Linear Algebra

Linear system of equations

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Thought for the day
Do What is Right
Not What is Easy.

Learning objective

After this lecture you should be able to:

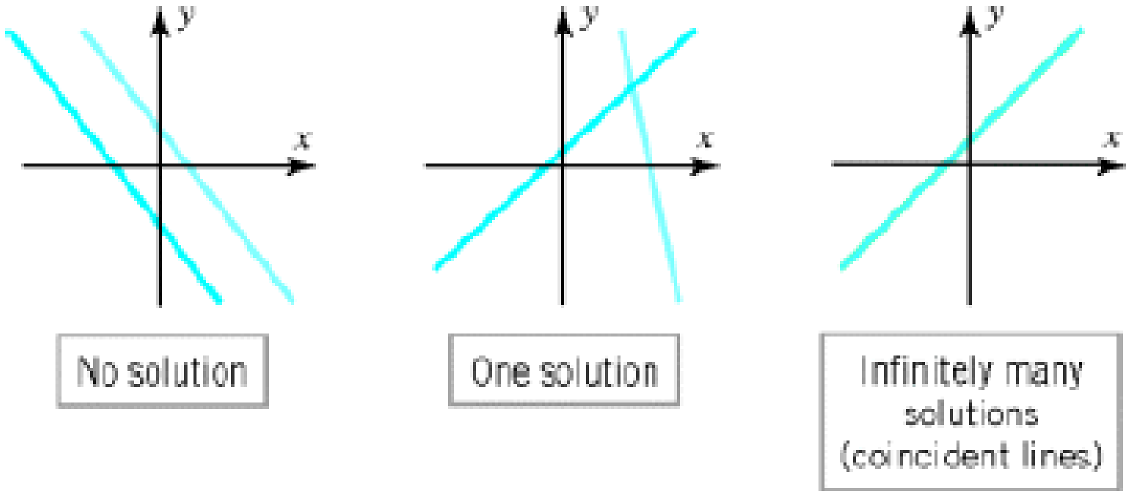
- solve systems of linear equations using matrix algebra and the inverse
- solve systems of linear equations by using Gaussian elimination (i.e. using elementary row operations),
- when relevant, interpret the solution geometrically

Linear systems with two unknowns

Linear systems in two unknowns arise in connection with intersections of lines. For example, consider the linear system

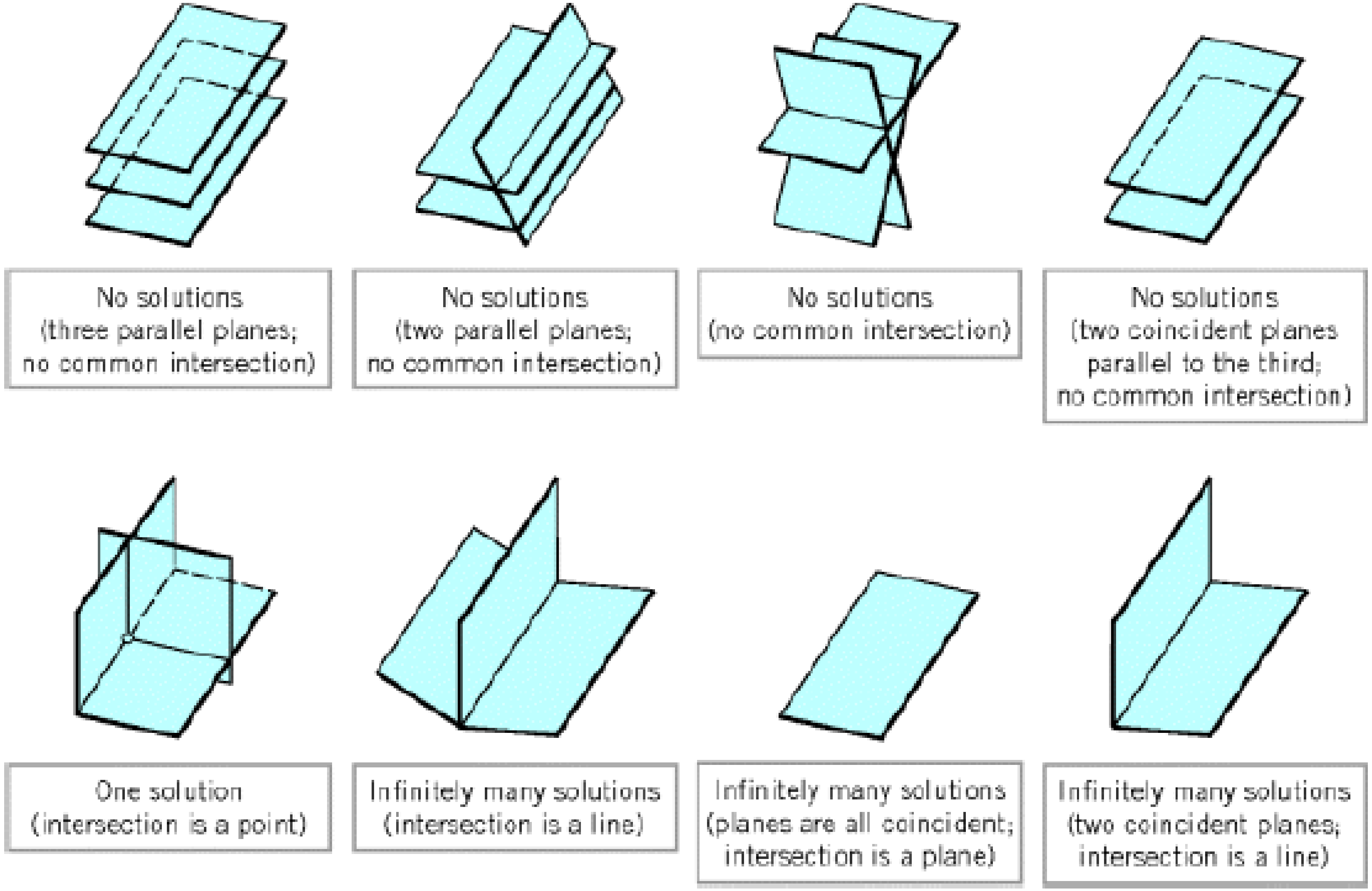
$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

there are three possibilities



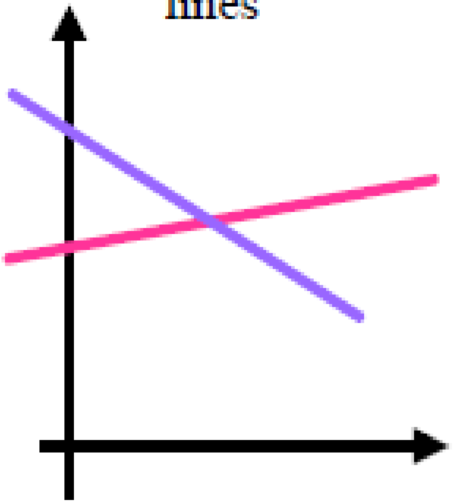
Linear systems with two unknowns

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

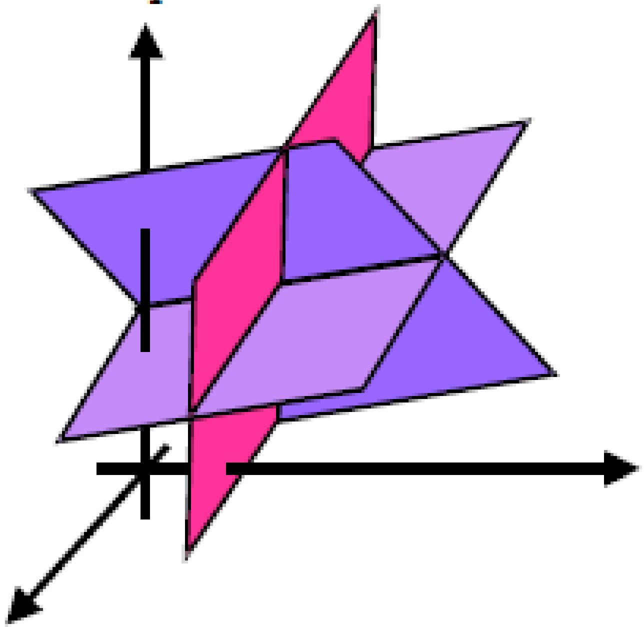


Visualising n linear equations in n variables

- 2 variables
- Solution is at the intersection of 2 lines



- 3 variables
- Solution is at the intersection of 3 planes



Solving 2 linear equations

The 2×2 system of linear equations

$$
\begin{aligned}
2x + y &= 5 \\
x + y &= 3
\end{aligned}
$$

can be solved by eliminating variables. Subtract (1/2) times the first equation from the second.

$$
\begin{aligned}
2x + y &= 5 \\
+ 0.5y &= 0.5
\end{aligned}
$$

Now solve the second for y to get $y = 1$. Back substitute the value of y in the first equation to get x.

$$
\begin{aligned}
2x &= 5 - 1 \\
x &= 2
\end{aligned}
$$

Example

$$\begin{aligned} 5x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 - 2x_2 + 7x_3 &= 2 \\ 4x_1 + 3x_2 - 6x_3 &= 1 \end{aligned}$$

is a system of three equations in three unknowns.

