Linear Algebra
Matrix Theory-Section 1: Matrix Basics

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Linear Algebra

- Meaning of linear?
- Developed from studies of system of linear equations
- Fairly extensive subject-That covers
  - Vectors
  - Matrices
  - Determinants
  - System of linear equations
  - Vector spaces
  - Linear transformations
  - Eigen value problems

Usage

- Due to the introduction of computers and information systems
  - Linear algebra and matrix methods have been widespread usage in many areas of science and engineering
- Matrices have been employed effectively in many applications
  - Signal processing
  - Controls
  - Finite elements
  - Communications
  - Computer vision
  - Electromagnetics
  - Social and health sciences

Usage

- MATLAB-It is a matrix base analysis type package
- VECTOR- It is an ordered array of numbers of algebraic variables
  - $c = \begin{bmatrix} c1 \\ c2 \\ c3 \end{bmatrix}$
- MATRIX-is a doubly ordered array of elements.

Learning objectives

- Understand what a matrix is and why they are useful
- Compute the following matrix operations:
  - addition
  - subtraction
  - scalar multiplication
  - transpose
  - matrix multiplication

Where are matrices used?

- Almost everywhere in engineering problems involving linear equations.
- Computer graphics (transformations)
- Random processes (Markov chains)
- Linear programming (e.g. optimization models for supply chains)
- Solving systems of differential equations e.g. in fluid dynamics
- In Google's Page Rank algorithm

Ice breaking question

- What did you all bring other than your books, food etc

THREE CRITERIA FOR SUCCESS

1. EFFORT MUST BE ENJOYABLE
2. ACHIEVE THE OBJECTIVE
3. ENJOY WHAT YOU ACHIEVED

Basic terminology

- Before we see why matrices are useful it is helpful to define some basic terminology, so that we can talk about the characteristics of any given matrix, in particular its size and element values.

What is a matrix?

- A matrix is a rectangular array of elements

$$\mathbf{A} = \begin{bmatrix} a_{11} ,\ldots, & a_{1n} \\ a_{21} ,\ldots, & a_{2n} \\ \ldots\ \ldots & \ldots\ \ldots \\ a_{m1} ,\ldots, & a_{mn} \end{bmatrix} = \{A_{ij}\}$$

- The elements may be of any type (e.g. integer, real, complex, logical, or even other matrices).
- In this course we will only consider matrices that have integer, real, or complex elements.

Where are matrices used?

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Matrix

A matrix is any doubly subscripted array of elements arranged in rows and columns.

$$\mathbf{A} = \begin{bmatrix} a_{11}\ ,\ldots,\ a_{1n} \\ a_{21}\ ,\ldots,\ a_{2n} \\ \ldots\ \ldots\ \ldots\ \ldots \\ a_{m1}\ ,\ldots,\ a_{mn} \end{bmatrix} = \{A_{ij}\}$$

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Order of a matrix (size)

- Order 4x3: 4 rows and 3 columns

$$\begin{bmatrix} -5 & 0 & 1 \\ 2 & 3 & -4 \\ -9 & 2 & 6 \\ 3 & 1 & 4 \end{bmatrix}$$

- Order 3x4:

$$\begin{bmatrix} -5 & 0 & 1 & 2 \\ 3 & -4 & -9 & 2 \\ 3 & 1 & 4 & 2 \end{bmatrix}$$

- REMEMBER: rows first, then columns

Special matrices: row and column

- A 1xn matrix is called a **row** matrix
  - E.g.
    $$\begin{array}{cc} & 6\ \text{columns} \rightarrow \\ 1\ \text{rows}\ \downarrow & \begin{bmatrix} 2 & 1 & 1 & -2 & 1 & -5 \end{bmatrix} \end{array}$$
- An mx1 matrix is called a **column** matrix.
  - E.g.
    $$\begin{array}{cc} & 1\ \text{columns} \rightarrow \\ 3\ \text{rows}\ \downarrow & \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix} \end{array}$$
  - So vectors can be though of as matrices

Specifying Elements

$$A = \begin{bmatrix} -5 & 0 & 1 \\ -2 & -3 & -4 \\ -9 & 2 & 6 \\ 3 & 1 & 4 \end{bmatrix}$$

$$a_{32} = 2$$

$$a_{41} = 3$$

REMEMBER: rows first, then columns

Some simple operations

- Note vectors are a subset of matrices
- Addition, subtraction and scalar multiplication are all operations that can the be done with matrices, in the same was as was done for vectors.

Matrix Arithmetic

Definition 1.1 (Equality of matrices). Matrices A and B are said to be equal if they have the same size and their corresponding elements are equal; i.e., A and B have dimension $m \times n$, and $A = [a_{ij}], B = [b_{ij}]$, with $a_{ij} = b_{ij}$ for $1 \leq i \leq m, 1 \leq j \leq n$.

Definition 1.2 (Addition of matrices). Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be of the same size. Then $A + B$ is the matrix obtained by adding corresponding elements of A and B; that is,

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}].$$

Definition 1.3 (Scalar multiple of a matrix). Let $A = [a_{ij}]$ and t be a number (*scalar*). Then tA is the matrix obtained by multiplying all elements of A by t; that is,

$$tA = t[a_{ij}] = [ta_{ij}].$$

Definition 1.4 (Negative of a matrix). Let $A = [a_{ij}]$. Then $-A$ is the matrix obtained by replacing the elements of A by their negatives; that is,

$$-A = -[a_{ij}] = [-a_{ij}].$$

Matrix Arithmetic

Definition 1.5 (Subtraction of matrices). Matrix subtraction is defined for two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size, in the usual way; that is,

$$A - B = [a_{ij}] - [b_{ij}] = [a_{ij} - b_{ij}].$$

Definition 1.6 (The zero matrix). Each $m \times n$ matrix, all of whose elements are zero, is called the *zero* matrix (of size $m \times n$) and is denoted by the symbol 0.

$$tA = t[a_{ij}] = [ta_{ij}].$$

Lower, Upper Triangular matrix

Lower triangular matrix: An $m \times n$ matrix for which $a_{ij} = 0$ for $j > i$ is called a lower triangular matrix.

A lower triangular matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -3 & 4 & 0 & 0 \\ 5 & 1 & 2 & 0 \\ -3 & -4 & 2 & 9 \end{bmatrix}$$

A square matrix $A = [a_{ij}]$ is *upper triangular* or simply *triangular* if all entries below the (main) diagonal are equal to 0—that is, if $a_{ij} = 0$ for $i > j$. Generic upper triangular matrices of orders 2, 3, 4 are as follows:

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix}, \quad \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ & b_{22} & b_{23} \\ & & b_{33} \end{bmatrix}, \quad \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ & c_{22} & c_{23} & c_{24} \\ & & c_{33} & c_{34} \\ & & & c_{44} \end{bmatrix}$$

Diagonal matrix

Diagonal matrix: A square matrix whose nondiagonal elements are equal to zero is called a diagonal matrix.

A 3×3 diagonal matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

The matrix operations

The matrix operations of addition, scalar multiplication, negation and subtraction satisfy the usual laws of arithmetic. (In what follows, s and t are arbitrary scalars and A, B, C are matrices of the same size.)

1. $(A + B) + C = A + (B + C)$;
2. $A + B = B + A$;
3. $0 + A = A$;
4. $A + (-A) = 0$;
5. $(s + t)A = sA + tA$, $(s - t)A = sA - tA$;
6. $t(A + B) = tA + tB$, $t(A - B) = tA - tB$;
7. $s(tA) = (st)A$;
8. $1A = A$, $0A = 0$, $(-1)A = -A$;
9. $tA = 0 \Rightarrow t = 0$ or $A = 0$.

Matrix Addition and Subtraction

- Add or subtract corresponding elements
- $[1 \quad -1 \quad 0] + [2 \quad 3 \quad -2] = [3 \quad 2 \quad -2]$
- $\begin{bmatrix} 3 & 5 \\ 2 & 4 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 4 \\ 1 & -2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 6 \\ -3 & 1 \end{bmatrix}$
- A new matrix **C** may be defined as the additive combination of matrices **A** and **B** where: **A** + **B=C** is defined by:

$$\{C_{ij}\} = \{A_{ij}\} + \{B_{ij}\}$$

- Note: all three matrices are of the same dimension

Matrix addition

- Note that matrices being added or subtracted must be of the same order,
- E.g. $\begin{bmatrix} 5 & 0 & 1 \\ 2 & 3 & -4 \\ -9 & 2 & 6 \\ 3 & 1 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix} = invalid$

Matrix Subtraction

C = A - B

Is defined by

$$\{C_{ij}\} = \{A_{ij}\} - \{B_{ij}\}$$

Matrix Addition and Subtraction

Find $C = A + B$ and $D = A - B$ if:

$$A = \begin{bmatrix} 2 & 3 & -4 \\ -2 & 1 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 6 & -5 \end{bmatrix}$$

Solution:

$$C = A + B = \begin{bmatrix} 0 & 4 & -1 \\ 2 & 7 & 1 \end{bmatrix} \text{ and } D = A - B = \begin{bmatrix} 4 & 2 & -7 \\ -6 & -5 & 11 \end{bmatrix}$$

The MATLAB® command to add or subtract two matrices is $\mathbf{C} = \mathbf{A} \pm \mathbf{B}$.

Properties of Matrix Addition

The following properties hold for matrix addition. They can be easily proven using the definition of addition.

a. Commutative property: $A + B = B + A$

b. Associative property: $A + (B + C) = (A + B) + C$

Matrix Addition

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 6 & 8 \\ 1 & -3 & -7 \end{bmatrix}$

Then

$$A + B = \begin{bmatrix} 1+4 & -2+6 & 3+8 \\ 0+1 & 4+(-3) & 5+(-7) \end{bmatrix} = \begin{bmatrix} 5 & 4 & 11 \\ 1 & 1 & -2 \end{bmatrix}$$

$$3A = \begin{bmatrix} 3(1) & 3(-2) & 3(3) \\ 3(0) & 3(4) & 3(5) \end{bmatrix} = \begin{bmatrix} 3 & -6 & 9 \\ 0 & 12 & 15 \end{bmatrix}$$

$$2A - 3B = \begin{bmatrix} 2 & -4 & 6 \\ 0 & 8 & 10 \end{bmatrix} + \begin{bmatrix} -12 & -18 & -24 \\ -3 & 9 & 21 \end{bmatrix} = \begin{bmatrix} -10 & -22 & -18 \\ -3 & 17 & 31 \end{bmatrix}$$

Multiplication by a Scalar

$$3\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 6 & 3 \end{bmatrix}$$

Matrix Transpose and Hermitian

Matrix transpose and Hermitian: The transpose of matrix A is obtained by interchanging the rows and columns of A. It is denoted by A^T. If matrix A has dimensions $n \times m$, its transpose will be $m \times n$. Therefore:

$$B = A^T \text{ if } b_{ij} = a_{ji}$$

$$\begin{bmatrix} 0 & -2 & 1 \\ 2 & 1.2 & 3 \\ 0 & -3 & 1 \\ 0 & 1.5 & 2 \\ 0 & 1.4 & 3 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ -2 & 1.2 & -3 & 1.5 & 1.4 \\ 1 & 3 & 1 & 2 & 3 \end{bmatrix}$$

The matrix A^H is obtained by transposing and then conjugating every element of matrix A. That is:

$$B = A^H \text{ if } b_{ij} = \bar{a}_{ji}$$

where "bar" stands for complex conjugation. It should be noted that $A^H = A^T$ if A is real. Finally, a real square matrix is symmetric if $A = A^T$. A complex square matrix is said to be Hermitian if $A = A^H$.

Matrix Transpose and Hermitian

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad [1, -3, -5]^T = \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix}$$

$$\text{If } A = \begin{bmatrix} 2 & 7 & 8 & -3 & 1 \\ 1 & 4 & 6 & 2 & -4 \\ -2 & 3 & 2 & 3 & 5 \end{bmatrix} \text{ then } B = A^T = \begin{bmatrix} 2 & 1 & -2 \\ 7 & 4 & 3 \\ 8 & 6 & 2 \\ -3 & 2 & 3 \\ 1 & -4 & 5 \end{bmatrix}$$

$$\text{If } C = \begin{bmatrix} 2-j & 1+j3 \\ j4 & 4 \end{bmatrix} \text{ then } C^H = \begin{bmatrix} 2+j & -j4 \\ 1-j3 & 4 \end{bmatrix}$$

Symmetric Matrices

A matrix A is *symmetric* if $A^T = A$. Equivalently, $A = [a_{ij}]$ is symmetric if *symmetric elements* (mirror elements with respect to the diagonal) are equal—that is, if each $a_{ij} = a_{ji}$.

