UNIVERSITY OF JAFFNA

FACULTY OF ENGINEERING

END SEMESTER EXAMINATION - JANUARY 2022

MC 3010 : DIFFERENTIAL EQUATIONS AND NUMERICAL METHODS

(Duration: TWO Hours)

Instructions

1. This is a **Closed-book exam** exam.
2. A formula sheet is provided at the end of the paper.
3. Answer in the space provided.
4. Read all the problems first before beginning to answer any of them. Start with the one you feel most comfortable with, and only move on to the next problem when you are certain you have completed it perfectly.
5. If you have any doubt as to the interpretation of the wording of a question, make your own decision, but clearly state in the script.
6. This examination accounts for **60%** of module assessment. Total maximum mark attainable is **100**.
7. Write your **registration number** in the space provided. Also write your registration number on each additional sheet attached.

Registration Number	**201 . . . |E| . . . .**
Parts	Marks
1	
2	
3	
TOTAL	

Part 1[Multivariable Calculus][35 marks]: You are advised to spend 40 minutes. This part have SIX questions and best of FOUR will be considered for your Final Mark. All questions carry equal marks.

1. Find all stationary points of $f(x,y) = \dfrac{1}{3}x^3 + \dfrac{1}{3}y^3 - x^2 - y^2$ and classify them. ...
....
....
....
....
....
....
....
....

Classification of stationary points.					
Points	$\dfrac{\partial^2 f}{\partial x^2}$	$\dfrac{\partial^2 f}{\partial y^2}$	$\dfrac{\partial^2 f}{\partial x \partial y}$	Δ	Classification

2. Find the length of the circular helix defined by: $\mathbf{r}(t) = (\cos t, \sin t, t),\ t \in [0, 2\pi]$.

   (a) Velocity vector: ....
   ....

   (b) Speed: ....
   ....

   (c) Length L: ....
   ....

3. Find both components of the acceleration of the curve: $\mathbf{r}(t) = (2\cos t, 2\sin t, t)$ according to the following steps.

   (a) The velocity $\mathbf{v}(t) =$ ....
   ....

   (b) The acceleration $\mathbf{a}(t) =$ ....
   ....

   (c) The unit velocity vector $\tau(t) =$ ....
   ....

(d) The tangential acceleration vector $\mathbf{a}_{tan}(t) =$ ....
....

(e) The normal acceleration vector $\mathbf{a}_{norm}(t) =$ ....
....

4. The time T of a pendulum of length l under certain conditions is given by $T = 2\pi\sqrt{\dfrac{l}{g'}}$ where $g' = g\left(\dfrac{r}{r+h}\right)^2$. Find the error in T due to error $p\%$ and $q\%$ in h and l respectively according to the following steps. (g and r are constants)

$\dfrac{100\delta h}{h} =$ ....

$\dfrac{100\delta l}{l} =$ ....

$T =$ ....

$\log T =$ ....

$\dfrac{\delta T}{T} =$ ....

$\dfrac{100\delta T}{T} =$ ....

Percentage of error in $T =$ ....

5. Suppose electrical charge is distributed over a region which has area in the xy plane and its charge density is defined by the function e^{x^2}. Then the total charge of the plate is defined by the expression $Q = \int_0^4 \int_{\sqrt{y}}^2 e^{x^2} dx dy$.

   It is a integration over non- rectangular region. The inner integral here is difficult, perhaps impossible. However, progress can be made by changing the order of integration, so that we integrate with respect to y first, then with respect to x.

   - Draw a region of integration. . . . .

     . . . .
     . . . .
     . . . .
     . . . .
     . . . .
     . . . .
     . . . .

   - Draw a strip in the x direction
     (inner integral). . . . .

   - draw a strip in the y direction. . . . .

   - Evaluate new limits. . . . .

   - Evaluate the integral. . . . .
     . . . .
     . . . .
     . . . .
     . . . .
     . . . .
     . . . .
     . . . .

