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**MC 2020 : Linear Algebra**

**Tutorial-05** **November 2023**

1. (a) Write the equation $x_1^2+4x_1x_2+4x_3^2-10x_1x_3+5x_2^2$ in quadratic form and as $X^TAX$ for some symmetric matrix $A$. Classify it as positive definite, indefinite or negative definite.
   (b) Find a real symmetric matrix $C$ such that $Q = X^TCX$, where $Q = (x_1 - x_2)^2$.
   (c) Find the principal directions of the ellipse $17x^2+12xy+8y^2 = 5$ and use it to express them in the form: $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2} = r$; where $r$ is the right-hand side of the equation.

2. The matrix $B$ is defined by $B = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

   (a) Show that $V_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are eigen vectors of $B$ and find the two corresponding eigen values.
   (b) Given that the third eigen value of $B$ is 4 , find the corresponding eigenvector $V_3$.

3. The matrix $A$ is defined by $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

   (a) Find the eigen values and the corresponding eigen vectors.
   (b) Define the matrices $P$ , $P^{-1}$ and show that $D = P^{-1}AP$ is a diagonal matrix.
   (c) Find trace$(P^{-1}AP)$ .
   (d) Find $A^3$ ( Hint : $A^k = PD^kP^{-1}$).

4. Let $A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{pmatrix}$

   (a) Compute the three upper-left determinants ( principal minors ) of $A$.
   (b) Using eigen values determine whether $A$ is positive definite, positive semidefinite, negative definite, negative semidefinite, or indefinite.

5. (a) For which real values of $\lambda$ do the following vectors form a linearly dependent set in $R^3$?
   $v_1 = (\lambda, -\frac{1}{2}, -\frac{1}{2}), v_2 = (-\frac{1}{2}, \lambda, -\frac{1}{2})$ and $v_3 = (-\frac{1}{2}, -\frac{1}{2}, \lambda)$
   (b) Use the Wronskian to show that the following functions $f_1(x) = e^x$, $f_2(x) = xe^x$, and $f_3(x) = x^2e^x$ are linear independent vectors in $C^\infty(-\infty, \infty)$.

6. (a) Find the eigen values $\lambda_1$ and $\lambda_2$ of the matrix $A = \begin{pmatrix} 3 & 1 \\ 5 & -1 \end{pmatrix}$ also find the eigenvalues of $A^T$ and $A^{-1}$.

   (b) Verify that

      i. $\lambda_1 + \lambda_2 = trace(A)$.

      ii. $\lambda_1 . \lambda_2 = det(A)$.

      iii. Eigenvalues of $A^T$ are also $\lambda_1$ and $\lambda_2$.

      iv. Eigenvalues of $A^{-1}$ are $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$.

   (c) Find the eigenvectors of $A$.

7. (a) Show that $\mathbf{r} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector of $\mathbf{A} = \begin{pmatrix} 7 & 4 & -4 \\ 4 & 1 & 8 \\ -4 & 8 & 1 \end{pmatrix}$ and determine the corresponding eigenvalue.

   (b) State two other eigenvectors of $\mathbf{A}$ which, together with $\mathbf{r}$, give three mutually perpendicular eigenvectors and state the corresponding eigenvalues.

   (c) What is the value of $\det(\mathbf{A})$?

8. The matrix $B$ is defined by $B = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$

   (a) i. Find the eigenvalues of $B$.

      ii. Find an eigenvector corresponding to each eigenvalues.

      iii. Verify that these eigenvectors are orthogonal.

   (b) Given that $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$, $U = \begin{pmatrix} 6 & 5 \\ -3 & k \end{pmatrix}$, find the values of $k$ such that $U^{-1}AU = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$

9. (a) Given matrices $A$ and $P$ below,
$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -4 \\ -3 & -4 & 8 \end{pmatrix}$, $P = \begin{pmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$
Show that $P^T AP$ is a diagonal matrix.

   (b) Compute the eigen values of $A$.

10. Transform the shape with vertices $A = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $D = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ by the matrix $M = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$. Find the determinant of $M$?

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