A matrix A is *skew-symmetric* if $A^T = -A$ or, equivalently, if each $a_{ij} = -a_{ji}$. Clearly, the diagonal elements of such a matrix must be zero, because $a_{ii} = -a_{ii}$ implies $a_{ii} = 0$.

(Note that a matrix A must be square if $A^T = A$ or $A^T = -A$.)

Examples: Symmetric Matrices

Let $A = \begin{bmatrix} 2 & -3 & 5 \\ -3 & 6 & 7 \\ 5 & 7 & -8 \end{bmatrix}, B = \begin{bmatrix} 0 & 3 & -4 \\ -3 & 0 & 5 \\ 4 & -5 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

a. By inspection, the symmetric elements in A are equal, or $A^T = A$. Thus, A is symmetric.

b. The diagonal elements of B are 0 and symmetric elements are negatives of each other, or $B^T = -B$. Thus, B is skew-symmetric.

c. Because C is not square, C is neither symmetric nor skew-symmetric.

Matrix Multiplication

- Matrices A and B can be multiplied if:

[r x c] and [s x d]

c = s

$$\begin{array}{cc} & c \\ r & \begin{matrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \\ A_{10} & A_{11} & A_{12} \end{matrix} \end{array} \times \begin{array}{cc} d & \\ \begin{matrix} B_{13} & B_{14} \\ B_{15} & B_{16} \\ B_{17} & B_{18} \end{matrix} & s \end{array} = \mathbf{r} \times \mathbf{d} \text{ matrix}$$

- The resulting matrix will have the dimensions:

[r x c] and [s x d]

r x d

Computation: A x B = C

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad [2 \text{ x } 2]$$

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \quad [2 \text{ x } 3]$$

$$C = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \end{bmatrix}$$

[2 x 3]

Matrix multiplication

- Multiplication method:

Sum over product of respective <u>rows</u> and <u>columns</u>

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \text{ X } \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

A

Define output matrix

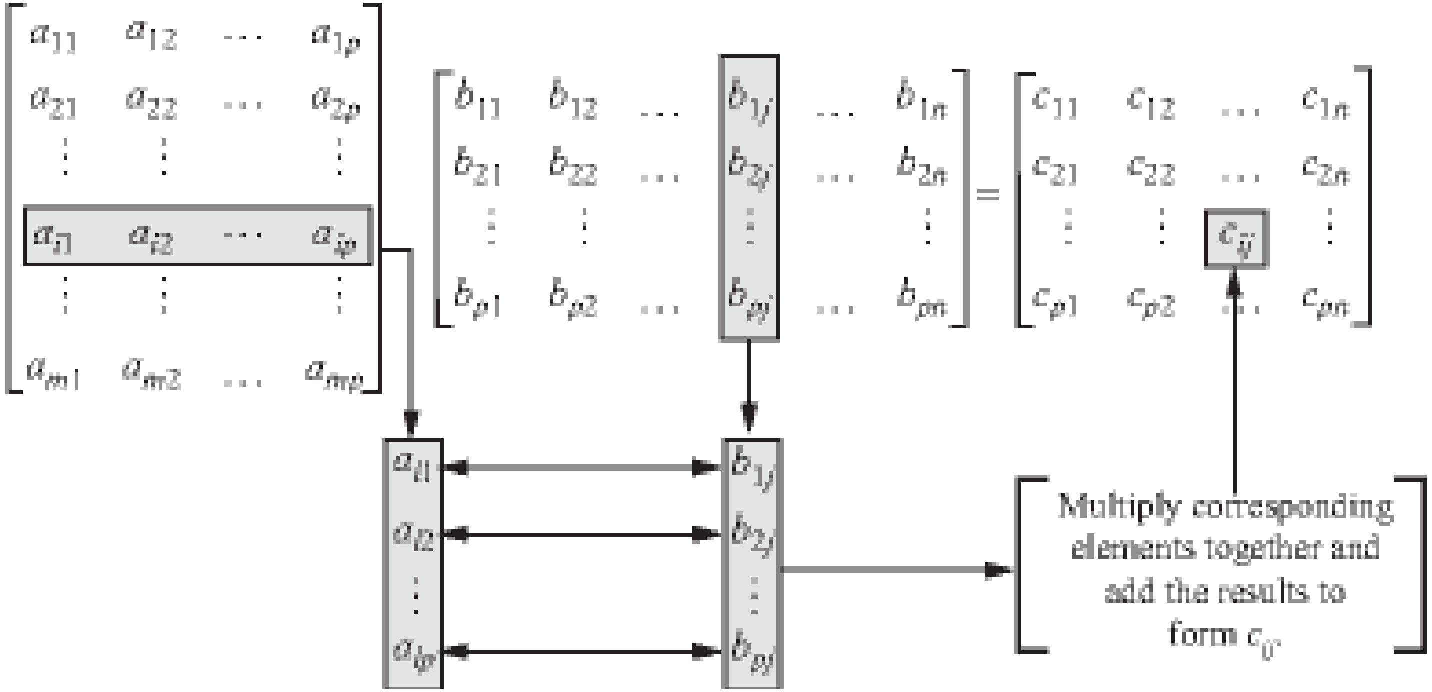
$$= \begin{bmatrix} (1\times 2)+(0\times 3) & (1\times 1)+(0\times 1) \\ (2\times 2)+(3\times 3) & (2\times 1)+(3\times 1) \end{bmatrix}$$

$$= \begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{13} & \mathbf{5} \end{pmatrix}$$

- Matlab does all this for you!
- Simply type: C = A * B

Matrix multiplication

Definition 1.7 (Matrix product). Let $A = [a_{ij}]$ be a matrix of size $m \times p$ and $B = [b_{jk}]$ be a matrix of size $p \times n$ (i.e., the number of columns of A equals the number of rows of B). Then AB is the $m \times n$ matrix $C = [c_{ik}]$ whose (i, j)th element is defined by the formula



$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj} = a_{i1} b_{1j} + \cdots + a_{ip} b_{pj}.$$

Matrix multiplication: Ex 1

$$[7, -4, 5]\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = 7(3) + (-4)(2) + 5(-1) = 21 - 8 - 5 = 8$$

$$[6, -1, 8, 3]\begin{bmatrix} 4 \\ -9 \\ -2 \\ 5 \end{bmatrix} = 24 + 9 - 16 + 15 = 32$$

Matrix multiplication: Ex 2

Find AB where $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & -4 \\ 5 & -2 & 6 \end{bmatrix}$.

Because A is 2×2 and B is 2×3, the product AB is defined and AB is a 2×3 matrix. To obtain the first row of the product matrix AB, multiply the first row $[1, 3]$ of A by each column of B,

$$\begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ 6 \end{bmatrix}$$

respectively. That is,

$$AB = \begin{bmatrix} 2+15 & 0-6 & -4+18 \\ & & \end{bmatrix} = \begin{bmatrix} 17 & -6 & 14 \\ & & \end{bmatrix}$$

To obtain the second row of AB, multiply the second row $[2, -1]$ of A by each column of B. Thus,

$$AB = \begin{bmatrix} 17 & -6 & 14 \\ 4-5 & 0+2 & -8-6 \end{bmatrix} = \begin{bmatrix} 17 & -6 & 14 \\ -1 & 2 & -14 \end{bmatrix}$$

Matrix multiplication is not commutative

Suppose $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 0 & -2 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 5+0 & 6-4 \\ 15+0 & 18-8 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 15 & 10 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 5+18 & 10+24 \\ 0-6 & 0-8 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ -6 & -8 \end{bmatrix}$$

The above example shows that matrix multiplication is not commutative—that is, in general,

$$AB \neq BA$$

Properties of Matrix multiplication

a. Distributive law: $A(B+C) = AB + AC$

b. Associative property: $A(BC) = (AB)C$

c. $AI = IA = A$, where I is the identity matrix of appropriate size and A is a square matrix.

d. $A^{m+n} = A^m A^n = A^n A^m$ where $A^n = \overbrace{A \times A \times \cdots \times A}^{n \text{ times}}$

Vector Products

Two vectors:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Inner product = scalar

$$\mathbf{x}^T \mathbf{y} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3 = \sum_{i=1}^{3} x_i y_i$$

Outer product = matrix

$$xy^T = \begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} = \begin{vmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 \\ x_2 y_1 & x_2 y_2 & x_2 y_3 \\ x_3 y_1 & x_3 y_2 & x_3 y_3 \end{vmatrix}$$

Outer product XY^T is a matrix

(nx1) (1xn)

Properties of Matrix transpose

The following properties hold for transpose and Hermitian operations. The proofs are left as an exercise.

a. $(A^T)^T = A$ or $(A^H)^H = A$

b. $(AB)^T = B^T A^T$ or $(AB)^H = B^H A^H$

Symmetric matrix: An $n\times n$ real matrix A is said to be symmetric if $A^T = A$.

Skew-symmetric matrix: An $n\times n$ real matrix A is said to be skew-symmetric if $A^T = -A$.

Trace of a matrix: The trace of a square matrix is the sum of the diagonal elements of the matrix, that is:

$$\text{Trace}(A) = \sum_{i=1}^{n} a_{ii}$$

The MATLAB command to compute the trace of matrix A is **trace(A)**.

Properties of Matrix transpose

$$\text{If } A = \begin{bmatrix} 4 & 4 & -5 \\ 3 & 4 & 1 \\ 6 & 7 & -3 \end{bmatrix} \text{ then:}$$

$$\text{Trace}(A) = \sum_{i=1}^{n} a_{ii} = 4 + 4 - 3 = 5$$

We state the properties of the trace of a matrix without proof. The proofs are left as an exercise. The following properties hold for square matrices:

a. $\text{Trace}(A \pm B) = \text{Trace}(A) \pm \text{Trace}(B)$
b. $\text{Trace}(A^T) = \text{Trace}(A)$
c. $\text{Trace}(AB) = \text{Trace}(B^T A^T)$
d. $\text{Trace}(\alpha A) = \alpha \text{Trace}(A)$
e. $\text{Trace}(AB) = \text{Trace}(BA)$

Examples

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & -4 & -4 \\ 5 & 6 & 7 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -5 & 1 \\ 0 & 3 & -2 \\ 1 & 2 & -4 \end{bmatrix}$$

diagonal of $A = \{1, -4, 7\}$ and $\operatorname{tr}(A) = 1 - 4 + 7 = 4$

diagonal of $B = \{2, 3, -4\}$ and $\operatorname{tr}(B) = 2 + 3 - 4 = 1$

Moreover,

$$\operatorname{tr}(A+B) = 3 - 1 + 3 = 5, \qquad \operatorname{tr}(2A) = 2 - 8 + 14 = 8, \qquad \operatorname{tr}(A^T) = 1 - 4 + 7 = 4$$

$$\operatorname{tr}(AB) = 5 + 0 - 35 = -30, \qquad \operatorname{tr}(BA) = 27 - 24 - 33 = -30$$

As expected $\operatorname{tr}(A+B) = \operatorname{tr}(A) + \operatorname{tr}(B)$, $\quad \operatorname{tr}(A^T) = \operatorname{tr}(A)$, $\quad \operatorname{tr}(2A) = 2\operatorname{tr}(A)$

Furthermore, although $AB \neq BA$, the traces are equal.

Identity matrix

The n-square *identity* or *unt* matrix, denoted by I_n, or simply I, is the n-square matrix with 1's on the diagonal and 0's elsewhere. The identity matrix I is similar to the scalar 1 in that, for any n-square matrix A,

$$\boxed{I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.}$$

$$AI = IA = A$$

More generally, if B is an $m \times n$ matrix, then $BI_n = I_m B = B$.

For any scalar k $\quad (kI)A = k(IA) = kA$

Remark

The *Kronecker delta function* δ_{ij} is defined by $\quad \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

Thus, the identity matrix may be defined by $I = [\delta_{ij}]$.

Identity matrix

Identity matrix: A diagonal matrix with all diagonal elements equal to one.

Identity matrices of different sizes

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The MATLAB® command to create a $N \times N$ identity matrix is **eye(N)**.

Identity matrix

Worked example
$\boldsymbol{A}\,\boldsymbol{I_3} = \boldsymbol{A}$
for a 3x3 matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0+0 & 0+2+0 & 0+0+3 \\ 4+0+0 & 0+5+0 & 0+0+6 \\ 7+0+0 & 0+8+0 & 0+0+9 \end{bmatrix}$$

Power of Matrices, Polynomials in Matrices

Let A be an n-square matrix over a field K. *Powers* of A are defined as follows:

$$A^2 = AA, \qquad A^3 = A^2A, \qquad \ldots, \qquad A^{n+1} = A^nA, \qquad \ldots, \qquad \text{and} \qquad A^0 = I$$

Polynomials in the matrix A are also defined. Specifically, for any polynomial

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

where the a_i are scalars in K, $f(A)$ is defined to be the following matrix:

$$f(A) = a_0I + a_1A + a_2A^2 + \cdots + a_nA^n$$

Power of Matrices, Polynomials in Matrices: Example

Suppose $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$.

