Faculty of Engineering
University of Jaffna, Sri Lanka
MC 2020 Linear Algebra - February 2023

Assignment - 04 **Duration: 60 Minutes**

1. (a) Write the equation $x_1^2 + 4x_1x_2 + 4x_3^2 - 10x_1x_3 + 5x_2^2$ in quadratic form and as X^TAX for some symmetric matrix A. Classify it as positive definite, indefinite or negative definite. 10

5 ~~10~~ (b) Find a real symmetric matrix C such that $Q = X^TCX$, where $Q = (x_1 - x_2)^2$.

20 ~~15~~ (c) Find the principal directions of the ellipse $17x^2 + 12xy + 8y^2 = 5$ and use it to express them in the form: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = r$; where r is the right-hand side of the equation.

2. The matrix B is defined by $B = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

5 (a) Show that $V_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are eigen vectors of B and find the two corresponding eigen values.

5 (b) Given that the third eigen value of B is 4 , find the corresponding eigenvector V_3.

3. The matrix A is defined by $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

10 (a) Find the eigen values and the corresponding eigen vectors.

10 (b) Define the matrices P , P^{-1} and show that $D = P^{-1}AP$ is a diagonal matrix.

5 (c) Find trace$(P^{-1}AP)$.

10 (d) Find A^3 (Hint : $A^k = PD^kP^{-1}$).

4. Let $A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{pmatrix}$

10 (a) Compute the three upper-left determinants (principal minors) of A.

10 (b) Using eigen values determine whether A is positive definite, positive semidefinite, negative definite, negative semidefinite, or indefinite.

***** **END** *****

Assignment - 04

1.

(a) $x_1^2 + 4x_1x_2 + 4x_3^2 - 10x_1x_3 + 5x_2^2$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -5 \\ 2 & 5 & 0 \\ -5 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = X^T A X$$

$$A = \begin{bmatrix} 1 & 2 & -5 \\ 2 & 5 & 0 \\ -5 & 0 & 4 \end{bmatrix}$$

$M_1 = |1| = 1 > 0$

$M_2 = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5-4 = 1 > 0$

$M_3 = \begin{vmatrix} 1 & 2 & -5 \\ 2 & 5 & 0 \\ -5 & 0 & 4 \end{vmatrix} = 1(20-0) - 2(8-0) - 5(0+25)$

$= -121 < 0.$ 07

A is a indefinite.

b) $Q = (x_1 - x_2)^2$

$= x_1^2 - 2x_1x_2 + x_2^2$ 05

$$C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$C - \lambda I = \begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix}$$

Charaacteristic equation,

$\det (C - \lambda I) = 0$

$(1-\lambda)(1-\lambda) - 1 = 0.$

$(1-\lambda+1)(1-\lambda-1) = 0$

$+\lambda(\lambda-2)=0$

$\Rightarrow \lambda_1 = 0 \;,\quad \lambda_2 = 2$

All eigen values are not negative and one value is zero, so it is a semi positive definite matrix.

c) $17x^2 + 12xy + 8y^2 = 5$

$$Q = \begin{bmatrix} 17 & 6 \\ 6 & 8 \end{bmatrix} \;, \quad K = 5$$

$$Q - \lambda I = \begin{bmatrix} 17-\lambda & 6 \\ 6 & 8-\lambda \end{bmatrix}$$

$$\det(Q-\lambda I) = (17-\lambda)(8-\lambda) - 36 = 0$$

$$136 - 17\lambda - 8\lambda + \lambda^2 - 36 = 0$$

$$\lambda^2 - 25\lambda + 100 = 0$$

$$\lambda = 5, 20$$

So, eigenvectors are,

For $\lambda_1 = 5$; $\underline{s}_1$

$$As_1 = \lambda s_1$$

$$\begin{bmatrix} 17 & 6 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} s_a \\ s_b \end{bmatrix} = 5 \begin{bmatrix} s_a \\ s_b \end{bmatrix}$$

$$17s_a + 6s_b = 5s_a$$

$$6s_a + 8s_b = 5s_b$$

$$s_b = -2s_a .$$

$\underline{\underline{m}}$

Set $S_a = 1$, $\Rightarrow S_b = -2$

$$\underline{S}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \hat{\underline{S}}_1 = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

