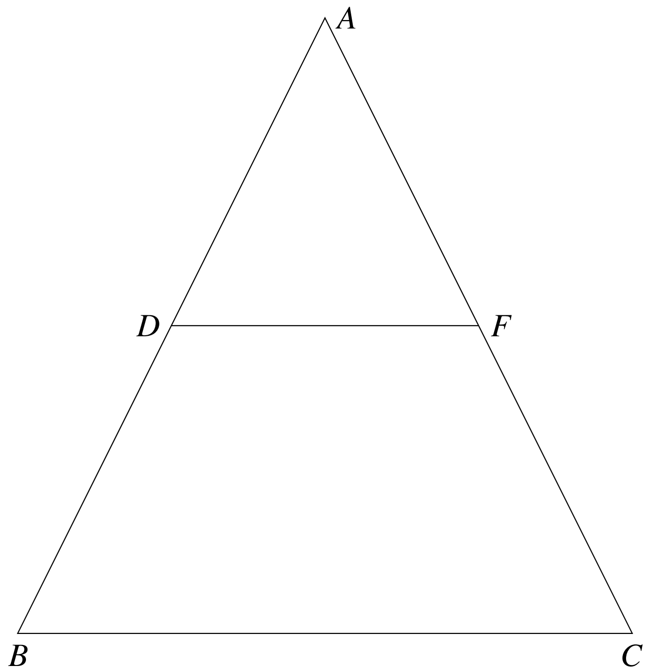


Fig. 1.5.1: Line segment DF joining mid-points of 2 sides of $\triangle ABC$

can be written as follows:

$$\mathbf{D} = \frac{A+B}{2} \tag{1.5.1}$$

$$\mathbf{F} = \frac{C+A}{2} \tag{1.5.2}$$

The line DF can be written in the form of direction vector as,

$$\begin{aligned}\mathbf{m}_{DF} &= \mathbf{D} - \mathbf{F} \\ &= \frac{A+B}{2} - \frac{C+A}{2} \\ &= \frac{\mathbf{B}-\mathbf{C}}{2} \\ &= \frac{\mathbf{m}_{BC}}{2}\end{aligned} \tag{1.5.3}$$

where $\mathbf{m}_{BC}$ is the direction vector of line BC. Consider equation (1.5.3),

$$\implies \mathbf{D} - \mathbf{F} = \frac{\mathbf{B}-\mathbf{C}}{2} \tag{1.5.4}$$

Applying norm on both sides of equation (1.5.4), we get

$$\|\mathbf{D} - \mathbf{F}\| = \frac{1}{2}\|\mathbf{B} - \mathbf{C}\| \tag{1.5.5}$$

From equation(1.5.3), **DF** $\parallel$ **BC** and from (1.5.5), the line-segment $\mathbf{DF} = \frac{1}{2}\mathbf{BC}$

1.6. A line through the mid-point of a side of a triangle parallel to another side bisects the third side.

Solution:

Given: Consider a $\triangle ABC$ with sides AB, BC, AC. Let **D** be the mid-point of AB, **E** be a point on AC and $DE \parallel BC$

Need to prove: $\mathbf{E} = \frac{A+C}{2}$

Proof:

$$\mathbf{D} = \frac{A+B}{2} \tag{1.6.1}$$

$$DE \parallel BC \tag{1.6.2}$$

Let direction vectors be $\mathbf{m}_{DE}$ and $\mathbf{m}_{BC}$ for line segment DE and BC respectively.

$$\mathbf{m}_{DE} = \mathbf{D} - \mathbf{E} = \frac{\mathbf{A}+\mathbf{B}}{2} - \mathbf{E} \tag{1.6.3}$$

$$\mathbf{m}_{BC} = \mathbf{B} - \mathbf{C} \tag{1.6.4}$$

From (1.6.2) we can write the following with **k** being a real value,

$$\mathbf{m}_{DE} = k\mathbf{m}_{BC} \tag{1.6.5}$$

Using (1.6.3) and (1.6.4) in the above equation,

$$\frac{\mathbf{A}+\mathbf{B}}{2} - \mathbf{E} = k(\mathbf{B} - \mathbf{C}) \tag{1.6.6}$$

Let $\mathbf{E} = \frac{m\mathbf{A}+C}{m+1}$ and Substitute **E** in (1.6.5)

$$\frac{\mathbf{A}+\mathbf{B}}{2} - \frac{m\mathbf{A}+C}{m+1} = k(\mathbf{B} - \mathbf{C}) \tag{1.6.7}$$

$$\left(\frac{1}{2} - \frac{m}{m+1}\right)\mathbf{A} + \left(\frac{1}{2} - k\right)\mathbf{B} + \left(k - \frac{1}{m+1}\right)\mathbf{C} = 0 \tag{1.6.8}$$

Since **A**, **B** and **C** are points on a triangle and hence they are Linearly dependent which implies :

$\left(\frac{1}{2} - \frac{m}{m+1}\right)$=0 and $\left(\frac{1}{2} - k\right) = 0$ and $\left(k - \frac{1}{m+1}\right)$ =0

Therefore we get k = $\frac{1}{2}$ and m = 1. Substituting m value in **E** we get,

$$\mathbf{E} = \frac{\mathbf{A}+\mathbf{C}}{2} \tag{1.6.9}$$

Hence Proved

1.7. ABC is a triangle right angled at C. A line through the mid-point M of hypotenuse AB and parallel to BC intersects AC at D. Show that

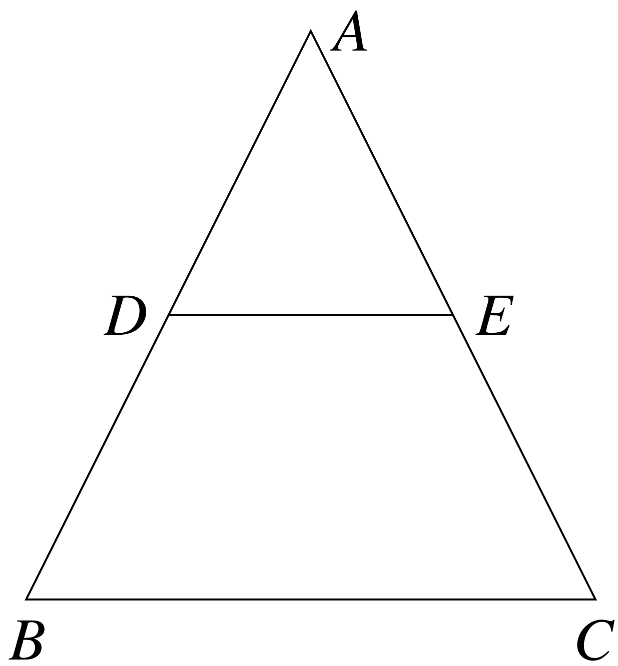


Fig. 1.6.1: Triangle

(i) D is the mid-point of AC (ii) $MD \perp AC$ (iii) $CM = MA = \frac{1}{2}AB$

Solution: In $\triangle ABC$, $\mathbf{M}$ is midpoint of AB and

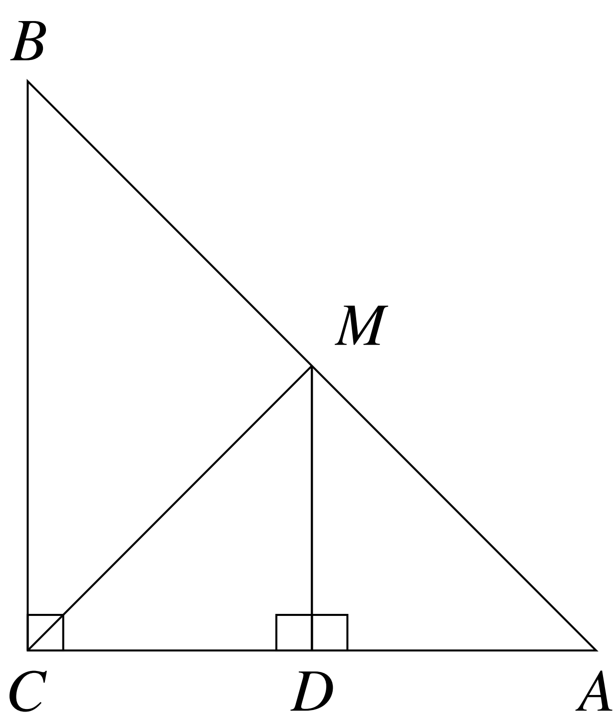


Fig. 1.7.1: Right Angled Triangle by Latex-Tikz

MD is parallel to BC, hence,

$$\mathbf{M} = \frac{\mathbf{A}+\mathbf{B}}{2} \tag{1.7.1}$$

$$MD \parallel BC \tag{1.7.2}$$

Let $\mathbf{m_{MD}}$ and $\mathbf{m_{BC}}$ are direction vectors of MD and BC respectively. Then,

$$\mathbf{m}_{MD} = \mathbf{M} - \mathbf{D} = \frac{\mathbf{A}+\mathbf{B}}{2} - \mathbf{D} \tag{1.7.3}$$

$$\mathbf{m}_{BC} = \mathbf{B} - \mathbf{C} \tag{1.7.4}$$

Now from (1.7.2) we get,

$$\mathbf{m}_{MD} = k\mathbf{m}_{BC} \tag{1.7.5}$$

$$\implies \frac{\mathbf{A}+\mathbf{B}}{2} - \mathbf{D} = k(\mathbf{B} - \mathbf{C}) \tag{1.7.6}$$

Let $\mathbf{D} = \frac{m\mathbf{A}+\mathbf{C}}{m+1}$, then from (1.7.6) we get,

$$\frac{\mathbf{A}+\mathbf{B}}{2} - \frac{m\mathbf{A}+\mathbf{C}}{m+1} = k(\mathbf{B} - \mathbf{C}) \tag{1.7.7}$$

$$\implies \left(\frac{1}{2} - \frac{m}{m+1}\right)\mathbf{A} + \left(\frac{1}{2} - k\right)\mathbf{B} + \left(k - \frac{1}{m+1}\right)\mathbf{C} = 0 \tag{1.7.8}$$

Since $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are linearly dependent as they form $\triangle ABC$ then

$$\frac{1}{2} - \frac{m}{m+1} = 0 \tag{1.7.9}$$

$$\frac{1}{2} - k = 0 \tag{1.7.10}$$

$$k - \frac{1}{m+1} = 0 \tag{1.7.11}$$

Solving (1.7.9), (1.7.10) and (1.7.11) we get $k = \frac{1}{2}$ and $m = 1$. Hence, substituting value of m in $\mathbf{D}$ we get,

$$\mathbf{D} = \frac{\mathbf{A}+\mathbf{C}}{2} \tag{1.7.12}$$

Hence Proved.
From figure 1.7.1,

$$(\mathbf{M} - \mathbf{D})(\mathbf{A} - \mathbf{D}) = \left(\frac{\mathbf{A}+\mathbf{B}}{2} - \frac{\mathbf{A}+\mathbf{C}}{2}\right)(\mathbf{A} - \mathbf{C}) \tag{1.7.13}$$

$$\implies (\mathbf{M} - \mathbf{D})(\mathbf{A} - \mathbf{D}) = \frac{1}{2}(\mathbf{B} - \mathbf{C})(\mathbf{A} - \mathbf{C}) \tag{1.7.14}$$

$$\implies (\mathbf{M} - \mathbf{D})(\mathbf{A} - \mathbf{D}) = 0 \quad \because BC \perp AC \tag{1.7.15}$$

From (1.7.15), it is proved that MD⊥AC
Again we get,

$$\mathbf{C}-\mathbf{M}=\mathbf{C}-\mathbf{D}+\mathbf{D}-\mathbf{M} \tag{1.7.16}$$

$$\implies \mathbf{C}-\mathbf{M}=\mathbf{A}-\mathbf{D}+\mathbf{D}-\mathbf{M} \quad \text{[From (1.7.12)]} \tag{1.7.17}$$

$$\implies \mathbf{C}-\mathbf{M}=\mathbf{A}-\mathbf{M} \tag{1.7.18}$$

$$\implies \mathbf{C}-\mathbf{M}=\mathbf{A}-\frac{\mathbf{A}+\mathbf{B}}{\mathbf{2}} \quad \text{[From (1.7.1)]} \tag{1.7.19}$$

$$\implies \mathbf{C}-\mathbf{M}=\frac{1}{2}(\mathbf{A}-\mathbf{B}) \tag{1.7.20}$$

$$\implies \|\mathbf{C}-\mathbf{M}\|=\frac{1}{2}\|\mathbf{A}-\mathbf{B}\| \tag{1.7.21}$$

Hence from (1.7.18) and (1.7.21) proved,
CM = MA = $\frac{1}{2}$ AB

1.8. Sides opposite to equal angles of a triangle are equal.
Solution: Given that two angles of traingle are equal,

$$\angle ABC=\angle ACB \tag{1.8.1}$$

$$\cos\angle ABC=\cos\angle ACB \tag{1.8.2}$$

$$\frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{C})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{B}-\mathbf{C}\|}=\frac{(\mathbf{C}-\mathbf{A})^T(\mathbf{C}-\mathbf{B})}{\|\mathbf{C}-\mathbf{A}\|\,\|\mathbf{C}-\mathbf{B}\|} \tag{1.8.3}$$

$$\frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{C})}{\|\mathbf{B}-\mathbf{A}\|}=\frac{(\mathbf{C}-\mathbf{A})^T(\mathbf{C}-\mathbf{B})}{\|\mathbf{C}-\mathbf{A}\|} \tag{1.8.4}$$

It can be showed that,

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{C})=\|\mathbf{A}-\mathbf{B}\|^2-(\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{B}) \tag{1.8.5}$$

$$(\mathbf{C}-\mathbf{A})^T(\mathbf{C}-\mathbf{B})=\|\mathbf{A}-\mathbf{C}\|^2-(\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{B}) \tag{1.8.6}$$

Substituting (1.8.5) and (1.8.6) in (1.8.4),

$$\frac{\|\mathbf{A}-\mathbf{B}\|^2-(\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{B})}{\|\mathbf{B}-\mathbf{A}\|}=\frac{\|\mathbf{A}-\mathbf{C}\|^2-(\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{B})}{\|\mathbf{C}-\mathbf{A}\|} \tag{1.8.7}$$

$$\|\mathbf{A}-\mathbf{B}\|-\frac{(\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{B})}{\|\mathbf{B}-\mathbf{A}\|}=\|\mathbf{A}-\mathbf{C}\|-\frac{(\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{B})}{\|\mathbf{C}-\mathbf{A}\|} \tag{1.8.8}$$

$$\|\mathbf{A}-\mathbf{B}\|-\|\mathbf{A}-\mathbf{C}\|\cos\angle BAC=\|\mathbf{A}-\mathbf{C}\|-\|\mathbf{A}-\mathbf{B}\|\cos\angle CAB \tag{1.8.9}$$

$$\|\mathbf{A}-\mathbf{B}\|+\|\mathbf{A}-\mathbf{B}\|\cos\angle CAB=\|\mathbf{A}-\mathbf{C}\|+\|\mathbf{A}-\mathbf{C}\|\cos\angle BAC \tag{1.8.10}$$

$$\|\mathbf{A}-\mathbf{B}\|(1+\cos\angle CAB)=\|\mathbf{A}-\mathbf{C}\|(1+\cos\angle BAC) \tag{1.8.11}$$

$$\implies \|\mathbf{A}-\mathbf{B}\|=\|\mathbf{A}-\mathbf{C}\| \tag{1.8.12}$$

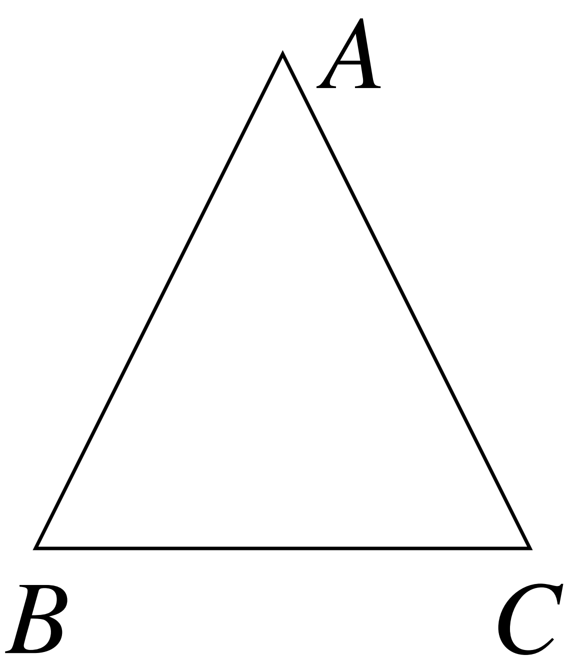


Fig. 1.8.1: $\triangle ABC$

1.9. AB is a line-segment. P and Q are points on opposite sides of AB such that each of them is equidistant from the points A and B. Show that the line PQ is the perpendicular bisector of AB.
Solution: In order to prove that line PQ is the perpendicular bisector of AB, two conditions need to be met:

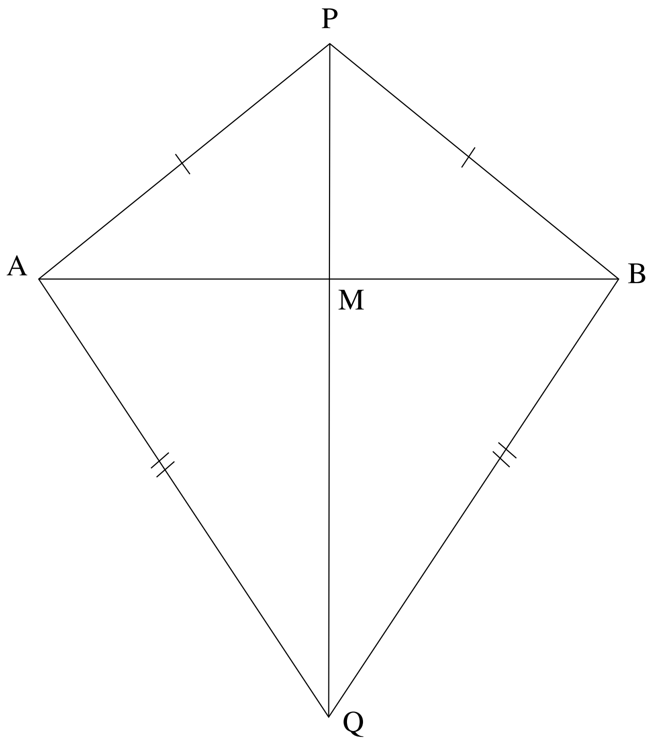
a) $PQ \perp AB$
b) $AM=BM$

These conditions can be proved as follow: It is given that the points **P** and **Q** are equidistant from the points **A** and **B**. Thus we can write:

$$\|\mathbf{P}-\mathbf{A}\|=\|\mathbf{P}-\mathbf{B}\| \tag{1.9.1}$$

$$\|\mathbf{Q}-\mathbf{A}\|=\|\mathbf{Q}-\mathbf{B}\| \tag{1.9.2}$$

Squaring both sides of equations 1.9.1 and



expanding further, we can write:

$$(\mathbf{P}-\mathbf{A})^T(\mathbf{P}-\mathbf{A}) = (\mathbf{P}-\mathbf{B})^T(\mathbf{P}-\mathbf{B}) \quad (1.9.3)$$

$$\mathbf{P}^T\mathbf{P}-\mathbf{P}^T\mathbf{A}-\mathbf{A}^T\mathbf{P}+\mathbf{A}^T\mathbf{A} = \mathbf{P}^T\mathbf{P}-\mathbf{P}^T\mathbf{B}-\mathbf{B}^T\mathbf{P}+\mathbf{B}^T\mathbf{B} \quad (1.9.4)$$

$$\therefore \mathbf{A}^T\mathbf{A}-\mathbf{B}^T\mathbf{B} = -2\mathbf{P}^T\mathbf{B}+2\mathbf{P}^T\mathbf{A} \quad (1.9.5)$$

Similarly, Squaring both sides of equations 1.9.2 and expanding further, we can write:

$$(\mathbf{Q}-\mathbf{A})^T(\mathbf{Q}-\mathbf{A}) = (\mathbf{Q}-\mathbf{B})^T(\mathbf{Q}-\mathbf{B}) \quad (1.9.6)$$

$$\mathbf{Q}^T\mathbf{Q}-\mathbf{Q}^T\mathbf{A}-\mathbf{A}^T\mathbf{Q}+\mathbf{A}^T\mathbf{A} = \mathbf{Q}^T\mathbf{Q}-\mathbf{Q}^T\mathbf{B}-\mathbf{B}^T\mathbf{Q}+\mathbf{B}^T\mathbf{B} \quad (1.9.7)$$

$$\therefore \mathbf{A}^T\mathbf{A}-\mathbf{B}^T\mathbf{B} = -2\mathbf{Q}^T\mathbf{B}+2\mathbf{Q}^T\mathbf{A} \quad (1.9.8)$$

From equations 1.9.5 and 1.9.8, we can write:

$$2\mathbf{P}^T(\mathbf{A}-\mathbf{B}) = 2\mathbf{Q}^T(\mathbf{A}-\mathbf{B}) \quad (1.9.9)$$

$$\mathbf{P}^T(\mathbf{A}-\mathbf{B})-\mathbf{Q}^T(\mathbf{A}-\mathbf{B}) = 0 \quad (1.9.10)$$

$$(\mathbf{P}-\mathbf{Q})^T(\mathbf{A}-\mathbf{B}) = 0 \quad (1.9.11)$$

Thus, Segment PQ is perpendicular to segment AB ($PQ \perp AB$).

From the figure, equations 1.9.1 can also be written as:

$$\|(\mathbf{P}-\mathbf{M})+(\mathbf{M}-\mathbf{A})\| = \|(\mathbf{P}-\mathbf{M})+(\mathbf{M}-\mathbf{B})\| \quad (1.9.12)$$

Squaring and expanding both the sides, we get:

$$((\mathbf{P}-\mathbf{M})+(\mathbf{M}-\mathbf{A}))^T((\mathbf{P}-\mathbf{M})+(\mathbf{M}-\mathbf{A})) = ((\mathbf{P}-\mathbf{M})+(\mathbf{M}-\mathbf{B}))^T((\mathbf{P}-\mathbf{M})+(\mathbf{M}-\mathbf{B})) \quad (1.9.13)$$

$$(\mathbf{P}-\mathbf{M})^T(\mathbf{P}-\mathbf{M})+(\mathbf{P}-\mathbf{M})^T(\mathbf{M}-\mathbf{A})+(\mathbf{M}-\mathbf{A})^T(\mathbf{P}-\mathbf{M})+(\mathbf{M}-\mathbf{A})^T(\mathbf{M}-\mathbf{A}) = (\mathbf{P}-\mathbf{M})^T(\mathbf{P}-\mathbf{M})+(\mathbf{P}-\mathbf{M})^T(\mathbf{M}-\mathbf{B})+(\mathbf{M}-\mathbf{B})^T(\mathbf{P}-\mathbf{M})+(\mathbf{M}-\mathbf{B})^T(\mathbf{M}-\mathbf{B}) \quad (1.9.14)$$

$$\|(\mathbf{M}-\mathbf{A})\|^2+2(\mathbf{M}-\mathbf{A})^T(\mathbf{P}-\mathbf{M}) = \|(\mathbf{M}-\mathbf{B})\|^2+2(\mathbf{M}-\mathbf{B})^T(\mathbf{P}-\mathbf{M}) \quad (1.9.15)$$

Since, $PQ \perp AB$. Hence, we can write:

$$(\mathbf{M}-\mathbf{A})^T(\mathbf{P}-\mathbf{M}) = (\mathbf{M}-\mathbf{B})^T(\mathbf{P}-\mathbf{M}) = 0 \quad (1.9.16)$$

From equation 1.9.15 and 1.9.16, we get:

$$\|(\mathbf{M}-\mathbf{A})\| = \|(\mathbf{M}-\mathbf{B})\| \quad (1.9.17)$$

Thus, M is the midpoint of segment AB ($AM = BM$). Thus, Segment PQ is perpendicular bisector of segment AB.

1.10. Show that $\cos 60° = \frac{1}{2}$.

Solution: Consider an equilateral $\triangle ABC$ as shown in below figure:
In equilateral triangle all sides have equal length

$$\implies \|\mathbf{A}-\mathbf{B}\| = \|\mathbf{B}-\mathbf{C}\| = \|\mathbf{A}-\mathbf{C}\| = k \quad (1.10.1)$$

Let $\mathbf{B} = 0$. Then substituting in (1.10.1) will give

$$\|\mathbf{A}\| = \|\mathbf{C}\| \quad (1.10.2)$$

$$\|\mathbf{A}\| = \|\mathbf{A}-\mathbf{C}\| \quad (1.10.3)$$

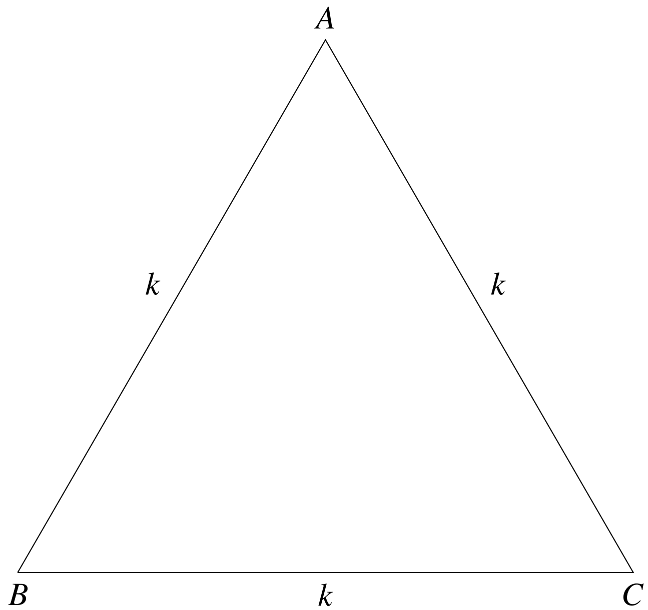


Fig. 1.10.1: Equilateral $\triangle ABC$

Taking square on both sides in (1.10.3).

$$\implies \|\mathbf{A}\|^2 = \|\mathbf{A}-\mathbf{C}\|^2 \tag{1.10.4}$$

$$\implies \|\mathbf{A}\|^2 = \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T\mathbf{C} \tag{1.10.5}$$

$$\implies \|\mathbf{A}\|^2 = \|\mathbf{A}\|^2 + \|\mathbf{A}\|^2 - 2\mathbf{A}^T\mathbf{C} \tag{1.10.6}$$

$$\implies 0 = \|\mathbf{A}\|^2 - 2\mathbf{A}^T\mathbf{C} \tag{1.10.7}$$

$$\implies 2\mathbf{A}^T\mathbf{C} = \|\mathbf{A}\|^2 \tag{1.10.8}$$

$$\implies \mathbf{A}^T\mathbf{C} = \frac{\|\mathbf{A}\|^2}{2} \tag{1.10.9}$$

let $\theta = \angle ABC$.
Taking the inner product of sides AB and BC.

$$(\mathbf{A}-\mathbf{B})^T(\mathbf{B}-\mathbf{C}) = \|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{B}-\mathbf{C}\|\cos\theta \tag{1.10.10}$$

$$\implies \cos\theta = \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{B}-\mathbf{C})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{1.10.11}$$

Substitute $\mathbf{B} = 0$ in (1.10.11)

$$\implies \cos\theta = \frac{\mathbf{A}^T\mathbf{C}}{\|\mathbf{A}\|\,\|\mathbf{C}\|} \tag{1.10.12}$$

Substitute (1.10.2),(1.10.9) in (1.10.12)

$$\implies \cos\theta = \frac{\frac{\|\mathbf{A}\|^2}{2}}{\|\mathbf{A}\|^2} \tag{1.10.13}$$

$$\implies \cos\theta = \frac{1}{2} \tag{1.10.14}$$

In equilateral triangle, $\angle ABC = 60°$

$$\implies \cos 60° = \frac{1}{2} \tag{1.10.15}$$

Hence proved.

1.11. Show that $\sin 60° = \frac{\sqrt{3}}{2}$.

Solution: Consider an equilateral triangle **ABC**. Since, $\triangle$**ABC** is an equilateral, all of its angles are 60°. Now, The direction vector of all the sides are given as,

$$\mathbf{AB} = \|\mathbf{A}-\mathbf{B}\| \tag{1.11.1}$$

$$\mathbf{BC} = \|\mathbf{B}-\mathbf{C}\| \tag{1.11.2}$$

$$\mathbf{AC} = \|\mathbf{A}-\mathbf{C}\| \tag{1.11.3}$$

Now for an equilateral triangle,

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{B}-\mathbf{C}\| = \|\mathbf{A}-\mathbf{C}\| \tag{1.11.4}$$

Let, **B** be the origin.Hence, $\mathbf{B} = 0$. Hence substituting in the equation (1.11.4) we get,

$$\|\mathbf{A}\| = \|\mathbf{C}\| = \|\mathbf{A}-\mathbf{C}\| \tag{1.11.5}$$

Squaring $\|\mathbf{A}-\mathbf{C}\|$ we get,

$$\|\mathbf{A}-\mathbf{C}\|^2 = \|\mathbf{A}\|^2 + \|\mathbf{C}\|^2 - 2\mathbf{A}^T\mathbf{C} \tag{1.11.6}$$

Substituting from equation (1.11.5) in above equation,

$$\implies \|\mathbf{A}\|^2 = 2\,\|\mathbf{A}\|^2 - 2\mathbf{A}^T\mathbf{C} \tag{1.11.7}$$

$$\implies \|\mathbf{A}\|^2 = 2\mathbf{A}^T\mathbf{C} \tag{1.11.8}$$

In figure 1.11.1, taking inner products of side **AB** and **BC** we get,

$$(\mathbf{AB})^T\mathbf{BC} = \|\mathbf{AB}\|\,\|\mathbf{BC}\|\cos\theta \tag{1.11.9}$$

Substituting these results in (1.11.9) and solving for $\cos\theta$ we get,

$$\cos\theta = \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{B}-\mathbf{C})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{1.11.10}$$

$$\implies \cos\theta = \frac{\mathbf{A}^T\mathbf{C}}{\|\mathbf{A}\|\,\|\mathbf{C}\|} \tag{1.11.11}$$

Imposing the results of (1.11.8) in (1.11.11) we

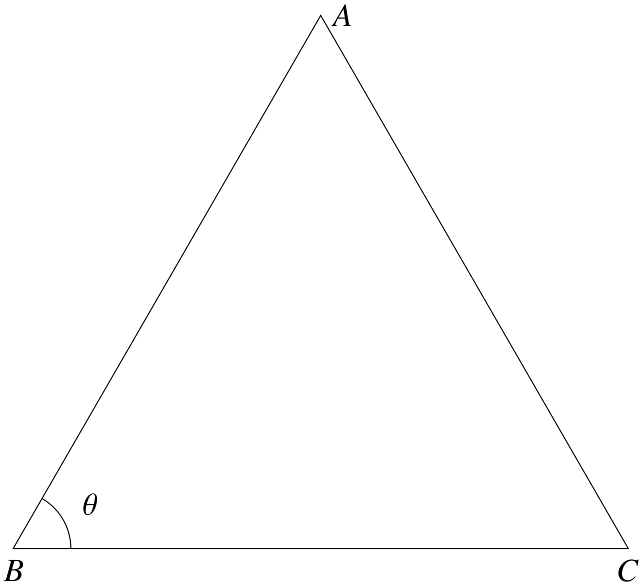


Fig. 1.11.1: Equilateral Triangle

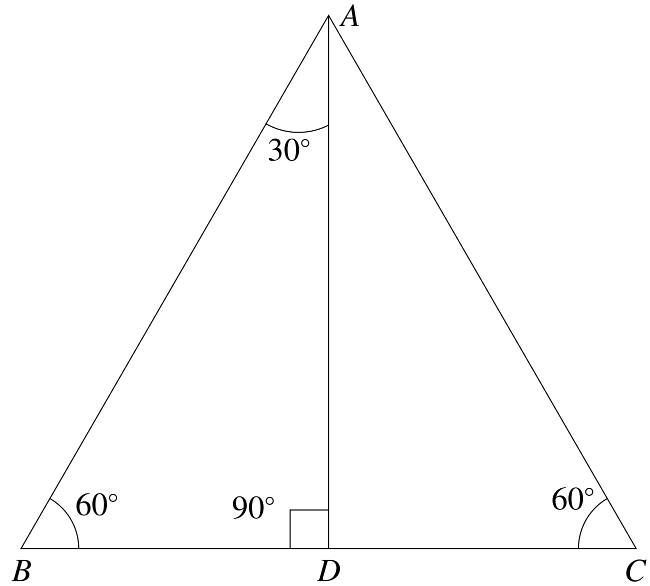


Fig. 1.12.1: Equilateral $\triangle ABC$

get,

$$\implies \cos\theta = \frac{\mathbf{A}^T\mathbf{C}}{2\mathbf{A}^T\mathbf{C}} \tag{1.11.12}$$

$$\implies \cos\theta = \frac{1}{2} \tag{1.11.13}$$

$\therefore \theta = 60°$

$$\therefore \cos 60° = \frac{1}{2} \tag{1.11.14}$$

Now using the property,

$$\cos^2\theta + \sin^2\theta = 1 \tag{1.11.15}$$

$\therefore$ at $\theta = 60°$,

$$\implies \sin 60° = \sqrt{1 - \cos^2 60°} \tag{1.11.16}$$

$$\implies \sin 60° = \frac{\sqrt{3}}{2}. \tag{1.11.17}$$

1.12. Find $\sin 30°$ and $\cos 30°$.
Solution: Consider an equilateral $\triangle ABC$ as shown in figure: 1.12.1. Let the angle bisector of $\angle$A intersect side **BC** at a point D between B and C.