Then $A^2 = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix}$ and $A^3 = A^2A = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} -11 & 38 \\ 57 & -106 \end{bmatrix}$

Suppose $f(x) = 2x^2 - 3x + 5$ and $g(x) = x^2 + 3x - 10$. Then

$$f(A) = 2\begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & -18 \\ -27 & 61 \end{bmatrix}$$

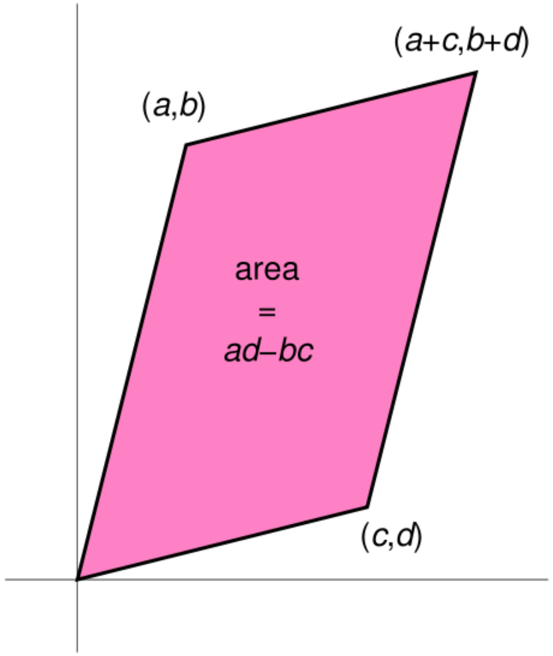
$$g(A) = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix} + 3\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} - 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, A is a zero of the polynomial $g(x)$.

Determinants

- Determinants can only be found for square matrices.
- For a 2x2 matrix A, **det(A) = ad-bc.** Lets have at closer look at that:

$$\det(A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$



- In Matlab: ***det(A)*** = `det(A)`

Invertible (Nonsingular) Matrices

A square matrix A is said to be *invertible* or *nonsingular* if there exists a matrix B such that

$$AB = BA = I$$

where I is the identity matrix. Such a matrix B is unique. That is, if $AB_1 = B_1A = I$ and $AB_2 = B_2A = I$, then

$$B_1 = B_1I = B_1(AB_2) = (B_1A)B_2 = IB_2 = B_2$$

We call such a matrix B the *inverse* of A and denote it by A^{-1}. Observe that the above relation is symmetric; that is, if B is the inverse of A, then A is the inverse of B.

Suppose that $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$.

$$AB = \begin{bmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 6-5 & 15-15 \\ -2+2 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, A and B are inverses.

Invertible Matrices

Now suppose A and B are invertible. Then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. More generally, if A_1, A_2, …, A_k are invertible, then their product is invertible and

$$(A_1 A_2 \ldots A_k)^{-1} = A_k^{-1} \ldots A_2^{-1} A_1^{-1}$$

the product of the inverses in the reverse order.

Matrix inverse

- **Definition.** A matrix ***A*** is called **nonsingular** or **invertible** if there exists a matrix ***B*** such that:

$$\boxed{A\,B = B\,A = I_n}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}+\frac{1}{3} & \frac{-1}{3}+\frac{1}{3} \\ \frac{-2}{3}+\frac{2}{3} & \frac{1}{3}+\frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- **Notation.** A common notation for the inverse of a matrix ***A*** is $\boldsymbol{A}^{-1}$. So:

$$\boxed{A\,A^{-1} = A^{-1}\,A = I_n\,.}$$

- The inverse matrix is unique when it exists. So if ***A*** is invertible, then $\boldsymbol{A}^{-1}$ is also invertible and then $(A^T)^{-1} = (A^{-1})^T$

• In Matlab: $\boldsymbol{A}^{-1}$ = **inv(*A*)**

•Matrix division: A/B= $\boldsymbol{A}*\boldsymbol{B}^{-1}$

Matrix inverse

- **For a XxX square matrix:** $A = \begin{pmatrix} x_{1,1} & \dots & x_{1,j} \\ \vdots & \ddots & \vdots \\ x_{i,1} & \dots & x_{i,j} \end{pmatrix}$

- **The inverse matrix is:** $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} \operatorname{cof}(A, x_{1,1}) & \dots & \operatorname{cof}(A, x_{1,j}) \\ \vdots & \ddots & \vdots \\ \operatorname{cof}(A, x_{i,1}) & \dots & \operatorname{cof}(A, x_{i,j}) \end{pmatrix}^T$

- **E.g.: 2x2 matrix** $\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$

Inverse

Let A be an arbitrary 2×2 matrix, say $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We want to derive a formula for A^{-1}, the inverse of A. Specifically, we seek $2^2 = 4$ scalars, say x_1, y_1, x_2, y_2, such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} ax_1 + by_1 & ax_2 + by_2 \\ cx_1 + dy_1 & cx_2 + dy_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Setting the four entries equal to the corresponding entries in the identity matrix yields four equations, which can be partitioned into two 2×2 systems as follows:

$$\begin{aligned} ax_1 + by_1 &= 1, \qquad & ax_2 + by_2 &= 0 \\ cx_1 + dy_1 &= 0, \qquad & cx_2 + dy_2 &= 1 \end{aligned}$$

Inverse

Suppose we let $|A| = ad - bc$ (called the *determinant* of A). Assuming $|A| \neq 0$, we can solve uniquely for the above unknowns x_1, y_1, x_2, y_2, obtaining

$$x_1 = \frac{d}{|A|}, \qquad y_1 = \frac{-c}{|A|}, \qquad x_2 = \frac{-b}{|A|}, \qquad y_2 = \frac{a}{|A|}$$

Accordingly,

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d/|A| & -b/|A| \\ -c/|A| & a/|A| \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

a. Interchange the two elements on the diagonal.

b. Take the negatives of the other two elements.

c. Multiply the resulting matrix by $1/|A|$ or, equivalently, divide each element by $|A|$.

Example

Find the inverse of $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$.

First evaluate $|A| = 2(5) - 3(4) = 10 - 12 = -2$. Because $|A| \neq 0$, the matrix A is invertible and

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix}$$

Now evaluate $|B| = 1(6) - 3(2) = 6 - 6 = 0$. Because $|B| = 0$, the matrix B has no inverse.

Remark: The above property that a matrix is invertible if and only if A has a nonzero determinant is true for square matrices of any order.

Orthogonal Matrices

A real matrix A is *orthogonal* if $A^T = A^{-1}$ — that is, if $AA^T = A^TA = I$. Thus, A must necessarily be square and invertible.

EXAMPLE

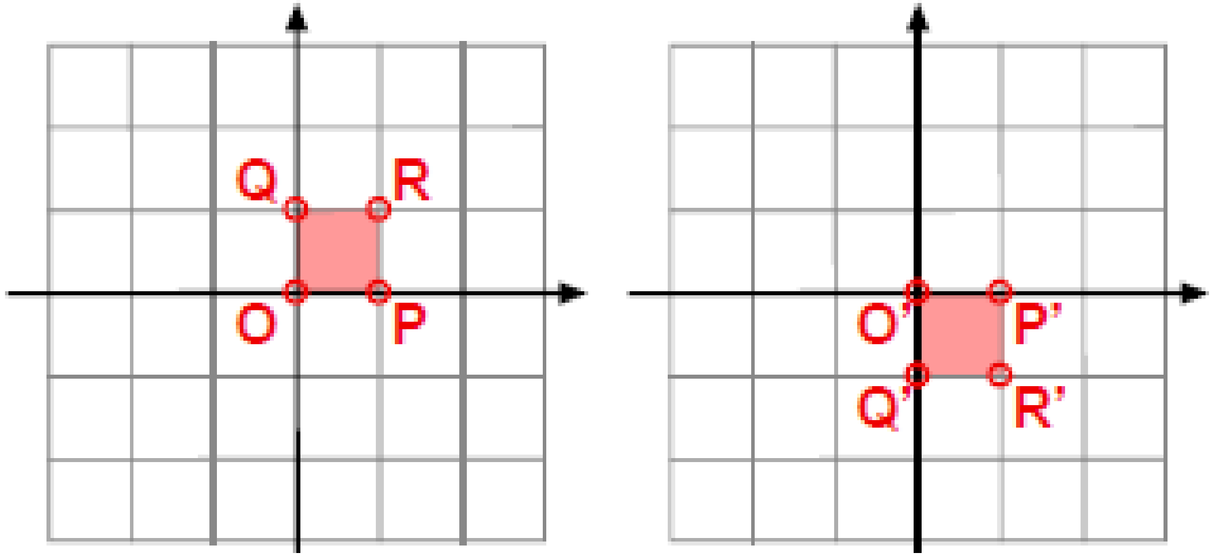
Let: $A = \begin{bmatrix} \frac{1}{9} & \frac{8}{9} & -\frac{4}{9} \\ \frac{4}{9} & -\frac{4}{9} & -\frac{7}{9} \\ \frac{8}{9} & \frac{1}{9} & \frac{4}{9} \end{bmatrix}$. Multiplying A by A^T yields I; that is, $AA^T = I$.

Transformations: Reflection

The general forms of reflections in the x–axis and y–axis are:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

e.g. an x–axis reflection is given by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



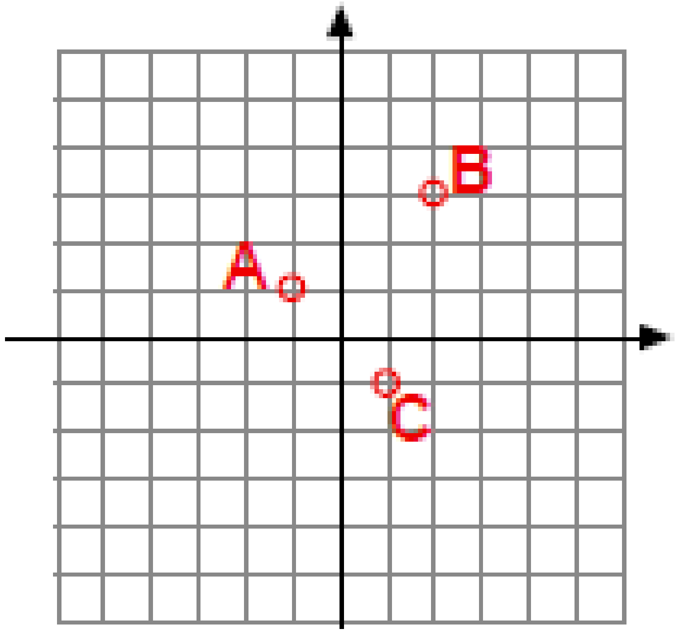
Matrix multiplication as a transformation

- A coordinate system can be used to uniquely identify points in m-dimensional space as column vectors ($m \times 1$ matrices).
- e.g. in 2-dimensional space the points **A**, **B**, and **C** are represented as:

$$\mathbf{A} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



Matrix multiplication as a transformation

Suppose we want to "transform" a point $\begin{bmatrix} x \\ y \end{bmatrix}$ to get a new point $\begin{bmatrix} x_{new} \\ y_{new} \end{bmatrix}$ using the following equations:

$$2x - y = x_{new}$$
$$x + y = y_{new}$$

We can write this nicely as a matrix multiplication:

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_{new} \\ y_{new} \end{bmatrix}$$

$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$ is our "transformation matrix"

Matrix multiplication as a transformation

First transform point A=$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ go get **A'**

We can write this nicely as a matrix multiplication:

$$\mathbf{A'} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \times -1 + 1 \times -1 \\ 1 \times -1 + 1 \times 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

Row 1, column 1 of our RHS matrix is the dot product of
- row 1 of our first matrix with
- column 1 of our second matrix

Row 2, column 1 of our RHS matrix is the dot product of
- row 2 of our first matrix with
- column 1 of our second matrix

Matrix multiplication as a transformation

Similarly for B and C

$$B' = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{B} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$C' = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{C} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

Row 1, column 1 of our RHS matrix is the dot product of
- row 1 of our first matrix with
- column 1 of our second matrix

