Linear Algebra

Changing Coordinating system

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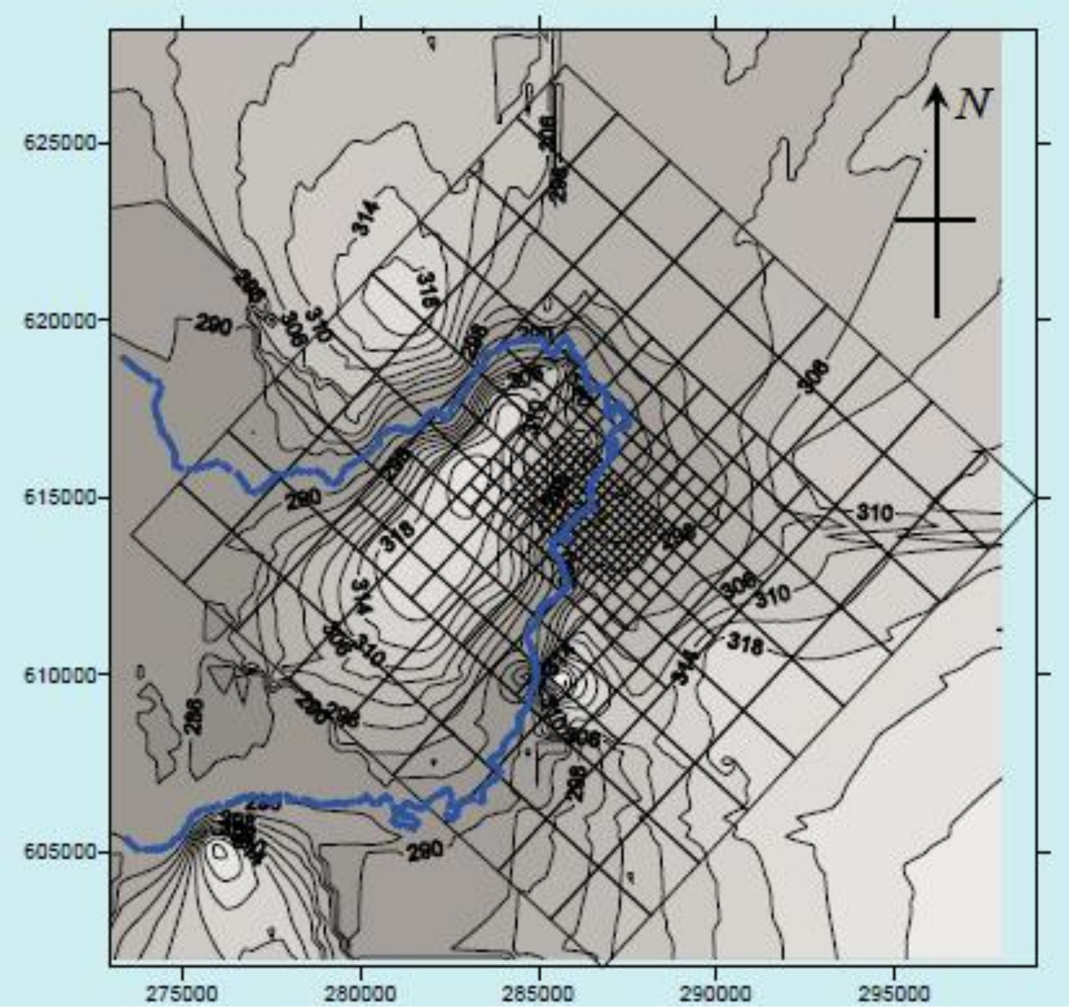
Changing Coordinate Systems

- Sometimes it is useful to use a different coordinate system from the usual (x,y)-plane.
- This can be helpful if we are deforming a material (e.g. sail), or changing an image view (e.g. chalk drawings).
- We often express a deformation as a change of coordinates and a scaling transformation using a matrix (e.g. A).
- But how do we express an object, defined by points (x, y), in the new coordinate system?

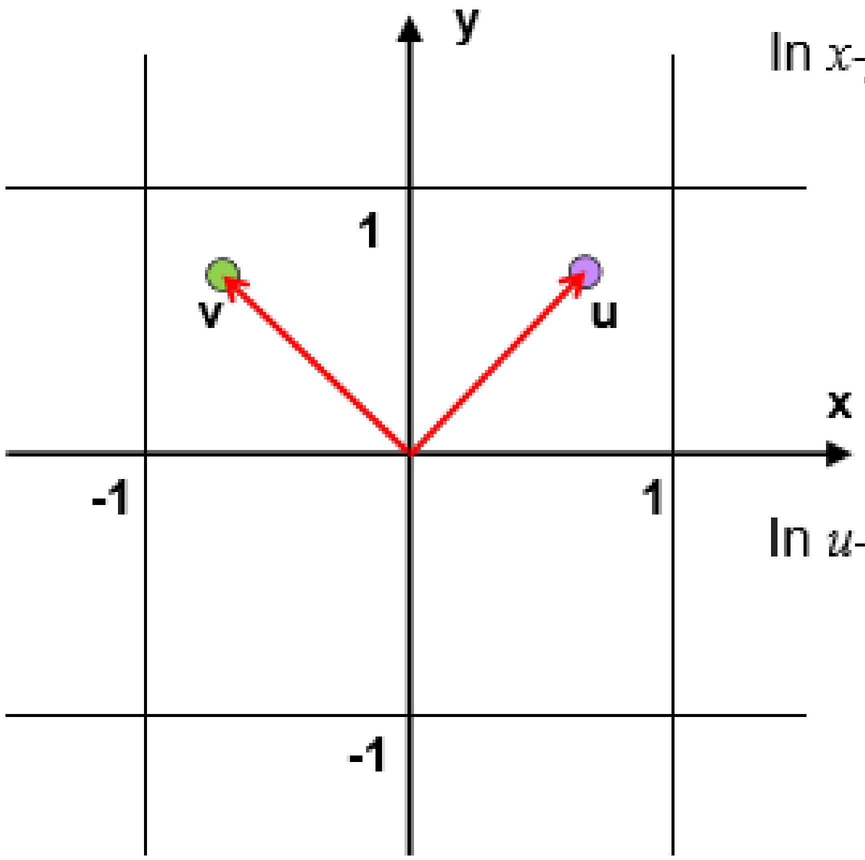
Why would we want to change coordinate systems?

Consider the following figure:



We are interested producing a mathematical model of the Ohaaki geothermal field. The area of geothermal interest is covered by a square grid that is rotated approximately 45° from North. For easier calculations we will use a rotated coordinate system. However, our data (e.g. well locations) will be given to us in north and east coordinates. This means that we will have to transform the data coordinate systems to include them in our model.

Changing Coordinate Systems



In x-y coordinates:

$$\mathbf{u} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

In u-v coordinates:

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We need a transformation matrix to convert from u-v to x-y coordinates.

Changing Coordinate Systems

- To determine the transformation matrix:
  - find <u>unit</u> vectors $\mathbf{u}, \mathbf{v}$ that define the directions of the new axes.
  - construct: $S = [\mathbf{u} \;\; \mathbf{v}] = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$
  - compute the transformation matrix, S^{-1}

Changing Coordinate Systems

- To apply the transformation matrix:
  - to calculate the new coordinates (p_u, p_v) from the old coordinates (p_x, p_y) use:

$$\begin{bmatrix} p_u \\ p_v \end{bmatrix} = S^{-1} \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

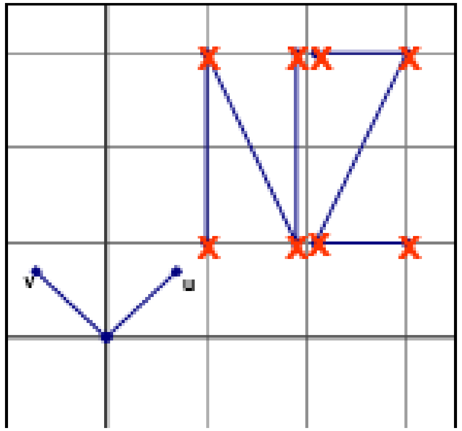
  - For the inverse transformation, you can apply S to the new coordinates (p_u, p_v) to calculate the old coordinates (p_x, p_y):

$$\begin{bmatrix} p_x \\ p_y \end{bmatrix} = S \begin{bmatrix} p_u \\ p_u \end{bmatrix}$$

Example – Changing Coordinate Systems

- Consider the set of points defining the letters "NZ" in the standard *xy*-coordinate system.

$$P = \begin{bmatrix} 1 & 1 & 1.9 & 1.9 & 2.1 & 3 & 2.1 & 3 \\ 1 & 3 & 1 & 3 & 3 & 3 & 1 & 1 \end{bmatrix}$$



Example – Changing Coordinate Systems

- We want to express the points in a new coordinate system with axes defined by:

$$\mathbf{u} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

- These are unit vectors so we can use them to form the S matrix.

Example – Changing Coordinate Systems

- Form the S matrix and its inverse:

$$S = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \Rightarrow S^{-1} = \frac{1}{\frac{1}{2}+\frac{1}{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\Rightarrow S^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

[Aside: since **u** and **v** are perpendicular, unit vectors, then S is an "orthonormal" matrix, and therefore $S^{-1}=S^T$.]

Example – Changing Coordinate Systems

- Then we can find the *uv*-coordinates:

$$P_{uv} = S^{-1} P_{xy}$$

$$\Rightarrow P_{uv} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1.9 & 1.9 & 2.1 & 3 & 2.1 & 3 \\ 1 & 3 & 1 & 3 & 3 & 3 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow P_{uv} = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 4 & 2.9 & 4.9 & 5.1 & 6 & 3.1 & 4 \\ 0 & 2 & -0.9 & 1.1 & 0.9 & 0 & -1.1 & -2 \end{bmatrix}$$

Key points: coordinate systems

- Know how to construct and apply transformation matrix, S
  - columns of S are <u>unit</u> vectors defining the new coordinates
  - determine S^{-1}
  - transform sets of points by pre-multiplying by S^{-1}

$$P_{uv} = S^{-1} P_{xy}$$