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{B}-\mathbf{C}\| = \|\mathbf{A}-\mathbf{C}\| \tag{1.12.1}$$

$$\angle ABD = \angle BAC = \angle ACD = 60° \tag{1.12.2}$$

$$\angle BAD = \angle CAD = 30° \tag{1.12.3}$$

Using angle bisector theorem ie. triangle will divide the opposite side into two segments that are proportional to the other two sides of the triangle.

$$\frac{\|\mathbf{A}-\mathbf{B}\|}{\|\mathbf{B}-\mathbf{D}\|} = \frac{\|\mathbf{A}-\mathbf{C}\|}{\|\mathbf{B}-\mathbf{C}\| - \|\mathbf{B}-\mathbf{D}\|} \tag{1.12.4}$$

Using (1.12.1)

$$\frac{\|\mathbf{A}-\mathbf{B}\|}{\|\mathbf{B}-\mathbf{D}\|} = \frac{\|\mathbf{A}-\mathbf{B}\|}{\|\mathbf{A}-\mathbf{B}\| - \|\mathbf{B}-\mathbf{D}\|} \tag{1.12.5}$$

$$\implies \|\mathbf{A}-\mathbf{B}\| = 2\|\mathbf{B}-\mathbf{D}\| \tag{1.12.6}$$

Taking the inner product of sides BA and AD.

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{D}) = \|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{D}\|\cos\theta \tag{1.12.7}$$

$$\cos\angle BAD = \frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{D})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{D}\|} \tag{1.12.8}$$

Now To Find AD.

$$\begin{aligned}
&(\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{A}) \\
&= (\mathbf{B}-\mathbf{D}+\mathbf{D}-\mathbf{A})^T(\mathbf{B}-\mathbf{D}+\mathbf{D}-\mathbf{A}) \\
&= [(\mathbf{B}-\mathbf{D})^T + (\mathbf{D}-\mathbf{A})^T][(\mathbf{B}-\mathbf{D}) + (\mathbf{D}-\mathbf{A})] \\
&= (\mathbf{B}-\mathbf{D})^T(\mathbf{B}-\mathbf{D}) + (\mathbf{B}-\mathbf{D})^T(\mathbf{D}-\mathbf{A}) + \\
&\quad (\mathbf{D}-\mathbf{A})^T(\mathbf{B}-\mathbf{D}) + (\mathbf{D}-\mathbf{A})^T(\mathbf{D}-\mathbf{A})
\end{aligned} \tag{1.12.9}$$

$$\because \triangle ABD = \angle BAD + \angle ADB + \angle DBA \tag{1.12.10}$$

$$180^\circ = 30^\circ + \angle ADB + 60^\circ. \tag{1.12.11}$$

$$\implies \angle ADB = 180^\circ - (60^\circ + 30^\circ) = 90^\circ. \tag{1.12.12}$$

$$\implies (\mathbf{B} - \mathbf{D})^T (\mathbf{D} - \mathbf{A}) = 0 \tag{1.12.13}$$

$$\implies (\mathbf{D} - \mathbf{A})^T (\mathbf{B} - \mathbf{D}) = 0 \tag{1.12.14}$$

which gives

$$\begin{aligned}(\mathbf{B} - \mathbf{A})^T (\mathbf{B} - \mathbf{A}) &= \\ (\mathbf{B} - \mathbf{D})^T (\mathbf{B} - \mathbf{D}) &+ (\mathbf{D} - \mathbf{A})^T (\mathbf{D} - \mathbf{A}) \\ \|\mathbf{B} - \mathbf{A}\|^2 &= \|\mathbf{B} - \mathbf{D}\|^2 + \|\mathbf{D} - \mathbf{A}\|^2 \\ \implies \|\mathbf{D} - \mathbf{A}\|^2 &= \|\mathbf{B} - \mathbf{A}\|^2 - \|\mathbf{B} - \mathbf{D}\|^2\end{aligned} \tag{1.12.15}$$

Using Eq (1.12.6) we get,

$$\|\mathbf{D} - \mathbf{A}\|^2 = \|\mathbf{B} - \mathbf{A}\|^2 - \frac{1}{4}\|\mathbf{B} - \mathbf{A}\|^2 \tag{1.12.16}$$

$$\|\mathbf{D} - \mathbf{A}\| = \frac{\sqrt{3}}{2}\|\mathbf{B} - \mathbf{A}\| \tag{1.12.17}$$

$$\implies \|\mathbf{B} - \mathbf{A}\| = \frac{2}{\sqrt{3}}\|\mathbf{D} - \mathbf{A}\| \tag{1.12.18}$$

Let $\mathbf{A} = 0$. Substituting in (1.12.6) and (1.12.18)

$$\|\mathbf{B}\| = 2\|\mathbf{B} - \mathbf{D}\| \tag{1.12.19}$$

$$\|\mathbf{B}\| = \frac{2}{\sqrt{3}}\|\mathbf{D}\| \tag{1.12.20}$$

Square on both sides in (1.12.19) and (1.12.20), we get,

$$\|\mathbf{B}\|^2 = 4\|\mathbf{B} - \mathbf{D}\|^2 \tag{1.12.21}$$

$$\frac{1}{4}\|\mathbf{B}\|^2 = \|\mathbf{B}\|^2 + \|\mathbf{D}\|^2 - 2\mathbf{B}^T\mathbf{D} \tag{1.12.22}$$

$$\|\mathbf{B}\|^2 = \frac{4}{3}\|\mathbf{D}\|^2 \tag{1.12.23}$$

Solving (1.12.22) and (1.12.23) we get,

$$\frac{1}{3}\|\mathbf{D}\|^2 = \frac{4}{3}\|\mathbf{D}\|^2 + \|\mathbf{D}\|^2 - 2\mathbf{D}^T\mathbf{D} \tag{1.12.24}$$

$$0 = \|\mathbf{D}\|^2 - 2\mathbf{B}^T\mathbf{D} \tag{1.12.25}$$

$$\implies \mathbf{B}^T\mathbf{D} = \|\mathbf{D}\|^2 \tag{1.12.26}$$

Substitute $\mathbf{A} = 0$ in (1.12.8) we get,

$$\cos \angle BAD = \frac{\mathbf{B}^T\mathbf{D}}{\|\mathbf{B}\|\,\|\mathbf{D}\|} \tag{1.12.27}$$

$$\because \|\mathbf{B}\| = \frac{2}{\sqrt{3}}\|\mathbf{D}\| \tag{1.12.28}$$

$$\implies \cos \angle BAD = \frac{\mathbf{B}^T\mathbf{D}}{\frac{2}{\sqrt{3}}\|\mathbf{D}\|\,\|\mathbf{D}\|} = \frac{\mathbf{B}^T\mathbf{D}}{\frac{2}{\sqrt{3}}\|\mathbf{D}\|^2} \tag{1.12.29}$$

Substitute (1.12.26) in (1.12.29)

$$\cos \angle BAD = \frac{\frac{\sqrt{3}}{2}\|\mathbf{D}\|^2}{\|\mathbf{D}\|^2} = \frac{\sqrt{3}}{2} \tag{1.12.30}$$

$$\because \angle BAD = 30^\circ \tag{1.12.31}$$

$$\implies \cos 30^\circ = \frac{\sqrt{3}}{2} \tag{1.12.32}$$

$$\because \cos^2\theta + \sin^2\theta = 1 \tag{1.12.33}$$

$$\sin 30^\circ = \sqrt{1 - \cos^2 30^\circ} \tag{1.12.34}$$

$$\implies \sin 30^\circ = \frac{1}{2}. \tag{1.12.35}$$

1.13. Triangles on the same base (or equal bases) and between the same parallels are equal in area.
Solution: Let ABC and ABD are the given triangles with the same base **AB** and between the same parallel lines **AB** and **CD**.

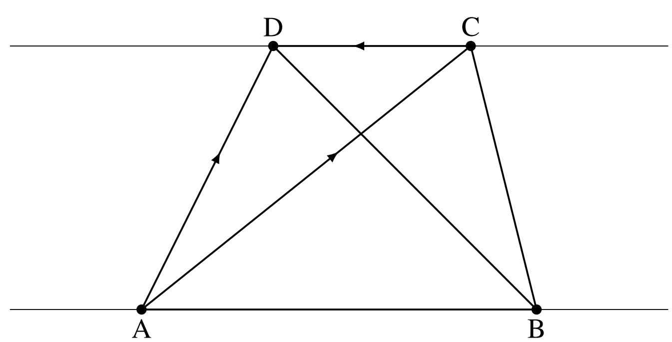


Fig. 1.13.1: Triangles on same base

The area of △ ABC is given by

$$Area(\triangle ABC) = \frac{1}{2}\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} - \mathbf{C})\| \tag{1.13.1}$$

Since $\mathbf{CD} \parallel \mathbf{AB}$,

$$\mathbf{C} - \mathbf{D} = k(\mathbf{A} - \mathbf{B}) \tag{1.13.2}$$

Hence, the area of $\triangle$ ABD is given by

$$\frac{1}{2}\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{D})\| \tag{1.13.3}$$

$$=\frac{1}{2}\|(\mathbf{A}-\mathbf{B})\times\{(\mathbf{A}-\mathbf{C})+(\mathbf{C}-\mathbf{D})\}\| \tag{1.13.4}$$

$$=\frac{1}{2}\|(\mathbf{A}-\mathbf{B})\times\{(\mathbf{A}-\mathbf{C})+k(\mathbf{A}-\mathbf{B})\}\| \tag{1.13.5}$$

$$=\frac{1}{2}\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})+k(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{B})\| \tag{1.13.6}$$

$$=\frac{1}{2}\|(\mathbf{A}-\mathbf{B})\times(\mathbf{A}-\mathbf{C})\|\ [\because \mathbf{A}\times\mathbf{A}=0] \tag{1.13.7}$$

From (1.13.1) and (1.13.7), we can infer that the area of two triangles are one and the same. Hence, it is proved that the triangles on the same base and between the same parallels are equal in area.

1.14. Triangles on the same base (or equal bases) and having equal areas lie between the same parallels.

Solution: Given that,

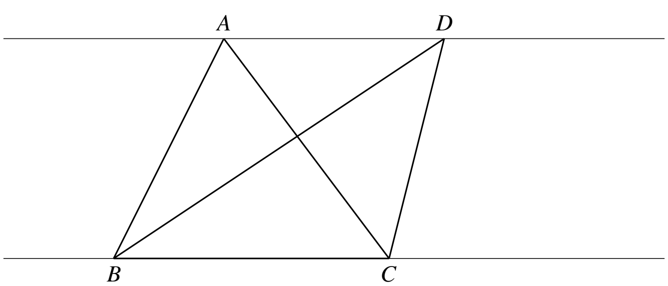


Fig. 1.14.1: $\triangle ABC$ and $\triangle BCD$ having common base BC

$$\text{Area of } \triangle ABC = \text{Area of } \triangle BCD \tag{1.14.1}$$

We know that, area of triangle can be obtained by cross product.

$$\text{Area of } \triangle ABC = \frac{1}{2}\|(\mathbf{B}-\mathbf{A})\times(\mathbf{B}-\mathbf{C})\| \tag{1.14.2}$$

$$=\frac{1}{2}\left\|((\mathbf{B}-\mathbf{D})+(\mathbf{D}-\mathbf{A}))\times(\mathbf{B}-\mathbf{C})\right\| \tag{1.14.3}$$

$$=\text{Area of } \triangle BCD+\frac{1}{2}\|(\mathbf{D}-\mathbf{A})\times(\mathbf{B}-\mathbf{C})\| \tag{1.14.4}$$

From (1.14.1) and (1.14.4) we get,

$$\|(\mathbf{D}-\mathbf{A})\times(\mathbf{B}-\mathbf{C})\|=0 \tag{1.14.5}$$

We know that, two nonzero vectors are **parallel** if and only if their cross product is zero.

$$\implies (\mathbf{A}-\mathbf{D})=k(\mathbf{B}-\mathbf{C}) \tag{1.14.6}$$

Hence proved triangles on the same base and having equal areas lie between the same parallels.

1.15. In $\triangle ABC$, the bisector AD of $\angle A$ is perpendicular to side BC. Show that $AB = AC$ and $\triangle ABC$ is isosceles.

Solution:

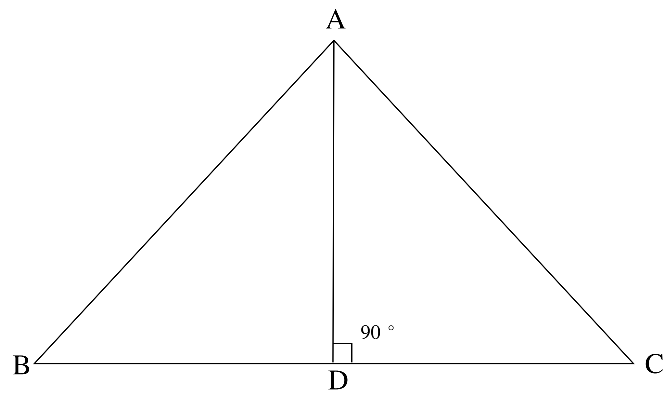


Fig. 1.15.1: Isosceles Triangle with **AD** $\perp$ **BC**

Given, line AD is perpendicular to line BC which implies the inner product is zero

$$(\mathbf{B}-\mathbf{D})^T(\mathbf{A}-\mathbf{D})=(\mathbf{D}-\mathbf{A})^T(\mathbf{B}-\mathbf{D})\quad=0 \tag{1.15.1}$$

$$(\mathbf{C}-\mathbf{D})^T(\mathbf{A}-\mathbf{D})=(\mathbf{D}-\mathbf{A})^T(\mathbf{C}-\mathbf{D})\quad=0 \tag{1.15.2}$$

Consider$\triangle BAD$ and $\triangle CAD$;
Taking inner product of sides BA and AD

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{D})=\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{D}\|\cos BAD \tag{1.15.3}$$

The angle BAD from the above equation is:

$$\cos BAD=\frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{D})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{D}\|} \tag{1.15.4}$$

Taking inner product of sides CA and AD

$$(\mathbf{C}-\mathbf{A})^T(\mathbf{A}-\mathbf{D})=\|\mathbf{C}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{D}\|\cos CAD \tag{1.15.5}$$

The angle CAD from the above equation is:

$$\cos CAD = \frac{(\mathbf{C}-\mathbf{A})^T(\mathbf{A}-\mathbf{D})}{\|\mathbf{C}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{D}\|} \tag{1.15.6}$$

from equation (1.15.4) and (1.15.6)

$$\angle BAD = \angle CAD \tag{1.15.7}$$

Now using pythagorus law;

$$\|\mathbf{B}-\mathbf{A}\|^2 = \|\mathbf{A}-\mathbf{D}\|^2 + \|\mathbf{B}-\mathbf{D}\|^2 \tag{1.15.8}$$

$$\|\mathbf{C}-\mathbf{A}\|^2 = \|\mathbf{A}-\mathbf{D}\|^2 + \|\mathbf{C}-\mathbf{D}\|^2 \tag{1.15.9}$$

using (1.15.1) and (1.15.2) in above equation we can conclude ;

$$\|\mathbf{B}-\mathbf{A}\| = \|\mathbf{C}-\mathbf{A}\| \tag{1.15.10}$$

Thus, $\triangle ABC$ is isosceles triangle.
Hence proved.

1.16. E and F are respectively the mid-points of equal sides AB and AC of $\triangle ABC$. Show that $BF = CE$.
Solution: According to figure 1.16.1

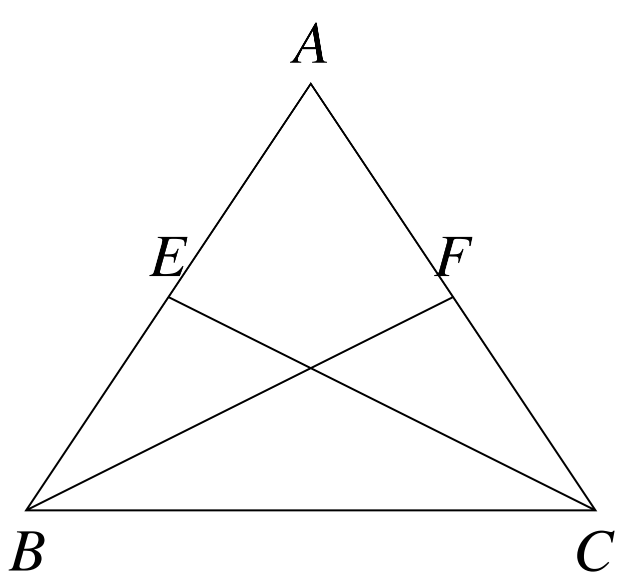


Fig. 1.16.1: Isosceles Triangle with mid-points E and F on equal sides

$$(\mathbf{A}-\mathbf{B}) + (\mathbf{B}-\mathbf{F}) = (\mathbf{A}-\mathbf{F}) \tag{1.16.1}$$

$$\therefore (\mathbf{B}-\mathbf{F}) = (\mathbf{A}-\mathbf{F}) - (\mathbf{A}-\mathbf{B}) \tag{1.16.2}$$

$$\therefore (\mathbf{B}-\mathbf{F}) = \frac{1}{2}(\mathbf{A}-\mathbf{C}) - (\mathbf{A}-\mathbf{B}) \tag{1.16.3}$$

similarly,

$$(\mathbf{A}-\mathbf{C}) + (\mathbf{C}-\mathbf{E}) = (\mathbf{A}-\mathbf{E}) \tag{1.16.4}$$

$$\therefore (\mathbf{C}-\mathbf{E}) = (\mathbf{A}-\mathbf{E}) - (\mathbf{A}-\mathbf{C}) \tag{1.16.5}$$

$$\therefore (\mathbf{C}-\mathbf{E}) = \frac{1}{2}(\mathbf{A}-\mathbf{B}) - (\mathbf{A}-\mathbf{C}) \tag{1.16.6}$$

Since AB = AC
$\therefore (AB)^2 = (AC)^2$

$$\therefore \|(\mathbf{A}-\mathbf{B})\|^2 = \|(\mathbf{A}-\mathbf{C})\|^2 \tag{1.16.7}$$

$$\|(\mathbf{A}-\mathbf{B})\|^2 + \frac{1}{4}\|(\mathbf{A}-\mathbf{C})\|^2 - (\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{B}) = \|(\mathbf{A}-\mathbf{C})\|^2 + \frac{1}{4}\|(\mathbf{A}-\mathbf{B})\|^2 - (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C}) \tag{1.16.8}$$

$$\left(\frac{1}{2}(\mathbf{A}-\mathbf{C}) - (\mathbf{A}-\mathbf{B})\right)^T\left(\frac{1}{2}(\mathbf{A}-\mathbf{C}) - (\mathbf{A}-\mathbf{B})\right) = \left(\frac{1}{2}(\mathbf{A}-\mathbf{B}) - (\mathbf{A}-\mathbf{C})\right)^T\left(\frac{1}{2}(\mathbf{A}-\mathbf{B}) - (\mathbf{A}-\mathbf{C})\right) \tag{1.16.9}$$

$$\|(\mathbf{B}-\mathbf{F})\|^2 = \|(\mathbf{C}-\mathbf{E})\|^2 \tag{1.16.10}$$

$$\therefore \|\mathbf{B}-\mathbf{F}\| = \|\mathbf{C}-\mathbf{E}\| \tag{1.16.11}$$

Hence, BF is equal to CE

1.17. In an isosceles $\triangle ABC$ with AB = AC, D and E are points on BC such that $BE = CD$. Show that $AD = AE$.
Solution:
In the given $\triangle ABC$, let D and E be any arbitrary points on the side BC such that $BE = CD$.
We are given that the sides $AB = AC$, and $BE =$

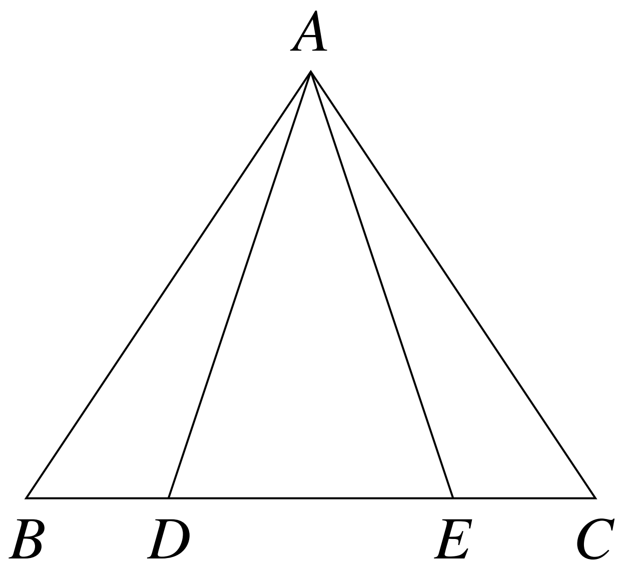


Fig. 1.17.1: Isosceles Triangle with sides AB = AC

CD. These two can be represented as

$$\|\mathbf{B}-\mathbf{A}\| = \|\mathbf{C}-\mathbf{A}\| \tag{1.17.1}$$

$$\|\mathbf{B}-\mathbf{E}\| = \|\mathbf{D}-\mathbf{C}\| \tag{1.17.2}$$

Since the given trianle is an isoceles triangle, the angles formed by AB and AC on BC will be the same. That is

$$\angle ABC = \angle ACB = \alpha \tag{1.17.3}$$

The side AD can be represented as

$$(\mathbf{D}-\mathbf{A}) = (\mathbf{B}-\mathbf{A}) - (\mathbf{B}-\mathbf{D}) \tag{1.17.4}$$

Squaring both the sides we get

$$\|\mathbf{D}-\mathbf{A}\|^2 = \|\mathbf{B}-\mathbf{A}\|^2 + \|\mathbf{B}-\mathbf{D}\|^2 - 2\|\mathbf{B}-\mathbf{A}\|\|\mathbf{B}-\mathbf{D}\|\cos\alpha \tag{1.17.5}$$

The side AE can be represented as

$$(\mathbf{A}-\mathbf{E}) = (\mathbf{C}-\mathbf{A}) - (\mathbf{C}-\mathbf{E}) \tag{1.17.6}$$

Squaring both the sides we get

$$\|\mathbf{A}-\mathbf{E}\|^2 = \|\mathbf{C}-\mathbf{A}\|^2 + \|\mathbf{C}-\mathbf{E}\|^2 - 2\|\mathbf{C}-\mathbf{A}\|\|\mathbf{C}-\mathbf{E}\|\cos\alpha \tag{1.17.7}$$

From (1.17.2), we can further $(\mathbf{E}-\mathbf{B})$ write it as

$$(\mathbf{E}-\mathbf{B}) = (\mathbf{D}-\mathbf{B}) + (\mathbf{E}-\mathbf{D}) \tag{1.17.8}$$

Squaring both the sides we get

$$\|\mathbf{E}-\mathbf{B}\|^2 = \|\mathbf{D}-\mathbf{B}\|^2 + \|\mathbf{E}-\mathbf{D}\|^2 + 2\|\mathbf{D}-\mathbf{B}\|\|\mathbf{E}-\mathbf{D}\|\cos\theta \tag{1.17.9}$$

Since, both the vectors are on the same direction, the angle between them is $\theta = 0°$.

$$\|\mathbf{E}-\mathbf{B}\|^2 = \|\mathbf{D}-\mathbf{B}\|^2 + \|\mathbf{E}-\mathbf{D}\|^2 + 2\|\mathbf{D}-\mathbf{B}\|\|\mathbf{E}-\mathbf{D}\| \tag{1.17.10}$$

$$\|\mathbf{E}-\mathbf{B}\|^2 = (\|\mathbf{D}-\mathbf{B}\| + \|\mathbf{E}-\mathbf{D}\|)^2 \tag{1.17.11}$$

From (1.17.2), we can further $(\mathbf{C}-\mathbf{D})$ write it as

$$(\mathbf{C}-\mathbf{D}) = (\mathbf{C}-\mathbf{E}) + (\mathbf{E}-\mathbf{D}) \tag{1.17.12}$$

Squaring both the sides we get

$$\|\mathbf{C}-\mathbf{D}\|^2 = \|\mathbf{C}-\mathbf{E}\|^2 + \|\mathbf{E}-\mathbf{D}\|^2 + 2\|\mathbf{C}-\mathbf{E}\|\|\mathbf{E}-\mathbf{D}\|\cos\theta \tag{1.17.13}$$

Since, both the vectors are on the same direction, the angle between them is $\theta = 0°$.

$$\|\mathbf{C}-\mathbf{D}\|^2 = \|\mathbf{C}-\mathbf{E}\|^2 + \|\mathbf{E}-\mathbf{D}\|^2 + 2\|\mathbf{C}-\mathbf{E}\|\|\mathbf{E}-\mathbf{D}\| \tag{1.17.14}$$

$$\|\mathbf{C}-\mathbf{D}\|^2 = (\|\mathbf{C}-\mathbf{E}\| + \|\mathbf{E}-\mathbf{D}\|)^2 \tag{1.17.15}$$

From equations (1.17.2), (1.17.11), (1.17.15), we get

$$\begin{aligned}(\|\mathbf{D}-\mathbf{B}\| + \|\mathbf{E}-\mathbf{D}\|)^2 &= (\|\mathbf{C}-\mathbf{E}\| + \|\mathbf{E}-\mathbf{D}\|)^2\\ \implies \|\mathbf{D}-\mathbf{B}\| + \|\mathbf{E}-\mathbf{D}\| &= \|\mathbf{C}-\mathbf{E}\| + \|\mathbf{E}-\mathbf{D}\|\\ \implies \|\mathbf{D}-\mathbf{B}\| &= \|\mathbf{C}-\mathbf{E}\|\end{aligned} \tag{1.17.16}$$

Using equations (1.17.1), (1.17.16), we apply them in the equation (1.17.5), (1.17.7). After applying it we see that the R.H.S components are getting equated to each other, then we can equate the L.H.S as well. We get

$$\|\mathbf{D}-\mathbf{A}\|^2 = \|\mathbf{A}-\mathbf{E}\|^2$$
$$\|\mathbf{D}-\mathbf{A}\| = \|\mathbf{A}-\mathbf{E}\| \tag{1.17.17}$$

Therefore, we can say that $AD = AE$.

1.18. P is a point equidistant from two lines l and m intersecting at point A. Show that the line AP bisects the angle between them.

Solution:

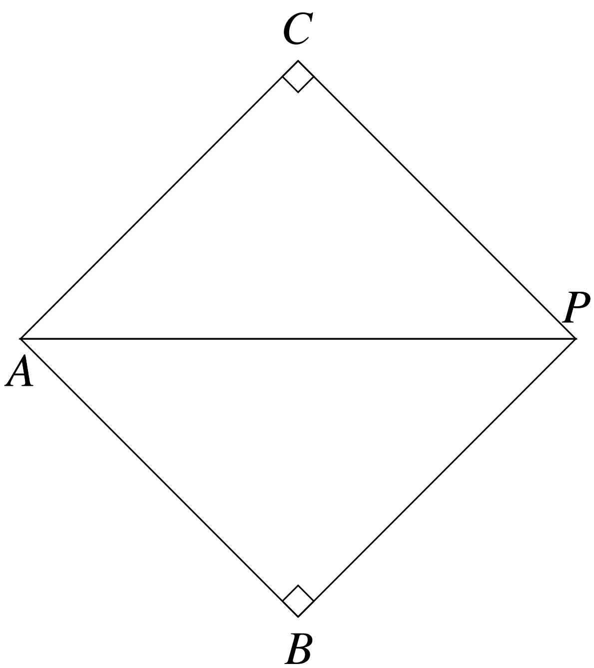


Fig. 1.18.1: figure

a) Here, the following information is given:

$$\|\mathbf{P}-\mathbf{B}\| = \|\mathbf{P}-\mathbf{C}\| \tag{1.18.1}$$

b) The lines PB is the perpendicular to line AB and PC is the perpendicular to line AC:

$$\left(\mathbf{P}-\mathbf{B}\right)^T\left(\mathbf{A}-\mathbf{B}\right) = 0 \implies \cos\angle PBA = 0 \tag{1.18.2}$$

$$\left(\mathbf{P}-\mathbf{C}\right)^T\left(\mathbf{A}-\mathbf{C}\right) = 0 \implies \cos\angle PCA = 0 \tag{1.18.3}$$

We know that

$$\|\mathbf{P}-\mathbf{A}\|^2 = (\mathbf{P}-\mathbf{A})^T(\mathbf{P}-\mathbf{A}) \tag{1.18.4}$$

$$(\mathbf{P}-\mathbf{A})^T(\mathbf{P}-\mathbf{A}) = (\mathbf{P}-\mathbf{B}+\mathbf{B}-\mathbf{A})^T(\mathbf{P}-\mathbf{B}+\mathbf{B}-\mathbf{A}) \tag{1.18.5}$$

$$\begin{aligned}\|\mathbf{P}-\mathbf{A}\|^2 &= \|\mathbf{P}-\mathbf{B}\|^2 + \|\mathbf{B}-\mathbf{A}\|^2 \\ &+2\|\mathbf{A}-\mathbf{P}\|\|\mathbf{B}-\mathbf{A}\|\cos\angle PBA\end{aligned} \tag{1.18.6}$$

using (1.18.2)

$$\implies \|\mathbf{P}-\mathbf{A}\|^2 = \|\mathbf{P}-\mathbf{B}\|^2 + \|\mathbf{B}-\mathbf{A}\|^2 \tag{1.18.7}$$

Similarly

$$(\mathbf{P}-\mathbf{A})^T(\mathbf{P}-\mathbf{A}) = (\mathbf{P}-\mathbf{C}+\mathbf{C}-\mathbf{A})^T(\mathbf{P}-\mathbf{C}+\mathbf{C}-\mathbf{A}) \tag{1.18.8}$$

$$\begin{aligned}\|\mathbf{P}-\mathbf{A}\|^2 &= \|\mathbf{P}-\mathbf{C}\|^2 + \|\mathbf{C}-\mathbf{A}\|^2 \\ &+2\|\mathbf{A}-\mathbf{P}\|\|\mathbf{C}-\mathbf{A}\|\cos\angle PCA\end{aligned} \tag{1.18.9}$$

using (1.18.3)

$$\implies \|\mathbf{P}-\mathbf{A}\|^2 = \|\mathbf{P}-\mathbf{C}\|^2 + \|\mathbf{C}-\mathbf{A}\|^2 \tag{1.18.10}$$

From (1.18.7) and (1.18.10) and substituting (1.18.1)

$$\|\mathbf{P}-\mathbf{C}\|^2 + \|\mathbf{C}-\mathbf{A}\|^2 = \|\mathbf{P}-\mathbf{B}\|^2 + \|\mathbf{B}-\mathbf{A}\|^2 \tag{1.18.11}$$

$$\implies \|\mathbf{C}-\mathbf{A}\|^2 = \|\mathbf{B}-\mathbf{A}\|^2 \implies \|\mathbf{C}-\mathbf{A}\| = \|\mathbf{B}-\mathbf{A}\| \tag{1.18.12}$$

We know that

$$\cos\angle BAP = \frac{(\mathbf{P}-\mathbf{A})^T(\mathbf{A}-\mathbf{B})}{\|\mathbf{P}-\mathbf{A}\|\|\mathbf{A}-\mathbf{B}\|} \tag{1.18.13}$$

$$= \frac{(\mathbf{P}-\mathbf{B}+\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{B})}{\|\mathbf{P}-\mathbf{A}\|\|\mathbf{A}-\mathbf{B}\|} \tag{1.18.14}$$

$$= \frac{(\mathbf{P}-\mathbf{B})^T(\mathbf{A}-\mathbf{B}) - \|\mathbf{B}-\mathbf{A}\|^2}{\|\mathbf{P}-\mathbf{A}\|\|\mathbf{A}-\mathbf{B}\|} \tag{1.18.15}$$

Using (1.18.2)

$$\cos\angle BAP = \frac{\|\mathbf{B}-\mathbf{A}\|}{\|\mathbf{P}-\mathbf{A}\|} \tag{1.18.16}$$

Similarly,

$$\cos\angle CAP = \frac{(\mathbf{P}-\mathbf{A})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{P}-\mathbf{A}\|\|\mathbf{A}-\mathbf{C}\|} \tag{1.18.17}$$

$$= \frac{(\mathbf{P}-\mathbf{C}+\mathbf{C}-\mathbf{A})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{P}-\mathbf{A}\|\|\mathbf{A}-\mathbf{C}\|} \tag{1.18.18}$$

$$= \frac{(\mathbf{P}-\mathbf{C})^T(\mathbf{A}-\mathbf{C}) - \|\mathbf{C}-\mathbf{A}\|^2}{\|\mathbf{P}-\mathbf{A}\|\|\mathbf{A}-\mathbf{C}\|} \tag{1.18.19}$$

Using (1.18.3)

$$\cos\angle CAP = \frac{\|\mathbf{C}-\mathbf{A}\|}{\|\mathbf{P}-\mathbf{A}\|} \tag{1.18.20}$$

Using (1.18.12) in (1.18.16) and (1.18.20),

$$\cos\angle BAP = \cos\angle CAP \tag{1.18.21}$$

$$\implies \angle BAP = \angle CAP \tag{1.18.22}$$

The line AP bisects the angle between them.

1.19. D is a point on side BC of $\triangle ABC$ such that

$AD = AC$. Show that $AB > AD$

Solution: See Fig. 1.19.1

Let $\mathbf{D}$ divide BC internally in ratio $1 : k$ where $0 < k < 1$. Then

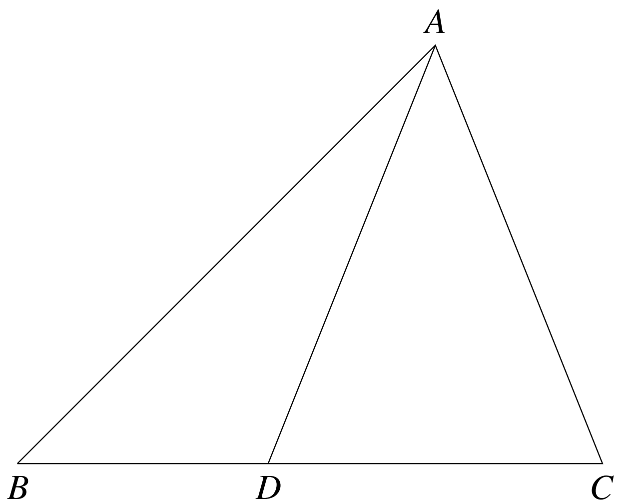


Fig. 1.19.1

$$\mathbf{D} = \frac{k\mathbf{B} + \mathbf{C}}{k+1} \tag{1.19.1}$$

The direction vector along AD is $\mathbf{D} - \mathbf{A}$.

$$\implies \quad \mathbf{D} - \mathbf{A} = \frac{k(\mathbf{B} - \mathbf{A}) + (\mathbf{C} - \mathbf{A})}{k+1} \tag{1.19.2}$$

$$\implies \quad \|\mathbf{D} - \mathbf{A}\|^2 = \frac{k^2 \|\mathbf{B} - \mathbf{A}\|^2 + \|\mathbf{C} - \mathbf{A}\|^2}{(k+1)^2} \tag{1.19.3}$$

Since $\|\mathbf{D} - \mathbf{A}\| = \|\mathbf{C} - \mathbf{A}\|$, so that

$$\|\mathbf{D} - \mathbf{A}\|^2 \left\{1 - \frac{1}{(k+1)^2}\right\} = \frac{k^2 \|\mathbf{B} - \mathbf{A}\|^2}{(k+1)^2} \tag{1.19.4}$$

$$\implies \quad \|\mathbf{B} - \mathbf{A}\|^2 = \left\{1 + \frac{2}{k}\right\} \|\mathbf{D} - \mathbf{A}\|^2 \tag{1.19.5}$$

$$\implies \quad \|\mathbf{B} - \mathbf{A}\|^2 > \|\mathbf{D} - \mathbf{A}\|^2 \tag{1.19.6}$$

$$\implies \quad AB > AD \tag{1.19.7}$$

1.20. AB is a line segment and line l is its perpendicular bisector. If a point P lies on l, show that P is equidistant from A and B.

Solution:

We have to prove that P is equidistant from A and B i.e. length of lines AP and BP are equal. Let DP be the perpendicular bisector of line AB. So,

$$\mathbf{A} - \mathbf{D} = \mathbf{D} - \mathbf{B} \tag{1.20.1}$$

$$\|\mathbf{A} - \mathbf{D}\| = \|\mathbf{D} - \mathbf{B}\| = k \tag{1.20.2}$$

$$\|\mathbf{D} - \mathbf{P}\| = l \tag{1.20.3}$$

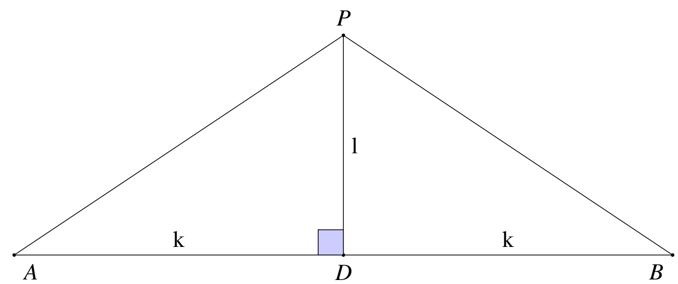


Fig. 1.20.1: $PD \perp AB$ by Latex-Tikz

Finding the length of line AP,

$$\begin{aligned}(\mathbf{A}-\mathbf{P})^T(\mathbf{A}-\mathbf{P}) &= (\mathbf{A}-\mathbf{D}+\mathbf{D}-\mathbf{P})^T(\mathbf{A}-\mathbf{D}+\mathbf{D}-\mathbf{P}) \\ &= [(\mathbf{A} - \mathbf{D})^T + (\mathbf{D} - \mathbf{P})^T][(\mathbf{A} - \mathbf{D}) + (\mathbf{D} - \mathbf{P})] \\ &= (\mathbf{A} - \mathbf{D})^T(\mathbf{A} - \mathbf{D}) + (\mathbf{A} - \mathbf{D})^T(\mathbf{D} - \mathbf{P}) + \\ &\quad (\mathbf{D} - \mathbf{P})^T(\mathbf{A} - \mathbf{D}) + (\mathbf{D} - \mathbf{P})^T(\mathbf{D} - \mathbf{P})\end{aligned} \tag{1.20.4}$$

Since, line AB is perpendicular to line DP the inner product is zero.

$$(\mathbf{A} - \mathbf{D})^T(\mathbf{D} - \mathbf{P}) = (\mathbf{D} - \mathbf{P})^T(\mathbf{A} - \mathbf{D}) = 0 \tag{1.20.5}$$

Thus,

$$(\mathbf{A}-\mathbf{P})^T(\mathbf{A}-\mathbf{P}) = (\mathbf{A}-\mathbf{D})^T(\mathbf{A}-\mathbf{D})+(\mathbf{D}-\mathbf{P})^T(\mathbf{D}-\mathbf{P})$$

$$\implies \|\mathbf{A} - \mathbf{P}\|^2 = \|\mathbf{A} - \mathbf{D}\|^2 + \|\mathbf{D} - \mathbf{P}\|^2$$

$$\implies \|\mathbf{A} - \mathbf{P}\| = \sqrt{k^2 + l^2} \tag{1.20.6}$$

Next finding the length of line BP,

$$\begin{aligned}(\mathbf{B}-\mathbf{P})^T(\mathbf{B}-\mathbf{P}) &= (\mathbf{B}-\mathbf{D}+\mathbf{D}-\mathbf{P})^T(\mathbf{B}-\mathbf{D}+\mathbf{D}-\mathbf{P}) \\ &= [(\mathbf{B} - \mathbf{D})^T + (\mathbf{D} - \mathbf{P})^T][(\mathbf{B} - \mathbf{D}) + (\mathbf{D} - \mathbf{P})] \\ &= (\mathbf{B} - \mathbf{D})^T(\mathbf{B} - \mathbf{D}) + (\mathbf{B} - \mathbf{D})^T(\mathbf{D} - \mathbf{P}) + \\ &\quad (\mathbf{D} - \mathbf{P})^T(\mathbf{B} - \mathbf{D}) + (\mathbf{D} - \mathbf{P})^T(\mathbf{D} - \mathbf{P})\end{aligned} \tag{1.20.7}$$

Again since the inner product of lines AB and

DP is zero,

$$\implies (\mathbf{B}-\mathbf{P})^T(\mathbf{B}-\mathbf{P}) = (\mathbf{B}-\mathbf{D})^T(\mathbf{B}-\mathbf{D})+(\mathbf{D}-\mathbf{P})^T(\mathbf{D}-\mathbf{P})$$
$$\implies \|\mathbf{B}-\mathbf{P}\|^2 = \|\mathbf{B}-\mathbf{D}\|^2 + \|\mathbf{D}-\mathbf{P}\|^2$$
$$\implies \|\mathbf{B}-\mathbf{P}\| = \sqrt{k^2+l^2} \quad (1.20.8)$$

From equations (1.20.6) and (1.20.8) we get,

$$\|\mathbf{A}-\mathbf{P}\| = \|\mathbf{B}-\mathbf{P}\| \quad (1.20.9)$$

Lengths of line AP and BP are equal. Hence, P is equidistant from A and B.

1.21. Line-segment AB is parallel to another line-segment CD. O is the mid-point of AD. Show that
   a) $\triangle AOB \cong \triangle DOC$
   b) O is also the mid-point of BC.

1.22. In quadrilateral $ACBD, AC = AD$ and AB bisects $\angle A$. Show that $\triangle ABC \cong \triangle ABD$. What can you say about BC and BD?

1.23. $ABCD$ is a quadrilateral in which $AD = BC$ and $\angle DAB = \angle CBA$. Prove that
   a) $\triangle ABD \cong \triangle BAC$
   b) $BD = AC$
   c) $\angle ABD = \angle BAC$.

Solution:

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{B}-\mathbf{A}\| \quad (1.23.1)$$
$$(\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{C}) = \|\mathbf{A}-\mathbf{B}\|^2 - (\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{B}) \quad (1.23.2)$$

ABCD is a quadrilateral, where AD=BC and

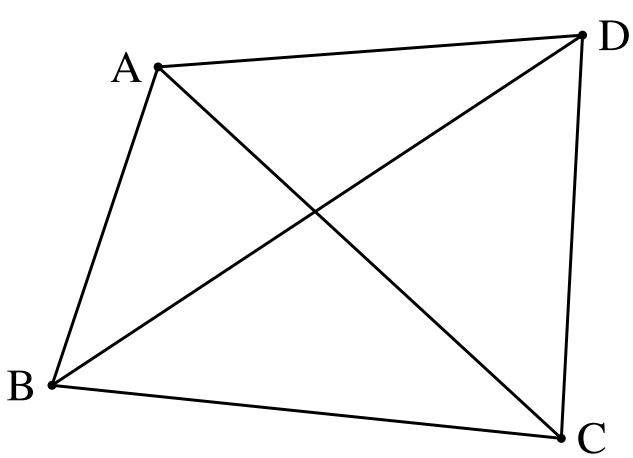


Fig. 1.23.1: Quadrilateral ABCD with AD = BC and ∠ DAB = ∠CBA

$\angle DAB = \angle CBA$.

To show $\triangle ABD \cong \triangle BAC$, we use

$$\angle DAB = \angle CBA \quad (Given) \quad (1.23.3)$$
$$AD = BC \quad (Given) \quad (1.23.4)$$
$$AB = BA \quad (Common\ Side) \quad (1.23.5)$$

Thus, by SAS Congruency Criteria, $\triangle ABD \cong \triangle BAC$.
Also, we are given that

$$\angle DAB = \angle CBA \quad (1.23.6)$$
$$\implies \cos\angle DAB = \cos\angle CBA \quad (1.23.7)$$
$$\frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{D})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{D}\|} = \frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{C})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{B}-\mathbf{C}\|} \quad (1.23.8)$$

Since,

$$\|\mathbf{A}-\mathbf{D}\| = \|\mathbf{B}-\mathbf{C}\| \quad (1.23.9)$$
$$\implies \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{D})}{\|\mathbf{A}-\mathbf{B}\|} = \frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{C})}{\|\mathbf{B}-\mathbf{A}\|} \quad (1.23.10)$$
$$\implies (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{D}) = (\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{C}) \quad (1.23.11)$$

$$\implies \|\mathbf{A}-\mathbf{B}\|^2 - (\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{D}) = \|\mathbf{A}-\mathbf{B}\|^2 - (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C}) \quad (1.23.12)$$

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{D}) = (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C}) \quad (1.23.13)$$

$$\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{B}-\mathbf{D}\|\cos\angle ABD = \|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{C}\|\cos\angle BAC \quad (1.23.14)$$

$$\|\mathbf{B}-\mathbf{D}\|\cos\angle ABD = \|\mathbf{A}-\mathbf{C}\|\cos\angle BAC \quad (1.23.15)$$

a) To Prove $\|\mathbf{B}-\mathbf{D}\| = \|\mathbf{A}-\mathbf{C}\|$.
   From (1.23.13),

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{D}) = (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C}) \quad (1.23.16)$$

$$\|\mathbf{B}-\mathbf{D}\|^2 - (\mathbf{D}-\mathbf{B})^T(\mathbf{D}-\mathbf{A}) = \|\mathbf{A}-\mathbf{C}\|^2 - (\mathbf{C}-\mathbf{B})^T(\mathbf{C}-\mathbf{A}) \quad (1.23.17)$$

$$\|\mathbf{B}-\mathbf{D}\|^2 - (\|\mathbf{A}-\mathbf{D}\|^2 - (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{D})) = \|\mathbf{A}-\mathbf{C}\|^2 - (\|\mathbf{B}-\mathbf{C}\|^2 - (\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{C})) \tag{1.23.18}$$

We know that

$$\|\mathbf{A}-\mathbf{D}\| = \|\mathbf{B}-\mathbf{C}\| \tag{1.23.19}$$

$$\|\mathbf{B}-\mathbf{D}\|^2 + (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{D}) = \|\mathbf{A}-\mathbf{C}\|^2 + (\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{C}) \tag{1.23.20}$$

$$\|\mathbf{B}-\mathbf{D}\|^2 + \|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{D}\|\cos\angle DAB = \|\mathbf{A}-\mathbf{C}\|^2 + \|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{B}-\mathbf{C}\|\cos\angle CBA \tag{1.23.21}$$

Since, we are given that $\angle DAB = \angle CBA$ and $\|\mathbf{A}-\mathbf{D}\| = \|\mathbf{B}-\mathbf{C}\|$. Then by (1.23.21)

$$\|\mathbf{B}-\mathbf{D}\|^2 = \|\mathbf{A}-\mathbf{C}\|^2 \tag{1.23.22}$$

$$\|\mathbf{B}-\mathbf{D}\| = \|\mathbf{A}-\mathbf{C}\| \tag{1.23.23}$$

Hence, BD = AC.

1.24. l and m are two parallel lines intersected by another pair of parallel lines p and q to form the quadrilateral $ABCD$. Show that $\triangle ABC \cong \triangle CDA$.

1.25. Line l is the bisector of $\angle A$ and B is any point on l. BP and BQ are perpendiculars from B to the arms of $\angle A$. Show that:

a) $\triangle APB \cong \triangle AQB$

b) $BP = BQ$ or B is equidistant from the arms of $\angle A$.

Solution: See Fig. 1.25.1.
Given:-

$$\angle BAP = \angle BAQ = \alpha \tag{1.25.1}$$

$$\angle AQB = \angle APB \tag{1.25.2}$$

In $\triangle ABQ$

$$\angle ABQ + \angle AQB + \angle BAQ = 180^\circ \tag{1.25.3}$$

In $\triangle ABP$

$$\angle ABP + \angle APB + \angle BAP = 180^\circ \tag{1.25.4}$$

Subtracting (1.25.3) and (1.25.4) and using (1.25.1) and (1.25.2) we get,

$$\angle ABQ = \angle ABP \tag{1.25.5}$$

Since,line BP and BQ are perpendicular to

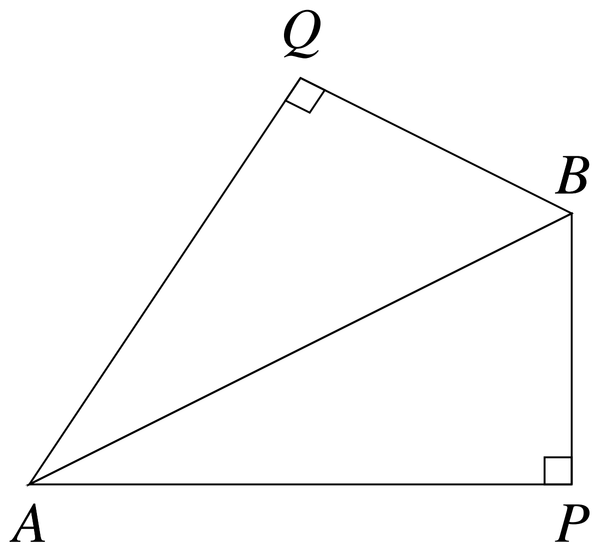


Fig. 1.25.1: figure

AP and AQ respectively..So,their respective dot product will be zero.We get,

$$(\mathbf{B}-\mathbf{Q})^T(\mathbf{A}-\mathbf{Q}) = 0 \tag{1.25.6}$$

$$(\mathbf{B}-\mathbf{P})^T(\mathbf{A}-\mathbf{P}) = 0 \tag{1.25.7}$$

We know that,$(\mathbf{B}-\mathbf{P})^T(\mathbf{B}-\mathbf{P}) = \|\mathbf{B}-\mathbf{P}\|^2$
Also let

$$\|\mathbf{B}-\mathbf{A}\|^2 = k^2 \tag{1.25.8}$$

$$(\mathbf{B}-\mathbf{P})^T(\mathbf{B}-\mathbf{P}) = (\mathbf{B}-\mathbf{A}+\mathbf{A}-\mathbf{P})^T(\mathbf{B}-\mathbf{A}+\mathbf{A}-\mathbf{P}) \tag{1.25.9}$$

$$\begin{aligned}\|\mathbf{B}-\mathbf{P}\|^2 &= (\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{A})\\ &+(\mathbf{A}-\mathbf{P})^T(\mathbf{A}-\mathbf{P})\\ &+(\mathbf{A}-\mathbf{P})^T(\mathbf{B}-\mathbf{A})\\ &+(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{P})\\ &= \|\mathbf{B}-\mathbf{A}\|^2 + \|\mathbf{A}-\mathbf{P}\|^2\\ &+2\,\|\mathbf{A}-\mathbf{P}\|\,\|\mathbf{B}-\mathbf{A}\|\cos\alpha\end{aligned} \tag{1.25.10}$$

$$\begin{aligned}(\mathbf{A}-\mathbf{P})^T(\mathbf{B}-\mathbf{A}) &= (\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{P})\\ &= \|\mathbf{A}-\mathbf{P}\|\,\|\mathbf{B}-\mathbf{A}\|\cos\alpha\end{aligned} \tag{1.25.11}$$

Substituting (1.25.11), (1.25.8) in (1.25.10) we get,

$$\|\mathbf{B}-\mathbf{P}\|^2 = k^2 + \|\mathbf{A}-\mathbf{P}\|^2 + 2k\,\|\mathbf{A}-\mathbf{P}\|\cos\alpha \tag{1.25.12}$$

Similarly,we get

$$\|\mathbf{B}-\mathbf{Q}\|^2 = k^2 + \|\mathbf{A}-\mathbf{Q}\|^2 + 2k\|\mathbf{A}-\mathbf{Q}\|\cos\alpha \tag{1.25.13}$$

$$\cos\alpha = \frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{P}-\mathbf{A})}{k\|\mathbf{P}-\mathbf{A}\|} = \frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{Q}-\mathbf{A})}{k\|\mathbf{Q}-\mathbf{A}\|} \tag{1.25.14}$$

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{P}-\mathbf{A}) = (\mathbf{B}-\mathbf{P}+\mathbf{P}-\mathbf{A})^T(\mathbf{P}-\mathbf{A}) \tag{1.25.15}$$

$$\implies (\mathbf{B}-\mathbf{P})^T(\mathbf{P}-\mathbf{A}) + \|\mathbf{P}-\mathbf{A}\|^2 \tag{1.25.16}$$

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{Q}-\mathbf{A}) = (\mathbf{B}-\mathbf{Q}+\mathbf{Q}-\mathbf{A})^T(\mathbf{Q}-\mathbf{A}) \tag{1.25.17}$$

$$\implies (\mathbf{B}-\mathbf{Q})^T(\mathbf{Q}-\mathbf{A}) + \|\mathbf{Q}-\mathbf{A}\|^2 \tag{1.25.18}$$

Substituting (1.25.6) and (1.25.7) in (1.25.18) and (1.25.16) respectively we get,

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{P}-\mathbf{A}) = \|\mathbf{P}-\mathbf{A}\|^2 \tag{1.25.19}$$

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{Q}-\mathbf{A}) = \|\mathbf{Q}-\mathbf{A}\|^2 \tag{1.25.20}$$

Substituting (1.25.19) and (1.25.20) in (1.25.14) we get,

$$\|\mathbf{P}-\mathbf{A}\| = \|\mathbf{Q}-\mathbf{A}\| \tag{1.25.21}$$

Substituting (1.25.21) in (1.25.12) and (1.25.13) we get,

$$\|\mathbf{B}-\mathbf{P}\| = \|\mathbf{B}-\mathbf{Q}\| \tag{1.25.22}$$

From (1.25.22) we can say that **B** is equidistant from the arms of $\angle A$,where **P** and **Q** are the points on the arms of $\angle A$ Using (1.25.1),(1.25.2),(1.25.22) and by AAS(Angle Angle Side) property of congruency we can say that:-

$$\triangle APB \cong \triangle AQB \tag{1.25.23}$$

1.26. $ABCE$ is a quadrilateral and D is a point on BC such that, $AC = AE, AB = AD$ and $\angle BAD = \angle EAC$. Show that $BC = DE$.
Solution: In, $\triangle ABC$ and $\triangle ADE$

$$\angle BAD = \angle EAC \quad (given) \tag{1.26.1}$$

Adding $\angle DAC$ on both side, We get:

$$\angle BAD + \angle DAC = \angle EAC + \angle DAC \tag{1.26.2}$$

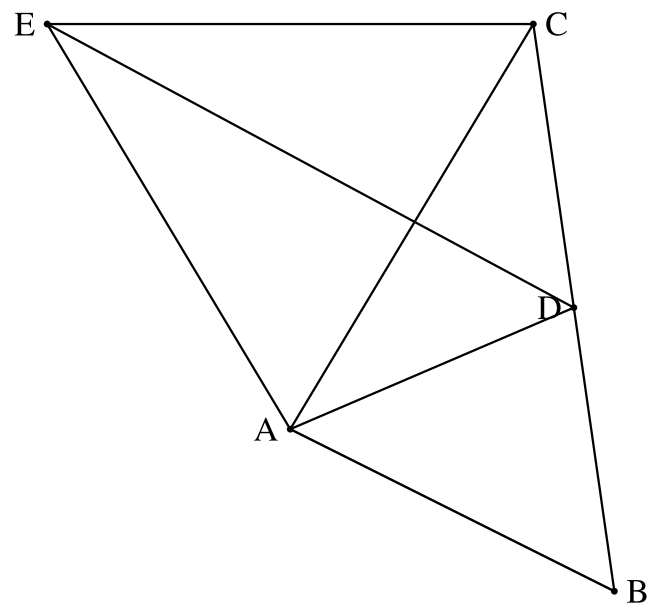


Fig. 1.26.1: Quadrilateral ABCE

We have,

$$\angle BAC = \angle DAE \tag{1.26.3}$$

$$\implies \cos\angle DAE = \cos\angle BAC \tag{1.26.4}$$

$$\frac{(\mathbf{A}-\mathbf{D})^T(\mathbf{A}-\mathbf{E})}{\|\mathbf{A}-\mathbf{D}\|\,\|\mathbf{A}-\mathbf{E}\|} = \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{C}\|} \tag{1.26.5}$$

We are given AE=AC and we know AD=AB always. Thus,

$$\|\mathbf{A}-\mathbf{E}\| = \|\mathbf{A}-\mathbf{C}\| \tag{1.26.6}$$

$$\|\mathbf{A}-\mathbf{D}\| = \|\mathbf{A}-\mathbf{B}\| \tag{1.26.7}$$

Then, from (1.26.5), we have,

$$(\mathbf{A}-\mathbf{D})^T(\mathbf{A}-\mathbf{E}) = (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C}) \tag{1.26.8}$$

Taking Transpose on both the side:

$$((\mathbf{A}-\mathbf{D})^T(\mathbf{A}-\mathbf{E}))^T = ((\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C}))^T \tag{1.26.9}$$

$$(\mathbf{A}-\mathbf{E})^T(\mathbf{A}-\mathbf{D}) = (\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{B}) \tag{1.26.10}$$

We need to prove: $\|\mathbf{B}-\mathbf{C}\| = \|\mathbf{D}-\mathbf{E}\|$

$$\implies \|\mathbf{B}-\mathbf{C}\|^2 - \|\mathbf{D}-\mathbf{E}\|^2 = (\mathbf{B}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) - (\mathbf{D}-\mathbf{E})^T(\mathbf{D}-\mathbf{E}) \tag{1.26.11}$$

$$= ((\mathbf{A}-\mathbf{C})-(\mathbf{A}-\mathbf{B}))^T((\mathbf{A}-\mathbf{C})-(\mathbf{A}-\mathbf{B})) - ((\mathbf{A}-\mathbf{E})-(\mathbf{A}-\mathbf{D}))^T((\mathbf{A}-\mathbf{E})-(\mathbf{A}-\mathbf{D})) \tag{1.26.12}$$

$$= \|\mathbf{A}-\mathbf{C}\|^2+\|\mathbf{A}-\mathbf{B}\|^2-(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C})-(\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{B}) -\|\mathbf{A}-\mathbf{E}\|^2-\|\mathbf{A}-\mathbf{D}\|^2+(\mathbf{A}-\mathbf{D})^T(\mathbf{A}+\mathbf{E})+(\mathbf{A}-\mathbf{E})^T(\mathbf{A}-\mathbf{D}) \tag{1.26.13}$$

Thus, from (1.26.6), (1.26.7), (1.26.8) and (1.26.10)

$$\|\mathbf{B}-\mathbf{C}\|^2 - \|\mathbf{D}-\mathbf{E}\|^2 = 0 \tag{1.26.14}$$
$$\implies \|\mathbf{B}-\mathbf{C}\|^2 = \|\mathbf{D}-\mathbf{E}\|^2 \tag{1.26.15}$$
$$\therefore \|\mathbf{B}-\mathbf{C}\| = \|\mathbf{D}-\mathbf{E}\| \tag{1.26.16}$$

Hence, BC=DE

1.27. In right triangle ABC, right angled at C, M is the mid-point of hypotenuse AB. C is joined to M and produced to a point D such that $DM = CM$. Point D is joined to point B. Show that:

a) $\triangle AMC \cong \triangle BMD$
b) $\angle DBC$ is a right angle.
c) $\triangle DBC \cong \triangle ACB$
d) $CM = \frac{1}{2}AB$

Solution: In $\triangle ABC$, **M** is midpoint of hypotenuse AB, thus

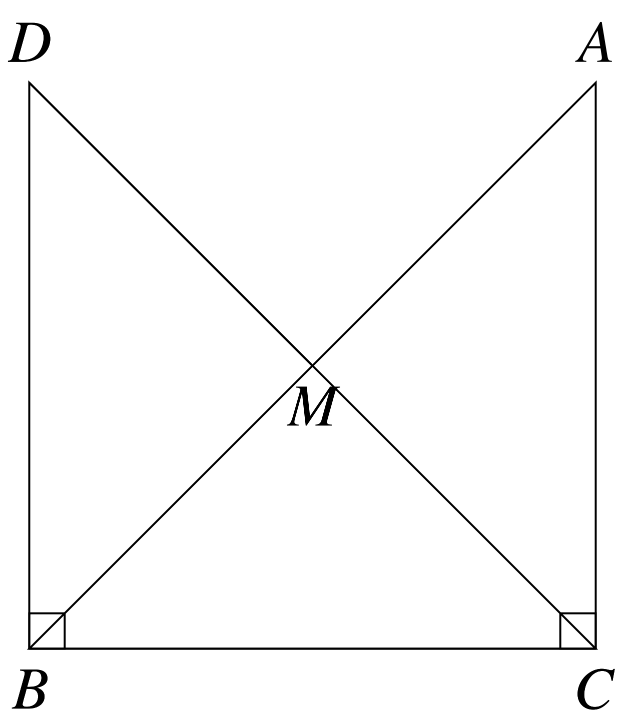


Fig. 1.27.1: Triangle ABC and DBC

$$\mathbf{M} = \frac{\mathbf{A}+\mathbf{B}}{2} \tag{1.27.1}$$
$$2\mathbf{M} = (\mathbf{A}+\mathbf{B}) \tag{1.27.2}$$
$$(\mathbf{A}-\mathbf{M}) = (\mathbf{M}-\mathbf{B}) \tag{1.27.3}$$
$$\|\mathbf{A}-\mathbf{M}\| = \|\mathbf{M}-\mathbf{B}\| \tag{1.27.4}$$
$$\mathbf{M} = \frac{\mathbf{C}+\mathbf{D}}{2} \tag{1.27.5}$$
$$2\mathbf{M} = (\mathbf{C}+\mathbf{D}) \tag{1.27.6}$$
$$(\mathbf{C}-\mathbf{M}) = (\mathbf{M}-\mathbf{D}) \tag{1.27.7}$$
$$\|\mathbf{C}-\mathbf{M}\| = \|\mathbf{M}-\mathbf{D}\| \tag{1.27.8}$$
$$\mathbf{M} = \frac{\mathbf{A}+\mathbf{B}}{2} = \frac{\mathbf{C}+\mathbf{D}}{2} \tag{1.27.9}$$
$$\mathbf{A}-\mathbf{C} = \mathbf{A}-\mathbf{M}+\mathbf{M}-\mathbf{C} \tag{1.27.10}$$
$$\mathbf{A}-\mathbf{C} = \mathbf{M}-\mathbf{B}+\mathbf{D}-\mathbf{M} \tag{1.27.11}$$
$$(\mathbf{A}-\mathbf{C}) = k(\mathbf{D}-\mathbf{B}) \quad \text{[k value is 1]} \tag{1.27.12}$$

