UNIVERSITY OF JAFFNA
FACULTY OF ENGINEERING

END SEMESTER EXAMINATION– FEBRUARY 2023

LINEAR ALGEBRA

MC 2020

Writing Time: TWO Hours

Instructions

1. This paper contains **FIVE (5)** questions:
2. Answer **all** questions in the answer book provided.
3. If you have any doubt as to the interpretation of the wording of a question, make your own decision, but clearly state in the script.
4. This examination accounts for **50%** of module assessment. Total maximum mark attainable is **100**.

Question 1[20 marks]

1. The matrix $A = \begin{bmatrix} 7 & 5 & 5 \\ 5 & -2 & 4 \\ 5 & 4 & -2 \end{bmatrix}$
   (a) Find the eigenvalues.
   (b) Find the corresponding eigen vectors.
   (c) Find the corresponding normalised eigen vectors.
   (d) Find a matrix P and a diagonal matrix D such that $P^{-1}AP = D$.
   (e) Find the value of $det(\mathbf{A})$ using the results obtained above.
   (f) Diagonalisation makes calculating the powers of matrices much simpler and faster. Evaluate $\mathbf{A^5}$ using the result you obtained from 1(c) above..
2. Plot the ellipse: $5x^2 - 4xy + 2y^2 = 1$ according to the following steps.
   (a) The corresponding quadratic forms,
   (b) Eigen values and corresponding unit eigen vectors,
   (c) Intercepts along the eigen vectors,
   (d) Plot the elipse.

Question 2[20 marks]

1. The following data is given for the velocity of the rocket as a function of time.

t	(s)	0	14	15	20	30	35
v(t)	(m/s)	0	227.04	362.78	517	30	35

   To find the velocity at $t = 21$ s, you are asked to use a quadratic polynomial,$v(t) = at^2 + bt + c$ to approximate the velocity profile. Find the correct set of equations that will find values of a, b and c.

2. Find all values of k for which the system of linear equations

$$\begin{aligned} x + y + kz &= -2 \\ 3x + 4y - z &= -3k \\ kx - y + z &= 2 \end{aligned}$$

   has

   (i) no solution

   (ii) an infinite number of solutions, and state the solutions;

   (iii) a unique solution for each value of k (do not state the solution).

3. Suppose a furniture shop makes three different types of table, type-1, type-2 and type-3.

   - The type-1 takes 4 hours to assemble, 3 hours to paint and 2 hours to check.
   - The type-2 takes 5 hours to assemble, 2 hours to paint and 3 hours to check.
   - The type-3 takes 3 hours to assemble, 3 hours to paint and 2 hours to check.

   The shop is limited to 47 hours for assembley, 33 hours for painting and 27 hours for checking.

   (a) Prepare a table to summarise the above information.

   (b) How many of each type should we produced to use up all the resources.

      i. System of equation in matrix form:

      ii. The LU factorisation:

      iii. Solve using forward and backward substitution:

   (c) What happens if we increase the number of hours for painting up to 38.

      i. Prepare a new table:

      ii. Solve using forward and backward substitution:

   (d) Explain which solution method (LU factorisation, Gaussian elimination) is the best approach for the above problem. Justify your answer:

Question 3[20 marks]

1. Let the set S be a subset of vector space R^3, defined by:

$$S = \{v = \begin{bmatrix} x & y & z \end{bmatrix}^T \in R^3 | x = y\}$$

show that S is a subspace of R^3.

2. Let $v_1 = t + 2$,$v_2 = t^2 + 1$ and $S = \{v_1, v_2\}$. Show that S does not span of P_2 (polynomial of degree 2)

3. Let $X = 3v_1 + 2v_2 - v_3$ with respect to basis vectors

$$E = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 5 \\ 1 & 2 & 4 \end{bmatrix}$$

Find the coordinates of vector X with respect to a new basis defined by:

$$F = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Use the Wronskian to check whether the given functions $f_1 = 4$, $f_2 = 4e^{-x}$ and $f_3 = 2e^{2x}$ are linearly dependent or independent.

5. Let $u = \begin{bmatrix} 1 & 3 & -5 & 2 \end{bmatrix}^T$,$v = \begin{bmatrix} 4 & -2 & 2 & 1 \end{bmatrix}^T \in R^4$

   (a) Compute the following norms of vector v in R^4.

      i. L_1

      ii. L_2

      iii. L_∞

   (b) Compute the 2-norm distance $d(x, y)$ between the vectors u and v

Question 4[20 marks]

1. A factory has a large number of machines that can be in one of two states, operating or broken. The probability that an operating machine breaks down by the end of the day is 0.05 and the probability that a broken machine is repaired by the end of the day is 0.80.

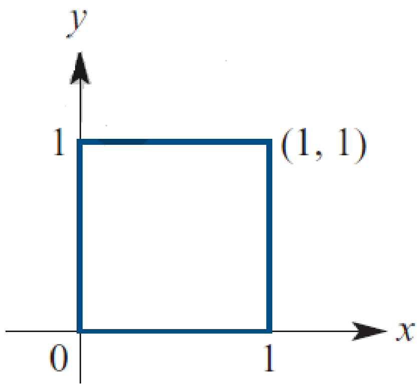
   (a) Find a transition matrix **T** that can be used to represent the above information?

   (b) If a machine is operating at the end of day 1, Find the probability that it is broken by the end of day 6.

2. A country is divided into three demographic regions. It is found that each year 5% of the residents of region 1 move to region 2, and 5% move to region 3. Of the residents of region 2, 15% move to region 1 and 10% move to region 3. And of residents of region 3, 10% move to region 1 and 5% move to region 2. What percentage of the population resides in each of the three regions after a long period of time?

Question 5[20 marks]

1. The square shown is subject to successive transformations.



- The first transformation has matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and
- the second transformation has matrix $\begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}$

Draw a image of the square after these two transformations?

2. Let A be the transformation that rotates the plane by 60 degrees around the origin.

(i) Write down the matrix that represents A.

(ii) Compute the matrix $A^3 = AAA$ and verify that it rotates the plane through 180 degrees.

(iii) By considering the geometry, describe the transformation that would be represented by the matrix A^{60}

——— *End of Examination* ———