Mathematics

Vectors

March 2024

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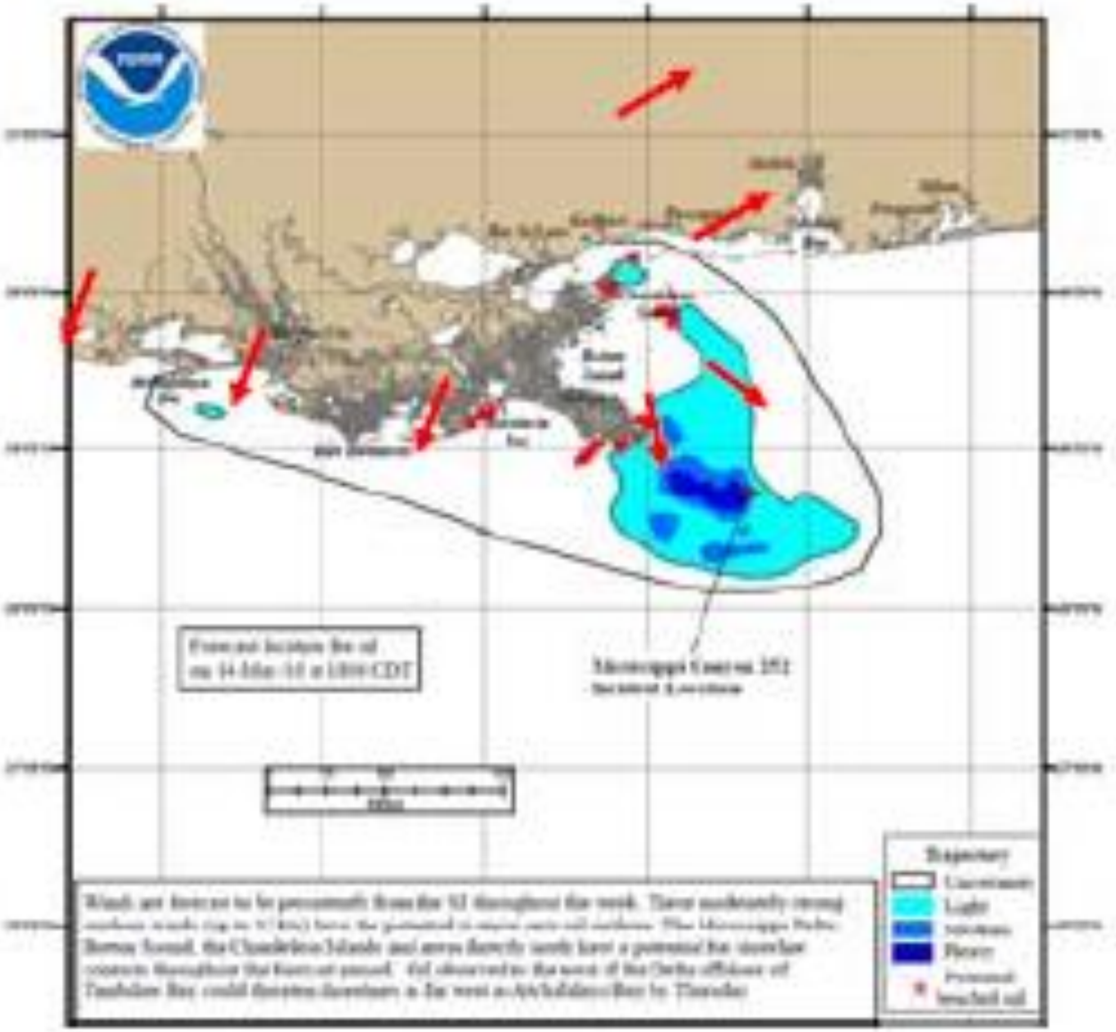
Vectors lecture 1 objectives

- After this lecture you should have a clear understanding of:
  - Use of vectors in engineering
  - Coordinate systems;
  - The properties of vectors;
  - Vector length, scalar multiplication, addition, and subtraction;
  - The dot product and its properties;
  - Projections
  - Orthogonol vectors
  - Resolving vector components;

Fluid Dynamics

Vectors are used to model engineering systems including aerospace, heating, ventilation, biomedical, oil and gas, marine and many others.

Oil spill modelling (Deep Water Horizon 2010)

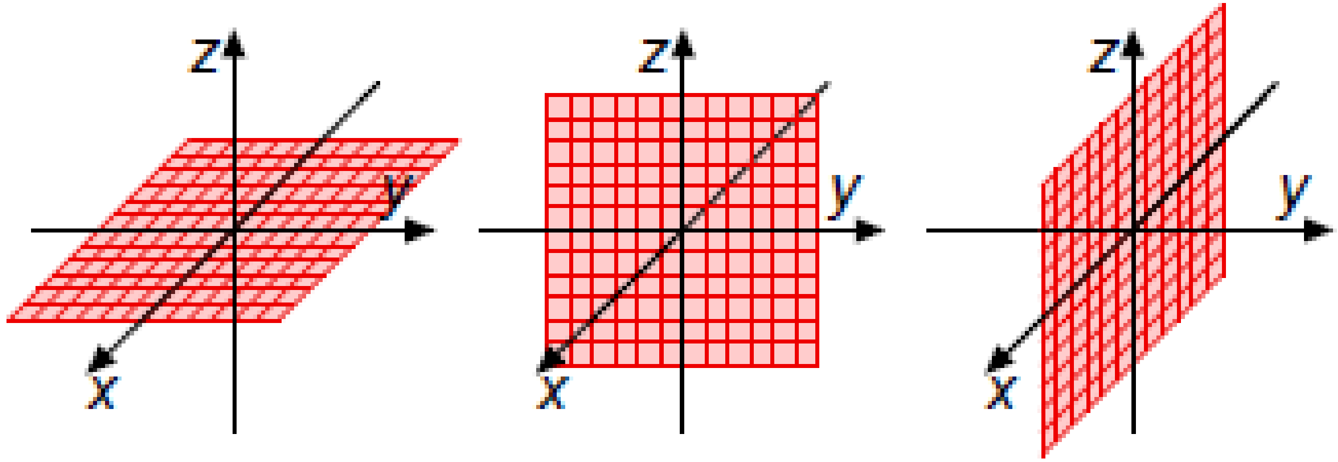


Coordinate systems

- A **coordinate system** is a system for specifying points in *n*–dimensional space using coordinates.
- A **coordinate** is a set of *n* variables which fix a geometric object.
- The simplest coordinate system consists of *n* coordinate axes oriented perpendicularly to each other, known as **Cartesian coordinates** (René Descartes 1637).
- In this module we will focus our attention on 2–dimensional and 3–dimensional Cartesian coordinate systems.

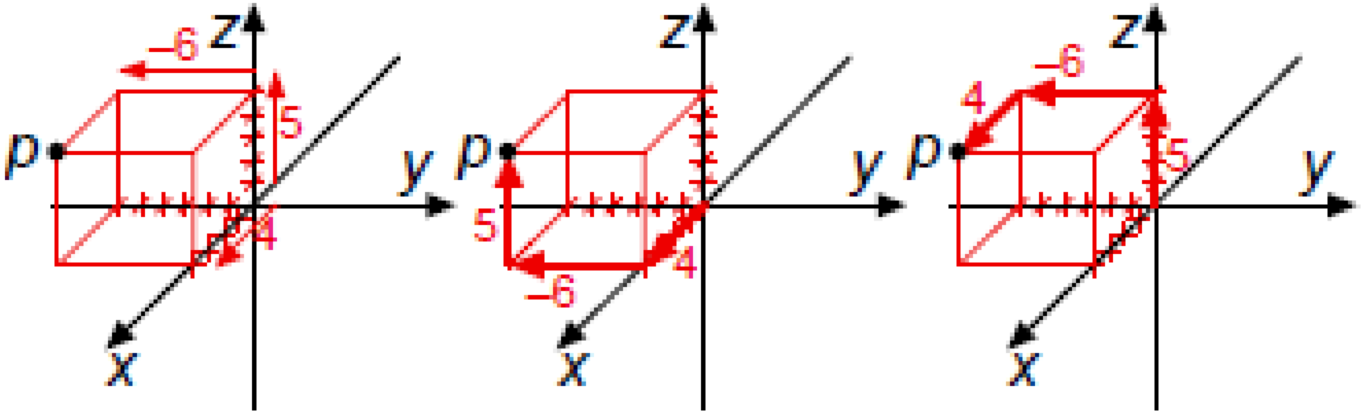
Coordinate planes in 3D

- The 3–dimensional Cartesian coordinate system defines 3 natural coordinate planes by taking coordinate axes in pairs.
- Illustrated below are the *xy*–, *yz*–, and *zx*–coordinate planes, respectively.



