Partial Differentiation

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When a function is single valued, the derivative with respect to the independent variable gives the slope of the curve defined by that function.

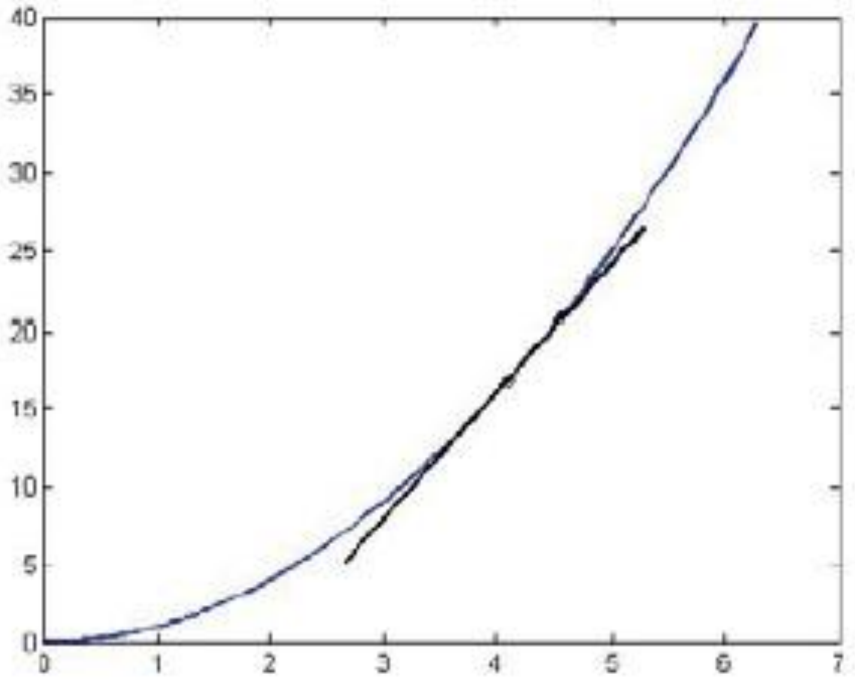


Figure. The derivative of a single variable function gives the slope at any point

When a function is multivariable, however, the notion of a derivative is not as straightforward. The slope of the surface defined by the function depends upon the direction in which the gradient is being measured.

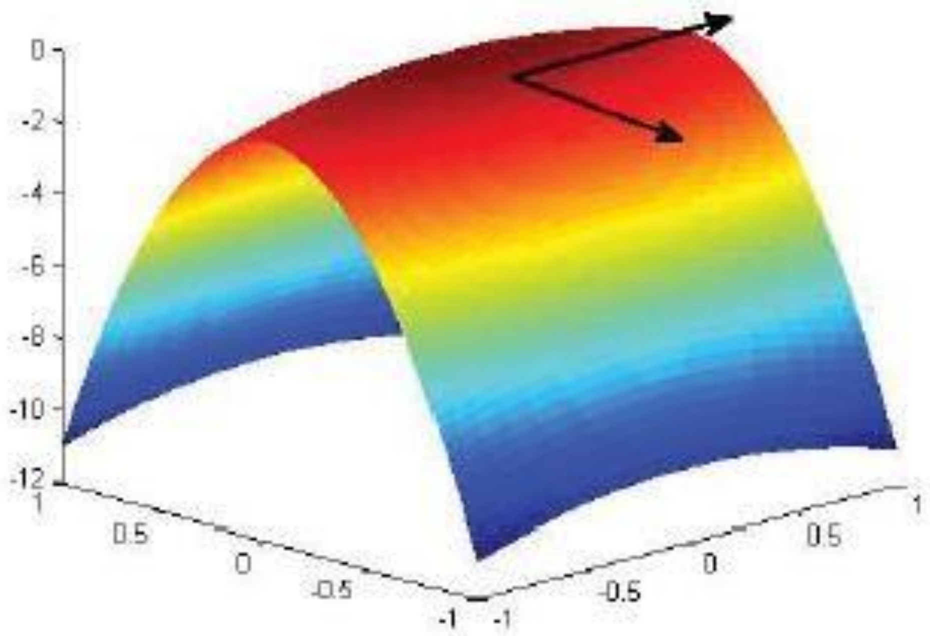


Figure. The slope of a multivariable function is direction-dependent

We can use *partial derivatives* to determine the gradient in the direction of a particular independent variable. This is simply done by treating all other independent variables as constant and differentiating with respect to the independent variable of interest in the usual way.

The notation for the partial derivative of a multivariable function $f(x, y)$ with respect to x (i.e. measuring the slope in the x-direction of the surface defined by $f(x, y)$) is

$$\frac{\partial f}{\partial x}$$

i.e. ∂ rather than the straight d 's used for derivatives of a single variable function with respect to x.

LECTURE EXAMPLES:

Find $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ if $f(x,y)=1-x^2$

To find $\dfrac{\partial f}{\partial x}$ differentiate f with respect to x leaving y alone.

i.e. $\dfrac{\partial f}{\partial x} = -2x$

$\dfrac{\partial f}{\partial y} = 0$ (no y)

Does this make sense?

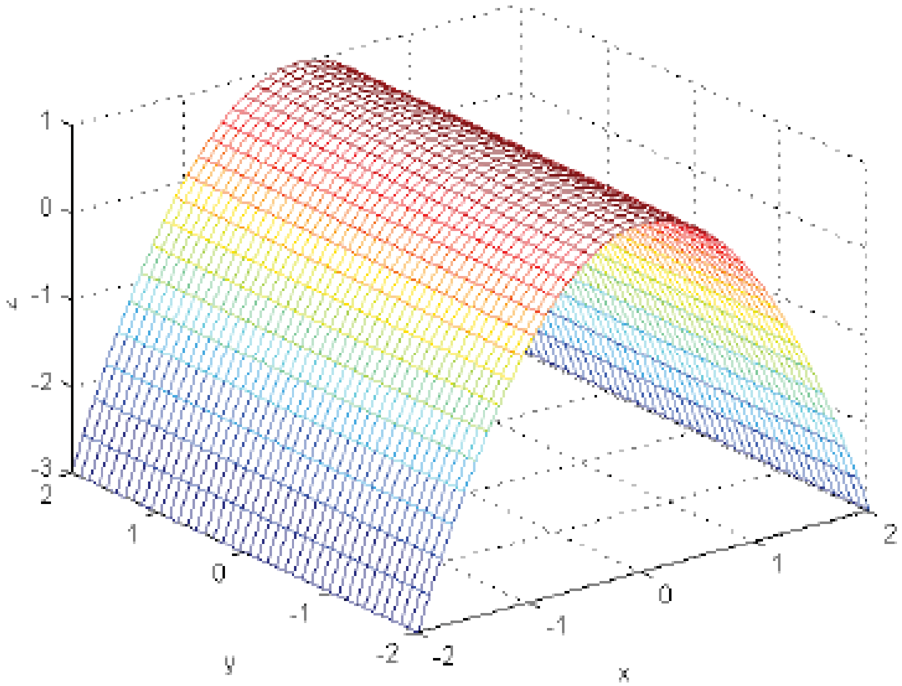


Figure: surface for $z = f(x,y) = 1 - x^2$

$\dfrac{\partial f}{\partial y}$ = slope of function in y direction.