How to Solve Systems of Equations

The only way to solve a general system of linear equations is by eliminating unknowns from equations to simplify things.

There are no universally applicable formulae that solve all equations.

The method of solution usually adopted is called *Gaussian elimination*, named after the German mathematician Carl Friedrich Gauss.

The first step in the solution process is to abandon the equations for a while and replace them by *matrices*.

Solving 3 linear equations

$$2x + y + z = 5$$
$$x + y + 2z = 3$$
$$2x - y + 2z = 3$$

Subtract (1/2) times the first equation from the second, subtract (1) times the first equation from the third.

$$2x + y + z = 5$$
$$0.5y + 1.5z = 0.5$$
$$-2y + z = -2$$

Subtract (-4) times the second equation from the third

$$2x + y + z = 5$$
$$0.5y + 1.5z = 0.5$$
$$7z = 0$$

Now solve the last for z to get $z = 0$. Back substitute the value of z in the second equation to get $y = 1$, and then in the first to get $x = 2$.

System of m linear equations with n unknowns

A **linear system of m equations in n unknowns** $x_1, \cdots, x_n$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots&\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

The system is called *linear* because each variable x_j appears in the first power only, just as in the equation of a straight line. $a_{11}, \cdots, a_{mn}$ are given numbers, called the **coefficients** of the system. $b_1, \cdots, b_m$ on the right are also given numbers. If all the b_j are zero, then (1) is called a **homogeneous system**. If at least one b_j is not zero, then (1) is called a **nonhomogeneous system**.

Matrix Form of the Linear System

$$\mathbf{Ax} = \mathbf{b}$$

where the **coefficient matrix** $\mathbf{A} = [a_{jk}]$ is the $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

are column vectors. We assume that the coefficients a_{jk} are not all zero, so that $\mathbf{A}$ is not a zero matrix. Note that $\mathbf{x}$ has n components, whereas $\mathbf{b}$ has m components.

Matrix method for solving n x n linear equations

- As with the 2×2 example we can solve a general system of n linear equations by writing it in matrix form, $A\mathbf{x} = \mathbf{b}$, and pre-multiplying the matrix equation by the inverse of the matrix of coefficients, i.e. A^{-1}.

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$
$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}, \text{ but } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$
$$\mathbf{I}\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$
$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Note that if A is a singular matrix (with determinant zero) then the inverse of A doesn't exist and we can't use this approach.

Matrix method for solving n x n linear equations

Solving $Ax = b$ by finding $x = A^{-1}b$ is conceptually a very nice approach

HOWEVER

this method is almost never used by hand because it is time consuming to find the inverse and it can be prone to errors (recall how much effort was required just to find the determinant of a 3x3 matrix!)

We need a better way of solving systems of linear equations, which brings us to Gaussian Elimination.

Augmented matrix

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

is called the **augmented matrix** of the system (1). The dashed vertical line could be omitted, as we shall do later. It is merely a reminder that the last column of $\tilde{\mathbf{A}}$ did not come from matrix $\mathbf{A}$ but came from vector $\mathbf{b}$. Thus, we *augmented* the matrix $\mathbf{A}$.

Stream lining the solving process-3 unknowns

- Consider a general system of 3 × 3 linear equations in x_1, x_2, x_3

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
a_{31}x_1 + a_{23}x_2 + a_{33}x_3 &= b_3
\end{aligned}
$$

- To make book-keeping simpler, we represent the system by an **augmented matrix**:

$$
\left[\begin{array}{ccc|c}
a_{11} & a_{12} & a_{13} & b_1 \\
a_{21} & a_{22} & a_{23} & b_2 \\
a_{31} & a_{32} & a_{33} & b_3
\end{array}\right]
$$

This table is called an Augmented matrix

Definition

Let A be an $m \times n$ matrix. An *elementary row operation* on A is one of the following three things:

- Interchange of two rows of A.
- Multiplication of all entries in a row of A by a *non-zero* number.
- Addition of a multiple of one row of A to a *different row* of A.

Much of matrix theory involves use of the row operations, and when we perform such operations, we should try to keep a symbolic record of what we have done, as this helps to explain our working method.

Elementary row operations

- The following operations can be applied to our augmented matrix without changing the solution
  - Multiplying a row through by a non-zero constant

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 2a_{21} & 2a_{22} & 2a_{23} & 2b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] r_2 \to 2r_2$$

  - Swapping two rows

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{31} & a_{32} & a_{33} & b_3 \\ 2a_{21} & 2a_{22} & 2a_{23} & 2b_2 \end{array}\right] r_2 \leftrightarrow r_3$$

  - Adding/subtracting a multiple of one row to another

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{31} & a_{32} & a_{33} & b_3 \\ 2a_{21}+3a_{11} & 2a_{22}+3a_{12} & 2a_{23}+3a_{13} & 2b_2+3b_1 \end{array}\right] r_3 \to r_3 + 3r_1$$

Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 6 \\ 3 & 1 & 1 \end{pmatrix} \xrightarrow[r_2 \leftrightarrow r_3]{} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \\ 2 & -1 & 6 \end{pmatrix}$$

$$\xrightarrow[r_2 \times 2]{} \begin{pmatrix} 1 & 2 & 3 \\ 6 & 2 & 2 \\ 2 & -1 & 6 \end{pmatrix} \xrightarrow[r_2 - 3r_3]{} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & -16 \\ 2 & -1 & 6 \end{pmatrix}$$

Echelon Matrices

Definition

We say that an $m \times n$ matrix is a *row echelon matrix* if it has the following three properties.

- All zero rows, if there are any, are at the bottom of the matrix.
- The leading entry of each non-zero row equals 1.
- If rows numbered i and $i+1$ are two successive non-zero rows, the leading entry of row $i+1$ is in a column *strictly to the right* of the column containing the leading entry of row i.

Echelon Matrices

Definition Continued

We sometimes add the following fourth condition to the list:

- If a column of the matrix contains the leading entry of some non-zero row, then *all other entries* in that column are 0.

Definition

We say that a matrix having all four properties listed in the definitions above is a *reduced row echelon matrix*.
We frequently omit the word *row* when describing these matrices and then speak more simply of an *echelon matrix* and of a *reduced echelon matrix*.

Example

$$A = \begin{pmatrix} 1 & -1 & 2 & 0 & 3 \\ 0 & 1 & -3 & 1/2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is a 4×5 echelon matrix.

It has leading entries in columns 1, 2 and 4.

A is not a reduced echelon matrix, as columns 2 and 4 do not have the correct form for this property.

Example

$$B = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

is a 3×4 reduced echelon matrix.

Example

$$C = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 1 & 3 \end{pmatrix}$$

is a not an echelon matrix, as the leading entries of rows 2 and 3 lie in the same column.

Reduction to Echelon Form

Much of the theory of linear equations rests on the theorem below, which we will not formally prove in these lectures.
It is not difficult to devise a proof. Indeed, a proof is really just a description of how to execute a practical procedure of row operations, so that we say the proof is algorithmic.

Theorem

Let A be any $m \times n$ matrix. Then we can transform A to a row echelon matrix by a sequence of elementary row operations.
After this, if required, we can transform this echelon matrix to a reduced echelon matrix by a further sequence of row operations.