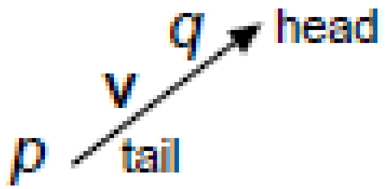
Points in 3D

- A point in a 3–dimensional coordinate system can be characterised by a set of 3 real numbers, representing the distances along the x–, y–, and z–coordinate axes.
- e.g. the point p at $x = 4$, $y = –6$, and $z = 5$.
  Note: the order of movement is not important



Vector

- A **vector** is essentially a directed arrow.
- A vector has the properties of **length** and **direction**, but **not position**.
- Two vectors are **equal** if they have the same length and the same direction.
- A vector from point p to point q has its **tail** at p and its **head** at q.



- A vector with its tail at the coordinate origin is sometimes called a **position vector**.

Specifying vectors and components

- Vectors are generally denoted by lowercase, non-italicised, bold characters
  e.g. **u**, **v**, **w** will be commonly used.
- Components of vectors are usually denoted by lowercase, italicised, non-bold characters with an index as a subscript
  e.g. u_3 refers to the 3rd component of vector **u**.
- More generally, an *n*–dimensional vector, **v**, can be written as

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Length of a vector

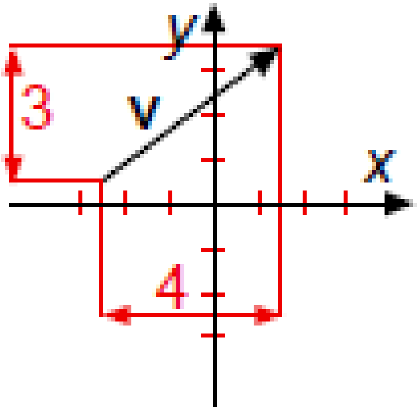
- The **length** of a vector **v**, denoted |**v**|, can be readily calculated from its components, v_i

if $$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

then $$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + \ldots v_n^2}$$

- e.g. for $$\mathbf{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$|\mathbf{v}| = \sqrt{4^2 + 3^2} = 5$$



Zero vector

- A **zero vector**, denoted **0**, is a vector of length zero.
- All the components of a zero vector are zero.
- The zero vector is the **additive identity**
  i.e. **v** + **0** = **v**
- Listed below are zero vectors for 1–, 2–, 3–, and *n*–dimensions

$$0 = \begin{bmatrix} 0 \end{bmatrix} \quad 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad 0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Unit vector

- A **unit vector** is a vector of length 1, sometimes called a **direction vector**.
- An arbitrary vector, **v**, can be converted to a unit vector, $\hat{\mathbf{v}}$, by dividing each of its components by its length

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \begin{bmatrix} v_1/|\mathbf{v}| \\ v_2/|\mathbf{v}| \\ \vdots \\ v_n/|\mathbf{v}| \end{bmatrix}$$

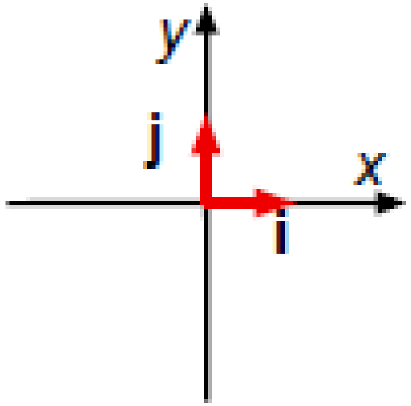
- e.g. $\begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$ is a vector of unit length with the same direction as the vector $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

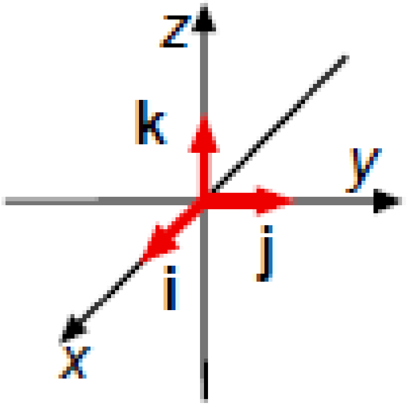
Base vectors…

A coordinate system provides a natural set of **base vectors** of unit length with direction in each of the coordinate axes.

In a 2–dimensional coordinate system with coordinate axes *x* and *y*, the base vectors are denoted **i** and **j**, respectively.

In a 3–dimensional coordinate system with coordinate axes *x*, y, and z, the base vectors are denoted **i** , **j**, and **k**





Base vectors

Another way of writing the base vectors is

- in two dimensions

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- in three dimensions

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Base vectors

Any vector can be expressed in terms of its coordinate base vectors

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix} = 4\mathbf{i} + 3\mathbf{j} \qquad \begin{bmatrix} 4 \\ -6 \\ 5 \end{bmatrix} = 4\mathbf{i} - 6\mathbf{j} + 5\mathbf{k}$$

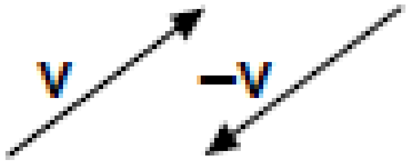
For an n–dimensional coordinate system with coordinates x_1, x_2, … x_n, and

base vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n$

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \ldots v_n\mathbf{e}_n$$

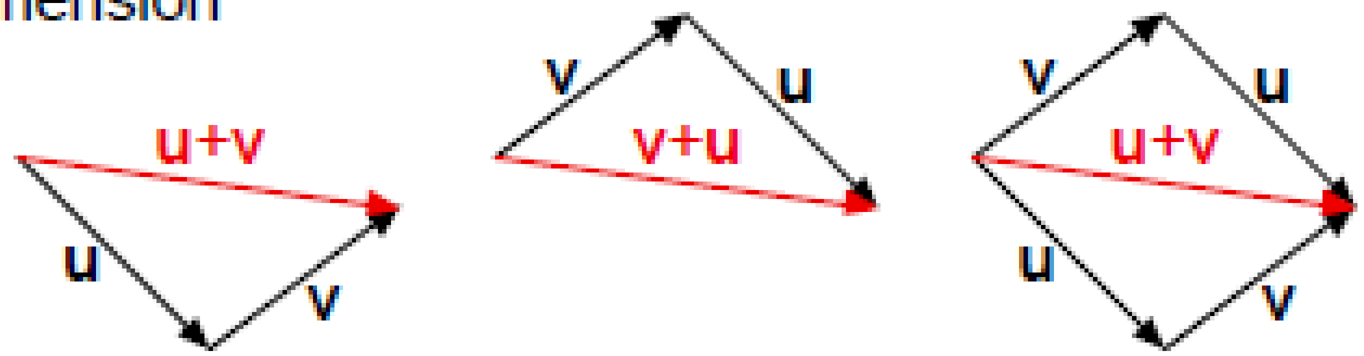
Vector negation

- Multiplying an *n*–dimensional vector by –1 results in a vector of the same length but reversed direction.



Vector addition…

- Vector addition is the operation of adding two or more vectors together into a vector sum.
- For vectors **u** and **v**, the vector sum **u** + **v** is obtained by placing them head to tail and drawing the vector from the free tail to the free head.
- Note that **u** + **v** = **v** + **u**
- Vectors being added must have the same dimension



…Vector addition

- In Cartesian coordinates, vector addition can be performed simply by adding the corresponding components of the vectors

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}+\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}=\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix}$$

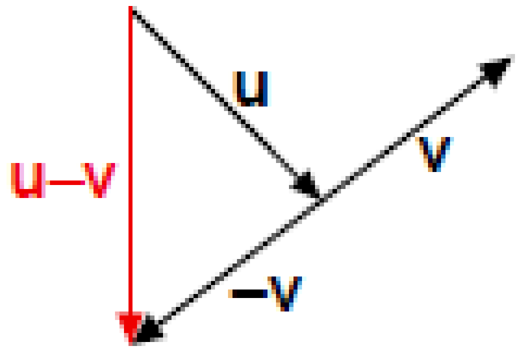
- e.g.

$$\begin{bmatrix} 4 \\ -4 \end{bmatrix}+\begin{bmatrix} 4 \\ 3 \end{bmatrix}=\begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

$$(4\mathbf{i}-4\mathbf{j})+(4\mathbf{i}+3\mathbf{j})=(8\mathbf{i}-\mathbf{j})$$

Vector subtraction…

- A vector difference is the result of subtracting one vector from another.
- A vector difference is denoted using the normal minus sign, i.e., the vector difference of vectors **u** and v is written **u** – v.
- A vector difference is equivalent to a vector sum with the orientation of the second vector reversed, i.e. **u** – **v** = **u** + (-v)



…Vector subtraction

- In Cartesian coordinates, vector subtraction can be performed simply by subtracting the corresponding components of the vectors

$$\mathbf{u}-\mathbf{v}=\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}-\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}=\begin{bmatrix} u_1-v_1 \\ u_2-v_2 \\ \vdots \\ u_n-v_n \end{bmatrix}$$