For $\lambda_2 = 20$, $\underline{S_2}$

$$A S_2 = \lambda S_2$$

$$\begin{bmatrix} 17 & 6 \\ 6 & 8 \end{bmatrix}\begin{bmatrix} S_a \\ S_b \end{bmatrix} = 20\begin{bmatrix} S_a \\ S_b \end{bmatrix}$$

$$17 S_a + 6 S_b = 20 S_a$$
$$6 S_a + 8 S_b = 20 S_b.$$

$$\Rightarrow S_a = 2 S_b.$$

Set $S_b = 1$; $S_a = 2$

$$\underline{S_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \hat{\underline{S}}_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$I = \pm\sqrt{\frac{k}{\lambda}}$$

$$I_1 = \pm\sqrt{\frac{5}{5}} = \pm\sqrt{1} = \pm 1$$

$$I_2 = \pm\sqrt{\frac{5}{20}} = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$$

$$a^2 \longrightarrow u^2.$$
$$a = \pm 1$$

$$b^2 \longrightarrow v^2$$
$$b = \pm\frac{1}{2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = r.$$

$$\frac{x^2}{1} + \frac{y^2}{(\frac{1}{2})^2} = 5.$$

$$x^2 + 4y^2 = 5$$

-20-

2) $B = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix}$

a) $v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$$B\underline{s} = \lambda \underline{s}$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$1 - 3 = \lambda$$
$$\lambda = -2$$

$v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ -0.5

$$B\underline{s} = \lambda \underline{s}$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$1 = \lambda$$
$$\Rightarrow \lambda = 1$$

b) $\lambda_3 = 4$

$$A\underline{s} = \lambda \underline{s}$$

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} s_a \\ s_b \\ s_c \end{bmatrix} = 4 \begin{bmatrix} s_a \\ s_b \\ s_c \end{bmatrix}$$

$$s_a + 3s_b = 4s_a$$
$$2s_a + 2s_c = 4s_b$$
$$s_a + s_b + 2s_c = 4s_c$$

$$3S_a = 3S_b$$
$$\Rightarrow S_a = S_b.$$

$$2S_a + 2S_c = 4S_c$$
$$\Rightarrow 2S_a = 2S_c$$
$$\Rightarrow S_a = S_c$$

Set $S_a = 1$ 0.5
$$\Rightarrow S_b = 1, \quad S_c = 1$$

$$\underline{v_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3) $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

a) $\det(A - \lambda I) = 0.$

$$\begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(3-\lambda) - 2 = 0$$
$$12 - 4\lambda - 3\lambda + \lambda^2 - 2 = 0$$
$$\lambda^2 - 7\lambda + 10 = 0$$
$$(\lambda-5)(\lambda-2) = 0 \quad 10$$
$$\lambda = 2, 5$$

Eigen vectors are,

For $\lambda = 2$

$$A\underline{s} = \lambda \underline{s}$$

$$\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} s_a \\ s_b \end{bmatrix} = 2\begin{bmatrix} s_a \\ s_b \end{bmatrix}$$

$$\equiv$$

$$4s_a + s_b = 2s_a$$
$$2s_a + 3s_b = 2s_b$$

$$\Rightarrow s_b = -2s_a$$
$$s_a = -\tfrac{1}{2}s_b.$$

Set $s_a = 1$, $s_b = -2$

$$\underline{s}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \hat{\underline{s}}_1 = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

$$\lambda = 5$$

$$A\underline{s} = \lambda \underline{s}$$

$$\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} s_a \\ s_b \end{bmatrix} = 5\begin{bmatrix} s_a \\ s_b \end{bmatrix}$$

$$4s_a + s_b = 5s_a$$
$$2s_a + 3s_b = 5s_b$$

$$s_a = s_b$$

Set $s_a = 1$, $s_b = 1$

$$\underline{s}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} , \quad \hat{\underline{s}}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

b) $P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$

$$\det P = 1 - (-2) = 3$$

$$P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= D.$$

-10

c) $\text{trace}(P^{-1}AP) = \text{trace}(D)$

$$= 2+5$$

$$= 7$$

-05

d) $A^k = PD^kP^{-1}$

$k=3$

$A^3 = PD^3P^{-1}$

$$D^3 = D \cdot D \cdot D = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 25 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 \\ 0 & 125 \end{bmatrix}$$

$$PD^3P^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 125 \end{bmatrix} \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 125 \\ -16 & 125 \end{bmatrix} \begin{bmatrix} 1/3 & -1/3 \\ 2/3 & 1/3 \end{bmatrix}$$

-10

$$= \begin{bmatrix} 86 & 39 \\ 78 & 47 \end{bmatrix} = A^3 \quad //$$

4) $A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{pmatrix}$

a) $PMD_1 = |2| = 2$

$PMD_2 = \begin{vmatrix} 2 & 2 \\ 2 & 5 \end{vmatrix} = 10 - 4 = 6$

$PMD_3 = \begin{vmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{vmatrix}$

-10

$= 30$

b) $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 3 \\ 0 & 3 & 8-\lambda \end{vmatrix} = 0$$

$-\lambda^3 + 15\lambda^2 - 53\lambda + 30 = 0$

$\lambda_1 = 10 \quad = 10$

$\lambda_2 = \frac{-\sqrt{13}+5}{2} = 0.6972$

$\lambda_3 = \frac{\sqrt{13}+5}{2} = 4.3028$

All the eigen values are positive. So positive definite.