Linear Algebra – MC2020

Markov process

Markov Process. Powers of a Matrix. Stochastic Matrix

Suppose that the 2004 state of land use in a city of 60 mi^2 of built-up area is

C: Commercially Used 25%   I: Industrially Used 20%   R: Residentially Used 55%.

Find the states in 2009, 2014, and 2019, assuming that the transition probabilities for 5-year intervals are given by the matrix A and remain practically the same over the time considered.

$$\begin{array}{cccc} & \text{From C} & \text{From I} & \text{From R} \\ \end{array}$$
$$\mathbf{A} = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix} \begin{array}{l} \text{To C} \\ \text{To I} \\ \text{To R} \end{array}$$

A is a **stochastic matrix**, that is, a square matrix with all entries nonnegative and all column sums equal to 1. Our example concerns a **Markov process**,[1] that is, a process for which the probability of entering a certain state depends only on the last state occupied (and the matrix A), not on any earlier state.

A **Markov process** is useful for analyzing dependent random events - that is, events whose likelihood depends on what happened last.

Solution. From the matrix A and the 2004 state we can compute the 2009 state,

$$\begin{array}{c} \text{C} \\ \text{I} \\ \text{R} \end{array} \begin{bmatrix} 0.7 \cdot 25 + 0.1 \cdot 20 + 0 \cdot 55 \\ 0.2 \cdot 25 + 0.9 \cdot 20 + 0.2 \cdot 55 \\ 0.1 \cdot 25 + 0 \cdot 20 + 0.8 \cdot 55 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.1 & 0 \\ 0.2 & 0.9 & 0.2 \\ 0.1 & 0 & 0.8 \end{bmatrix} \begin{bmatrix} 25 \\ 20 \\ 55 \end{bmatrix} = \begin{bmatrix} 19.5 \\ 34.0 \\ 46.5 \end{bmatrix}.$$

To explain: The 2009 figure for C equals 25% times the probability 0.7 that C goes into C, plus 20% times the probability 0.1 that I goes into C, plus 55% times the probability 0 that R goes into C. Together,

$$25 \cdot 0.7 + 20 \cdot 0.1 + 55 \cdot 0 = 19.5\ [\%]. \qquad \text{Also} \qquad 25 \cdot 0.2 + 20 \cdot 0.9 + 55 \cdot 0.2 = 34\ [\%].$$

Similarly, the new R is 46.5%. We see that the 2009 state vector is the column vector

$$\mathbf{y} = [19.5 \quad 34.0 \quad 46.5]^\mathsf{T} = \mathbf{Ax} = \mathbf{A}\,[25 \quad 20 \quad 55]^\mathsf{T}$$

where the column vector $\mathbf{x} = [25 \quad 20 \quad 55]^\mathsf{T}$ is the given 2004 state vector. Note that the sum of the entries of $\mathbf{y}$ is 100 [%]. Similarly, you may verify that for 2014 and 2019 we get the state vectors

$$\mathbf{z} = \mathbf{Ay} = \mathbf{A}(\mathbf{Ax}) = \mathbf{A}^2\mathbf{x} = [17.05 \quad 43.80 \quad 39.15]^\mathsf{T}$$
$$\mathbf{u} = \mathbf{Az} = \mathbf{A}^2\mathbf{y} = \mathbf{A}^3\mathbf{x} = [16.315 \quad 50.660 \quad 33.025]^\mathsf{T}.$$

Answer. In 2009 the commercial area will be 19.5% (11.7 mi^2), the industrial 34% (20.4 mi^2), and the residential 46.5% (27.9 mi^2). For 2014 the corresponding figures are 17.05%, 43.80%, and 39.15%. For 2019 they are 16.315%, 50.660%, and 33.025%. (In Sec. 8.2 we shall see what happens in the limit, assuming that those probabilities remain the same. In the meantime, can you experiment or guess?)

Model Questions

1. Suppose that the initial population of city detail given below. There is 35000 population in urban area and 65000 population in suburban area. If every year 40% of the urban area population moves to the suburbans, while 30% of the suburban population moves to the urban area, determine:

   a) The transition matrix, T that can be used to represent this information.
   b) The estimated number of people living in the suburban after 4 years, assuming that the total population remains constant. Give your answer to the nearest 100 people.

2. A country is divided into three demographic regions which are region 1, region2 and region 3. Initially there are 12000, 8000 and 6000 residents are in the regions 1,2 and 3 respectively. It is found that each year 5% of the residents of region 1 move to region 2, and 5% move to region 3. Of the residents region 2, 15% move to region 1 and 10% move to region 3. Of the residents region 3, 10% move to region 1 and 5% move to region 2. determine:

   a) The transition matrix, T that can be used to represent this information.
   b) What percentage of the population residents in each of the tree region after 2 years.