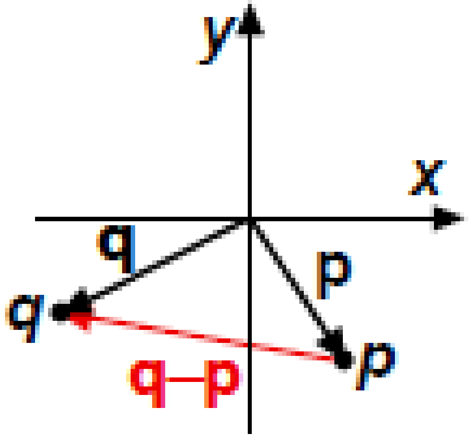
- e.g.

$$\begin{bmatrix} 4 \\ -4 \end{bmatrix}-\begin{bmatrix} 4 \\ 3 \end{bmatrix}=\begin{bmatrix} 0 \\ -7 \end{bmatrix}$$

$$(4\mathbf{i}-4\mathbf{j})-(4\mathbf{i}+3\mathbf{j})=(0\mathbf{i}-7\mathbf{j})$$

Vector between two points

- If p and q are two points with position vectors **p** and **q**, respectively, then the vector joining p to q is **q** – **p**.
- The finite line segment starting at p and terminating at q is denoted $\overline{pq}$
- Remember as *final* – *initial*, also **q** + (-**p**)

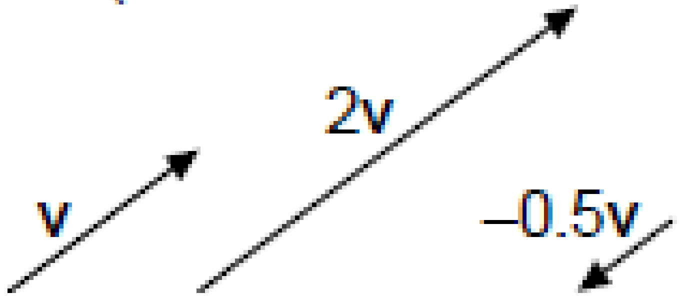


Vector scalar multiplication

- Multiplying an *n*–dimensional vector by a scalar results in an *n*–dimensional vector with each of its elements multiplied by the scalar.
- e.g.

$$-2\begin{bmatrix}4\\-6\\5\end{bmatrix}=\begin{bmatrix}-8\\12\\-10\end{bmatrix} \qquad c\mathbf{v}=c\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=\begin{bmatrix}cv_1\\cv_2\\\vdots\\cv_n\end{bmatrix}$$

- Note: the length scales by *c* but the direction remains the same (or is reversed if $c < 0$).



Dot product (or scalar product)

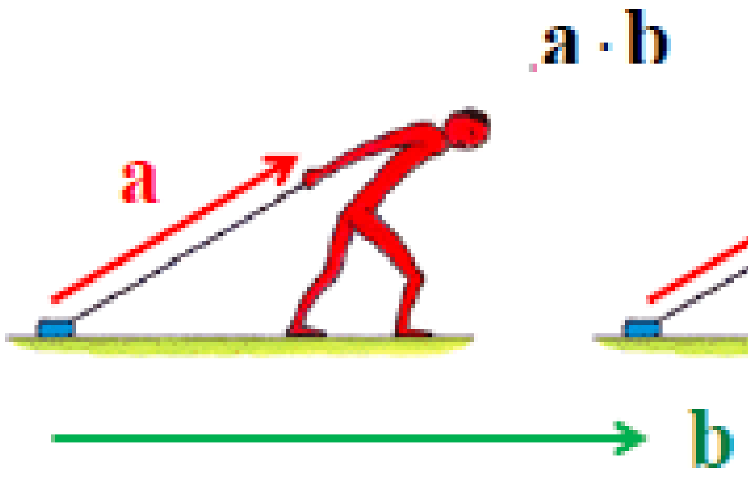
The **dot product** can be defined for two *n*–dimensional vectors **u** and **v** by

$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\ |\mathbf{v}|\ \cos(\theta)$

where θ is the angle between the vectors and | **u** | and | **v** | are the lengths of **u** and **v**, respectively.

Note that the dot product of two vectors results in a **scalar**.

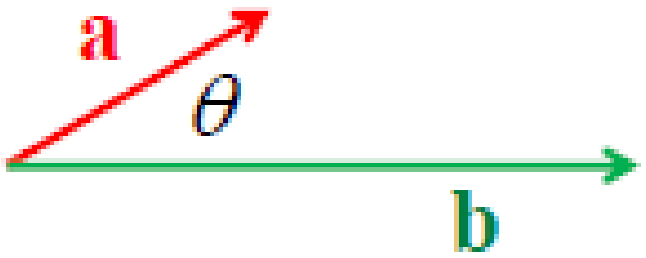
Applications…The scalar or dot product

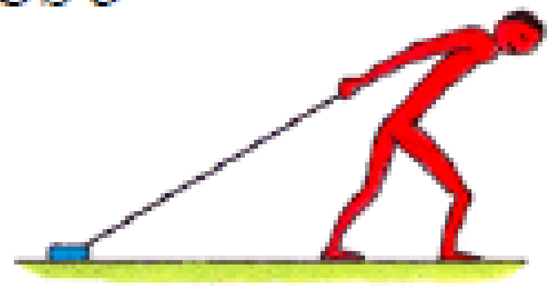


The man is pulling the block with a constant force **a** so that it moves along the horizontal ground. The **work done** in moving the block through a distance **b** is then given by the distance moved through multiplied by the magnitude of the component of the force in the direction of motion.

Applications…The scalar or dot product

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$





IMPORTANT Each of the lengths $|\mathbf{a}|$ and $|\mathbf{b}|$ is a number and cos θ is a number, so $\mathbf{a}\cdot\mathbf{b}$ is not a vector but a number or scalar. This is why it's called the scalar product. When writing down two vectors multiplied in this way, you *must* include the dot between them. Writing $\mathbf{ab}$ is meaningless.

…Dot product…

In Cartesian coordinates, the dot product can be performed by summing the products of the corresponding components of the vectors

$$\mathbf{u}\cdot\mathbf{v}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}\cdot\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=u_1v_1+u_2v_2+\dots u_nv_n$$

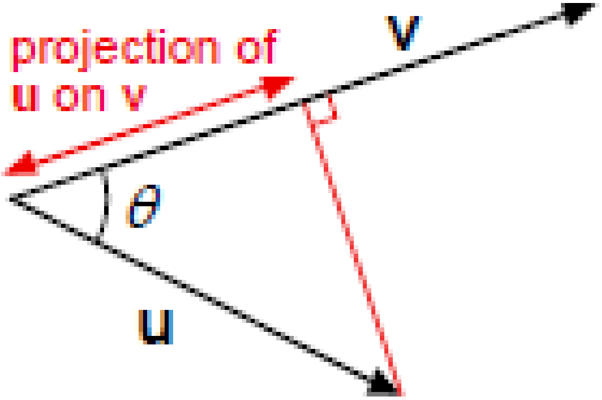
$$\begin{bmatrix}4\\-4\end{bmatrix}\cdot\begin{bmatrix}4\\3\end{bmatrix}=(4\times4)+((-4)\times3)=4$$

e.g.

$$\begin{bmatrix}4\\-6\\5\end{bmatrix}\cdot\begin{bmatrix}3\\2\\-1\end{bmatrix}=(4\times3)+((-6)\times2)+(5\times(-1))=-5$$

Dot product

- The dot product, **u•v**, has the geometric interpretation as the length of **v** times the length of the projection of **u** onto **v** when the two vectors are placed so that their tails coincide.



- Again think of the "work done" example
- We will come back to this

Perpendicular vectors…

Two vectors are **perpendicular** if the angle between them is a right angle (i.e. $\pi/2$ radians or 90°).

Since **u•v** = |**u**| |**v**| cos(θ) and cos(π/2) = 0, it follows that two non–zero vectors **u** and **v** are perpendicular if, and only if, **u•v** = 0.

e.g.

$$\mathbf{u}=\begin{bmatrix}2\\-1\end{bmatrix},\ \mathbf{v}=\begin{bmatrix}1\\2\end{bmatrix}$$

$$\mathbf{u}\cdot\mathbf{v}=\begin{bmatrix}2\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\2\end{bmatrix}=(2\times1)+((-1)\times2)=0$$

So **u** and **v** are perpendicular.

Dot product of base vectors

Recall that in a Cartesian coordinate system the base vectors are mutually perpendicular and of unit length.

