Linear Algebra

Linear Independence

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Nature of solutions to $\boldsymbol{Ax}=\boldsymbol{b}$

Revision – in the context of LU factorisation.

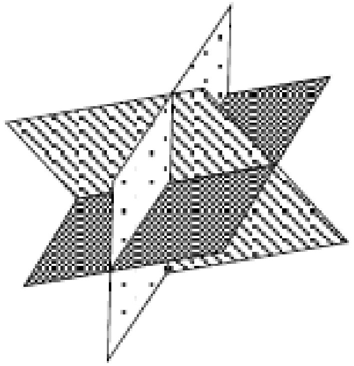
- Systems of linear equations do not always have a single solution.

- For a 3x3 system we can represent the solution to a system with intersecting planes in 3D.

Nature of solutions to *Ax=b*

- <u>Case 1 – Unique solution:</u>

  This is the simple case where the 3 planes intersect at a point – giving a single solution.



- **Example**

$$\begin{bmatrix} 2 & -1 & 3 \\ -8 & 3 & -8 \\ -2 & -2 & 7 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 5 \\ -18 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 5 \\ -18 \\ -1 \end{bmatrix}$$

Nature of solutions to *Ax=b*

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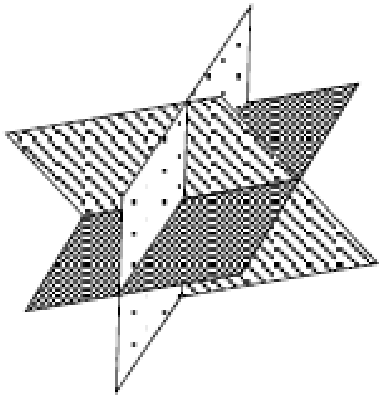


- **Example**

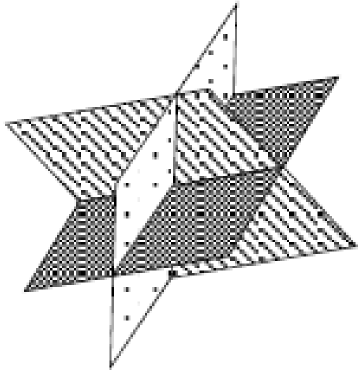
$$L\mathbf{y} = \mathbf{b} \quad \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -18 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix}$$

Nature of solutions to *Ax=b*

- <u>Case 1 – Unique solution:</u>

  This is the simple case where the 3 planes intersect at a point – giving a single solution.



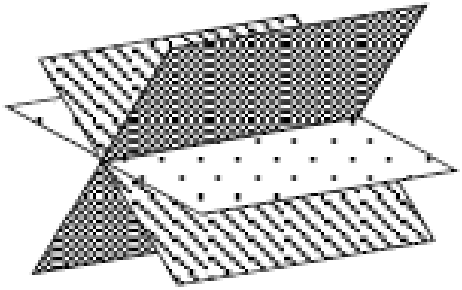
- **Example**

$$U\mathbf{x} = \mathbf{y} \quad \Rightarrow \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

Nature of solutions to $Ax=b$

- <u>Case 2 – Infinite solutions:</u>

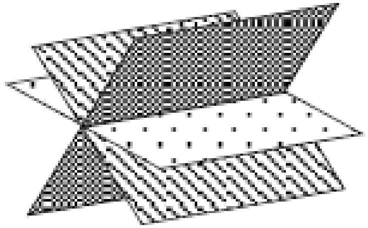


The three planes intersect along a line.

This gives an infinite number of solutions to the system.

Nature of solutions to *Ax=b*

- <u>Case 2 – Infinite solutions:</u>
- **Example**



$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & -2 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

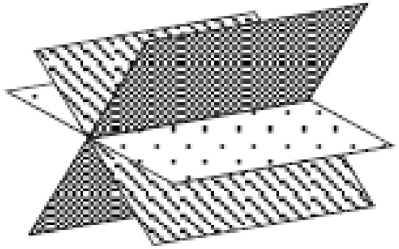
zero row

$$L\mathbf{y} = \mathbf{b} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ \boxed{0} \end{bmatrix}$$

Nature of solutions to *Ax=b*

- <u>Case 2 – Infinite solutions:</u>
- **Example**



$$U\mathbf{x} = \mathbf{y} \qquad \Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

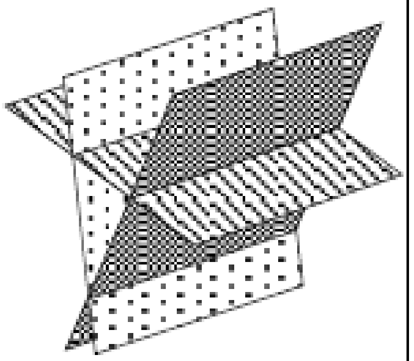
consistent

2 equations (top 2 rows) in 3 unknowns (x_1, x_2, x_3).

This gives an infinite number of solutions to the system.

Nature of solutions to *Ax=b*

- <u>Case 3 – No solution:</u>



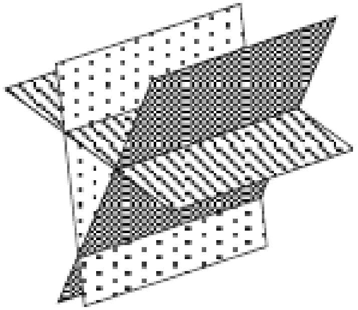
In this case, there is no point at which all 3 planes intersect.

This corresponds to the intersecting lines being parallel.

This system is known as being inconsistent.

Nature of solutions to *Ax=b*

- <u>Case 3 – No solution:</u>
- **Example**



$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & -2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

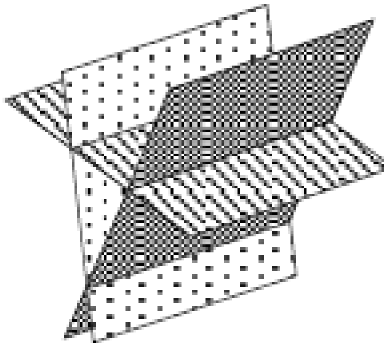
zero row

$$L\mathbf{y} = \mathbf{b} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Nature of solutions to *Ax=b*

- Case 3 – No solution:
- **Example**



$$U\mathbf{x} = \mathbf{y} \quad \Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ \boxed{0 \quad 0 \quad 0} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ \boxed{-1} \end{bmatrix}$$

inconsistent

This system cannot be solved,
i.e. no solutions!

Rank of a matrix

- The *rank* of a matrix is the *number of non-zero rows* in *U*.
- If the rank of a square matrix is the same as the number of rows, we say that the matrix is of *full rank*.
- A full rank matrix has a *unique solution*.

Example – Rank of a matrix

- The matrix in Case 1 was:

$$\begin{array}{ccc} A & = & L \qquad\qquad U \end{array}$$

$$\begin{bmatrix} 2 & -1 & 3 \\ -8 & 3 & -8 \\ -2 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & -2 \end{bmatrix}$$

- *U* has no zero rows, therefore *A* is of full rank, i.e. rank=3.
- Therefore, the system has a unique solution.

Example – Rank of a matrix

- The matrix in Case 2 is:

$$\underset{A}{\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & -2 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix}}$$

- U has one zero row.
- Therefore, the A matrix has rank = 2,

  i.e. A is not of full rank.
- Therefore, system has no unique solution.

Determined and free variables

- For engineering applications, we usually expect the solution of a system to be unique.
- That means that each of the variables has a unique value – they are *determined* variables.
- However, if there is not a unique solution, then some variables will be expressed in terms of another group variables.
- The second group of variables do not have unique values – they are the *free* variables.

Determined and free variables

- Look at the example system from Case 2:

$$U\mathbf{x} = \mathbf{y} \quad \Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Backward substitution:

$$\Rightarrow 3x_2 = 2 + 4x_3 \Rightarrow x_2 = \frac{2}{3} + \frac{4}{3}x_3$$

$$\Rightarrow x_1 - x_2 + 2x_3 = 1 \quad \Rightarrow x_1 - \left(\frac{2}{3} + \frac{4}{3}x_3\right) + 2x_3 = 1 \quad \Rightarrow x_1 = \frac{5}{3} - \frac{2}{3}x_3$$

$$\Rightarrow \mathbf{x} = \begin{bmatrix} \frac{5}{3} - \frac{2}{3}x_3 \\ \frac{2}{3} + \frac{4}{3}x_3 \\ x_3 \end{bmatrix}$$