From the plot we can see that the slope in the y direction is 0.

For the function:

$$f(x, y) = 2xy + y^2$$

find the value of the partial derivatives at the point $P = (1, -2)$.

First find the partial derivatives:

$$\frac{\partial f}{\partial x} = 2y, \quad \frac{\partial f}{\partial y} = 2x + 2y$$

Substituting in our point we get:

$$\frac{\partial f}{\partial x}(1, -2) = 2 \times -2 = -4$$

$$\frac{\partial f}{\partial y}(1, -2) = 2 \times 1 + 2 \times -2 = -2.$$

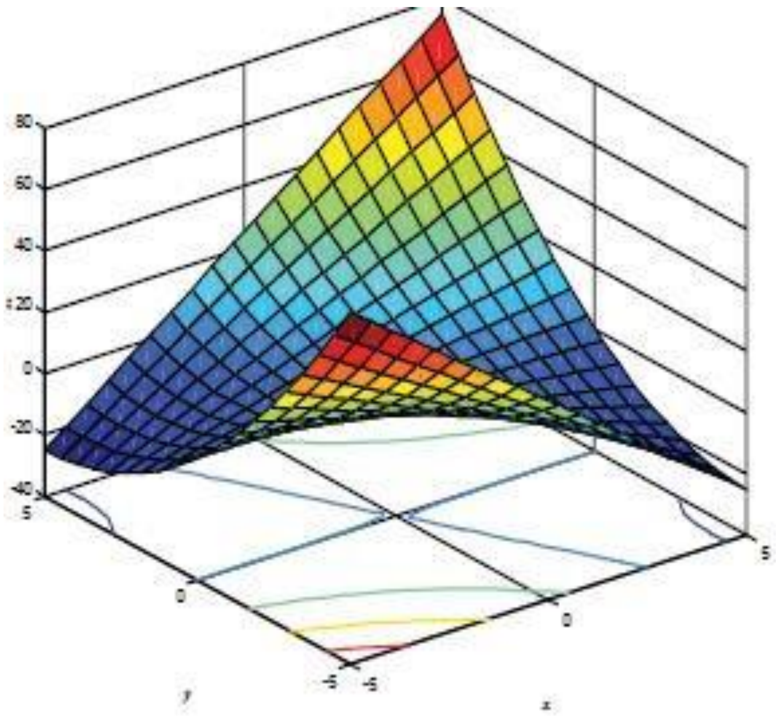


Figure: surface for $z = f(x, y) = 2xy + y^2$

Find $\frac{\partial f}{\partial y}$ for the function:

$$f(x, y) = \ln(x)y^2$$

We are treating x as a constant, so $\ln(x)$ is a constant. This gives:

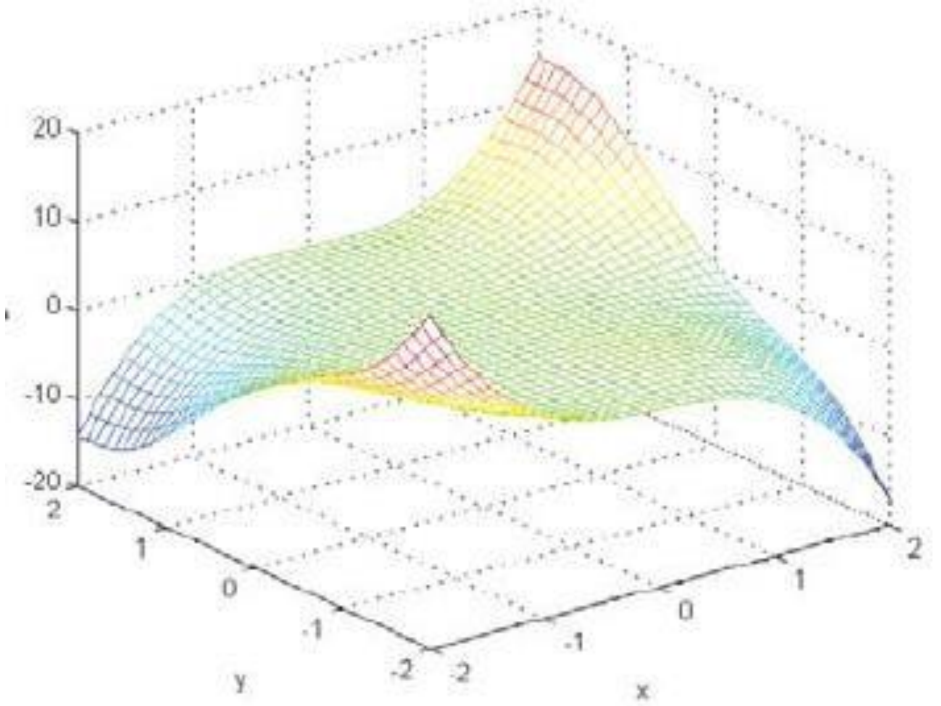
$$\frac{\partial f}{\partial y} = 2\ln(x)y$$

Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ if $f(x,y) = x^3 y + x \sin y^2$

$$\frac{\partial f}{\partial x} = 3x^2 y + \sin y^2 \qquad \text{(Treat } y \text{ as a constant)}$$

As we require the slope in the x direction, we don't care what y is doing.

$$\frac{\partial f}{\partial y} = x^3 + x\cos(y^2)2y \qquad \text{(Treat } x \text{ as a constant)}$$



Plot of $z = f(x,y) = x^3 y + x \sin y^2$

Note that the surface looks flat near (0,0) and sure enough $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$ there.

