UNIVERSITY OF JAFFNA

FACULTY OF ENGINEERING

ASSIGNMENT TEST 01 ANSWER SCRIPT - MAY 2023

MC 3010 : DIFFERENTIAL EQUATIONS AND NUMERICAL METHODS

(Duration: ONE Hour)

1. Complete the following computation and state what type of error is present in this situation.
$$\int_0^{0.5} e^{x^2} dx \simeq \int_0^{0.5} (1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!}) dx$$
If true value $p = 0.544987104$, find the absolute error.

   - approximate value is
$$\begin{aligned} \hat{p} &= \int_0^{0.5} \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!}\right) dx \\ &= \left(x + \frac{x^3}{3} + \frac{x^5}{10} + \frac{x^7}{42} + \frac{x^9}{216}\right)_0^{0.5} \\ &= 0.544986721 \end{aligned}$$
absolute error $= \mid p - \hat{p} \mid \quad = \quad \mid 0.544987104 - 0.544986721 \mid = 3.83 \times 10^{-7}$

2. Given the Taylor polynomial Expansions
$$tan^{-1}(h) = h - \frac{h^3}{3} + \frac{h^5}{5} + \mathcal{O}(h^7)$$
and
$$ln(h+1) = h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} + \mathcal{O}(h^5)$$
determine the order of approximation for their sum and product.

   - Sum
$$\begin{aligned} tan^{-1}(h) + ln(h+1) &= h - \frac{h^3}{3} + \frac{h^5}{5} + \mathcal{O}(h^7) + h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} + \mathcal{O}(h^5) \\ &= 2h - \frac{h^2}{2} - \frac{h^4}{4} + \frac{h^5}{5} + \mathcal{O}(h^5) + \mathcal{O}(h^7) \end{aligned}$$
Since $\frac{h^5}{5} + \mathcal{O}(h^5) = \mathcal{O}(h^5)$ and $\mathcal{O}(h^5) + \mathcal{O}(h^7) = \mathcal{O}(h^5)$, this reduces to
$$tan^{-1}(h) + ln(h+1) = 2h - \frac{h^2}{2} - \frac{h^4}{4} + \mathcal{O}(h^5)$$
and the order of approximation is $\mathcal{O}(h^5)$

- Product

$$
\begin{aligned}
tan^{-1}(h)\cdot ln(h+1) &= \left(h-\frac{h^3}{3}+\frac{h^5}{5}+\mathcal{O}(h^7)\right)\cdot\left(h-\frac{h^2}{2}+\frac{h^3}{3}-\frac{h^4}{4}+\mathcal{O}(h^5)\right)\\
&= \left(h-\frac{h^3}{3}+\frac{h^5}{5}\right)\cdot\left(h-\frac{h^2}{2}+\frac{h^3}{3}-\frac{h^4}{4}\right)+\left(h-\frac{h^3}{3}+\frac{h^5}{5}\right)\cdot\mathcal{O}(h^5)\\
&+ \left(h-\frac{h^2}{2}+\frac{h^3}{3}-\frac{h^4}{4}\right)\cdot\mathcal{O}(h^7)+\mathcal{O}(h^5)\cdot\mathcal{O}(h^7)\\
&= h^2-\frac{h^3}{2}-\frac{h^5}{12}+\frac{4h^6}{45}-\frac{h^7}{60}+\frac{h^8}{15}-\frac{h^9}{20}+\mathcal{O}(h^7)\mathcal{O}(h^5)
\end{aligned}
$$

Since $\mathcal{O}(h^5)\mathcal{O}(h^7)=\mathcal{O}(h^{12})$ and $-\frac{h^5}{12}+\frac{4h^6}{45}-\frac{h^7}{60}+\frac{h^8}{15}-\frac{h^9}{20}+\mathcal{O}(h^{12})=\mathcal{O}(h^5)$, this reduces to

$$tan^{-1}(h)\cdot ln(h+1)\cdot = h^2-\frac{h^3}{2}+\mathcal{O}(h^5)$$

and the order of approximation is $\mathcal{O}(h^5)$

3. Let $f(x)=x+\frac{2}{x}$. Use quadratic Lagrange interpolation based on the nodes $x_0=1, x_1=2, x_2=2.5$ to approximate $f(1.5)$