Thus in 3–dimensions:

$$\mathbf{i}\cdot\mathbf{i}=1 \quad \mathbf{i}\cdot\mathbf{j}=0 \quad \mathbf{i}\cdot\mathbf{k}=0$$
$$\mathbf{j}\cdot\mathbf{i}=0 \quad \mathbf{j}\cdot\mathbf{j}=1 \quad \mathbf{j}\cdot\mathbf{k}=0$$
$$\mathbf{k}\cdot\mathbf{i}=0 \quad \mathbf{k}\cdot\mathbf{j}=0 \quad \mathbf{k}\cdot\mathbf{k}=1$$

Finding angles between vectors…

- Since **u•v** = |**u**| |**v**| cos(θ)
  and **u•v** = $u_1v_1 + u_2v_2 + \ldots u_nv_n$
  the dot product can be used to find the angle, θ, between vectors
  u and **v**

$$\cos(\theta) = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{u_1v_1 + u_2v_2 + \ldots u_nv_n}{|\mathbf{u}||\mathbf{v}|}$$

- e.g.

$$\mathbf{u} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$|\mathbf{u}| = \sqrt{4^2 + (-4)^2} = \sqrt{32}, |\mathbf{v}| = \sqrt{4^2 + 3^2} = 5$$

$$\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} = (4\times 4) + ((-4)\times 3) = 4$$

$$\cos(\theta) = \frac{\mathbf{u}\bullet\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{4}{\sqrt{32}\times 5} \approx 0.14142 \;\Rightarrow\; \theta \approx 81.87^\circ$$

Example 1 (Exercise)

$$\mathbf{u} = \begin{bmatrix} 4 \\ -6 \\ 5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

$$|\mathbf{u}| =$$

$$|\mathbf{v}| =$$

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 4 \\ -6 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \qquad =$$

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \qquad \approx \qquad \Rightarrow \theta \approx$$

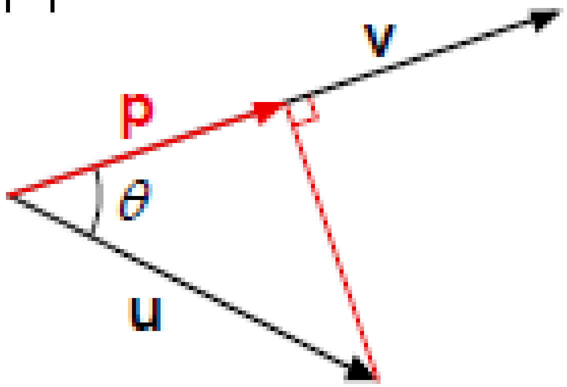
Projection of vector on vector…

- Given a vector **u**, find its projection **p** in the direction of vector **v**.

$$|\mathbf{p}| = |\mathbf{u}|\cos(\theta)$$
$$= |\mathbf{u}||\hat{\mathbf{v}}|\cos(\theta), \text{ since } |\hat{\mathbf{v}}| = 1$$
$$= \mathbf{u}\cdot\hat{\mathbf{v}}$$

$$\mathbf{p} = |\mathbf{p}|\hat{\mathbf{v}} = (\mathbf{u}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}} = \left(\mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{(\mathbf{u}\cdot\mathbf{v})}{|\mathbf{v}|^2}\mathbf{v}$$

$$\boxed{\mathbf{p} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}}, \text{ since } |\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v}$$



Example 2

Find the projection of **u** on **v**, where

$$\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\mathbf{u} \cdot \mathbf{v} = \qquad\qquad\qquad =$$

$$\mathbf{v} \cdot \mathbf{v} = \qquad\qquad\qquad =$$

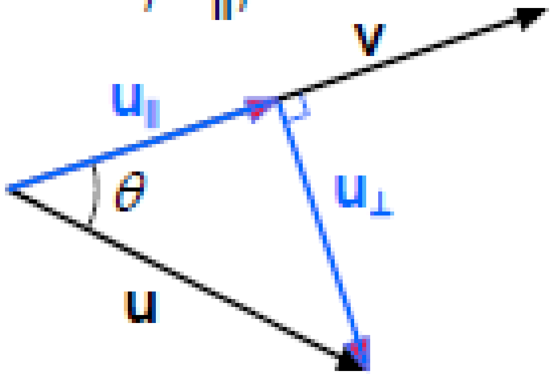
$$\mathbf{p} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v} = \qquad \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

Resolving vector components…

- The projection formula can be used to resolve a vector, **u**, to a direction, **v** into parallel, $\mathbf{u}_{\parallel}$, and perpendicular, $\mathbf{u}_{\perp}$.

$$\mathbf{u}_{\parallel} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$$

$$\mathbf{u}_{\perp} = \mathbf{u} - \mathbf{u}_{\parallel}$$



- e.g. resolve **u** into vectors parallel and perpendicular to **v**, where $\mathbf{u} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$\mathbf{u}_{\parallel} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \frac{-4}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\mathbf{u}_{\perp} = \mathbf{u} - \mathbf{u}_{\parallel} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Example 3

Resolve the vector, **u**, parallel, $\mathbf{u}_{\|}$, and perpendicular, $\mathbf{u}_{\perp}$, to the vector, **v**, where

$$\mathbf{u} = \begin{bmatrix} 2 \\ -6 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\mathbf{u}_{\|} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \frac{6}{3}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$$

$$\mathbf{u}_{\perp} = \mathbf{u} - \mathbf{u}_{\|} = \begin{bmatrix} 2 \\ -6 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 4 \end{bmatrix}$$

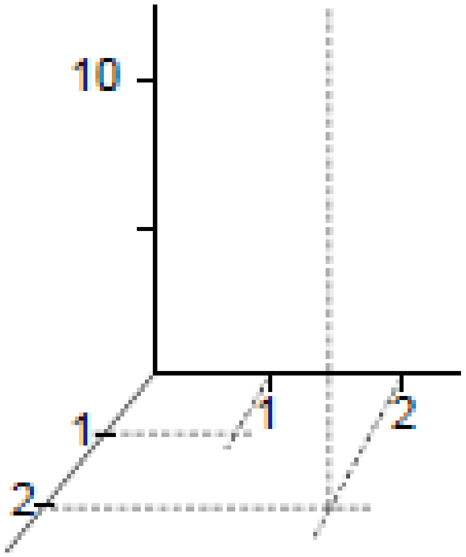
Example 4 (Exercise)

Resolve the vector, **u**, parallel, $\mathbf{u}_{\|}$, and perpendicular, $\mathbf{u}_{\perp}$, to the vector, **v**, where

$$\mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ 9 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{u}_{\|} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} = \quad \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

$$\mathbf{u}_{\perp} = \mathbf{u} - \mathbf{u}_{\|} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} - \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$



Summary

You should now have a clear understanding of:

- Use of vectors in engineering
- Coordinate systems
- The properties of vectors
- Vector length, scalar multiplication, addition, and subtraction
- The dot product and its properties
- Two ways of computing a dot product
- Projections
- Orthogonal vectors
- Resolving vector components

Vectors lecture 2 objectives

After this lecture you should have a clear understanding of:

- The cross product and its properties;
- How to calculate a cross product;
- Calculate areas of parallelograms;
- The use of cross product in applications.

Cross product

For 3–dimensional vectors **u** and **v**, the **cross product** is defined as

$$\mathbf{u}\times\mathbf{v} = |\mathbf{u}|\ |\mathbf{v}|\ \sin(\theta)\ \hat{\mathbf{n}}$$

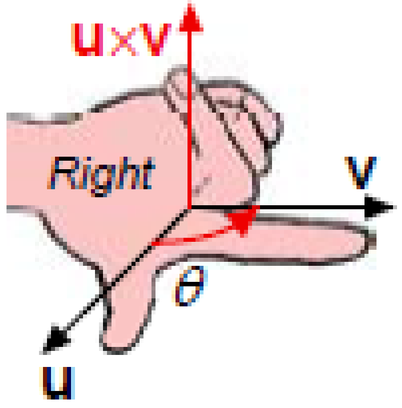
where θ is the angle between the vectors,

$|\mathbf{u}|$ and $|\mathbf{v}|$ are the lengths of **u** and **v**, and

$\hat{\mathbf{n}}$ is the unit vector normal to the plane

containing **u** and **v**.

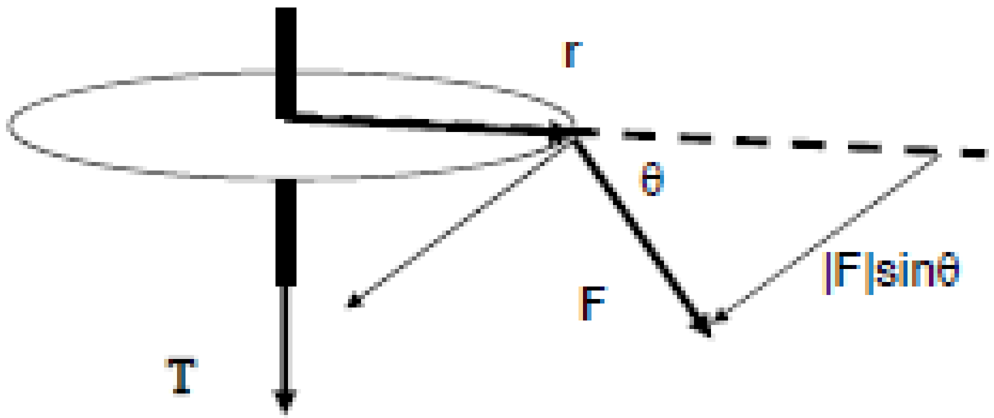
The cross product uses the right hand rule to determine the direction of the result. i.e. **u**, **v**, and **u**×**v** form a right handed system.