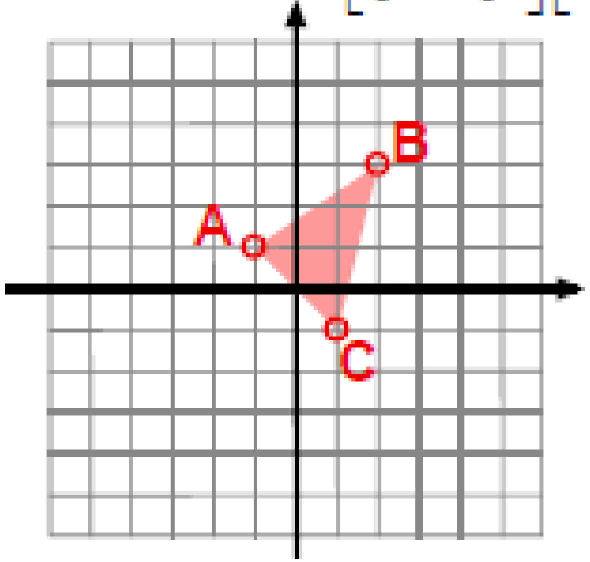
Row 2, column 1 of our RHS matrix is the dot product of
- row 2 of our first matrix with
- column 1 of our second matrix

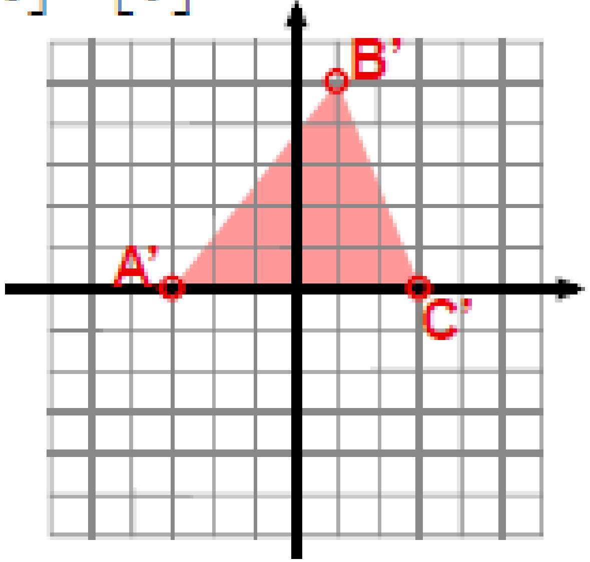
Matrix transformation

$$A' = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

$$B' = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$C' = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

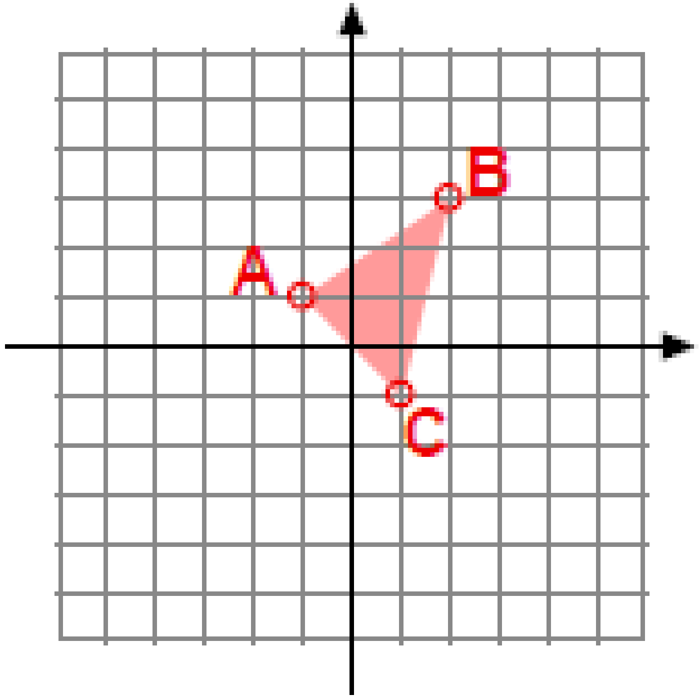




Even more efficiently…

- The position of n points in space can be characterised by a grouping of n column vectors forming an $m \times n$ **matrix**.
- e.g. in 2-dimensional space the triangle with vertices **A**, **B**, and **C** can be represented by the matrix:

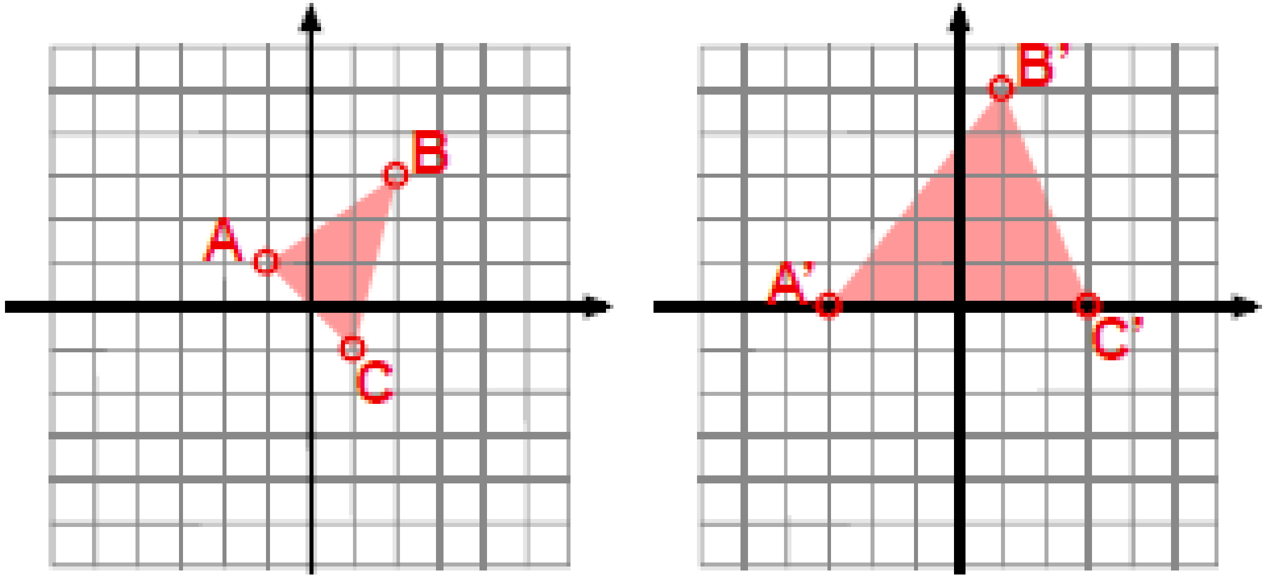
$$\begin{array}{c}\begin{bmatrix} -1 & 2 & 1 \\ 1 & 3 & -1 \end{bmatrix} \\ \begin{matrix} A & \quad B & \quad C \end{matrix}\end{array}$$



Matrix transformation

So very succinctly in matrix form:

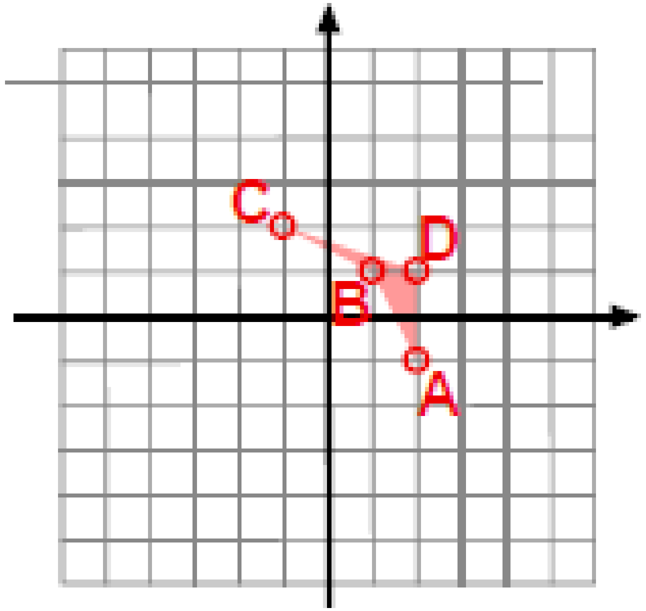
$$\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}\begin{array}{c}\begin{bmatrix}-1 & 2 & 1\\ 1 & 3 & -1\end{bmatrix}\\ \begin{array}{ccc}A & B & C\end{array}\end{array}=\begin{array}{c}\begin{bmatrix}-3 & 1 & 3\\ 0 & 5 & 0\end{bmatrix}\\ \begin{array}{ccc}A' & B' & C'\end{array}\end{array}$$

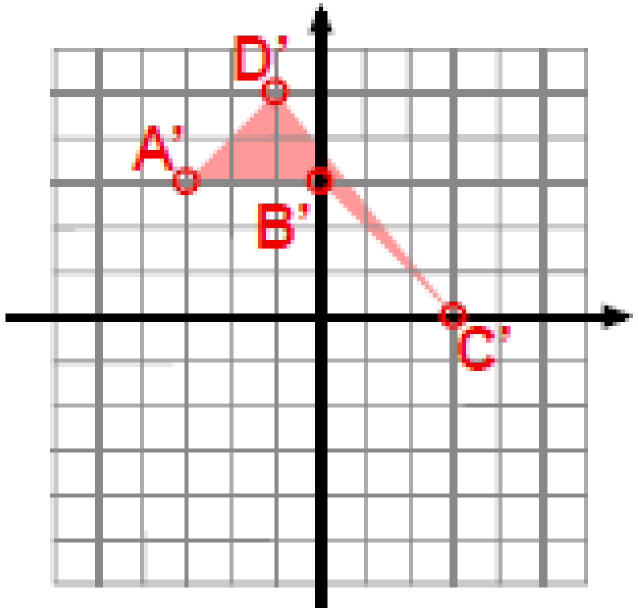


Example 1

$$\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 & 2 \\ -1 & 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 3 & -1 \\ 3 & 3 & 0 & 5 \end{bmatrix}$$

Reversed ordering of vertices shows the shape has been "inverted"





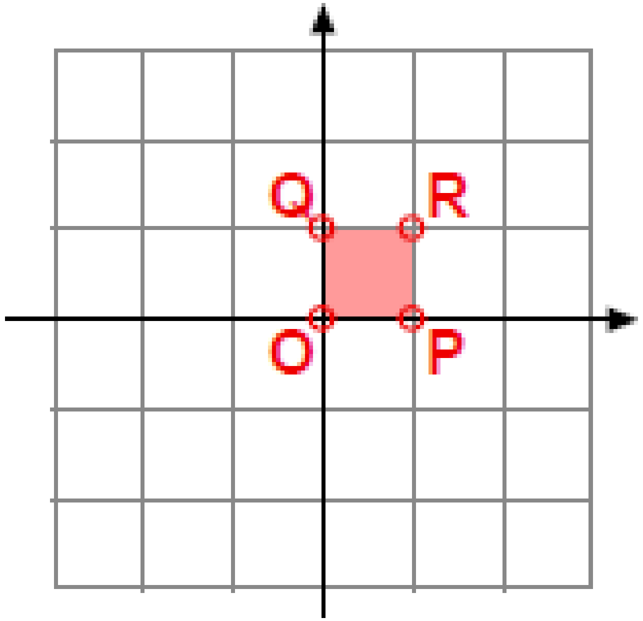
The unit square

It is instructive to look at the effect of transformations on a simple shape such as the unit square.

The matrix for the unit square is the 2×4 matrix

$$\begin{array}{c} \begin{array}{cccc} O & P & Q & R \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Notice how each column of the matrix contains the coordinates of a point on the square



Transforming the unit square

- Transforming the unit square gives information about the transformation matrix

$$\begin{array}{cc} & \begin{array}{cccc} O & P & Q & R \end{array} \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} O' & P' & Q' & R' \end{array} \\ \begin{bmatrix} 0 & a_{11} & a_{12} & a_{11}+a_{12} \\ 0 & a_{21} & a_{22} & a_{21}+a_{22} \end{bmatrix} \end{array}$$

- i.e. transforming the unit square results in [zeros, original matrix, sum of columns]

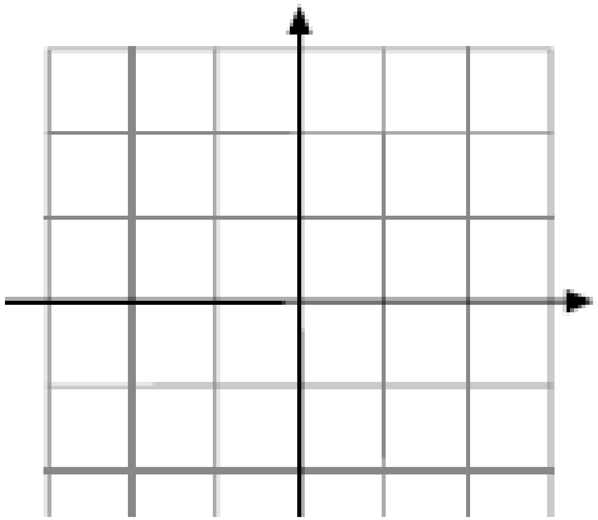
$$\begin{array}{cc} & \begin{array}{cccc} O & P & Q & R \end{array} \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} O' & P' & Q' & R' \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 4 & 7 \end{bmatrix} \end{array}$$