free variable – infinitely many solutions

Linear combinations and (in)dependence of vectors

Linear combinations of vectors

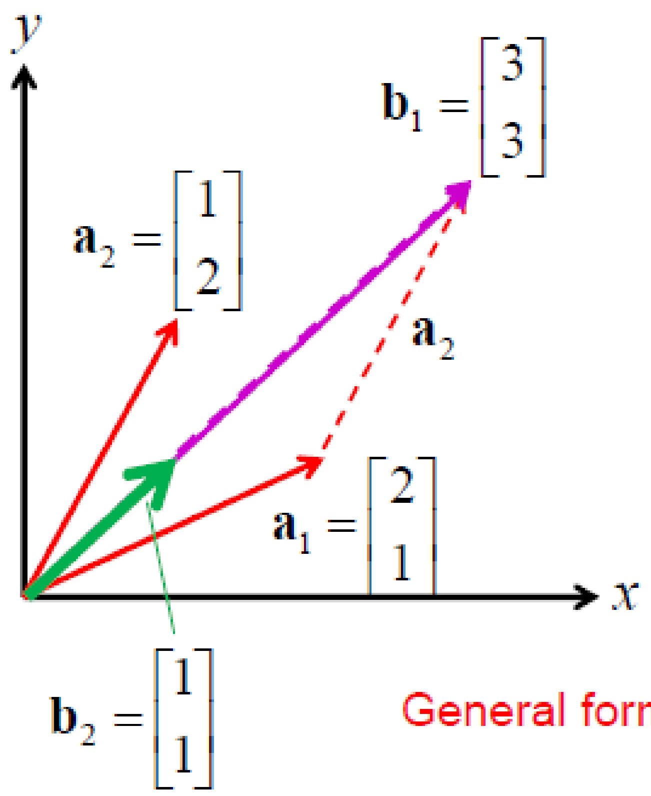
- A *linear combination* of the vectors:

$$\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$$

is a sum of scalar multiples of the vectors:

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

Linear combinations of vectors: graphical examples



Linear combinations:

$$\boxed{\begin{gathered} 1.\mathbf{a}_1 + 1.\mathbf{a}_2 = \mathbf{b}_1 \\ \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \end{gathered}}$$

$$\boxed{\begin{gathered} \frac{1}{3}\mathbf{a}_1 + \frac{1}{3}\mathbf{a}_2 = \mathbf{b}_2 \\ \frac{1}{3}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{gathered}}$$

General form in 2D: $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$

or: $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{b}$ $\quad A\mathbf{x} = \mathbf{b}$

Linear combinations of vectors

- A *linear combination* of the vectors:

$$\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$$

is a sum of scalar multiples of the vectors:

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

- We can write this as a system of equations:

$$A\mathbf{x} = \mathbf{b} \qquad \text{where} \qquad A = [\mathbf{a}_1 \mid \mathbf{a}_2 \mid \ldots \mid \mathbf{a}_n]$$

- So, for a given solution vector **x**:

b is a linear combination of the columns of *A*

Linear dependence of vectors

- We say that the vectors: $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are *linearly dependent* if:

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0}$$

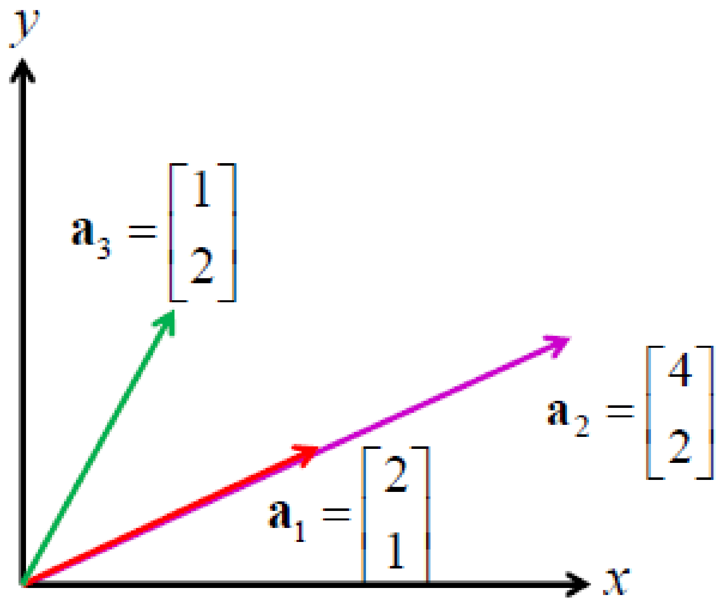
$$\text{for some} \quad x_i \neq 0$$

- This implies that one vector can be expressed in terms of the others, e.g.:

$$\mathbf{a}_1 = -\left(x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + \ldots + x_n\mathbf{a}_n\right)/\, x_1 \quad \text{for } x_1 \neq 0$$

a_1 depends on the other columns

Linear dependence of vectors: graphical examples



Linear dependence:

$\mathbf{a}_2 = 2.\mathbf{a}_1 \qquad \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$2.\mathbf{a}_1 + (-1).\mathbf{a}_2 = \mathbf{0}$

General form in 2D:

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{0}$$

or: $[\mathbf{a}_1 \quad \mathbf{a}_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad A\mathbf{x} = \mathbf{0}$

The only solution to $x_1\mathbf{a}_1 + x_3\mathbf{a}_3 = \mathbf{0}$ is when $x_1 = x_3 = 0$

So we cannot express $\mathbf{a}_3$ in terms of $\mathbf{a}_1$

i.e. $\mathbf{a}_1$ and $\mathbf{a}_3$ are linearly independent.

Linear dependence of vectors

- Test for vector dependence by making the vectors the columns of a matrix:

$$A = [\mathbf{a}_1 \mid \mathbf{a}_2 \mid \ldots \mid \mathbf{a}_n]$$

  and looking for non-zero solutions to:

$$A\mathbf{x} = \mathbf{0}$$

- If there exists a solution where any x_i is not zero, the vectors are *linearly dependent*.
- Conversely, if the only solution is $\mathbf{x} \equiv \mathbf{0}$ then the vectors are *linearly independent*.

Linear dependence of vectors – example 1

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

We can see by inspection that the columns are related, e.g.

$$\mathbf{a}_1 = \mathbf{a}_2 + \mathbf{a}_3$$

i.e. A has linearly dependent columns (and rows)

$$\text{or} \quad (1)\mathbf{a}_1 + (-1)\mathbf{a}_2 + (-1)\mathbf{a}_3 = \mathbf{0}$$

$$\text{i.e.} \quad A\mathbf{x} = \mathbf{0} \qquad \text{with} \quad \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \neq \mathbf{0}$$

Linear dependence of vectors – example 2

- Does the A matrix in Case 2 have linearly independent columns?

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & -2 \end{bmatrix} \Rightarrow \mathbf{U} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Solve: $A\mathbf{x} = \mathbf{0}$

$$L\mathbf{y} = \mathbf{0} \quad \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Linear dependence of vectors – example 2

- Does the *A* matrix in Case 2 have linearly independent columns?

$$U\mathbf{x} = \mathbf{y} \quad \Rightarrow \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \mathbf{x} = \begin{bmatrix} -\frac{2}{3}x_3 \\ \frac{4}{3}x_3 \\ x_3 \end{bmatrix}$$

$$\text{e.g. choose } x_3 = 3 \qquad \Rightarrow \mathbf{x} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}$$

(choose anything except zero)

Linear dependence of vectors – example 2

- Does the A matrix in Case 2 have linearly independent columns? **X no!**

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & -2 \end{bmatrix} \qquad A\mathbf{x} = \mathbf{0} \qquad \Rightarrow \mathbf{x} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}$$

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{0}$$

$$-2\mathbf{a}_1 + 4\mathbf{a}_2 + 3\mathbf{a}_3 = \mathbf{0} \qquad \mathbf{a}_3 = \frac{2}{3}\mathbf{a}_1 - \frac{4}{3}\mathbf{a}_2$$

$$-2\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \frac{4}{3}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

linearly dependent ! **linear combination**

Linear dependence – Properties of square matrices

Columns of A are linearly	independent	dependent
Rows of A are linearly	independent	Dependent
Solutions for $A\mathbf{x} = \mathbf{b}$	One unique	Many solutions or no solution
Rank of A	Full rank	Rank < number of rows
$\det(A)$	$\neq 0$	$= 0$
Does A^{-1} exist?	Yes	No
Is A singular?	No	Yes

Key points: LU factorisation and properties of matrices

- Know how to compute the solution of *A* when rank is less then the number of rows
  - If RHS value is 0 then there are infinitely many solutions; we can give the general form.
  - If RHS value is not zero, then there is no solution

- The vectors $\mathbf{a}_i$ are linearly dependent if:

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0} \quad \text{with some } x_i \neq 0$$

- We can get the components in ***x*** by inspection, or by solving

$$A\mathbf{x} = \mathbf{0}$$

Exercises

1. Determine whether the following sets of vectors are linearly dependent or linearly independent:

   (a) $\begin{bmatrix} 2 & 0 & 5 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 7 & 0 & -2 \end{bmatrix}$

   (b) $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

2. Determine whether the rows of the following matrices are linearly dependent or linearly independent.

(a) $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 1 & 2 \\ -1 & 2 & 1 \\ 2 & -1 & 1 \end{bmatrix}$