Example

Transform the matrix

$$A = \begin{pmatrix} 2 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & -1 & 2 & 3 \end{pmatrix}$$

to a reduced echelon matrix.

We begin by first transforming to an echelon matrix. Look in column 1 for a non-zero entry.

If we can find such an entry equal to 1 or -1, this is a good choice of initial so-called *pivoting position*. We then move the pivoting position to the top row by a row interchange, as indicated below.

Example Continued

If we can find no pivoting position with entry 1 or -1, we may be forced to multiply all entries in a row by some number to produce a leading entry equal to 1.
This can be annoying, as it usually introduces fractions, which complicate the arithmetic.

$$A \xrightarrow[r_1 \leftrightarrow r_2]{} \begin{pmatrix} 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 2 \\ 1 & -1 & 2 & 3 \end{pmatrix}$$

This operation produces an entry equal to 1 in the top left hand corner.

Example Continued

This is our new choice of pivoting position, and it is usually the one where we prefer to start.

We now use the pivoting position in the top left hand corner to produce two zeros below this position.

$$A \xrightarrow[r_3 - r_1]{r_2 - 2r_1} \begin{pmatrix} 1 & 2 & 4 & 3 \\ 0 & -3 & -5 & -4 \\ 0 & -3 & -2 & 0 \end{pmatrix}$$

Column 1 now has the correct form.

Next, we look at the two leading entries located in column 2. We want to use one of them as a pivot.

Example Continued

Neither of these two leading entries is 1, but as the entries are equal, we can perform the operation below without introducing fractions.

$$\xrightarrow[r_3 - r_2]{} \begin{pmatrix} 1 & 2 & 4 & 3 \\ 0 & -3 & -5 & -4 \\ 0 & 0 & 3 & 4 \end{pmatrix}$$

The matrix we have produced is almost an echelon matrix.

We just have to make the leading entries in rows 2 and 3 equal to 1, which we can do as follows.

Example Continued

$$\xrightarrow[r_3 \times \frac{1}{3}]{r_2 \times -\frac{1}{3}} \begin{pmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & \frac{5}{3} & \frac{4}{3} \\ 0 & 0 & 1 & \frac{4}{3} \end{pmatrix}$$

To obtain a reduced echelon matrix, we work with the lowest leading entry (located in column 3) to remove the non-zero entries above it:

Example Concluded

$$\xrightarrow[r_1 - 2r_2]{} \begin{pmatrix} 1 & 0 & 0 & -\frac{5}{9} \\ 0 & 1 & 0 & -\frac{8}{9} \\ 0 & 0 & 1 & \frac{4}{3} \end{pmatrix}$$

This is the reduced echelon matrix we wanted to construct.

There are many different ways to perform the row operations, but it is a non-obvious fact that, however we proceed, subject to making no mistakes, we should achieve the same reduced echelon matrix.

It is possible to obtain many different, but nonetheless correct, *echelon* matrices starting from a given matrix.

Example

Transform the matrices below into an echelon matrix.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 2 & -2 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 & 4 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 6 \end{pmatrix}$$

Example

Transform the matrix below into a reduced echelon matrix.

$$A = \begin{pmatrix} 1 & 1 & 1 & 3 & -2 \\ 2 & 1 & -1 & 1 & -2 \\ 3 & 2 & -2 & -2 & 2 \\ -2 & -1 & 3 & 2 & 2 \end{pmatrix}$$

We choose the entry 1 in the top left corner as a pivot and use it to replace the leading entries 2, 3 and -2 in column 1 by zeros. To save time and space, we combine three row operations into a single frame.

Example Continued

$$\xrightarrow[\substack{r_3-3r_1\\ r_4+2r_1}]{r_2-2r_1} \begin{pmatrix} 1 & 1 & 1 & 3 & -2 \\ 0 & -1 & -3 & -5 & 2 \\ 0 & -1 & -5 & -11 & 8 \\ 0 & 1 & 5 & 8 & -2 \end{pmatrix}$$

Column 1 now has the correct form.

We proceed to use the leading entry of the new row 2, which is in column 2, to remove the two entries below it. This is expressed in the following diagram.

Example Continued

$$\xrightarrow[r_4+r_2]{r_3-r_2} \begin{pmatrix} 1 & 1 & 1 & 3 & -2 \\ 0 & -1 & -3 & -5 & 2 \\ 0 & 0 & -2 & -6 & 6 \\ 0 & 0 & 2 & 3 & 0 \end{pmatrix}$$

We leave column 2 for the time being and use the leading entry of the new row 3 (which is -2) as pivot.

We perform the next row operation to remove the 2 below the pivot.

Example Continued

$$\xrightarrow[r_4+r_3]{} \begin{pmatrix} 1 & 1 & 1 & 3 & -2 \\ 0 & -1 & -3 & -5 & 2 \\ 0 & 0 & -2 & -6 & 6 \\ 0 & 0 & 0 & -3 & 6 \end{pmatrix}$$

Next, we make all leading entries equal to 1.

Example Continued

$$\xrightarrow[\substack{r_3 \times -\frac{1}{2} \\ r_4 \times -\frac{1}{3}}]{r_2 \times -1} \begin{pmatrix} 1 & 1 & 1 & 3 & -2 \\ 0 & 1 & 3 & 5 & -2 \\ 0 & 0 & 1 & 3 & -3 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

We have reached an echelon matrix.

To achieve a reduced echelon matrix, we use the bottom leading entry, in row 4, as pivot, and make the three entries above it 0, as follows:

Example Continued

$$\xrightarrow[\substack{r_2-5r_4 \\ r_1-3r_4}]{r_3-3r_4} \begin{pmatrix} 1 & 1 & 1 & 0 & 4 \\ 0 & 1 & 3 & 0 & 8 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

Next, we use the leading entry in row 3 to remove the two entries above it:

Example Continued

$$\xrightarrow[r_1 - r_3]{r_2 - 3r_3} \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

Finally, we use the leading entry of row 2 to remove the entry above it.

Example Concluded

$$\xrightarrow[r_1 - r_2]{} \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

This is a reduced echelon matrix, and is the only such matrix we could obtain, however we may have performed the row operations.

Perform row operations to produce an upper triangular matrix

- We eliminate x_1 from the second and third row by using elementary row operations

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{array}\right]$$

- Similarly, we eliminate x_2 from row three by using elementary row operations, forming an upper triangular matrix:

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a''_{33} & b''_3 \end{array}\right]$$

Back substitution

- The last row represents an equation in a single variable

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & | & b_1 \\ 0 & a'_{22} & a'_{23} & | & b'_2 \\ 0 & 0 & a''_{33} & | & b''_3 \end{bmatrix}$$

$a''_{33}\, x_3 = b''_3$

which can be solved: $x_3 = b''_3 \,/\, a''_{33}$

- The second row represents an equation in two variables

$a'_{22}\, x_2 + a'_{23}\, x_3 = b'_2$

- Since the variable x_3 has already been found in the previous step, x_2 can be computed:

$x_2 = (b'_2 - a'_{23}\, x_3) \,/\, a'_{22}$

Back substitution

- The first row represents an equation in three variables
  $a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$
- Since the variables x_2 and x_3 have already been found in the previous steps, x_1 can now be computed:
  $x_1 = (b_1 - a_{12} x_2 - a_{13} x_3) / a_{11}$
- This process of solving an upper triangular matrix equation is called back substitution.

- You don't need to memorise the general formula above but you DO need to understand the process

Example 1

- Solve the system of equations

$$\begin{aligned} x+y+z &= 4 \\ x-y-z &= -2 \\ 2x+8y+z &= 19 \end{aligned}$$

- Represent the system as an augmented matrix, and do all calculations on elements of this matrix.

- $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & -1 & -2 \\ 2 & 8 & 1 & 19 \end{array}\right]$$

- IMPORTANT: We record our row operations to the right of the matrix, to make it clear what we are doing.

