MC3010: Differential Equations & Numerical Methods

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MC3010: Differential Equations & Numerical Methods

Course subsections:

- Multivariable Calculus
- Differential Equations
- Numerical Methods

Course Plan

Course Title	**Differential Equations and Numerical Methods**
Course Code	MC3010
Course Credit	03
Course Status	Core
Prerequisite	Mathematics MC1020, Computing EC1010
Synopsis	This course integrates mathematical models to provide students with an introduction to the advanced skills required for engineering. Topics included are: Ordinary Differential Equations, Multivariable Calculus, and Numerical Methods. The introduction to Ordinary Differential Equations, first order separable ODEs, first order and second order linear ODEs with constant coefficients. Multivariable Calculus includes representation of functions of several variables, surfaces and curves in space, partial differentiation, directional derivatives, gradient, divergence and curl, line integrals, surface integrals, Green's theorem. The topic of Numerical method involves non-linear equations, systems of linear equation, interpolation and curve fitting, numerical differentiation and numerical integration, and ordinary differential equations.

By the end of this course, students should be able to;

ILO 01 : Discuss the concepts of differential equations, and Multivariable calculus

ILO 02 : Compare and analyse the methods in differential equations and Multivariable calculus

ILO 03 : Translate engineering problems in differential equations and Multivariable calculus and solve it

ILO 04 : Apply appropriate numerical methods to solve engineering problems and implement it using suitable software.

Course Plan cont'd

Week	Topics	ILO	Delivery	Assessment/ Evaluation
01	Numerical errors	04	• Lecture • Tutorial	Assignment
02	Roots of equations	04	• Lecture • Tutorial	Assignment
03	System of linear equations	04	• Lecture • Tutorial	Assignment
04	Numerical differentiation	04	• Lecture • Tutorial	Assignment
05	Numerical integration	04	• Lecture • Tutorial	Assignment
06	Partial Differentiation	01	• Lecture • Tutorial	Assignment
07	Curves and Integrals	03	• Lecture • Tutorial	Assignment
08	Multivariable Integration	01	• Lecture • Tutorial	Assignment
	Mid – Semester Examination (20%)			
09	Introduction to Differential Equations	02	• Lecture • Tutorial	Assignment
10	Higher order ordinary differential equations	02	• Lecture • Tutorial	Assignment
11	Higher order ordinary differential equations	02	• Lecture • Tutorial	Assignment
12	Nonhomogeneous equations and operator methods	02	• Lecture • Tutorial	Assignment
13	Nonhomogeneous equations and operator methods	02	• Lecture • Tutorial	Assignment
14	Laplace transform	02	• Lecture • Tutorial	Assignment
15	Numerical solution of ordinary differential equations	04	• Lecture • Tutorial	

Teaching - Learning Approach		Hours per semester
	Lectures	29
	Tutorials (2 hours = 1 Lecture)	16
	Total	**45**

Course Plan cont’d

		Percentage
Assessment	Assignments	30
	Mid Semester Assessment	20
	End Semester Examination	50
Total		100

Assessment	**Week**	**Tentative Date**
Assignment – 01	Week 03	01st March 2024
Assignment – 02	Week 06	22nd March 2024
Mid-Semester Exam	Week 08	05th April 2024
Assignment – 03	Week 10	26th April 2024
Assignment – 04	Week 13	17th May 2024

Further Reading…

Resources

1. Advanced engineering mathematics", by H.K.Dass, S.Chand, 1988
2. Advanced engineering mathematics", by Ladis D.Kovach, Addison-wesley, 1982.
3. Advanced Engineering Mathematics by Kreyszig
4. Modern Engineering Mathematics by James
5. Mathematics for Engineers by Croft and Davison
6. Mathematical Methods for Engineers and Scientists by Fitzgerald and Peckham.
7. Applied Numerical Methods WITH MATLAB for Engineers and Scientist, Steven C. Chapra

Numerical Methods : Syllabus Outline

1. Numerical Errors
2. Roots of Equations
3. System of Linear Equations
4. Numerical Differentiation
5. Numerical Integration
6. Numerical Solution for Ordinary Differential Equations

Introduction to Numerical Computing

What do we mean by "Numerical Computing's“?

- Numerical computation is an essential tool for solving complex problems.
- Numerical computing/ analysis is concerned with the solutions of mathematically formulated problems via computer.
- It is a branch of computer science and mathematics that involves the use of algorithms and computer software to solve mathematical problems that are too complex to be solved analytically.

Introduction to Numerical Computing cont'd

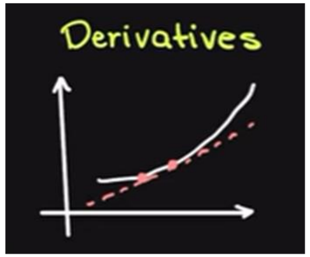
Numerical Methods	Analytical Methods
- Approximate	- Exact