6. Find the Jacobian $J\left(\frac{u,v}{x,y}\right)$ for $u = \exp(x)\sin y$ and $v = x\log\sin y$ according to the following steps:

$\frac{\partial u}{\partial x}$= .... $\frac{\partial u}{\partial y}$ ....

$\frac{\partial v}{\partial x}$= .... $\frac{\partial v}{\partial y}$ ....

$J\left(\frac{u,v}{x,y}\right)$= ....
....
....
....
....
....

Part 2[Differential Equations][35 marks:] You are advised to spend 40 minutes. This part have SIX questions and best of FOUR will be considered for your Final Mark. All questions carry equal marks.

1. For the following differential equations, state their order and say whether they are linear/non-linear or homogeneous/non-homogeneous. In the case of the linear differential equations, state whether they have constant coefficients or not.

	Linear, Non-Linear	Homogeneous, Non-Homogeneous	Order (e.g. 1st, 2nd,)	Constant Coefficients
$\frac{d^2y}{dx^2} = 0$				
$\frac{dy}{dx} = 2x$				
$\frac{d^2y}{dx^2} + \ln x = 0$				
$\frac{d^2y}{dx^2} + \ln y = 0$				
$\frac{d^2y}{dx^2} = \frac{dy}{dx}$				
$\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 3y$				
$y^2 + \frac{d^2y}{dx^2} = 3$				

2. A population of bacteria grows at a rate that is proportional to the size of the population multiplied by an antibiotic effectiveness factor. The effectiveness factor of the antibiotic is inversely proportional to the square of time:

   (a) Write down an Ordinary Differential Equation for the number of bacteria, denoted by n. . . . .
   . . . .
   . . . .

   (b) Solve the resulting Ordinary Differential Equation, assuming that at time $t = 1$ there is one bacterium. . . . .
   . . . .
   . . . .
   . . . .
   . . . .
   . . . .
   . . . .
   . . . .
   . . . .

(c) Hence determine if the number of bacteria reaches a steady state.

3. A large cylindrical tank has radius 2 m. Water flows out of a tap at the bottom of the tank at a rate proportional to the square root of the depth of the water within it. Initially the tank is full to a depth of 9 m. After 15 minutes the depth of water is 4 m. How long will it take for the tank to empty?

4. Solve the differential equation $x\dfrac{dy}{dx} - y = x$, subject to the initial condition $y(1) = 2$, according to the following steps:

   (a) Put the differential equation in the standard form $\dfrac{dy}{dx} + p(x)y = r(x)$ and so determine $p(x)$ and $r(x)$.

   (b) Determine $h(x)$ using the formula: $h(x) = \displaystyle\int p(x)dx$

   (c) Find y by using the linear differential equation formula: $y = e^{-h(x)} \displaystyle\int e^{h(x)} r(x)dx$

. . . .
. . . .
. . . .
. . . .

(d) Apply the initial condition to find the constant of integration, and hence write down the solution for y.

. . . .
. . . .
. . . .
. . . .

5. Solve the differential equation $\left(D^2 - 3D + 2\right) y = e^x$ according to the following steps:

(a) Find the complementary function

. . . .
. . . .
. . . .
. . . .
. . . .
. . . .

(b) Find a particular integral

. . . .
. . . .
. . . .
. . . .
. . . .
. . . .

(c) Find the solution.

. . . .
. . . .
. . . .
. . . .
. . . .
. . . .

6. The vertical motion of a particle is modelled by:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 9;\ y(0) = 0,\ \frac{dy}{dx}(0) = 0$$

Use Laplace transforms to solve according to the following steps:

(a) Taking the Laplace transform of both sides of the differential equation by applying the formulae for the Laplace transforms.

(b) Put in the given initial conditions $y(0)$ and $\frac{dy}{dx}(0)$

(c) Re-arrange the equation to make $\mathcal{L}(y)$ the subject.

(d) Determine y by using, where necessary, partial fractions, and taking the inverse of each term.

