UNIVERSITY OF JAFFNA
FACULTY OF ENGINEERING

Assignment Test 02 – December 2022

Linear Algebra

MC 2020

Reading Time: Five Minutes

Writing Time: 105 Minutes

Question 1[15 marks]

1. (a) The matrix $\mathbf{S} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
   i. Give a geometrical interpretation of the transformation represented by S.
   ii. Show that $\mathbf{S}^2 = \mathbf{I}$.
   iii. Give a geometrical interpretation of the transformation represented by $\mathbf{S}^{-1}$.

   (b) The matrix $\mathbf{T} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
   i. Give a geometrical interpretation of the transformation represented by T.
   ii. Show that $\mathbf{T}^2 = \mathbf{I}$.
   iii. Give a geometrical interpretation of the transformation represented by $\mathbf{T}^{-1}$.

   (c) Calculate det S and det T and comment on their values in the light of the transformations they represent.

-20 —

Question 2[30 marks]

2. (a) Let A be a transformation that rotates the plane by 60 degrees around the origin.
   i. Write down the matrix that represents A.
   ii. Compute the matrix $A^3 = AAA$ and verify that it rotates the plane through 180 degrees.
   iii. By considering the geometry, describe the transformation that would be represented by the matrix A^{60}.

-30 —

   (b) Transform the shape with vertices $A = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $D = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ by the matrix $M = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$. Sketch the shape $ABCD$ and its image $A'B'C'D'$. Find the determinant of M and interpret the result.

Question 3[30 marks]

3. (a) My Tunes has 10000 users signed up to their music download service. These are casual users and premium users. Every year 10% of people who are casual users sign up for premium service, and 30% of premium subscribers let their subscriptions lapse and so become casual users. No new users are admitted to the service and nobody leaves.

   i. Write down a transition matrix A for the Markov chain that models My Tunes users.
   ii. Suppose at the start of this year there are 8000 casual users and 2000 premium users. How many of each type will there be at the end of the year?
   iii. What proportion of users will be premium users in the long run?

   (b) The transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix T where $T^{-1} = \begin{pmatrix} 2 & 3 & 3 \\ -1 & 4 & 5 \\ 2 & 1 & 1 \end{pmatrix}$. The point with position vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is transformed by T to the point with position vector $\begin{pmatrix} -12 \\ -7 \\ 8 \end{pmatrix}$.

30

   i. Find the values of the constants a, b and c.
   A line l_1 which passes through the origin is transformed by T to the line l_2.
   A vector equation $r = t\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, where t is a parameter.
   ii. Find a vector equation of l_2.

Question 4[25 marks]

4. (a) Consider the system of equations

$$\begin{aligned} x + y + kz &= -2 \\ 3x + 4y - z &= -3k \\ kx - y + z &= 2 \end{aligned}$$

   Find the value of k, when the system

   i. has no solution
   ii. has infinite solution
   iii. has unique solution

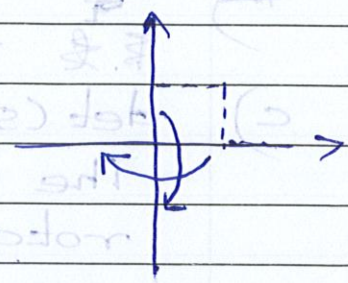
20

   (b) The transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the matrix T where $T = \begin{pmatrix} 3 & -2 & -2 \\ -2 & -8 & 4 \\ -2 & 4 & 0 \end{pmatrix}$. The plane Π_1 is transformed by T to the plane Π_2. The plane Π_1 has Cartesian equation $x - 2y + z = 0$.
   Find a Cartesian equation of Π_2.

Assignment-02

Solutions.

4) a) $S = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$



i) Rotation of 180° about $(0,0)$

ii) $S^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

iii) Rotation of 180° about $(0,0)$

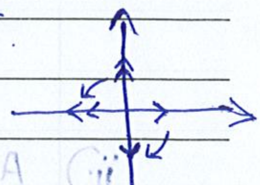
$S^{-1} = \frac{1}{1}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

b) $T = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

i) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ maps to $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ maps to $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

Hence it is reflection in $y = -x$



ii) $T^2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

iii) Reflection in $y = -x$

$T^{-1} = \frac{1}{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

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iii) $\cancel{\overline{B}}.\cancel{x}$ $T\cancel{x} = \cancel{x_2}$

c) $\det(S) = 1 - 0 = 1$
The area of a shape is unchanged by a rotation.