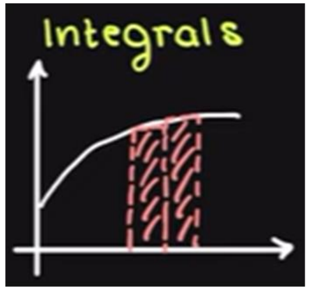
Root finding ← Algebra

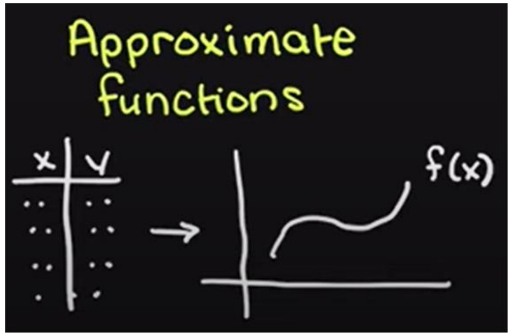
Ax = B ← Linear Algebra

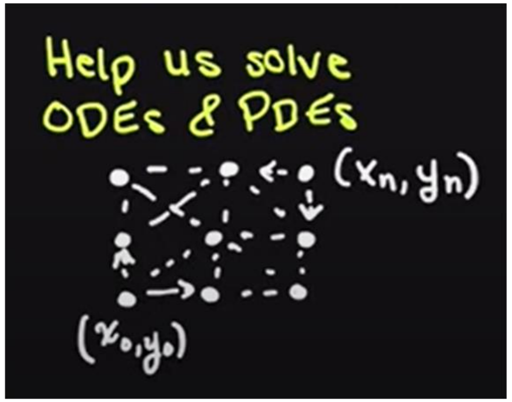
Integration/differentiation ← Calculus

Differential Equations ← Differential Equations









Introduction to Numerical Computing cont'd

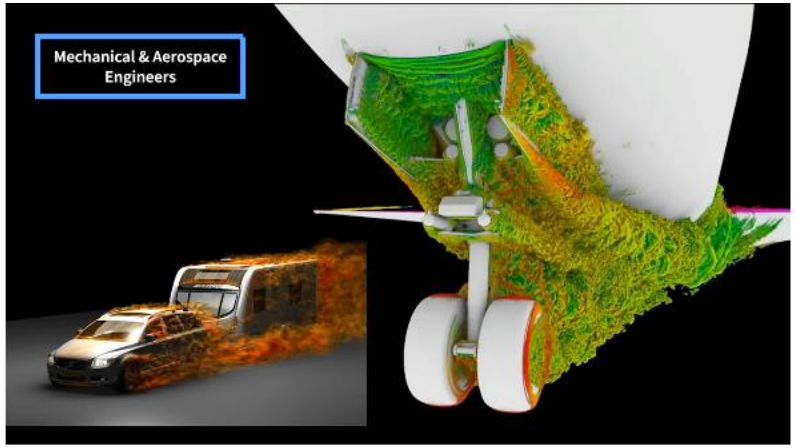
1. Limitations of analytical methods in practical applications; i.e. can't solve a mathematically formulated problem with a pen and paper alone
2. Nonexistence of exact solution

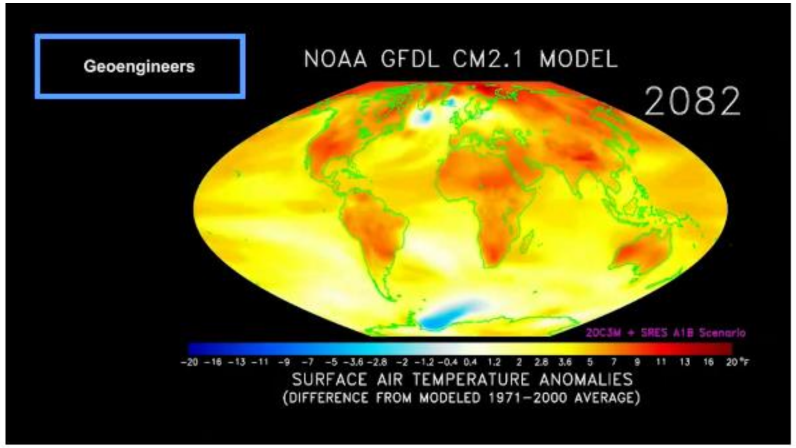
Approach Numerical methods

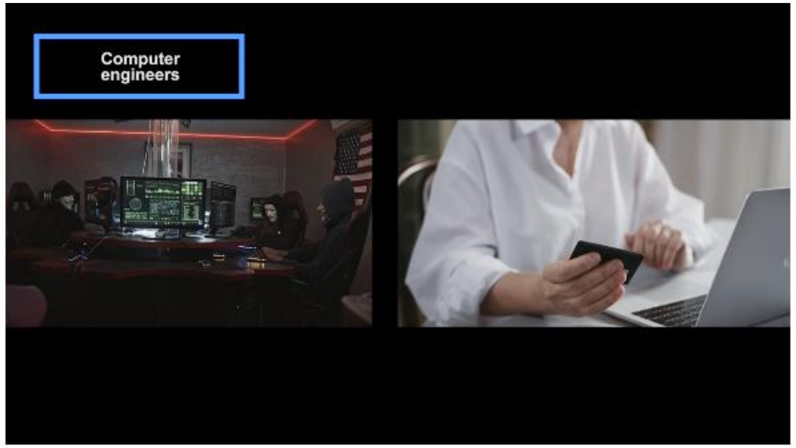
Approach involves formulation of model of physical situations that can be solve with arithmetic operations