- $x_0=1\Rightarrow f(x_0)=3$
  $x_1=2\Rightarrow f(x_1)=3$
  $x_2=2.5\Rightarrow f(x_2)=3.3$

$$
\begin{aligned}
L_0(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}=\frac{(x-2)(x-2.5)}{(1-2)(1-2.5)}=\frac{(x-2)(x-2.5)}{1.5}\\
L_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}=\frac{(x-1)(x-2.5)}{(2-1)(2-2.5)}=-\frac{(x-1)(x-2.5)}{0.5}\\
L_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}=\frac{(x-1)(x-2)}{(2.5-1)(2.5-2)}=\frac{(x-1)(x-2)}{(1.5)(0.5)}
\end{aligned}
$$

$$
\begin{aligned}
P_2(x) &= L_0(x)f_0+L_1(x)f_1+L_2(x)f_2\\
&= \frac{(x-2)(x-2.5)}{1.5}\cdot 3-\frac{(x-1)(x-2.5)}{0.5}\cdot 3+\frac{(x-1)(x-2)}{(1.5)(0.5)}\cdot 3.3\\
&= 2(x-2)(x-2.5)-6(x-1)(x-2.5)+4.4(x-1)(x-2)
\end{aligned}
$$

$$
\begin{aligned}
P_2(1.5) &= 2(1.5-2)(1.5-2.5)-6(1.5-1)(1.5-2.5)+4.4(1.5-1)(1.5-2)\\
&= 1+3-1.1\\
&= 2.9
\end{aligned}
$$

4. Apply Newton-Raphson method to find an approximate solution of the equation $e^x - 3x = 0$ correct up-to three decimal figures (assume $x_0 = 0.4$)

- $f(x) = e^x - 3x = 0 \quad \implies f'(x) = e^x - 3$
  $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

n	x_n	$f(x_n$	$f'(x_n)$	x_{n+1}
0	0.4	0.29182	-1.50818	0.593495
1	0.5935	0.02981	-1.18969	0.61856
2	0.6186	5.2736×10^{-4}	-1.14367	0.61906
3	0.6191	-4.4241×10^{-5}	-1.14274	0.61906

  approximate solution $= 0.619$

5. Find the root of the equation $2x - log_{10}x = 7$ which lies between 3 and 4, correct to four places of decimal, using bisection method.

- $f(3) = -1.4771 < 0$
  $f(4) = 0.3979 > 0$
  $f(3)f(4) < 0 \therefore$ roots lies between $[3, 4]$

n	a	$f(a)$	b	$f(b)$	c	$f(c)$
1	3	-1.4771	4	0.3979	3.5	-0.5441
2	3.5	-0.5441	4	0.3979	3.75	-0.0740
3	3.75	-0.0740	4	0.3979	3.875	0.1617
4	3.75	-0.0740	3.875	0.1617	3.8125	0.0438
5	3.75	-0.0740	3.8125	0.0438	3.78125	-0.0151
6	3.78125	-0.0151	3.8125	0.0438	3.79688	0.0143
7	3.78125	-0.0151	3.79688	0.0143	3.7891	-3.36 $\times 10^{-4}$

  root of the equation is 3.7891
  $\therefore \quad x = 3.7891$

6. In a vibration experiment, the position (x) of a block of mass is given as a function of time(t). The recorded data for the first 2 seconds are given in the table below.

t(s)	0	0.5	1	1.5	2
x(mm)	200	123	27	-56	-100

   The velocity of the block is the derivative of the position w.r.t time. Use Forward divided difference, backward divided difference or central divided difference approximation method to find the velocity at time $t = 1.5s$

- forward divided difference

$$v = \frac{dx}{dt} = \frac{f(t+h)-f(t)}{h} = \frac{f(2)-f(1.5)}{0.5} = \frac{-100+56}{0.5} = -88mm/s$$

- backward divided difference

$$v = \frac{dx}{dt} = \frac{f(t)-f(t-h)}{h} = \frac{f(1.5)-f(1)}{0.5} = \frac{-56-27}{0.5} = -166mm/s$$

- central divided difference

$$v = \frac{dx}{dt} = \frac{f(t+h)-f(t-h)}{2h} = \frac{f(2)-f(1)}{2\times 0.5} = \frac{-100-27}{1} = -127mm/s$$

7. Calculate the integral value, to using trapezoidal rule . The step size $h = 0.25$

$$\int_0^1 (1+e^{-x}cos(4x))dx$$

- $f(x) = 1 + e^{-x}cos(4x)$

$$\int_0^1 f(x)\ \ dx \approx \frac{h}{2}(f(x_0)+2f(x_1)+2f(x_2)+2f(x_3)+f(x_4)) \quad \longrightarrow \quad (*)$$

$f(x_0) = f(0) = 2$
$2f(x_1) = 2f(0.25) = 2.8417571781$
$2f(x_2) = 2f(0.5) = 1.4951883694$
$2f(x_3) = 2f(0.75) = 1.0647213143$
$f(x_4) = f(1) = 0.7595379500$

$(*) \Rightarrow$

$$\int_0^1 f(x)dx \approx \frac{0.25}{2}(2+2.8417571781+1.4951883694+1.0647213143+0.7595379500)$$
$$= \ = 1.020127919$$

8. Find the number $2m$(no.of.sub interval) and the step size h so that the error $E_s(f,h)$ for the Simpson rule is less than 5×10^{-9} for the approximation $\int_2^7 \frac{dx}{x}$.