Exercises

1. Obtain $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point (2, 1) for the following functions:

(a)	$3x + 7y - 2$	(b)	$-2x + 3y + 4$
(c)	$2x^2 - 3y^2 - 2xy - x - y + 1$	(d)	$\frac{1}{8}x^3 + y^3 - 2y - 1$
(e)	$x^4y^2 - 1$	(f)	$(x-1)(y-2)$
(g)	$\frac{1}{xy}$	(h)	$\frac{x}{y}$
(i)	$\frac{x-y}{x+y}$	(j)	$\frac{3}{x^2+y^2}$
(k)	$(x^2+y^2)^{\frac{1}{2}}$	(l)	$(2x-3y+2)^3$
(m)	$e^{x^2+y^2}$	(n)	$\cos(x^2-y^2)$
(o)	$\sin\frac{x}{y}$	(p)	$\arctan\frac{y}{x}$

2. (a) Let $z = \sin(x - y)$; show that $\dfrac{\partial z}{\partial x} \bigg/ \dfrac{\partial z}{\partial y} = -1$

(b) Let $z = g(x - y)$; show that $\dfrac{\partial z}{\partial x} \bigg/ \dfrac{\partial z}{\partial y} = -1$

3. Show that, if $z = g\left(\dfrac{x}{y}\right)$, then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0,$$

and check the result in the case $z = \sin\dfrac{x}{y}$.

Multivariable Chain rule

If we have a single variable quantity w which depends on a variable x, which in turn depends on another variable t, then we can use the chain rule to calculate how w varies with t. Thus for $w = w(x)$ and $x = x(t)$ then:

$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$$

This is known as the chain rule or sometimes the function of a function rule.

This becomes more complicated if the function is multivariable, say if w depends on x, y and z which in turn depend on t. This is the sort of problem you might wish to deal with if you wished to calculate the rate of change of temperature experienced by an aircraft flying along a particular path. In this case $w(x, y, z)$ represents the temperature data supplied by meteorologists and $(x(t), y(t), z(t))$ is the flight path. In this case:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

It becomes even more complicated when x, y and z depend on more than one variable, u and v say. This will be the case for example when we wish to work out how some quantity is varying on a 3D surface. So given a surface in the form $(x(u,v), y(u,v), z(u,v))$ how do we calculate the rate of change of w on the surface if we know how w varies with x, y and z. This can be done through an extended chain rule:

$$\frac{dw}{du} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{dw}{dv} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

We can use this chain rule to deal with changes in our coordinate system. For example we may have a function defined in the usual rectangular Cartesian (x, y) coordinate system, say $w = f(x, y)$, but we wish to change to a polar coordinate system (r, θ) because of the geometry of our problem. Then:

$$\frac{dw}{dr} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y}\sin\theta$$

$$\frac{dw}{d\theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} = -\frac{\partial w}{\partial x}r\sin\theta + \frac{\partial w}{\partial y}r\cos\theta$$

LECTURE EXAMPLE:

Consider the function

$$w(x, y) = x^2 + y^2$$

and find the rates of change in polar coordinates (r, θ) where

$$x = r\cos\theta, y = r\sin\theta$$

Using chain rule

$$\frac{dw}{dr} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

$$w = x^2 + y^2 \Rightarrow \frac{\partial w}{\partial x} = 2x, \frac{\partial w}{\partial y} = 2y$$

$$x = r\cos\theta, y = r\sin\theta \Rightarrow \frac{\partial x}{\partial r} = \cos\theta, \frac{\partial y}{\partial r} = \sin\theta$$

Substituting in gives

$$\frac{dw}{dr}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

$$\Rightarrow \frac{dw}{dr}=2x\cos\theta+2y\sin\theta$$

$$\Rightarrow \frac{dw}{dr}=2r\cos\theta\cos\theta+2r\sin\theta\sin\theta$$

$$\boxed{\Rightarrow \frac{dw}{dr}=2r\cos^2\theta+2r\sin^2\theta}$$

Substituting

$$\frac{dw}{d\theta}=\frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta}$$

$$\Rightarrow \frac{dw}{d\theta}=2x(-r\sin\theta)+2y(r\cos\theta)$$

$$\boxed{\Rightarrow \frac{dw}{d\theta}=-2r^2\cos\theta\sin\theta+2r^2\sin\theta\cos\theta=0}$$

Exercises

1. Let $r(t)$ and $\theta(t)$ be polar coordinates which are functions of a parameter t.

(a) Express $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ in terms of $\dfrac{dr}{dt}$, $\dfrac{d\theta}{dt}$, r, and θ.

(b) Prove that

$$\cos\theta\frac{d^2x}{dt^2}+\sin\theta\frac{d^2y}{dt^2}=\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2$$

$$\cos\theta\frac{d^2y}{dt^2}-\sin\theta\frac{d^2x}{dt^2}=\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)^2$$

(These two equations express the radial and tangential components of acceleration, given on the left, in terms of polar coordinates.)

2. Use the chain rule to find $\frac{df}{du}$ and $\frac{df}{dv}$ in terms of u and v in each of the following cases.

(a) $f(x,y) = 2x - y, x = uv, y = u^2 - v^2$

(b) $f(x,y) = \dfrac{y}{x}, x = u + v, y = u - v$

(c) $f(x,y) = y^2, x = u^2 + v^2, y = \dfrac{v}{u}$

(d) $f(x,y) = \dfrac{x}{x+y}, x = v, y = u - v$

3. By using the chain rule twice, obtain

$\dfrac{\partial^2 f}{\partial u^2}$, $\dfrac{\partial^2 f}{\partial v^2}$, and $\dfrac{\partial^2 f}{\partial u \partial v}$ in each of the following cases:

(a) $f(x, y) = \dfrac{y}{x}, x = u + v, y = u - v$

(b) $f(x, y) = x^2 + y^2, x = uv, y = u^2 - v^2$

(c) $f(x, y) = y^2, x = uv, y = v$

Here we have used $x = r\cos\theta,\ y = r\sin\theta$

4. Let r and θ be the usual polar coordinates, and $z = f(x, y)$. Show that:

(a) $$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

(b) $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial^2 z}{\partial \theta^2}$$

Higher Partial Derivatives

In analogy to taking higher derivatives of single variable functions *f(x)*

$$\frac{d^2 f}{dx^2},$$

when a function has two or more independent variables we can take higher partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

as well as *mixed partial derivatives*

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$\frac{\partial^2 f}{\partial x \partial y}$ is the rate of change in the x direction of $\frac{\partial f}{\partial y}$

$\frac{\partial^2 f}{\partial y \partial x}$ is the rate of change in the y direction of $\frac{\partial f}{\partial x}$.