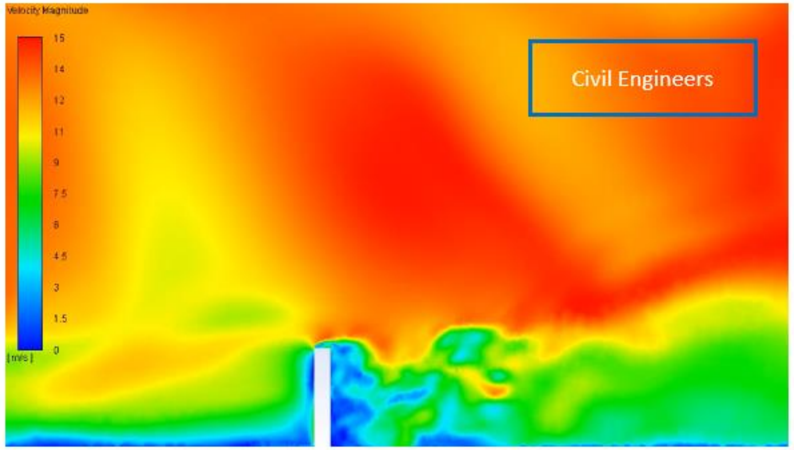
1. Formulation of mathematical model
2. Construction of an appropriate numerical method
3. Implement the method to obtain a solution
4. Validation of the solution

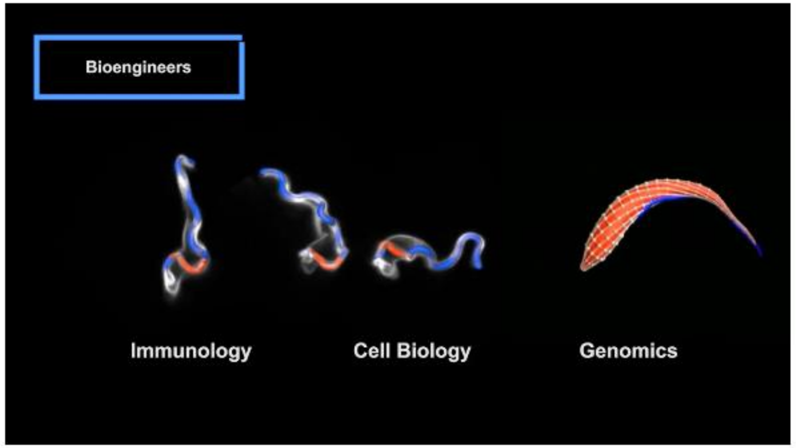
Applications of Numerical Methods for Engineers





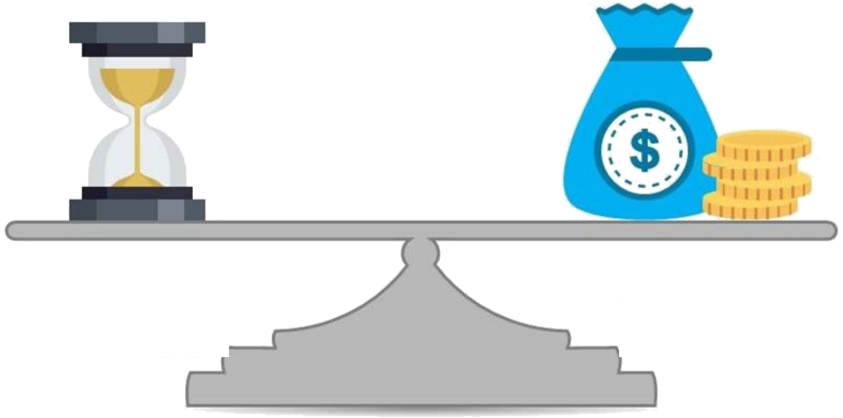






Accuracy and Efficiency

- Accuracy: Very few of our computations will yield the exact answer to a problem, so we will have to understand how much error is made, and how to control (or even diminish) that error.
- Efficiency: Does the algorithm take an inordinate amount of computer time? This might seem to be an odd question to concern ourselves with—after all, computers are fast, right?—but there are slow ways to do things and fast ways to do things. All else being equal (it rarely is), we prefer the fast ways. We say that these two issues compete with each other because, the steps that can be taken to make an algorithm more accurate usually make it more costly, that is, less efficient.



NUMERICAL COMPUTATION

Lecture – 01: Error Analysis

1. Error Analysis

While investigating the accuracy of the results of a certain numerical method, two key questions arise:

(1) What are the possible sources of error?

(2) To what degree do these errors affect the ultimate result?

Numerical methods involve approximating solutions to mathematical problems using algorithms and computer programs. However, since these approximations are not exact, there is always a certain amount of error involved.

Error analysis is an essential component of numerical methods, as it helps to evaluate the accuracy and reliability of the numerical solutions obtained.

Sources of Numerical Errors

- Round-off error occurs when a number is rounded to a certain number of decimal places
- Truncation error occurs when an approximation is made by truncating an infinite series or using a numerical method with a limited number of steps.
- Interpolation error occurs when an approximation is made using an interpolating function that does not perfectly fit the data.
- Integration error occurs when an approximation is made using a numerical method to approximate the area under a curve.

1.1 Absolute Error

- Absolute error is a measure of the difference between the true value and the approximate value obtained by a numerical method.
- Suppose that $\hat{p}$ is an approximation to p. The absolute error can be defined as:

$$E_p = | p - \hat{p} |$$

1.2 Relative Error

Relative error is a measure of the difference between the true value and the approximate value obtained by a numerical method, expressed as a percentage or a fraction of the true value.

The absolute error can be defined as:

$$R_p = \frac{|p - \hat{p}|}{|p|} \quad \text{, where } p \neq 0$$

Example 1:

Find the error and relative error in the following three cases.

