UNIVERSITY OF JAFFNA

FACULTY OF ENGINEERING

END SEMESTER EXAMINATION - DECEMBER 2022

MC 3010 : DIFFERENTIAL EQUATIONS AND NUMERICAL METHODS

(Duration: TWO Hours)

Instructions

1. Answer all parts.
2. A formula sheet is provided at the end of the paper.
3. This examination accounts for **50%** of module assessment. Total maximum mark attainable is **100**.

Part 1[Multivariable Calculus][30 marks]: You are advised to spend 35 minutes. This part have SIX questions and best of FIVE will be considered for your Final Mark. All questions carry equal marks.

1. Find all stationary points of $f(x, y) = 10xy \exp\left(-(x^2 + y^2)\right)$ and classify them.
2. The semicircle defined by: $\mathbf{r}(t) = (2\cos t, 2\sin t, 0),\ t \in [0, \pi]$. Find the following:

   (a) Velocity vector (b) Speed (c) Length

3. Find the work done by the force: $F(t) = \left(y(t)^2, -x(t)^2, 0\right)$ on the particle moving at constant speed along the straight line segment from $(0, 0, 0)$ to $(1, 2, 0)$.
4. A drone flies along a path described by: $\mathbf{r}(t) = (3\sin t, 5\cos t, 4\sin t + 6)$. Find the following:

   (a) the unit tangent vector. (b) the tangential acceleration.

5. The period T of a simple pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$. Find the maximum error in T due to possible errors up to 1% and 2% in l and g respectively.
6. Consider the integral $Q = \int_{-3}^{0}\int_{-y}^{3} e^{x^2}\,dxdy$.

   - Draw a region of integration, clearly labelling the lines used.
   - Evaluate the integral by changing the order of integration.

Part 2[Differential Equations][30 marks:] You are advised to spend 35 minutes. This part have SEVEN questions and best of FIVE will be considered for your Final Mark. All questions carry equal marks.

1. Classify the following Ordinary Differential Equations by completing the following table.

	LINEAR, NONLINEAR	HOMOGENEOUS, NON-HOMOGENEOUS	ORDER (e.g. 1st, 2nd,)
$\frac{d^3y}{dx^3} - y = 0$			
$\left(\frac{dy}{dx}\right)^2 = y$			
$\frac{d^2x}{dt^2} - \cos(t) = 0$			
$\frac{dy}{dt} + \frac{1}{y} = 0$			

2. A 1 kg beach ball is thrown directly downwards off a tall building, with an initial downwards velocity of 1 m/s. The drag force on the beach ball is proportional to velocity squared with a constant of proportionality of 1/4. For velocities close to 1 we can approximate v^2 by v. If we apply Newtons second law this gives us the equation:
$$\frac{dv}{dt} = g - \frac{1}{4}v, \ v(0) = 1.$$
Solve this equation using the integrating factor method.

3. Consider the following differential equation:
$$\frac{d^2y}{dt^2} - 4y = 0, \ y(0) = 2, \ \frac{dy}{dt}(0) = 0.$$
   (a) Write down the characteristic equation of this differential equation.
   (b) Find the roots of the characteristic equation.
   (c) Write down the general solution of this differential equation.
   (d) Find the particular solution corresponding to the given initial conditions.

4. Find the Laplace transform of $2\sin 2t \cos 4t$.

5. Use the Laplace transforms method to solve the initial value problem
$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 16y = 0, \ y(0) = 1, \ \frac{dy}{dt}(0) = 0.$$
Please show **all** of your workings.

6. Find the particular Integral of differential equation $\left(D^2 - 3D + 2\right) y = e^x$, where the operator $D = \dfrac{d}{dx}$.

7. Solve $(D-2)^2 y = 50e^{2x}$, where the operator $D = \dfrac{d}{dx}$.

Part 3[Numerical Methods][40 marks] You are advised to spend 50 minutes. This part have EIGHT questions and best of FIVE will be considered for your Final Mark. All questions carry equal marks.

