# MC2020 - Linear Algebra
# Eigen values and Eigen vectors

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# Eigenvalues and eigenvectors

- The *eigenvalues*, $\lambda$, and *eigenvectors*, $\mathbf{s}$, of a matrix $A$ are the constants and associated vectors that satisfy the equation:

$$A\mathbf{s} = \lambda\mathbf{s}$$

transformation stretches $\mathbf{s}$, but does not change its direction

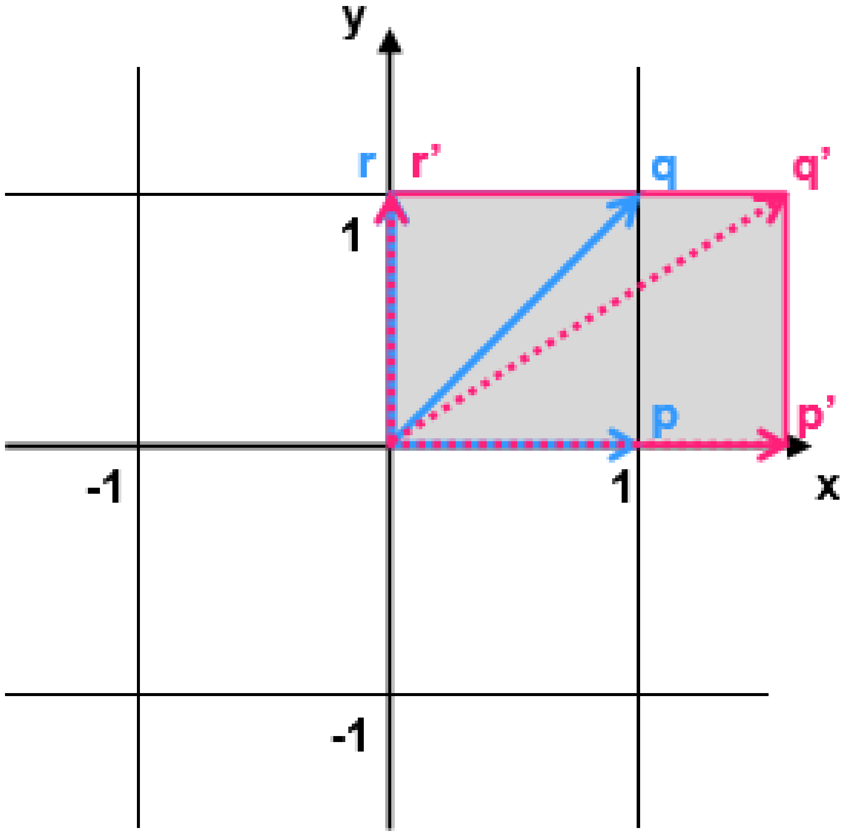
or $$(A - \lambda I)\mathbf{s} = \mathbf{0}$$

$\lambda$ = stretch ratio

- Let's consider some transformation matrices and see if we can identify the eigenvectors (and corresponding eigenvalues)…

# Eigenvalues and eigenvectors – spot the eigenvectors $A\mathbf{s} = \lambda \mathbf{s}$

- x-stretch: $A = \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}$




$$\mathbf{p} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{p}' = A\mathbf{p} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}$$

$A\mathbf{p} = 1.5\mathbf{p}$ eigenvector!

$$\mathbf{q} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{q}' = A\mathbf{q} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$$

$A\mathbf{q} \neq \lambda \mathbf{q}$ not an eigenvector

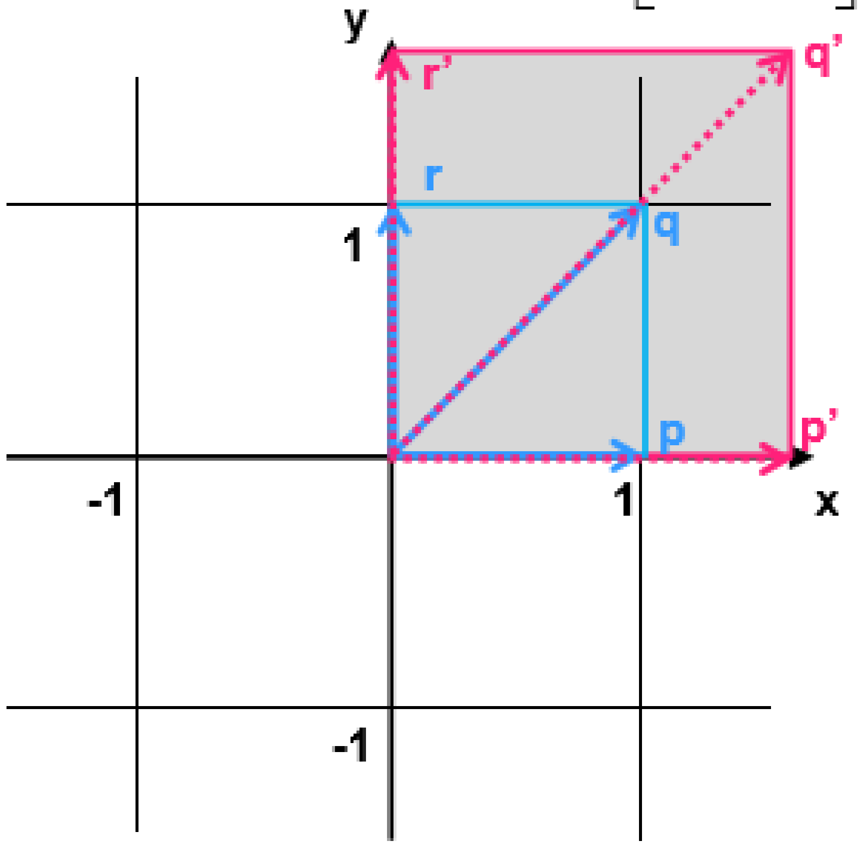
$$\mathbf{r} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{r}' = A\mathbf{r} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$A\mathbf{r} = 1.0\mathbf{r}$ eigenvector!

# Eigenvalues and eigenvectors – spot the eigenvectors $A\mathbf{s} = \lambda\mathbf{s}$

- uniform scale: $A = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}$

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$A\mathbf{p} = 1.5\mathbf{p}$ eigenvector!

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$A\mathbf{q} = 1.5\mathbf{q}$ eigenvector!

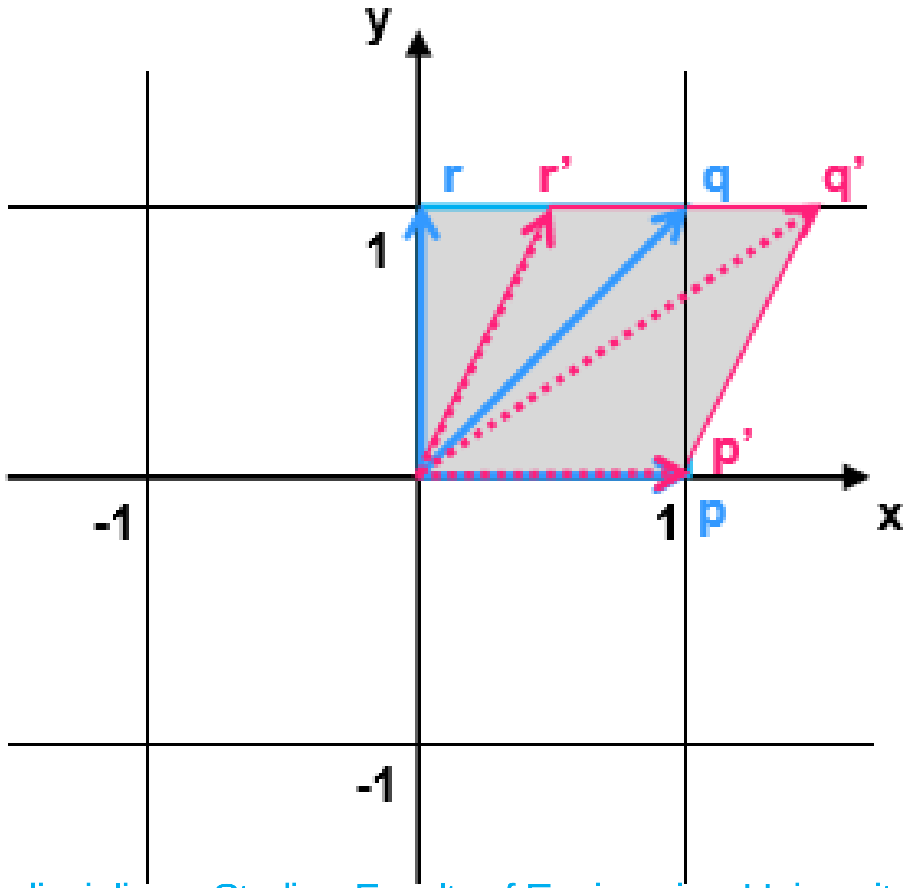
$$\mathbf{r} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{r}' = A\mathbf{r} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}$$

$A\mathbf{r} = 1.5\mathbf{r}$ eigenvector!

# Eigenvalues and eigenvectors – spot the eigenvectors $\quad As = \lambda s$

- simple shear (in x): $A = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$

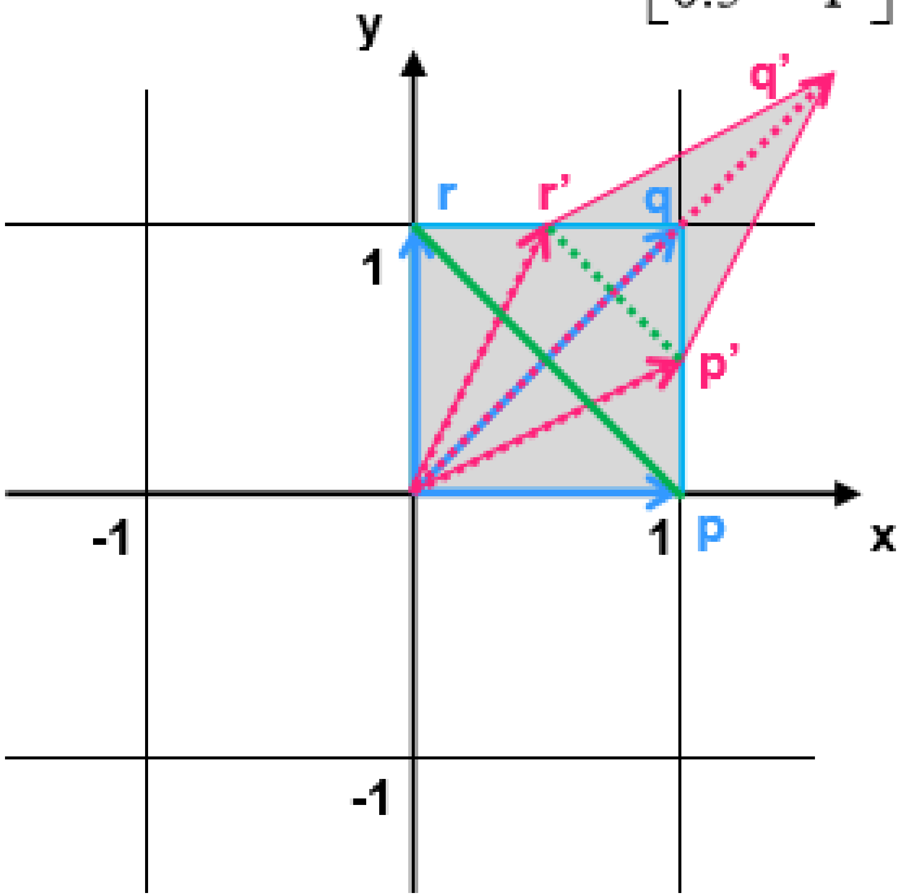



**p** is an eigenvector!

**q,r** are not eigenvectors

# Eigenvalues and eigenvectors – spot the eigenvectors $\quad A\mathbf{s} = \lambda \mathbf{s}$

- pure (x-y) shear: $A = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$




**q** is an eigenvector!

**p,r** not eigenvectors

another eigenvector!

# Determining eigenvalues and eigenvectors

- We determine the eigenvalues and eigenvectors of a square matrix, $A$, by analysing the system of equations:

$$(A - \lambda I)\mathbf{s} = \mathbf{0} \qquad \left(\text{came from } A\mathbf{s} = \lambda\mathbf{s}\right)$$

  - Step 1: calculate the eigenvalues

  - Step 2: for each eigenvalue, calculate the *associated* eigenvector

# Determining eigenvalues and eigenvectors

- Step 1: calculate the eigenvalues
  - Solve the equation:

$$\det(A - \lambda I) = 0$$

  - This gives the *characteristic polynomial of $A$* in terms of $\lambda$ (the order of the polynomial is equal to the number of rows of $A$).
  - The solutions (roots) of this characteristic polynomial are the eigenvalues of $A$.

# Determining eigenvalues and eigenvectors

- Step 2: for each eigenvalue ($\lambda_i$), calculate the associated eigenvector ($\mathbf{s}_i$)
  - Solve the equations: $$(A - \lambda I)\mathbf{s} = \mathbf{0}$$
  - This will give a relationship between the components of $\mathbf{s}$.
  - Set a sensible value for one of the components of $\mathbf{s}$ (the “free variable” - don’t set to zero) and calculate the other components.
  - Normalise $\mathbf{s}$ to unit length

## Normalizing a vector

We can find a unit vector from any vector through a process called normalization. When normalizing a vector we want to form a unit vector with a same direction of a vector. Consider the length formula for a vector:

$$\mathbf{v} = [v_1, v_2, \ldots, v_n]^T$$

$$\Rightarrow \|\mathbf{v}\| = \sqrt{v_1^{\,2} + v_2^{\,2} + \ldots + v_n^{\,2}}$$

If we divide the vector by its length:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left[\frac{v_1}{\|\mathbf{v}\|}, \frac{v_2}{\|\mathbf{v}\|}, \ldots, \frac{v_n}{\|\mathbf{v}\|}\right]^T$$

and find the length of this new vector:

$$\hat{\mathbf{v}} = \left[\frac{v_1}{\|\mathbf{v}\|}, \frac{v_2}{\|\mathbf{v}\|}, \ldots, \frac{v_n}{\|\mathbf{v}\|}\right]^T$$

$$\Rightarrow \|\hat{\mathbf{v}}\| = \sqrt{\left(\frac{v_1}{\|\mathbf{v}\|}\right)^2 + \left(\frac{v_2}{\|\mathbf{v}\|}\right)^2 + \ldots + \left(\frac{v_n}{\|\mathbf{v}\|}\right)^2}$$

$$= \sqrt{\frac{v_1^{\ 2}}{\|\mathbf{v}\|^2} + \frac{v_2^{\ 2}}{\|\mathbf{v}\|^2} + \ldots + \frac{v_n^{\ 2}}{\|\mathbf{v}\|^2}}$$

$$= \sqrt{\frac{v_1^{\ 2} + v_2^{\ 2} + \ldots + v_n^{\ 2}}{v_1^{\ 2} + v_2^{\ 2} + \ldots + v_n^{\ 2}}} = 1$$

Therefore we can normalize a vector by dividing the vector by its length:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

# Example – determining eigenvalues and eigenvectors

- Find the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix}$$

# Example – determining eigenvalues and eigenvectors

- Calculate the eigenvalues:

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow \det\begin{pmatrix} 5-\lambda & -2 \\ 1 & 2-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (5-\lambda)(2-\lambda) + 2 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 12 = 0$$

$$\Rightarrow \lambda_1 = 3, \quad \lambda_2 = 4$$

# Example – determining eigenvalues and eigenvectors

- For each eigenvalue ($\lambda_i$), calculate the associated eigenvector using:

$$(A - \lambda I)\mathbf{s} = \mathbf{0}$$

$$\lambda_1 = 3 \qquad \Rightarrow \begin{bmatrix} 5-3 & -2 \\ 1 & 2-3 \end{bmatrix} \begin{bmatrix} s_a \\ s_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} s_a \\ s_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \begin{matrix} 2s_a - 2s_b = 0 \\ s_a - s_b = 0 \end{matrix}$$

$$\Rightarrow s_a = s_b$$

$$\text{Set} \quad s_b = 1 \quad \Rightarrow s_a = 1 \qquad \Rightarrow \mathbf{s}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \Rightarrow \hat{\mathbf{s}}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

# Example – determining eigenvalues and eigenvectors

- For each eigenvalue ($\lambda_i$), calculate the associated eigenvector using:

$$(A-\lambda I)\mathbf{s}=\mathbf{0}$$

$$\lambda_2 = 4 \quad \Rightarrow \begin{bmatrix} 5-4 & -2 \\ 1 & 2-4 \end{bmatrix}\begin{bmatrix} s_a \\ s_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} s_a \\ s_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \begin{matrix} s_a - 2s_b = 0 \\ s_a - 2s_b = 0 \end{matrix}$$

$$\Rightarrow s_a = 2s_b$$

$$\text{Set} \quad s_b = 1 \quad \Rightarrow s_a = 2 \qquad \Rightarrow \mathbf{s}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \Rightarrow \hat{\mathbf{s}}_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

# Example – determining eigenvalues and eigenvectors

- So the matrix:

$$A = \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix}$$

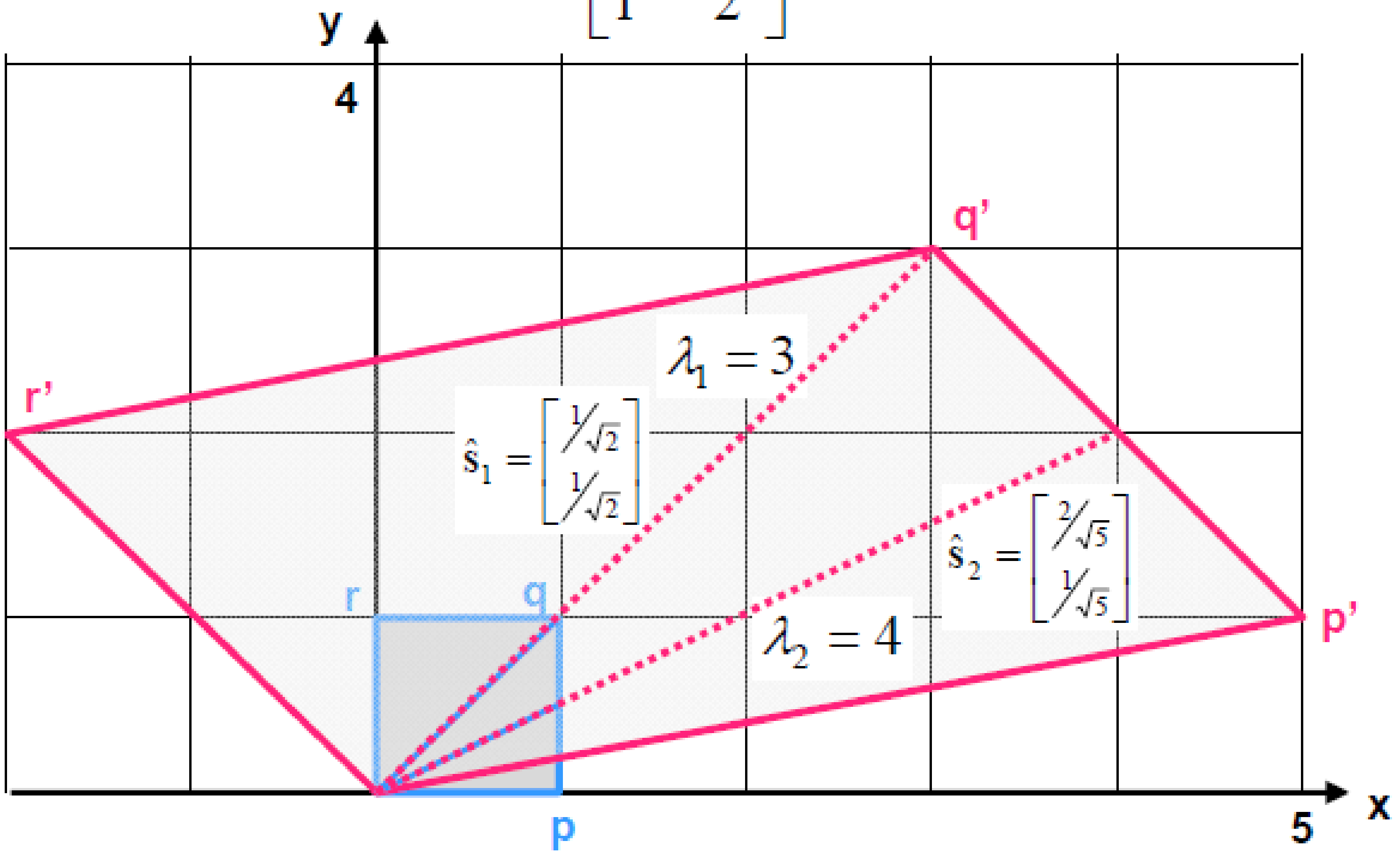
has the eigenvalues and unit eigenvectors:

$$\lambda_1 = 3, \quad \hat{\mathbf{s}}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\lambda_2 = 4, \quad \hat{\mathbf{s}}_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

# Example – determining eigenvalues and eigenvectors

$$A = \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix}$$



1. (a) Find the eigenvalues and eigenvectors of the following matrices;

(i) $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(iv) $\begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

(vi) $\begin{bmatrix} 6 & 2 \\ -9 & 0 \end{bmatrix}$

(vii) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

(viii) $\begin{bmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{bmatrix}$

(ix) $\begin{bmatrix} 2 & -4 \\ -2 & 0 \end{bmatrix}$

(b) Use **MATLAB** to find the eigenvalues and eigenvectors.

2. Find the eigenvalues and eigenvectors of

(a) $\begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 1 & 5 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 3 & 1 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

3. (a) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

(b) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

(c) Use **MATLAB** to find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}$$

4. Show that the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

has eigenvalues $\cos\theta \pm i\sin\theta$.

5. Consider the vector

$$\mathrm{v} = \begin{bmatrix} 1 \\ a \\ -1 \end{bmatrix}$$

and the matrix

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$$

For what value of $a$ is v an eigenvector of $A$? Find the eigenvalue corresponding to v for this value of $a$.

6. (a) Find all the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 3 & 4 \end{bmatrix}$$

(b) Find all the eigenvalues of the inverse of $A$.

(c) What are the eigenvectors of the inverse of $A$?

7. Find the eigenvalues of the symmetric matrix

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

where $a$, $b$ and $c$ are real numbers, and explain why the eigenvalues are never complex valued.

**8.** (a) Find the largest eigenvalue of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -8 & 4 & -6 \\ 8 & 1 & 9 \end{bmatrix}$$

(b) Find the largest eigenvalue of the matrix $A^2$.

(c) Find the largest eigenvalue of $A^3$.

(d) Deduce the largest eigenvalue of $A^4$.

(e) Guess the smallest eigenvalue of $A^4$, and then check the result using **MATLAB**.

(f) What are the eigenvectors of $A^4$?

**9.** (a) Find all of the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -8 & 4 & -6 \\ 8 & 1 & 9 \end{bmatrix}$$

(b) Find the eigenvalues of the matrix inv($A$) and compare with those of $A$.

(c) Find the eigenvectors of inv($A$) and compare with those of $A$.

# Eigenvalues and eigenvectors – Diagonal matrices are simple

- For a diagonal matrix:

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{bmatrix}$$

the eigenvalues and eigenvectors are:

$$\lambda_1 = d_1,\ \hat{s}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \end{bmatrix}, \qquad \lambda_2 = d_2,\ \hat{s}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \qquad \ldots \quad , \qquad \lambda_n = d_n,\ \hat{s}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

# Eigenvalues and eigenvectors – Diagonal matrices example

- Find the eigenvalues and eigenvectors of the matrix:

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

- Try using the standard method.
  Find the eigenvalues:

$$\det(A - \lambda I) = 0 \quad \Rightarrow \begin{vmatrix} 3-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(2-\lambda) = 0 \qquad \Rightarrow \lambda_1 = 3, \quad \lambda_2 = 2$$

# Eigenvalues and eigenvectors – Diagonal matrices example

- For each eigenvalue, find the associated eigenvector: $(A - \lambda I)\mathbf{s} = \mathbf{0}$

$$\lambda_1 = 3 \qquad \Rightarrow \begin{bmatrix} 3-3 & 0 \\ 0 & 2-3 \end{bmatrix} \begin{bmatrix} s_a \\ s_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} 0 \times s_a + 0 \times s_b = 0 \\ s_b = 0 \end{matrix} \qquad \Rightarrow s_a \text{ is free, } s_b = 0$$

$$\text{Set } s_a = 1 \qquad \Rightarrow \hat{\mathbf{s}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

# Eigenvalues and eigenvectors – Diagonal matrices example

- For each eigenvalue, find the associated eigenvector: $(A-\lambda I)\mathbf{s}=\mathbf{0}$

$$\lambda_2 = 2 \quad \Rightarrow \begin{bmatrix} 3-2 & 0 \\ 0 & 2-2 \end{bmatrix} \begin{bmatrix} s_a \\ s_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} s_a = 0 \\ 0 \times s_b = 0 \end{matrix} \qquad \Rightarrow s_a = 0, s_b \text{ is free}$$

$$\text{Set } s_b = 1 \qquad \Rightarrow \hat{\mathbf{s}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

# Eigenvalues and eigenvectors – Diagonal matrices example

- So for the matrix:

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

  we have the eigenvalues and unit eigenvectors:

$$\lambda_1 = 3, \hat{\mathbf{s}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \lambda_2 = 2, \hat{\mathbf{s}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

  - So the standard method for finding eigenvalues and eigenvectors agrees with our simple method for diagonal matrices.

# Key points: eigenvalues and eigenvectors

- Understand the physical interpretation of eigenvalues and eigenvectors

- For an NxN matrix, know the order of the characteristic polynomial, and up to many eigenvalues to expect to find

- Know how to determine the eigenvalues and eigenvectors for a square matrix

# Eigenvalues and eigenvectors

Application 1:

Quadratic forms and ellipses

$$\mathbf{x}^T Q\mathbf{x} = k$$

# Quadratic forms and ellipses

- We can express any ellipse

$$ax^2 + bxy + cy^2 = k \qquad \text{(general form of an ellipse)}$$

using a *quadratic form:*

$$\mathbf{x}^T Q \mathbf{x} = k$$

$$\text{where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad Q = \begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix}$$

# Quadratic forms and ellipses

- How do we get the components of $Q$?

- Try expanding the quadratic form and comparing to the general ellipse:

$$\mathbf{x}^T Q\mathbf{x} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} q_1 x + q_2 y & q_2 x + q_3 y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= q_1 x^2 + q_2 xy + q_2 xy + q_3 y^2 \quad = q_1 x^2 + 2q_2 xy + q_3 y^2$$

$$\Rightarrow q_1 x^2 + 2q_2 xy + q_3 y^2 = ax^2 + bxy + cy^2$$ (equate coeffs)

$$\Rightarrow q_1 = a, \quad q_2 = \frac{b}{2}, \quad q_3 = c$$

# Quadratic forms and ellipses

- So, in general, a quadratic form representation for an ellipse is:

$$ax^2 + bxy + cy^2 = k \quad \text{(general form of an ellipse)}$$

$$\Downarrow$$

$$\mathbf{x}^T Q \mathbf{x} = k$$

$$\text{where } \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad Q = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$$

# Quadratic forms and ellipses

- This can also be extended to ellipsoids (3D ellipses):

$$ax^2 + bxy + cxz + dy^2 + eyz + fz^2 = k$$

(general form of an ellipsoid)

$$\mathbf{x}^T Q \mathbf{x} = k$$

$$\text{where } \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad Q = \begin{bmatrix} a & b/2 & c/2 \\ b/2 & d & e/2 \\ c/2 & e/2 & f \end{bmatrix}$$

# Quadratic forms and ellipses

- When we use a quadratic form to express an ellipse, we can plot the ellipse using the eigenvalues and eigenvectors of $Q$.
- The *principal axes* (the longest and shortest lines across the ellipse) are the eigenvectors of $Q$.
- The intercepts of the ellipse on the principal axes can be calculated from the eigenvalues of $Q$ using:

$$I = \pm\sqrt{\frac{k}{\lambda}}$$

# Example 1 – Quadratic forms and ellipses

- Let's look at a simple ellipse:

$$\tfrac{1}{4}x^2 + \tfrac{1}{9}y^2 = 1$$

- Quadratic form:

$$\mathbf{x}^T Q\mathbf{x} = k \qquad \Rightarrow \mathbf{x}^T \begin{bmatrix} 1/4 & 0 \\ 0 & 1/9 \end{bmatrix} \mathbf{x} = 1$$

- How do we plot this ellipse?

# Example 1 – Quadratic forms and ellipses

- Let's look at a simple ellipse:

$$\tfrac{1}{4}x^2 + \tfrac{1}{9}y^2 = 1$$

- From high school we can plot this:

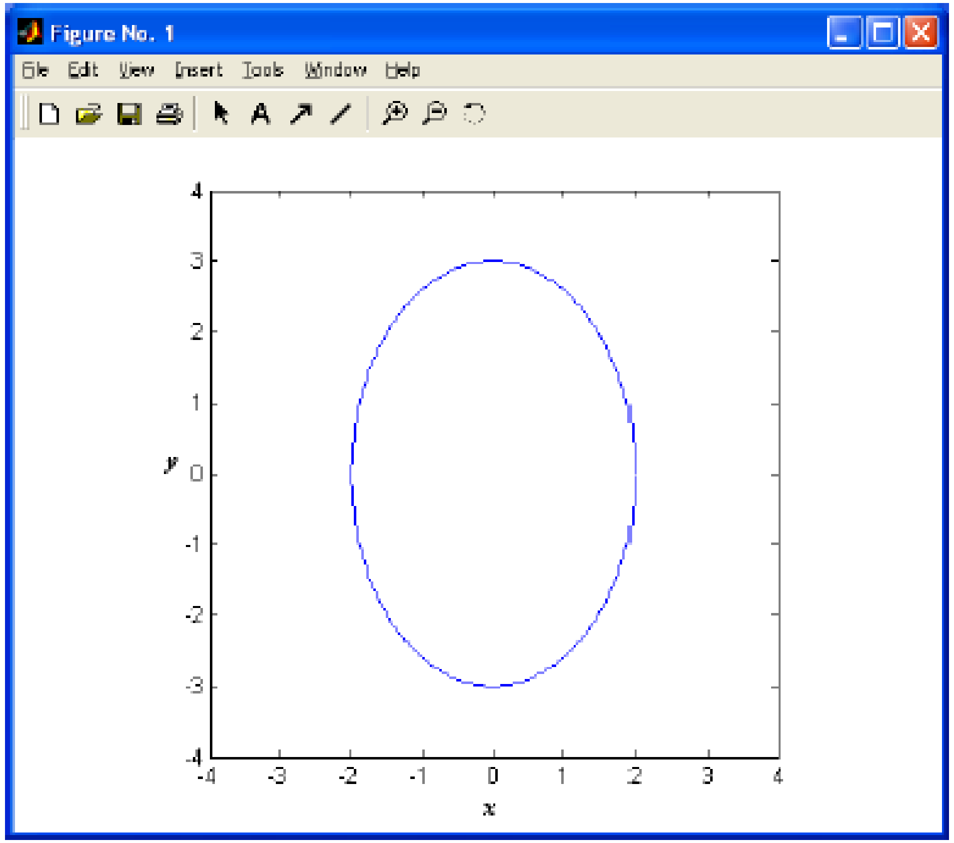
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow x_{\text{intercepts}} = \pm a$$

$$y_{\text{intercepts}} = \pm b$$

# Example 1 – Quadratic forms and ellipses

- So the plot is:



# Example 1 – Quadratic forms and ellipses

- Alternatvely, we can use the quadratic form:

$$\tfrac{1}{4}x^2 + \tfrac{1}{9}y^2 = 1 \;\Rightarrow\; \mathbf{x}^T \begin{bmatrix} 1/4 & 0 \\ 0 & 1/9 \end{bmatrix} \mathbf{x} = 1$$

- Here $Q$ is diagonal, so the eigenvalues and eigenvectors are easy:

$$\begin{bmatrix} 1/4 & 0 \\ 0 & 1/9 \end{bmatrix} \Rightarrow \quad \lambda_1 = \tfrac{1}{4},\ \hat{\mathbf{s}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \lambda_2 = \tfrac{1}{9},\ \hat{\mathbf{s}}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

# Example 1 – Quadratic forms and ellipses

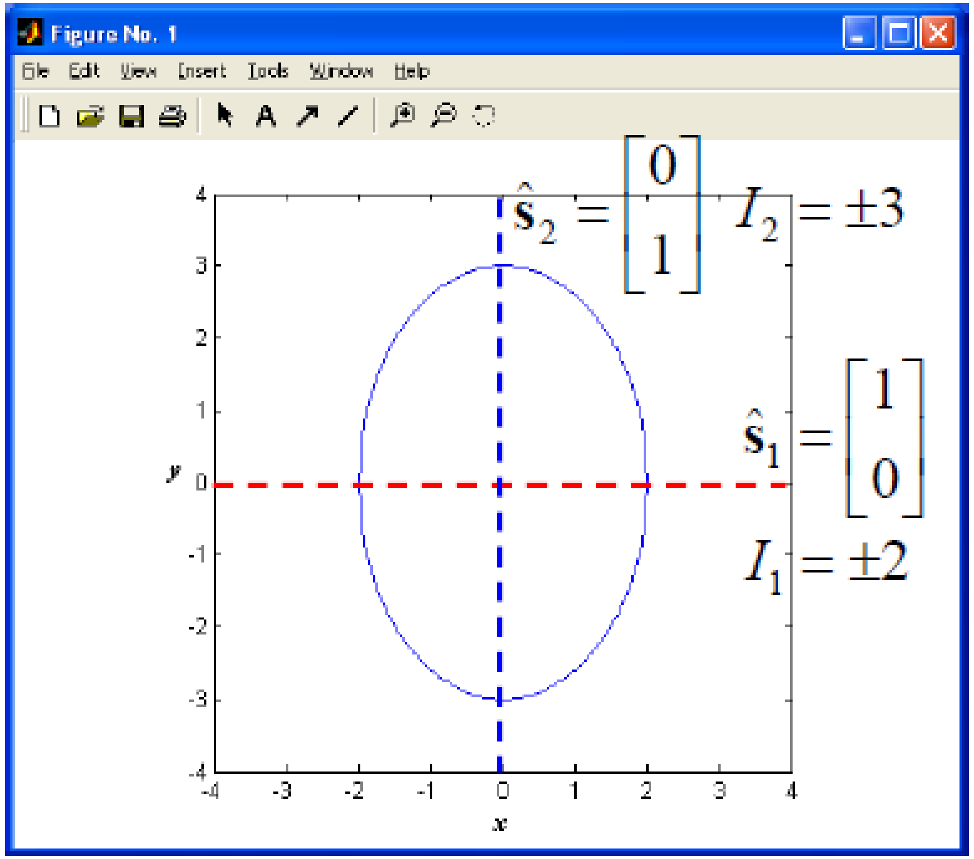
- This gives the intercepts

$$I_1 = \pm\sqrt{\frac{1}{1/4}} = \pm\sqrt{4} = \pm 2$$

$$I_2 = \pm\sqrt{\frac{1}{1/9}} = \pm\sqrt{9} = \pm 3$$

# Example 1 – Quadratic forms and ellipses

- We get the same plot:



# Example 2 – Quadratic forms and ellipses

- Use the quadratic form to plot the ellipse:

$$3x^2 + 2xy + 3y^2 = 8 \qquad \Rightarrow \mathbf{x}^T \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \mathbf{x} = 8$$

- $Q$ has eigenvalues and unit eigenvectors:

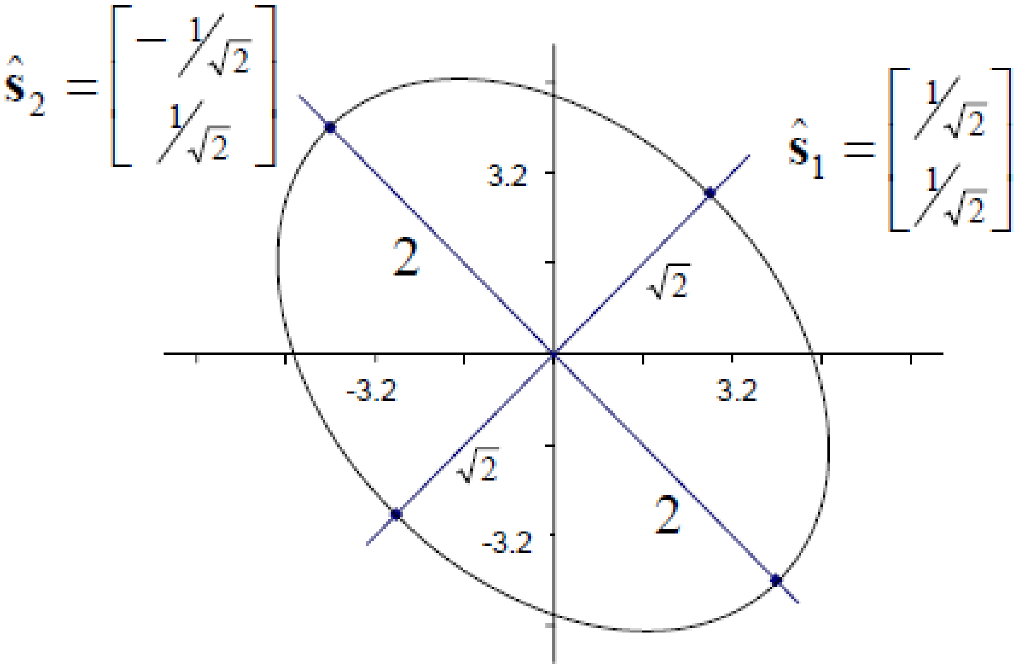
$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \Rightarrow \quad \lambda_1 = 4, \hat{\mathbf{s}}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \lambda_2 = 2, \hat{\mathbf{s}}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

- Intercepts along the eigenvectors:

$$I_1 = \pm\sqrt{\frac{8}{4}} = \pm\sqrt{2} \qquad\qquad I_2 = \pm\sqrt{\frac{8}{2}} = \pm 2$$

# Example 2 – Quadratic forms and ellipses

- which gives the plot:




**Solution! Ellipse rotated into eigenvector directions**

# Key points: eigen-analysis

- Know how to compute the eigenvalues and eigenvectors of a square matrix.
  - Solve characteristic equation for eigenvalues:

$$\det(A - \lambda I) = 0$$

  - For each eigenvalue, determine the associated eigenvector by solving:

$$(A - \lambda I)\mathbf{s} = \mathbf{0}$$

- Determine quadratic form of an equation of an ellipse, and then use eigen-analysis to plot the ellipse.

# Eigenvalues and eigenvectors

## Application 2:

## Diagonalisation of matrices

$$D = S^{-1}AS$$

# Diagonalisation of matrices

- Suppose we have a square matrix, $A$, with $n$ distinct eigenvalues and their associated eigenvectors:

$$\lambda_1, \lambda_2, \ldots, \lambda_n \qquad \mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n$$

- We can express $A$ as:

$$A = SDS^{-1} = [\mathbf{s}_1 \mid \mathbf{s}_2 \mid \ldots \mid \mathbf{s}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} [\mathbf{s}_1 \mid \mathbf{s}_2 \mid \ldots \mid \mathbf{s}_n]^{-1}$$

- This is a *diagonalisation* of $A$ because this factorisation has the diagonal matrix of eigenvalues.

- Inversely: $D = S^{-1}AS$

# Diagonalisation of matrices – Example

$$A = SDS^{-1}$$

- Find the diagonalisation of:

$$A = \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix}$$

- From earlier we know:

$$\lambda_1 = 3, \quad \hat{\mathbf{s}}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\lambda_2 = 4, \quad \hat{\mathbf{s}}_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

# Diagonalisation of matrices – Example

$$A = SDS^{-1}$$

- Therefore, the diagonalisation is:

$$A = \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 2/\sqrt{5} \\ 1/\sqrt{2} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 2/\sqrt{5} \\ 1/\sqrt{2} & 1/\sqrt{5} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1/\sqrt{2} & 2/\sqrt{5} \\ 1/\sqrt{2} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -\sqrt{2} & 2\sqrt{2} \\ \sqrt{5} & -\sqrt{5} \end{bmatrix}$$

# Diagonalisation of matrices – Example

$$A = SDS^{-1}$$

- Check the expansion:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -\sqrt{2} & 2\sqrt{2} \\ \sqrt{5} & -\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{2}} & \frac{8}{\sqrt{5}} \\ \frac{3}{\sqrt{2}} & \frac{4}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} -\sqrt{2} & 2\sqrt{2} \\ \sqrt{5} & -\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix} = A$$ ☺

# Eigenvalues and eigenvectors – symmetric matrices

- If $A$ is a *real, symmetric* matrix ($A = A^T$):
  - the eigenvalues of $A$ are *real and distinct*
  - the eigenvectors of $A$ are *orthogonal* (i.e. at right angles)

# Diagonalisation of symmetric matrices

- If $A$ is a *symmetric* matrix and we use *normalised* (unit) eigenvectors then:

$$\boxed{S^{-1} = S^T}$$ ("orthonormal")

- Thus, for symmetric $A$:

$$A = SDS^T = [\mathrm{s}_1 \mid \mathrm{s}_2 \mid \ldots \mid \mathrm{s}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \begin{bmatrix} \mathrm{s}_1^T \\ \mathrm{s}_2^T \\ \vdots \\ \mathrm{s}_n^T \end{bmatrix}$$

$$D = S^T A S$$ (only when A is real & symmetric)

# Eigenvalues and eigenvectors

## Application 3:

## Matrix exponentiation

Calculate $A^k$

e.g. $P' = A.A.A.A.A.A.P = A^5P$

# Diagonalisation of matrices – matrix exponentiation

- Diagonalisation makes calculating the powers of matrices much simpler/faster:

$$A = SDS^{-1} \qquad \Rightarrow A^k = SD^kS^{-1}$$

where:

$$D^k = \begin{bmatrix} \lambda_1{}^k & 0 & \cdots & 0 \\ 0 & \lambda_2{}^k & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n{}^k \end{bmatrix} \qquad \Rightarrow A^k = S\begin{bmatrix} \lambda_1{}^k & 0 & \cdots & 0 \\ 0 & \lambda_2{}^k & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n{}^k \end{bmatrix}S^{-1}$$

# Diagonalisation of matrices – Matrix exponentiation example

- Evaluate:

$$A^5 = \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix}^5$$

- Using the diagonalisation (see above):

$$A = \begin{bmatrix} 1/\sqrt{2} & 2/\sqrt{5} \\ 1/\sqrt{2} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -\sqrt{2} & 2\sqrt{2} \\ \sqrt{5} & -\sqrt{5} \end{bmatrix}$$

$$\Rightarrow A^5 = \begin{bmatrix} 1/\sqrt{2} & 2/\sqrt{5} \\ 1/\sqrt{2} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 3^5 & 0 \\ 0 & 4^5 \end{bmatrix} \begin{bmatrix} -\sqrt{2} & 2\sqrt{2} \\ \sqrt{5} & -\sqrt{5} \end{bmatrix}$$

$$\Rightarrow A^5 = \begin{bmatrix} 1/\sqrt{2} & 2/\sqrt{5} \\ 1/\sqrt{2} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 243 & 0 \\ 0 & 1024 \end{bmatrix} \begin{bmatrix} -\sqrt{2} & 2\sqrt{2} \\ \sqrt{5} & -\sqrt{5} \end{bmatrix}$$

# Eigenvalues and eigenvectors – many applications

- Face recognition (principal component analysis)

Description: http://en.wikipedia.org/wiki/Eigenface (machine learning)

Applet:

http://cognitrn.psych.indiana.edu/nsfgrant/FaceMachine/faceMachine.html

# Diagonalisation of matrices – matrix exponentiation example

- Evaluate:

$$A^5 = \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix}^5$$

- Using the diagonalisation (see above):

$$\Rightarrow A^5 = \begin{bmatrix} 1/\sqrt{2} & 2/\sqrt{5} \\ 1/\sqrt{2} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 243 & 0 \\ 0 & 1024 \end{bmatrix} \begin{bmatrix} -\sqrt{2} & 2\sqrt{2} \\ \sqrt{5} & -\sqrt{5} \end{bmatrix}$$

we get: $\Rightarrow A^5 = \begin{bmatrix} 243/\sqrt{2} & 2048/\sqrt{5} \\ 243/\sqrt{2} & 1024/\sqrt{5} \end{bmatrix} \begin{bmatrix} -\sqrt{2} & 2\sqrt{2} \\ \sqrt{5} & -\sqrt{5} \end{bmatrix}$

$$\Rightarrow A^5 = \begin{bmatrix} 1805 & -1562 \\ 781 & -538 \end{bmatrix}$$

(many fewer calculations, compared to many matrix multiplications)

# Eigenvalues and eigenvectors

## Application 4:

Solving systems of coupled 1st order ODEs

$$\mathbf{x}' = A\mathbf{x}$$

# Diagonalisation of matrices – solving systems of coupled ODEs

- Matrix diagonalisation can be used to solve systems of ODEs:

  e.g. $\mathbf{x}' = A\mathbf{x}$

- Use the substitution: $\boxed{\mathbf{x} = S\mathbf{y}}$

  $$\Rightarrow S\mathbf{y}' = AS\mathbf{y} \qquad \Rightarrow \mathbf{y}' = S^{-1}AS\mathbf{y}$$

- Recall: $D = S^{-1}AS$

  $$\Rightarrow \mathbf{y}' = D\mathbf{y}$$ uncoupled (diagonal) system

- Solve uncoupled system to find $\mathbf{y}$ (easy); convert/apply initial conditions; invert for $\mathbf{x}$

# Diagonalisation of matrices – solving systems of ODEs example

- We will step through solving a system of coupled 1st order ODEs:

$$\boxed{\begin{aligned} \frac{dx_1}{dt} &= 5x_1 - 2x_2 \\ \frac{dx_2}{dt} &= -2x_1 + 2x_2 \\ x_1(0) = 1, &\quad x_2(0) = 0 \end{aligned}}$$

- Write the system in <u>matrix form</u>:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \mathbf{x}' = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}\mathbf{x} \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{x}' = A\mathbf{x} \qquad A = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$$

# Diagonalisation of matrices – solving systems of ODEs example

- Find the eigenvalues of $A$:

$$\det(A-\lambda I)=0$$

$$\begin{vmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix}=0$$

$$\Rightarrow (5-\lambda)(2-\lambda)-4=0$$

$$\Rightarrow \lambda^2-7\lambda+6=0$$

$$\Rightarrow \lambda_1=1, \quad \lambda_2=6$$

# Diagonalisation of matrices – solving systems of ODEs example

- For each eigenvalue, find the <u>unit</u> eigenvector:

$$\lambda_1 = 1 \qquad \Rightarrow \begin{bmatrix} 5-1 & -2 \\ -2 & 2-1 \end{bmatrix} \begin{bmatrix} s_a \\ s_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2s_a + s_b = 0 \qquad \Rightarrow 2s_a = s_b$$

$$\text{Set } s_a = 1 \qquad \Rightarrow s_b = 2$$

$$\Rightarrow \mathbf{s}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \hat{\mathbf{s}}_1 = \frac{\mathbf{s}_1}{|\mathbf{s}_1|} = \frac{1}{\sqrt{1^2 + 2^2}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

# Diagonalisation of matrices – solving systems of ODEs example

- For each eigenvalue, find the unit eigenvector:

$$\lambda_2 = 6 \qquad \Rightarrow \begin{bmatrix} 5-6 & -2 \\ -2 & 2-6 \end{bmatrix} \begin{bmatrix} s_a \\ s_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-s_a - 2s_b = 0 \qquad \Rightarrow s_a = -2s_b$$

$$\text{Set } s_b = 1 \qquad \Rightarrow s_a = -2$$

$$\Rightarrow \mathbf{s}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \hat{\mathbf{s}}_2 = \frac{\mathbf{s}_2}{|\mathbf{s}_2|} = \frac{1}{\sqrt{(-2)^2 + 1^2}} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

# Diagonalisation of matrices – solving systems of ODEs example

- This gives:

$$S = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

- Use the substitution: $\boxed{\mathbf{x} = S\mathbf{y}}$

$$\mathbf{x}' = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}\mathbf{x} \qquad \Rightarrow S\mathbf{y}' = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} S\mathbf{y}$$

$$\Rightarrow \mathbf{y}' = S^{-1}\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} S\mathbf{y}$$

# Diagonalisation of matrices – solving systems of ODEs example

- Recall: $A = SDS^{-1} \quad \Rightarrow D = S^{-1}AS$

- So: $\mathbf{y}' = S^{-1}\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix} S\mathbf{y} \quad \Rightarrow \mathbf{y}' = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}\mathbf{y}$

- The equations in $\mathbf{y}$ are uncoupled (diagonalised) and have the solution:

$$\Rightarrow \begin{array}{l} y_1 = c_1 e^{t} \\ y_2 = c_2 e^{6t} \end{array}$$

# Diagonalisation of matrices – solving systems of ODEs example

- Convert initial conditions:

$$\mathbf{x} = S\mathbf{y} \quad \Rightarrow \mathbf{y} = S^{-1}\mathbf{x}$$

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Rightarrow \mathbf{y}(0) = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \mathbf{y}(0) = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

- This gives: $y_1 = \dfrac{1}{\sqrt{5}} e^{t}$, $y_2 = -\dfrac{2}{\sqrt{5}} e^{6t}$

$$\Rightarrow \mathbf{y} = \begin{bmatrix} \dfrac{1}{\sqrt{5}} e^{t} \\ -\dfrac{2}{\sqrt{5}} e^{6t} \end{bmatrix}$$

# Diagonalisation of matrices – solving systems of ODEs example

- Convert back from $\mathbf{y}$ to $\mathbf{x}$:

$$\mathbf{x} = S\mathbf{y} \qquad \Rightarrow \mathbf{x} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \frac{1}{\sqrt{5}}\begin{bmatrix} e^{t} \\ -2e^{6t} \end{bmatrix}$$

$$\Rightarrow \mathbf{x} = \frac{1}{5}\begin{bmatrix} e^{t} + 4e^{6t} \\ 2e^{t} - 2e^{6t} \end{bmatrix}$$

Solution!

# Diagonalisation of matrices – solving systems of coupled ODEs

- Matrix diagonalisation can be used to solve systems of ODEs:

  e.g. $\mathbf{x}' = A\mathbf{x}$

- Use the substitution: $\boxed{\mathbf{x} = S\mathbf{y}}$

  $\Rightarrow S\mathbf{y}' = AS\mathbf{y} \qquad \Rightarrow \mathbf{y}' = S^{-1}AS\mathbf{y}$

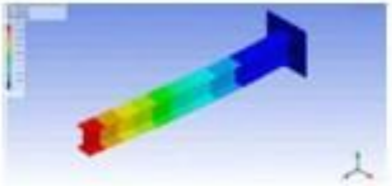
- Recall: $D = S^{-1}AS$

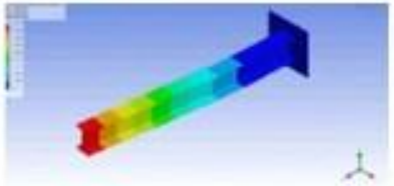
  $\Rightarrow \mathbf{y}' = D\mathbf{y}$ uncoupled (diagonal) system

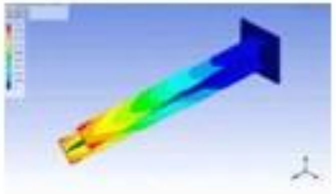
- Solve uncoupled system to find $\mathbf{y}$ (easy); convert/apply initial conditions; invert for $\mathbf{x}$

# Eigenvalues and eigenvectors – many applications

- Face recognition (principal component analysis)
  http://en.wikipedia.org/wiki/Eigenface (machine learning)
  http://cognitrn.psych.indiana.edu/nsfgrant/FaceMachine/faceMachine.html
- Electrical engineering: circuit, controllers, MEMS design
- Chemical engineering: stability of reactions
- Structural engineering (vibrations and modal analysis)
  http://en.wikipedia.org/wiki/Vibration







- Numerical methods: stability of iterative solutions to ODEs/PDEs
- many, MANY more…

# Key points: eigen-analysis

- Know how to compute the eigenvalues and eigenvectors of a square matrix:

  $$\det(A - \lambda I) = 0 \qquad (A - \lambda I)\mathbf{s} = \mathbf{0}$$

- Determine quadratic form of equation for an ellipse, and then use eigen-analysis to plot the ellipse.

- Use eigen-analysis to diagonalise a matrix:

  $$A = SDS^{-1} \qquad D = S^{-1}AS$$

- Use eigen-analysis to compute powers of matrices.

- Use eigen-analysis to solve systems of coupled first order ODEs by substituting $\mathbf{x}=S\mathbf{y}$ into $\mathbf{x}'=A\mathbf{x}$

# Exercises

**1.** When loads are applied to the boundary of a small element of linearly elastic sheet metal the stresses are:

$$\sigma_{xx} = 140 \text{ N/mm}^2, \ \sigma_{yy} = -50 \text{ N/mm}^2, \ \sigma_{xy} = \sigma_{yx} = 80 \text{ N/mm}^2$$

Find the eigenvectors of the matrix

$$A = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{bmatrix}.$$

These vectors give the directions of principal stress, along which the shear stress vanishes. The eigenvalues are the values of these principal stresses.

**2.** Write the following in quadratic form and as $\mathbf{x}^T A\mathbf{x}$ for some symmetric matrix $A$. Classify them as positive definite, indefinite or negative definite.

**(a)** $x_1^{\ 2} + 4x_1x_2 + 4x_3^{\ 2} - 10x_1x_3 + 5x_2^{\ 2}$

**(b)** $4x_1^{\ 2} - 2x_1x_2 + 4x_2^{\ 2} + 2x_1x_3 + 6x_2x_3 + 5x_3^{\ 2}$

**(c)** $x_1^{\ 2} = 4x_1x_2 + x_2^{\ 2} + 2x_1x_3 - 2x_2x_3 + 2x_3^{\ 2}$

**(d)** $2x_1x_2 - x_1^2 - 2x_2^2 - 2x_2x_3 - 2x_3^2$

**3.** Find a real symmetric matrix $C$ such that $Q = \mathbf{x}^T C\mathbf{x}$, where $Q$ equals the expressions below. Test for positive definiteness.

**(a)** $(x_1 - x_2)^2$

**(b)** $x_1^2 + 2x_1x_2 + 3x_2^2 + 6x_2x_3 + 2x_3^2$

**(c)** $(x_1 - x_2 + 2x_3 - 2x_4)^2$

**(d)** $(x_1 + x_2)^2 + (x_3 + x_4)^2$

4. (a) Find the principal directions of the following ellipses and use it to express them in the form:

$$\frac{u^2}{a^2}+\frac{v^2}{b^2}=r$$

where $r$ is the right-hand side of the equation.

(i) $17x^2+12xy+8y^2=5$

(ii) $41x^2-24xy+34y^2=25$

(iii) $5x^2-2xy+5y^2=2$

(iv) $41x^2-24xy+34y^2=25$

(b) Use **EXCEL** to find the principal directions.

(c) Use **MATLAB** to find the principal directions.

**5.** Compute the directions of the principal axes of the following ellipses and write them in the form

$$\frac{y_1^{\,2}}{a^2} + \frac{y_2^{\,2}}{b^2} = 1$$

where $y_1$ and $y_2$ are linear functions of $x_1$ and $x_2$:

**(a)** $3.4x_1^{\,2} + 2.4x_1x_2 + 1.6x_2^{\,2} = 1$

**(b)** $5x_1^{\,2} - 4x_1x_2 + 2x_2^{\,2} = 1$

6. Compute the directions of the principal axes of the following hyperbola and write them in the form

$$\frac{y_1^{\,2}}{a^2} - \frac{y_2^{\,2}}{b^2} = 1$$

where $y_1$ and $y_2$ are linear functions of $x_1$ and $x_2$:

(a) $x_1^{\,2} + 4x_1x_2 - 2x_2^{\,2} = 1$

(b) $4x_1^{\,2} + 6x_1x_2 - 4x_2^{\,2} = 1$

7. Find out what ellipse (or pair of straight lines) is represented by the following quadratic forms. Transform it to principal axes. Express $\mathbf{x}^T = [x_1 \quad x_2]$ in terms of the new coordinate vector $\mathbf{y}^T = [y_1 \quad y_2]$.

(a) $3x_1^2 + 4\sqrt{3}x_1x_2 + 7x_2^2 = 9$

(b) $4x_1x_2 + 3x_2^2 = 1$

**8.** Let

$$A = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 4 & 3 \\ 0 & 3 & 8 \end{bmatrix}$$

**(a)** Compute the first two leading principal minors of $A$ by hand, and the third leading principal minor using **MATLAB**.
**(b)** Compute the eigenvalues of $A$ using **MATLAB**.
**(c)** Determine whether $A$ is positive definite, positive semidefinite, negative definite, negative semidefinite, or indefinite.

**9.** Let

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 3 & 8 \end{bmatrix}$$

**(a)** Compute the three upper-left determinants (principal minors) of $A$.
**(b)** Compute the eigenvalues of $A$.
**(c)** Determine whether $A$ is positive definite, positive semidefinite, negative definite, negative semidefinite, or indefinite.