Note that for reasonably well behave functions the order of differentiation does not matter, so that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

LECTURE EXAMPLE:

Find all 2nd partials of $f(x,y) = x^3y + x\sin(y^2)$

We have $\frac{\partial f}{\partial x} = 3x^2y + \sin(y^2)$:

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = 6xy$$

$$\frac{\partial^2 f}{\partial y \partial x} = 3x^2 + 2y\cos(y^2)$$

Similarly $\frac{\partial f}{\partial y} = x^3 + 2xy\cos(y^2)$

$$\Rightarrow \frac{\partial^2 f}{\partial y^2} = 2x\cos(y^2) - 4xy^2\sin(y^2)$$

$$\frac{\partial^2 f}{\partial x \partial y} = 3x^2 + 2y\cos y^2$$

Note that this example illustrates the rule $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

Exercises

1. Find $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y}$ in each of the following cases:

(a) $ax + by + c$

(b) $x^2 + 2y^2 + 3x - y + 1$

(c) $\sin(x - y)$

(d) $\frac{y}{x}$

(e) e^{2x+3y}

(f) $\frac{1}{x} + \frac{1}{y}$

(g) $\sin 3x + \cos 2y$

(h) $(3x - 4y)^4$

(i) $\frac{1}{x + y}$

(j) $\ln xy$

(k) $\frac{1}{(x^2 + y^2)^{\frac{1}{2}}}$

2. Confirm that, if $r = (x^2 + y^2)^{\frac{1}{2}}$ and

$z = \log r$, then

$$\frac{\partial z}{\partial x} = \frac{x}{r^2} \qquad \text{and} \qquad \frac{\partial^2 z}{\partial x^2} = \frac{1}{r^2} - \frac{2x^2}{r^4}$$

Show that $z = \log r$ is a solution of the equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

(This is called Laplace's partial differential equation in two dimensions.)

3. Find the first and second partial derivatives of the following functions and evaluate them at (1, 2):

(a) x^2y^4

(b) $\cos(x+y)$

(c) $\sqrt{x^2+y^2}$

(d) $2x\sin xy$

Gradient Operators

Consider a scalar function $f = f(x, y, z)$

We know that:

$$\left.\frac{\partial f}{\partial x}\right|_P = \text{rate of change of } f \text{ in direction } x \text{ at } P;$$

$$\left.\frac{\partial f}{\partial y}\right|_P = \text{rate of change of } f \text{ in direction } y \text{ at } P;$$

$$\left.\frac{\partial f}{\partial z}\right|_P = \text{rate of change of } f \text{ in direction } z \text{ at } P.$$

Definition

The vector:

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \equiv \text{grad}\, f \equiv \nabla f$$

is called the gradient of f.

Note: ∇ operates on a scalar function and yields a vector.

LECTURE EXAMPLES:

Find the gradient of the function

$$f(x, y) = \frac{1}{3}x^3 - \frac{1}{3}y^3 - x + y + 3$$

We know the gradient is

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

$$\frac{\partial f}{\partial x} = x^2 - 1 \text{ and } \frac{\partial f}{\partial y} = -y^2 + 1$$

$$\Rightarrow \nabla f = \left(x^2 - 1, -y^2 + 1 \right)$$

Evaluate ∇f at the point (1,2) where:

$$f(x,y) = \frac{1}{(x^2 + xy^2)} = (x^2 + xy^2)^{-1}$$

$$\frac{\partial f}{\partial x} = -1(x^2 + xy^2)^{-2}(2x + y^2) = \frac{-(2x + y^2)}{(x^2 + xy^2)^2} \ ,$$

$$\frac{\partial f}{\partial y} = -1(x^2 + xy^2)^{-2}(2xy) = \frac{-2xy}{(x^2 + xy^2)^2} \ ,$$

Thus at (1,2): $\nabla f = \left(-\frac{6}{25}, -\frac{4}{25}\right)$

Divergence Operator

The *divergence operator* gives a scalar when applied to a vector field (compare with gradient, which gives a vector when applied to a scalar function). If $\mathbf{v} = (v_1, v_2, v_3)$:

$$\nabla \bullet \mathbf{v} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) \bullet (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

$$\Rightarrow \operatorname{div} \mathbf{v} = \nabla \bullet \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Note: that $\nabla \bullet (\nabla f)$ is written $\nabla^2 f$ i.e $\nabla^2 = \nabla \bullet \nabla$. It is called the Laplacian operator and is pronounced "del squared"

$$\nabla \bullet (\nabla f) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Note that div v measures how much a quantity represented by a vector field is expanding or contracting. For example, if the vector field represents the flow of a gas, then

- If $\nabla \bullet \mathrm{v} > 0$ the gas is expanding
- If $\nabla \bullet \mathrm{v} < 0$ the gas is contracting
- If $\nabla \bullet \mathrm{v} = 0$ the gas is incompressible

LECTURE EXAMPLES:

If $\mathbf{v} = x^2\mathbf{i} + (2yz + xz)\mathbf{j} - y\mathbf{k}$

$$\nabla \bullet \mathbf{v} = 2x + 2z \qquad \text{(scalar)}$$

$$\nabla(\nabla \bullet \mathbf{v}) = \left(\frac{\partial(\nabla \bullet \mathbf{v})}{\partial x}, \frac{\partial(\nabla \bullet \mathbf{v})}{\partial y}, \frac{\partial(\nabla \bullet \mathbf{v})}{\partial z} \right) = (2,0,2) \qquad \text{(vector)}$$

If $f = x^3 y^2 z$ (scalar)

$$\nabla f = (3x^2y^2z, 2yx^3z, x^3y^2) \qquad \text{(vector)}$$

$$\nabla \bullet (\nabla f) = 6xy^2z + 2x^3z \qquad \text{(scalar)}$$

Curl Operator

The curl quantifies the rotation of a quantity represented by a vector field **v** and is defined as

$$\text{curl}\,\mathbf{v} = \nabla \times \mathbf{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (v_1, v_2, v_3)$$

$$\text{Recall } \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}:$$

$$\nabla \times \mathbf{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (v_1, v_2, v_3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k}$$

The curl gives a vector when applied to a vector, so in summary:

$$\boxed{\begin{aligned} &\nabla : scalar \to vector \\ &\nabla \bullet : vector \to scalar \\ &\nabla \times : vector \to vector \end{aligned}}$$

LECTURE EXAMPLE:

$$\mathbf{v} = xy\mathbf{i} - y^2 z\,\mathbf{j} + (x - y + z)\mathbf{k} \equiv v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

$$\nabla \times \mathbf{v} = (-1 + y^2)\mathbf{i} + (0 - 1)\mathbf{j} + (0 - x)\mathbf{k} = (y^2 - 1)\mathbf{i} - \mathbf{j} - x\mathbf{k}$$

Physical Interpretation of Curl

Consider the rotation of a rigid body about a fixed axis. The rotation is described by the vector $\mathbf{w}$ ($\mathbf{w}$ is oriented along the axis and has magnitude equal to the angular speed of the body).