Example 1

- Eliminate *x* by subtracting the first row from the second row, and subtracting *twice* the first row from the third row

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & -1 & -2 \\ 2 & 8 & 1 & 19 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & -2 & -6 \\ 0 & 6 & -1 & 11 \end{array}\right] \begin{array}{l} \\ r_2 \to r_2 - r_1 \\ r_3 \to r_3 - 2r_1 \end{array}$$

Example 1

- Eliminate y by adding 3 times the second row to the third row, forming an upper triangular matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & -2 & -6 \\ 0 & 6 & -1 & 11 \end{array}\right] \begin{array}{l} \\ r_2 \to r_2 - r_1 \\ r_3 \to r_3 - 2r_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & -2 & -6 \\ 0 & 0 & -7 & -7 \end{array}\right] \begin{array}{l} \\ \\ r_3 \to r_3 + 3r_r \end{array}$$

Example 1

Back substitute $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & -2 & -6 \\ 0 & 0 & -7 & -7 \end{array}\right]$

$z = -7 / -7 = 1$

$y = (-6 + 2) / (-2) = 2$

$x = (4 - 2 - 1) / 1 = 1$

The solution is thus $x = 1, y = 2, z = 1$.

Example 2 (Exercise)

- Solve the system of equations

$$\begin{aligned} 2x+y+z &= 5 \\ -2x+z &= -1 \\ -4x-y+2z &= -4 \end{aligned}$$

- Represent the system as an augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ -2 & 0 & 1 & -1 \\ -4 & -1 & 2 & -4 \end{array}\right]$$

Example 2

- Eliminate x from row 2 and row 3 by using row operations

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ -2 & 0 & 1 & -1 \\ -4 & -1 & 2 & -4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ & & & \\ & & & \end{array}\right] \quad \begin{array}{l} \\ r_2+r_1 \\ r_3+2r_1 \end{array}$$

Example 2

- Eliminate y from row 3 by using row operations

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & & & \\ 0 & & & \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & & & \\ 0 & & & \end{array}\right] \qquad r_3 - r_2$$

Example 2

- Back substitute to solve for z, y and x.

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

Example 2 (Check answer)

- Solve the system of equations

$$
\begin{aligned}
2x+y+z &= 5\\
-2x+z &= -1\\
-4x-y+2z &= -4
\end{aligned}
$$

- Check the solution

$x = 1$ $\quad y = 2$ $\quad z = 1$

Geometric interpretation: 1 solution

- Row operations produce the form

$$\left[\begin{array}{ccc|c} \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet \end{array}\right]$$

  where none of the diagonal elements are zero
- We get the usual form of the augmented matrix and can find a unique solution by back substitution

Geometric interpretation: 0 solutions

- Row operations produce the form

$$\left[\begin{array}{ccc|c} \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & \bullet \end{array}\right]$$

  where b_3 is not zero
- We end up with $0\,z =$ non-zero
  i.e. no solution can exist because at least 2 planes, or 3 lines of intersection, are parallel

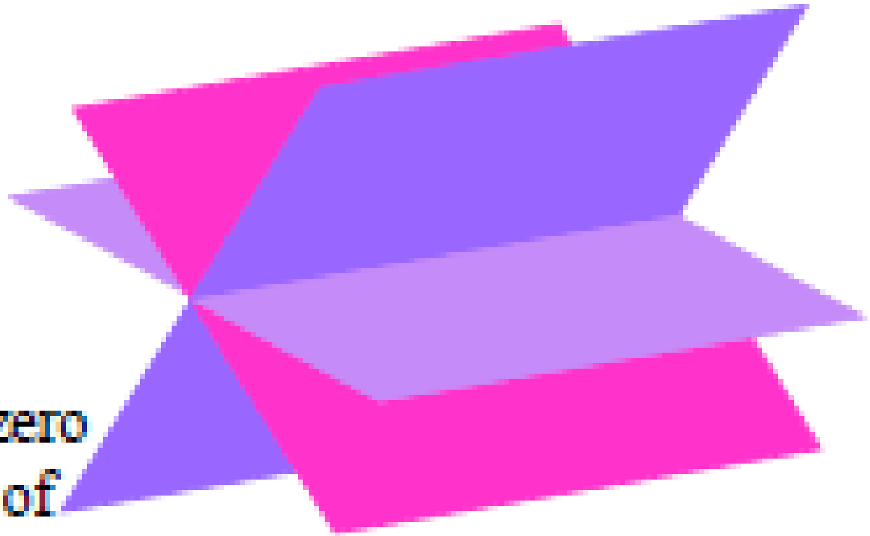
Geometric interpretation: ∞ solutions

- Row operations produce the form

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  i.e. b_3 is zero

- We end up with $0\,z$ = zero i.e. an infinite number of solutions exist because this is always true for any value of z



Examples

- Performing row operations brings us to the augmented matrices shown. Discuss the nature of the solutions

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solving linear equations with LU factorization

Learning objectives

- solve systems of linear equations using LU factorisation
- evaluate which solution method is the best approach under any given circumstance (matrix algebra, Gaussian elimination,, LU factorisation)

LU factorisation –systems with diagonal and triangular matrices

- Solving a system of equations where the matrix is diagonal is very easy.

$$D\mathbf{x} = \mathbf{b} \;\Rightarrow x_i = \frac{b_i}{d_{ii}}$$

- Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}$$

LU factorisation – solving systems with diagonal and triangular matrices

- Solving lower and upper triangular matrices is also very straightforward.
- For a lower triangular matrix, L, we start by finding the first unknown using simple arithmetic.
- We then perform *forward substitution* to solve the system.

LU factorisation –systems with diagonal and triangular matrices

- Example

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ -1 & 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$$

$$\Rightarrow \boxed{x_1 = 2}$$

$$\Rightarrow 3 \times 2 - x_2 = 5$$

$$\Rightarrow \boxed{x_2 = 1}$$

$$\Rightarrow -1 \times 2 + 4 \times 1 - 2x_3 = 4$$

$$\Rightarrow -2x_3 = 2 \Rightarrow x_3 = -1 \qquad \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

LU factorisation – solving systems with diagonal and triangular matrices

- For a upper triangular matrix, U, we start by finding the last unknown using simple arithmetic.
- We then perform *backward substitution* to solve the system.

LU factorisation –systems with diagonal and triangular matrices

- Example

$$\begin{bmatrix} 2 & 2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 9 \end{bmatrix}$$

$$\Rightarrow \boxed{x_3 = 3}$$

$$\Rightarrow x_2 + 2\times 3 = 8$$

$$\Rightarrow \boxed{x_2 = 2}$$

$$\Rightarrow 2\times x_1 + 2\times 2 - 1\times 3 = 3$$

$$\Rightarrow 2x_1 = 2 \Rightarrow x_1 = 1 \qquad \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

LU factorisation – solving systems with diagonal and triangular matrices

- So we can solve diagonal and triangular systems very easily.
- However, these systems are rare in practice.
- So how is this helpful?

LU factorisation

"Any square matrix can be factorised into a product of a lower triangular matrix (L) and an upper triangular matrix (U)."

Warning: this statement comes with a proviso; we might needed to swap some rows around first – see later.

This means if A is square,

we can find an L and a U such that A=LU

- This is called an *LU factorisation* of a matrix.
- How do we find this factorisation?