Part 3[Numerical Methods][30 marks] You are advised to spend 40 minutes. This part have FIVE questions and best of FOUR will be considered for your Final Mark. All questions carry equal marks.

1. Given the Taylor polynomial expansions

$$\frac{1}{1-h} = 1 + h + h^2 + h^3 + \mathcal{O}(h^4)$$

and

$$\cos(h) = 1 - \frac{h^2}{2!} + \frac{h^4}{4!} + \mathcal{O}(h^6).$$

Determine the order of approximation for their sum and product.
Sum: ....
....
....
....
....
Product: ....
....
....
....
....

2. Set up a Newton iteration for computing square root x of a given positive number c and apply it to $c = 2$ with initial value $x_0 = 1$. ....
....
....
....
....
....
....
....
....
....

3. Consider the function $f(x) = x^3$. Calculate its first derivative at point x = 3 numerically with the forward, backward, and central finite difference formulas and using the h values of 1, 0.25 and compare the results with the exact (analytical) derivative. Further investigate the effect that the spacing, h, between the points has on the truncation and round-off errors.

		h=0.25		h=1	
	method	Estimation	Error	Estimation	Error
1	Forward 2-Point				
2	Forward 3-Point				
3	Central 2-Point				
4	Central 4-Point				
5	Backward 2-Point				
6	Backward 3-Point				

Comparison of the results

(a) Method:...

(b) Points:...

(c) h-value:...

4. Use the three-point Gauss-Legendre rule to approximate $\int_1^5 \frac{dt}{t} = \ln(5) - \ln(1) \approx 1.609438$ and compare the result with simpson and trapezoidal rule for $h = 1$.

(a) Three-point Gauss-Legendre rule..
....
....
....
....
....
....
....
....

(b) Simpson rule ....
....

(c) Trapezoidal rule

Comparison:

5. Use Euler's method to solve the initial value problem $\dfrac{dy}{dt} = \dfrac{t-y}{2}$, on $[0,3]$ with $y(0) = 1$. Compare solutions for $h = 0.25$, and 1.

6. Use Runge-Kutta method of order $N = 4$ to solve the initial value problem $\frac{dy}{dt} = \frac{t-y}{2}$, on $[0, 3]$ with $y(0) = 1$. Compare solutions for $h = 0.25$, and 1. ....

——— End of Examination ———

Formula sheet

1. Table of Laplace Transform

$f(t)$	$\mathcal{L}[f(t)]$	$f(t)$	$\mathcal{L}[f(t)]$
1	$\frac{1}{s}$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$	$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$	$e^{at}\sinh kt$	$\frac{k}{(s-a)^2-k^2}$
$\frac{e^{at}-e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$	$e^{at}\cosh kt$	$\frac{s-a}{(s-a)^2-k^2}$
$\delta(t-t_0)$	e^{-st_0}	$t\sin kt$	$\frac{2ks}{(s^2+k^2)^2}$
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$	$t\cos kt$	$\frac{s^2-k^2}{(s^2+k^2)^2}$
e^{at}	$\frac{1}{s-a}$	$t\sinh kt$	$\frac{2ks}{(s^2-k^2)^2}$
$f^n(t)$	$s^n F(s) - s^{(n-1)}f(0) - \cdots - f^{(n-1)}(0)$	$t\cosh kt$	$\frac{s^2+k^2}{(s^2-k^2)^2}$
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$	$\frac{\sin at}{t}$	$\arctan \frac{a}{s}$
$t^n \ (n=0,1,2,\dots)$	$\frac{n!}{s^{n+1}}$	$\frac{1}{\sqrt{\pi t}}e^{-a^2/4t}$	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$
$t^x \ (x \geq -1 \in \mathbb{R})$	$\frac{\Gamma(x+1)}{s^{x+1}}$	$\frac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
$\sin kt$	$\frac{k}{s^2+k^2}$	$\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$	$\frac{e^{-a\sqrt{s}}}{s}$
$\cos kt$	$\frac{s}{s^2+k^2}$		
$\sinh kt$	$\frac{k}{s^2-k^2}$		
$\cosh kt$	$\frac{s}{s^2-k^2}$		