Applications of the cross product

Moment or torque (**T**) is a vector defined by a force (**F**) and a lever arm r, where

$$\mathbf{T} = r \times \mathbf{F}$$

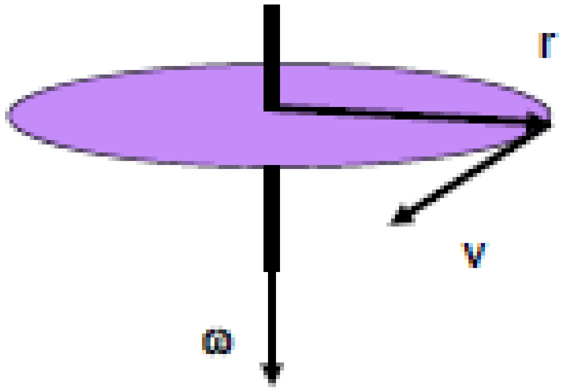


Applications of the cross product

- DC Electric motor produces a force on charge q moving with velocity v in a magnetic field **B**.

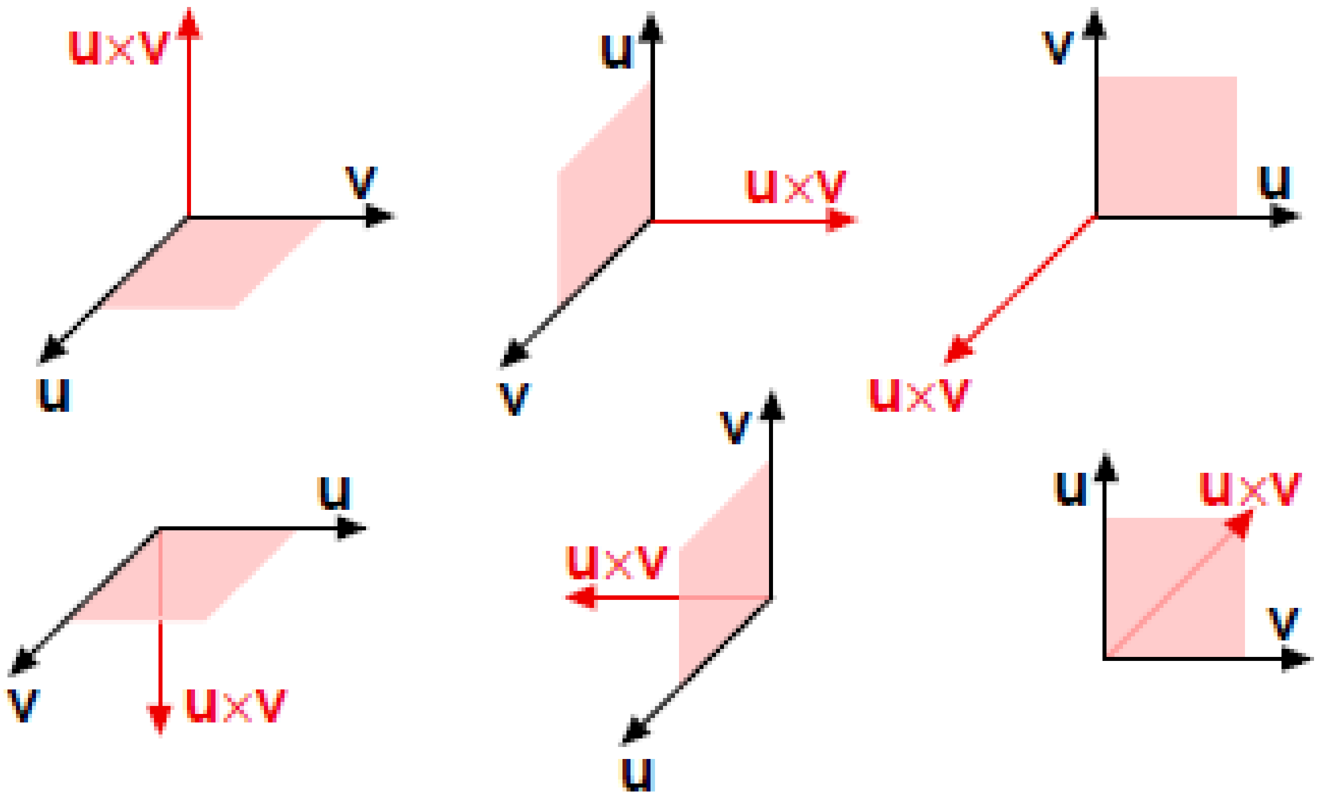
  Force = $q\mathbf{v}\times\mathbf{B}$

- The velocity **v** of a point on a spinning wheel at radius **r** is a vector defined by $\mathbf{v}=\boldsymbol{\omega}\times\mathbf{r}$, where $\boldsymbol{\omega}$ is the angular velocity vector.



Right-hand rule works like a screwdriver

Direction of **u**×**v** is the direction a screw goes if you turn it from **u** to **v** using the right hand



Cross product properties

- The cross product is anti–commutative,
  $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$.
  This is a consequence of the right hand rule.
- The cross product is distributive,
  $\mathbf{u}\times(\mathbf{v} + \mathbf{w}) = \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w}$
- For a scalar c, $(c\mathbf{u})\times\mathbf{v} = c(\mathbf{u}\times\mathbf{v})$
- Note that the cross product operates on two 3–dimensional vectors to produce a 3–dimensional vector perpendicular to the first two.

Computing u x v : the hard way

Suppose $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. Find $\mathbf{u} \times \mathbf{v}$.

$$|\mathbf{u}| = \sqrt{1+4+1} = \sqrt{6}$$

$$|\mathbf{v}| = \sqrt{4+1+1} = \sqrt{6}$$

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{2+2-1}{\sqrt{6}\sqrt{6}} = \frac{1}{2}$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \frac{\sqrt{3}}{2}$$

$$|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta = \sqrt{6}\sqrt{6}\frac{\sqrt{3}}{2} = 3\sqrt{3}$$

Direction of $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and to $\mathbf{v}$.

Computing u x v: the hard way

Recall $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

Direction of $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ and to $\mathbf{v}$

$|\mathbf{w}| = 3\sqrt{3}$

Let $\mathbf{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and solve $\mathbf{w} \cdot \mathbf{u} = 0, \mathbf{w} \cdot \mathbf{v} = 0, |\mathbf{w}| = 3\sqrt{3}$.

$$\begin{aligned} x + 2y - z &= 0 \\ 2x + y + z &= 0 \\ x^2 + y^2 + z^2 &= 27 \end{aligned}$$

$$3x + 3y = 0 \Rightarrow x = -y = -z$$
$$x^2 + y^2 + z^2 = 27 \Rightarrow 3x^2 = 27 \Rightarrow x = \pm 3$$

Right-hand rule says $x = 3$. So $\mathbf{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix}$

Easy cross product calculation…

In Cartesian coordinates, the cross product can be calculated using the following method

$$\textit{if} \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \textit{ and } \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \quad \text{then}$$

$$\mathbf{u} \times \mathbf{v} = \mathbf{i}(u_2 v_3 - u_3 v_2) - \mathbf{j}(u_1 v_3 - u_3 v_1) + \mathbf{k}(u_1 v_2 - u_2 v_1)$$

$$= \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Easy cross product calculation…

Your formula book uses a slightly different notation

$$\text{if}\quad \mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix},\ \text{and}\quad \mathbf{w} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \quad \text{then}$$

$$\mathbf{v} \times \mathbf{w} = \mathbf{i}(br - cq) + \mathbf{j}(cp - ar) + \mathbf{k}(aq - bp)$$

$$= \begin{bmatrix} br - cq \\ cp - ar \\ aq - bp \end{bmatrix}$$

Example 1: the easy way

If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ then

$$\mathbf{u} \times \mathbf{v} = \mathbf{i}\left(u_2 v_3 - u_3 v_2\right) - \mathbf{j}\left(u_1 v_3 - u_3 v_1\right) + \mathbf{k}\left(u_1 v_2 - u_2 v_1\right)$$

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{u} \times \mathbf{v} = \cancel{\begin{matrix} 1\mathbf{i} & +2\mathbf{j} & -1\mathbf{k} \\ & \times & \\ 2\mathbf{i} & +1\mathbf{j} & +1\mathbf{k} \end{matrix}}$$