Let x = 3.141592 and $\hat{x}$= 3.14.

(1a) $$E_x = |x - \hat{x}| = |3.141592 - 3.14| = 0.001592,$$

and the relative error is

$$R_x = \frac{|x - \hat{x}|}{|x|} = \frac{0.001592}{3.141592} = 0.00507.$$

Let $y = 1{,}000{,}000$ and $\hat{y} = 999{,}996$; then the error is

(1b) $$E_y = |y - \hat{y}| = |1{,}000{,}000 - 999{,}996| = 4,$$

and the relative error is

$$R_y = \frac{|y - \hat{y}|}{|y|} = \frac{4}{1{,}000{,}000} = 0.000004.$$

Let $z = 0.000012$ and $\hat{z} = 0.000009$; then the error is

(1c) $$E_z = |z - \hat{z}| = |0.000012 - 0.000009| = 0.000003,$$

and the relative error is

$$R_z = \frac{|z - \hat{z}|}{|z|} = \frac{0.000003}{0.000012} = 0.25.$$

1.3 Truncation Error

Truncation error arises in numerical methods when an infinite process, such as a series or an integral, is approximated with a finite number of terms or intervals.

For example; suppose we use the Taylor series representation of the sine function

$$\boxed{\sin\alpha = \sum_{n=odd}^{\infty} \frac{(-1)^{(n-1)/2}}{n!}\alpha^n = \alpha - \frac{1}{3!}\alpha^3 + \frac{1}{5!}\alpha^5 - \cdots + \frac{(-1)^{(m-1)/2}}{m!}\alpha^m + E_m}$$

Em is the tail end of the expansion, neglected in the process

$Let\ \propto = x$:

$$\sin x = \mathrm{x} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots \ldots \ldots \ldots \ldots$$

It is the difference between the exact value of a mathematical expression and its approximate value obtained through numerical methods. Truncation Error denoted by E^{TR}.

The value of $\sin\frac{\pi}{6}$ can be determined exactly with Eq

If only the first term is used:

$$\sin\left(\frac{\pi}{6}\right) = \frac{\pi}{6} = 0.5235988 \quad E^{TR} = 0.5\text{-}0.5235988 = \text{-}0.0235988$$

If two terms of the Taylor's series are used:

$$\sin\left(\frac{\pi}{6}\right) = \frac{\pi}{6} - \frac{(\pi/6)^3}{3!} = 0.4996742 \quad E^{TR} = 0.5\text{-}0.4996742 = 0.0003258$$

Note :-

The number $\widehat{p}$ is said to ***approximate*** p to d significant digits if d is the largest nonnegative integer for which

$$\frac{|p-\widehat{p}|}{|p|} < \frac{10^{1-d}}{2}.$$

Example : 2

If $y = 1{,}000{,}000$ and $\widehat{y} = 999{,}996$, then $|y-\widehat{y}|/|y| = 0.000004 < 10^{-5}/2$. Therefore, $\widehat{y}$ approximates y to six significant digits.

If $z = 0.000012$ and $\widehat{z} = 0.000009$, then $|z-\widehat{z}|/|z| = 0.25 < 10^{-0}/2$. Therefore, $\widehat{z}$ approximates z to one significant digit. ■

Example :3

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

which can be used to approximate the cosine of an angle x in radians.

Let us find, for instance, the value of $\cos(0.5)$ using three terms of the series. We have

$$\cos(0.5) \approx 1 - \frac{1}{8} + \frac{1}{384} \approx 0.8776041667 = x^*.$$

Since $|\cos(0.5) - x^*|/|\cos(0.5)| = 2.4619 \times 10^{-5} < 5 \times 10^{-5}$, x^* approximates $\cos(0.5)$ to five significant digits.

Note that one can use the expression of the remainder R_{n+1} of the Taylor series to get a bound on the truncation error. In this case

$$R_6(x) = \frac{f^{(6)}(z)x^6}{6!}.$$

Now $f^{(6)}(z) = -\cos z$, so the remainder term is $-x^6 \cos z/6!$. Since $|-\cos z| \leq 1$ on the interval $(0, 0.5)$, so we see that the remainder is bounded by $(0.5)^6(1)/6! = 0.217 \times 10^{-4}$. The actual absolute error is 0.216×10^{-4}, versus 0.217×10^{-4} remainder-formula bound.

Example : 4

Given that

$$I = \int_0^{1/4} e^{-x^2} dx = 0.244887887180,$$

approximate the integral by replacing e^{-x^2} with the truncated Maclaurin series $p_6(x) = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6$.

We have

$$\begin{aligned}\int_0^{1/4} (1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6) dx &= \left[x - \frac{1}{3}x^3 + \frac{1}{10}x^5 - \frac{1}{42}x^7 \right]_0^{1/4} \\ &= \frac{1}{4} - \frac{1}{192} + \frac{1}{10240} - \frac{1}{688130} \\ &= 0.2447999705027 = I^*.\end{aligned}$$

Since $\frac{|I - I^*|}{|I|} = 3.590 \times 10^{-4} < 5 \times 10^{-4}$, I^* approximates I to four significant digits.

1.4 Round-off Error

- Round-off error occurs when numerical values are approximated or rounded to a finite number of digits.
- In numerical methods, computations are performed using finite-precision arithmetic, which means that numbers are represented with a limited number of bits. As a result, some numbers cannot be represented exactly and must be rounded to the nearest representable value. This rounding process can introduce small errors into the computations, which are known as round-off errors.