2. Newton's iteration formula: $x_{n+1} = x_n - \frac{f(x_n}{f'(x_n)}$

3. Trapezoidal rule: $\int_a^b f(x)dx \approx \frac{h}{2}(f(a)+f(b)) + h\sum_{k=1}^{M-1} f(x_k)$, by sampling $f(x)$ at the $2M+1$ equally spaced points $x_k = a+kh$, for $k=0,1,2,\dots,2M$ by sampling $f(x)$ at the $M+1$ equally spaced points $x_k = a+kh$, for $k=0,1,2,\dots,M$.

4. Simpson rule: $\int_a^b f(x)dx \approx \frac{h}{3}(f(a)+f(b)) + \frac{2h}{3}\sum_{k=1}^{M-1} f(x_{2k} + \frac{4h}{3}\sum_{k=1}^{M-1} f(x_{2k-1})$, by sampling $f(x)$ at the $2M+1$ equally spaced points $x_k = a+kh$, for $k = 0, 1, 2, \ldots, 2M$.

5. To approximate the integral: $\int_a^b f(x)dx \approx \frac{b-a}{2}\sum_{k=1}^{N} w_{N,k} f(t_{N,k})$ by sampling $f(x)$ at the N unequally spaced points $\{t_{N,k}\}_{k=t}^{N}$, the changes of variable $t = \frac{a+b}{2} + \frac{b-a}{2}x$ and $dt = \frac{b-a}{2}dx$ are used. The abscissas $\{x_N, k\}_{k=1}^{N}$ and the corresponding weights $\{w_{N,k}\}_{k=1}^{N}$ must be obtained from a table of known values.

$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{N} w_{N,k} f(x_N, k) + E_N(f)$			
N	Abscissas, $x_{N,k}$	Weights, $w_{N,k}$	Truncation error, $E_N(f)$
2	-0.5773502692 0.5773502692	1 1	$\frac{f^4(c)}{135}$
3	±0.7745966692 0.0000000	0.5555555556 0.888888888	$\frac{f^6(c)}{15750}$
4	±0.8611363116 ±0.3399810436	0.3478548451 0.6521451549	$\frac{f^8(c)}{3472875}$

6. The table below lists difference formulas, of various accuracy, that can be used for numerical evaluation of first derivative.

First Derivative		
No	Method	Formula
1	2-point forward difference	$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h}$
2	3-point forward difference	$f'(x_i) = \frac{-3f(x_i) + 4f(x_{i+1}) - f(x_{i+2})}{2h}$
3	2-point backward difference	$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h}$
4	3-point backward difference	$f'(x_i) = \frac{f(x_{i-2}) - 4f(x_{i-1}) + 3f(x_i)}{2h}$
5	2-point central difference	$f'(x_i = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$
6	4-point central difference	$f'(x_i) = \frac{f(x_{i-2}) - 8f(x_{i-1}) + 8f(x_{i+1}) - f(x_{i+2})}{12h}$

7. Eulers method: $y_{k+1} = y_k + hf(t_k, y_k)$, $t_{k+1} = t_k + h$ for $k = 0, 1, \ldots, M-1$

8. Runge-Kutta method of order $N = 4$: $y_{k+1} = y_k + \dfrac{h(f_1 + 2f_2 + 2f_3 + f_4)}{6}$, where $f_1 = f\left(t_k, y_k\right)$, $f_2 = f\left(t_k + \dfrac{h}{2}, y_k + \dfrac{h}{2} f_1\right)$, $f_3 = f\left(t_k + \dfrac{h}{2}, y_k + \dfrac{h}{2} f_2\right)$, $f_4 = f\left(t_k + h, y_k + h f_3\right)$.