MC 2020 : LINEAR ALGEBRA

1. (a) Let A be the transformation that rotates the plane by 60 degrees around the origin. (Hint: The general form of a rotation by k about the origin is $\begin{pmatrix} \cos(k) & -\sin(k) \\ \sin(k) & \cos(k) \end{pmatrix}$.

    i. Write down the matrix that represents A.

    ii. Compute the matrix $A^3 = AAA$ and verify that it rotates the plane through 180 degrees.

    iii. By considering the geometry, describe the transformation that would be represented by the matrix A^{60}.

2. (a) A railway has 600 wagons carrying goods from point A to point B. At the end of each week it finds that 30% of wagons that started the week at A are at B and 20% of wagons that have started at B are now at A.

    i. How many wagons are at A and B at the end of two weeks if 300 wagons started at A and 300 started at B?

    ii. If there are 400 at A and 200 wagons at B at the end of a week, how many wagons were there at and at B at the start of the week?.

    iii. How many wagons would need to be at A and at B at the start of the week if there were to be the same numbers at the end of the week?

3. (a) For the following case prove or disprove whether S is a subspace of V.

V is an inner product space and $S = [u \in V \text{and} < u, u_0 >= 0]$, where $u_0 \in V$.

b)Let u and v be non-zero vectors in V. Prove or disprove the following claim.

(u+v) and (u-v) are linearly dependent if u and v are linearly dependent.

c) Show that vector $V = [1\ 8\ 11]^T$ is a linear combination of $V_1 = [1\ 2\ 1]^T$ and $V_2 = [-1\ 1\ 4]^T$.

d) Use the Wronskian to show that $f_1 = x$ and $f_2 = \sin x$ are linearly independent.

e) Consider $f(t) = 3t - 5$ and $g(t) = t^2$ in the polynomial space $\mathbf{P}(t)$ with inner product

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt.$$

Find

i. $\|f\|$

ii. $\|g\|$

4. (a) Let $A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{pmatrix}$

i. Compute the three upper-left determinants (principal minors) of A.

ii. Compute the eigenvalues of A.

iii. Determine whether A is positive definite, positive semidefinite,negative definite, negative semidefinite, or indefinite.

(b) Use the quadratic form and eigen value/eigen vector analysis to plot the ellipse: $5x^2 - 4xy + 2y^2 = 1$

5. (a) i. Find all of the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ -8 & 4 & -6 \\ 8 & 1 & 9 \end{pmatrix}$

ii. Find the eigenvalues of the matrix inv(A) and compare with those of A.

iii. Find the eigenvectors of inv(A) and compare with those of A.

(b) Consider a matrix P defining the coordinates of the letters NZ expressed in the standard xy coordinates: $P = \begin{pmatrix} 1 & 1 & 1.9 & 1.9 & 2.1 & 3 & 2.1 & 3 \\ 1 & 3 & 1 & 3 & 3 & 3 & 1 & 1 \end{pmatrix}$. The plot of these points is shown below. Calculate the coordinates of the same points in the uv coordinate system shown (Hint:Find (a) unit vectors u, v, (b) S=[u v], (c) S^{-1}, (d) $S^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$.

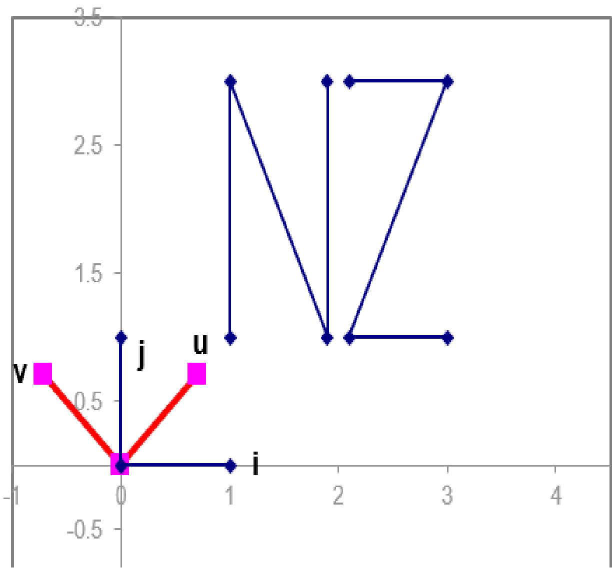

Figure 1: