Linear Algebra
Revision- sample papers
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Find all possible values of k for which $\begin{pmatrix} 1 & k & 0 \\ 2 & -3 & k \\ 1 & k & 2 \end{pmatrix}$ is singular.

Find all possible solutions for $\det\begin{pmatrix} x & 6 & -9 \\ 2 & 4 & x \\ -1 & -2 & 3 \end{pmatrix} = 0.$

Matrices $\mathbf{A}$ and $\mathbf{B}$ are given by $\mathbf{A} = \begin{pmatrix} a & 2 & 3 \\ 0 & 4 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -2 & b \\ 3 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

Given that both $\mathbf{A} + \mathbf{B}$ and $\mathbf{A} - \mathbf{B}$ are singular, find the values of a and b.

Matrices $\mathbf{A}$ and $\mathbf{B}$ are given by $\mathbf{A} = \begin{pmatrix} 1 & k & 0 \\ 3 & 1 & 4 \\ -1 & 3 & -2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -2 & k & -5 \\ -2 & 0 & -3 \\ k & 1 & 2 \end{pmatrix}$.

a Find each of $\det \mathbf{A}$ and $\det \mathbf{B}$ in terms of k.

b Without finding $\mathbf{AB}$, determine all values of k for which $\mathbf{AB}$ is singular.

Matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$.

a Find $\det \mathbf{M}$.

b A three-dimensional object A with volume $2.5\,\text{cm}^3$ is transformed by a linear transformation with matrix $\mathbf{T}$. Write down the volume of the image of A when:

i $\mathbf{T} = \mathbf{M}$

ii $\mathbf{T} = \mathbf{A}^{-1}\mathbf{M}\mathbf{A}$ for some non-singular matrix $\mathbf{A}$

iii $\mathbf{T} = \mathbf{M}\mathbf{B}^2$ where $\mathbf{B}$ is a 3×3 matrix with determinant 0.2.

Transformation T is given by matrix $\mathbf{M}$ where $\mathbf{M} = \begin{pmatrix} 2 & 1 & p \\ 3 & -2 & 1 \\ q & 4 & -1 \end{pmatrix}$.

a Given $\mathbf{M}^{-1} = \frac{1}{8}\begin{pmatrix} -2 & 5 & 3 \\ 2 & -1 & 1 \\ r & 1-r & 3-r \end{pmatrix}$, find p, q and r.

b Given that T maps point A to $(1, 1, -1)$, find the coordinates of A.

Two invertible transformations $\mathbf{A}$ and $\mathbf{B}$ are defined by matrices $\mathbf{A} = \begin{pmatrix} 1 & a & 2 \\ 0 & 3 & 1 \\ 2 & 2 & 1 \end{pmatrix}$ and

$$\mathbf{B} = \begin{pmatrix} 0 & 4 & b \\ 1 & 1 & 2 \\ 1 & 2 & -1 \end{pmatrix}.$$

Point C has image $(-2, 1, 8)$ under both matrices.

Find a, b and the coordinates of point C.

Matrix $\mathbf{M}$ is given by $\begin{pmatrix} 2 & 4 & 8 \\ x & x^2 & x^3 \\ y & y^2 & y^3 \end{pmatrix}$.

a Explain why the structure of the matrix shows that these must be factors of the determinant of $\mathbf{M}$:

i x

ii $(x - y)$

iii $(x - 2)$.

b Find the determinant of $\mathbf{M}$, factorising your answer as far as possible.

Matrix $\mathbf{M}$ and $\mathbf{N}$ are given by $\mathbf{M} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} a+bk & b & c \\ d+ek & e & f \\ g+hk & h & i \end{pmatrix}$.

a Define the column operation which converts $\mathbf{M}$ to $\mathbf{N}$.

b By considering $\mathbf{M} = \mathbf{NR}$ for some matrix $\mathbf{R}$, show that $\det \mathbf{M} = \det \mathbf{N}$.