There are two common ways used to representing a given real number *x* by a floating-point machine number, denoted by *fl(x)*, rounding and chopping. Consider a positive real number *x* in the normalized decimal form.

$$x = (2.b_1 b_2 \dots \dots . b_k b_{k+1} \dots .)_{10} \times 10^n$$

We say that the number *x* chopped to *k* digits when all digits following the *k*th digits are discarded; that is, the digits $b_{k+1} b_{k+2} \dots$ are chopped off to obtain

$$fl(x) = (2.b_1 b_2 \dots \dots . b_k)_{10} \times 10^n$$

Conversely, *x* is rounded to *k* digits when *fl(x)* is obtain by choosing *fl(x)* nearest to *x*; that is, adding one to b_k if $b_{k+1} \geq 5$ and chop off but the first *k* digits if $b_{k+1} < 5$

Example : 5

Consider $e = 2.71828182\ \dots\ = 0.271828182\ \dots\ \times 10^1$. If we use 5-digit chopping ($m = 5$), the floating-point form is $\text{FL}(e) = 0.27182 \times 10^1 = 2.7182$. We next use rounding. Since the digit immediately to the right of d_5 is $d_6 = 8 > 5$, we add 1 to d_5 to obtain

$$\text{FL}(e) = 0.27183 \times 10^1 = 2.7183$$

so that we have rounded up. The same result is obtained by following the strategy of adding $5 \times 10^{p-(m+1)}$ to e and chopping. Note that $p = 1$ and $m = 5$, so that $5 \times 10^{p-(m+1)} = 5 \times 10^{-5} = 0.00005$. Adding this to e, we have

$$e + 0.00005 = 2.71828182\ \dots + 0.00005 = 2.71833\ \dots = 0.271833\ \dots \times 10^1$$

Five-digit chopping yields $\text{FL}(e) = 0.27183 \times 10^1 = 2.7183$, which agrees with the result of rounding up.

Loss of Significance

Loss of significance is a phenomenon that occurs when a mathematical operation results in a significant reduction in the accuracy of the computed value, due to the limited precision of the available numerical representation.

This can happen when the input values to a mathematical operation are very close to each other or when they are subtracted.

For example, consider the two numbers $p = 3.1415926536$ and $q = 3.1415957341$ which are nearly equal, and both carry 11 decimal digits of precision. Suppose that their difference is formed :

$p - q \ = -0.0000030805$ contains only five decimal digits of precision.

Example :6

Compare the results of calculating $f(500)$ and $g(500)$ using six digits and rounding. The functions are $f(x) = x\left(\sqrt{x+1} - \sqrt{x}\right)$ and $g(x) = \dfrac{x}{\sqrt{x+1} + \sqrt{x}}$.
For the first function,

$$f(500) = 500\left(\sqrt{501} - \sqrt{500}\right)$$
$$= 500(22.3830 - 22.3607) = 500(0.0223) = 11.1500.$$

For $g(x)$,

$$g(500) = \frac{500}{\sqrt{501} + \sqrt{500}}$$
$$= \frac{500}{22.3830 + 22.3607} = \frac{500}{44.7437} = 11.1748.$$

The second function, $g(x)$, is algebraically equivalent to $f(x)$, as shown by the computation

$$f(x) = \frac{x\left(\sqrt{x+1}-\sqrt{x}\right)\left(\sqrt{x+1}+\sqrt{x}\right)}{\sqrt{x+1}+\sqrt{x}}$$

$$= \frac{x\left(\left(\sqrt{x+1}\right)^2-\left(\sqrt{x}\right)^2\right)}{\sqrt{x+1}+\sqrt{x}}$$

$$= \frac{x}{\sqrt{x+1}+\sqrt{x}}.$$

The answer, $g(500) = 11.1748$, involves less error and is the same as that obtained by rounding the true answer $11.174755300747198\ldots$ to six digits. ■

Example: 7

Compare the results of calculating $f(0.01)$ and $P(0.01)$ using six digits and rounding, where

$$f(x) = \frac{e^x - 1 - x}{x^2} \qquad \text{and} \qquad P(x) = \frac{1}{2} + \frac{x}{6} + \frac{x^2}{24}.$$

The function $P(x)$ is the Taylor polynomial of degree $n = 2$ for $f(x)$ expanded about $x = 0$.

For the first function

$$f(0.01) = \frac{e^{0.01} - 1 - 0.01}{(0.01)^2} = \frac{1.010050 - 1 - 0.01}{0.001} = 0.5.$$

For the second function

$$\begin{aligned} P(0.01) &= \frac{1}{2} + \frac{0.01}{6} + \frac{0.001}{24} \\ &= 0.5 + 0.001667 + 0.000004 = 0.501671. \end{aligned}$$

The answer $P(0.01) = 0.501671$ contains less error and is the same as that obtained by rounding the true answer $0.50167084168057542\ldots$ to six digits. ■

Example: 8

Let $P(x) = x^3 - 3x^2 + 3x - 1$ and $Q(x) = ((x - 3)x + 3)x - 1$.

