Department of Inter Disciplinary Studies,
Faculty of Engineering,
University of Jaffna, Sri Lanka
MC 2020 : Linear Algebra

Tutorial-05 **November 2023**

1. What do you mean by the term "Vector Space"?

   (a) Let V be the set of all real numbers x such that $x > 0$. Define an operation of addition by $x \oplus y = xy, \forall x, y \in V$. Define another operation of scalar multiplication by $\alpha \odot x = x^{\alpha}, \forall \alpha \in \mathbb{R}$ and $x \in V$. Prove that under the operation $\oplus$ and $\odot$, the set of V is a vector space.

   (b) Let V be the set of all fifth degree polynomials with standard addition and multiplication. Is it a vector space? Justify your answer.

2. Let V denote the set of vectors in $\mathbb{R}^2$. If $x = [x_1 \quad x_2]^T$ and $y = [y_1 \quad y_2]^T$ are elements of V and $\alpha \in \mathbb{R}$, define:

$$x + y = [x_1 + y_1 \quad x_2y_2]^T \text{ and } \alpha[x_1 \quad x_2]^T = [\alpha x_1 \quad x_2]^T$$

   Is V a vector space with these operations?

3. Let V be the set of all complex valued functions f on the real line such that

$$f(t) = \overline{f(t)};\ \forall t \in \mathbb{R}.$$

   Show that, V with operations

$$(f + g)(t) = f(t) + g(t)$$

$$(cf)(t) = cf(t)$$

   is a vector space over the field of real numbers.

4. Let V be a vector space and S be set and $F(S, V) = \{f \mid f : S \to V\}$. Define

$$f + g : S \to V \text{ by } (f + g)(s) = f(s) + g(s),\ \forall s \in S$$

   and

$$r.f : S \to V \text{ by } (r.f)(s) = r.f(s),\ \forall s \in S$$

   Then show the following:

   (a) $F(S, V)$ is a vector space;

   (b) if W is a subspace of V then $F(S, W)$ is subspace of $F(S, V)$.

5. Let V be a vector space and S be set and $s \in S$ be any fixed element. Define

$$e_s : F(S,V) \to V \text{ by } e_s(f) = f(s)$$

Show that $N = \{f \in F(S,V) \mid e_s(f) = 0\}$ is a subspace of $F(S,V)$.

6. Let V denote the set of all 2×2 matrices and let S denote the set of all symmetric 2×2 matrices. Show that V is a vector space and S is a subspace of V.

7. Let the distance between two vectors x and y in a linear normed vector space be defined as:

$$d(x,y) = \|x - y\|_p$$

Show that this distance function satisfies the distance properties.

8. Let $S = \{(a_1 \quad a_2) : a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in S$ and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2)$$

and

$$c(a_1, a_2) = (ca_1, ca_2).$$

Then S is not a vector space.

9. Show that each of following is a vector spaces over $\mathbb{R}$ under usual addition and scalar multiplication.
   (a) The set $A = \{(x, y, z, w) | x + y + z + w = 0; x, y, z, w \in \mathbb{R}\}$
   (b) The set of polynomials with coefficient from $\mathbb{R}$
   (c) The set of solutions to a homogeneous linear system $Ax = 0$
   (d) The set $\{f : \mathbb{R} \to \mathbb{R} : \dfrac{df}{dx} = 0\}$

10. Determine whether the following are vector space over $\mathbb{R}$.
    (a) Under the usual addition and scalar multiplication in $\mathbb{R}^3$

    $$S_1 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$$

    (b) Under the usual addition and scalar multiplication on matrices

    $$S_2 = \{\begin{bmatrix} a & 1 \\ b & c \end{bmatrix} \mid a, b, c \in \mathbb{R}\}$$

    (c) Under the usual addition and scalar multiplication in $\mathbb{R}^3$

    $$S_3 = \{(x, y, z) \mid x + y + z = 1\}$$

    (d) The set $\{f : \mathbb{R} \to \mathbb{R} : \dfrac{df}{dx} = 1\}$, under the usual addition and scalar multiplication for functions.

11. Show that the following sets are subspaces of the given spaces.

    (a) $\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a+b=0\}$ of $M_{2\times 2}$
    (b) $\{P(x) \in \mathbb{P}_2 \,:\, P(7)=0\}$ of $\mathbb{P}_2$
    (c) $\{(x,y,z,w) \,:\, 2x+y+w=0,\ y+2z=0\}$ of $\mathbb{R}^4$
    (d) $\{(x,y,z) \,:\, 3x+2y+z=0\}$ of $\mathbb{R}^3$
    (e) $\{a_o+a_1x+a_2x^2+a_3x^3 \,:\, a_o+a_1=0, a_2-a_3=0\}$ of $\mathbb{P}_3$

12. Determine whether the following set is vector space over $\mathbb{R}$ under the usual addition and scalar multiplication of polynomials. Justify your answer.

$$W = \{a_2 + a_1x + a_0x^2 \,|\, a_0, a_1, a_2 \in \mathbb{R}^+\}$$

13. Given non-zero real numbers a and b, consider

$$X = \{(x \ \ y) \in \mathbb{R}^2 \,|\, ax+by=0\}.$$

    Show that X is a subspace of vector space $\mathbb{R}^2$

14. Given non-zero real numbers a, b and c, consider

$$X_c = \{(x \ \ y) \in \mathbb{R}^2 \,|\, ax+by=c\}.$$

    Show that X_c is not a subspace of $\mathbb{R}^2$.

15. Determine whether or not the following subsets of $\mathbb{R}^4$ are subspace:

    (a) $\{(a,b,c,d) : a+b=c+d\}$
    (b) $\{(a,b,c,d) : a+b=1\}$
    (c) $\{(a,b,c,d) : a^2+b^2=0\}$
    (d) $\{(a,b,c,d) : a^2+b^2=1\}$

16. Determine which of the following are subspace of some Euclidean spaces over the field $\mathbb{R}$:

    (a) $H = \{(a+b, 3a-b, 2a+b) : a, b \in \mathbb{R}\}$
    (b) $H = \{(x,y,z) : x+y-z=0, 2x-3y+z=0\}$
    (c) $H = \{(x,y,z) : x+y-z=0, 2x-3y+z=1\}$
    (d) $H = \{(x,y) : xy \geq 0\}$

17. Determine which of the following sets of functions are subspace of V:

    (a) All f such that $f(x^2) = (f(x))^2$
    (b) All f such that $f(0) = f(1)$
    (c) All f such that $f(3) = 1 + f(-5)$
    (d) All f such that $f(-1) = 0$

18. Let $V = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is a function}\}$ with $(f+g)(x) = f(x)+g(x)$ and $(cf)(x) = cf(x)$. Let $S = \{f \in V : \dfrac{d^2 f}{dx^2} + f = 0\}$, show that S is a subspace of V.

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