The velocity field is given by $\mathbf{v} = \mathbf{w} \times \mathbf{r}$ (use right hand rule to show $\mathbf{v}$ points in correct direction)

If $\mathbf{w} = \omega\mathbf{k}$ is the angular speed (i.e rotation about z axis), then:

$$\mathbf{v} = \mathbf{w} \times \mathbf{r} = -\omega y\mathbf{i} + \omega x\mathbf{j} \qquad \text{(exercise)}$$

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega\mathbf{k}$$

So for the case of a rigid body, $\nabla \times \mathbf{v}$ has the direction of the axis of rotation and has magnitude equal to twice the angular speed of rotation.

So for an arbitrary $\mathbf{v}$, $\nabla \times \mathbf{v}$ at a point can be interpreted as the amount by which a particle at the point is rotating.

For a fluid flow, curl $\mathbf{v}$ measures how much the fluid is swirling or eddying, where curl $\mathbf{v}$ is a vector pointing along the axis of rotation and the right-hand grip rule shows direction of rotation.

If $\nabla \times \mathbf{v} = 0$ then the flow is said to be irrotational (rotational otherwise).

LECTURE EXAMPLE:

Find the curl of $\mathbf{v} = 2x\mathbf{i} + 2yz\mathbf{j} + y^2\mathbf{k}$, and so determine if $\mathbf{v}$ is irrotational.

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x & 2yz & y^2 \end{vmatrix} = (2y - 2y)\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$$

i.e. $\mathbf{v}$ represents an irrotational flow.

Exercises

1. If:

$$f(x, y, z) = x^2 + y^2 + z^2$$

then the level surface:

$$f(x, y, z) = 1$$

is a sphere. Find the unit normal vector at a point $P = (x_0, y_0, z_0)$ on the surface.

Unit normal: $\hat{\mathbf{n}} = \dfrac{\nabla f}{|\nabla f|} = \dfrac{1}{\sqrt{4x_0^{\,2} + 4y_0^{\,2} + 4z_0^{\,2}}}(2x_0\mathbf{i} + 2y_0\mathbf{j} + 2z_0\mathbf{k})$

2. Consider the level surface given by:

$$f(x, y, z) = z - \cos\left(x^2 + yz\right) = 2$$

At the point $P = \left(0, \frac{\pi}{4}, 2\right)$ on the surface find the normal vector and the equation for the tangent plane.

3. Obtain the tangent plane and a normal vector for the following surfaces at the points given:

(a)	$z = x^2 + y^2$	at (1, 1, 2)
(b)	$z = xy$	at (2, 2, 4)
(c)	$z = \dfrac{x}{y}$	at (2, 1, 2)
(d)	$z = (29 - x^2 - y^2)^{\frac{1}{2}}$	at (3, 4, 2)
(e)	$z = x^2 + y^2 - 2x - 2y$	at (1, 1, -2)
(f)	$z = e^{xy}$	at (0, 0, 1)

4. Find the directional derivative $\dfrac{df}{ds}$ of each of the following functions according to the data. Also, for the given point, find the direction of steepest ascent.

(a) $f(x, y) = x^2 + y^2$ at (1, 2), direction $\theta = 30°$

(b) $f(x, y) = x^2 y^2$ at (2, 1), direction $\theta = -45°$

(c) $f(x, y) = x^2 y - xy^2 + 2$ at (-1, 1), direction $\theta = 120°$

(d) $f(x, y) = \sin xy$ at ($^1/_2$, π), direction $\theta = -90°$

(e) $f(x, y) = \cos(x^2 - y)$ at (0, $-\pi$), direction $\theta = 0$

(f) $f(x, y) = e^{x-y}$ at (1, 1), direction $\theta = -45°$

5. Find the directional derivative of:

$$f(x, y, z) = x^2 + 2y^2 + 3z^2$$

at the point $P = (3, 2, 1)$ in the direction of the vector $\mathbf{b} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$.

6. Find the directional derivative of:

$$f(x, y, z) = x^2 yz$$

at the point $P = (1, 2, 1)$ in the direction of the vector $\mathbf{b} = \mathbf{i} + \mathbf{k}$.

7. Obtain ∇f, where $f(x, y)$ is given by the following. Give its components, its direction, and its magnitude at the points specified:

(a) $\dfrac{1}{x+y}$ at (1, -2)

(b) $\dfrac{y}{x}$ at (2, 0)

(c) $y^2 - 3x^2 + 1$ at (0, 0)

(d) $\dfrac{1}{x} - \dfrac{1}{y}$ at (2, 1)

8. Use the gradient vector to obtain a *unit* vector perpendicular to the following curves at the points given:

(a) $2x - 3y + 1 = 0$ at any point

(b) $x^2 + y^2 = 5$ at (2, 1)

(c) $x^2 + y^2 = r^2$ at (x_0, y_0) on the circle

(d) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ at (x_0, y_0) on the ellipse

(e) $y = 3x^2 - 2$ at (2, 10)

9. Divergence of a vector field. Find the divergence of the following vectors:

(a) $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

(b) $y^2 e^z \mathbf{i} + x^2 z^2 \mathbf{k}$

(c) $\cos(x)\cosh(y)\mathbf{i} + \sin(x)\sinh(y)\mathbf{j}$

(d) $x^2 y\mathbf{i} + x^2 y\mathbf{j} + y^2 z\mathbf{k}$

(e) $v_1(y,z)\mathbf{i} + v_2(x,z)\mathbf{j}$

10. Prove the following formulae

(a) $\text{div}\,(f\mathbf{v}) = f\,\text{div}\,\mathbf{v} + \mathbf{v}.\nabla f$

(b) $\text{div}\,(f\nabla g) = f\nabla^2 g + \nabla f.\nabla g$

11. (a) Calculate the divergence of the velocity vector field **v** defined by

$$v = \cos(y)\sin(x)\,\mathbf{i} - e^x y^2\,\mathbf{j} + \frac{z}{x}\mathbf{k}$$

(b) Can **v** represent the flow of an incompressible fluid? Why?