Reminder: Gaussian elimination

- Recall that a general system of 3 × 3 linear equations in

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

- Can be represent by an augmented matrix:

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

- If we limit ourselves to a particular form of row operation, using subtraction factors, we can use Gaussian elimination to find a U and at the same time construct an L, such that that A=LU

LU factorisation

- Given a square matrix A we set:

$$U = A \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ * & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ * & \cdots & * & 1 \end{bmatrix}$$

- We then use *Gaussian elimination* to reduce U to an upper triangular matrix.
- i.e. we use row subtraction operations to eliminate entries in U, and we record the subtraction multipliers in the corresponding entries of L.
- e.g. if we subtract 5 x row 1 from row 2 to eliminate component u_{21} then $\Rightarrow l_{21} = 5$

Using subtraction factors

- We eliminate x_1 by subtracting a_{21}/a_{11} times the first row from the second row, and subtracting a_{31}/a_{11} times the first row from the third row (primes indicate changed values)

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & a'_{32} & a'_{33} & b'_3 \end{array}\right]$$

- Similarly, we eliminate x_2 by subtracting a'_{32}/a'_{22} times the second row from the third row (double primes indicate changed values), forming an upper triangular matrix:

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a''_{33} & b''_3 \end{array}\right]$$

Example – LU factorisation

- Find the LU factorisation of:

$$A = \begin{bmatrix} 2 & -1 & 3 \\ -8 & 3 & -8 \\ -2 & -2 & 7 \end{bmatrix}$$

- Set up matrices:

$$U = \begin{bmatrix} 2 & -1 & 3 \\ -8 & 3 & -8 \\ -2 & -2 & 7 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix}$$

Example – LU factorisation

- Now perform Gaussian elimination:

$$U=\begin{bmatrix}2 & -1 & 3\\ -8 & 3 & -8\\ -2 & -2 & 7\end{bmatrix} \quad r_2-\boxed{(-4)}r_1 \qquad L=\begin{bmatrix}1 & 0 & 0\\ -4 & 1 & 0\\ * & * & 1\end{bmatrix}$$

$$\Rightarrow U=\begin{bmatrix}2 & -1 & 3\\ 0 & -1 & 4\\ -2 & -2 & 7\end{bmatrix} \quad r_3-\boxed{(-1)}r_1 \qquad L=\begin{bmatrix}1 & 0 & 0\\ -4 & 1 & 0\\ -1 & * & 1\end{bmatrix}$$

$$\Rightarrow U=\begin{bmatrix}2 & -1 & 3\\ 0 & -1 & 4\\ 0 & -3 & 10\end{bmatrix} \quad r_3-\boxed{(3)}r_2 \qquad L=\begin{bmatrix}1 & 0 & 0\\ -4 & 1 & 0\\ -1 & 3 & 1\end{bmatrix}$$

$$\Rightarrow U=\begin{bmatrix}2 & -1 & 3\\ 0 & -1 & 4\\ 0 & 0 & -2\end{bmatrix}$$

Example – LU factorisation

- So we have the LU factorisation:

$$A = LU$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 3 \\ -8 & 3 & -8 \\ -2 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & -2 \end{bmatrix}$$

- Check by evaluating LU !

LU factorisation – Solving a system of equations

- We can use LU factorisation to solve a system of equations.
- Given the system:

$$A\mathbf{x} = \mathbf{b}$$

- And the LU factorisation:

$$A = LU$$

- we can rearrange the system:

$$A\mathbf{x} = \mathbf{b} \;\Rightarrow L\boxed{U\mathbf{x}} = \mathbf{b}$$

$$\Rightarrow L\mathbf{y} = \mathbf{b} \quad \text{where} \quad U\mathbf{x} = \mathbf{y}$$

LU factorisation – Solving a system of equations

So to solve the system:

$$A\mathbf{x} = \mathbf{b}$$

1. Perform *LU* factorisation:

$$A = LU$$

2. Use forward substitution to solve:

$$L\mathbf{y} = \mathbf{b}$$

3. Use backward substitution to solve:

$$U\mathbf{x} = \mathbf{y}$$

Example – LU factorisation solving a system of equations

Solve the system:

$$\begin{bmatrix} 2 & -1 & 3 \\ -8 & 3 & -8 \\ -2 & -2 & 7 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -5 \\ 20 \\ 3 \end{bmatrix}$$

- We know from the previous example that the LU factorisation is:

$$\begin{bmatrix} 2 & -1 & 3 \\ -8 & 3 & -8 \\ -2 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & -2 \end{bmatrix}$$

Example – LU factorisation solving a system of equations

First we use forward substitution to solve for y:

$$L\mathbf{y} = \mathbf{b}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 20 \\ 3 \end{bmatrix}$$

$$\Rightarrow y_1 = -5$$

$$\Rightarrow -4 \times -5 + y_2 = 20$$
$$\Rightarrow y_2 = 0$$

$$\Rightarrow -1 \times -5 + 3 \times 0 + y_3 = 3 \quad \Rightarrow \mathbf{y} = \begin{bmatrix} -5 \\ 0 \\ -2 \end{bmatrix}$$
$$\Rightarrow y_3 = -2$$

Example – LU factorisation solving a system of equations

Now we use backward substitution to solve for **x**

$$U\mathbf{x} = \mathbf{y}$$

$$\Rightarrow \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ -2 \end{bmatrix}$$

$$\Rightarrow x_3 = 1$$

$$\Rightarrow -x_2 + 4 \times 1 = 0$$
$$\Rightarrow x_2 = 4$$

$$\Rightarrow 2x_1 - 1 \times 4 + 3 \times 1 = -5 \quad \Rightarrow \mathbf{x} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$$
$$\Rightarrow x_1 = -2$$

Solve

$$
\begin{aligned}
x_1 + 2x_2 + 4x_3 + x_4 &= 21 \\
2x_1 + 8x_2 + 6x_3 + 4x_4 &= 52 \\
3x_1 + 10x_2 + 8x_3 + 8x_4 &= 79 \\
4x_1 + 12x_2 + 10x_3 + 6x_4 &= 82.
\end{aligned}
$$

Use the triangular factorization method and the fact that

$$
A = \begin{bmatrix} 1 & 2 & 4 & 1 \\ 2 & 8 & 6 & 4 \\ 3 & 10 & 8 & 8 \\ 4 & 12 & 10 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 4 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 4 & -2 & 2 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & -6 \end{bmatrix} = LU.
$$

Use the forward-substitution method to solve $LY = B$:

$$
\begin{aligned}
y_1 \qquad\qquad\qquad &= 21 \\
2y_1 + y_2 \qquad\qquad &= 52 \\
3y_1 + y_2 + \quad y_3 \qquad &= 79 \\
4y_1 + y_2 + 2y_3 + y_4 &= 82.
\end{aligned}
$$

Compute the values $y_1 = 21$, $y_2 = 52 - 2(21) = 10$, $y_3 = 79 - 3(21) - 10 = 6$, and $y_4 = 82 - 4(21) - 10 - 2(6) = -24$, or $Y = \begin{bmatrix} 21 & 10 & 6 & -24 \end{bmatrix}'$. Next write the system $UX = Y$:

$$
\begin{aligned}
x_1 + 2x_2 + 4x_3 + x_4 &= 21 \\
4x_2 - 2x_3 + 2x_4 &= 10 \\
-2x_3 + 3x_4 &= 6 \\
-6x_4 &= -24.
\end{aligned}
$$

Now use back substitution and compute the solution $x_4 = -24/(-6) = 4$, $x_3 = (6 - 3(4))/(-2) = 3$, $x_2 = (10 - 2(4) + 2(3))/4 = 2$, and $x_1 = 21 - 4 - 4(3) - 2(2) = 1$, or $X = [1 \quad 2 \quad 3 \quad 4]'$. ■

LU factorisation – Computing determinants

LU factorisations make finding determinants very easy.

One of the properties of determinants is that for A=LU:

$$\det(A) = \det(L) \times \det(U)$$

Another useful property is that the determinant of triangular matrices is the product of the diagonal entries (you can use cofactor expansion to show this)

LU factorisation – Computing determinants

When we construct L we set all of the diagonals to be 1, so:

$$\det(L) = 1 \times 1 \times \ldots \times 1 = 1$$

(assumes no row swaps)

Which means the determinant is:

$$\det(A) = \det(U) = \prod_{i=1}^{n} u_{ii}$$

The product of the diagonal entries

EXAMPLES:

Compute the determinant of the matrix using LU factorisation:

$$A = \begin{bmatrix} 1 & -1 & -3 \\ -2 & 4 & 9 \\ 2 & -4 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -3 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \det(A) = \det(L) \times \det(U)$$

$$\Rightarrow \det(A) = (1 \times 1 \times 1) \times (1 \times 2 \times 1)$$

$$\Rightarrow \det(A) = 2$$

EXAMPLES:

Compute the determinant of the matrix using LU factorisation:

$$A = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & -\frac{11}{2} \end{bmatrix}$$

$$\Rightarrow \det(A) = \det(L) \times \det(U)$$

$$\Rightarrow \det(A) = (-1 \times 1 \times 1 \times 1) \times \left(2 \times 2 \times -\tfrac{11}{2}\right)$$

$$\Rightarrow \det(A) = 22$$

LU factorisation – Computing determinants

- Very easy to calculate determinants using LU factorisation.
- Properties of determinants:

$$PA = LU \quad \Rightarrow \det(P) \times \det(A) = \det(L) \times \det(U)$$

$$\Rightarrow \det(A) = \det(L) \times \det(U) / \det(P)$$

- The determinant of a triangular matrix is the product of the diagonal entries

- Diagonals of L are all 1, so:

$$\det(L) = 1 \times 1 \times \ldots \times 1 = 1$$

- Determinant of U is:

$$\det(U) = u_{11} \times u_{22} \times \ldots \times u_{nn} = \prod_{i=1}^{N} u_{ii}$$ **Product of the diagonal entries**

- Multiply by (-1) for each row swap

$$\det(P) = (-1)^R$$ (R = number of row swaps)

- Therefore:

$$\det(A) = \det(L) \times \det(U) / \det(P) = (-1)^R \prod_{i=1}^{N} u_{ii}$$

Comparison with the Co-Factor method

The classical means of evaluating determinants is via *co-factor expansion* along any row or column of the matrix:

$$\det(A) = \sum_i (-1)^{i+j} a_{ij} C_{ij} \leftarrow \text{expansion down column } j$$

or

$$\det(A) = \sum_j (-1)^{i+j} a_{ij} C_{ij} \leftarrow \text{expansion along row } i$$

Here, C_{ij} is the ij^{th} *co-factor* of A, defined to be the determinant of the *sub-matrix* formed after deleting the i^{th} row and the j^{th} column of A.

The co-factors depend recursively on their own co-factors, which involve successively smaller sub-matrices.

If A is an $n \times n$ matrix, then $\det(A)$ relies on n co-factors, *each of which* rely on $n-1$ of their own co-factors, *each of which* rely on $n-2$ of their co-factors etc. Apparently, the work requirements scale like $n \times (n-1) \times (n-2) \times \cdots \times 1 = n!$, which grows extremely rapidly with respect to dimension n.

n	Number of multiplications need to find $\det(A)$
2	2
3	9
4	40
5	205
6	1236
7	8659
8	69280
9	623529
10	6235300
11	68588311
$\vdots$	$\vdots$
19	209020565553571999
20	4180411311071440000

It would take a computer with a 3 GHz processor slightly more than 44 years to find the determinant of a 20×20 matrix. However it only takes 2667 calculations to find the determinant of a 20×20 matrix using LU factorisation. This would take $0.1\mu s$ for the same computer to calculate.

LU factorisation in MATLAB

MATLAB can easily find the LU factors for a matrix. Using the matrix from EXAMPLE 3.8., enter the matrix with the command:

```
>> A = [0 2 4;2 1 1;1 2 -2];
```

We can find the LU factorisation by entering the command:

```
>> [L,U]=lu(A)

L =                                          U =
         0    1.0000         0                   2.0000    1.0000    1.0000
    1.0000         0         0                        0    2.0000    4.0000
    0.5000    0.7500    1.0000                        0         0   -5.5000
```

LU factorisation – Solving multiple systems of equations

- Using LU factorisation makes solving a system of equations very simple.
- LU factorisations are especially helpful when solving the same system for more than one RHS (a situation that often occurs in practice).
- When solving for many RHS you only have to find the LU factorisation once.
- After the finding the LU factorisation, the solution for each RHS only involves forward and backward substitutions.

Example – LU and multiple systems

Suppose a vehicle company manufactures three different types of car, a station wagon, coupe and sedan. The number of each car produced is:

$$x_1 = \text{number of station wagons;}$$

$$x_2 = \text{number of coupes;}$$

$$x_3 = \text{number of sedans.}$$

Example – LU and multiple systems

- The station wagon takes 4 hours to assemble, 3 hours to paint and 2 hours to check.

Example – LU and multiple systems

- The coupe takes 5 hours to assemble, 2 hours to paint and 3 hours to check.

Example – LU and multiple systems

- The sedan takes 3 hours to assemble, 3 hours to paint and 2 hours to check.

Example – LU and multiple systems

The plant is limited to 47 hours for assembly, 33 hours for painting and 27 hours for checking.

This gives the table:

	Station wagon	Coupe	Sedan	Total
Assembly	4	5	3	47
Painting	3	2	3	33
Checking	2	3	2	27

How many of each type should we produce to use up all the resources?

Example – LU and multiple systems

- The matrix form is:

$$\begin{bmatrix} 4 & 5 & 3 \\ 3 & 2 & 3 \\ 2 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 47 \\ 33 \\ 27 \end{bmatrix}$$

- The LU factorisation is:

$$\begin{bmatrix} 4 & 5 & 3 \\ 3 & 2 & 3 \\ 2 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{2}{7} & 1 \end{bmatrix} \begin{bmatrix} 4 & 5 & 3 \\ 0 & -\frac{7}{4} & \frac{3}{4} \\ 0 & 0 & \frac{5}{7} \end{bmatrix}$$

Example – LU and multiple systems

- Solve this using forward and backward substitution:

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{2}{7} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 47 \\ 33 \\ 27 \end{bmatrix}$$

$$y_1 = 47$$

$$\frac{3}{4} \times 47 + y_2 = 33 \Rightarrow y_2 = -\frac{9}{4}$$

$$\frac{47}{2} + \frac{9}{14} + y_3 = 27 \Rightarrow y_3 = \frac{20}{7}$$

Example – LU and multiple systems

- Solve this using forward and backward substitution:

$$\begin{bmatrix} 4 & 5 & 3 \\ 0 & -\frac{7}{4} & \frac{3}{4} \\ 0 & 0 & \frac{5}{7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 47 \\ -\frac{9}{4} \\ \frac{20}{7} \end{bmatrix}$$

$$\frac{5}{7}x_3 = \frac{20}{7} \Rightarrow x_3 = 4$$

$$\Rightarrow -\frac{7}{4}x_2 + \frac{3}{4}\times 4 = -\frac{9}{4} \Rightarrow x_2 = 3$$

$$\Rightarrow 4x_1 + 5\times 3 + 3\times 4 = 47 \Rightarrow x_1 = 5$$

Example – LU and multiple systems

- So we make 5 station wagons, 3 coupes and 4 sedans.

- What happens if we increase the number of hours for painting up to 38?

Example – LU and multiple systems

- Now we have:

	Station wagon	Coupe	Sedan	Total
Assembly	4	5	3	47
Painting	3	2	3	38
Checking	2	3	2	27

- Which gives the new system:

$$\begin{bmatrix} 4 & 5 & 3 \\ 3 & 2 & 3 \\ 2 & 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 47 \\ 33 \\ 27 \end{bmatrix} \Rightarrow \begin{bmatrix} 4 & 5 & 3 \\ 3 & 2 & 3 \\ 2 & 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 47 \\ \boxed{38} \\ 27 \end{bmatrix}$$

Example – LU and multiple systems

- Since only the RHS has changed we can use the <u>same</u> LU factorisation!!

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{4} & 1 & 0 \\ \frac{1}{2} & -\frac{2}{7} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 47 \\ \boxed{38} \\ 27 \end{bmatrix}$$

$$y_1 = 47$$

$$\frac{3}{4} \times 47 + y_2 = \boxed{38} \Rightarrow y_2 = \boxed{\frac{11}{4}}$$

$$\frac{47}{2} + \frac{9}{14} + y_3 = 27 \Rightarrow y_3 = \boxed{\frac{30}{7}}$$

Example – LU and multiple systems

- Backward substitution:

$$\begin{bmatrix} 4 & 5 & 3 \\ 0 & -\frac{7}{4} & \frac{3}{4} \\ 0 & 0 & \frac{5}{7} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 47 \\ \frac{11}{4} \\ \frac{30}{7} \end{bmatrix}$$

$$\frac{5}{7}x_3 = \boxed{\frac{30}{7}} \Rightarrow x_3 = \boxed{6} \quad \text{(was 4)}$$