Exercise: Transforming the unit square

Transform the unit square by $\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$

$$\begin{array}{cc} & \begin{array}{cccc} O & P & Q & R \end{array} \\ \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \end{array}$$

And sketch the result

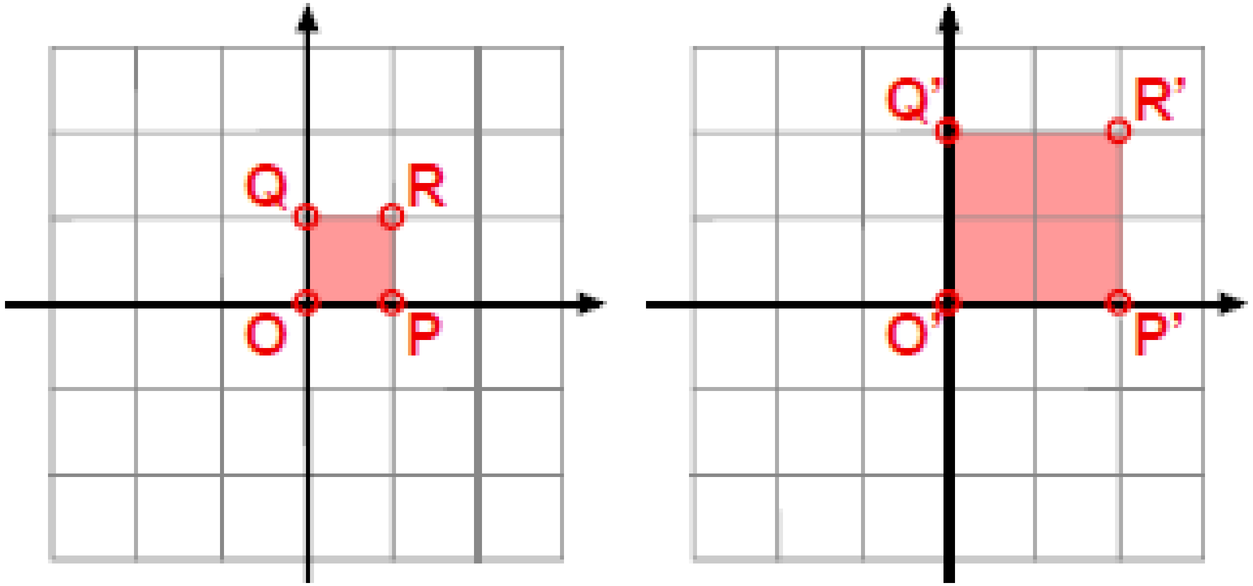


Transformations: uniform scaling

The general form of uniform scaling by k is:

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

e.g. uniform scaling by 2 is given by $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

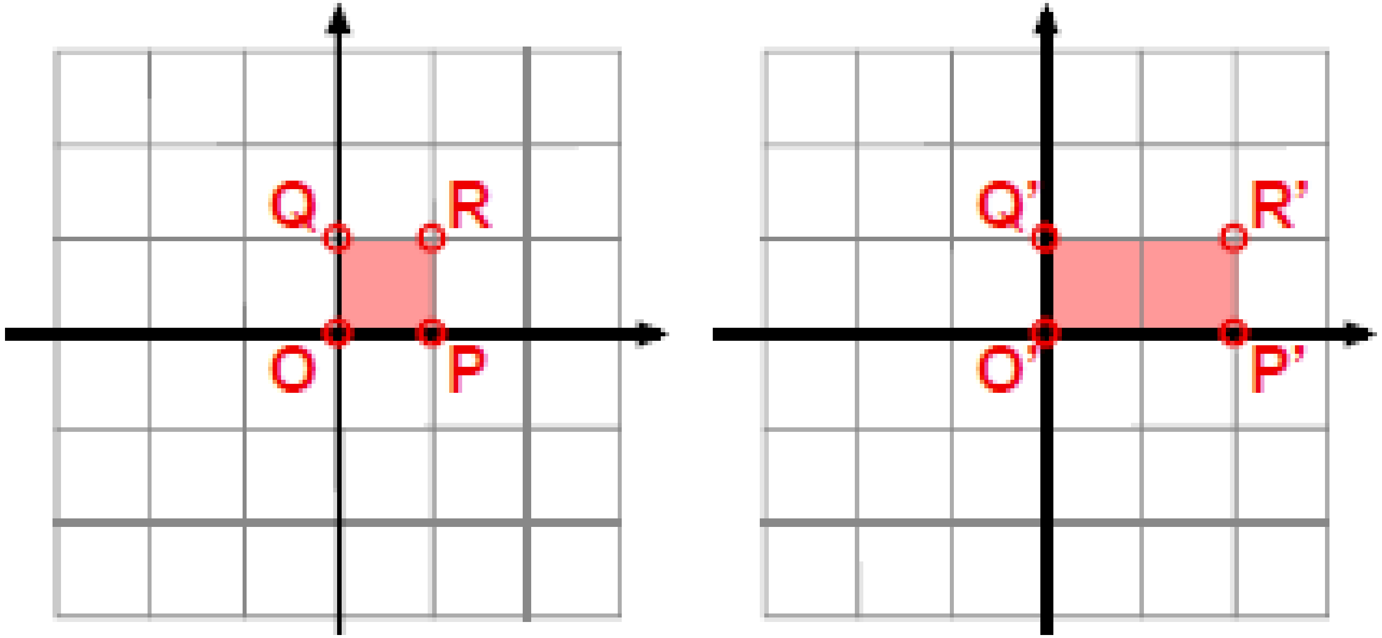


Transformation: x stretch

The general form of a stretch by k in the x–direction is:

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

e.g. x–direction stretch by 2 is given by $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

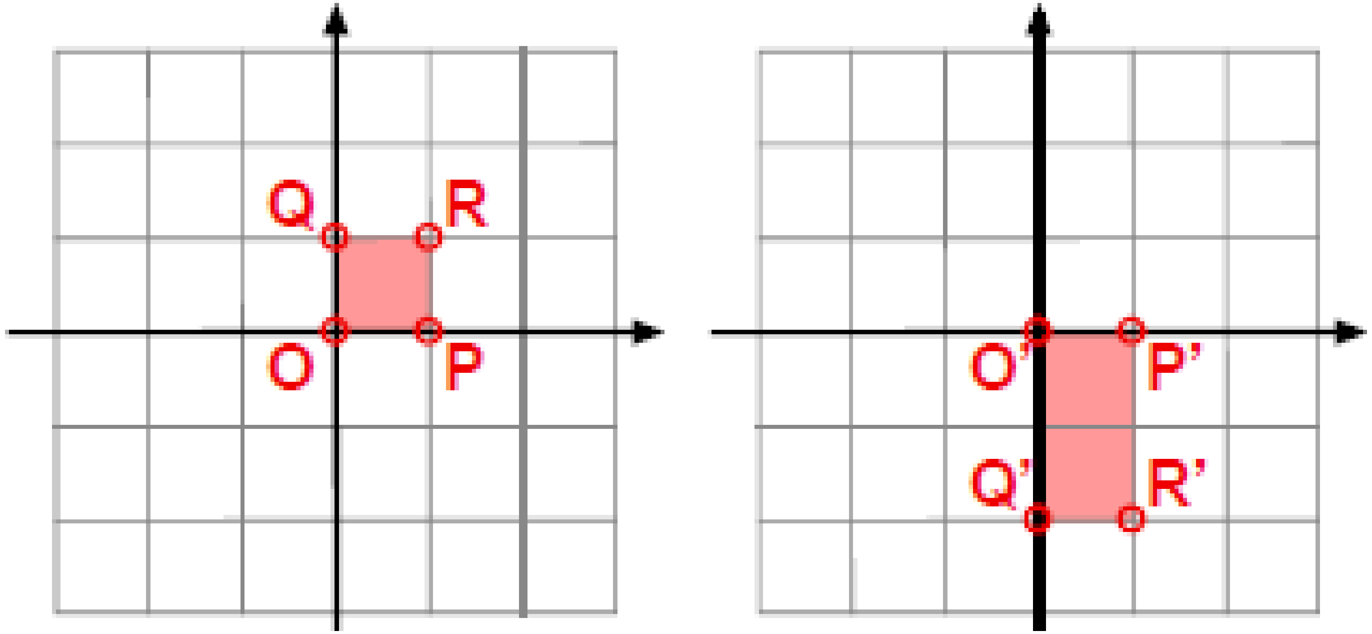


Transformation: y stretch

The general form of a stretch by k in the y–direction is:

$$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

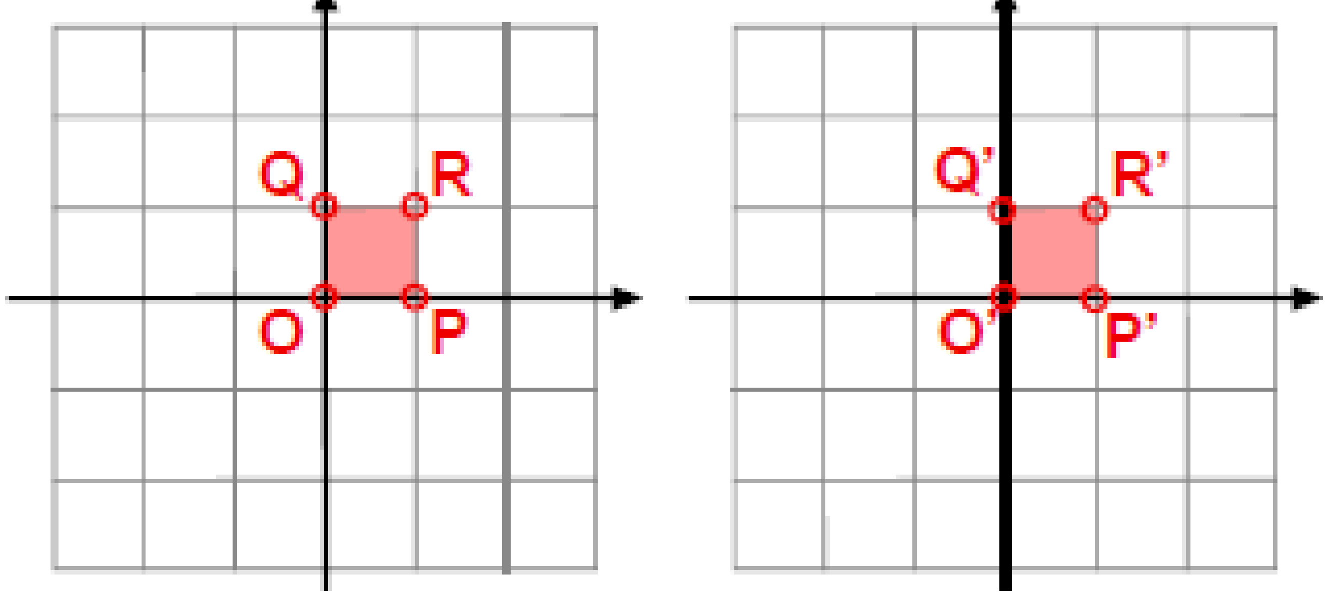
e.g. y–direction stretch by -2 is given by $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$



Transformations: the identity

Recall the general form of uniform scaling by k is:
$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

What happens if we “scale” by 1? i.e.
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

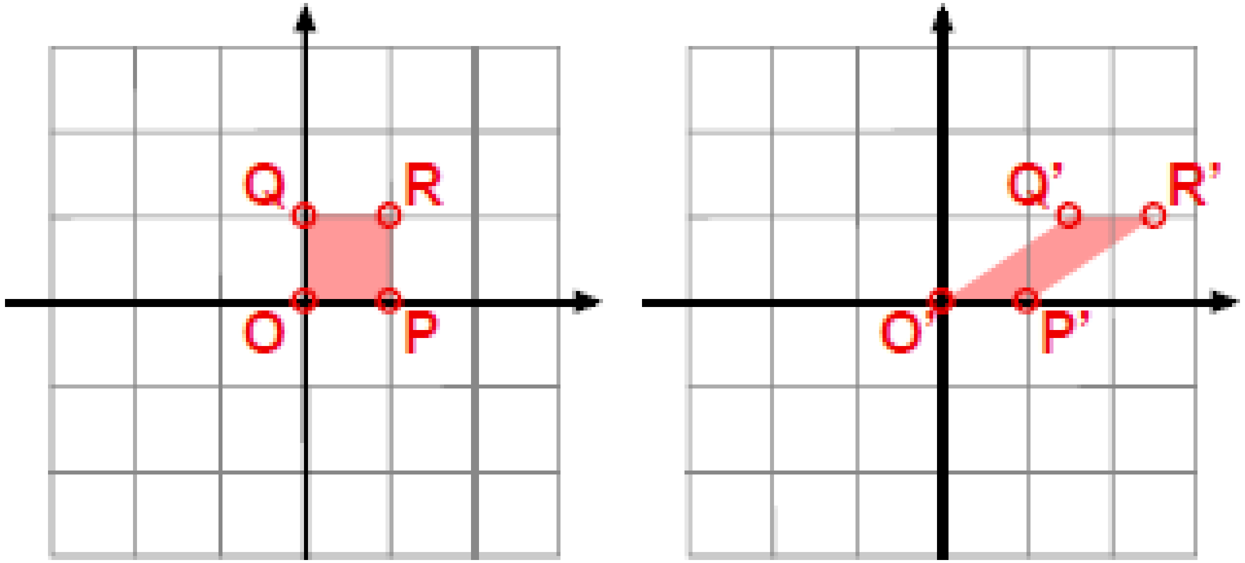


Transformations: x shear

The general form of a shear by k in the x–direction is:

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

e.g. x–direction shear by 1.5 is given by $\begin{bmatrix} 1 & 1.5 \\ 0 & 1 \end{bmatrix}$

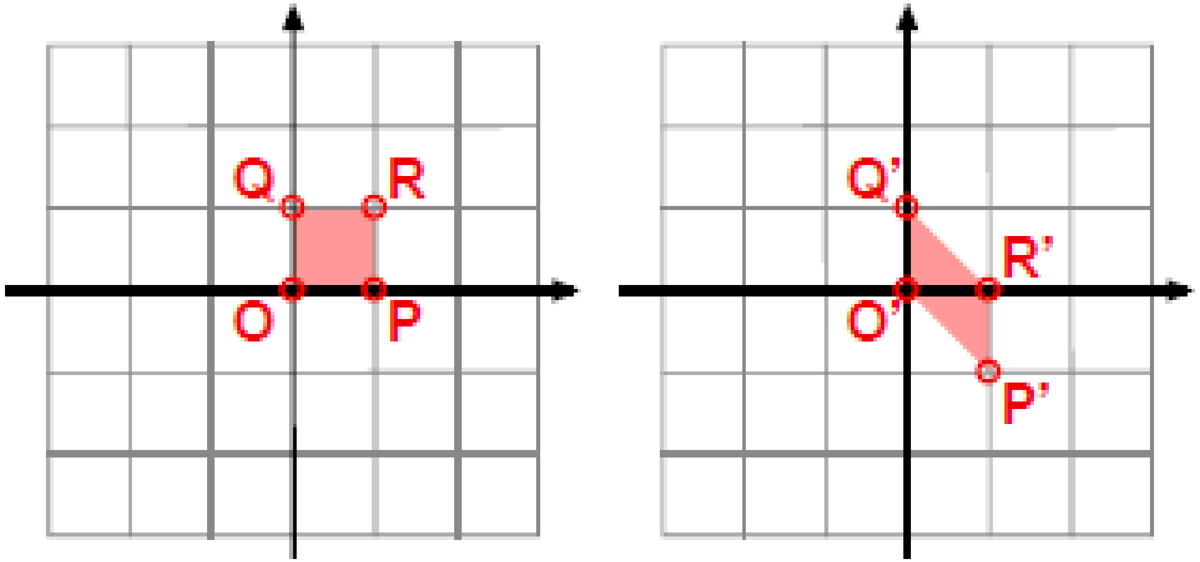


Transformations: y shear

The general form of a shear by k in the y–direction is:

$$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$

e.g. y–direction shear by -1 is given by $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$



Triangular matrices

The x shear and y shear transformations are examples of triangular matrices. A 2×2 matrix is

- upper **triangular** if the entries below the leading diagonal are all zero

- lower **triangular** if the entries above the leading diagonal are all zero

e.g. Upper triangular

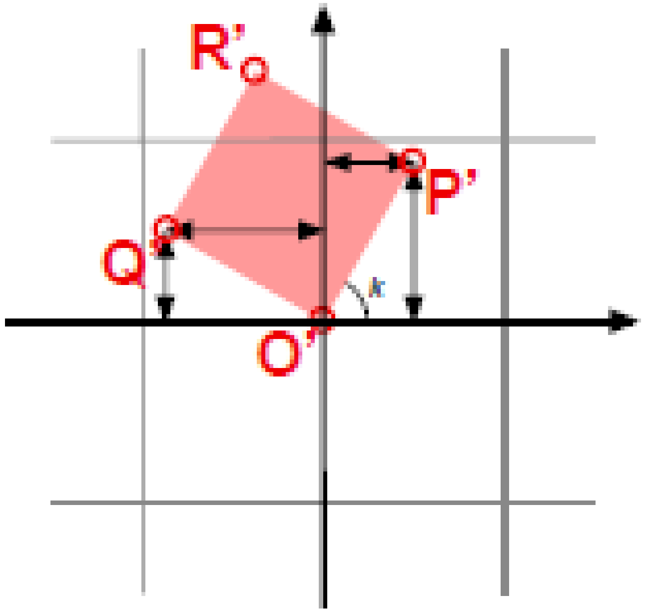
$$\begin{bmatrix} 1 & 1.5 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

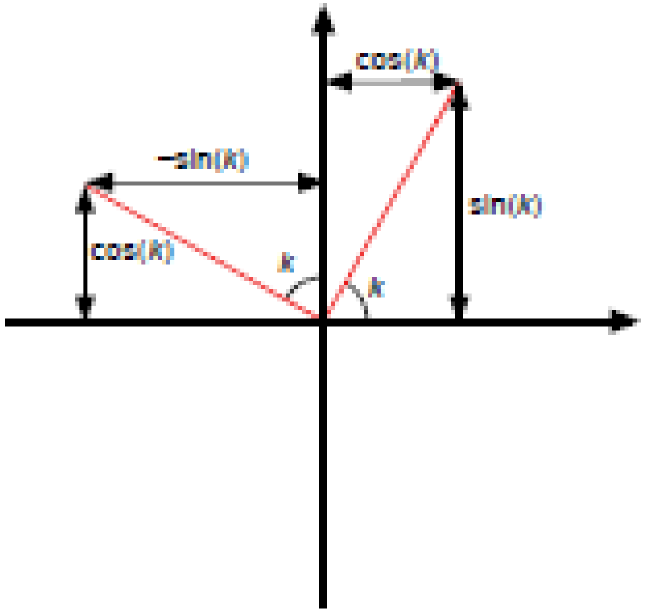
Lower triangular

Transformations: Rotation

The general form of a rotation by k about the origin is:

$$\begin{bmatrix} \cos(k) & -\sin(k) \\ \sin(k) & \cos(k) \end{bmatrix}$$





Exercise

Find the matrix representing rotation of 90°

$$\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Find the matrix representing rotation of -30° (i.e. clockwise)

$$\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Combined transformations

- If the matrices **A** and **B** represent two transformations, then the matrix product **AB** represents the combined transformation of *first* applying **B** and *then* applying **A**.
- i.e. let the matrix **U** represent the unit square.
  Then **BU** represents the transformation **B** applied to the unit square.
  Now **A**(**BU**) represents the transformation **A** applied to the unit square transformed by **B**.
  This sequence of transformations is the same as the combined transformation (**AB**) applied to **U** (matrix multiplication is associative)
  ABU = **A**(**BU**) = (**AB**)**U**

Combined transformations

Let the matrices A and B represent x–shear by 1 and y–stretch by 2 transformations

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$BU = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$A(BU) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$$(AB)U = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

Combined transformations

Note that combined transformations are not commutative.
i.e. $\mathbf{AB} \neq \mathbf{BA}$

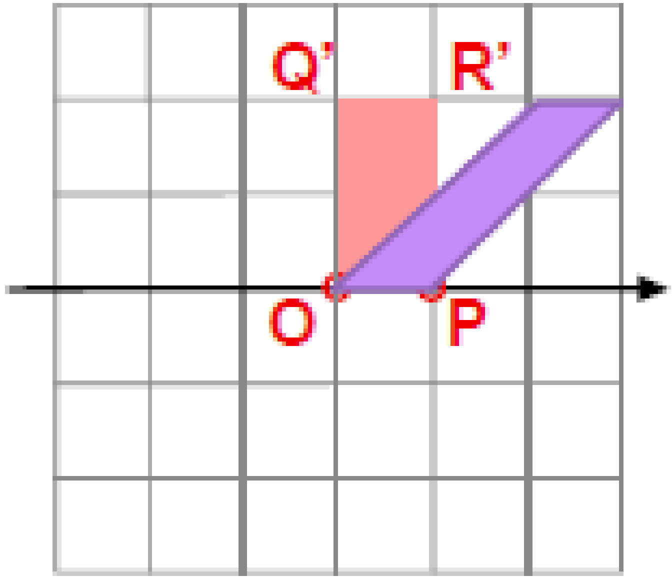
$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}, \mathbf{BA} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

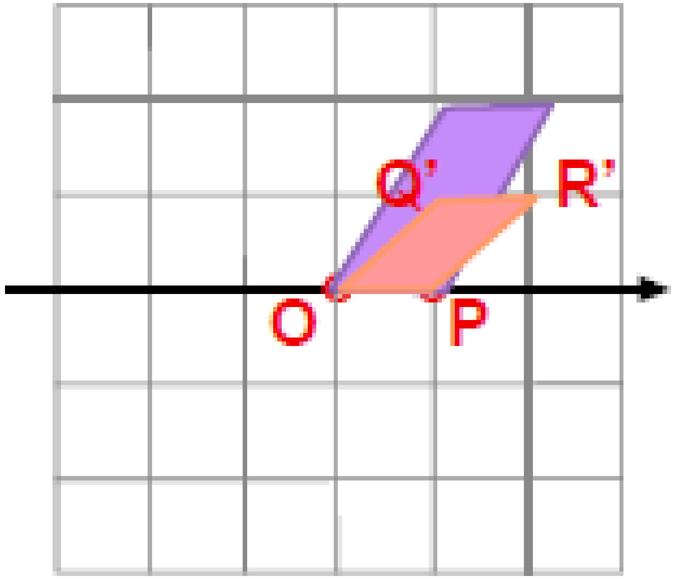
$$\mathbf{ABU} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 2 \end{bmatrix}, \mathbf{BAU} = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

AB≠BA

$$\mathbf{AB}=\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}=\begin{bmatrix}1&2\\0&2\end{bmatrix},\mathbf{BA}=\begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}1&1\\0&1\end{bmatrix}=\begin{bmatrix}1&1\\0&2\end{bmatrix}$$



Expand vertically then shear



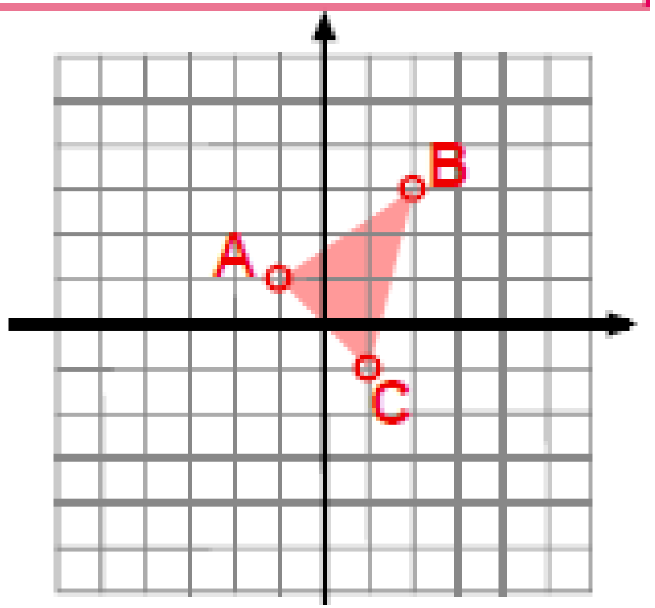
Shear then expand vertically

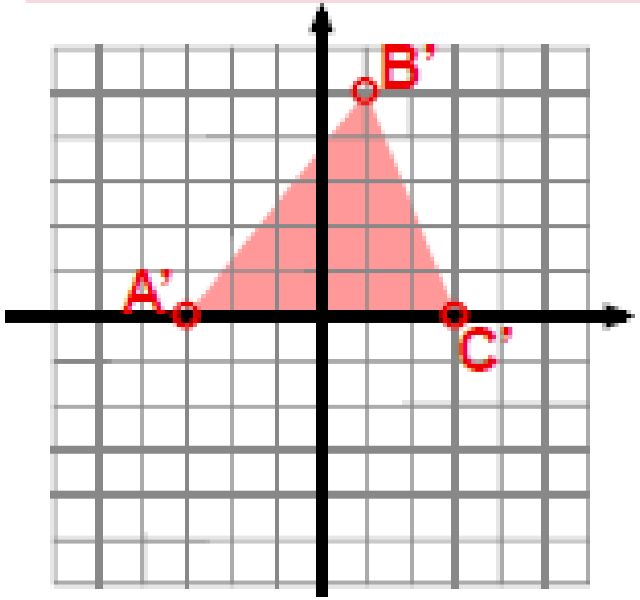
Determinants give scale

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 1 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 3 \\ 0 & 5 & 0 \end{bmatrix}$$