$\det T = 0 - 1 = -1$
The area of a shape is unchanged by a reflection (the minus sign indicating that it has been reflected)

2) a) $A = \begin{pmatrix} \cos(k) & -\sin(k) \\ \sin(k) & \cos(k) \end{pmatrix}$

Anticlockwise

Clockwise

i) $A = \begin{pmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{pmatrix}$ $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$= \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$ $= \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$

ii) $AAA = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$

$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} \cos 180 & -\sin 180 \\ \sin 180 & \cos 180 \end{pmatrix}$ $\begin{pmatrix} \cos 180 & \sin 180 \\ -\sin 180 & \cos 180 \end{pmatrix}$

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iii) $(A^6)^{10} = A^{60} = (A^3A^3)^{10}$

$$= \left[\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\right]^{10}$$

$$= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}^{10}$$

$$= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$= I$$

I is an (2×2) identity matrix.

b) $M = \begin{bmatrix}-1 & 1\\2 & 1\end{bmatrix}$

$$\begin{bmatrix}-1 & 1\\2 & 1\end{bmatrix}\begin{array}{c}\begin{array}{cccc}A & B & C & D\end{array}\\\begin{bmatrix}2 & 1 & -1 & 2\\-1 & 1 & 2 & 1\end{bmatrix}\end{array}$$

$$= \begin{bmatrix}-2-1 & -1+1 & 1+2 & -2+1\\4-1 & +2+1 & -2+2 & 4+1\end{bmatrix}$$

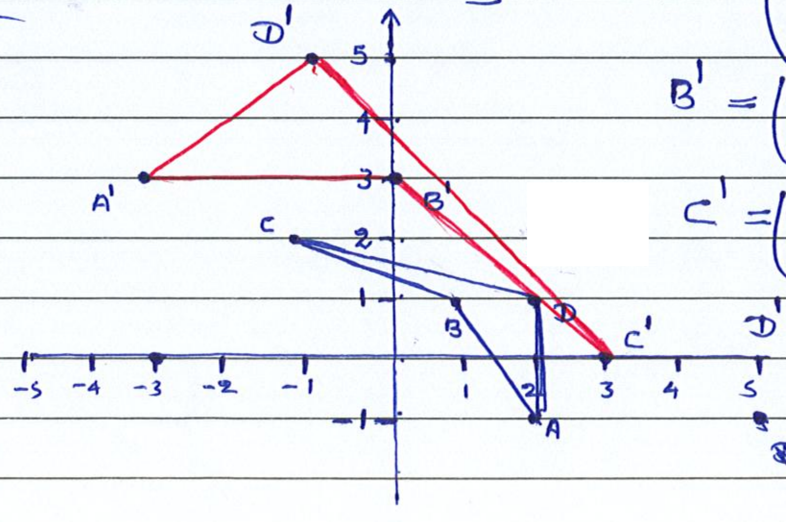
$$= \begin{array}{c}\begin{array}{cccc}A' & B' & C' & D'\end{array}\\\begin{bmatrix}-3 & 0 & 3 & -1\\3 & +3 & 0 & 5\end{bmatrix}\end{array}$$

$A' = \begin{pmatrix}-3\\3\end{pmatrix}$

$B' = \begin{pmatrix}0\\3\end{pmatrix}$

$C' = \begin{pmatrix}3\\0\end{pmatrix}$

$D' = \begin{pmatrix}-1\\5\end{pmatrix}$



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b) $\text{Det}(M) = \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix}$

$= -1-2$

$= -3$

So the shape will expand 3 times and it was the mirrow shadow of the first shape.

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Linear Algebra

from .

3 (a)

$$\begin{array}{ccc} & \text{Causual users} & \text{Premium users} \\ & & \end{array}$$

i)

$$\begin{array}{c} \\ \end{array}\begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix}\begin{array}{l} \text{Causual users} \\ \text{Premium users} \end{array}$$

ii)

$$\begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix}\begin{pmatrix} 8000 \\ 2000 \end{pmatrix} = \begin{pmatrix} 7800 \\ 2200 \end{pmatrix}$$

$$\begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix}\begin{pmatrix} 3000 \\ 2000 \end{pmatrix} = \begin{pmatrix} 3300 \\ 1700 \end{pmatrix}$$

iii)

$$\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix}\begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}$$

$Pq = q$

$(I-P)q = 0$

$$\begin{pmatrix} 0.1 & -0.3 \\ -0.1 & 0.3 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

$\pi_1 = 0.9\pi_1 + 0.3\pi_2$

$\pi_2 = 0.1\pi_1 + 0.7\pi_2$

$0.1\pi_1 = 0.3\pi_2 \Rightarrow \pi_2 = \frac{1}{3}\pi_1$ — (1)

$\pi_1 + \pi_2 = 1$ — (2)