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} \boxed{2 \times 1 \quad - \quad (-1) \times 1} \\ \boxed{-(1 \times 1 \quad - \quad 2 \times (-1))} \\ \boxed{1 \times 1 \quad - \quad 2 \times 2} \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix}$$

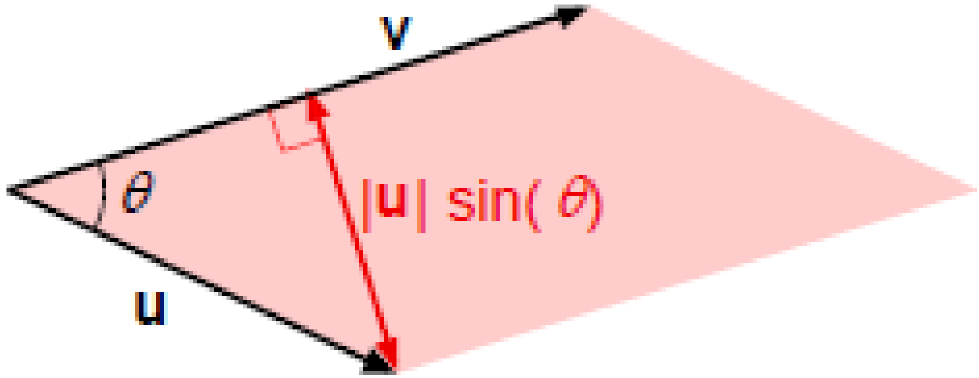
Example 2: Exercise

$$\text{if } \mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}, \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \quad \text{find } \mathbf{u} \times \mathbf{v}$$

Area of a parallelogram

- If a parallelogram is defined by the vectors *u* and *v* then the area of the parallelogram is given by $|u \times v| = |v \times u|$
- i.e.

$$
\begin{aligned}
area &= base \times height \\
&= |v||u| \sin(\theta) \\
&= |v \times u|
\end{aligned}
$$



Example 3

Calculate the area of the parallelogram whose edges are

$$\mathbf{p} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \quad \text{and} \quad \mathbf{q} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$$

$$\mathbf{p} \times \mathbf{q} = \begin{array}{c} \mathbf{1i} \ \ \mathbf{+3j} \ \mathbf{+6k} \\ \mathbf{x} \\ \mathbf{-2i} \ \mathbf{+0j} \ \mathbf{+4k} \end{array}$$

$$|\mathbf{p} \times \mathbf{q}| = \left| \begin{bmatrix} 12 \\ -16 \\ 6 \end{bmatrix} \right| = \sqrt{436} = 20.88$$

Example 4 (Exercise)

Find the area of the parallelogram with edges represented by the vectors:

$$2\mathbf{i} - \mathbf{j} + 3\mathbf{k} \text{ and } 7\mathbf{i} + \mathbf{j} + \mathbf{k}$$

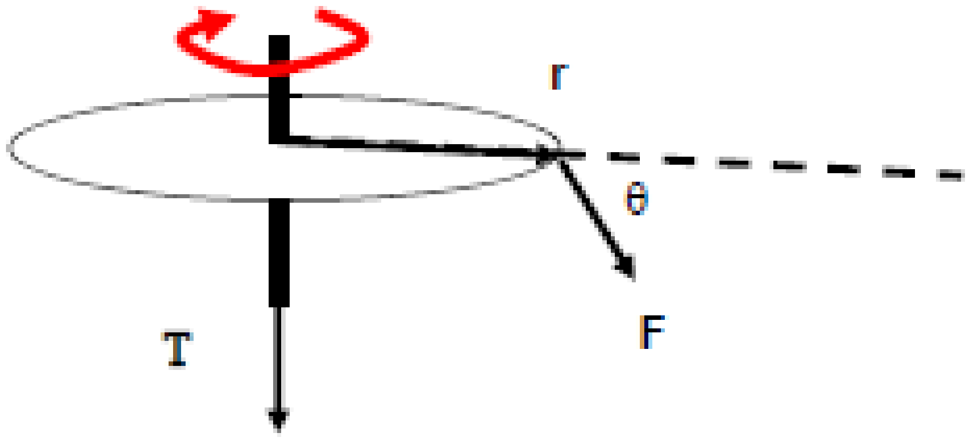
Overview of multiplicative operations

	Scalar Multiplication	Scalar Product	Vector Product
Notation	$s\mathbf{a}$	$\mathbf{a} \cdot \mathbf{b}$	$\mathbf{a} \times \mathbf{b}$
Multiply	a scalar and a vector	two vectors	two vectors
Result	a parallel vector	a scalar	a perpendicular vector
Involves		cosine	sine

Recall application of the cross product

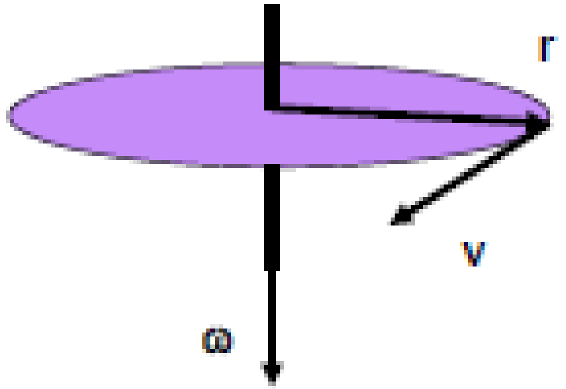
Moment or torque (**T**) is a vector defined by a force (**F**) and a lever arm r, whereby

$$\mathbf{T} = r \times \mathbf{F}$$



Angular velocity

The velocity **v** of a point on a spinning wheel at radius **r** is a vector defined by **v**=$\omega \times$**r**, where ω is the angular velocity vector.



Vectors lecture 3 objectives

After this lecture you should have a clear understanding of:

- The equation of a line;
- The equation of a plane;
- The normal to a plane;
- The intersection of a plane and the axes;
- Intersections between planes and lines.

You will need to be able to

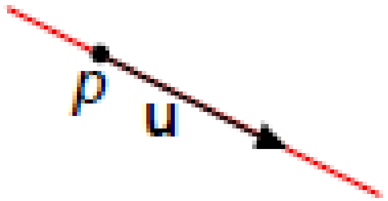
- Find the equation of the line through a point in a given direction;
- Find the line joining two given points;
- Find the equation of a plane through **p** with normal vector **n**;
- Find the equation of a plane through three given points;
- Find the point where a plane and line meet (if at all);
- Find the intersection line of two planes.

Lines

- A **line** is a straight one-dimensional figure having no thickness and extending infinitely in both directions. A line is sometimes called a **straight line**.
- A line is uniquely determined by either:
  - a point and a direction vector; or
  - two points.

Line defined by a point and vector

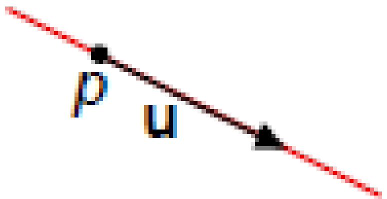
- We can define the line that passes through a point *p* in the direction of a vector **u** parametrically as
  $\mathbf{x} = \mathbf{p} + \mathbf{u}t$
  where t is a parameter that defines the position on the line (i.e. $\mathbf{x} = \mathbf{p}$ at $t = 0$, and $\mathbf{x} = \mathbf{p} + \mathbf{u}$ at $t = 1$).



Example 1

Find an equation that defines the line that passes through the point p in the direction of a vector $\mathbf{u}$ where

$$\mathbf{p} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



Example 1 (cont.)

Find an equation that defines the line that passes through the point *p* in the direction of a vector **u** where

$$\mathbf{p} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\mathbf{x} = \mathbf{p} + \mathbf{u}\, t$$

$$= \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix} t$$

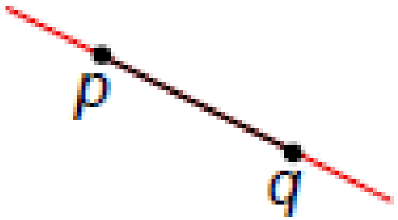
$$x = 1 - 3t,\ y = 3 + 2t,\ z = 4 + 5t$$

Line defined by two points…

- The line passing through points p and q is denoted

$$\overleftrightarrow{pq}$$

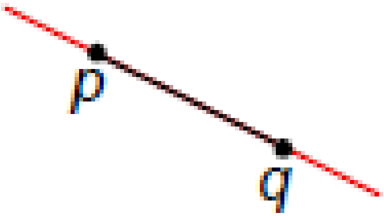
- Recall that if p and q are two points with position vectors $\mathbf{p}$ and $\mathbf{q}$, respectively, then the vector $\mathbf{u}$ joining p to q is $\mathbf{q} - \mathbf{p}$.
- We can thus define the line that passes through points p and q parametrically as
  $\mathbf{x} = \mathbf{p} + (\mathbf{q} - \mathbf{p})t$
  where t is a parameter that defines the position on the line (i.e. $\mathbf{x} = \mathbf{p}$ at $t = 0$, and
  $\mathbf{x} = \mathbf{q}$ at $t = 1$).