Area = 0.5|(c-a)x(b-a)|
=0.5|2i-2j x 3i+2j|
=0.5|10k|
= 5 units

Area = 0.5|(c'-a')x(b'-a')|
=0.5|6i x 4i+5j|
=0.5|30k|
= 15 units

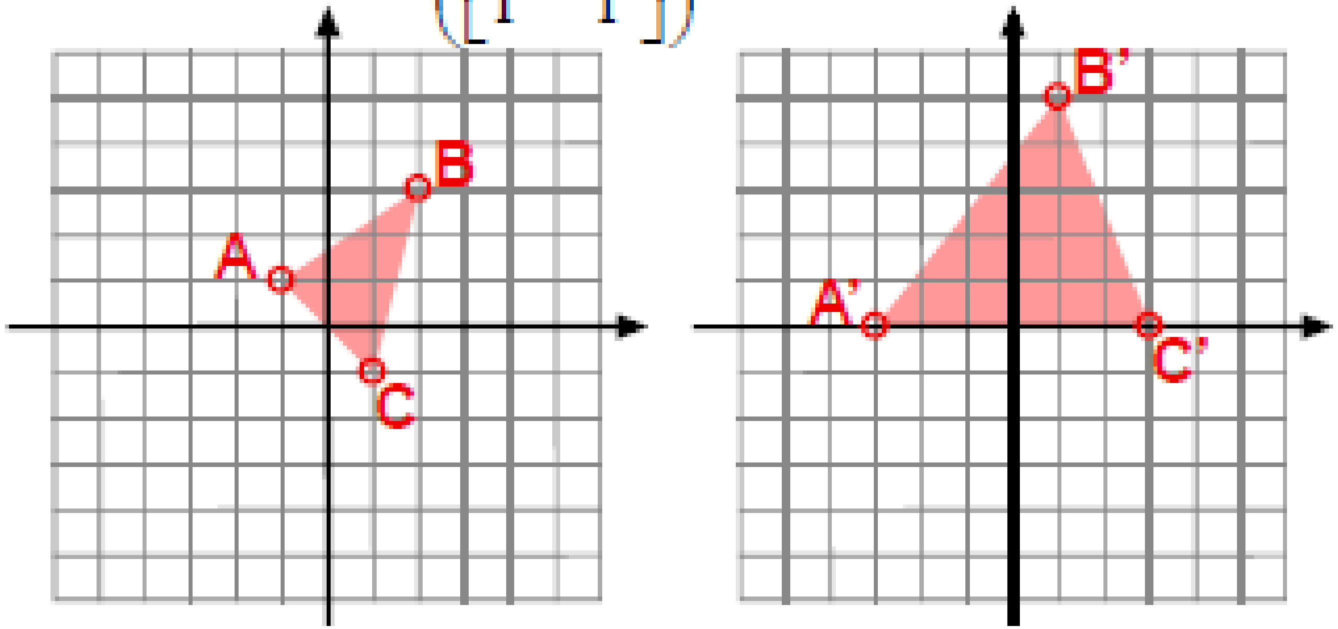




Determinant gives scale

- Note: the determinant of the transformation matrix gives the scale of the area change.
- e.g. area [A, B, C] = 5, area [A´, B´, C´] = 15

$$\det\left(\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\right) = 2 + 1 = 3$$



Example

Transform the shape with vertices

$$\mathbf{A} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \ \mathbf{D} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

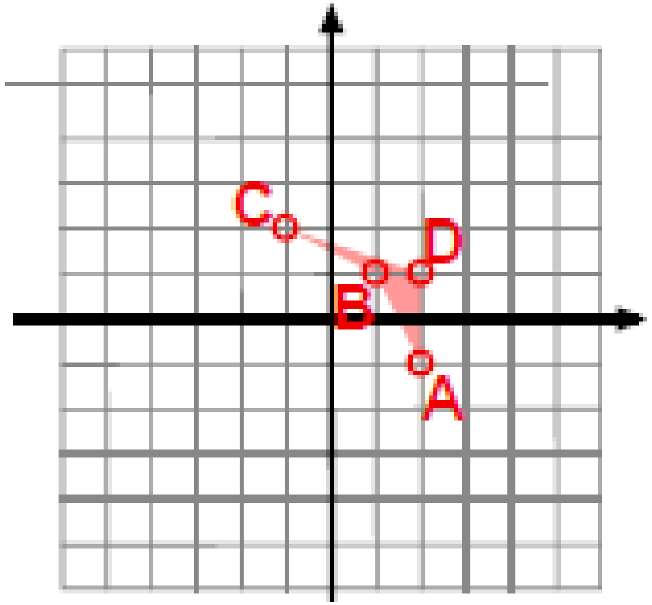
by the matrix

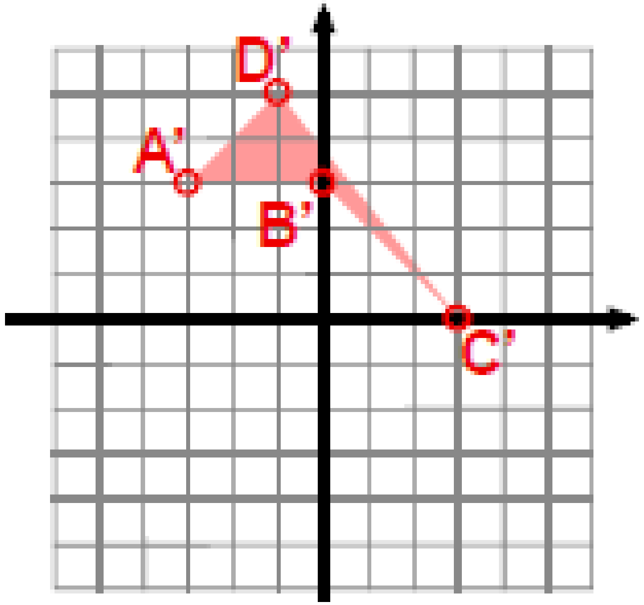
$$M = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$$

Find the determinant of M and interpret the result.

Example

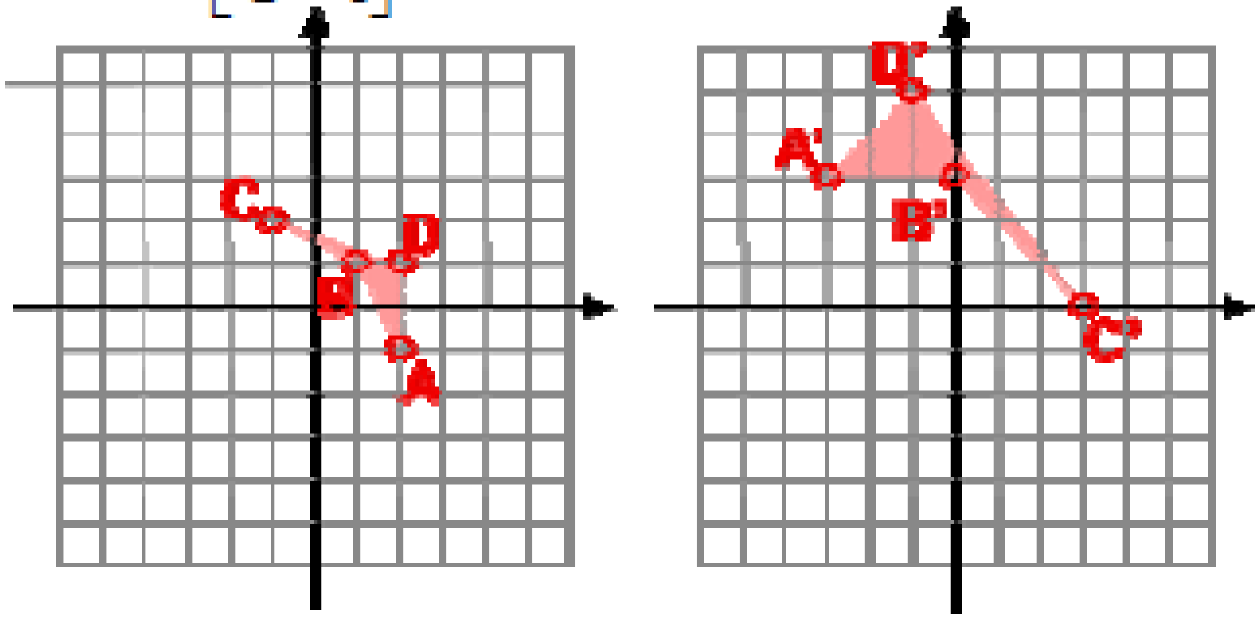
$$\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 & 2 \\ -1 & 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 3 & -1 \\ 3 & 3 & 0 & 5 \end{bmatrix}$$





Example

$$M = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} \qquad \det M = -3$$



Transformed area is three times larger.
Negative determinant means area is inverted

Co factor expansion:
Determinant of a 3 times 3 matrix

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Can work along any row or column.
To illustrate we will use row 1.

Co factor expansion reduces the problem from finding a determinant of a 3 by 3 matrix to that of finding three 2 by 2 determinants.

Co factor expansion:
Determinant of a 3 times 3 matrix

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Co factor expansion: Example

$$\det\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix} = 1\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 0 & 0 \\ 1 & -1 \end{vmatrix}$$

$$= 1(1) - 2(-1) + 3(0)$$

$$= 3$$

This means if we transformed each point of a three dimensional shape by this matrix, the transformed shape would have triple the volume

Determinants: Cramer’s Rule

A determinant of order n is a scalar associated with an $n \times n$ (hence ***square!***) matrix $\mathbf{A} = [a_{jk}]$, and is denoted by

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

For $n = 1$, this determinant is defined by $D = a_{11}.$

Determinants: Cramer's Rule

For $n \geqq 2$ by

$$D = a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn} \qquad (j = 1, 2, \cdots, \text{or } n)$$

or

$$D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk} \quad (k = 1, 2, \cdots, \text{or } n).$$

Here,

$$C_{jk} = (-1)^{j+k}M_{jk}$$

and M_{jk} is a determinant of order $n - 1$, namely, the determinant of the submatrix of **A** obtained from **A** by omitting the row and column of the entry a_{jk}, that is, the jth row and the kth column.

M_{jk} is called the **minor** *of* a_{jk} *in* D, and C_{jk} the **cofactor** *of* a_{jk} *in* D.

Minors and Cofactors of a Third-Order Determinant

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \qquad M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \qquad M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

and the cofactors are $C_{21} = -M_{21}$, $C_{22} = +M_{22}$, and $C_{23} = -M_{23}$.

Expansions of a Third-Order Determinant

$$D = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix}$$

$$= 1(12 - 0) - 3(4 + 4) + 0(0 + 6) = -12.$$

This is the expansion by the first row. The expansion by the third column is

$$D = 0\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix} - 4\begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0 - 12 + 0 = -12.$$

Verify that the other four expansions also give the value -12.

Determinant of a Triangular Matrix

$$\begin{vmatrix} -3 & 0 & 0 \\ 6 & 4 & 0 \\ -1 & 2 & 5 \end{vmatrix} = -3\begin{vmatrix} 4 & 0 \\ 2 & 5 \end{vmatrix} = -3 \cdot 4 \cdot 5 = -60.$$

Theorem 1

Behavior of an *n*th-Order Determinant under Elementary Row Operations

(a) *Interchange of two rows multiplies the value of the determinant by* -1.

(b) *Addition of a multiple of a row to another row does not alter the value of the determinant.*

(c) *Multiplication of a row by a nonzero constant* c *multiplies the value of the determinant by* c. (This holds also when $c = 0$, but no longer gives an elementary row operation.)