12. Find curl **v** where, with respect to the right-handed Cartesian coordinates, **v** equals

(a) $y\mathbf{i} + 2x\mathbf{j}$ (b) $z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$

(c) $v_1(x)\mathbf{i} + v_2(y)\mathbf{j} + v_3(z)\mathbf{k}$ (d) $\sin(y)\mathbf{i} + \cos(x)\mathbf{j}$

(e) $(x^2 + y^2 + z^2)^{-\frac{3}{2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ (f) $y^2\mathbf{i} + z^2\mathbf{j} + x^2\mathbf{k}$

(g) $e^x(\cos(z)\mathbf{j} + \sin(z)\mathbf{k})$ (h) $xyz(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

(i) $\ln(x^2 + y^2)\mathbf{i} + \arctan\left(\frac{y}{x}\right)\mathbf{j}$ (j) $e^{xyz}(\mathbf{i} + \mathbf{j} + \mathbf{k})$

13. Show that, granted sufficient differentiability,

$$\operatorname{curl}(\mathbf{u} + \mathbf{v}) = \operatorname{curl}\mathbf{u} + \operatorname{curl}\mathbf{v}$$

14. Show that, granted sufficient differentiability,

$$\operatorname{curl}(\operatorname{grad} f) = 0$$

Stationary Points

Recall that for a function of one variable, points at which $f'(x) = 0$ were extreme points and we used f'' to tell us if the point was a maximum $(f'' < 0)$, a minimum $(f'' > 0)$ or a point of inflection $(f'' = 0)$.

The ideas of local maxima and minima still carry over to functions of several variables (think of walking in a mountain range – you can look and see where the peaks (local maxima) and valleys (local minima) occur). Stationary points of multivariable functions can be found by extending the idea of a *tangent line*, which is applicable to single variable functions, to that of a *tangent plane*.

Tangent Plane and Stationary Points

For a single variable function the *tangent line* is horizontal at local maxima or minima located at x_0 say

$$z = \left(\frac{df}{dx}\right)_{x=x_0}(x - x_0) + z_0 = z_0 \Rightarrow \left(\frac{df}{dx}\right)_{x=x_0} = 0$$

Analogously, for a multivariable function the *tangent plane* is horizontal at local maxima or minima. A tangent plane is given by at the point (x_0, y_0) is given by:

$$z = \left(\frac{\partial f}{\partial x}\right)_{x=x_0, y=y_0}(x - x_0) + \left(\frac{\partial f}{\partial y}\right)_{x=x_0, y=y_0}(y - y_0) + z_0 .$$

For this to be horizontal we must have

$$\boxed{\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0}$$

LECTURE EXAMPLE:

1. Find the local extrema of $f(x,y)=x^2-xy+y^2+3x$

$$\frac{\partial f}{\partial x}=2x-y+3$$

$$\frac{\partial f}{\partial y}=-x+2y$$

$$\frac{\partial f}{\partial x}=0\Rightarrow 2x-y+3=0$$

$$\frac{\partial f}{\partial y}=0\Rightarrow -x+2y=0$$

Solve $\Rightarrow x=-2,\ \ y=-1$ is the only stationary point. However, is this a local maximum, local minimum or neither?

Classifying stationary points - Two Variables

To determine whether a stationary point is a local maximum/minimum or neither we need to consider the second partial derivatives. However there are three 2^{nd} partial derivatives, so which do we use?

The 2^{nd} derivative test for a multivariable function $f(x, y)$ is not too dissimilar to 2^{nd} derivative test for a single-valued function $g(x)$, we just need to check things in more than one direction.

For a function of one variable $g(x)$

$$g'' < 0 \text{ max}$$
$$g'' > 0 \text{ min}$$
$$g'' = 0 \text{ no conclusion}$$

- For $f(x,y)$ to have maximum, it should be clear that both $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ are <0 (i.e. maximum in each direction).
- For a minimum, must have $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ are > 0 (i.e. minimum in each direction).

So for either a min. or a max. $\frac{\partial^2 f}{\partial x^2} \times \frac{\partial^2 f}{\partial y^2}$ must be > 0. However this alone is not sufficient - the term $\frac{\partial^2 f}{\partial x \partial y}$ is needed as well (Taylor series expansions are needed to prove this)

$$\Delta = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0 .$$

In general, let (a,b) be a critical point of $f(x,y)$

i.e: $\dfrac{\partial f}{\partial x}(a,b) = 0 = \dfrac{\partial f}{\partial y}(a,b)$

and let $\Delta = \det\begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix}$ (each derivative evaluated at (a,b))

Then

1) If $\Delta > 0$ and $\left.\dfrac{\partial^2 f}{\partial x^2}\right|_{(a,b)} > 0$ then (a,b) is a (local) minimum.

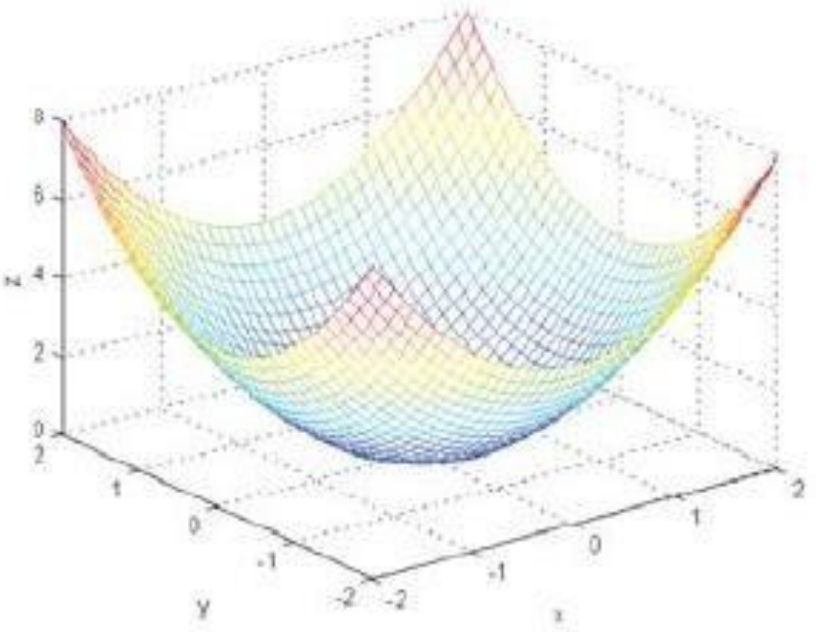
2) If $\Delta > 0$ <u>and</u> $\left.\dfrac{\partial^2 f}{\partial x^2}\right|_{(a,b)} < 0$ then (a,b) is a local maximum.