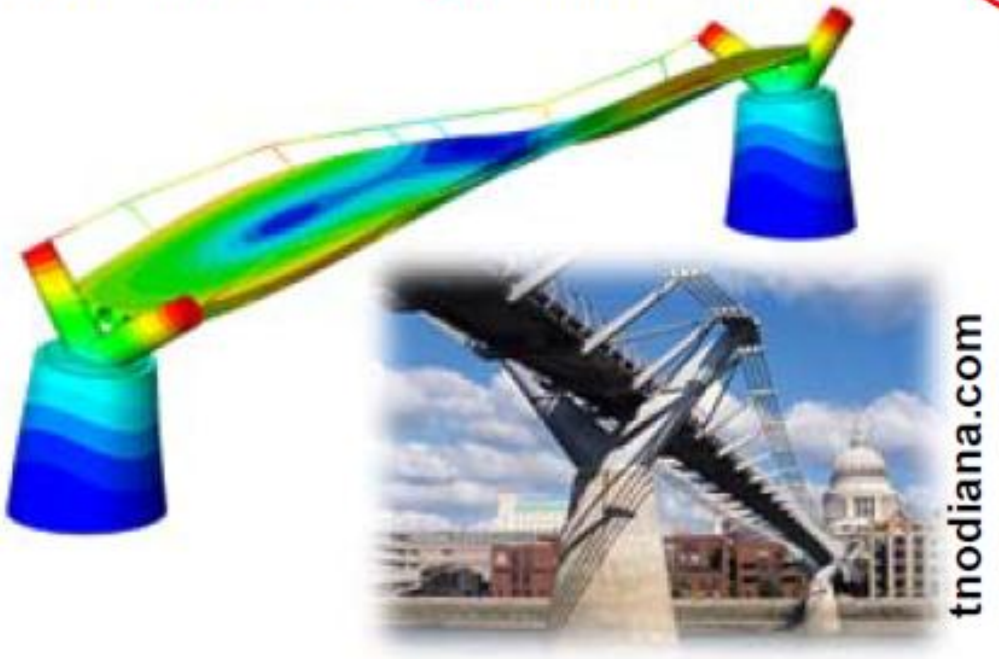
$$\Rightarrow -\frac{7}{4}x_2 + \frac{3}{4}\boxed{\times 6} = \boxed{\frac{11}{4}} \Rightarrow x_2 = \boxed{1} \quad \text{(was 3)}$$

$$\Rightarrow 4x_1 + 5\times\boxed{1} + 3\times\boxed{6} = 47 \Rightarrow x_1 = \boxed{6} \quad \text{(was 5)}$$

LU factorisation – Solving for multiple RHSs

e.g. Finite Element Method (FEM): $Ku = f$ $(Ax=b)$

K: geometry/stiffness; u: displacements; f: loads



LU factorisation – Finding A^{-1} (multiple RHSs)

- If we have the *LU* factorisation of *A*, we can find A^{-1} column-by-column by solving $LU\mathbf{a}_i=\mathbf{e}_i$ where the multiple RHS vectors, $\mathbf{e}_i$, is simply each column if the identity.

$$A\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad A\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad A\mathbf{x}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Can calculate the x_m simultaneously (parallel programming)

$$A^{-1} = \begin{bmatrix} \vdots & \vdots & & \vdots \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

Extra for experts – zero pivots and row swaps

- When we are using Gaussian elimination to find an LU factorisation we sometimes encounter a *zero pivot*.
- This means that there is a zero diagonal entry (the *pivot*) in the column for which we are trying to eliminate entries below the diagonal. So we cannot continue row operations.
- Thus we must swap rows to get rid of the *zero pivot*.
- If we swap rows in the U matrix we must swap the same rows in the L matrix.
- We usually swap the rows in L after we have finished the factorisation – it is easier!

Example – Zero pivots and row swaps

Find the LU factorisation for the system:

$$
\begin{aligned}
2x_2 + 4x_3 &= -2 \\
2x_1 + x_2 + x_3 &= 1 \\
x_1 + 2x_2 - 2x_3 &= 7
\end{aligned}
$$

Form the matrices:

$$
U = \begin{bmatrix} \underset{\text{zero pivot}}{\boxed{0}} & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 2 & -2 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix}
$$

Example – Zero pivots and swaps

- Use Gaussian elimination:

$$U = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 2 & -2 \end{bmatrix} \; r_1 \Leftrightarrow r_2 \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix}$$

$$\Rightarrow U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 1 & 2 & -2 \end{bmatrix} \; \begin{matrix} \\ r_2 - 0r_1 \\ r_3 - \frac{1}{2}r_1 \end{matrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & * & 1 \end{bmatrix}$$

$$\Rightarrow U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & \frac{3}{2} & -\frac{5}{2} \end{bmatrix} \; \begin{matrix} \\ \\ r_3 - \frac{3}{4}r_2 \end{matrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{3}{4} & 1 \end{bmatrix}$$

$$\Rightarrow U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & -\frac{11}{2} \end{bmatrix}$$

Example – Zero pivots and row swaps

- Remembering the row swap we have:

$$A = LU$$

rows swapped

$$\Rightarrow \begin{bmatrix} 0 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & -\frac{11}{2} \end{bmatrix}$$

Example – Zero pivots and row swaps

- Check the factorisation:

$$LU = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & -\frac{11}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0\times 2+1\times 0+0\times 0 & 0\times 1+1\times 2+0\times 0 & 0\times 1+1\times 4+0\times -\frac{11}{2} \\ 1\times 2+0\times 0+0\times 0 & 1\times 1+0\times 2+0\times 0 & 1\times 1+0\times 4+0\times -\frac{11}{2} \\ \frac{1}{2}\times 2+\frac{3}{4}\times 0+1\times 0 & \frac{1}{2}\times 1+\frac{3}{4}\times 2+1\times 0 & \frac{1}{2}\times 1+\frac{3}{4}\times 4+1\times -\frac{11}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 2 & -2 \end{bmatrix} = A$$

Example – Zero pivots and row swaps

- Row swaps changes the forward substitution step:

$$Ly = \mathbf{b}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}$$

$$\Rightarrow y_1 = 1$$

$$\Rightarrow y_2 = -2$$

$$\Rightarrow \tfrac{1}{2} \times 1 + \tfrac{3}{4} \times -2 + y_3 = 3$$

$$\Rightarrow y_3 = 4$$

Check the LU factorisation:

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & -\frac{11}{2} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 1 & 2 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 2 & -2 \end{bmatrix}$$

So *LU* is similar to *A*, but with row swaps (i.e. *A*≠*LU*)

Need to swap (permute) the rows of *A*

$$PA = LU$$

The permutation matrix, *P*, is the identity matrix with the same row swaps that were made to *U*

Perform row swaps and row subtraction ops:

$$A = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 2 & -2 \end{bmatrix} \; r_1 \Leftrightarrow r_2 \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Start with Identity matrix

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 1 & 2 & -2 \end{bmatrix} \begin{matrix} \\ r_2 - (0) r_1 \\ r_3 - \left(\frac{1}{2}\right) r_1 \end{matrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & * & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & \frac{3}{2} & -\frac{5}{2} \end{bmatrix} \begin{matrix} \\ \\ r_3 - \left(\frac{3}{4}\right) r_2 \end{matrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{3}{4} & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & -\frac{11}{2} \end{bmatrix} = U$$

[Note: if no row swaps are required, then *P=I* the Identity matrix.]