The $f^{(4)}(x) = \frac{24}{x^5}$ and the maximum value of taken over [2,7] occurs at the end point $x = 2$.

- $\mid f^{(4)}(c) \mid \leq \mid f^{(4)}(2) \mid = \frac{24}{32} = \frac{3}{4}$

  for $2 \leq c \leq 7$
  $E_s(f,h) = \frac{-(b-a)f^{(4)}(c)h^4}{180}$

$$| E_s(f,h) |=| \frac{-(b-a)f^{(4)}(c)h^4}{180} | \quad \leq \quad \frac{(7-2)\frac{3}{4}h^4}{180} = \frac{h^4}{48}$$

but $h = \dfrac{5}{2m}$

$$| E_s(f,h) | \quad \leq \quad \frac{625}{768m^4} \quad \leq \quad 5 \times 10^{-9}$$

$2m \leq 225.9$

$2m = 226$ (m must be an integer)

$$h = \frac{5}{2m} = \frac{5}{226} = 0.0221239$$

$\therefore \quad 2m = 226, \ h = 0.0221239$

9. Show that two integrals are equivalent and calculate $G_2(f)$ (Two-point Gauss-Legendre)

$$\tfrac{1}{\pi}\textstyle\int_0^\pi \cos(0.6\sin(t))dt = 0.5\int_{-1}^1 \cos(0.6\sin((x+1)\frac{\pi}{2}))dx$$

- $t = \left(\dfrac{a+b}{2}\right) + \left(\dfrac{b-a}{2}\right)x = \left(\dfrac{0+\pi}{2}\right) + \left(\dfrac{\pi-0}{2}\right)x = \pi\left(\dfrac{x+1}{2}\right)$

$$t = \pi\frac{(x+1)}{2}$$

$$\frac{dt}{dx} = \frac{\pi}{2}$$

$$\begin{aligned}
\frac{1}{\pi}\int_0^\pi \cos(0.6\sin(t))dt &= \frac{1}{\pi}\int_{-1}^1 \cos(0.6\sin(\pi\frac{(x+1)}{2}))\frac{\pi}{2}\ dx \\
&= \frac{1}{2}\int_{-1}^1 \cos(0.6\sin((\pi\frac{(x+1)}{2}))\ dx \\
&= 0.5\int_{-1}^1 \cos(0.6\sin((x+1)\frac{\pi}{2}))\ dx
\end{aligned}$$

$\therefore$ two integrals are equivalent.

$$f(x) = 0.5\cos(0.6\sin((x+1)\frac{\pi}{2}))$$

$$\begin{aligned}
\int_{-1}^1 f(x)dx &= f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \\
&= \int_{-1}^1 f(x)dx \\
&= f(-0.57735) + f(0.57735) \\
&= 0.4662152735 + 0.4662152735 \\
&= 0.9324305
\end{aligned}$$

10. Compare the truncation error term for the two-point Gauss-Legendre rule $(\frac{f^{(4)}(c)}{135})$ and Simpson's rule $(\frac{-h^5 f^{(4)}(c)}{90})$ on the closed interval[-1,1] and state which method do you think best? Why?

- 4th derivative does not change too much,then,

  $$\left| \frac{f^{(4)}(c)}{135} \right| \quad < \quad \left| \frac{-h^5 f^{(4)}(c)}{90} \right|$$

  Based on the error analysis, we can conclude that the two-point Gauss-Legendre rule is generally more accurate than Simpson's rule. This is because the Gauss-Legendre rule has a smaller error term and is not affected by the choice of step size.The truncation error for Simpson's rule depends on the step size h, and as h decreases (finer intervals), the error decreases as h^5. However, for larger step sizes, the error can become significant.

  The truncation error term for Simpson's rule explicitly depends on the fifth power of the step size (h^5), indicating that the error decreases rapidly as the step size is reduced. In contrast, the truncation error term for the two-point Gauss-Legendre rule does not explicitly depend on the step size, which means its error behavior is not directly controlled by the step size.

  If a high degree of accuracy is desired and the step size can be made sufficiently small, Simpson's rule is a better choice. As the step size decreases, the truncation error diminishes rapidly due to the fifth power dependence on h. If the accuracy requirement is not as stringent or the step size cannot be made very small, the two-point Gauss-Legendre rule can be a suitable option.