3) If $\Delta < 0$ then (a,b) is a SADDLE point (neither a max or a min)

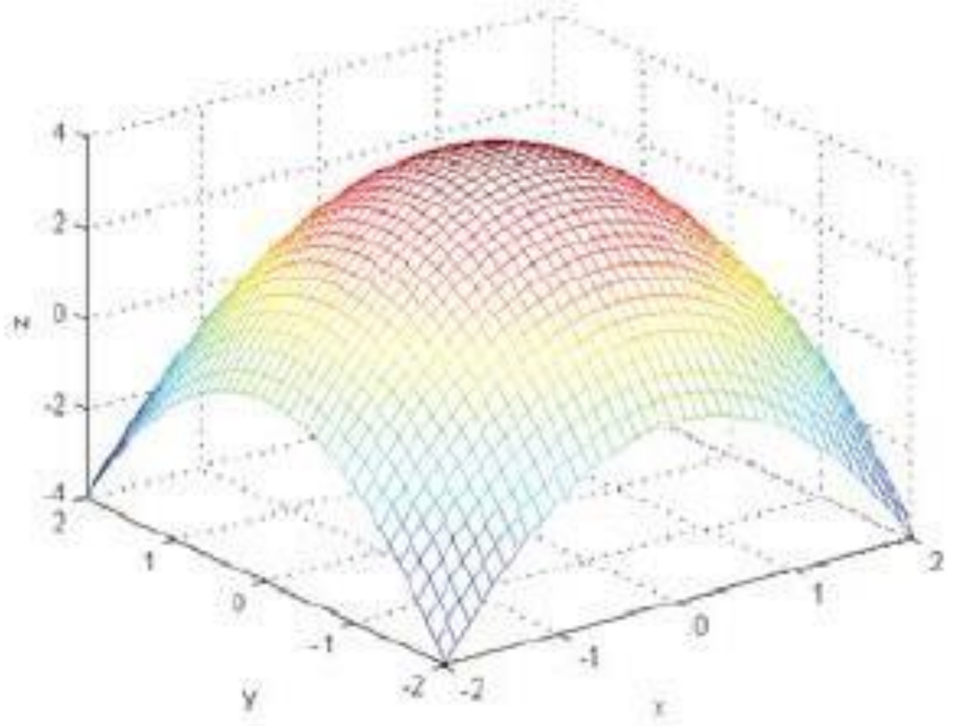
4) If $\Delta = 0$ then no conclusion can be drawn form this information alone.

Note that $\Delta = \dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2$

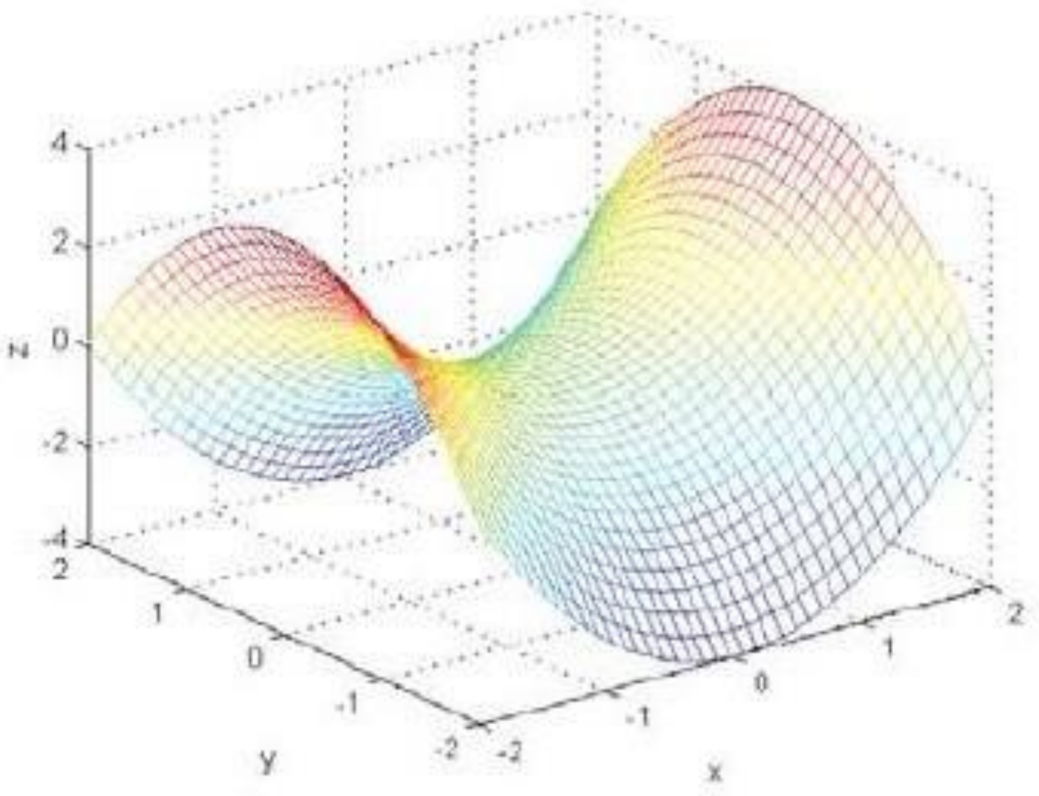
$\Delta = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$	$\frac{\partial^2 f}{\partial x^2}$	Classification
> 0	> 0	Local minimum
> 0	< 0	Local maximum
< 0	-----	Saddle Point
$= 0$	-----	No conclusion



Plot of $z = f(x, y) = x^2 + y^2$ showing a minimum at (0,0)



Plot of $z = f(x, y) = 4 - x^2 - y^2$ showing a maximum at (0,0)



Plot of $z = f(x, y) = x^2 - y^2$ showing a saddle at (0,0)

LECTURE EXAMPLE:

Find all stationary points of $f(x,y)=x^3y-y^2-3x^2y$ and classify those away from the origin.

Find the first derivatives:

$$\frac{\partial f}{\partial x}=3x^2y-6xy$$
$$\frac{\partial f}{\partial y}=x^3-2y-3x^2$$

From 1st equation we have $x=0$ or $y=0$ or $x=2$.