Express the determinant $\begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ p^3 & q^3 & r^3 \end{vmatrix}$ as the product of four linear factors.

For what values of k does the system of equations

$$\begin{cases} -x + (2k-5)\,y - 2z = 2 \\ (1+k)\,x - y + (k-1)z = 5 \\ x + y + 2z = 1 \end{cases}$$

have no unique solution?

Matrix $\mathbf{M}$ is given by $\mathbf{M} = \begin{pmatrix} a & 1 & 2 \\ 3 & a & -2 \\ -1 & 1 & 1 \end{pmatrix}$.

a Show that $\mathbf{M}$ is non-singular for every real value a.

b Find $\mathbf{M}^{-1}$ in terms of a.

c Use $\mathbf{M}^{-1}$ to solve the equations $\begin{cases} x + y + 2z = 4 \\ 3x + y - 2z = 1 \\ -x + y + z = -2 \end{cases}$

a Show that the system of equations $\begin{cases} kx + 3y - z = -2 \\ -3x + (k+4)y + z = -8 \\ x + 3y + (k-2)z = 4 \end{cases}$

has no unique solution for $k = 1$.

b Find all values of k for which there is no unique solution.

c For $k = 3$, find x, y and z.

A system of equations is given by $\begin{cases} 3x + y + z = 8 \\ -7x + 3y + z = 2 \\ x + y + 3z = 0 \end{cases}$

a Show that the system has a unique solution and find this solution.

b The three equations represent planes. Describe the configuration of the three planes.

a Show that there is no unique solution to the equation system given by $\begin{cases} 2x - y + z = 6 \\ 3x + y + 5z = -7 \\ x - 3y - 3z = 8 \end{cases}$

b Show that the system is inconsistent.

c Interpret the system geometrically.

Consider this system of equations:

$$\begin{cases} 2x + y - 2z = 0 \\ x - 2y - z = 2 \\ 3x + 4y - 3z = d \end{cases}$$

a Show that the system does not have a unique solution.

b Find the value of d for which the system is consistent.

c The three equations represent planes. For the value of d found in part **b**, describe the configuration of the three planes.

d For the value of d found in part **b**, solve the system of equations.

a Show that the system of equations

$$\begin{cases} x + y = 0 \\ x - 4y - 2z = 0 \\ \frac{1}{2}x + 3y + z = 0 \end{cases}$$

is consistent.

The three equations in part **a** represent three planes.

b Describe the geometrical configuration of the planes.

c Find the solution of the system.

a Find the inverse of the matrix $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$.

b Hence find, in terms of d, the coordinates of the point of intersection of the planes $x - y = 4$, $y + z = 1$ and $x - z = d$.

Consider the system of equations

$$\begin{cases} x - 2y - z = -2 \\ 2x + y - 3z = \quad 9 \\ x + 3y - az = 3 \end{cases}$$

a Find the value of a for which the system does not have a unique solution.

b For the value of a found in part **a**, determine whether the system is consistent, and describe the geometric configuration of the three planes represented by the system.

a Find the value of p for which the system of equations $\begin{cases} x - y - z = -2 \\ 2x + 3y - 7z = a + 4 \\ x + 2y + pz = a^2 \end{cases}$ does not have a unique solution.

b For the value of p found in part **a**, find the two values of a for which the system is consistent.

c Describe the geometric configuration of the three planes represented by the three equations.

d Find the solution of the system for the value of p from part **a** and the larger of the two values of a from part **b**.

Find the value of k that results in the system of equations shown having a non-unique solution.

$$\begin{cases} x + y + z = k \\ x - y + z = k^2 \\ kx + 2ky + 3kz = k^3 \end{cases}$$

Choose from these options.

A 0

B 1

C 2

D 3

Let $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix}$.

a Use a row operation to show that $(x+y+z)$ is a factor of Δ.

b Hence, or otherwise, express Δ as a product of linear factors.