Use three-digit rounding arithmetic to compute approximations to $P(2.19)$ and $Q(2.19)$. Compare them with the true values, $P(2.19) = Q(2.19) = 1.685159$.

$$\begin{aligned} P(2.19) &\approx (2.19)^3 - 3(2.19)^2 + 3(2.19) - 1 \\ &= 10.5 - 14.4 + 6.57 - 1 = 1.67. \\ Q(2.19) &\approx ((2.19 - 3)2.19 + 3)2.19 - 1 = 1.69. \end{aligned}$$

The errors are 0.015159 and −0.004841, respectively. Thus the approximation $Q(2.19) \approx 1.69$ has less error. Exercise 6 explores the situation near the root of this polynomial. ■

$O(h^n)$ Order of Approximation

The order of approximation is typically expressed as a power of the step size.

For example, if the error in the approximation decreases with the square of the step size, the method is said to be second-order accurate. The higher the order of approximation, the faster the convergence to the exact solution, and the smaller the error for a given step size.

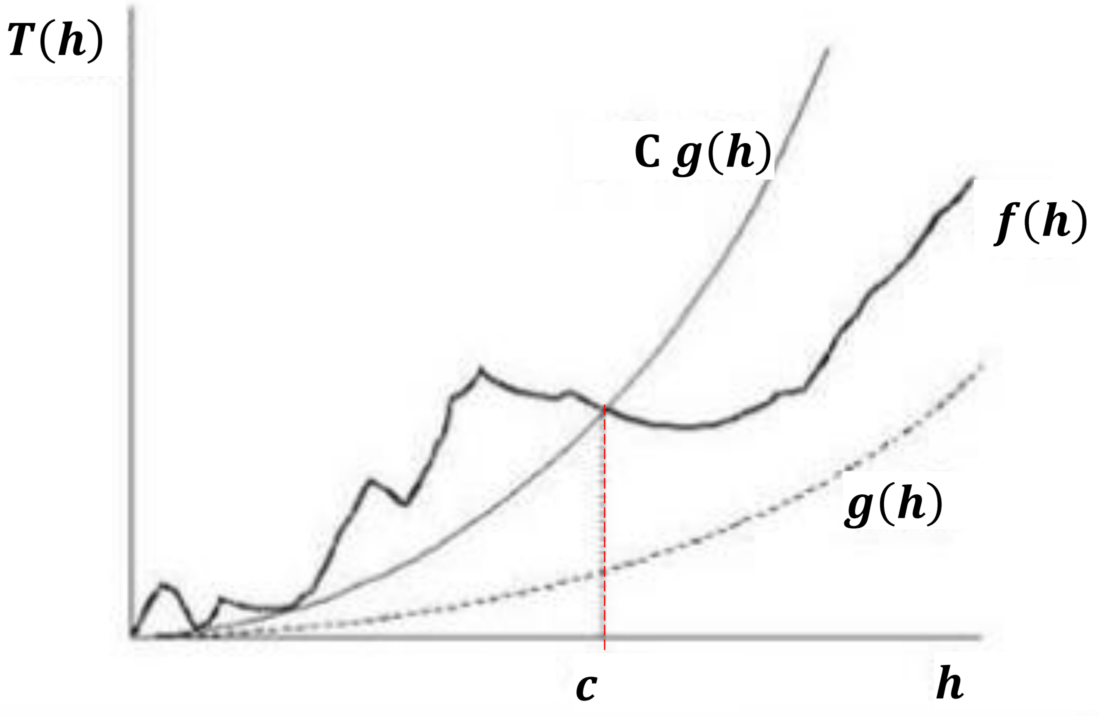
Definition

The function $f(h)$ is said to be big Oh of $g(h)$, denoted $f(h) = O(g(h))$

If there exist constants C and c such that

$$|f(h)| \leq C|g(h)|$$

whenever $h \geq c$



Example : 9

Consider the function $f(x) = x^2 + 1$ and $g(x) = x^3$.

Since $x^2 \leq x^3$ and $1 \leq x^3$ for $x \geq 1$,

it follows that $x^2 + 1 \leq 2x^3$ for $x \geq 1$. Therefore, $f(x) = \mathrm{O}(\mathrm{g}(\mathrm{x}))$.

The big Oh notation provides a useful way of describing the rate of growth of a function in terms of well-know elementary functions $(x^n, x^{1/n}, a^x, \log_a x, etc)$. The rate of convergence of sequences can be described in a similar manner.

Definition

Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two sequences. The sequence $\{x_n\}$ is said to be of order big Oh of $\{y_n\}$, denoted $x_n = O(y_n)$, if there exist constants C and N such that

$$|x_n| \leq C|y_n| \qquad \text{whenever } n \geq N$$

Example : 10

$$\frac{n^2-1}{n^3} = O\left(\frac{1}{n}\right), \text{ since } \frac{n^2-1}{n^3} \leq \frac{n^2}{n^3} = \frac{1}{n} \quad \textit{whenever } n \geq 1$$

Definition

Assume that *f(h)* is approximated by the function *p(h)* and that there exist a real constant $M > 0$ and a positive integer n so that

$$\frac{|f(h)-p(h)|}{|h^n|} \leq M \text{ for sufficiently small } h$$

We say that *p(h)* **approximates** *f(h)*with order of approximation $O(h^n)$ and write

$$f(h) = p(h) + O(h^n)$$

Also, by rewriting, $|f(h) - p(h)| \leq M|h^n|$; $M|h^n|$ is known as the error bound. We see that $O(h^n)$ stands in a place of error bound.