Now from equation (1.27.12) we can say that

$$AC \parallel DB \tag{1.27.13}$$
$$\|\mathbf{A}-\mathbf{C}\| = \|\mathbf{D}-\mathbf{B}\| \tag{1.27.14}$$

Now it is given that AC $\perp$ BC, using this we can prove that DB $\perp$ BC.

$$(\mathbf{A}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{1.27.15}$$
$$(\mathbf{A}-\mathbf{M}+\mathbf{M}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{1.27.16}$$
$$(\mathbf{M}-\mathbf{B}+\mathbf{D}-\mathbf{M})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{1.27.17}$$
$$(\mathbf{D}-\mathbf{B})^T(\mathbf{B}-\mathbf{C}) = 0 \tag{1.27.18}$$
$$\implies DB \perp BC \tag{1.27.19}$$
$$\mathbf{A}-\mathbf{B} = \mathbf{A}-\mathbf{C}+\mathbf{C}-\mathbf{B} \tag{1.27.20}$$
$$\mathbf{A}-\mathbf{B} = \mathbf{B}-\mathbf{D}+\mathbf{C}-\mathbf{B} \quad \text{[From (1.27.14)]} \tag{1.27.21}$$
$$\mathbf{A}-\mathbf{B} = \mathbf{C}-\mathbf{D} \tag{1.27.22}$$
$$\mathbf{A}-\mathbf{B} = \mathbf{C}-\mathbf{M}+\mathbf{M}-\mathbf{D} \tag{1.27.23}$$
$$\mathbf{A}-\mathbf{B} = \mathbf{C}-\mathbf{M}+\mathbf{C}-\mathbf{M} \quad \text{[From (1.27.8)]} \tag{1.27.24}$$

$$\mathbf{A}-\mathbf{B} = 2(\mathbf{C}-\mathbf{M}) \tag{1.27.25}$$
$$\mathbf{C}-\mathbf{M} = \frac{1}{2}(\mathbf{A}-\mathbf{B}) \tag{1.27.26}$$
$$\|\mathbf{C}-\mathbf{M}\| = \frac{1}{2}\|\mathbf{A}-\mathbf{B}\| \tag{1.27.27}$$

Hence from (1.27.27) proved,
CM = $\frac{1}{2}$ AB

1.28. In an isosceles $\triangle ABC$, with $AB = AC$, the bisectors of $\angle B$ and $\angle C$ intersect each other at O. Join A to O. Show that :
   a) $OB = OC$
   b) AO bisects $\angle A$

1.29. In $\triangle ABC$, AD is the perpendicular bisector of BC. Show that $\triangle ABC$ is an isosceles triangle in which $AB = AC$.
   Solution: Consider the above ΔABC. Given

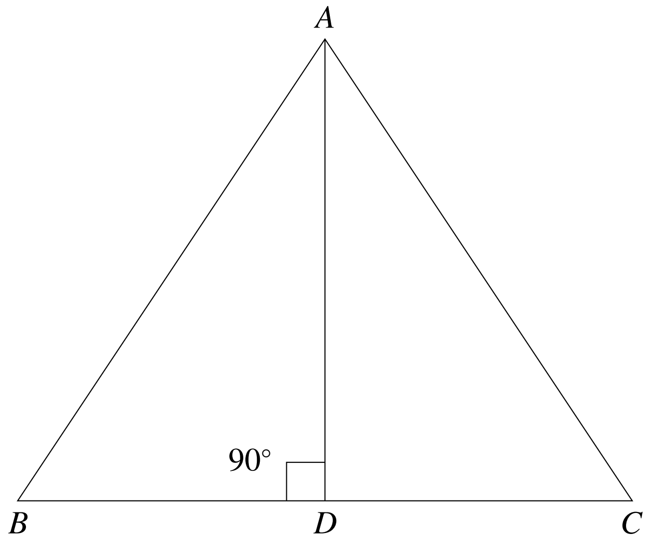


Fig. 1.29.1

that AD is the perpendicular bisector of BC. So, BD = DC and ∠ADB = ∠ADC = 90.Since D is the midpoint of BC

$$\mathbf{D} = (\mathbf{B} + \mathbf{C})/2$$
$$2\mathbf{D} = (\mathbf{B} + \mathbf{C})$$
$$(\mathbf{B} - \mathbf{D}) = (\mathbf{D} - \mathbf{C})$$
$$\|\mathbf{B} - \mathbf{D}\| = \|\mathbf{D} - \mathbf{C}\| \quad (1.29.1)$$

Since, AD is the perpendicular bisector of BC

$$(\mathbf{A} - \mathbf{D})^T(\mathbf{B} - \mathbf{C}) = (\mathbf{B} - \mathbf{C})^T(\mathbf{A} - \mathbf{D}) = 0$$
$$(\mathbf{A} - \mathbf{D})^T(\mathbf{B} - \mathbf{D} + \mathbf{D} - \mathbf{C}) = 0$$
$$(\mathbf{A} - \mathbf{D})^T(\mathbf{B} - \mathbf{D}) + (\mathbf{A} - \mathbf{D})^T(\mathbf{D} - \mathbf{C}) = 0$$
$$\implies (\mathbf{A} - \mathbf{D})^T(\mathbf{B} - \mathbf{D}) = (\mathbf{A} - \mathbf{D})^T(\mathbf{D} - \mathbf{C}) = 0 \quad (1.29.2)$$

$$(\mathbf{B} - \mathbf{C})^T(\mathbf{A} - \mathbf{D}) = 0$$
$$(\mathbf{B} - \mathbf{D} + \mathbf{D} - \mathbf{C})^T(\mathbf{A} - \mathbf{D}) = 0$$
$$[(\mathbf{B} - \mathbf{D})^T + (\mathbf{D} - \mathbf{C})^T](\mathbf{A} - \mathbf{D}) = 0$$
$$(\mathbf{B} - \mathbf{D})^T(\mathbf{A} - \mathbf{D}) + (\mathbf{D} - \mathbf{C})^T(\mathbf{A} - \mathbf{D}) = 0$$
$$\implies (\mathbf{B} - \mathbf{D})^T(\mathbf{A} - \mathbf{D}) = (\mathbf{D} - \mathbf{C})^T(\mathbf{A} - \mathbf{D}) = 0 \quad (1.29.3)$$

Find the length of AB,

$$\|\mathbf{A} - \mathbf{B}\|^2 = (\mathbf{A} - \mathbf{B})^T(\mathbf{A} - \mathbf{B})$$
$$= (\mathbf{A} - \mathbf{D} + \mathbf{D} - \mathbf{B})^T(\mathbf{A} - \mathbf{D} + \mathbf{D} - \mathbf{B})$$
$$= [(\mathbf{A} - \mathbf{D})^T + (\mathbf{D} - \mathbf{B})^T][(\mathbf{A} - \mathbf{D}) + (\mathbf{D} - \mathbf{B})]$$
$$= (\mathbf{A} - \mathbf{D})^T(\mathbf{A} - \mathbf{D}) + (\mathbf{A} - \mathbf{D})^T(\mathbf{D} - \mathbf{B}) +$$
$$(\mathbf{D} - \mathbf{B})^T(\mathbf{A} - \mathbf{D}) + (\mathbf{D} - \mathbf{B})^T(\mathbf{D} - \mathbf{B}) \quad (1.29.4)$$

From Eq (1.29.2) and Eq (1.29.3),

$$(\mathbf{A} - \mathbf{B})^T(\mathbf{A} - \mathbf{B})$$
$$= (\mathbf{A} - \mathbf{D})^T(\mathbf{A} - \mathbf{D}) + (\mathbf{D} - \mathbf{B})^T(\mathbf{D} - \mathbf{B})$$
$$\implies \|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{A} - \mathbf{D}\|^2 + \|\mathbf{D} - \mathbf{B}\|^2 \quad (1.29.5)$$

Similarily, find the length of AC

$$\|\mathbf{A} - \mathbf{C}\|^2 = (\mathbf{A} - \mathbf{C})^T(\mathbf{A} - \mathbf{C})$$
$$= (\mathbf{A} - \mathbf{D} + \mathbf{D} - \mathbf{C})^T(\mathbf{A} - \mathbf{D} + \mathbf{D} - \mathbf{C})$$
$$= [(\mathbf{A} - \mathbf{D})^T + (\mathbf{D} - \mathbf{C})^T][(\mathbf{A} - \mathbf{D}) + (\mathbf{D} - \mathbf{C})]$$
$$= (\mathbf{A} - \mathbf{D})^T(\mathbf{A} - \mathbf{D}) + (\mathbf{A} - \mathbf{D})^T(\mathbf{D} - \mathbf{C}) +$$
$$(\mathbf{D} - \mathbf{C})^T(\mathbf{A} - \mathbf{D}) + (\mathbf{D} - \mathbf{C})^T(\mathbf{D} - \mathbf{C}) \quad (1.29.6)$$

From Eq (1.29.2) and Eq (1.29.3)

$$(\mathbf{A} - \mathbf{C})^T(\mathbf{A} - \mathbf{C})$$
$$= (\mathbf{A} - \mathbf{D})^T(\mathbf{A} - \mathbf{D}) + (\mathbf{D} - \mathbf{C})^T(\mathbf{D} - \mathbf{C})$$
$$\implies \|\mathbf{A} - \mathbf{C}\|^2 = \|\mathbf{C} - \mathbf{D}\|^2 + \|\mathbf{D} - \mathbf{A}\|^2 \quad (1.29.7)$$

Eq (1.29.1), Eq (1.29.5) and Eq (1.29.7)

$$\|\mathbf{A} - \mathbf{B}\|^2 = \|\mathbf{A} - \mathbf{C}\|^2 \quad (1.29.8)$$

Since the lengths AB and AC are equal, We can conclude that the ΔABC is an isosceles triangle with AB = AC.

1.30. ABC is an isosceles triangle in which altitudes BE and CF are drawn to equal sides AC and AB respectively . Show that these altitudes are

equal.
Solution: See Fig. 1.30.1

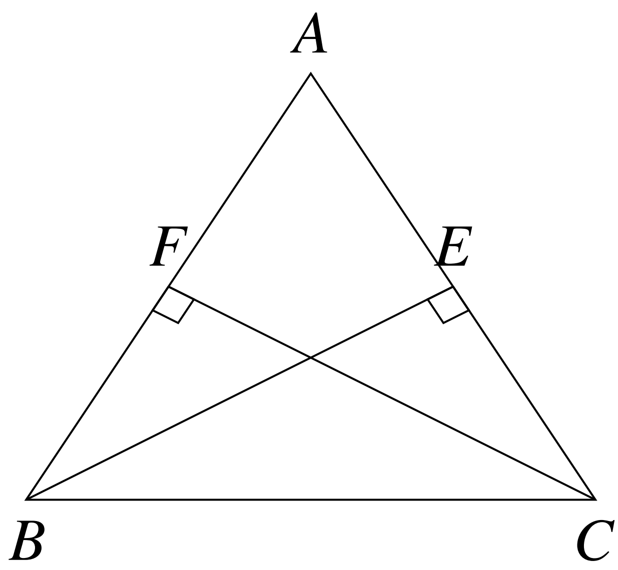


Fig. 1.30.1: Isosceles Triangle with altitudes drawn to equal sides

Let $\mathbf{m}_{AC}$ and $\mathbf{m}_{BE}$ be direction vector of side AC and altitude BE respectively.

$$\mathbf{m}_{AC} = \mathbf{A} - \mathbf{C} \tag{1.30.1}$$
$$\mathbf{m}_{BE} = \mathbf{B} - \mathbf{E} \tag{1.30.2}$$

Here, BE ⊥ AC because BE is the altitude to side AC.So,

$$\mathbf{m}_{AC}^T\mathbf{m}_{BE} = 0 \tag{1.30.3}$$
$$(\mathbf{A} - \mathbf{C})^T(\mathbf{B} - \mathbf{E}) = 0 \tag{1.30.4}$$
$$(\mathbf{A} - \mathbf{E} + \mathbf{E} - \mathbf{B} + \mathbf{B} - \mathbf{C})^T(\mathbf{B} - \mathbf{E}) = 0 \tag{1.30.5}$$
$$(\mathbf{A} - \mathbf{E})^T(\mathbf{B} - \mathbf{E}) + \|\mathbf{B} - \mathbf{E}\|^2 + \tag{1.30.6}$$
$$(\mathbf{B} - \mathbf{C})^T(\mathbf{B} - \mathbf{E}) = 0 \tag{1.30.7}$$
$$\|\mathbf{B} - \mathbf{E}\|^2 + (\mathbf{B} - \mathbf{C})^T(\mathbf{B} - \mathbf{E}) = 0 \tag{1.30.8}$$

Let $\mathbf{m}_{AB}$ and $\mathbf{m}_{CF}$ be direction vector of side AB and altitude CF respectively.

$$\mathbf{m}_{AB} = \mathbf{A} - \mathbf{B} \tag{1.30.9}$$
$$\mathbf{m}_{CF} = \mathbf{C} - \mathbf{F} \tag{1.30.10}$$

Here, CF ⊥ AB because CF is the altitude to side AB.So,

$$\mathbf{m}_{AB}^T\mathbf{m}_{CF} = 0 \tag{1.30.11}$$
$$(\mathbf{A} - \mathbf{B})^T(\mathbf{C} - \mathbf{F}) = 0 \tag{1.30.12}$$
$$(\mathbf{A} - \mathbf{F} + \mathbf{F} - \mathbf{C} + \mathbf{C} - \mathbf{B})^T(\mathbf{C} - \mathbf{F}) = 0 \tag{1.30.13}$$
$$(\mathbf{A} - \mathbf{F})^T(\mathbf{C} - \mathbf{F}) + \|\mathbf{C} - \mathbf{F}\|^2 + \tag{1.30.14}$$
$$(\mathbf{C} - \mathbf{B})^T(\mathbf{C} - \mathbf{F}) = 0 \tag{1.30.15}$$
$$\|\mathbf{C} - \mathbf{F}\|^2 + (\mathbf{C} - \mathbf{B})^T(\mathbf{C} - \mathbf{F}) = 0 \tag{1.30.16}$$

Comparing equation (1.30.8) and (1.30.16)

$$\|\mathbf{C} - \mathbf{F}\|^2 + (\mathbf{C} - \mathbf{B})^T(\mathbf{C} - \mathbf{F}) = \\ \|\mathbf{B} - \mathbf{E}\|^2 + (\mathbf{B} - \mathbf{C})^T(\mathbf{B} - \mathbf{E}) \tag{1.30.17}$$

$$\|\mathbf{C} - \mathbf{F}\|^2 + (\mathbf{C} - \mathbf{B})^T(\mathbf{C} - \mathbf{A} + \mathbf{A} - \mathbf{F}) = \\ \|\mathbf{B} - \mathbf{E}\|^2 + (\mathbf{B} - \mathbf{C})^T(\mathbf{B} - \mathbf{A} + \mathbf{A} - \mathbf{E}) \tag{1.30.18}$$

$$\|\mathbf{C} - \mathbf{F}\|^2 + (\mathbf{C} - \mathbf{B})^T(\mathbf{C} - \mathbf{A}) + (\mathbf{C} - \mathbf{B})^T(\mathbf{A} - \mathbf{F}) = \\ \|\mathbf{B} - \mathbf{E}\|^2 + (\mathbf{B} - \mathbf{C})^T(\mathbf{B} - \mathbf{A}) + (\mathbf{B} - \mathbf{C})^T(\mathbf{A} - \mathbf{E}) \tag{1.30.19}$$

$$\|\mathbf{C} - \mathbf{F}\|^2 + 2(\mathbf{C} - \mathbf{B})^T(\mathbf{C} - \mathbf{A}) = \\ \|\mathbf{B} - \mathbf{E}\|^2 + 2(\mathbf{B} - \mathbf{C})^T(\mathbf{B} - \mathbf{A}) \tag{1.30.20}$$

$$\|\mathbf{C} - \mathbf{F}\|^2 + 2(\|\mathbf{C} - \mathbf{B}\| \, \|\mathbf{C} - \mathbf{A}\|)\cos\theta = \\ \|\mathbf{B} - \mathbf{E}\|^2 + 2(\|\mathbf{B} - \mathbf{C}\| \, \|\mathbf{B} - \mathbf{A}\|)\cos\theta \tag{1.30.21}$$

$$\|\mathbf{C} - \mathbf{F}\|^2 = \|\mathbf{B} - \mathbf{E}\|^2 \tag{1.30.22}$$
$$\|\mathbf{C} - \mathbf{F}\| = \|\mathbf{B} - \mathbf{E}\| \tag{1.30.23}$$

Hence, the altitudes drawn to equal sides of isosceles triangle is equal.

1.31. ABC is a triangle in which altitudes BE and CF to sides AC and AB are equal. Show that

a) $\triangle ABE \cong \triangle ACF$
b) $AB = AC$, i.e., ABC is an isosceles triangle.

Solution:
Let consider we have a triangle $\triangle$ABC. There are two altitudes BE and CF being drawn from the vertices B and C respectively. And we have ,

$$\|\mathbf{E}-\mathbf{B}\| = \|\mathbf{F}-\mathbf{C}\| \tag{1.31.1}$$

Let $\mathbf{m}_{AB}$ and $\mathbf{m}_{CF}$ are the direction vectors of AB and CF respectively. Since AB $\perp$ CF hence,

$$\mathbf{m}_{AB}\mathbf{m}_{CF} = 0 \tag{1.31.2}$$

$$(\mathbf{B}-\mathbf{E})^{\mathbf{T}}(\mathbf{A}-\mathbf{C}) = \mathbf{0} \tag{1.31.3}$$

$$(\mathbf{B}-\mathbf{A}+\mathbf{A}-\mathbf{C}+\mathbf{C}-\mathbf{E})^{\mathbf{T}}(\mathbf{A}-\mathbf{C}) = \mathbf{0} \tag{1.31.4}$$

$$(\mathbf{B}-\mathbf{A})^{\mathbf{T}}(\mathbf{A}-\mathbf{C}) + \|\mathbf{A}-\mathbf{C}\|^2 + (\mathbf{C}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{C}) = \mathbf{0} \tag{1.31.5}$$

Similarly, AC $\perp$ BE hence,

$$\mathbf{m}_{AC}\mathbf{m}_{BE} = 0 \tag{1.31.6}$$

$$(\mathbf{C}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) = \mathbf{0} \tag{1.31.7}$$

$$(\mathbf{C}-\mathbf{A}+\mathbf{A}-\mathbf{B}+\mathbf{B}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) = \mathbf{0} \tag{1.31.8}$$

$$(\mathbf{C}-\mathbf{A})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) + \|\mathbf{A}-\mathbf{B}\|^2 + (\mathbf{B}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) = \mathbf{0} \tag{1.31.9}$$

In $\triangle$ABC, taking inner product of sides AB and AC we can write :

$$\implies \cos\angle BAC = \frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{C}\|} \tag{1.31.10}$$

and,

$$(\mathbf{C}-\mathbf{A})^T(\mathbf{A}-\mathbf{B}) = \|\mathbf{C}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{B}\|\cos\angle ABC \tag{1.31.11}$$

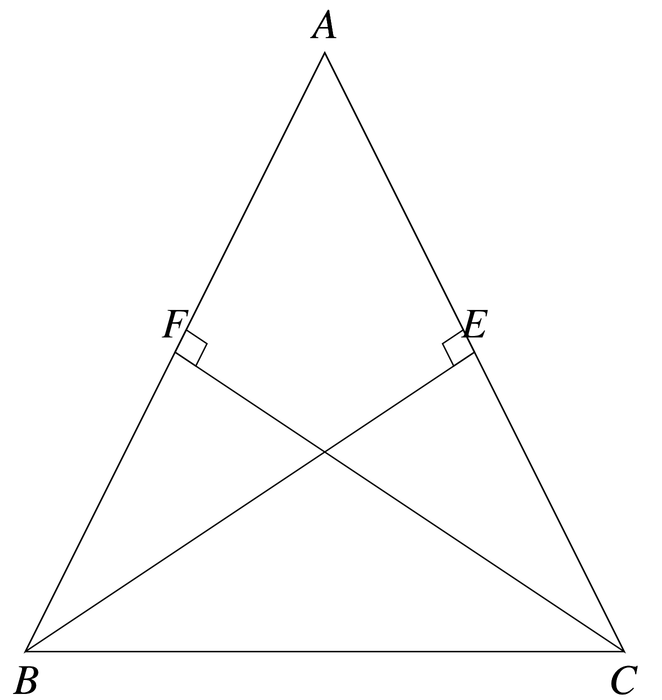


Fig. 1.31.1: Isosceles triangle ABC

From equation 1.31.10, and 1.31.11, we have ,

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{C}) = (\mathbf{C}-\mathbf{A})^T(\mathbf{A}-\mathbf{B}) \tag{1.31.12}$$

using equation 1.31.12 in 1.31.5 and 1.31.9 we can write,

$$\|\mathbf{A}-\mathbf{C}\|^2 + (\mathbf{C}-\mathbf{E})^T(\mathbf{A}-\mathbf{C}) = \|\mathbf{A}-\mathbf{B}\|^2 + (\mathbf{B}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) \tag{1.31.13}$$

$$\|\mathbf{A}-\mathbf{C}\|^2 + (\mathbf{C}-\mathbf{E})^T(\mathbf{A}-\mathbf{C}) = \|\mathbf{A}-\mathbf{B}\|^2 + (\mathbf{B}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) \tag{1.31.14}$$

$$\|\mathbf{A}-\mathbf{C}\|^2+(\mathbf{C}-\mathbf{A}+\mathbf{A}-\mathbf{B}+\mathbf{B}-\mathbf{E})^T(\mathbf{A}-\mathbf{C}) = \|\mathbf{A}-\mathbf{B}\|^2+(\mathbf{B}-\mathbf{A}+\mathbf{A}-\mathbf{C}+\mathbf{C}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) \tag{1.31.15}$$

$$\|\mathbf{A}-\mathbf{C}\|^2+\|\mathbf{A}-\mathbf{C}\|^2+(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C})+(\mathbf{B}-\mathbf{E})^T(\mathbf{A}-\mathbf{C}) = \|\mathbf{A}-\mathbf{B}\|^2+\|\mathbf{A}-\mathbf{B}\|^2+(\mathbf{A}-\mathbf{C})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) + (\mathbf{C}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) \tag{1.31.16}$$

since BE ⊥ AC and CF ⊥ AB, hence :

$$(\mathbf{B}-\mathbf{E})^T(\mathbf{A}-\mathbf{C}) = \mathbf{0} \tag{1.31.17}$$

and,

$$(\mathbf{C}-\mathbf{F})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) = \mathbf{0} \tag{1.31.18}$$

Now equation 1.31.16 become :

$$2\left\|\mathbf{A}-\mathbf{C}\right\|^2 + (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C}) = 2\left\|\mathbf{A}-\mathbf{B}\right\|^2 + (\mathbf{A}-\mathbf{C})^{\mathbf{T}}(\mathbf{A}-\mathbf{B}) \tag{1.31.19}$$

Using equation 1.31.12 in equation 1.31.19,

$$\left\|\mathbf{A}-\mathbf{C}\right\| = \left\|\mathbf{A}-\mathbf{B}\right\| \tag{1.31.20}$$

1.32. ABC and DBC are two isosceles triangles on the same base BC. Show that $\angle ABD = \angle ACD$. **Solution:** In an Isosceles triangle the angles opposite to sides of equal length are equal.Therefore the angles $\angle ABC = \angle ACB$ and $\angle DBC = \angle DCB$.Let the vertex B be at origin and not lose generality.Since the two triangles are isosceles,$\|\mathbf{B}-\mathbf{D}\| = \|\mathbf{C}-\mathbf{D}\|$ and $\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{A}-\mathbf{C}\|$. The triangles $\triangle ABC$ and $\triangle DBC$ are isosceles triangles,so

$$\left\|\mathbf{A}-\mathbf{B}\right\| = \left\|\mathbf{A}-\mathbf{C}\right\| \tag{1.32.1}$$
$$\left\|\mathbf{D}-\mathbf{B}\right\| = \left\|\mathbf{D}-\mathbf{C}\right\| \tag{1.32.2}$$

From equation (1.32.1),we get

$$\begin{aligned}(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{B}) &= (\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{C})\\ (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{D}+\mathbf{D}-\mathbf{B}) &= \\ (\mathbf{A}-\mathbf{C})^T&(\mathbf{A}-\mathbf{D}+\mathbf{D}-\mathbf{C})\\ (\mathbf{A}-\mathbf{B})^T(\mathbf{D}-\mathbf{B}) &= \\ (\mathbf{A}-\mathbf{C})^T(\mathbf{D}-\mathbf{C}) &+ (\mathbf{B}-\mathbf{C})^T(\mathbf{A}-\mathbf{D})\end{aligned} \tag{1.32.3}$$

Doing the below calculation,we get

$$\begin{aligned}(\mathbf{C}-\mathbf{B})^T(\mathbf{A}-\mathbf{B}) - (\mathbf{C}-\mathbf{B})^T(\mathbf{D}-\mathbf{B}) &= \\ (\mathbf{A}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) &- (\mathbf{D}-\mathbf{C})^T(\mathbf{B}-\mathbf{C})\\ (\mathbf{C}-\mathbf{B})^T(\mathbf{A}-\mathbf{D}) &= (\mathbf{A}-\mathbf{D})^T(\mathbf{B}-\mathbf{C})\end{aligned} \tag{1.32.4}$$

Since $(\mathbf{A}-\mathbf{D})^T(\mathbf{B}-\mathbf{C})$=$(\mathbf{B}-\mathbf{C})^T(\mathbf{A}-\mathbf{D})$, the

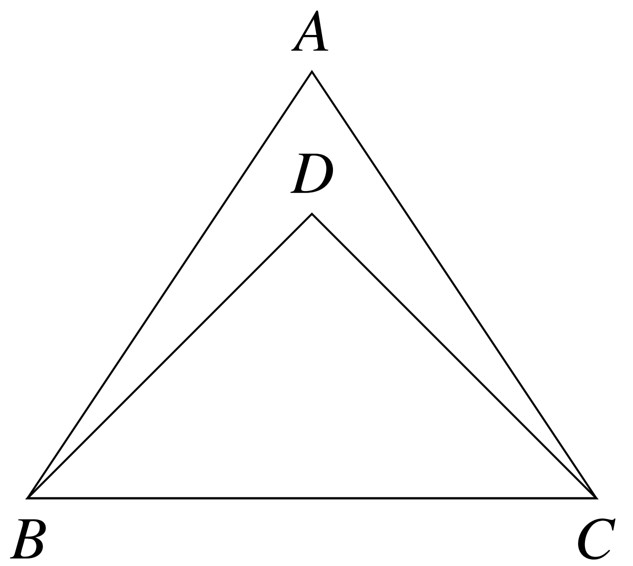


Fig. 1.32.1: Isosceles triangles with common base BC

equation (1.32.4) can be written as

$$(\mathbf{B}-\mathbf{C})^T(\mathbf{A}-\mathbf{D}) = (\mathbf{C}-\mathbf{B})^T(\mathbf{A}-\mathbf{D}) \tag{1.32.5}$$
$$(\mathbf{B}-\mathbf{C})^T(\mathbf{A}-\mathbf{D}) = -(\mathbf{B}-\mathbf{C})^T(\mathbf{A}-\mathbf{D}) \tag{1.32.6}$$
$$2(\mathbf{B}-\mathbf{C})^T(\mathbf{A}-\mathbf{D}) = 0 \tag{1.32.7}$$
$$(\mathbf{B}-\mathbf{C})^T(\mathbf{A}-\mathbf{D}) = 0 \tag{1.32.8}$$

Taking the inner product of $A-B, B-D$ and $A-C, D-C$,we get

$$\cos\angle ABD = \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{D}-\mathbf{B})}{\left\|\mathbf{A}-\mathbf{B}\right\|\left\|\mathbf{D}-\mathbf{B}\right\|} \tag{1.32.9}$$
$$\cos\angle ACD = \frac{(\mathbf{A}-\mathbf{C})^T(\mathbf{D}-\mathbf{C})}{\left\|\mathbf{A}-\mathbf{C}\right\|\left\|\mathbf{D}-\mathbf{C}\right\|} \tag{1.32.10}$$

Subtracting the above equations we get

$$\cos\angle ABD - \cos\angle ACD \tag{1.32.11}$$
$$= \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{D}-\mathbf{B}) - (\mathbf{A}-\mathbf{C})^T(\mathbf{D}-\mathbf{C})}{\left\|\mathbf{A}-\mathbf{C}\right\|\left\|\mathbf{D}-\mathbf{C}\right\|} \tag{1.32.12}$$
$$= \frac{(\mathbf{B}-\mathbf{C})^T(\mathbf{A}-\mathbf{D})}{\left\|\mathbf{A}-\mathbf{C}\right\|\left\|\mathbf{D}-\mathbf{C}\right\|} \tag{1.32.13}$$

Using the equation (1.32.8),we get

$$\cos\angle ABD - \cos\angle ACD = 0 \quad (1.32.14)$$

$$\cos\angle ABD = \cos\angle ACD \quad (1.32.15)$$

$$\angle ABD = \angle ACD \quad (1.32.16)$$

1.33. $\triangle ABC$ and $\triangle DBC$ are two isosceles triangles on the same base BC and vertices A and D are on the same side of BC. If AD is extended to intersect BC at P, show that

a) $\triangle ABD \cong \triangle ACD$
b) $\triangle ABP \cong \triangle ACP$
c) AP bisects $\angle A$ as well as $\angle D$.
d) AP is the perpendicular bisector of BC.