Check the *PA=LU* factorisation:

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & -\frac{11}{2} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 1 & 2 & -2 \end{bmatrix}$$

$$PA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 4 \\ 1 & 2 & -2 \end{bmatrix} = LU$$ ☺

Example

Find the LU factorisation for the system:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1 \\ 4x_1 + 2x_2 \quad\quad &= -2 \\ -2x_1 + 2x_2 + x_3 &= 7 \end{aligned}$$

Form the initial matrices:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 0 \\ -2 & 2 & 1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix}$$

Perform row swaps and row subtraction ops:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 0 \\ -2 & 2 & 1 \end{bmatrix} \begin{matrix} \\ r_2 - (2)r_1 \\ r_3 - (-1)r_1 \end{matrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Start with Identity matrix

zero pivot

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & \boxed{0} & -2 \\ 0 & 3 & 2 \end{bmatrix} \; r_2 \Leftrightarrow r_3 \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & * & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{matrix} \\ \\ r_3 - (0)r_2 \end{matrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix} = U$$

Need to swap rows in *U* & *P*,
AND swap any row multipliers that have already been recorded in *L*

Check the *PA*=*LU* factorisation:

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ -2 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 0 \\ -2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ -2 & 2 & 1 \\ 4 & 2 & 0 \end{bmatrix} = LU$$ ☺

Zero pivots and row swaps: summary

- Swap rows to avoid zero pivots.
- In $A \rightarrow U$, always swap with a row below the diagonal.
- In P, swap the same rows (in the same sequence).
- In L, swap the rows of any multipliers that have already been recorded (do not swap the diagonal 1s or the 0s above the diagonal).
- When solving systems, forward/back substitution are similar (calculate variables in increasing/decreasing order)

Key points: LU factorisation and solving $A\mathbf{x}=\mathbf{b}$

- Perform LU factorisation ($A=LU$)
  - Use Gaussian elimination on $A \to U$, and put row subtraction multipliers in L
- If we encounter zero pivots, perform row swaps to avoid them
  - Remember to construct the permutation matrix
  - Remember to swap subtraction multipliers in L
- To solve $A\mathbf{x}=\mathbf{b}$ (can easily do multiple $\mathbf{b}$'s)
  - Solve $L\mathbf{y}=P\mathbf{b}$ (forward substitution)
  - Solve $U\mathbf{x}=\mathbf{y}$ (backward substitution)

Exercises

1. (a) Solve the system of equations $Ax = \mathbf{b}$ using LU factorisation, where

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 & 5 \\ 3 & -1 & 6 & 10 & 22 \\ 0 & -4 & 3 & 0 & -13 \\ 1 & 7 & 5 & 13 & 35 \\ 2 & -2 & -5 & 4 & 13 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 15 \\ 62 \\ -29 \\ 83 \\ 25 \end{bmatrix}$$

(b) Use MATLAB to perform the Gaussian elimination.

2. Write the following systems of equations in matrix form.

(a)
$$\begin{aligned} 3x_1 + x_2 - x_3 + x_4 &= 1 \\ 3x_3 - x_2 + x_4 &= -3 \\ 2x_2 + 5x_4 &= 1 \\ 2x_1 + x_3 - x_4 &= -2 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 1 \\ 2x_1 + 6x_3 &= 8 \\ x_3 - x_2 &= -6 \end{aligned}$$

(c)
$$\begin{aligned} 2x_1 - x_3 &= -1 \\ x_3 - 5x_2 &= 6 \\ 2x_2 - x_1 + 3x_3 &= -4 \end{aligned}$$

(d)
$$\begin{aligned} 3x_1 - 2x_2 + x_4 &= 1 \\ 5x_4 - 2x_3 &= -4 \\ 3x_2 + x_3 &= 2 \\ -2x_1 + x_2 - x_3 &= 2 \end{aligned}$$

3. (a) Solve the following systems of linear equations using LU factorisation.

(i)
$$\begin{aligned} 3x + y &= 11 \\ x - y &= 5 \end{aligned}$$

(ii)
$$\begin{aligned} 3y - 4z &= 1 \\ 9x - 4y + z &= 4 \\ x + y + z &= 15 \end{aligned}$$

(iii)
$$\begin{aligned} 3x - 12 &= 6 \\ -7x + 28y &= -14 \\ 5x - 20y &= 10 \end{aligned}$$

(iv)
$$\begin{aligned} z - x &= 2 \\ 2z - x - y &= 1 \\ 2x - y &= -4 \end{aligned}$$

(v)
$$\begin{aligned} x + y + 2z &= 4 \\ 4x + 6y + 9z &= 17 \\ 4x + 8y - 5z &= 3 \end{aligned}$$

(vi)
$$\begin{aligned} x - y + 3z &= 12 \\ 2y - 3x - 2z &= -14 \\ 2x - y + z &= 8 \end{aligned}$$

(vii)
$$\begin{aligned} x + y &= 5 \\ 2x + y + 2z &= 4 \\ z - x &= -5 \end{aligned}$$

4. Find an LU factorisation of each of the following matrices:

(a) $A = \begin{bmatrix} 1 & 3 & -1 & 0 \\ -1 & -5 & 5 & 1 \\ 2 & 8 & -2 & 1 \\ 2 & 10 & 10 & 12 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 3 & 9 & 4 \\ 5 & 10 & 5 & 5 \\ 2 & 1 & 3 & 2 \\ 1 & 6 & 7 & 1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 2 & 2 & 2 & 1 \\ 2 & 2 & 0 & 4 \\ 0 & 3 & 2 & 2 \end{bmatrix}$

(d) $A = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 4 & 3 & 0 & 3 \\ 2 & 9 & 12 & 5 \\ 4 & 3 & 5 & 6 \end{bmatrix}$

5. Find an LU factorisation for each of the following matrices.

(a) $\begin{bmatrix} 10 & 10 & 20 \\ 5 & 5 & 20 \\ 5 & 10 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 2 & 4 \\ 2 & 2 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 2 \\ 1 & 3 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 10 & 10 & 40 \\ 5 & 4 & 39 \\ 3 & 4 & 11 \end{bmatrix}$

5. Find an LU factorisation for each of the following matrices.

(a) $\begin{bmatrix} 10 & 10 & 20 \\ 5 & 5 & 20 \\ 5 & 10 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 2 & 4 \\ 2 & 2 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 2 & 2 \\ 1 & 3 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 10 & 10 & 40 \\ 5 & 4 & 39 \\ 3 & 4 & 11 \end{bmatrix}$

6. (a) Find an LU factorisation for each of the following matrices.
Show that $\det(L) \times \det(U) = \det(A)$.

(i) $\begin{bmatrix} 3 & 9 & 1 \\ 6 & 14 & 0 \\ 9 & 31 & 8 \end{bmatrix}$

(ii) $\begin{bmatrix} 5 & 5 & 2 \\ 5 & 3 & 3 \\ 10 & 16 & 17 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 4 & 5 & 6 \\ 6 & 12 & 15 & 18 \\ 9 & 4 & 0 & 0 \\ 12 & 4 & 15 & 19 \end{bmatrix}$

(iv) $\begin{bmatrix} 3 & 3 & 4 & 4 \\ 18 & 11 & 17 & 17 \\ 6 & 6 & 9 & 9 \\ 0 & 14 & 17 & 18 \end{bmatrix}$

(c) Use **MATLAB** to find the LU factorisation.

7. For each of the following a computer program has returned the following LU factors (after operating on a matrix A). Solve the system for the given right-hand side:

(a) $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$ $U = \begin{bmatrix} 2 & -1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$ $\mathbf{b} = \begin{bmatrix} 8 \\ 8 \\ 4 \end{bmatrix}$

(b) $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ $U = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ $\mathbf{b} = \begin{bmatrix} 6 \\ 3 \\ 7 \end{bmatrix}$

8. Using the LU factorisation for A below solve

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 5 & 5 & 2 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix} = LU$$

(a) $$Ax = \begin{bmatrix} 14 \\ -2 \\ -60 \end{bmatrix}$$

(b) $$Ax = \begin{bmatrix} 10 \\ -12 \\ -14 \end{bmatrix}$$

9. Find the LU factors of the matrix below, and use them to solve the given system of equations.

$$\begin{bmatrix} 0 & 1 & 2 & -1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & 4 \\ 1 & -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \\ 0 \\ 11 \end{bmatrix}$$

10. Find the LU factors of the matrix A. Then use your LU factors to solve the system of equations $Ax = \mathrm{b}$:

$$A = \begin{bmatrix} 1 & -1 & 2 & 3 & 5 \\ 3 & -1 & 6 & 10 & 22 \\ 0 & -4 & 3 & 0 & -13 \\ 1 & 7 & 5 & 13 & 35 \\ 2 & -2 & -5 & 4 & 13 \end{bmatrix} \text{ and } \mathrm{b} = \begin{bmatrix} 22 \\ 93 \\ -48 \\ 125 \\ 40 \end{bmatrix}.$$

11. Given

$$A = \begin{bmatrix} 6 & 4 & 3 \\ 4 & 3 & 2 \\ 3 & 4 & 2 \end{bmatrix} \text{ and } \mathrm{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

(a) Find an LU factorisation of A.
(b) Use this to solve $A\mathrm{x} = \mathrm{b}$.