1. Sometimes the loss of significance error can be avoided by rearranging terms in the function using a known identity from trigonometry or algebra. Find an equivalent formula for the following functions that avoids a loss of significance.
   (a) $\ln(x+1) - \ln(x)$.
   (b) $\cos^2(x) - \sin^2(x)$ for $x = \frac{\pi}{4}$

2. Consider the Taylor polynomial expansions $e^h = 1 + h + \dfrac{h^2}{2!} + \dfrac{h^3}{3!} + O(h^4)$ and $\cos(h) = 1 - \dfrac{h^2}{2!} + \dfrac{h^4}{4!} + O(h^6)$. Determine the order of approximation for their product.

3. Solve the equation $x^3 + 3x^2 - 12 = 0$, correct to two decimal using the Newton's method and $x_0 = 1.5$ as a first approximation.

4. Approximate the derivative of $f(x) = 0.2xe^{1.5x}$ at $x = 3$ using the forward, backward, and central difference method with step sizes $h = 0.1$, $h = 0.5$ and compare your answers.

5. The arc length of the curve $y = f(x)$ over the interval $a \leq x \leq 1$ is
$$l = \int_a^b \sqrt{1 + (f'(x)^2)}dx.$$
   (i) Approximate the arc length of $f(x) = x^3$ for $0 \leq x \leq 1$ using the composite trapezoidal rule with $M = 10$.
   (ii) Approximate the arc length of function given in (i) using the Simpson rule with $M = 10$.

6. Find the value of $\displaystyle\int_1^5 \frac{1}{x}dx$ by using the three points Gauss quadrature rule.

7. State whether the three points Gauss quadrature formula is exact for $f(x) = x^5$ and $f(x) = x^6$

8. Given $3\dfrac{dy}{dx} + 5y^2 = \sin x$, $y(0.3) = 5$ and using a step size of $h = 0.3$, find the value of $y(0.9)$ using the Runge-Kutta 4-th order method.

Formula sheet

1. Table of Laplace Transform

$f(t)$	$\mathcal{L}[f(t)]$	$f(t)$	$\mathcal{L}[f(t)]$
1	$\frac{1}{s}$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\sin kt$	$\frac{k}{(s-a)^2+k^2}$
$\mathcal{U}(t-a)$	$\frac{e^{-as}}{s}$	$e^{at}\cos kt$	$\frac{s-a}{(s-a)^2+k^2}$
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$	$e^{at}\sinh kt$	$\frac{k}{(s-a)^2-k^2}$
$\frac{e^{at}-e^{bt}}{a-b}$	$\frac{1}{(s-a)(s-b)}$	$e^{at}\cosh kt$	$\frac{s-a}{(s-a)^2-k^2}$
$\delta(t-t_0)$	e^{-st_0}	$t\sin kt$	$\frac{2ks}{(s^2+k^2)^2}$
$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$	$t\cos kt$	$\frac{s^2-k^2}{(s^2+k^2)^2}$
e^{at}	$\frac{1}{s-a}$	$t\sinh kt$	$\frac{2ks}{(s^2-k^2)^2}$
$f^n(t)$	$s^n F(s) - s^{(n-1)}f(0) - \cdots - f^{(n-1)}(0)$	$t\cosh kt$	$\frac{s^2+k^2}{(s^2-k^2)^2}$
$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$	$\frac{\sin at}{t}$	$\arctan\frac{a}{s}$
$t^n\ (n=0,1,2,\dots)$	$\frac{n!}{s^{n+1}}$	$\frac{1}{\sqrt{\pi t}}e^{-a^2/4t}$	$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$
$t^x\ (x \geq -1 \in \mathbb{R})$	$\frac{\Gamma(x+1)}{s^{x+1}}$	$\frac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t}$	$e^{-a\sqrt{s}}$
$\sin kt$	$\frac{k}{s^2+k^2}$	$\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$	$\frac{e^{-a\sqrt{s}}}{s}$
$\cos kt$	$\frac{s}{s^2+k^2}$		
$\sinh kt$	$\frac{k}{s^2-k^2}$		
$\cosh kt$	$\frac{s}{s^2-k^2}$		