Solution:
$\triangle ABC$ and $\triangle DBC$ are two isosceles triangles on the same base BC and the vertices A and D are on the same side of BC. If AD is extended to intersect BC at P, show that
a) $\triangle ABD \cong \triangle ACD$
b) $\triangle ABP \cong \triangle ACP$
c) AP bisects $\angle A$ as well as $\angle D$
d) AP is the parpendicular bisector of BC

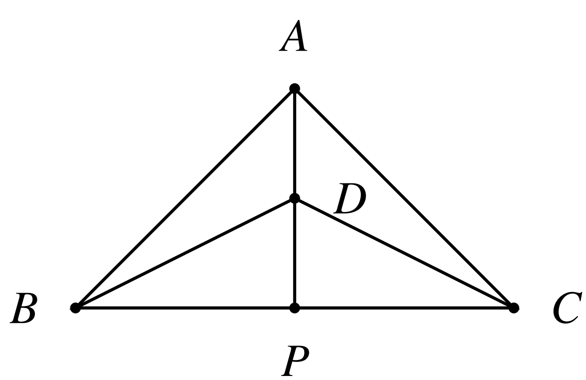


Fig. 1.33.1: Iso-sceles Triangles by Latex-Tikz

The above problem statement is depicted in the figure 1.33.1 where the vertices are: A, B and C for $\triangle ABC$ and D, B and C for $\triangle DBC$. For $\triangle ABC$ the sides AB, BC and CA are represented by the vectors $\mathbf{A}-\mathbf{B}$, $\mathbf{B}-\mathbf{C}$ and $\mathbf{C}-\mathbf{A}$ and for $\triangle DBC$ the sides DB, BC and CD are represented by $\mathbf{D}-\mathbf{B}$, $\mathbf{B}-\mathbf{C}$ and $\mathbf{C}-\mathbf{D}$. From the problem statement we get that:

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{A}-\mathbf{C}\| \quad (1.33.1)$$

$$\|\mathbf{D}-\mathbf{B}\| = \|\mathbf{D}-\mathbf{C}\| \quad (1.33.2)$$

$$\mathbf{P}-\mathbf{C} = K_2(\mathbf{B}-\mathbf{C}) \quad (1.33.3)$$

$$\mathbf{P}-\mathbf{B} = K_1(\mathbf{B}-\mathbf{C}) \quad (1.33.4)$$

$$\begin{aligned}
&\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{A}-\mathbf{C}\| \\
&\implies \|\mathbf{A}-\mathbf{B}\|^2 = \|\mathbf{A}-\mathbf{C}\|^2 \\
&\implies \|(\mathbf{A}-\mathbf{P})+(\mathbf{P}-\mathbf{B})\|^2 = \|(\mathbf{A}-\mathbf{P})+(\mathbf{P}-\mathbf{C})\|^2 \\
&\implies \|\mathbf{A}-\mathbf{P}\|^2+\|\mathbf{P}-\mathbf{B}\|^2+2(\mathbf{A}-\mathbf{P})^T(\mathbf{P}-\mathbf{B}) = \\
&\quad \|\mathbf{A}-\mathbf{P}\|^2+\|\mathbf{P}-\mathbf{C}\|^2+2(\mathbf{A}-\mathbf{P})^T(\mathbf{P}-\mathbf{C}) \\
&\implies \|\mathbf{P}-\mathbf{B}\|^2+2(\mathbf{A}-\mathbf{P})^T(\mathbf{P}-\mathbf{B}) = \\
&\quad \|\mathbf{P}-\mathbf{C}\|^2+2(\mathbf{A}-\mathbf{P})^T(\mathbf{P}-\mathbf{C}) \\
&\implies \|\mathbf{P}-\mathbf{B}\|^2-\|\mathbf{P}-\mathbf{C}\|^2 \\
&\quad +2(\mathbf{A}-\mathbf{P})^T((\mathbf{P}-\mathbf{B})-(\mathbf{P}-\mathbf{C})) = 0 \\
&\implies ((\mathbf{P}-\mathbf{B})+(\mathbf{P}-\mathbf{C}))^T((\mathbf{P}-\mathbf{B})-(\mathbf{P}-\mathbf{C})) \\
&\quad +2(\mathbf{A}-\mathbf{P})^T((\mathbf{P}-\mathbf{B})-(\mathbf{P}-\mathbf{C})) = 0 \\
&\implies (\mathbf{B}+\mathbf{C}-2\mathbf{P})^T(\mathbf{B}-\mathbf{C}) \\
&\quad +2(\mathbf{A}-\mathbf{P})^T(\mathbf{P}-\mathbf{B}-\mathbf{P}+\mathbf{C}) = 0 \\
&\implies (\mathbf{B}+\mathbf{C}-2\mathbf{P})^T(\mathbf{B}-\mathbf{C})+2(\mathbf{A}-\mathbf{P})^T(\mathbf{B}-\mathbf{C}) = 0 \\
&\implies ((\mathbf{B}+\mathbf{C}-2\mathbf{P})^T-2(\mathbf{A}-\mathbf{P})^T)(\mathbf{B}-\mathbf{C}) = 0 \\
&\implies ((\mathbf{B}+\mathbf{C}-2\mathbf{P})-(\mathbf{2A}-\mathbf{2P}))^T(\mathbf{B}-\mathbf{C}) = 0 \\
&\implies (\mathbf{B}+\mathbf{C}-\mathbf{2A})^T(\mathbf{B}-\mathbf{C}) = 0 \\
&\implies (\mathbf{B}-\mathbf{C})^T(\mathbf{B}+\mathbf{C}-\mathbf{2A}) = 0
\end{aligned} \quad (1.33.5)$$

Now, from 1.33.5

$$(\mathbf{B}-\mathbf{C})^T\left(\left(\frac{\mathbf{B}+\mathbf{C}}{\mathbf{2}}\right)-\mathbf{A}\right) = 0$$

$$\implies \left(\left(\frac{\mathbf{B}+\mathbf{C}}{\mathbf{2}}\right)-\mathbf{A}\right) \perp (\mathbf{B}-\mathbf{C}) \quad (1.33.6)$$

So, we can conclude that $\mathbf{A}-\mathbf{P}$ bisects $\mathbf{B}-\mathbf{C}$

perpendicularly. Similarly,

$$\|\mathbf{D}-\mathbf{B}\| = \|\mathbf{D}-\mathbf{C}\|$$
$$\implies \|\mathbf{D}-\mathbf{B}\|^2 = \|\mathbf{D}-\mathbf{C}\|^2$$
$$\implies \|(\mathbf{D}-\mathbf{P})+(\mathbf{P}-\mathbf{B})\|^2 = \|(\mathbf{D}-\mathbf{P})+(\mathbf{P}-\mathbf{C})\|^2$$
$$\implies \|\mathbf{D}-\mathbf{P}\|^2+\|\mathbf{P}-\mathbf{B}\|^2+2(\mathbf{D}-\mathbf{P})^T(\mathbf{P}-\mathbf{B}) = \|\mathbf{D}-\mathbf{P}\|^2+\|\mathbf{P}-\mathbf{C}\|^2+2(\mathbf{D}-\mathbf{P})^T(\mathbf{P}-\mathbf{C})$$
$$\implies \|\mathbf{P}-\mathbf{B}\|^2+2(\mathbf{D}-\mathbf{P})^T(\mathbf{P}-\mathbf{B}) = \|\mathbf{P}-\mathbf{C}\|^2+2(\mathbf{D}-\mathbf{P})^T(\mathbf{P}-\mathbf{C})$$
$$\implies \|\mathbf{P}-\mathbf{B}\|^2-\|\mathbf{P}-\mathbf{C}\|^2 + 2(\mathbf{D}-\mathbf{P})^T((\mathbf{P}-\mathbf{B})-(\mathbf{P}-\mathbf{C})) = 0$$
$$\implies ((\mathbf{P}-\mathbf{B})+(\mathbf{P}-\mathbf{C}))^T((\mathbf{P}-\mathbf{B})-(\mathbf{P}-\mathbf{C})) + 2(\mathbf{D}-\mathbf{P})^T((\mathbf{P}-\mathbf{B})-(\mathbf{P}-\mathbf{C})) = 0$$
$$\implies (\mathbf{B}+\mathbf{C}-2\mathbf{P})^T(\mathbf{B}-\mathbf{C}) + 2(\mathbf{D}-\mathbf{P})^T(\mathbf{P}-\mathbf{B}-\mathbf{P}+\mathbf{C}) = 0$$
$$\implies (\mathbf{B}+\mathbf{C}-2\mathbf{P})^T(\mathbf{B}-\mathbf{C})+2(\mathbf{D}-\mathbf{P})^T(\mathbf{B}-\mathbf{C}) = 0$$
$$\implies ((\mathbf{B}+\mathbf{C}-2\mathbf{P})^T-2(\mathbf{D}-\mathbf{P})^T)(\mathbf{B}-\mathbf{C}) = 0$$
$$\implies ((\mathbf{B}+\mathbf{C}-2\mathbf{P})-(\mathbf{2D}-\mathbf{2P}))^T(\mathbf{B}-\mathbf{C}) = 0$$
$$\implies (\mathbf{B}+\mathbf{C}-\mathbf{2D})^T(\mathbf{B}-\mathbf{C}) = 0$$
$$\implies (\mathbf{B}-\mathbf{C})^T(\mathbf{B}+\mathbf{C}-\mathbf{2D}) = 0 \tag{1.33.7}$$

Similarly, 1.33.7 dividing by 2, we get:

$$(\mathbf{B}-\mathbf{C})^T\left(\left(\frac{\mathbf{B}+\mathbf{C}}{\mathbf{2}}\right)-\mathbf{D}\right) = 0$$
$$\implies \left(\left(\frac{\mathbf{B}+\mathbf{C}}{\mathbf{2}}\right)-\mathbf{D}\right) \perp (\mathbf{B}-\mathbf{C}) \tag{1.33.8}$$

So, we can conclude that $\mathbf{D}-\mathbf{P}$ bisects $\mathbf{B}-\mathbf{C}$ perpendicularly.

1.34. AD is an altitude of an isosceles $\triangle ABC$ in which $AB = AC$. Show that

a) AD bisects BC

b) AD bisects $\angle A$.

Solution:

Given AD is altitude of the $\triangle ABC$. Therefore

$$(\mathbf{B}-\mathbf{C})^T(\mathbf{A}-\mathbf{D}) = 0 \tag{1.34}$$

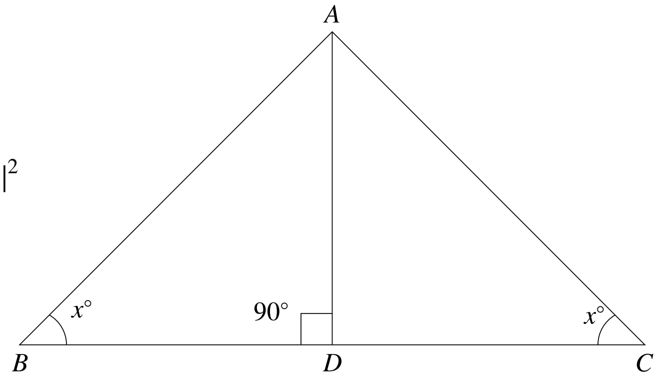


Fig. 1.34.1: $\triangle ABC$ with AD as altitude

$$\|\mathbf{A}-\mathbf{B}\|^2 = (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{B})$$
$$= (\mathbf{A}-\mathbf{B})^T((\mathbf{A}-\mathbf{D})-(\mathbf{B}-\mathbf{D}))$$
$$= (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{D})-(\mathbf{A}-\mathbf{B})^T(\mathbf{B}-\mathbf{D})$$
$$= \mathbf{A}^T(\mathbf{A}-\mathbf{D})-\mathbf{B}^T(\mathbf{A}-\mathbf{D})-(\mathbf{A}-\mathbf{B})^T(\mathbf{B}-\mathbf{D}) \tag{1.34}$$

Similarly for AC,

$$\|\mathbf{A}-\mathbf{C}\|^2 = (\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{C})$$
$$= (\mathbf{A}-\mathbf{C})^T((\mathbf{A}-\mathbf{D})-(\mathbf{C}-\mathbf{D}))$$
$$= (\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{D})-(\mathbf{A}-\mathbf{C})^T(\mathbf{C}-\mathbf{D})$$
$$= \mathbf{A}^T(\mathbf{A}-\mathbf{D})-\mathbf{C}^T(\mathbf{A}-\mathbf{D})-(\mathbf{A}-\mathbf{C})^T(\mathbf{C}-\mathbf{D}) \tag{1.34}$$

Given AB = AC. Therefore

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{A}-\mathbf{C}\| \tag{1.34}$$

By equating (1.34) and (1.34)

$$\mathbf{B}^T(\mathbf{A}-\mathbf{D})+(\mathbf{A}-\mathbf{B})^T(\mathbf{B}-\mathbf{D}) = \mathbf{C}^T(\mathbf{A}-\mathbf{D})+(\mathbf{A}-\mathbf{C})^T(\mathbf{C}-\mathbf{D})$$
$$\implies (\mathbf{B}-\mathbf{C})^T(\mathbf{A}-\mathbf{D})+(\mathbf{A}-\mathbf{B})^T(\mathbf{B}-\mathbf{D}) = (\mathbf{A}-\mathbf{C})^T(\mathbf{C}-\mathbf{D}) \quad \text{From (1.34)}$$
$$(\mathbf{A}-\mathbf{B})^T(\mathbf{B}-\mathbf{D}) = (\mathbf{A}-\mathbf{C})^T(\mathbf{C}-\mathbf{D}) \tag{1.34}$$

Since $\triangle ABC$ is isosceles angle ABD is equal to angle ACD

$$\frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{B}-\mathbf{D})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{B}-\mathbf{D}\|} = \frac{(\mathbf{A}-\mathbf{C})^T(\mathbf{C}-\mathbf{D})}{\|\mathbf{A}-\mathbf{C}\|\,\|\mathbf{C}-\mathbf{D}\|} \tag{1.34}$$

By refering the values from 1.34 and 1.34

$$\|\mathbf{B}-\mathbf{D}\| = \|\mathbf{C}-\mathbf{D}\| \tag{1.34}$$

Therefore BD = DC. In $\triangle ABD$

$$\cos\angle DAB = \frac{(\mathbf{D}-\mathbf{A})^T(\mathbf{A}-\mathbf{B})}{\|\mathbf{D}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{B}\|} \tag{1.34}$$

$$= \frac{(\mathbf{D}-\mathbf{A})^T(\mathbf{A}-\mathbf{C}+\mathbf{C}-\mathbf{B})}{\|\mathbf{D}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{B}\|} \tag{1.34}$$

$$= \frac{(\mathbf{D}-\mathbf{A})^T(\mathbf{A}-\mathbf{C})+(\mathbf{D}-\mathbf{A})^T(\mathbf{C}-\mathbf{B})}{\|\mathbf{D}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{B}\|} \tag{1.34}$$

By refering the values from (1.34) (1.34)

$$\frac{(\mathbf{D}-\mathbf{A})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{D}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{C}\|} = \cos\angle DAC \tag{1.34}$$

Therefore angle DAC is equal to angle DAB. Thus AD is the angular bisector of angle A.

1.35. Two sides AB and BC and median AM of one triangle ABC are respectively equal to sides PQ and QR and median PN of $\triangle PQR$. Show that:

a) $\triangle ABM \cong \triangle PQN$

b) $\triangle ABC \cong \triangle PQR$

Solution: Given Condition in the question are:

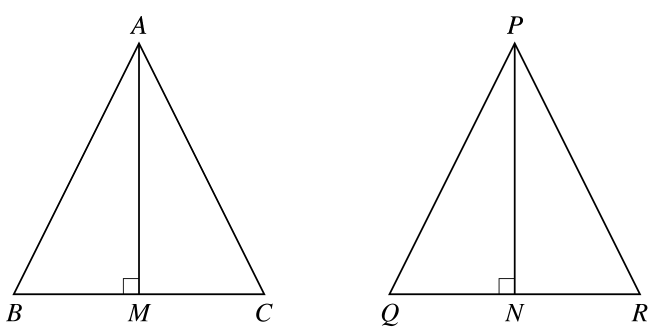


Fig. 1.35.1: $\triangle ABC$ and $\triangle PQR$

$$AB = PQ \tag{1.35}$$

$$BC = QR \tag{1.35}$$

$$AM = PN \tag{1.35}$$

As M and N are medians of triangle ABC and triangle PQR respectively, we deduce the following:

$$\mathbf{M} = \frac{\mathbf{B}+\mathbf{C}}{2} \tag{1.35}$$

$$\implies 2\mathbf{M} = (\mathbf{B}+\mathbf{C}) \tag{1.35}$$

$$\implies (\mathbf{B}-\mathbf{M}) = (\mathbf{M}-\mathbf{C}) \tag{1.35}$$

$$\implies \|\mathbf{B}-\mathbf{M}\| = \|\mathbf{M}-\mathbf{C}\| \tag{1.35}$$

Also (1.35)

$$\mathbf{N} = \frac{\mathbf{Q}+\mathbf{R}}{2} \tag{1.35}$$

$$\implies 2\mathbf{N} = (\mathbf{Q}+\mathbf{R}) \tag{1.35}$$

$$\implies (\mathbf{Q}-\mathbf{N}) = (\mathbf{N}-\mathbf{R}) \tag{1.35}$$

$$\implies \|\mathbf{Q}-\mathbf{N}\| = \|\mathbf{N}-\mathbf{R}\| \tag{1.35}$$

Refer(1.35)and (1.35)

$$\|\mathbf{B}-\mathbf{M}\| = \|\mathbf{Q}-\mathbf{N}\| \tag{1.35}$$

Hence in triangle ABM and triangle PQN sides AB,BM and MA are equal to PQ,QN and NP so by SSS congruency criteria

$$\triangle ABM \cong \triangle PQN \tag{1.35}$$

Now for proving congruence of triangle ABC and triangle PQR we know that the corresponding angles of congruent triangles are equal and to prove that we make a hypothesis and proceed as follows

$$\angle ABM = \angle PQN \tag{1.35}$$

and one of the proved condition

$$BM = QN \tag{1.35}$$

Refer(1.35)

$$\cos\angle ABM = \cos\angle PQN \tag{1.35}$$

$$\frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{M})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{B}-\mathbf{M}\|} = \frac{(\mathbf{Q}-\mathbf{P})^T(\mathbf{Q}-\mathbf{N})}{\|\mathbf{Q}-\mathbf{P}\|\,\|\mathbf{Q}-\mathbf{N}\|} \tag{1.35}$$

Equating (1.35)

$$\implies \frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{M})}{\|\mathbf{B}-\mathbf{A}\|} = \frac{(\mathbf{Q}-\mathbf{P})^T(\mathbf{Q}-\mathbf{N})}{\|\mathbf{Q}-\mathbf{P}\|} \tag{1.35}$$

It can be shown that

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{M}) = \|\mathbf{A}-\mathbf{B}\|^2 - (\mathbf{A}-\mathbf{M})^T(\mathbf{A}-\mathbf{B}) \quad (1.35)$$

$$(\mathbf{Q}-\mathbf{P})^T(\mathbf{Q}-\mathbf{N}) = \|\mathbf{P}-\mathbf{Q}\|^2 - (\mathbf{P}-\mathbf{N})^T(\mathbf{P}-\mathbf{Q}) \quad (1.35)$$

Substituting (1.35) and (1.35) in (1.35)

$$\|\mathbf{A}-\mathbf{B}\| - \frac{(\mathbf{A}-\mathbf{M})^T(\mathbf{A}-\mathbf{B})}{\|\mathbf{B}-\mathbf{A}\|} = \|\mathbf{P}-\mathbf{Q}\| - \frac{(\mathbf{P}-\mathbf{N})^T(\mathbf{P}-\mathbf{Q})}{\|\mathbf{Q}-\mathbf{P}\|} \quad (1.35)$$

$$\|\mathbf{A}-\mathbf{B}\| - \|\mathbf{A}-\mathbf{M}\|\cos\angle BAM = \|\mathbf{P}-\mathbf{Q}\| - \|\mathbf{P}-\mathbf{N}\|\cos\angle QPN \quad (1.35)$$

Refer (1.35) and (1.35)

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{P}-\mathbf{Q}\| \quad (1.35)$$

$$\|\mathbf{A}-\mathbf{M}\| = \|\mathbf{P}-\mathbf{N}\| \quad (1.35)$$

$$\therefore \cos\angle BAM = \cos\angle QPN \quad (1.35)$$

$$\implies \angle BAM = \angle QPN \quad (1.35)$$

Hence our hypothesis is right as we prove that corresponding angles of congurent triangles are equal. So we get

$$\angle ABM = \angle PQN \quad (1.35)$$

$$\therefore \angle ABC = \angle PQR \quad (1.35)$$

So by applying SAS criteria we conclude that

$$\triangle ABC \cong \triangle PQR \quad (1.35)$$

1.36. BE and CF are two equal altitudes of a triangle ABC. Using RHS congruence rule, prove that the triangle ABC is isosceles. **Solution:** BE and CF are two equal altitudes of a triangle ABC. Given:-
1) Altitudes are Equal means their magnitude are same

$$\|\mathbf{E}-\mathbf{B}\| = \|\mathbf{F}-\mathbf{C}\| \quad (1.36)$$

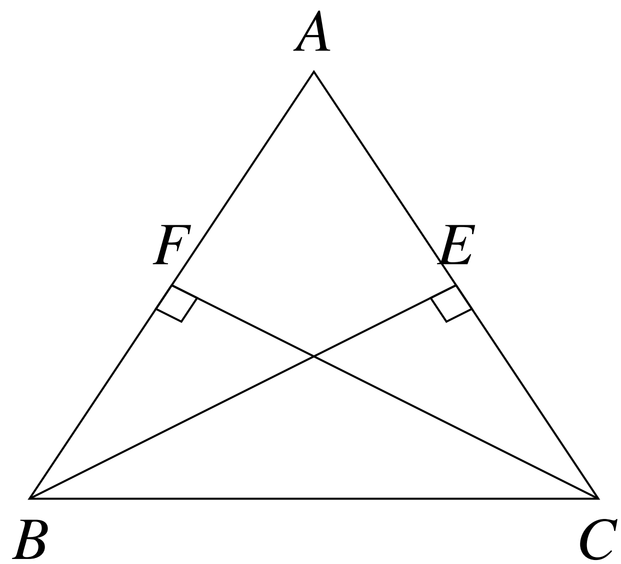


Fig. 1.36.1: Triangle with equal altitudes on two sides

2) Altitude makes right angle at the base therefore $\cos 90 = 0$ therefore FC $\perp$ BF and EB $\perp$ CE where $\mathbf{m}$ is the directional vectors.

$$\mathbf{m}_{FC}\mathbf{m}_{BF} = 0 \quad (1.36)$$
$$\mathbf{m}_{EB}\mathbf{m}_{CE} = 0 \quad (1.36)$$

From (1.36)

$$(\mathbf{B}-\mathbf{F})^T(\mathbf{F}-\mathbf{C}) = \mathbf{0} \quad (\mathbf{F}-\mathbf{C})^T(\mathbf{B}-\mathbf{F}) = \mathbf{0} \quad (1.36)$$

From (1.36) and using (1.36)

$$(\mathbf{B}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) \quad (1.36)$$
$$= (\mathbf{B}-\mathbf{F}+\mathbf{F}-\mathbf{C})^T(\mathbf{B}-\mathbf{F}+\mathbf{F}-\mathbf{C})) \quad (1.36)$$

$$= (\mathbf{B}-\mathbf{F})^T(\mathbf{B}-\mathbf{F}) + (\mathbf{F}-\mathbf{C})^T(\mathbf{F}-\mathbf{C}) \quad (1.36)$$

$$\|\mathbf{B}-\mathbf{C}\|^2 = \|\mathbf{B}-\mathbf{F}\|^2 + \|\mathbf{F}-\mathbf{C}\|^2 \quad (1.36)$$

Similarly
From (1.36)

$$(\mathbf{E}-\mathbf{B})^T(\mathbf{E}-\mathbf{C}) = \mathbf{0} \quad (\mathbf{E}-\mathbf{C})^T(\mathbf{B}-\mathbf{E}) = \mathbf{0} \quad (1.36)$$

From (1.36) and using (1.36)

$$(\mathbf{B}-\mathbf{C})^T(\mathbf{B}-\mathbf{C}) \quad (1.36)$$
$$= (\mathbf{B}-\mathbf{E}+\mathbf{E}-\mathbf{C})^T(\mathbf{B}-\mathbf{E}+\mathbf{E}-\mathbf{C}) \quad (1.36)$$

$$= (\mathbf{B}-\mathbf{E})^T(\mathbf{B}-\mathbf{E}) + (\mathbf{E}-\mathbf{C})^T(\mathbf{E}-\mathbf{C}) \quad (1.36)$$

$$\|\mathbf{B}-\mathbf{C}\|^2 = \|\mathbf{B}-\mathbf{E}\|^2 + \|\mathbf{E}-\mathbf{C}\|^2 \quad (1.36)$$

Equating (1.36) and (1.36) and using (1.36)

$$\|\mathbf{B}-\mathbf{F}\|^2 + \|\mathbf{F}-\mathbf{C}\|^2 = \|\mathbf{B}-\mathbf{E}\|^2 + \|\mathbf{E}-\mathbf{C}\|^2 \quad (1.36)$$

$$\|\mathbf{B}-\mathbf{F}\|^2 = \|\mathbf{E}-\mathbf{C}\|^2 \quad (1.36)$$

$$= \|\mathbf{B}-\mathbf{F}\| = \|\mathbf{E}-\mathbf{C}\| \quad (1.36)$$

Let $\angle FBC = \theta_1$ and $\angle EBC = \theta_2$

$$(\mathbf{B}-\mathbf{F})^T(\mathbf{B}-\mathbf{C}) = \|\mathbf{B}-\mathbf{F}\|\,\|\mathbf{B}-\mathbf{C}\|\cos\theta_1 \quad (1.36)$$

$$\cos\theta_1 = \frac{(\mathbf{B}-\mathbf{F})^T(\mathbf{B}-\mathbf{C})}{\|\mathbf{B}-\mathbf{F}\|\,\|\mathbf{B}-\mathbf{C}\|} \quad (1.36)$$

$$\cos\theta_1 = \frac{(\mathbf{B}-\mathbf{F})^T(\mathbf{B}-\mathbf{F}+\mathbf{F}-\mathbf{C})}{\|\mathbf{B}-\mathbf{F}\|\,\|\mathbf{B}-\mathbf{C}\|} \quad (1.36)$$

$$\cos\theta_1 = \frac{(\mathbf{B}-\mathbf{F})^T(\mathbf{B}-\mathbf{F}) + (\mathbf{B}-\mathbf{F})^T(\mathbf{F}-\mathbf{C})}{\|\mathbf{B}-\mathbf{F}\|\,\|\mathbf{B}-\mathbf{C}\|} \quad (1.36)$$

From (1.36)

$$\cos\theta_1 = \frac{(\mathbf{B}-\mathbf{F})^T(\mathbf{B}-\mathbf{F})}{\|\mathbf{B}-\mathbf{F}\|\,\|\mathbf{B}-\mathbf{C}\|} \quad (1.36)$$

$$\cos\theta_1 = \frac{\|\mathbf{B}-\mathbf{F}\|^2}{\|\mathbf{B}-\mathbf{F}\|\,\|\mathbf{B}-\mathbf{C}\|} \quad (1.36)$$

$$\cos\theta_1 = \frac{\|\mathbf{B}-\mathbf{F}\|}{\|\mathbf{B}-\mathbf{C}\|} \quad (1.36)$$

Similarly for $\angle EBC = \theta_2$

$$(\mathbf{C}-\mathbf{E})^T(\mathbf{B}-\mathbf{C}) = \|\mathbf{C}-\mathbf{E}\|\,\|\mathbf{B}-\mathbf{C}\|\cos\theta_2 \quad (1.36)$$

$$\cos\theta_2 = \frac{(\mathbf{C}-\mathbf{E})^T(\mathbf{B}-\mathbf{C})}{\|\mathbf{C}-\mathbf{E}\|\,\|\mathbf{B}-\mathbf{C}\|} \quad (1.36)$$

$$\cos\theta_2 = \frac{(\mathbf{C}-\mathbf{E})^T(\mathbf{B}-\mathbf{E}+\mathbf{E}-\mathbf{C})}{\|\mathbf{C}-\mathbf{E}\|\,\|\mathbf{B}-\mathbf{C}\|} \quad (1.36)$$

$$\cos\theta_2 = \frac{(\mathbf{C}-\mathbf{E})^T(\mathbf{B}-\mathbf{E}) + (\mathbf{C}-\mathbf{E})^T(\mathbf{E}-\mathbf{C})}{\|\mathbf{C}-\mathbf{E}\|\,\|\mathbf{B}-\mathbf{C}\|} \quad (1.36)$$

From (1.36)

$$\cos\theta_2 = \frac{(\mathbf{C}-\mathbf{E})^T(\mathbf{C}-\mathbf{E})}{\|\mathbf{C}-\mathbf{E}\|\,\|\mathbf{B}-\mathbf{C}\|} \quad (1.36)$$

$$\cos\theta_2 = \frac{\|\mathbf{C}-\mathbf{E}\|^2}{\|\mathbf{C}-\mathbf{E}\|\,\|\mathbf{B}-\mathbf{C}\|} \quad (1.36)$$

$$\cos\theta_2 = \frac{\|\mathbf{C}-\mathbf{E}\|}{\|\mathbf{B}-\mathbf{C}\|} \quad (1.36)$$

From (1.36) we know $\|\mathbf{B}-\mathbf{F}\| = \|\mathbf{E}-\mathbf{C}\|$ we conclude

$$\cos\theta_1 = \cos\theta_2 \implies \theta_1 = \theta_2 \quad (1.36)$$

So the sides opposite to equal angles are equal. Hence AB=AC hence the given Triangle is isosceles.

1.37. ABC is an isosceles triangle with $AB = AC$. Draw $AP \perp BC$ to show that $\angle B = \angle C$.

1.38. $\triangle ABC$ is an isosceles triangle in which $AB = AC$. Side BA is produced to D such that $AD = AB$. Show that $\angle BCD$ is a right angle.

Solution: We are given that AB = AC. So

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{A}-\mathbf{C}\| \quad (1.38)$$

Also BA is produced to D such that AB = AD. Therefore we have

$$\mathbf{A}-\mathbf{B} = \mathbf{D}-\mathbf{A} \quad (1.38)$$

Taking dot product of vectors $(\mathbf{B}-\mathbf{C})$ and $(\mathbf{D}-\mathbf{C})$ we get

$$\begin{aligned}&(\mathbf{B}-\mathbf{C})^T(\mathbf{D}-\mathbf{C})\\ &= (\mathbf{B}-\mathbf{A}+\mathbf{A}-\mathbf{C})^T(\mathbf{D}-\mathbf{A}+\mathbf{A}-\mathbf{C})\\ &= ((\mathbf{B}-\mathbf{A})+(\mathbf{A}-\mathbf{C}))^T((\mathbf{D}-\mathbf{A})+(\mathbf{A}-\mathbf{C}))\end{aligned}$$

using (1.38) we get

$$\begin{aligned}&(\mathbf{B}-\mathbf{C})^T(\mathbf{D}-\mathbf{C})\\ &= ((\mathbf{B}-\mathbf{A})+(\mathbf{A}-\mathbf{C}))^T((\mathbf{D}-\mathbf{A})+(\mathbf{A}-\mathbf{C}))\\ &= ((\mathbf{B}-\mathbf{A})+(\mathbf{A}-\mathbf{C}))^T((\mathbf{A}-\mathbf{B})+(\mathbf{A}-\mathbf{C}))\\ &= (-(\mathbf{A}-\mathbf{B})+(\mathbf{A}-\mathbf{C}))^T((\mathbf{A}-\mathbf{B})+(\mathbf{A}-\mathbf{C}))\\ &= -\|\mathbf{A}-\mathbf{B}\|^2 - (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C})\\ &\quad + (\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{B}) + \|\mathbf{A}-\mathbf{C}\|^2\end{aligned}$$

now $(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C})$ and $(\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{B})$ are both dot product of vectors $(\mathbf{A}-\mathbf{B})$ and

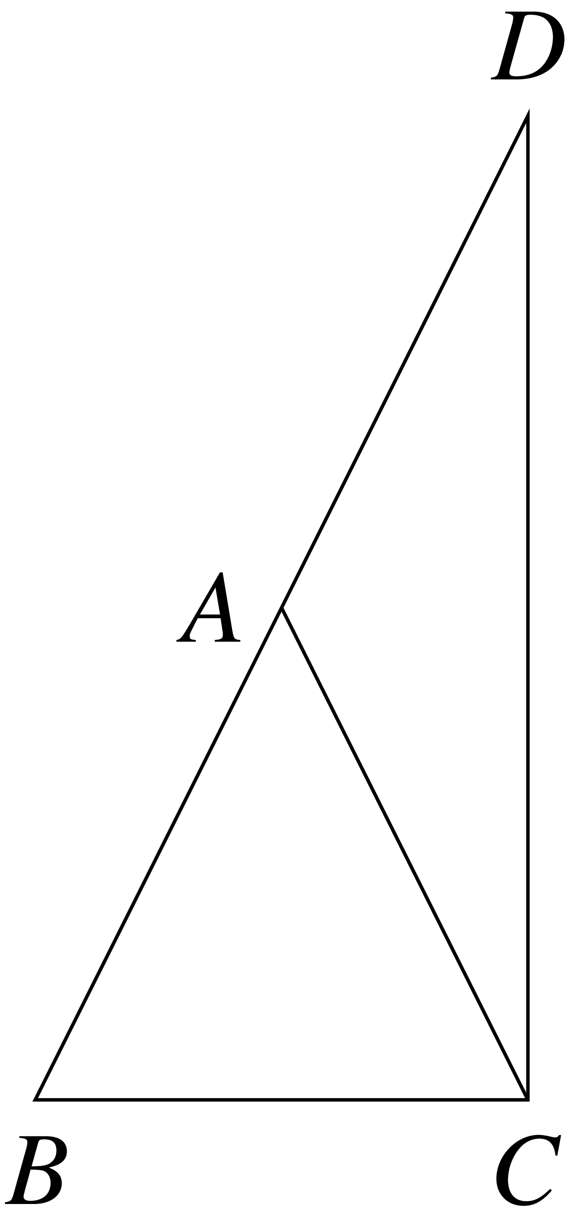


Fig. 1.38.1

$(\mathbf{A}-\mathbf{C})$,therefore

$$\begin{aligned}&(\mathbf{B}-\mathbf{C})^T(\mathbf{D}-\mathbf{C})\\ &= -\|\mathbf{A}-\mathbf{B}\|^2 - (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C})\\ &\qquad + (\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{B}) + \|\mathbf{A}-\mathbf{C}\|^2\\ &= -\|\mathbf{A}-\mathbf{B}\|^2 + \|\mathbf{A}-\mathbf{C}\|^2\end{aligned}$$

using (1.38) we get

$$(\mathbf{B}-\mathbf{C})^T(\mathbf{D}-\mathbf{C}) = -\|\mathbf{A}-\mathbf{B}\|^2 + \|\mathbf{A}-\mathbf{C}\|^2 = 0 \quad (1.38)$$

since $(\mathbf{B}-\mathbf{C})^T(\mathbf{D}-\mathbf{C}) = 0$,therefore BC $\perp$ CD and $\angle BCD$ is a right angle.