If $x=0$, 2nd equation gives $y=0$. So (0,0) is one critical point .If $y=0$, 2nd equation gives $x^2(x-3)=0 \Rightarrow x=0,3$. So (3,0) is 2nd critical point.

If $x = 2$, 2^{nd} equation gives $y = -2 \Rightarrow (2,-2)$ is 3^{rd} critical point.

This gives the stationary points:

$$(0,0), (3,0), (2,-2)$$

Now we need the second derivatives to classify the points:

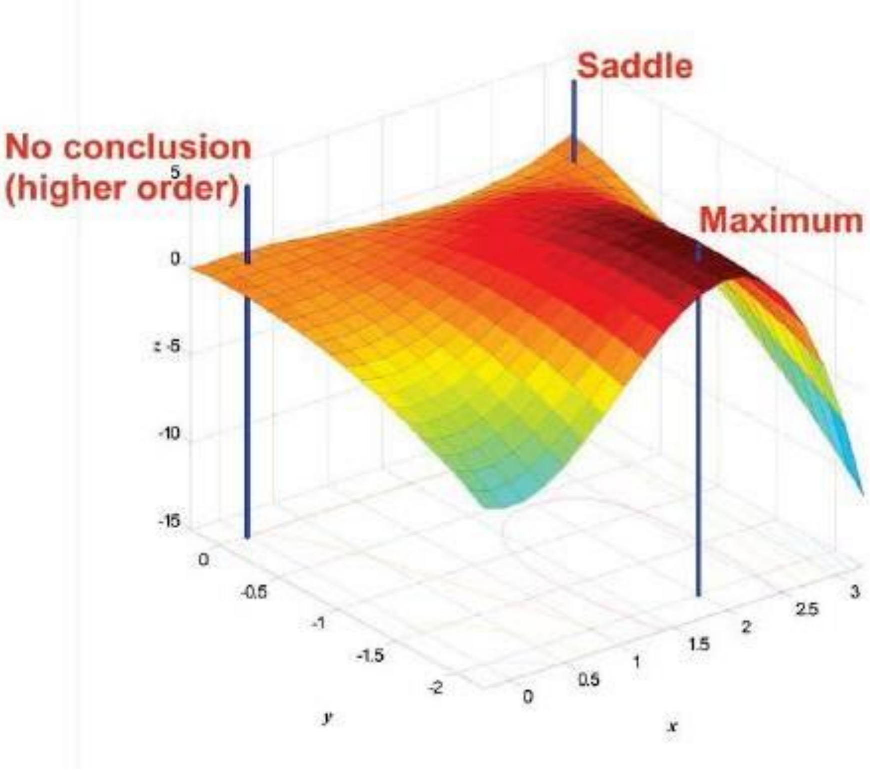
$$\frac{\partial f}{\partial x} = 3x^2 y - 6xy \Rightarrow \frac{\partial^2 f}{\partial x^2} = 6xy - 6y$$

$$\frac{\partial f}{\partial y} = x^3 - 2y - 3x^2 \Rightarrow \frac{\partial^2 f}{\partial y^2} = -2$$

$$\frac{\partial f}{\partial y} = x^3 - 2y - 3x^2 \Rightarrow \frac{\partial^2 f}{\partial x \partial y} = 3x^2 - 6x$$

We can classify the points using a table:

Point	$\frac{\partial^2 f}{\partial x^2} = 6xy - 6y$	$\frac{\partial^2 f}{\partial y^2} = -2$	$\frac{\partial^2 f}{\partial x \partial y} = 3x^2 - 6x$	Δ	Classification
$(0,0)$	0	-2	0	0	No conclusion
$(3,0)$	0	-2	9	-81	Saddle Point
$(2,-2)$	-12	-2	0	24	Local maximum



ADDITIONAL EXAMPLE:

Find all stationary points of:

$$f(x,y)=\frac{1}{3}x^3+\frac{1}{3}y^3-x^2-y^2$$

and classify them.

Find the partial first derivatives:

$$\frac{\partial f}{\partial x}=x^2-2x,\frac{\partial f}{\partial y}=y^2-2y$$

Set equal to 0 and solve for stationary points:

$$\frac{\partial f}{\partial x}=x^2-2x=0\Rightarrow x=0,2$$

$$\frac{\partial f}{\partial y}=y^2-2y=0\Rightarrow y=0,2$$

$$\Rightarrow \text{stationary points}=(0,0),(0,2),(2,0),(2,2)$$

Find the second partial derivatives:

$$\frac{\partial f}{\partial x}=x^2-2x\Rightarrow\frac{\partial^2 f}{\partial x^2}=2x-2$$

$$\frac{\partial f}{\partial y}=y^2-2y\Rightarrow\frac{\partial^2 f}{\partial y^2}=2y-2$$

$$\frac{\partial f}{\partial y}=y^2-2y\Rightarrow\frac{\partial^2 f}{\partial x\partial y}=0$$

Set up a table to classify points:

Point	$\frac{\partial^2 f}{\partial x^2} = 2x - 2$	$\frac{\partial^2 f}{\partial y^2} = 2y - 2$	$\frac{\partial^2 f}{\partial x \partial y} = 0$	Δ	Classification
(0,0)	-2	-2	0	4	Local maximum
(0,2)	-2	2	0	-4	Saddle Point
(2,0)	2	-2	0	-4	Saddle Point
(2,2)	2	2	0	4	Local minimum

Exercises

Find and classify all critical points of:

(a) $x^4 + y^2 - 2(x-y)^2$

(b) $x^3 + y^3 - 3axy \;\; (a > 0)$

(c) $x^3 - 4x^2 - xy - y^2$

Find a classify all the critical points of the function:

$$2x^3 + 6xy^2 - 3y^3 - 150x$$

Summary

- **The difference between functions of a single variable and functions of several variables**

  In the real world functions will often depend on more than one variable, e.g. temperature in a room is a function with 3 variables – the (x, y, z) position, and possibly time t as well.

- **Partial derivatives and finding/classifying stationary points**

  We revised the ideas of partial differentiation of multivariable functions and also looked at finding the higher order partial derivatives.

  We then applied this knowledge to finding and classifying the stationary points of multivariable functions.

- **Vector fields and operators**

  We looked at how multivariable functions can represent scalar and vector fields.

  We also looked at the gradient, divergence and curl operators and examined their physical interpretations.