Theorem

Assume that $f(h) = p(h) + \boldsymbol{O}(h^n)$, $g(h) = q(h) + \boldsymbol{O}(h^m)$, and $r = \min\{m, n\}$. Then

$$f(h) + g(h) = p(h) + q(h) + \boldsymbol{O}(h^r),$$
$$f(h)g(h) = p(h)q(h) + \boldsymbol{O}(h^r),$$

and

$$\frac{f(h)}{g(h)} = \frac{p(h)}{q(h)} + \boldsymbol{O}(h^r) \quad \text{provided that } g(h) \neq 0 \text{ and } q(h) \neq 0.$$

Theorem (Taylor's Theorem)

Assume that $f \in C^{n+1}[a,b]$. If both x_0 and $x = x_0 + h$ lie in $[a,b]$, then

$$f(x_0 + h) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} h^k + \boldsymbol{O}(h^{n+1})$$

Note :-

For the computations, use

i. $\boldsymbol{O}(h^p) + \boldsymbol{O}(h^p) = \boldsymbol{O}(h^p)$

ii. $\boldsymbol{O}(h^p) + \boldsymbol{O}(h^q) = \boldsymbol{O}(h^r)\,; r = min\{p,q\}$

iii. $\boldsymbol{O}(h^p)\,\boldsymbol{O}(h^q) = \boldsymbol{O}(h^s)\,; s = p + q$

Example: 11

Consider the Taylor polynomial expansion

$$e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \boldsymbol{O}(h^4) \quad \text{and} \quad \cos(h) = 1 - \frac{h^2}{2!} + \frac{h^4}{4!} + \boldsymbol{O}(h^6).$$

Determine the order of approximation for their sum and product.

For the sum we have

$$\begin{aligned} e^h + \cos(h) &= 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \boldsymbol{O}(h^4) + 1 - \frac{h^2}{2!} + \frac{h^4}{4!} + \boldsymbol{O}(h^6) \\ &= 2 + h + \frac{h^3}{3!} + \boldsymbol{O}(h^4) + \frac{h^4}{4!} + \boldsymbol{O}(h^6). \end{aligned}$$

Since $\boldsymbol{O}(h^4) + \frac{h^4}{4!} = \boldsymbol{O}(h^4)$ and $\boldsymbol{O}(h^4) + \boldsymbol{O}(h^6) = \boldsymbol{O}(h^4)$, this reduces to

$$e^h + \cos(h) = 2 + h + \frac{h^3}{3!} + \boldsymbol{O}(h^4),$$

and the order of approximation is $\boldsymbol{O}(h^4)$.

The product is treated similarly:

$$
\begin{aligned}
e^h \cos(h) &= \left(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + O(h^4)\right)\left(1 - \frac{h^2}{2!} + \frac{h^4}{4!} + O(h^6)\right) \\
&= \left(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!}\right)\left(1 - \frac{h^2}{2!} + \frac{h^4}{4!}\right) \\
&\quad + \left(1 + h + \frac{h^2}{2!} + \frac{h^3}{3!}\right) O(h^6) + \left(1 - \frac{h^2}{2!} + \frac{h^4}{4!}\right) O(h^4) \\
&\quad + O(h^4) O(h^6) \\
&= 1 + h - \frac{h^3}{3} - \frac{5h^4}{24} - \frac{h^5}{24} + \frac{h^6}{48} + \frac{h^7}{144} \\
&\quad + O(h^6) + O(h^4) + O(h^4) O(h^6).
\end{aligned}
$$

Since $O(h^4)O(h^6) = O(h^{10})$ and

$$
-\frac{5h^4}{24} - \frac{h^5}{24} + \frac{h^6}{48} + \frac{h^7}{144} + O(h^6) + O(h^4) + O(h^{10}) = O(h^4),
$$

the preceding equation is simplified to yield

$$
e^h \cos(h) = 1 + h - \frac{h^3}{3} + O(h^4),
$$

and the order of approximation is $O(h^4)$.

Order of Convergence of a Sequence

Definition

Suppose that $\lim_{n\to\infty} x_n = x$ and $\{r_n\}_{n=1}^{\infty}$ is a sequence with $\lim_{n\to\infty} r_n = 0$. We say that $\{x_n\}_{n=1}^{\infty}$ ***converges*** to x with the order of convergence $\boldsymbol{O}(r_n)$, if there exists a constant $K > 0$ such that

$$\frac{|x_n - x|}{|r_n|} \leq K \quad \text{for } n \text{ sufficiently large.}$$

This is indicated by writing $x_n = x + \boldsymbol{O}(r_n)$, or $x_n \to x$ with order of convergence $\boldsymbol{O}(r_n)$.

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Let $x_n = \cos(n)/n^2$ and $r_n = 1/n^2$; then $\lim_{n\to\infty} x_n = 0$ with a rate of convergence $O(1/n^2)$. This follows immediately from the relation

$$\frac{|\cos(n)/n^2|}{|1/n^2|} = |\cos(n)| \le 1 \qquad \text{for all } n.$$

Propagation of Error

Let us investigate how error might be propagated in successive computations. Consider the addition of two numbers p and q (the true values) with the approximate values $\widehat{p}$ and $\widehat{q}$, which contain errors ϵ_p and ϵ_q, respectively. Starting with $p = \widehat{p} + \epsilon_p$ and $q = \widehat{q} + \epsilon_q$, the sum is

$$p + q = (\widehat{p} + \epsilon_p) + (\widehat{q} + \epsilon_q) = (\widehat{p} + \widehat{q}) + (\epsilon_p + \epsilon_q).$$

Hence, for addition, the error in the sum is the sum of the errors in the addends.

