Random Variables in Python

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Contents

***Abstract*—This manual provides a simple introduction to random variables. This is done by generating random variables in Python and computing metrics like the CDF and PDF for some random variables. In the process, basic concepts like transformation of random variables, central limit theorem, etc.. are introduced.**

1 Transformation of Variables

1.1 Using Definition

Problem 1.1. Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{1.1}$$

Problem 1.2. If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{1.2}$$

find α.

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Problem 1.3. Plot the CDF and PDf of

$$A = \sqrt{V} \tag{1.3}$$

Problem 1.4. Find an expression for $F_A(x)$ using the definition. Plot this expression and compare with the result of problem 1.3.

Problem 1.5. Find an expression for $p_A(x)$.

1.2 Using Jacobian

Problem 1.6. Evaluate the joint PDF of X_1, X_2, given by

$$p_{X_1,X_2}(x_1,x_2) = p_{X_1}(x_1)\,p_{X_2}(x_2) \tag{1.4}$$

Problem 1.7. Let

$$X_1 = \sqrt{V}\cos\theta \tag{1.5}$$

$$X_2 = \sqrt{V}\sin\theta. \tag{1.6}$$

Evaluate the Jacobian

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v} \\ \frac{\partial x_1}{\partial \theta} & \frac{\partial x_2}{\partial \theta} \end{vmatrix} \tag{1.7}$$

Problem 1.8. Find

$$p_{V,\Theta}(v,\theta) = |J|\,p_{X_1,X_2}(x_1,x_2) \tag{1.8}$$

Problem 1.9. Find $p_V(v)$.

Problem 1.10. Find $p_\Theta(\theta)$.

Problem 1.11. Are Y and Θ independent?

Problem 1.12. Find $p_A(x)$ using the Jacobian.

2 Conditional Probability

Problem 2.1. Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{2.1}$$

for

$$Y = AX + N, \tag{2.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1), X \in (-1,1)$ for $0 \le \gamma \le 10$ dB.

Problem 2.2. Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

Problem 2.3. For a function g,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{2.3}$$

Find $P_e = E[P_e(N)]$.

Problem 2.4. Plot P_e in problems 2.1 and 2.3 on the same graph w.r.t γ. Comment.

3 Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{3.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0,1). \tag{3.3}$$

Problem 3.1. Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{3.4}$$

on the same graph using a scatter plot.

Problem 3.2. For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

Problem 3.3. Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \tag{3.5}$$

with respect to the SNR from 0 to 10 dB.

Problem 3.4. Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.

4 Transform Domain

Let $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$.

Problem 4.1. Find $M_X(s) = E\left[e^{-sX}\right]$.

Problem 4.2. Let

$$N = n_1 - n_2, \quad n_1, n_2 \sim \mathcal{N}(0,1). \tag{4.1}$$

Find $M_N(s)$, assuming that n_1 and n_2 are independent.

Problem 4.3. Show that N is Gaussian. Find its mean and variance. Comment.

5 Random Number Generation

5.1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

Problem 5.1. Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Problem 5.2. Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat.

Problem 5.3. Verify that your CDF in the above problem is correct by plotting the theoretical $F_U(x)$.

5.2 Central Limit Theorem

Problem 5.4. Generate $U_i, i = 1, 2, \dots, 12$, a set of independent uniform random variables between 0 and 1 using a C program.

Problem 5.5. Generate 10^6 samples of the random variable

$$S = \sum_{i=1}^{12} U_i - 6 \tag{5.1}$$

and save in a file called s.dat

Problem 5.6. Load s.dat in python and plot the empirical PDF of S using the samples in s.dat. Does it look familiar? Comment.

5.3 From Uniform to Other

Problem 5.7. Generate samples of

$$V = -2\ln(1 - U) \tag{5.2}$$

and plot its CDF. Comment.

Problem 5.8. Generate the Rayleigh distribution from Uniform. Verify your result through graphical plots.