Example 2

Find an equation that defines the line that passes through points **p** and **q** where

$$\mathbf{p} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} -2 \\ 5 \\ 9 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



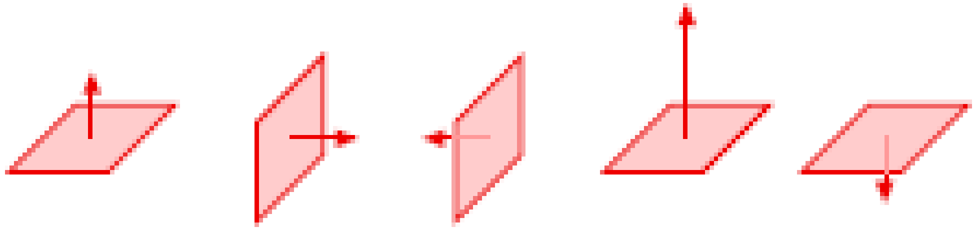
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \quad\quad\quad\quad \\ \\ \\ \end{bmatrix} + \begin{bmatrix} \quad\quad\quad\quad \\ \\ \\ \end{bmatrix} t$$

Planes…

- A **plane** is a 2–dimensional surface in a 3–dimensional space.
- More generally, a **hyperplane** is an (n–1)–dimensional vector subspace in an n–dimensional vector space.
- In this module we will be dealing only with planes in 3D.
- A plane is uniquely determined by either:
  - Three points; or
  - Two points and a direction vector; or
  - One point and two direction vectors; or
  - One point and a normal vector.

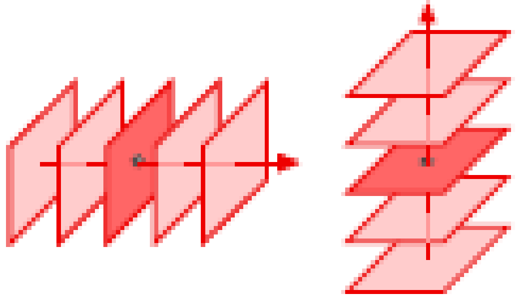
…Planes…

- One way to think of a plane is as a flat surface with a handle perpendicular to the surface.
- The handle, or normal vector, determines the orientation of the plane.
- The magnitude of the (non–zero) normal vector does not affect the orientation of the plane.



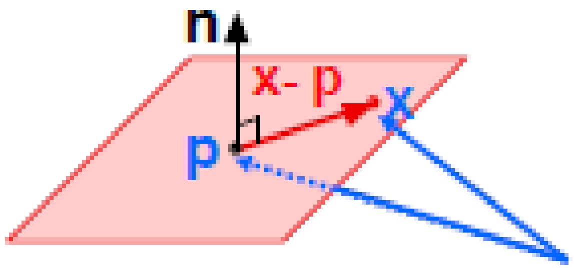
…Planes

- Specifying the normal vector does not uniquely determine the plane.
- We need to choose a point to fix an individual plane.



Plane defined by one point and a normal vector…

- The equation of a plane passing through the point *p* with a non–zero normal vector **n** can be defined by $\mathbf{n} \bullet (\mathbf{x} - \mathbf{p}) = 0$
  or, equivalently, $\mathbf{n} \bullet \mathbf{x} = \mathbf{n} \bullet \mathbf{p}$
  where **x** is any general point in the plane.
- Note that this equation states that any vector starting from the point *p* and ending on a point in the plane must be perpendicular to the normal vector **n**.



Example 3

Find the equation of the plane passing through the point $\boldsymbol{p}$ with normal vector $\mathbf{n}$, where

$$\mathbf{p} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \mathbf{n} = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$$

Example 4 (exercise)

Find the equation of the plane passing through the point p with normal vector $\mathbf{n}$, where

$$\mathbf{p} = \begin{bmatrix} -4 \\ 5 \\ -2 \end{bmatrix}, \text{ and } \mathbf{n} = \begin{bmatrix} -5 \\ -4 \\ 1 \end{bmatrix}$$

Plane defined by one point and a normal vector

- Note that the coefficients of *x*, *y*, and *z* in the equation for a plane are simply the components of the normal vector
- i.e. a normal vector to the plane defined by

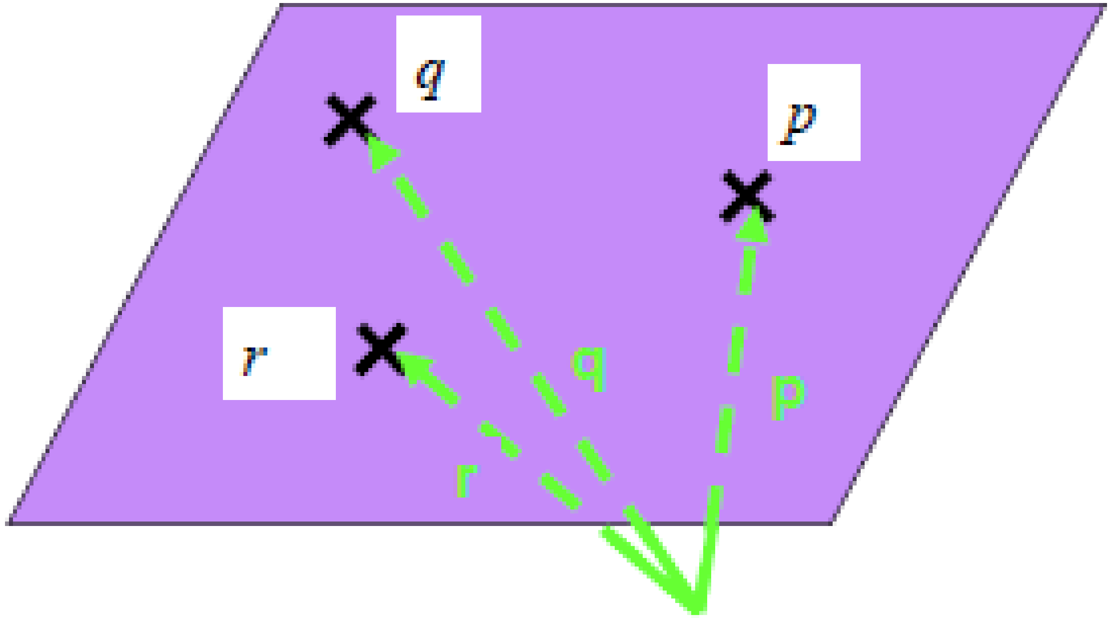
$$ax + by + cz = d \text{ is } \quad \mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

- e.g. a normal vector to the plane defined by

$$-5x - 4y + z = -2 \text{ is } \mathbf{n} = \begin{bmatrix} -5 \\ -4 \\ 1 \end{bmatrix}$$

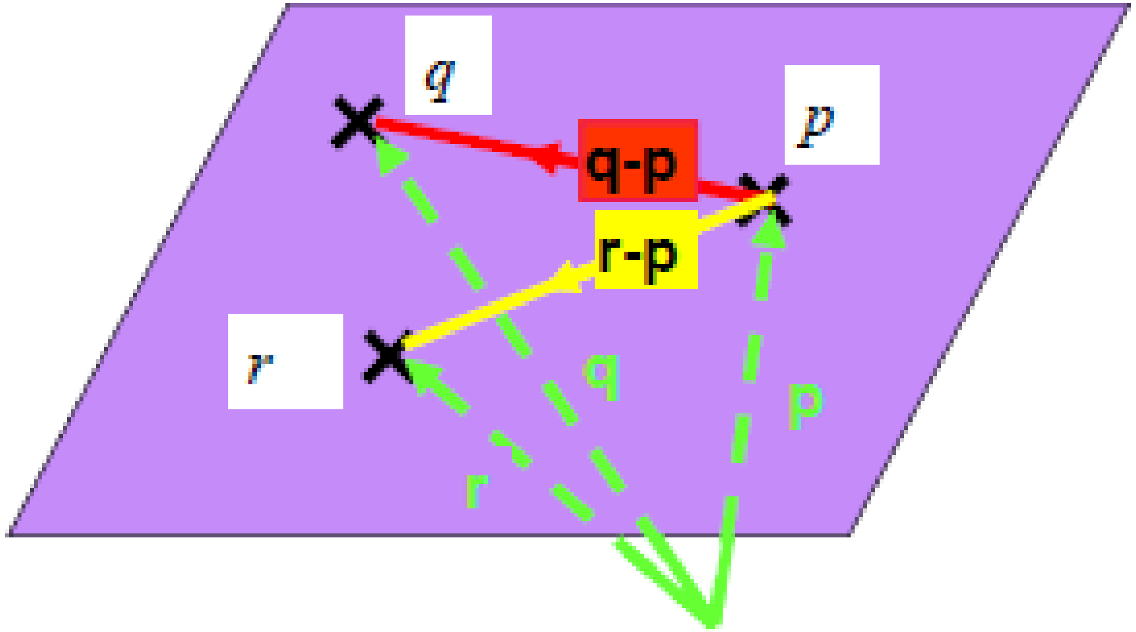
Plane defined by three points…

Find the equation of a plane passing through three points p, q, and r.