1.39. ABC is a right angled triangle in which $\angle A = 90°$ and $AB = AC$. Find $\angle B$ and $\angle C$.

1.40. Show that in a right angled triangle, the hypotenuse is the longest side.

1.41. Sides AB and AC of $\triangle ABC$ are extended to points P and Q respectively. Also, $\angle PBC < \angle QCB$. Show that $AC > AB$. **Solution:**

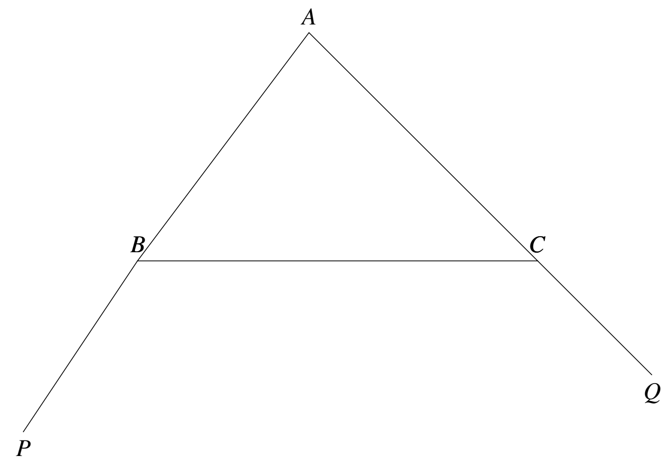


Fig. 1.41.1: $\triangle ABC$

Given conditions are : $\triangle ABC$ is a triangle having $\angle PBC < \angle QCB$

$$\frac{(\mathbf{P}-\mathbf{B})^T(\mathbf{B}-\mathbf{C})}{\|\mathbf{P}-\mathbf{B}\|\,\|\mathbf{B}-\mathbf{C}\|} < \frac{(\mathbf{Q}-\mathbf{C})^T(\mathbf{C}-\mathbf{B})}{\|\mathbf{Q}-\mathbf{C}\|\,\|\mathbf{C}-\mathbf{B}\|} \quad (1.41)$$

Let, $\mathbf{P}-\mathbf{B} = K_1(\mathbf{A}-\mathbf{B})$ and $\mathbf{Q}-\mathbf{C} = K_2(\mathbf{A}-\mathbf{C})$ Also, $\|\mathbf{C}-\mathbf{B}\| = \|\mathbf{B}-\mathbf{C}\|$

$$\implies \frac{(K_1(\mathbf{A}-\mathbf{B}))^T(\mathbf{B}-\mathbf{C})}{\|K_1(\mathbf{A}-\mathbf{B})\|} < \frac{(K_2(\mathbf{A}-\mathbf{C}))^T(\mathbf{C}-\mathbf{B})}{\|K_2(\mathbf{A}-\mathbf{C})\|} \quad (1.41)$$

$$\implies \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{B}-\mathbf{C})}{\|\mathbf{A}-\mathbf{B}\|} < \frac{(\mathbf{A}-\mathbf{C})^T(\mathbf{C}-\mathbf{B})}{\|\mathbf{A}-\mathbf{C}\|} \quad (1.41)$$

On, rewriting the equation above;

$$\implies \frac{\|\mathbf{A}-\mathbf{B}\|^2 - (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{A}-\mathbf{B}\|} < \frac{\|\mathbf{A}-\mathbf{C}\|^2 - (\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{B})}{\|\mathbf{A}-\mathbf{C}\|} \quad (1.41)$$

Upon rearranging the terms, we have

$$\implies \|\mathbf{A}-\mathbf{B}\| - \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{A}-\mathbf{B}\|} < \|\mathbf{A}-\mathbf{C}\| - \frac{(\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{B})}{\|\mathbf{A}-\mathbf{C}\|} \quad (1.41)$$

We know that

$$\cos\angle BAC = \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{C}\|} \quad (1.41)$$

Re-writing (1.41)

$$\|\mathbf{A}-\mathbf{B}\| - \|\mathbf{A}-\mathbf{C}\|\cos\angle BAC < \|\mathbf{A}-\mathbf{C}\| - \|\mathbf{A}-\mathbf{B}\|\cos\angle BAC \quad (1.41)$$

Upon rearranging,

$$\implies \|\mathbf{A}-\mathbf{B}\| + \|\mathbf{A}-\mathbf{B}\|\cos\angle BAC < \|\mathbf{A}-\mathbf{C}\| + \|\mathbf{A}-\mathbf{C}\|\cos\angle BAC \quad (1.41)$$

$$\implies \|\mathbf{A}-\mathbf{B}\|(1+\cos\angle BAC) < \|\mathbf{A}-\mathbf{C}\|(1+\cos\angle BAC) \quad (1.41)$$

Therefore, $\|\mathbf{A}-\mathbf{B}\| < \|\mathbf{A}-\mathbf{C}\|$
Hence Proved

1.42. Line segments AD and BC intersect at O and form $\triangle OAB$ and $\triangle ODC$. $\angle B < \angle A$ and $\angle C < \angle D$. Show that $AD < BC$.
1.43. AB and CD are respectively the smallest and longest sides of a quadrilateral $ABCD$. Show that $\angle A > \angle C$ and $\angle B > \angle D$.
1.44. In $\triangle PQR, PR > PQ$ and PS bisects $\angle QPR$. Prove that $\angle PSR > \angle PSQ$.
1.45. Show that of all line segments drawn from a given point not on it, the perpendicular line segment is the shortest.
1.46. $ABCD$ is a trapezium with $AB \parallel DC$. E and F are points on non-parallel sides AD and BC respectively such that EF is parallel to AB. Show that $\frac{AE}{ED} = \frac{BF}{FC}$.
1.47. ST is a line joining two points on PQ and PR in $\triangle PQR$. If $\frac{PS}{SQ} = \frac{PT}{TR}$ and $\angle PST = \angle PRQ$, prove that PQR is an isosceles triangle.
1.48. If $LM \parallel CB$ and $LN \parallel CD$, prove that $\frac{AM}{AB} = \frac{AN}{AD}$.
1.49. D is a point on AB and E, F are points on BC such that $DE \parallel AC$ and $DF \parallel AE$. Prove that $\frac{BF}{FE} = \frac{BE}{EC}$.
1.50. O is a point in the interior of $\triangle ABC$. D is a point on OA. If $DE \parallel OB$ and $DF \parallel OC$. Show that $EF \parallel BC$.
1.51. O is a point in the interior of $\triangle PQR$. $A, BandC$ are points on OP, OQ and OR respectively such that $AB \parallel PQ$ and $AC \parallel PR$. Show that $BC \parallel QR$.
1.52. $ABCD$ is a trapezium in which $AB \parallel DC$ and its diagonals intersect each other at the point O. Show that $\frac{AO}{BO} = \frac{CO}{DO}$
1.53. The diagonals of a quadrilateral $ABCD$ intersect each other at the point O such that $\frac{AO}{BO} = \frac{CO}{DO}$. Show that $ABCD$ is a trapezium.
1.54. $PQ \parallel RS$ and PS intersects QR at O. Show that $\triangle OPQ \sim \triangle ORS$.
1.55. CM and RN are respectively the medians of $\triangle ABC$ and $\triangle PQR$. If $\triangle ABC \sim \triangle PQR$, prove that
   a) $\triangle AMC \sim \triangle PNR$
   b) $\frac{CM}{RN} = \frac{AB}{PQ}$
   c) $\triangle CMB \sim \triangle RNQ$
1.56. Diagonals AC and BD of a trapezium $ABCD$ with $AB \parallel DC$ intersect each other at the point O. Using a similarity criterion for two triangles, show that $\frac{OA}{OC} = \frac{OB}{OD}$
1.57. In $\triangle PQR$, QP is extended to T and S is a point on QR such that $\frac{QR}{QS} = \frac{QT}{PR}$. If $\angle PRQ = \angle PQS$, show that that $\triangle PQS \sim \triangle TQR$.
1.58. S and T are points on sides PR and QR of $\triangle PQR$ such that $\angle P = \angle RTS$. Show that $\triangle RPQ \sim \triangle RTS$.
1.59. In $\triangle ABC$, D and E are points on the sides AB and AC respectively. If $\triangle ABE \cong \triangle ACD$, show that $\triangle ADE \sim \triangle ABC$.
1.60. Altitudes AD and CE of $\triangle ABC$ intersect each other at the point P. Show that:
   a) $\triangle AEP \sim \triangle CDP$
   b) $\triangle ABD \sim \triangle CBE$
   c) $\triangle AEP \sim \triangle ADB$
   d) $\triangle PDC \sim \triangle BEC$
1.61. E is a point on the side AD produced of a parallelogram $ABCD$ and BE intersects CD at F. Show that $\triangle ABE \sim \triangle CFB$.
1.62. ABC and AMP are two right triangles, right angled at B and M respectively. M lies on AC and AB is extended to meet P. Prove that:
   a) $\triangle ABC \sim \triangle AMP$
   b) $\frac{CA}{PA} = \frac{BC}{MP}$
1.63. CD and GH are respectively the bisectors of $\angle ACB$ and $\angle EGF$ such that D and H lie on sides AB and FE of $\triangle ABC$ and $\triangle EFG$

respectively. If $\triangle ABC \sim \triangle FEG$, show that:

1.64. $\frac{CD}{GH} = \frac{AC}{FG}$
1.65. $\triangle DCB \sim \triangle HGE$
1.66. $\triangle DCA \sim \triangle HGF$
1.67. E is a point on side CB produced of an isosceles $\triangle ABC$ with $AB = AC$. If $AD \perp BC$ and $EF \perp AC$, prove that $\triangle ABD \sim \triangle ECF$.
1.68. Sides AB and BC and median AD of a $\triangle ABC$ are respectively proportional to sides PQ and QR and median PM of $\triangle PQR$. Show that $\triangle ABC \sim \triangle PQR$.
1.69. D is a point on the side BC of a $\triangle ABC$ such that $\angle ADC = \angle BAC$. Show that $CA^2 = CB.CD$.
1.70. Sides AB and AC and median AD of a $\triangle ABC$ are respectively proportional to sides PQ and PR and median PM of another $\triangle PQR$. Show that $\triangle ABC \sim \triangle PQR$.
1.71. If AD and PM are medians of $\triangle sABC$ and PQR, respectively where $\triangle ABC \sim \triangle PQR$, prove that $\frac{AB}{PQ} = \frac{AD}{PM}$
1.72. The line segment XY is parallel to side AC of $\triangle ABC$ and it divides the triangle into two parts of equal areas. Find the ratio $\frac{AX}{AB}$
1.73. Diagonals of a trapezium $ABCD$ with $AB \parallel DC$ intersect each other at the point O. If $AB = 2CD$, find the ratio of the areas of $\triangle sAOB$ and COD.
1.74. ABC and DBC are two triangles on the same base BC. If AD intersects BC at O, show that $\frac{ar(ABC)}{ar(DBC)} = \frac{AO}{DO}$.
1.75. If the areas of two similar triangles are equal, prove that they are congruent.
1.76. D, E and F are respectively the mid-points of sides AB, BC and CA of $\triangle ABC$. Find the ratio of the areas of $\triangle DEF$ and $\triangle ABC$.
1.77. Prove that the ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding medians.
1.78. Prove that the area of an equilateral triangle described on one side of a square is equal to half the area of the equilateral triangle described on one of its diagonals.
1.79. ABC and BDE are two equilateral triangles such that D is the mid-point of BC. Find the ratio of the areas of triangles ABC and BDE.
1.80. The sides of two similar triangles are in the ratio 4 : 9. Find the ratio the area of these triangles are in the ratio
1.81. In $\triangle ABC, \angle ACB = 90°$ and $CD \perp AB$. Prove that $\frac{BC^2}{AC^2} = \frac{BD}{AD}$.
1.82. In $\triangle ABC$, if $AD \perp BC$, prove that $AB^2+CD^2 = BD^2 + AC^2$.
1.83. BL and CM are medians of a $\triangle ABC$ right angled at A. Prove that $4(BL^2 + CM^2) = 5BC^2$.
1.84. O is any point inside a rectangle $ABCD$. Prove that $OB^2 + OD^2 = OA^2 + OC^2$.
1.85. PQR is a triangle right angled at P and M is a point on QR such that $PM \perp QR$. Show that $PM^2 = QM.MR$.
1.86. ABD is a triangle right angled at A and $AC \perp BD$. Show that
   a) $AB^2 = BC.BD$
   b) $AC^2 = BC.DC$
   c) $AD^2 = BD.CD$
1.87. ABC is an isosceles triangle right angled at C. Prove that $AB^2 = 2AC^2$.
1.88. ABC is an isosceles triangle with $AC = BC$. If $AB^2 = 2AC^2$, prove that ABC is a right triangle.
1.89. ABC is an equilateral triangle of side $2a$. Find each of its altitudes.
1.90. Prove that the sum of the squares of the sides of a rhombus is equal to the sum of the squares of its diagonals.
1.91. O is a point in the interior of a $\triangle ABC, OD \perp BC, OE \perp ACandOF \perp AB$. Show that
   a) $OA^2 + OB^2 + BD^2 – OD2 – OE2 – OF2 = AF^2 + BD^2 + CE^2$.
   b) $AF^2 + BD^2 + CE^2 = AE^2 + CD^2 + BF^2$.
1.92. D and E are points on the sides CA and CB respectively of a $\triangle ABC$ right angled at C. Prove that $AE^2 + BD^2 = AB^2 + DE^2$.
1.93. The perpendicular from A on side BC of a $\triangle ABC$ intersects BC at D such that $DB = 3CD$. Prove that $2AB^2 = 2AC^2 + BC^2$.
1.94. In an equilateral $\triangle ABC$, D is a point on side BC such that $BD = \frac{1}{3}BC$. Prove that $9AD^2 = 7AB^2$.
1.95. In an equilateral triangle, prove that three times the square of one side is equal to four times the square of one of its altitudes.
1.96. PS is the bisector of $\angle QPR$ of $\triangle PQR$. Prove that $\frac{QS}{SR} = \frac{PQ}{PR}$
1.97. D is a point on hypotenuse AC of $\triangle ABC$, such that $BD \perp AC, DM \perp BC$ and $DN \perp AB$. Prove that :
   a) DM2 = DN . MC
   b) DN2 = DM . AN
1.98. ABC is a triangle in which $\angle ABC > 90°$ and $AD \perp CB$ produced. Prove that $AC^2 = AB^2 +$

$BC^2 + 2BC.BD$.

1.99. ABC is a triangle in which $\angle ABC < 90°$ and $AD \perp BC$. Prove that $AC^2 = AB^2 + BC^2 - 2BC.BD$.

1.100. AD is a median of a $\triangle ABC$ and $AM \perp BC$. Prove that :

a) $AC^2 = AD^2 + BC.DM + \left(\frac{BC}{2}\right)^2$

Solution:

In a right angled triangle, square of hypotenuse is equal to sum of squares of the other two sides. Also, a median divides a side into two equal halves.

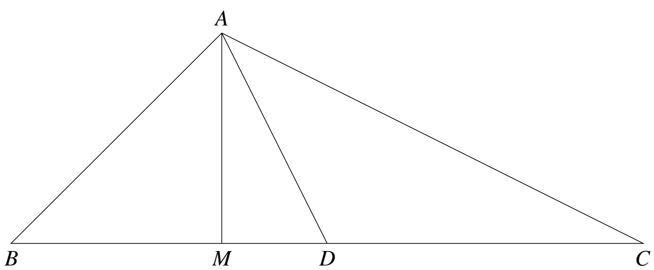


Fig. 1.100.1: $\triangle ABC$ with AD as median and $AM \perp BC$

See Fig.1.100.1, It is given that :

$$\|\mathbf{B} - \mathbf{D}\| = \|\mathbf{D} - \mathbf{C}\| \tag{1.100}$$

$$AM \perp BC \tag{1.100}$$

We have to prove that :

$$\|\mathbf{A} - \mathbf{C}\|^2 = \|\mathbf{A} - \mathbf{D}\|^2 + \|\mathbf{B} - \mathbf{C}\| \, \|\mathbf{D} - \mathbf{M}\| + \left\|\frac{\mathbf{B} - \mathbf{C}}{2}\right\|^2 \tag{1.100}$$

From $\triangle AMC$, we know that :

$$\|\mathbf{A} - \mathbf{C}\|^2 = \|\mathbf{A} - \mathbf{M}\|^2 + \|\mathbf{M} - \mathbf{C}\|^2 \tag{1.100}$$

Using Pythagoras theorem we can write it as:

$$= \|\mathbf{A} - \mathbf{D}\|^2 - \|\mathbf{M} - \mathbf{D}\|^2 + \|\mathbf{M} - \mathbf{C}\|^2 \tag{1.100}$$

$$= \|\mathbf{A} - \mathbf{D}\|^2 - \|\mathbf{M} - \mathbf{D}\|^2 + (\|\mathbf{M} - \mathbf{D}\| + \|\mathbf{D} - \mathbf{C}\|)^2 \tag{1.100}$$

Expanding $(a + b)^2$ and solving, we get :

$$= \|\mathbf{A} - \mathbf{D}\|^2 + 2\,\|\mathbf{M} - \mathbf{D}\| \, \|\mathbf{D} - \mathbf{C}\| + \|\mathbf{D} - \mathbf{C}\|^2 \tag{1.100}$$

As AD is a median, we can write :

$$\|\mathbf{B} - \mathbf{D}\| = \|\mathbf{D} - \mathbf{C}\| = \frac{\|\mathbf{B} - \mathbf{C}\|}{2} \tag{1.100}$$

Using (1.100), we can re-write (1.100) as

$$= \|\mathbf{A} - \mathbf{D}\|^2 + 2\,\|\mathbf{M} - \mathbf{D}\| \frac{\|\mathbf{B} - \mathbf{C}\|}{2} + \left\|\frac{\mathbf{B} - \mathbf{C}}{2}\right\|^2$$

$$= \|\mathbf{A} - \mathbf{D}\|^2 + \|\mathbf{M} - \mathbf{D}\| \, \|\mathbf{B} - \mathbf{C}\| + \left\|\frac{\mathbf{B} - \mathbf{C}}{2}\right\|^2 \tag{1.100}$$

Hence it is proved.

b) $AB^2 = AD^2 - BC.DM + \left(\frac{BC}{2}\right)^2$

Solution:

c) $AC^2 + AB^2 = 2AD^2 + \frac{1}{2}BC^2$

1.101. Prove that the sum of the squares of the diagonals of parallelogram is equal to the sum of the squares of its sides.

Solution:

Given a parallelogram ABCD we have to prove that

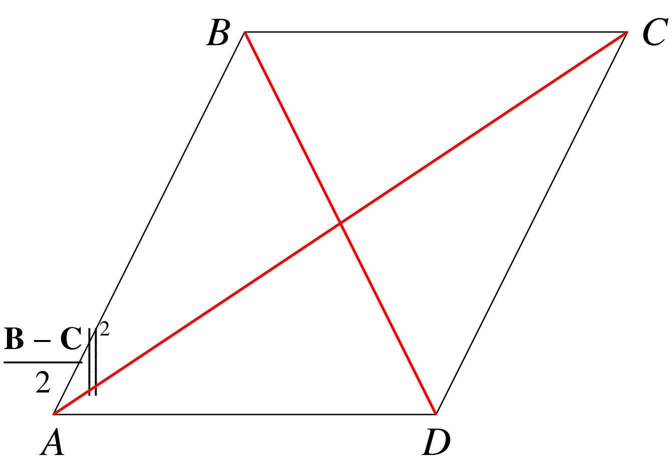


Fig. 1.101.1: parallelogram $ABCD$

$$\|\mathbf{A} - \mathbf{C}\|^2 + \|\mathbf{B} - \mathbf{D}\|^2 = \|\mathbf{A} - \mathbf{B}\|^2 + \|\mathbf{B} - \mathbf{C}\|^2 + \|\mathbf{D} - \mathbf{C}\|^2 + \|\mathbf{A} - \mathbf{D}\|^2 \tag{1.101}$$

The diagonals of parallelogram are

$$\mathbf{A} - \mathbf{C} = (\mathbf{A} - \mathbf{D}) + (\mathbf{D} - \mathbf{C}) \tag{1.101}$$

$$\mathbf{B} - \mathbf{D} = (\mathbf{A} - \mathbf{D}) - (\mathbf{A} - \mathbf{B}) \tag{1.101}$$

The sum of the squares of diagonals is

$$\|\mathbf{A}-\mathbf{C}\|^2+\|\mathbf{B}-\mathbf{D}\|^2=\|(\mathbf{A}-\mathbf{D})+(\mathbf{D}-\mathbf{C})\|^2 + \|(\mathbf{A}-\mathbf{D})-(\mathbf{A}-\mathbf{B})\|^2 \tag{1.101}$$

$$= \|\mathbf{A}-\mathbf{D}\|^2+\|\mathbf{D}-\mathbf{C}\|^2+2(\mathbf{A}-\mathbf{D})^T(\mathbf{D}-\mathbf{C})+ \|\mathbf{A}-\mathbf{D}\|^2+\|\mathbf{A}-\mathbf{B}\|^2-2(\mathbf{A}-\mathbf{D})^T(\mathbf{A}-\mathbf{B}) \tag{1.101}$$

$$= \|\mathbf{A}-\mathbf{D}\|^2+\|\mathbf{D}-\mathbf{C}\|^2+2\,\|\mathbf{A}-\mathbf{D}\|\,\|\mathbf{D}-\mathbf{C}\| \cos(180^\circ-\angle ADC)+\|\mathbf{A}-\mathbf{D}\|^2+\|\mathbf{A}-\mathbf{B}\|^2 -2\|\mathbf{A}-\mathbf{D}\|\,\|\mathbf{A}-\mathbf{B}\|\cos\angle DAB \tag{1.101}$$

In the parallelogram $ABCD$

$$\|\mathbf{A}-\mathbf{D}\|=\|\mathbf{B}-\mathbf{C}\| \tag{1.101}$$
$$\|\mathbf{A}-\mathbf{B}\|=\|\mathbf{D}-\mathbf{C}\| \tag{1.101}$$
$$\angle ADC+\angle DAB=180^\circ \tag{1.101}$$
$$\angle DAB=180^\circ-\angle ADC \tag{1.101}$$

From equation (1.101),(1.101),(1.101) and (1.101)

$$= \|\mathbf{B}-\mathbf{C}\|^2+\|\mathbf{D}-\mathbf{C}\|^2+2\,\|\mathbf{A}-\mathbf{D}\|\,\|\mathbf{A}-\mathbf{B}\|\cos \angle DAB+\|\mathbf{A}-\mathbf{D}\|^2+\|\mathbf{A}-\mathbf{B}\|^2-2\|\mathbf{A}-\mathbf{D}\|\,\|\mathbf{A}-\mathbf{B}\| \cos\angle DAB \tag{1.101}$$

Simplifying equation (1.101)

$$\|\mathbf{A}-\mathbf{C}\|^2+\|\mathbf{B}-\mathbf{D}\|^2=\|\mathbf{A}-\mathbf{B}\|^2+\|\mathbf{B}-\mathbf{C}\|^2+ \|\mathbf{D}-\mathbf{C}\|^2+\|\mathbf{A}-\mathbf{D}\|^2 \tag{1.101}$$

The equations (1.101) and (1.101) are equal,hence proved

1.102. D is a point on side BC of $\triangle ABC$ such that $\frac{BD}{CD}\frac{AB}{AC}$. Prove that AD is the bisector of $\angle BAC$.

2 Quadrilateral Exercises

2.1. Parallelograms on the same base (or equal bases) and between the same parallels are equal in area.

2.2. If a parallelogram and a triangle are on the same base and between the same parallels, then area of the triangle is half the area of the parallelogram.

2.3. The quadrilateral formed by joining the mid-points of the sides of a quadrilateral, in order, is a parallelogram.

2.4. Two parallel lines l and m are intersected by a transversal p. Show that the quadrilateral formed by the bisectors of interior angles is a rectangle.

2.5. Show that the bisectors of angles of a parallelogram form a rectangle.

2.6. $ABCD$ is a parallelogram in which P and Q are mid-points of opposite sides AB and CD. If AQ intersects DP at S and BQ intersects CP at R, show that:
   a) $APCQ$ is a parallelogram.
   b) $DPBQ$ is a parallelogram.
   c) $PSQR$ is a parallelogram.

2.7. l, m and n are three parallel lines intersected by transversals p and q such that l, m and n cut off equal intercepts AB and BC on p . Show that l, m and n cut off equal intercepts DE and EF on q also.

2.8. Parallelograms on the same base (or equal bases) and between the same parallels are equal in area.

2.9. Area of a parallelogram is the product of its base and the corresponding altitude.

2.10. Parallelograms on the same base (or equal bases) and having equal areas lie between the same parallels.

2.11. If a parallelogram and a triangle are on the same base and between the same parallels, then area of the triangle is half the area of the parallelogram.

2.12. In parallelogram $ABCD$, two points P and Q are taken on diagonal BD such that $DP = BQ$. show that
   a) $\triangle APD \cong \triangle CQB$
   b) $AP = CQ$
   c) $\triangle AQB \cong \triangle CPD$
   d) $AQ = CP$
   e) $APCQ$ is a parallelogram

2.13. $ABCD$ is a parallelogram and AP and CQ are perpendiculars from vertices A and C on diagonal BD . Show that
   a) $\triangle APB \cong \triangle CQD$
   b) $AP = CQ$

2.14. In $\triangle ABC$ and $\triangle DEF, AB = DE, AB \parallel DE, BC = EF$ and $BC \parallel EF$. Vertices A, B and C are joined to vertices D, E and F respectively. Show that

a) quadrilateral $ABED$ is a parallelogram
b) quadrilateral $BEFC$ is a parallelogram
c) $AD \parallel CF$ and $AD = CF$
d) quadrilateral $ACFD$ is a parallelogram
e) $AC = DF$
f) $\triangle ABC \cong \triangle DEF$.

2.15. $ABCD$ is a trapezium in which $AB \parallel CD$ and $AD = BC$. Show that

a) $\angle A = \angle B$
b) $\angle C = \angle D$
c) $\triangle ABC \cong \triangle BAD$
d) diagonal AC = diagonal BD

2.16. $ABCD$ is a quadrilateral in which P, Q, R and S are mid-points of the sides AB, BC, CD and DA AC is a diagonal. Show that

a) $SR \parallel AC$ and $SR = \frac{1}{2}AC$
b) $PQ = SR$
c) $PQRS$ is a parallelogram.

2.17. $ABCD$ is a rhombus and P, Q, R and S are the mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral $PQRS$ is a rectangle.

2.18. $ABCD$ is a rectangle and P, Q, R and S are mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral $PQRS$ is a rhombus.

2.19. $ABCD$ is a trapezium in which $AB \parallel DC, BD$ is a diagonal and E is the mid-point of AD. A line is drawn through $E \parallel AB$ intersecting BC at F. Show that F is the mid-point of BC.

2.20. In a parallelogram $ABCD$, E and F are the mid-points of sides AB and CD respectively . Show that the line segments AF and EC trisect the diagonal BD.

2.21. Show that the line segments joining the mid-points of the opposite sides of a quadrilateral bisect each other.

2.22. $ABCD$ is a parallelogram in which P and Q are mid-points of opposite sides AB and CD. If AQ intersects DP at S and BQ intersects CP at R, show that:

a) $APCQ$ is a parallelogram.
b) $DPBQ$ is a parallelogram.
c) $PSQR$ is a parallelogram.

2.23. l, m and n are three parallel lines intersected by transversals p and q such that l, m and n cut off equal intercepts AB and BC on p . Show that l, m and n cut off equal intercepts DE and EF on q also.

2.24. Diagonal AC of a parallelogram $ABCD$ bisects $\angle A$. show that

a) it bisects $\angle C$ also,
b) $ABCD$ is a rhombus.

2.25. $ABCD$ is a rhombus. Show that diagonal AC bisects $\angle A$ as well as $\angle C$ and diagonal BD bisects $\angle B$ as well as $\angle D$.

2.26. $ABCD$ is a rectangle in which diagonal AC bisects $\angle A$ as well as $\angle C$. Show that

a) $ABCD$ is a square
b) diagonal BD bisects $\angle B$ as well as $\angle D$.

2.27. If E, F, G and H are respectively the mid-points of the sides of a parallelogram $ABCD$, show that

$$ar(EFGH) = \frac{1}{2}ar(ABCD). \qquad (2.27.1)$$

2.28. P and Q are any two points lying on the sides DC and AD respectively of a parallelogram $ABCD$. Show that $ar(APB) = ar(BQC)$.

2.29. P is a point in the interior of a parallelogram $ABCD$. Show that

a) $ar(APB) + ar(PCD) = \frac{1}{2}ar(ABCD)$
b) $ar(APD) + ar(PBC) = ar(APB) + ar(PCD)$

2.30. $PQRS$ and $ABRS$ are parallelograms and X is any point on side BR. show that

a) $ar(PQRS) = ar(ABRS)$
b) $ar(AXS) = \frac{1}{2}ar(PQRS)$

2.31. A farmer was having a field in the form of a parallelogram $PQRS$. She took any point A on RS and joined it to points P and Q. In how many parts the fields is divided? What are the shapes of these parts? The farmer wants to sow wheat and pulses in equal portions of the field separately. How should she do it?

2.32. $ABCD$ is a quadrilateral and $BE \parallel AC$ and also BE meets DC produced at E. Show that area of $\triangle ADE$ is equal to the area of the quadrilateral $ABCD$.

2.33. E is any point on median AD of a $\triangle ABC$. Show that $ar(ABE) = ar(ACE)$.

2.34. In a $\triangle ABC, E$ is the mid-point of median AD. Show that $ar(BED) = \frac{1}{4}ar(ABC)$.

2.35. Show that the diagonals of a parallelogram divide it into four triangles of equal area.

2.36. ABC and ABD are two triangles on the same base AB. If line- segment CD is bisected by

AB at O, show that $ar(ABC) = ar(ABD)$.

2.37. D, E and F are respectively the mid-points of the sides BC, CA and AB of a $\triangle ABC$. show that
   a) $BDEF$ is a parallelogram.
   b) $ar(BDEF) = \frac{1}{2}ar(ABC)$

2.38. Diagonals AC and BD of quadrilateral $ABCD$ intersect at O such that $OB = OD$. If $AB = CD$, then show that
   a) $ar(DOC) = ar(AOB)$
   b) $ar(DCB) = ar(ACB)$
   c) $ar(DEF) = \frac{1}{4}ar(ABC)$

2.39. D and E are points on sides AB and AC respectively of $\triangle ABC$ such that $ar(DBC) = ar(EBC)$. Prove that $DE \parallel BC$.