The propagation of error in multiplication is more complicated. The product is

$$pq = (\widehat{p} + \epsilon_p)(\widehat{p} + \epsilon_q) = \widehat{p}\widehat{q} + \widehat{p}\epsilon_q + \widehat{q}\epsilon_p + \epsilon_p\epsilon_q.$$

Hence, if $\widehat{p}$ and $\widehat{q}$ are larger than 1 in absolute value, the terms $\widehat{p}\epsilon_q$ and $\widehat{q}\epsilon_p$ show that there is a possibility of magnification of the original errors ϵ_p and ϵ_q. Insights are

Exercise

1. Find the error E_x and relative error R_x. Also determine the number of significant digits in the approximation.

(a) $x = 2.71828182, \widehat{x} = 2.7182$

(b) $y = 98,350, \widehat{y} = 98,000$

(c) $z = 0.000068, \widehat{z} = 0.00006$

2. Complete the following computation:

$$\int_0^{1/4} e^{x^2}\,dx \approx \int_0^{1/4} \left(1 + x^2 + \frac{x^2}{2!} + \frac{x^6}{3!}\right) dx = \widehat{p}.$$

State what type of error is present in this situation. Compare your answer with the true value $p = 0.2553074606$.

3. (a) Consider the data $p_1 = 1.414$ and $p_2 = 0.09125$, which have four significant digits of accuracy. Determine the proper answer for the sum $p_1 + p_2$ and the product $p_1 p_2$.

(b) Consider the data $p_1 = 31.415$ and $p_2 = 0.027182$, which have five significant digits of accuracy. Determine the proper answer for the sum $p_1 + p_2$ and the product $p_1 p_2$.

4. Complete the following computation and state what type of error is present in this situation.

(a) $$\frac{\sin\left(\frac{\pi}{4}+0.00001\right)-\sin\left(\frac{\pi}{4}\right)}{0.00001}=\frac{0.70711385222-0.70710678119}{0.00001}=\cdots$$

(b) $$\frac{\ln(2+0.00005)-\ln(2)}{0.00005}=\frac{0.69317218025-0.69314718056}{0.00005}=\cdots$$

5. Sometimes the loss of significance error can be avoided by rearranging terms in the function using a known identity from trigonometry or algebra. Find an equivalent formula for the following functions that avoids a loss of significance.

(a) $\ln(x+1)-\ln(x)$ for large x

(b) $\sqrt{x^2+1}-x$ for large x

(c) $\cos^2(x)-\sin^2(x)$ for $x \approx \pi/4$

(d) $\sqrt{\frac{1+\cos(x)}{2}}$ for $x \approx \pi$

6. *Polynomial evaluation.* Let $P(x) = x^3 - 3x^2 + 3x - 1$, $Q(x) = ((x - 3)x + 3)x - 1$, and $R(x) = (x - 1)^3$.

(a) Use four-digit rounding arithmetic and compute $P(2.72)$, $Q(2.72)$, and $R(2.72)$. In the computation of $P(x)$, assume that $(2.72)^3 = 20.12$ and $(2.72)^2 = 7.398$.

(b) Use four-digit rounding arithmetic and compute $P(0.975)$, $Q(0.975)$, and $R(0.975)$. In the computation of $P(x)$, assume that $(0.975)^3 = 0.9268$ and $(0.975)^2 = 0.9506$.

7. Use three-digit rounding arithmetic to compute the following sums (sum in the given order):

(a) $\displaystyle\sum_{k=1}^{6} \frac{1}{3^k}$ **(b)** $\displaystyle\sum_{k=1}^{6} \frac{1}{3^{7-k}}$

8. Discuss the propagation of error for the following:

(a) The sum of three numbers:

$$p + q + r = (\widehat{p} + \epsilon_p) + (\widehat{q} + \epsilon_q) + (\widehat{r} + \epsilon_r).$$

(b) The quotient of two numbers: $\dfrac{p}{q} = \dfrac{\widehat{p} + \epsilon_p}{\widehat{q} + \epsilon_q}.$

(c) The product of three numbers:

$$pqr = (\widehat{p} + \epsilon_p)(\widehat{q} + \epsilon_q)(\widehat{r} + \epsilon_r).$$

9. Given the Taylor polynomial expansions

$$\frac{1}{1-h} = 1 + h + h^2 + h^3 + \boldsymbol{O}(h^4)$$

and

$$\cos(h) = 1 - \frac{h^2}{2!} + \frac{h^4}{4!} + \boldsymbol{O}(h^6),$$

determine the order of approximation for their sum and product.

10. Given the Taylor polynomial expansions

$$e^h = 1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \frac{h^4}{4!} + \boldsymbol{O}(h^5)$$

and

$$\sin(h) = h - \frac{h^3}{3!} + \boldsymbol{O}(h^5),$$

determine the order of approximation for their sum and product.

11. Given the Taylor polynomial expansions

$$\cos(h) = 1 - \frac{h^2}{2!} + \frac{h^4}{4!} + \boldsymbol{O}(h^6)$$

and

$$\sin(h) = h - \frac{h^3}{3!} + \frac{h^5}{5!} + \boldsymbol{O}(h^7),$$

determine the order of approximation for their sum and product.

Thank You