Theorem 2

Further Properties of nth-Order Determinants

(a)–(c) *in Theorem 1 hold also for columns.*

(d) ***Transposition*** *leaves the value of a determinant unaltered.*

(e) *A* ***zero row or column*** *renders the value of a determinant zero.*

(f) ***Proportional rows or columns*** *render the value of a determinant zero. In particular, a determinant with two identical rows or columns has the value zero.*

Evaluation of Determinants by Reduction to Triangular Form

$$D = \begin{vmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ -3 & 8 & 9 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 2 & 6 & -1 \\ 0 & 8 & 3 & 10 \end{vmatrix} \begin{matrix} \\ \text{Row 2} - 2\text{ Row 1} \\ \\ \text{Row 4} + 1.5\text{ Row 1} \end{matrix}$$

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & -11.4 & 29.2 \end{vmatrix} \begin{matrix} \\ \\ \text{Row 3} - 0.4\text{ Row 2} \\ \text{Row 4} - 1.6\text{ Row 2} \end{matrix}$$

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & -0 & 47.25 \end{vmatrix} \begin{matrix} \\ \\ \\ \text{Row 4} + 4.75\text{ Row 3} \end{matrix}$$

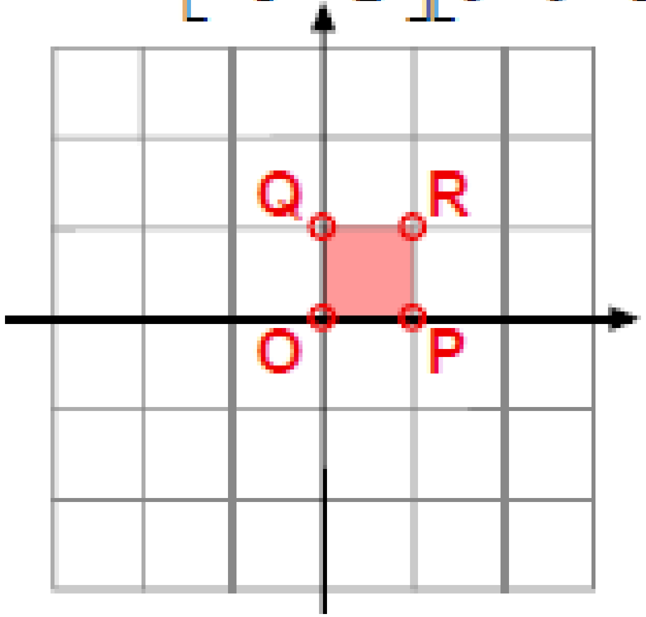
$$= 2 \cdot 5 \cdot 2.4 \cdot 47.25 = 1134.$$

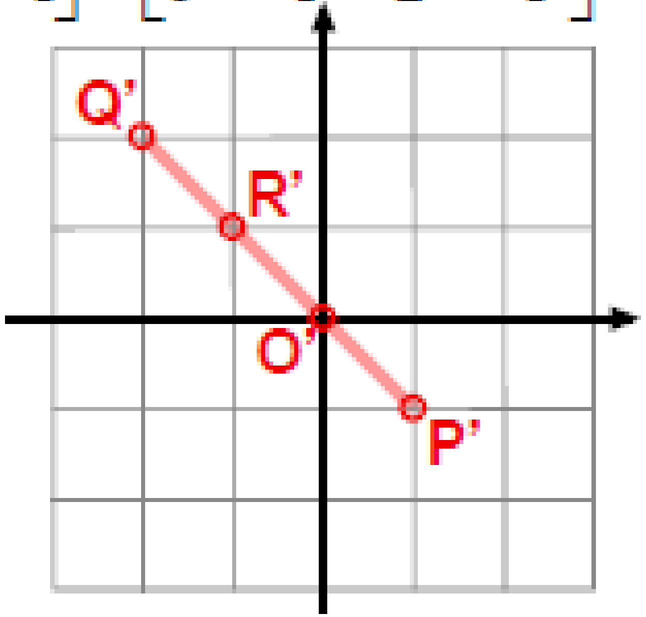
Singular transformation

What if the transformation is singular?

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}, \quad \det(A) = 0$$

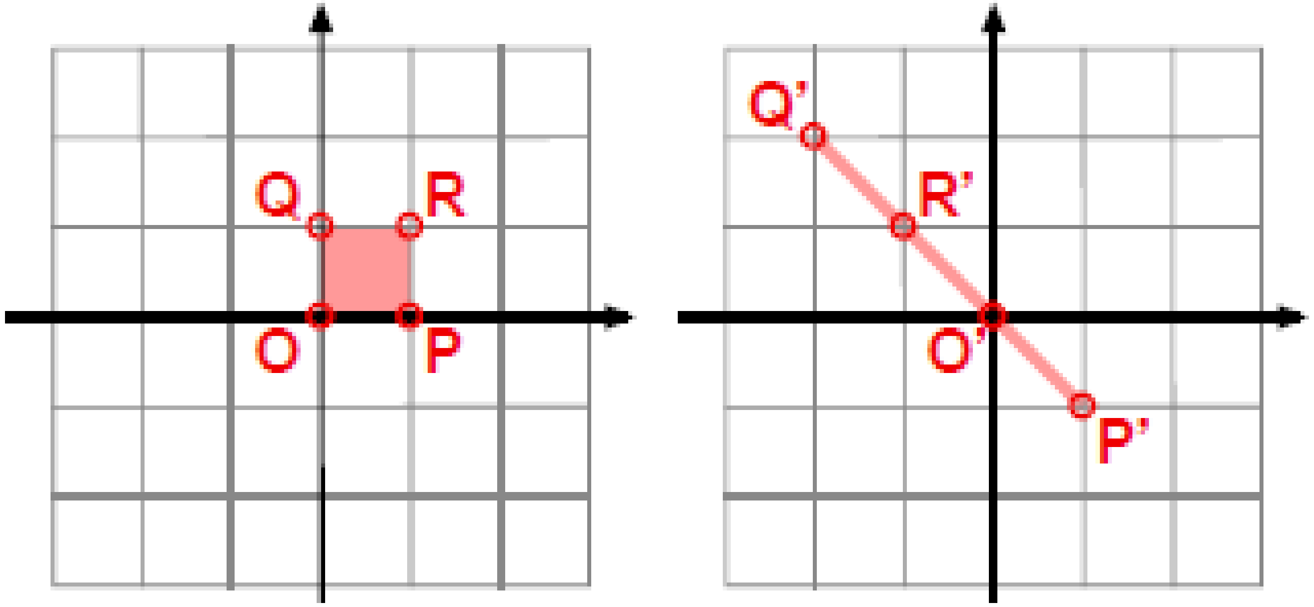
$$\mathbf{AU} = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 & -1 \\ 0 & -1 & 2 & 1 \end{bmatrix}$$





Singular transformations

- Singular transformation collapses the unit square into a straight line (or a single point).
- Singular transformations cannot be reversed because there isn't a one-to-one mapping between the original and transformed spaces.



Some Applications of Matrix Multiplication

Example 1

Computer Production. Matrix Times Matrix

Supercomp Ltd produces two computer models PC1086 and PC1186. The matrix **A** shows the cost per computer (in thousands of dollars) and **B** the production figures for the year 2010 (in multiples of 10,000 units.) Find a matrix **C** that shows the shareholders the cost per quarter (in millions of dollars) for raw material, labor, and miscellaneous.

$$\mathbf{A} = \begin{array}{c} \text{PC1086} \quad \text{PC1186} \\ \begin{bmatrix} 1.2 & 1.6 \\ 0.3 & 0.4 \\ 0.5 & 0.6 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Raw Components} \\ \text{Labor} \\ \text{Miscellaneous} \end{array} \qquad \mathbf{B} = \begin{array}{c} \text{Quarter} \\ 1 \quad 2 \quad 3 \quad 4 \\ \begin{bmatrix} 3 & 8 & 6 & 9 \\ 6 & 2 & 4 & 3 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \text{PC1086} \\ \text{PC1186} \end{array}$$

Solution.

$$\mathbf{C} = \mathbf{AB} = \begin{array}{c} \text{Quarter} \\ 1 \quad\; 2 \quad\; 3 \quad\; 4 \\ \begin{bmatrix} 13.2 & 12.8 & 13.6 & 15.6 \\ 3.3 & 3.2 & 3.4 & 3.9 \\ 5.1 & 5.2 & 5.4 & 6.3 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \text{Raw Components} \\ \text{Labor} \\ \text{Miscellaneous} \end{array}$$

Since cost is given in multiples of \$1000 and production in multiples of 10,000 units, the entries of **C** are multiples of \$10 millions; thus $c_{11} = 13.2$ means \$132 million, etc. ■

Some Applications of Matrix Multiplication

Example 2

Weight Watching. Matrix Times Vector

Suppose that in a weight-watching program, a person of 185 lb burns 350 cal/hr in walking (3 mph), 500 in bicycling (13 mph), and 950 in jogging (5.5 mph). Bill, weighing 185 lb, plans to exercise according to the matrix shown. Verify the calculations (W = Walking, B = Bicycling, J = Jogging).

$$
\begin{array}{c c}
 & \begin{array}{ccc} \text{W} & \text{B} & \text{J} \end{array} \\
\begin{array}{c} \text{MON} \\ \text{WED} \\ \text{FRI} \\ \text{SAT} \end{array} &
\begin{bmatrix} 1.0 & 0 & 0.5 \\ 1.0 & 1.0 & 0.5 \\ 1.5 & 0 & 0.5 \\ 2.0 & 1.5 & 1.0 \end{bmatrix}
\end{array}
\begin{bmatrix} 350 \\ 500 \\ 950 \end{bmatrix}
=
\begin{array}{c c}
\begin{bmatrix} 825 \\ 1325 \\ 1000 \\ 2400 \end{bmatrix} &
\begin{array}{c} \text{MON} \\ \text{WED} \\ \text{FRI} \\ \text{SAT} \end{array}
\end{array}
$$

Some Applications of Matrix Multiplication

Example 3

Markov Process. Powers of a Matrix. Stochastic Matrix

Suppose that the 2004 state of land use in a city of 60 mi^2 of built-up area is

C: Commercially Used 25%   I: Industrially Used 20%   R: Residentially Used 55%.

Find the states in 2009, 2014, and 2019, assuming that the transition probabilities for 5-year intervals are given by the matrix A and remain practically the same over the time considered.

$$A = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccc} \text{From C} & \text{From I} & \text{From R} \\ \end{array}$$

$$A = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix} \begin{array}{l} \text{To C} \\ \text{To I} \\ \text{To R} \end{array}$$

A is a stochastic matrix, that is, a square matrix with all entries nonnegative and all column sums equal to 1. Our example concerns a Markov process,[1] that is, a process for which the probability of entering a certain state depends only on the last state occupied (and the matrix A), not on any earlier state.

A **Markov process** is useful for analyzing dependent random events - that is, events whose likelihood depends on what happened last.

Some Applications of Matrix Multiplication

Solution. From the matrix A and the 2004 state we can compute the 2009 state,

$$\begin{array}{c} \text{C} \\ \text{I} \\ \text{R} \end{array} \begin{bmatrix} 0.7 \cdot 25 + 0.1 \cdot 20 + 0 \quad \cdot 55 \\ 0.2 \cdot 25 + 0.9 \cdot 20 + 0.2 \cdot 55 \\ 0.1 \cdot 25 + \quad 0 \cdot 20 + 0.8 \cdot 55 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} 25 \\ 20 \\ 55 \end{bmatrix} = \begin{bmatrix} 19.5 \\ 34.0 \\ 46.5 \end{bmatrix}.$$

To explain: The 2009 figure for C equals 25% times the probability 0.7 that C goes into C, plus 20% times the probability 0.1 that I goes into C, plus 55% times the probability 0 that R goes into C. Together,

$$25 \cdot 0.7 + 20 \cdot 0.1 + 55 \cdot 0 = 19.5\,[\%]. \qquad \text{Also} \qquad 25 \cdot 0.2 + 20 \cdot 0.9 + 55 \cdot 0.2 = 34\,[\%].$$

Similarly, the new R is 46.5%. We see that the 2009 state vector is the column vector

$$\mathbf{y} = [19.5 \quad 34.0 \quad 46.5]^{\mathsf{T}} = \mathbf{A}\mathbf{x} = \mathbf{A}\,[25 \quad 20 \quad 55]^{\mathsf{T}}$$

where the column vector $\mathbf{x} = [25 \quad 20 \quad 55]^{\mathsf{T}}$ is the given 2004 state vector. Note that the sum of the entries of $\mathbf{y}$ is 100 [%]. Similarly, you may verify that for 2014 and 2019 we get the state vectors

$$\mathbf{z} = \mathbf{A}\mathbf{y} = \mathbf{A}(\mathbf{A}\mathbf{x}) = \mathbf{A}^2\mathbf{x} = [17.05 \quad 43.80 \quad 39.15]^{\mathsf{T}}$$

$$\mathbf{u} = \mathbf{A}\mathbf{z} = \mathbf{A}^2\mathbf{y} = \mathbf{A}^3\mathbf{x} = [16.315 \quad 50.660 \quad 33.025]^{\mathsf{T}}.$$

Answer. In 2009 the commercial area will be 19.5% (11.7 mi^2), the industrial 34% (20.4 mi^2), and the residential 46.5% (27.9 mi^2). For 2014 the corresponding figures are 17.05%, 43.80%, and 39.15%. For 2019 they are 16.315%, 50.660%, and 33.025%. (In Sec. 8.2 we shall see what happens in the limit, assuming that those probabilities remain the same. In the meantime, can you experiment or guess?) ■