2.40. XY is a line parallel to side BC of a $\triangle ABC$. If $BE \parallel AC$ and $CF \parallel AB$ meet XY at E and F respectively, show that $ar(ABE) = ar(ACF)$.

2.41. The side AB of a parallelogram $ABCD$ is produced to any point P. A line through A and parallel to CP meets CB produced at Q and then parallelogram $PBQR$ is completed. Show that ar(ABCD) = $ar(PBQR)$.

2.42. Diagonals AC and BD of a trapezium $ABCD$ with $AB \parallel DC$ intersect each other at O. Prove that $ar(AOD) = ar(BOC)$.

2.43. $ABCDE$ is a pentagon. A line through B parallel to AC meets DC produced at F. Show that
   a) $ar(ACB) = ar(ACF)$
   b) $ar(AEDF) = ar(ABCDE)$.

2.44. A villager Itwaari has a plot of land of the shape of a quadrilateral. The Gram Panchayat of the village decided to take over some portion of his plot from one of the corners to construct a Health Centre. Itwaari agrees to the above proposal with the condition that he should be given equal amount of land in lieu of his land adjoining his plot so as to form a triangular plot. Explain how this proposal will be implemented.

2.45. $ABCD$ is a trapezium with $AB \parallel DC$. A line parallel to AC intersects AB at X and BC at Y. Prove that $ar(ADX) = ar(ACY)$.

2.46. $AP \parallel BQ \parallel CR$. Prove that $ar(AQC) = ar(PBR)$.

2.47. Diagonals AC and BD of a quadrilateral $ABCD$ intersect at O in such a way that $ar(AOD) = ar(BOC)$. Prove that $ABCD$ is a trapezium.

2.48. $AB \parallel DC \parallel RP$. $ar(DRC) = ar(DPC)$ and $ar(BDP) = ar(ARC)$. Show that both the quadrilaterals $ABCD$ and $DCPR$ are trapeziums.

2.49. Parallelogram $ABCD$ and rectangle $ABEF$ are on the same base AB and have equal areas. Show that the perimeter of the parallelogram is greater than that of the rectangle.

2.50. In $\triangle ABC$, D and E are two points on BC such that $BD = DE = EC$. Show that $ar(ABD) = ar(ADE) = ar(AEC)$.

2.51. $ABCD, DCFE$ and $ABFE$ are parallelograms. Show that ar$(ADE) = ar(BCF)$.

2.52. $ABCD$ is a parallelogram and BC is produced to a point Q such that $AD = CQ$. If AQ intersect DC at P, show that $ar(BPC) = ar(DPQ)$. ABC and BDE are two equilateral triangles such that D is the mid-point of BC. If AE intersects BC at$F$, show that
   a) $ar(BDE) = \frac{1}{4}ar(ABC)$
   b) $ar(BDE) = \frac{1}{2}ar(BAE)$
   c) $ar(ABC) = 2ar(BEC)$
   d) $ar(BFE) = ar(AFD)$
   e) $ar(BFE) = 2ar(FED)$
   f) $ar(FED) = \frac{1}{8}ar(AFC)$

2.53. Diagonals AC and BD of a quadrilateral $ABCD$ intersect each other at P. Show that $ar(APB) \times ar(CPD) = ar(APD) \times ar(BPC)$.

2.54. P and Q are respectively the mid-points of sides AB and BC of a $\triangle ABC$ and R is the mid-point of AP, show that
   a) $ar(PRQ) = \frac{1}{2}ar(ARC)$
   b) $ar(PBQ) = ar(ARC)$
   c) $ar(RQC) = \frac{3}{8}ar(ABC)$

2.55. ABC is a right triangle right angled at A. $BCED$, $ACFG$ and $ABMN$ are squares on the sides BC, CA and AB respectively. Line segment $AX \perp DE$ meets BC at Y. Show that
   a) $\triangle MBC \cong \triangle ABD$
   b) $ar(BYXD) = ar(ABMN)$
   c) $ar(CYXE) = 2ar(FCB)$
   d) $ar(BYXD) = 2ar(MBC)$
   e) $\triangle FCB \cong \triangle ACE$
   f) $ar(CYXE) = ar(ACFG)$
   g) $ar(BCED) = ar(ABMN) + ar(ACFG)$

2.56. L is a point on the diagonal AC of quadrilateral $ABCD$. If LM —— CB and LN —— CD, prove that $\frac{AM}{AB} = \frac{AN}{AD}$

2.57. The angles of quadrilateral are in the ratio 3 : 5 : 9 : 13. Find all the angles of the quadrilateral.

3 Circle Exercises

3.1. Equal chords of a circle (or of congruent circles) subtend equal angles at the centre.
3.2. If the angles subtended by two chords of a circle (or of congruent circles) at the centre (corresponding centres) are equal, the chords are equal.
3.3. The perpendicular from the centre of a circle to a chord bisects the chord.
3.4. The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.
3.5. There is one and only one circle passing through three non-collinear points.
3.6. Equal chords of a circle (or of congruent circles) are equidistant from the centre (or corresponding centres).
3.7. Chords equidistant from the centre (or corresponding centres) of a circle (or of congruent circles) are equal.
3.8. If two arcs of a circle are congruent, then their corresponding chords are equal and conversely if two chords of a circle are equal, then their corresponding arcs (minor, major) are congruent.
3.9. Congruent arcs of a circle subtend equal angles at the centre.
3.10. The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.
3.11. Angles in the same segment of a circle are equal.
3.12. Angle in a semicircle is a right angle.
3.13. If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, the four points lie on a circle.
3.14. The sum of either pair of opposite angles of a cyclic quadrilateral is 180°.
3.15. If sum of a pair of opposite angles of a quadrilateral is 180°, the quadrilateral is cyclic.
3.16. AB is a diameter of the circle, CD is a chord equal to the radius of the circle. AC and BD when extended intersect at a point E. Prove that $\angle AEB = 60°$.
3.17. $ABCD$ is a cyclic quadrilateral in which AC and BD are its diagonals. If $\angle DBC = 55°$ and $\angle BAC = 45°$, find $\angle BCD$
3.18. Two circles intersect at two points $AandB$. AD and AC are diameters to the two circles. Prove that B lies on the line segment DC.
3.19. Prove that the quadrilateral formed (if possible) by the internal angle bisectors of any quadrilateral is cyclic.
3.20. Equal chords of a circle (or of congruent circles) subtend equal angles at the centre.
3.21. If the angles subtended by two chords of a circle (or of congruent circles) at the centre (corresponding centres) are equal, the chords are equal.
3.22. The perpendicular from the centre of a circle to a chord bisects the chord.
3.23. The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.
3.24. There is one and only one circle passing through three non-collinear points.
3.25. Equal chords of a circle (or of congruent circles) are equidistant from the centre (or corresponding centres).
3.26. Chords equidistant from the centre (or corresponding centres) of a circle (or of congruent circles) are equal.
3.27. If two arcs of a circle are congruent, then their corresponding chords are equal and conversely if two chords of a circle are equal, then their corresponding arcs (minor, major) are congruent.
3.28. Congruent arcs of a circle subtend equal angles at the centre.
3.29. The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.
3.30. Angles in the same segment of a circle are equal.
3.31. Angle in a semicircle is a right angle.
3.32. If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the line segment, the four points lie on a circle.
3.33. The sum of either pair of opposite angles of a cyclic quadrilateral is 180°.
3.34. If sum of a pair of opposite angles of a quadrilateral is 180°, the quadrilateral is cyclic.
3.35. AB is a diameter of the circle, CD is a chord equal to the radius of the circle. AC and BD when extended intersect at a point E. Prove that $\angle AEB = 60°$.
3.36. Two circles intersect at two points A and B. AD and AC are diameters to the two circles. Prove that B lies on the line segment DC.

3.37. Prove that the quadrilateral formed (if possible) by the internal angle bisectors of any quadrilateral is cyclic.
3.38. If two equal chords of a circle intersect within the circle, prove that the segments of one chord are equal to corresponding segments of the other chord.
3.39. If two equal chords of a circle intersect within the circle, prove that the line joining the point of intersection to the centre makes equal angles with the chords.
3.40. If a line intersects two concentric circles (circles with the same centre) with centre O at A, B, C and D, prove that AB = CD.
3.41. A chord of a circle is equal to the radius of the circle. Find the angle subtended by the chord at a point on the minor arc and also at a point on the major arc.
3.42. If diagonals of a cyclic quadrilateral are diameters of the circle through the vertices of the quadrilateral, prove that it is a rectangle.
3.43. If the non-parallel sides of a trapezium are equal, prove that it is cyclic.
3.44. Two circles intersect at two points B and C. Through B, two line segments ABD and PBQ are drawn to intersect the circles at A, D and P, Q respectively. Prove that $\angle ACP = \angle QCD$.
3.45. If circles are drawn taking two sides of a triangle as diameters, prove that the point of intersection of these circles lie on the third side.
3.46. ABC and ADC are two right triangles with common hypotenuse AC. Prove that $\angle CAD = \angle CBD$.
3.47. Prove that a cyclic parallelogram is a rectangle.
3.48. Prove that the line of centres of two intersecting circles subtends equal angles at the two points of intersection.
3.49. Let the vertex of an angle ABC be located outside a circle and let the sides of the angle intersect equal chords AD and CE with the circle. Prove that $\angle ABC$ is equal to half the difference of the angles subtended by the chords AC and DE at the centre.
3.50. Prove that the circle drawn with any side of a rhombus as diameter, passes through the point of intersection of its diagonals.
3.51. $ABCD$ is a parallelogram. The circle through A, B and C intersect CD (produced if necessary) at E. Prove that $AE = AD$.
3.52. AC and BD are chords of a circle which bisect each other. Prove that (i) AC and BD are diameters, (ii) $ABCD$ is a rectangle.
3.53. Bisectors of angles A, B and C of a $\triangle ABC$ intersect its circumcircle at D, E and F respectively. Prove that the angles of the $\triangle DEF$ are $90° - \frac{A}{2}, 90° - \frac{B}{2}$ and $90° - \frac{C}{2}$.
3.54. Two congruent circles intersect each other at points A and B. Through A any line segment PAQ is drawn so that P, Q lie on the two circles. Prove that $BP = BQ$.
3.55. In any $\triangle ABC$, if the angle bisector of $\angle A$ and perpendicular bisector of BC intersect, prove that they intersect on the circumcircle of the $\triangle ABC$.
3.56. The lengths of tangents drawn from an external point to a circle are equal.
3.57. Prove that in two concentric circles, the chord of the larger circle, which touches the smaller circle, is bisected at the point of contact.
3.58. Two tangents TP and TQ are drawn to a circle with centre O from an external point T. Prove that $\angle PTQ = 2\angle OPQ$.
3.59. Prove that the tangents drawn at the ends of a diameter of a circle are parallel.
3.60. Prove that the perpendicular at the point of contact to the tangent to a circle passes through the centre.
3.61. A quadrilateral $ABCD$ is drawn to circumscribe a circle. Prove that $AB + CD = AD + BC$.
3.62. XY and $X'Y'$ are two parallel tangents to a circle with centre O and another tangent AB with point of contact C intersecting XY at A and $X'Y'$ at B. Prove that $\angle AOB = 90°$
3.63. Prove that the angle between the two tangents drawn from an external point to a circle is supplementary to the angle subtended by the line-segment joining the points of contact at the centre.
3.64. Prove that the parallelogram circumscribing a circle is a rhombus.
3.65. Prove that opposite sides of a quadrilateral circumscribing a circle subtend supplementary angles at the centre of the circle.
3.66. Find the area of a sector of angle p (in degrees) of a circle with radius R.
3.67. Two chords AB and CD intersect each other at the point P. Prove that :

a) $\triangle APC \sim \triangle DPB$
b) $AP.PB = CP.DP$

3.68. Two chords AB and CD of a circle intersect each other at the point P (when produced) outside the circle. Prove that

a) $\triangle PAC \sim \triangle PDB$

b) $PA.PB = PC.PD$

4 Miscellaneous Exercises

4.1. $ABCD$ is a cyclic quadrilateral in which AC and BD are its diagonals. If $\angle DBC = 55°$ and $\angle BAC = 45°$, find $\angle BCD$

4.2. Two circles of radii 5 cm and 3 cm intersect at two points and the distance between their centres is 4 cm. Find the length of the common chord.

4.3. A,B and C are three points on a circle with centre O such that $\angle BOC = 30°$ and $\angle AOB = 60°$. If D is a point on the circle other than the arc ABC, find $\angle ADC$.

4.4. $\angle PQR = 100°$, where P, Q and R are points on a circle with centre O. Find $\angle OPR$.

4.5. A, B, C, D are points on a circle such that $\angle ABC = 69°, \angle ACB = 31°$, find $\angle BDC$.

4.6. A, B, C and D are four points on a circle. AC and BD intersect at a point E such that $\angle BEC = 130°$ and $\angle ECD = 20°$. Find $\angle BAC$.

4.7. $ABCD$ is a cyclic quadrilateral whose diagonals intersect at a point E. If $\angle DBC = 70°, \angle BAC$ is $30°$, find $\angle BCD$. Further, if $AB = BC$, find $\angle ECD$.

4.8. Two chords AB and CD of lengths 5 cm and 11 cm respectively of a circle are parallel to each other and are on opposite sides of its centre. If the distance between AB and CD is 6 cm, find the radius of the circle.

4.9. The lengths of two parallel chords of a circle are 6 cm and 8 cm. If the smaller chord is at distance 4 cm from the centre, what is the distance of the other chord from the centre?

4.10. A tangent PQ at a point P of a circle of radius 5 cm meets a line through the centre O at a point Q so that $OQ = 12$ cm. Find the length of PQ.

4.11. PQ is a chord of length 8 cm of a circle of radius 5 cm. The tangents at P and Q intersect at a point T. Find the length TP.

4.12. From a point Q, the length of the tangent to a circle is 24 cm and the distance of Q from the centre is 25 cm. Find the radius of the circle is

4.13. If TP and TQ are the two tangents to a circle with centre O so that $\angle POQ = 110°$, then find $\angle PTQ$

4.14. If tangents PA and PB from a point P to a circle with centre O are inclined to each other at angle of 80 ° , then find $\angle POA$

4.15. The length of a tangent from a point A at distance 5 cm from the centre of the circle is 4 cm. Find the radius of the circle.

4.16. Two concentric circles are of radii 5 cm and 3 cm. Find the length of the chord of the larger circle which touches the smaller circle.

4.17. A $\triangle ABC$ is drawn to circumscribe a circle of radius 4 cm such that the segments BD and DC into which BC is divided by the point of contact D are of lengths 8 cm and 6 cm respectively. Find the sides AB and AC.

4.18. The cost of fencing a circular field at the rate of ₹24 per metre is ₹5280. The field is to be ploughed at the rate of ₹0.50 per m^2. Find the cost of ploughing the field.

4.19. The radii of two circles are 19 cm and 9 cm respectively. Find the radius of the circle which has circumference equal to the sum of the circumferences of the two circles.

4.20. The radii of two circles are 8 cm and 6 cm respectively. Find the radius of the circle having area equal to the sum of the areas of the two circles.

4.21. A circular archery target is marked with its five scoring regions from the centre outwards as Gold, Red, Blue, Black and White. The diameter of the region representing Gold score is 21 cm and each of the other bands is 10.5 cm wide. Find the area of each of the five scoring regions.

4.22. The wheels of a car are of diameter 80 cm each. How many complete revolutions does each wheel make in 10 minutes when the car is travelling at a speed of 66 km per hour?

4.23. Find the area of the sector of a circle with radius 4 cm and of angle 30 ° . Also, find the area of the corresponding major sector.

4.24. Find the area of the segment AYB, if radius of the circle is 21 cm and $\angle AOB = 120°$.

4.25. Find the area of a sector of a circle with radius 6 cm if angle of the sector is 60 ° .

4.26. Find the area of a quadrant of a circle whose circumference is 22 cm. 3. The length of the minute hand of a clock is 14 cm. Find the area swept by the minute hand in 5 minutes.

4.27. A chord of a circle of radius 10 cm subtends a right angle at the centre. Find the area of the corresponding :

a) minor segment
b) major sector.

4.28. In a circle of radius 21 cm, an arc subtends an angle of 60 ° at the centre. Find:

a) the length of the arc
b) area of the sector formed by the arc
c) area of the segment formed by the corresponding chord

4.29. A chord of a circle of radius 15 cm subtends an angle of 60 ° at the centre. Find the areas of the corresponding minor and major segments of the circle.

4.30. A chord of a circle of radius 12 cm subtends an angle of 120 ° at the centre. Find the area of the corresponding segment of the circle.

4.31. A horse is tied to a peg at one corner of a square shaped grass field of side 15 m by means of a 5 m long rope. Find

a) the area of that part of the field in which the horse can graze.
b) the increase in the grazing area if the rope were 10 m long instead of 5 m.

4.32. A brooch is made with silver wire in the form of a circle with diameter 35 mm. The wire is also used in making 5 diameters which divide the circle into 10 equal sectors. Find :

a) the total length of the silver wire required.
b) the area of each sector of the brooch

4.33. An umbrella has 8 ribs which are equally spaced. Assuming umbrella to be a flat circle of radius 45 cm, find the area between the two consecutive ribs of the umbrella.

4.34. A car has two wipers which do not overlap. Each wiper has a blade of length 25 cm sweeping through an angle of 115 ° . Find the total area cleaned at each sweep of the blades.

4.35. To warn ships for underwater rocks, a lighthouse spreads a red coloured light over a sector of angle 80 ° to a distance of 16.5 km. Find the area of the sea over which the ships are warned.

4.36. A round table cover has six equal designs. If the radius of the cover is 28 cm, find the cost of making the designs at the rate of ₹0.35 per cm^2 .

4.37. Two circular flower beds are located on opposite sides of a square lawn $ABCD$ of side 56 m. If the centre Of each circular flower bed is the point of intersection O of the diagonals of the square lawn, find the sum of the areas of the lawn and the flower beds.

4.38. Four circles are inscribed inside a square $ABCD$ of side 14 cm such that each one touches exernally two adjacent sides of the square and two other circles. Find the region between the circles and the square.

4.39. $ABCD$ is a square of side 10 cm and semicircles are drawn with each side of the square as diameter. Find the area enclosed by the circular arcs.

4.40. P is a point on the semi-circle formed with diameter QR. Find the area between the semicircle and $\triangle PQR$ if PQ = 24 cm, PR = 7 cm and O is the centre Of the circle.

4.41. AC and BD are two arcs on concentric circles with radii 14 cm and 7 cm respectively, such that $\angle AOC = 40°$. Find the area of the region $ABDC$.

4.42. Find the area between a square $ABCD$ of side 14cm and the semi circles APD and BPC.

4.43. Find the area of the region enclosed by a circular arc of radius 6 cm drawn with vertex O of an equilateral triangle OAB of side 12 cm as centre.

4.44. From each corner of a square of side 4 cm a quadrant of a circle of radius 1 cm is cut and also a circle of diameter 2 cm is cut. Find the area of the remaining portion of the square.

4.45. In a circular table cover of radius 32 cm, a design is formed leaving an equilateral $\triangle ABC$ in the middle. Find the area of the design.

4.46. $ABCD$ is a square of side 14 cm. With centres A, B, C and D, four circles are drawn such that each circle touches externally two of the remaining three circles. Find the area within the square that lies outside the circles.

4.47. The left and right ends of a racing track are semicircular. The distance between the two inner parallel line segments is 60 m and they are each 106 m long. If the track is 10 m wide, find :

a) the distance around the track along its inner edge

b) the area of the track.

4.48. AB and CD are two diameters of a circle (with centre O) perpendicular to each other and OD is the diameter of a smaller circle inside. If OA = 7 cm, find the area of the smaller circle.

4.49. The area of an equilateral $\triangle ABC$ is 17320.5 cm^2 . With each vertex of the triangle as centre, a circle is drawn with radius equal to half the length of the side of the triangle. Find the area of region within the triangle but outside the circles.

4.50. On a square handkerchief, nine circular designs are inscribed touching each other, each of radius 7 cm. Find the area of the remaining portion of the handkerchief.

4.51. $OACB$ is a quadrant of a circle with centre O and radius 3.5 cm. D is a point on OA. If OD = 2 cm, find the area of the

a) quadrant $OACB$,

b) the region between the quadrant and $\triangle OBD$.

4.52. A square $OABC$ is inscribed in a quadrant $OPBQ$. If OA = 20 cm, find the area between the square and the quadrant.

4.53. AB and CD are respectively arcs of two concentric circles of radii 21 cm and 7 cm and centre O. If $\angle AOB = 30°$, find the area of the region $ABCD$.

4.54. ABC is a quadrant of a circle of radius 14 cm and a semicircle is drawn with BC as diameter. Find the area of the crescent formed.

4.55. Find the area common between the two quadrants of circles of radius 8 cm each if the centres of the circles lie on opposite sides of a square.

4.56. Find the area of the sector of a circle with radius 4 cm and of angle 30°. Also, find the area of the corresponding major sector.

4.57. A pole has to be erected at a point on the boundary of a circular park of diameter 13 metres in such a way that the differences of its distances from two diametrically opposite fixed gates A and B on the boundary is 7 metres. Is it possible to do so? If yes, at what distances from the two gates should the pole be erected?

4.58. Draw a triangle whose sides are 8cm and 11cm and the perimeter is 32 cm and find its area.

4.59. A triangular park ABC has sides 120m, 80m and 50m. A gardener Dhania has to put a fence all around it and also plant grass inside. Draw this park. How much area does she need to plant? Find the cost of fencing it with barbed wire at the rate of ₹20 per metre leaving a space 3m wide for a gate on one side.

4.60. The sides of a triangular plot are in the ratio of 3 : 5 : 7 and its perimeter is 300 m. Draw the plot and find its area.

4.61. A tower stands vertically on the ground. From a point on the ground, which is 15m away from the foot of the tower, the angle of elevation of the top of the tower is found to be 60°. Find the height of the tower.

4.62. An electrician has to repair an electric fault pole of height 5m. She needs to reach a point 1.3m below the top of the pole to undertake the repair work. What should be the length of the ladder that she should use which, when inclined at an angle of 60° to the horizontal, would enable her to reach the required position? Also, how far from the foot of the pole should she place the foot of the ladder?

4.63. An observer 1.5m tall is 28.5m away from a chimney. The angle of elevation of the top of the chimney from her eyes is 45°. What is the height of the chimney?

4.64. From a point **P** on the ground the angle of elevation of the top of a 10m tall building is 30°. A flag is hoisted at the top of the building and the angle of elevation of the top of the flagstaff from **P** is 45°. Find the length of the flagstaff and the distance of the building from the point **P**.

4.65. The shadow of a tower standing on a level ground is found to be 40m longer when the Sun's altitude is 30° than when it is 60°. Find the height of the tower.

4.66. The angles of depression of the top and the bottom of an 8m tall building from the top of a multi-storeyed building are 30° and 45° respectively. Find the height of the multi-storeyed building and the distance between the two buildings.

4.67. A traffic signal board, indicating 'SCHOOL AHEAD', is an equilateral triangle with side 'a'. Find the area of the signal board, using Heron's formula. If its perimeter is 180 cm, what will be the area of the signal board?

4.68. The triangular side walls of a flyover have been used for advertisements. The sides of the walls are 122 m, 22 m and 120 m. The advertisements yield an earning of ₹5000 per m^2 per year. A company hired one o its walls for 3 months. How much rent did it pay?

4.69. There is a slide in a park. One of its side walls has been painted in some colour with a message "KEEP THE PARK GREEN AND CLEAN". If the sides of the wall are 15 m, 11 m and 6 m, find the area painted in colour.

4.70. Find the area of a triangle two sides of which are 18cm and 10cm and the perimeter is 42cm.

4.71. Sides of a triangle are in the ratio of 12 : 17 : 25 and its perimeter is 540cm. Find its area.

4.72. An isosceles triangle has perimeter 30 cm and each of the equal sides is 12 cm. Find the area of the triangle.

4.73. A girl walks 4km west, then she walks 3km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

4.74. A circus artist is climbing a 20m long rope, which is tightly stretched and tied from the top of a vertical pole to the ground. Find the height of the pole, if the angle made by the rope with the ground level is 30°.

4.75. A tree breaks due to storm and the broken part bends so that the top of the tree touches the ground making an angle of 30° with it. The distance between the foot of the tree to the point where the top touches the ground is 8m. Find the height of the tree.

4.76. A contractor plans to install two slides for the children to play in a park. For the children below the age of 5 years, she prefers to have a slide whose top is at a height of 1.5m, and is inclined at an angle of 30° to the ground, whereas for elder children she wants to have a steep slide at a height of 3m, and inclined at an angle of 60° to the ground. What should be the length of the slide in each case?

4.77. The angle of elevation of the top of a tower from a point on the ground, which is 30m away from the foot of the tower, is 30°. Find the height of the tower.

4.78. A kite is flying at a height of 60m above the ground. The string attached to the kite is temporarily tied to a point on the ground. The inclination of the string with the ground is 60°. Find the length of the string, assuming that there is no slack in the string.

4.79. A 1.5m tall boy is standing at some distance from a 30m tall building. The angle of elevation from his eyes to the top of the building increases from 30° to 60° as he walks towards the building. Find the distance he walked towards the building.

4.80. From a point on the ground, the angles of elevation of the bottom and the top of a transmission tower fixed at the top of a 20 m high building are 45° and 60° respectively. Find the height of the tower.

4.81. A statue, 1.6 m tall, stands on the top of a pedestal. From a point on the ground, the angle of elevation of the top of the statue is 60° and from the same point the angle of elevation of the top of the pedestal is 45°. Find the height of the pedestal.

4.82. The angle of elevation of the top of a building from the foot of the tower is 30° and the angle of elevation of the top of the tower from the foot of the building is 60°. If the tower is 50 m high, find the height of the building.

4.83. Two poles of equal heights are standing opposite each other on either side of the road, which is 80 m wide. From a point between them on the road, the angles of elevation of the top of the poles are 60° and 30°, respectively. Find the height of the poles and the distances of the point from the poles.

4.84. A TV tower stands vertically on a bank of a canal. From a point on the other bank directly opposite the tower, the angle of elevation of the top of the tower is 60°. From another point 20 m away from this point on the line joing this point to the foot of the tower, the angle of elevation of the top of the tower is 30°. Find the height of the tower and the width of the canal.

4.85. From the top of a 7 m high building, the angle of elevation of the top of a cable tower is 60° and the angle of depression of its foot is 45°. Determine the height of the tower.

4.86. As observed from the top of a 75 m high lighthouse from the sea-level, the angles of depression of two ships are 30° and 45°. If one ship is exactly behind the other on the same side of the lighthouse, find the distance between the two ships.

4.87. A 1.2 m tall girl spots a balloon moving with the wind in a horizontal line at a height of 88.2 m from the ground. The angle of elevation of the balloon from the eyes of the girl at any instant is 60°. After some time, the angle of elevation reduces to 30°. Find the distance travelled by the balloon during the interval.

4.88. A straight highway leads to the foot of a tower. A man standing at the top of the tower observes a car at an angle of depression of 30°, which is approaching the foot of the tower with a uniform speed. Six seconds later, the angle of depression of the car is found to be 60°. Find the time taken by the car to reach the foot of the tower from this point.

4.89. The angles of elevation of the top of a tower from two points at a distance of 4 m and 9 m from the base of the tower and in the same straight line with it are complementary. Prove that the height of the tower is 6 m.

4.90. E and F are points on the sides PQ and PR respectively of a $\triangle PQR$. For each of the following cases, state whether $EF \parallel QR$

a) $PE = 3.9cm, EQ = 3cm, PF = 3.6cm$ and $FR = 2.4cm$
b) $PE = 4cm, QE = 4.5cm, PF = 8cm$ and $RF = 9cm$
c) $PQ = 1.28cm, PR = 2.56cm, PE = 0.18cm$ and $PF = 0.36cm$

4.91. A girl of height 90 cm is walking away from the base of a lamp-post at a speed of 1.2 m/s. If the lamp is 3.6 m above the ground, find the length of her shadow after 4 seconds.

4.92. $\triangle ODC \sim \triangle OBA, \angle BOC = 125°$ and $\angle CDO = 70°$. Find $\angle DOC, \angle DCO$ and $\angle OAB$.

4.93. Nazima is fly fishing in a stream. The tip of her fishing rod is 1.8 m above the surface of the water and the fly at the end of the string rests on the water 3.6 m away and 2.4 m from a point directly under the tip of the rod. Assuming that her string (from the tip of her rod to the fly) is taut, how much string does she have out? If she pulls in the string at the rate of 5 cm per second, what will be the horizontal distance of the fly from her after 12 seconds?

4.94. A vertical pole of length 6 m casts a shadow 4 m long on the ground and at the same time a tower casts a shadow 28 m long. Find the height of the tower.

4.95. Let $\triangle ABC \sim \triangle DEF$ and their areas be, respectively, 64 cm^2 and 121 cm^2. If $EF = 15.4cm$, find BC.

4.96. A ladder is placed against a wall such that its foot is at a distance of 2.5 m from the wall and its top reaches a window 6 m above the ground. Find the length of the ladder.

4.97. Sides of triangles are given below. Determine which of them are right triangles. In case of a right triangle, write the length of its hypotenuse.

a) 7 cm, 24 cm, 25 cm
b) 3 cm, 8 cm, 6 cm
c) 50 cm, 80 cm, 100 cm
d) 13 cm, 12 cm, 5 cm

4.98. A ladder 10 m long reaches a window 8 m above the ground. Find the distance of the foot of the ladder from base of the wall.

4.99. A guy wire attached to a vertical pole of height 18 m is 24 m long and has a stake attached to the other end. How far from the base of the pole should the stake be driven so that the wire will be taut?

4.100. An aeroplane leaves an airport and flies due north at a speed of 1000 km per hour. At the same time, another aeroplane leaves the same airport and flies due west at a speed of 1200 km per hour. How far apart will be the two planes after $1\frac{1}{2}$ hours?

4.101. Two poles of heights 6 m and 11 m stand on a plane ground. If the distance between the feet of the poles is 12 m, find the distance between their tops.

4.102. In $\triangle ABC, AB = 6\sqrt{3}cm, AC = 12cm$ and $BC = 6cm$. Find the angle B.

4.103. A park, in the shape of a quadrilateral $ABCD$, has $\angle C = 90°, AB = 9m, BC = 12m, CD = 5m$ and $AD = 8m$. How much area does it occupy? 2. Find the area of a quadrilateral $ABCD$ in which $AB = 3cm, BC = 4cm, CD = 4cm, DA = 5cm$ and $AC = 5cm$.

4.104. A triangle and a parallelogram have the same base and the same area. If the sides of the triangle are 26 cm, 28 cm and 30 cm, and the parallelogram stands on the base 28 cm, find the height of the parallelogram.

4.105. A rhombus shaped field has green grass for 18 cows to graze. If each side of the rhombus is 30 m and its longer diagonal is 48 m, how much area of grass field will each cow be getting?

4.106. A field is in the shape of a trapezium whose parallel sides are 25 m and 10 m. The non-parallel sides are 14 m and 13 m. Find the area of the field.

4.107. $ABCD$ is a parallelogram, $AE \perp DC$ and $CF \perp AD$. If $AB = 16cm$, $AE = 8$ cm and $CF = 10$ cm, find AD.

4.108. Kamla has a triangular field with sides 240 m, 200 m, 360 m, where she grew wheat. In another triangular field with sides 240 m, 320 m, 400 m adjacent to the previous field, she wanted to grow potatoes and onions. She divided the field in two parts by joining the mid-point of the longest side to the opposite vertex and grew patatoes in one part and onions in the other part. Draw the figure for this problem. How much area (in hectares) has been used for wheat, potatoes and onions? (1 hectare = 10000 m^2).