$0.1q_1 = 0.3q_2$

$q_1 = 3q_2$

(1) $\Rightarrow \pi_1 + \frac{1}{3}\pi_1 = 1$

$\pi_1 = \frac{3}{4}$

$q_1 + q_2 = 1$

$4q_2 = 1$

$\pi_2 = \frac{1}{4}$

$q_2 = \frac{1}{4}$

$\frac{1}{4}$ proportion of users will be premium use

OR

After 1 year $AX = \begin{pmatrix} 7800 \\ 2200 \end{pmatrix}$ C P

After 2 year $A^2X = \begin{pmatrix} 7680 \\ 2320 \end{pmatrix}$

$A^3X = \begin{pmatrix} 7608 \\ 2392 \end{pmatrix}$

$A^4X = \begin{pmatrix} 7564.3 \\ 2435.3 \end{pmatrix}$

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Proportion of premium

Year 1 $= \frac{2200}{10000} = 0.22$

Year 2 $= \frac{2220}{10,000} = 0.232$

⋮

Year 4 $= \frac{2435.2}{10\,000} \simeq 0.24$

It is about 0.24 / 0.3

b) $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$T \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -12 \\ -7 \\ 8 \end{pmatrix}$$

i) $$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = T^{-1} \begin{pmatrix} -12 \\ -7 \\ 8 \end{pmatrix}$$

ii) $\begin{pmatrix} 1/4 \\ 17/4 \\ -15/4 \end{pmatrix} t$

$$= \begin{pmatrix} 2 & 3 & 3 \\ -1 & 4 & 5 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} -12 \\ -7 \\ 8 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix} t$

$$= \begin{pmatrix} 2\times -12 + 3\times -7 + 3\times 8 \\ -1\times -12 + 4\times -7 + 5\times 8 \\ 2\times -12 + -7 + 8 \end{pmatrix}$$

52
28
24

-31
8
23

$$= \begin{pmatrix} -21 \\ 24 \\ -23 \end{pmatrix}$$

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4) a) $x+y+kz=-2$

$3x+4y-z=-3k$

$kx+y+z=2$

$$\begin{bmatrix} 1 & 1 & k \\ 3 & 4 & -1 \\ k & -1 & +1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} -2 \\ -3k \\ 2 \end{pmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & k & -2 \\ 3 & 4 & -1 & -3k \\ k & -1 & 1 & 2 \end{array}\right]$$

$r_2 \to r_2 - 3r_1$

$r_3 \to r_3 - kr_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & k & -2 \\ 0 & 1 & -(3k+1) & -3(k-2) \\ 0 & -(1+k) & (1-k^2) & 2(k+1) \end{array}\right]$$

$r_3 \to r_3 + (1+k)r_2$

$(1-k^2)-(1+k)(3k+1)$ | $2(k+1)+(1+k)$ $-3(k$

$$\left[\begin{array}{ccc|c} 1 & 1 & k & -2 \\ 0 & 1 & -(3k+1) & -3(k-2) \\ 0 & 0 & -4k(1+k) & (k+1)(-3k+8) \end{array}\right]$$

$3k^2-5k-8$

a) If no solutions.

$(1+k)(-4k)=0$ and $(k+1)(-3k+8)\neq 0$

$(k=0 \text{ or } k=-1) \wedge (k\neq -1 \text{ or } k\neq 8/3)$

$\Rightarrow k=0$ 02

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b) If infinite solutions

$(1+k)(-4k)=0$ and $(k+1)(-3k+8)=0$

$(k=-1 \text{ or } k=0)$ and $(k=-1 \text{ or } k=\frac{8}{3})$

$\Rightarrow k=-1$.

c) If unique solution.

$(1+k)(-4k) \neq 0$

$k \neq -1 \wedge k \neq 0$

$\Rightarrow k \in (-\infty,-1) \cup (-1,0) \cup (0,\infty)$

OR

$(1+k)(-4k) \neq 0 \wedge (k+1)(-3k+8) \neq 0$

$(k \neq -1 \wedge k \neq 0) \wedge (k \neq -1 \wedge k \neq \frac{8}{3})$

$k \neq -1$

For this case $k=0$ also include.

Not

ප්‍රශ්නයේ අංකය [illegible] තීරයේ ලියන්න.
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4) b)

$x - 2y + z = 0$

~~$x = t,\ y = t_1,\ z = 2t_1 - t$~~

$x = s,\ y = t,\ z = 2t - s$

$T\underline{r}_1 = \underline{r}_2$

$$\begin{pmatrix} 3 & -2 & -2 \\ -2 & -8 & 4 \\ -2 & 4 & 0 \end{pmatrix}\begin{pmatrix} s \\ t \\ 2t-s \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$3s - 2t - 2(2t - s) = x$

$5s - 6t = x$ —— ①

$-2s - 8t + 4(2t - s) = y$

$-6s = y$ —— ②

$-2s + 4t = z$ —— ③

① × 4 + ③ × 6 $\Rightarrow$

$20s - 24t - 12s + 24t = 4x + 6z$

$8s = 4x + 6z$ —— ④

④, ② $\Rightarrow$ $\cancel{8}^{4} \times -\frac{y}{\cancel{6}_{3}} = 4x + 6z$

$-4y = 12x + 18z$

$6x + 2y + 9z = 0$