2. Newton's iteration formula: $x_{n+1} = x_n - \frac{f(x_n}{f'(x_n)}$

3. Trapezoidal rule: $\int_a^b f(x)dx \approx h\left[\frac{1}{2}f_0 + f_1 + \ldots + f_{n-1} + \frac{1}{2}f_n\right]$, where $h = \frac{(b-a)}{n}$. The x_j's and a and b are called notes.

4. Simpson's rule: $\int_a^b f(x)dx \approx \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 \ldots + 2f_{2n-2} + 4f_{2n-1} + f_{2n}\right]$, where $h = \frac{(b-a)}{2n}$. The x_j's and a and b are called notes.

5. The table below lists difference formulas, of various accuracy, that can be used for numerical evaluation of first derivative.

First Derivative		
No	Method	Formula
1	2-point forward difference	$f'(x_i) = \dfrac{f(x_{i+1}) - f(x_i)}{h}$
2	3-point forward difference	$f'(x_i) = \dfrac{-3f(x_i) + 4f(x_{i+1}) - f(x_{i+2})}{2h}$
3	2-point backward difference	$f'(x_i) = \dfrac{f(x_i) - f(x_{i-1})}{h}$
4	3-point backward difference	$f'(x_i) = \dfrac{f(x_{i-2}) - 4f(x_{i-1}) + 3f(x_i)}{2h}$
5	2-point central difference	$f'(x_i = \dfrac{f(x_{i+1}) - f(x_{i-1})}{2h}$
6	4-point central difference	$f'(x_i) = \dfrac{f(x_{i-2}) - 8f(x_{i-1}) + 8f(x_{i+1}) - f(x_{i+2})}{12h}$

6. To approximate the integral: $\displaystyle\int_a^b f(x)dx \approx \frac{b-a}{2}\sum_{k=1}^{N} w_{N,k} f(t_{N,k})$ by sampling $f(x)$ at the N unequally spaced points $\{t_{N,k}\}_{k=t}^{N}$, the changes of variable $t = \dfrac{a+b}{2} + \dfrac{b-a}{2}x$ and $dt = \dfrac{b-a}{2}dx$ are used. The abscissas $\{x_N, k\}_{k=1}^{N}$ and the corresponding weights $\{w_{N,k}\}_{k=1}^{N}$ must be obtained from a table of known values.

$\displaystyle\int_{-1}^{1} f(x)dx = \sum_{k=1}^{N} w_{N,k} f(x_N, k) + E_N(f)$			
N	Abscissas, $x_{N,k}$	Weights, $w_{N,k}$	Truncation error, $E_N(f)$
2	-0.5773502692	1	$\dfrac{f^4(c)}{135}$
	0.5773502692	1	
3	$\pm$0.7745966692	0.5555555556	$\dfrac{f^6(c)}{15750}$
	0.0000000	0.888888888	
4	$\pm$0.8611363116	0.3478548451	$\dfrac{f^8(c)}{3472875}$
	$\pm$0.3399810436	0.6521451549	

7. Eulers method: $y_{k+1} = y_k + hf(t_k, y_k)$, $t_{k+1} = t_k + h$ for $k = 0, 1, \ldots, M-1$

8. Runge-Kutta method of order $N = 4$: $y_{k+1} = y_k + \dfrac{h(f_1 + 2f_2 + 2f_3 + f_4)}{6}$, where $f_1 = f(t_k, y_k)$, $f_2 = f\left(t_k + \dfrac{h}{2}, y_k + \dfrac{h}{2}f_1\right)$, $f_3 = f\left(t_k + \dfrac{h}{2}, y_k + \dfrac{h}{2}f_2\right)$, $f_4 = f(t_k + h, y_k + hf_3)$.