Plane defined by three points…

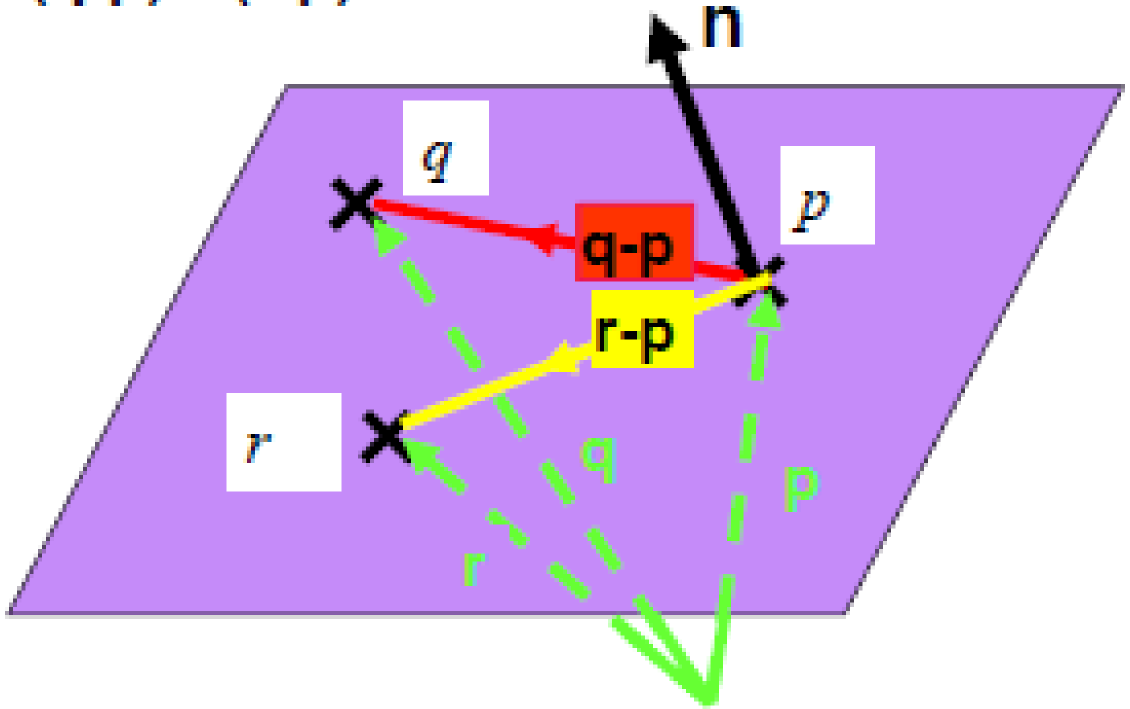
Choose one of the points, say *p*, and compute the vectors **r-p** and **q-p**



Plane defined by three points…

A normal to the plane can be found by computing

n = (q-p) x (r-p).



Example 6

Find the plane that passes through the points

$$\mathbf{p} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}, \text{and } \mathbf{r} = \begin{bmatrix} 3 \\ 1 \\ 8 \end{bmatrix}$$

First, choose point **p** and calculate the vectors joining **p** to **q**, (**q** – **p**), and **p** to **r**, (**r** – **p**)

$$(\mathbf{q}-\mathbf{p}) = \begin{bmatrix} \\ \\ \end{bmatrix} - \begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}, (\mathbf{r}-\mathbf{p}) = \begin{bmatrix} \\ \\ \end{bmatrix} - \begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

Example 6 (cont.)

Next find the normal to the plane by taking the cross product $\mathbf{n} = (\mathbf{q} - \mathbf{p}) \times (\mathbf{r} - \mathbf{p})$

$$\mathbf{n} = (\mathbf{q} - \mathbf{p}) \times (\mathbf{r} - \mathbf{p})$$

$$= \mathbf{i}(\qquad\qquad) - \mathbf{j}(\qquad\qquad) + \mathbf{k}(\qquad\qquad)$$

$$\mathbf{n} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

Example 6 (cont.)

Finally, use the formula for defining a plane from a point, *p*, and normal vector, **n**.

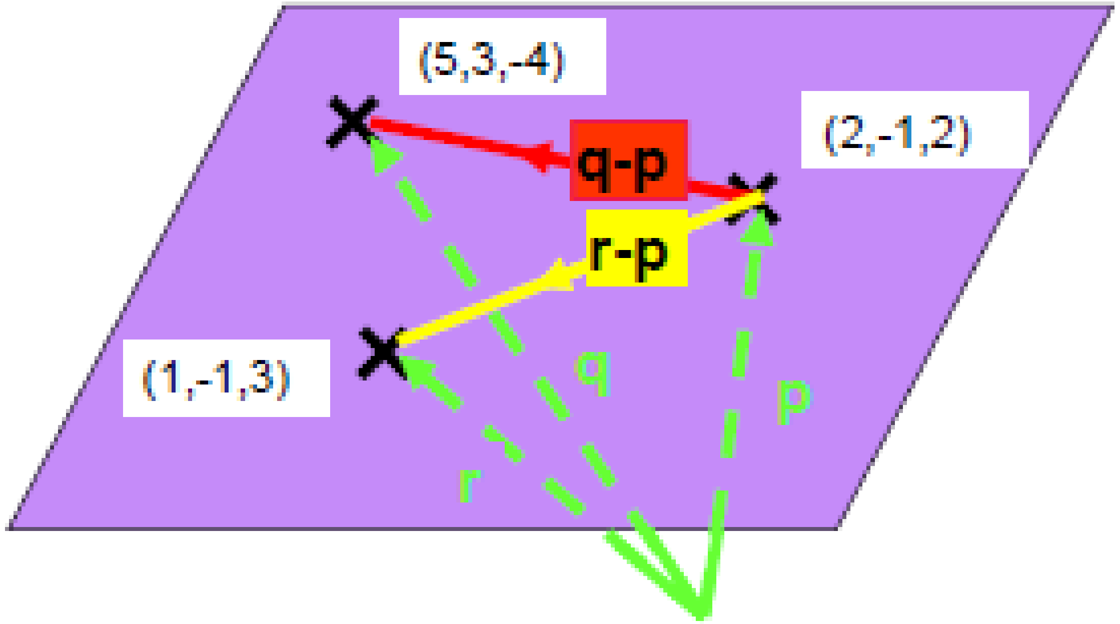
$$\mathbf{p} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}, \mathbf{n} = \begin{bmatrix} 14 \\ -10 \\ 3 \end{bmatrix}$$

$$14x - 10y + 3z = 56$$

Example 7 (Exercise)

Find the plane that passes through the points

$$\mathbf{p} = (2, -1, 2),\ \mathbf{q} = (5, 3, -4), \text{ and } \mathbf{r} = (1, -1, 3)$$



Find **n** using a cross product

$$\begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix} \times \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} =$$

Choose any point e.g. p=(2, -1, 2)

$$\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p} \Rightarrow$$

Example 8

Find the point $\boldsymbol{p}$ where the line through

the point $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ with direction $\begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix}$

meets the plane

$$3x + 2y - 4z = 218$$

Example 8 (cont.)

Find the equation of the line through

the point $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ with direction $\begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix}$

Note the concept of parametric definition.

The equation of the line is…

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + t \begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1-3t \\ 3+2t \\ 4+5t \end{bmatrix}$$

Substitute *x*, *y*, and *z* in the plane,

Solve for *t*,

Substitute *t* in the parametric formula to find *x*, *y*, and *z*:

Find the point p

$$3x + 2y - 4z = 218$$

$$3(1 - 3t) + 2(3 + 2t) - 4(4 + 5t) = 218$$

$$-25t = 225$$

$$t = -9$$

$$p = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + t\begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + (-9)\begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 28 \\ -15 \\ -41 \end{bmatrix}$$

Example 9 (Exercise)

At what point do the line

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2t \\ t-1 \\ 1 \end{bmatrix}$$

and the plane $x + 2y - z = 5$ meet?

The key here is to realize that values of the parameter define points on the line. The particular point we want is the one that is also on the plane.

Example 9 (HINT)

Suppose g is a typical point on the line.

$$\mathbf{g} = \begin{bmatrix} 2t \\ t-1 \\ 1 \end{bmatrix}$$

If g is on the plane then

$$\mathbf{g} \cdot \mathbf{n} = 5, \quad \text{where} \quad \mathbf{n} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Example 9 (cont.)

Solve for t