4.109. Students of a school staged a rally for cleanliness campaign. They walked through the lanes in two groups. One group walked through the lanes AB, BC and CA; while the other through AC, CD and DA. Then they cleaned the area enclosed within their lanes. If AB = 9 m, BC = 40 m, CD = 15 m, DA = 28 m and $\angle B = 90°$, which group cleaned more area and by how much? Draw the corresponding figure. Find the total area cleaned by the students (neglecting the width of the lanes).

4.110. Sanya has a piece of land which is in the shape of a rhombus. She wants her one daughter and one son to work on the land and produce different crops. She divided the land in two equal parts. If the perimeter of the land is 400 m and one of the diagonals is 160 m, how much area each of them will get for their crops? Draw the rhombus.

4.111. Three girls Reshma, Salma and Mandip are playing a game by standing on a circle of radius 5m drawn in a park. Reshma throws a ball to Salma, Salma to Mandip, Mandip to Reshma. If the distance between Reshma and Salma and between Salma and Mandip is 6m each, what is the distance between Reshma and Mandip?

4.112. A circular park of radius 20m is situated in a colony. Three boys Ankur, Syed and David are sitting at equal distance on its boundary each having a toy telephone in his hands to talk each other. Find the length of the string of each phone.

4.113. The longest side of a triangle is 3 times the shortest side and the third side is 2 cm shorter than the longest side. If the perimeter of the triangle is at least 61 cm, find the minimum length of the shortest side.

4.114. A rectangular park is to be designed whose breadth is 3 m less than its length. Its area is to be 4 square metres more than the area of a park that has already been made in the shape of an isosceles triangle with its base as the breadth of the rectangular park and of altitude 12 m. Find its length and breadth.

4.115. The area of a rectangular plot is 528 m^2 . The length of the plot (in metres) is one more than twice its breadth. We need to find the length and breadth of the plot.

4.116. The altitude of a right triangle is 7 cm less than its base. If the hypotenuse is 13 cm, find the other two sides.

4.117. The diagonal of a rectangular field is 60 metres more than the shorter side. If the longer side is 30 metres more than the shorter side, find the sides of the field.

4.118. Is it possible to design a rectangular mango grove whose length is twice its breadth, and the area is 800 m^2 ? If so, find its length and breadth.

4.119. Is it possible to design a rectangular park of perimeter 80 m and area 400 m^2 If so, find its length and breadth.

4.120. On an open ground, a motorist follows a track that turns to his left by an angle of 600 after every 500 m. Starting from a given turn, specify the displacement of the motorist at the third, sixth and eighth turn. Compare the magnitude of the displacement with the total path length covered by the motorist in each case.

4.121. A passenger arriving in a new town wishes to go from the station to a hotel located 10 km away on a straight road from the station. A dishonest cabman takes him along a circuitous path 23 km long and reaches the hotel in 28 min. What is

a) the average speed of the taxi,
b) the magnitude of average velocity ? Are the two equal ?

4.122. An aircraft is flying at a height of 3400 m above the ground. If the angle subtended at a ground observation point by the aircraft positions 10.0 s apart is 30°, what is the speed of the aircraft ?

4.123. Two identical billiard balls strike a rigid wall with the same speed but at different angles, and get reflected without any change in speed, as shown in Fig. 4.123.1. What is

a) the direction of the force on the wall due to each ball?

b) the ratio of the magnitudes of impulses imparted to the balls by the wall ?

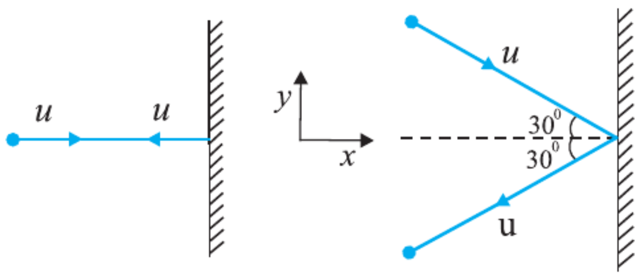


Fig. 4.123.1

4.124. See Fig. 4.124.1. A mass of 6 kg is suspended by a rope of length 2 m from the ceiling. A force of 50 N in the horizontal direction is applied at the midpoint P of the rope, as shown. What is the angle the rope makes with the vertical in equilibrium ? (Take g = 10 ms^{-2}).

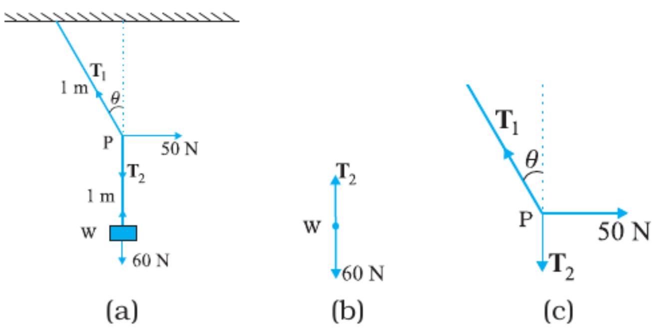


Fig. 4.124.1

4.125. See Fig. 4.125.1. A mass of 4 kg rests on a horizontal plane. The plane is gradually inclined until at an angle θ = 15° with the horizontal, the mass just begins to slide. What is the coefficient of static friction between the block and the surface ?

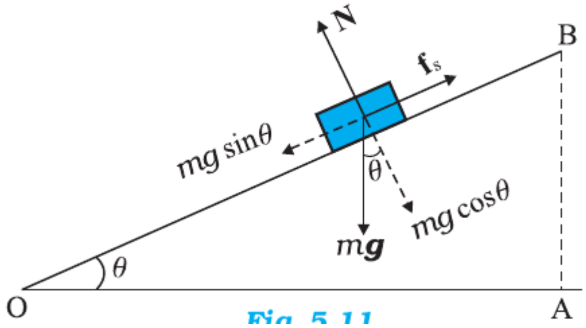


Fig. 4.125.1

4.126. A cyclist speeding at 18 km/h on a level road takes a sharp circular turn of radius 3 m without reducing the speed. The co-efficient of static friction between the tyres and the road is 0.1. Will the cyclist slip while taking the turn?

4.127. A circular racetrack of radius 300 m is banked at an angle of 15°. If the coefficient of friction between the wheels of a race-car and the road is 0.2, what is the

a) optimum speed of the racecar to avoid wear and tear on its tyres, and

b) maximum permissible speed to avoid slipping ?

4.128. An aircraft executes a horizontal loop at a speed of 720 km/h with its wings banked at 15°. What is the radius of the loop ?

4.129. A train runs along an unbanked circular track of radius 30 m at a speed of 54 km/h. The mass of the train is 10^6 kg. What provides the centripetal force required for this purpose - The engine or the rails ? What is the angle of banking required to prevent wearing out of the rail ?

4.130. A disc revolves with a speed of $33\frac{1}{3}$ rev/min and has a radius of 15 cm. Two coins are placed at 4 cm and 14 cm away from the centre of the record. If the co-efficient of friction between the coins and the record is 0.15, which of the coins will revolve with the record ?

4.131. A 70 kg man stands in contact against the inner wall of a hollow cylindrical drum of radius 3 m rotating about its vertical axis with 200 rev/min. The coefficient of friction between the wall and his clothing is 0.15. What is the minimum rotational speed of the cylinder to enable the man to remain stuck to the wall (without falling) when the floor is suddenly removed ?

4.132. A thin circular loop of radius R rotates about its vertical diameter with an angular frequency

ω. Show that a small bead on the wire loop remains at its lowermost point for $\omega \leq \sqrt{\frac{g}{R}}$. What is the angle made by the radius vector joining the centre to the bead with the vertical downward direction for $\omega = \sqrt{\frac{2g}{R}}$? Neglect friction.

4.133. A stone of mass 0.25 kg tied to the end of a string is whirled round in a circle of radius 1.5 m with a speed of 40 rev./min in a horizontal plane. What is the tension in the string ? What is the maximum speed with which the stone can be whirled around if the string can withstand a maximum tension of 200 N ?

4.134. A woman pushes a trunk on a railway platform which has a rough surface. She applies a force of 100 N over a distance of 10 m. Thereafter, she gets progressively tired and her applied force reduces linearly with distance to 50 N. The total distance through which the trunk has been moved is 20 m. Plot the force applied by the woman and the frictional force, which is 50 N versus displacement. Calculate the work done by the two forces over 20 m.

4.135. A bob of mass m is suspended by a light string of length L . It is imparted a horizontal velocity v_o such that it completes a semi-circular trajectory in the vertical plane with the string becoming slack only on reaching the topmost point, C. This is shown in Fig. 4.135.1. Obtain an expression for

a) vo;
b) the speeds at points B and C;
c) the ratio of the kinetic energies (K_B/K_C) at B and C.

Comment on the nature of the trajectory of the bob after it reaches the point C.

4.136. The bob A of a pendulum released from 30°to the vertical hits another bob B of the same mass at rest on a table as shown in Fig. 4.136.1. How high does the bob A rise after the collision? Neglect the size of the bobs and assume the collision to be elastic.

4.137. The bob of a pendulum is released from a horizontal position. If the length of the pendulum is 1.5 m, what is the speed with which the bob arrives at the lowermost point, given that it dissipated 5% of its initial energy against air resistance ?

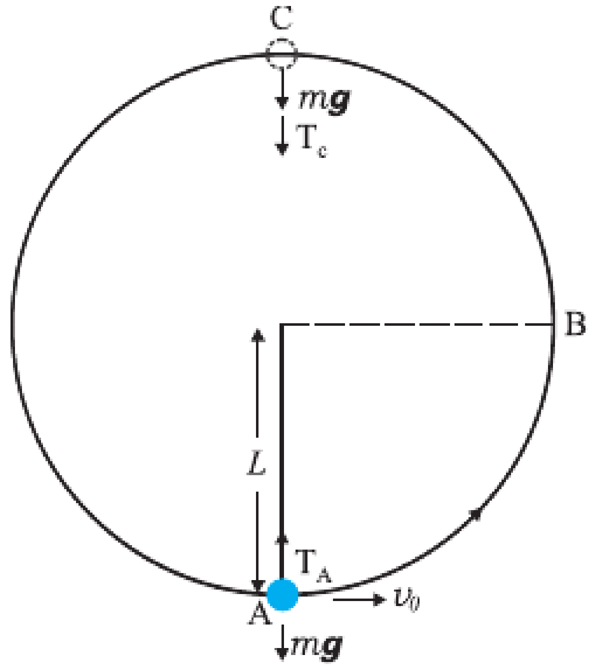


Fig. 4.135.1

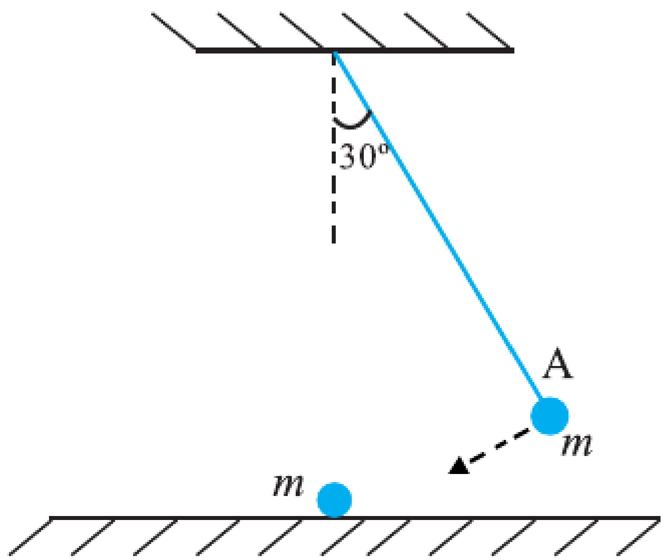


Fig. 4.136.1

4.138. The blades of a windmill sweep out a circle of area A.

(a) If the wind flows at a velocity v perpendicular to the circle, what is the mass of the air passing through it in time t ?
(b) What is the kinetic energy of the air ?
(c) Assume that the windmill converts 25% of the wind's energy into electrical energy, and that $A = 30m^2, v = 36km/h$ and the density

of air is $1.2 kgm^{-3}$. What is the electrical power produced ?

4.139. A bullet of mass 0.012 kg and horizontal speed $70 ms^{-1}$. strikes a block of wood of mass 0.4 kg and instantly comes to rest with respect to the block. The block is suspended from the ceiling by means of thin wires. Calculate the height to which the block rises. Also, estimate the amount of heat produced in the block.

4.140. Two inclined frictionless tracks, one gradual and the other steep meet at A from where two stones are allowed to slide down from rest, one on each track (Fig. 4.140.1). Will the stones reach the bottom at the same time? Will they reach there with the same speed? Explain. Given $\theta_1 = 30° \theta_2 = 60°$ and $h = 10m$, what are the speeds and times taken by the two stones?

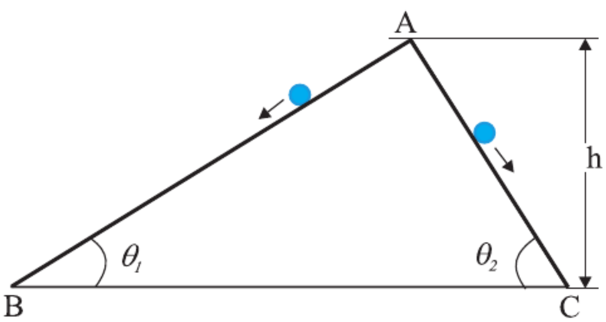


Fig. 4.140.1

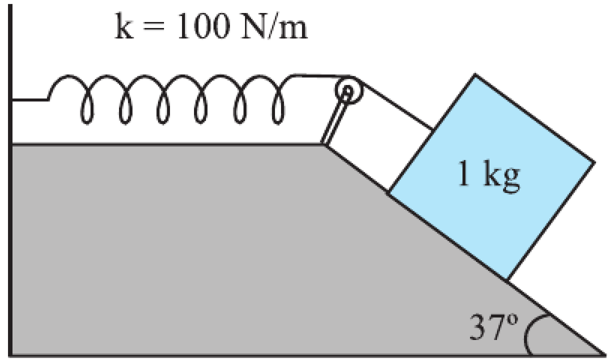


Fig. 4.141.1

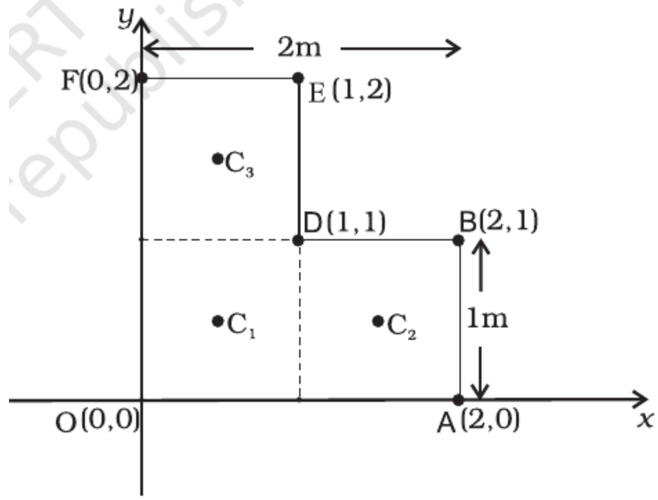


Fig. 4.144.1

4.141. A 1 kg block situated on a rough incline is connected to a spring of spring constant 100 N m–1 as shown in Fig. 4.141.1. The block is released from rest with the spring in the unstretched position. The block moves 10 cm down the incline before coming to rest. Find the coefficient of friction between the block and the incline. Assume that the spring has a negligible mass and the pulley is frictionless.

4.142. Find the centre of mass of three particles at the vertices of an equilateral triangle. The masses of the particles are 100g, 150g, and 200g respectively. Each side of the equilateral triangle is 0.5m long.

4.143. Find the centre of mass of a triangular lamina.

4.144. Find the centre of mass of a uniform L-shaped lamina (a thin flat plate) with dimensions as shown in Fig. 4.144.1. The mass of the lamina is 3 kg.

4.145. A metal bar 70 cm long and 4.00 kg in mass supported on two knife-edges placed 10 cm from each end. A 6.00 kg load is suspended at 30 cm from one end. Find the reactions at the knifeedges. (Assume the bar to be of uniform cross section and homogeneous.)

4.146. A 3m long ladder weighing 20 kg leans on a frictionless wall. Its feet rest on the floor 1 m from the wall as shown in Fig. 4.146.1. Find the reaction forces of the wall and the floor.

4.147. What is the moment of inertia of a rod of mass M, length l about an axis perpendicular to it through one end?

4.148. What is the moment of inertia of a ring about a tangent to the circle of the ring?

4.149. A cord of negligible mass is wound round the rim of a fly wheel of mass 20 kg and radius 20 cm. A steady pull of 25 N is applied on the cord as shown in Fig. 4.149.1. The flywheel is

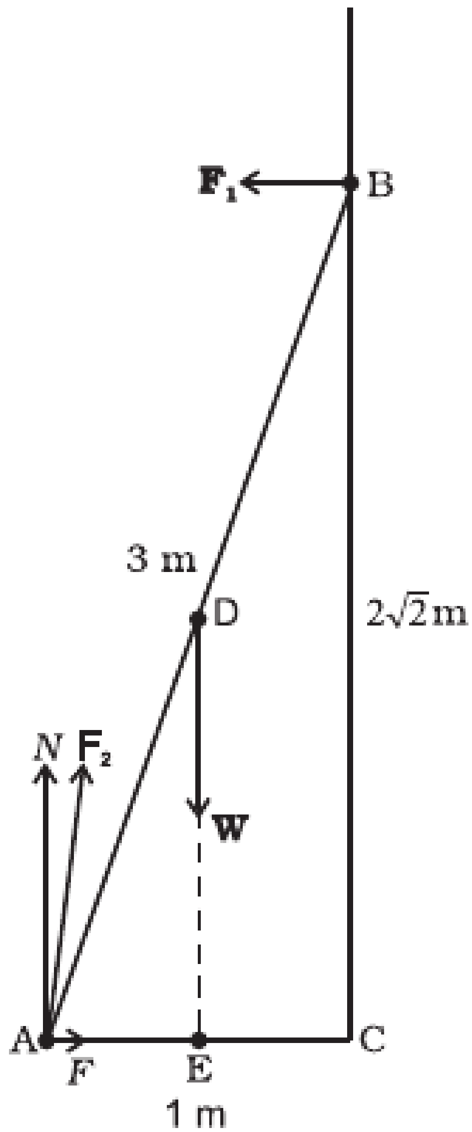


Fig. 4.146.1

mounted on a horizontal axle with frictionless bearings.

(a) Compute the angular acceleration of the wheel.

(b) Find the work done by the pull, when 2m of the cord is unwound.

(c) Find also the kinetic energy of the wheel at this point. Assume that the wheel starts from rest.

(d) Compare answers to parts (b) and (c).

4.150. Three bodies, a ring, a solid cylinder and a solid sphere roll down the same inclined plane without slipping. They start from rest. The radii of the bodies are identical. Which of the bodies reaches the ground with maximum velocity?

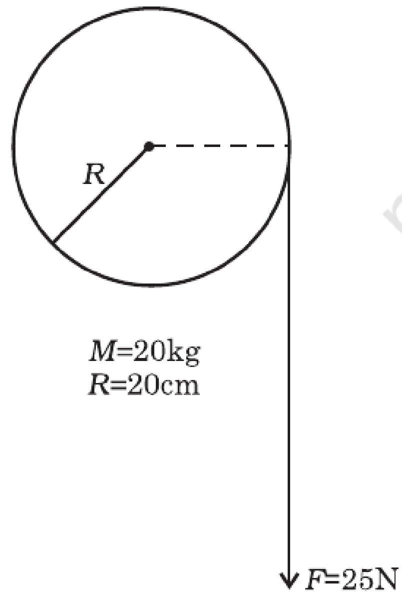


Fig. 4.149.1

4.151. In the HCl molecule, the separation between the nuclei of the two atoms is about 1.27 Å (1 Å = 10^{-10} m). Find the approximate location of the CM of the molecule, given that a chlorine atom is about 35.5 times as massive as a hydrogen atom and nearly all the mass of an atom is concentrated in its nucleus.

4.152. A non-uniform bar of weight W is suspended at rest by two strings of negligible weight as shown in Fig. 4.152.1. The angles made by the strings with the vertical are 36.9° and 53.1° respectively. The bar is 2 m long. Calculate the distance d of the centre of gravity of the bar from its left end.

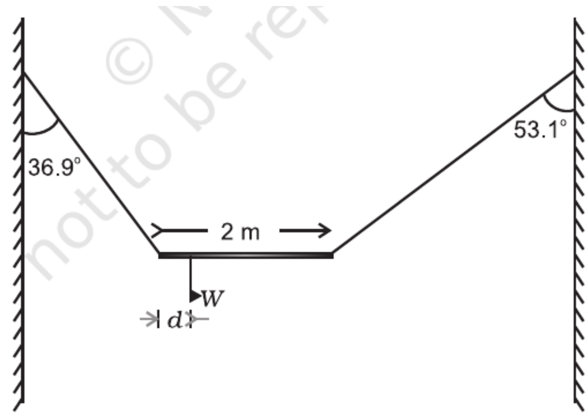


Fig. 4.152.1

4.153. A car weighs 1800 kg. The distance between its front and back axles is 1.8 m. Its centre of gravity is 1.05 m behind the front axle. Determine the force exerted by the level ground on each front wheel and each back wheel.

4.154.(a) Find the moment of inertia of a sphere about a tangent to the sphere, given the moment of inertia of the sphere about any of its diameters to be $\frac{2MR^2}{5}$ where M is the mass of the sphere and R is the radius of the sphere.

(b) Given the moment of inertia of a disc of mass M and radius R about any of its diameters to be $\frac{MR^2}{4}$, find its moment of inertia about an axis normal to the disc and passing through a point on its edge.

4.155. Torques of equal magnitude are applied to a hollow cylinder and a solid sphere, both having the same mass and radius. The cylinder is free to rotate about its standard axis of symmetry, and the sphere is free to rotate about an axis passing through its centre. Which of the two will acquire a greater angular speed after a given time.

4.156. A solid cylinder of mass 20 kg rotates about its axis with angular speed 100 $rads^{-1}$. The radius of the cylinder is 0.25 m. What is the kinetic energy associated with the rotation of the cylinder? What is the magnitude of angular momentum of the cylinder about its axis?

4.157.(a) A child stands at the centre of a turntable with his two arms outstretched. The turntable is set rotating with an angular speed of 40 rev/min. How much is the angular speed of the child if he folds his hands back and thereby reduces his moment of inertia to 2/5 times the initial value ? Assume that the turntable rotates without friction.

(b) Show that the child's new kinetic energy of rotation is more than the initial kinetic energy of rotation. How do you account for this increase in kinetic energy?

4.158. A rope of negligible mass is wound round a hollow cylinder of mass 3 kg and radius 40 cm. What is the angular acceleration of the cylinder if the rope is pulled with a force of 30 N ? What is the linear acceleration of the rope ? Assume that there is no slipping.

4.159. To maintain a rotor at a uniform angular speed of 200 $rads^{-1}$, an engine needs to transmit a torque of 180 N m. What is the power required by the engine ? (Note: uniform angular velocity in the absence of friction implies zero torque. In practice, applied torque is needed to counter frictional torque). Assume that the engine is 100% efficient.

4.160. From a uniform disk of radius R, a circular hole of radius R/2 is cut out. The centre of the hole is at R/2 from the centre of the original disc. Locate the centre of gravity of the resulting flat body.

4.161. A metre stick is balanced on a knife edge at its centre. When two coins, each of mass 5 g are put one on top of the other at the 12.0 cm mark, the stick is found to be balanced at 45.0 cm. What is the mass of the metre stick?

4.162. A solid sphere rolls down two different inclined planes of the same heights but different angles of inclination.

(a) Will it reach the bottom with the same speed in each case?

(b) Will it take longer to roll down one plane than the other?

(c) If so, which one and why?

4.163. A hoop of radius 2 m weighs 100 kg. It rolls along a horizontal floor so that its centre of mass has a speed of 20 cm/s. How much work has to be done to stop it?

4.164. The oxygen molecule has a mass of 5.30×10^{-26} kg and a moment of inertia of $1.94\times10^{-46}kgm^2$ about an axis through its centre perpendicular to the lines joining the two atoms. Suppose the mean speed of such a molecule in a gas is 500 m/s and that its kinetic energy of rotation is two thirds of its kinetic energy of translation. Find the average angular velocity of the molecule.

4.165. A solid cylinder rolls up an inclined plane of angle of inclination 30°. At the bottom of the inclined plane the centre of mass of the cylinder has a speed of 5 m/s.

(a) How far will the cylinder go up the plane?

(b) How long will it take to return to the bottom?

4.166. As shown in Fig. 4.166.1, the two sides of a step ladder BA and CA are 1.6 m long and hinged at A. A rope DE, 0.5 m is tied half way up. A weight 40 kg is suspended from a point F, 1.2 m from B along the ladder BA. Assuming the floor to be frictionless and neglecting the weight of the ladder, find the tension in the rope and forces exerted by the floor on the ladder. (Take $g = 9.8 m/s^2$) (Hint: Consider the equilibrium of each side of the ladder separately.)

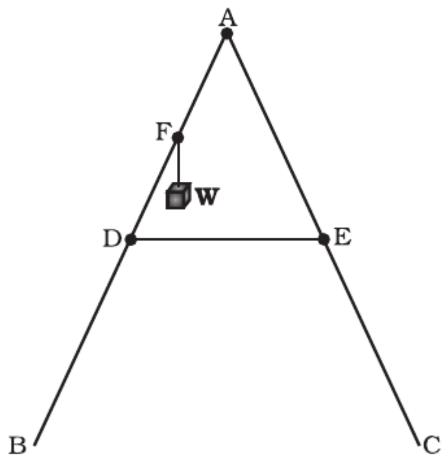


Fig. 4.166.1

4.167. A man stands on a rotating platform, with his arms stretched horizontally holding a 5 kg weight in each hand. The angular speed of the platform is 30 revolutions per minute. The man then brings his arms close to his body with the distance of each weight from the axis changing from 90cm to 20cm. The moment of inertia of the man together with the platform may be taken to be constant and equal to 7.6 kgm^2.

(a) What is his new angular speed? (Neglect friction.)

(b) Is kinetic energy conserved in the process? If not, from where does the change come about?

4.168. A bullet of mass 10 g and speed 500 m/s is fired into a door and gets embedded exactly at the centre of the door. The door is 1.0 m wide and weighs 12 kg. It is hinged at one end and rotates about a vertical axis practically without friction. Find the angular speed of the door just after the bullet embeds into it. (Hint: The moment of inertia of the door about the vertical axis at one end is $\frac{ML^2}{3}$.)

4.169. Two discs of moments of inertia I_1 and I_2 about their respective axes (normal to the disc and passing through the centre), and rotating with angular speeds ω_1 and ω_2 are brought into contact face to face with their axes of rotation coincident.

(a) What is the angular speed of the two-disc system?

(b) Show that the kinetic energy of the combined system is less than the sum of the initial kinetic energies of the two discs. How do you account for this loss in energy? Take $\omega_1 \neq \omega_2$.

4.170. Prove the result that the velocity v of translation of a rolling body (like a ring, disc, cylinder or sphere) at the bottom of an inclined plane of a height h is given by

$$v^2 = \frac{2gh}{1 + \frac{k^2}{R^2}} \tag{4.170.1}$$

using dynamical consideration (i.e. by consideration of forces and torques). Note k is the radius of gyration of the body about its symmetry axis, and R is the radius of the body. The body starts from rest at the top of the plane.

4.171. A disc rotating about its axis with angular speed ω_0 is placed lightly (without any translational push) on a perfectly frictionless table. The radius of the disc is R. What are the linear velocities of the points A, B and C on the disc shown in Fig. 4.171.1? Will the disc roll in the direction indicated ?

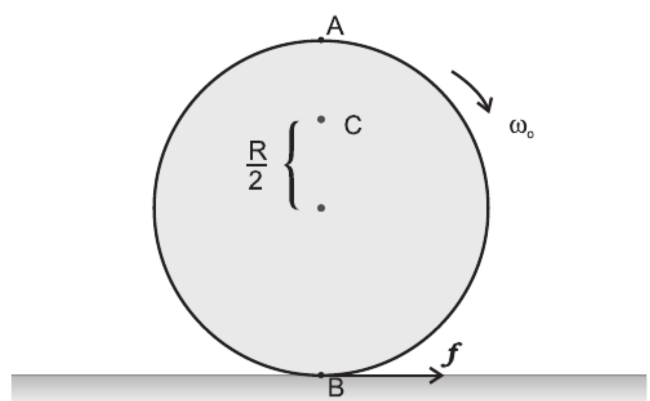


Fig. 4.171.1

4.172. Explain why friction is necessary to make the disc in Fig. 4.171.1 roll in the direction indicated.

(a) Give the direction of frictional force at B, and the sense of frictional torque, before perfect rolling begins.

(b) What is the force of friction after perfect rolling begins ?

4.173. A solid disc and a ring, both of radius 10 cm are placed on a horizontal table simultaneously, with initial angular speed equal to 10 $\pi rads^{-1}$. Which of the two will start to roll earlier ? The coefficient of kinetic friction is $\mu_k = 0.2$.

4.174. A cylinder of mass 10 kg and radius 15 cm is rolling perfectly on a plane of inclination 30°. The coefficient of static friction $\mu_s = 0.25$.
(a) How much is the force of friction acting on the cylinder ?
(b) What is the work done against friction during rolling ?
(c) If the inclination θ of the plane is increased, at what value of θ does the cylinder begin to skid, and not roll perfectly ?

4.175. Three equal masses of m kg each are fixed at the vertices of an equilateral triangle ABC.
(a) What is the force acting on a mass 2m placed at the centroid G of the triangle?
(b) What is the force if the mass at the vertex A is doubled ? Take AG = BG = CG = 1 m (see Fig. 4.175.1)

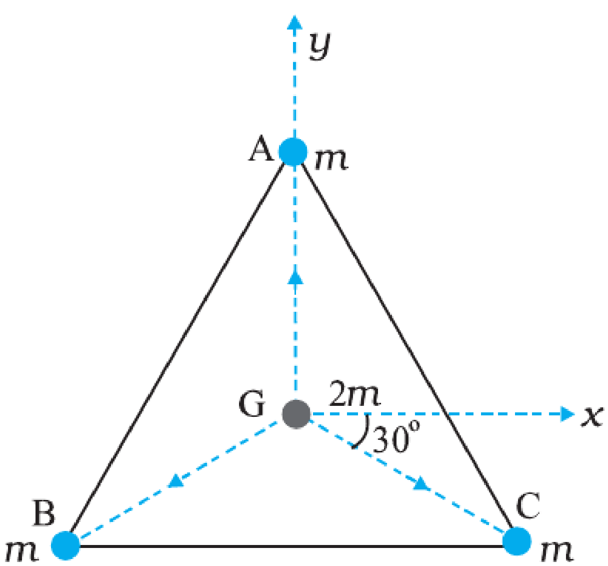


Fig. 4.175.1

4.176. Find the potential energy of a system of four particles placed at the vertices of a square of side l. Also obtain the potential at the centre of the square.

4.177. Two uniform solid spheres of equal radii R, but mass M and 4 M have a centre to centre separation 6 R, as shown in Fig. 4.177.1. The two spheres are held fixed. A projectile of mass m is projected from the surface of the sphere of mass M directly towards the centre of the second sphere. Obtain an expression for the minimum speed v of the projectile so that it reaches the surface of the second sphere.

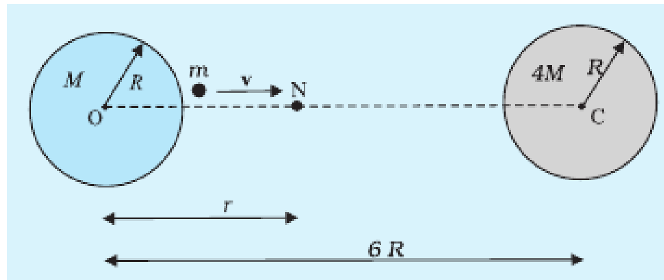


Fig. 4.177.1