Multivariable Integration

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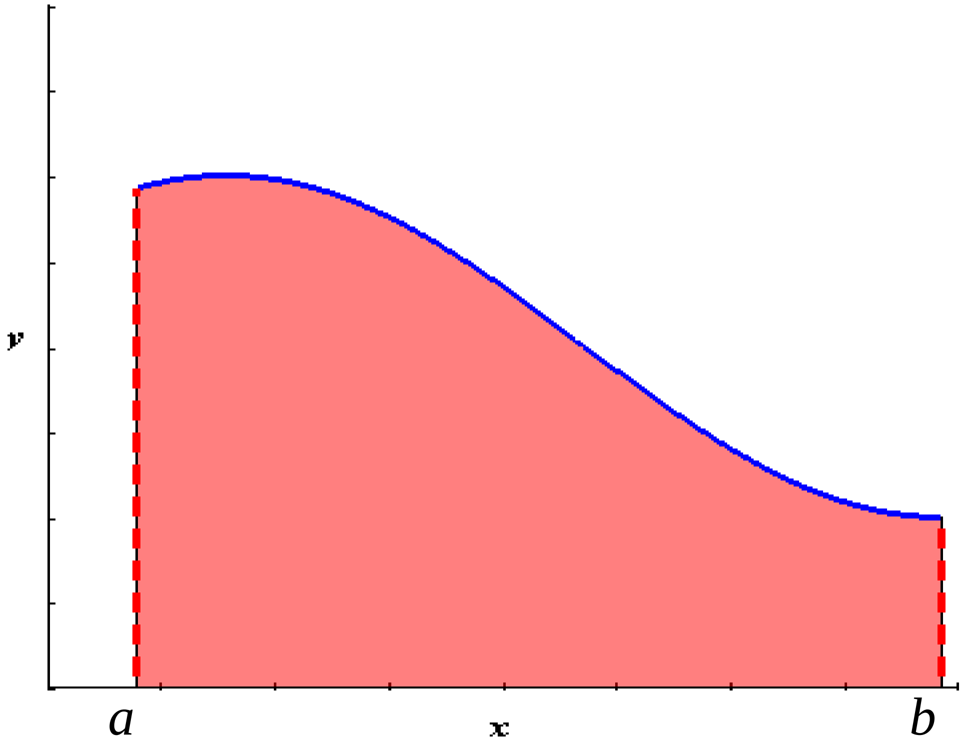
Introduction

- 1-D Integrals
- Multivariable integrals
- Changing the order of integration
- Changing variables in double integrals
- Summary

1-D Integrals

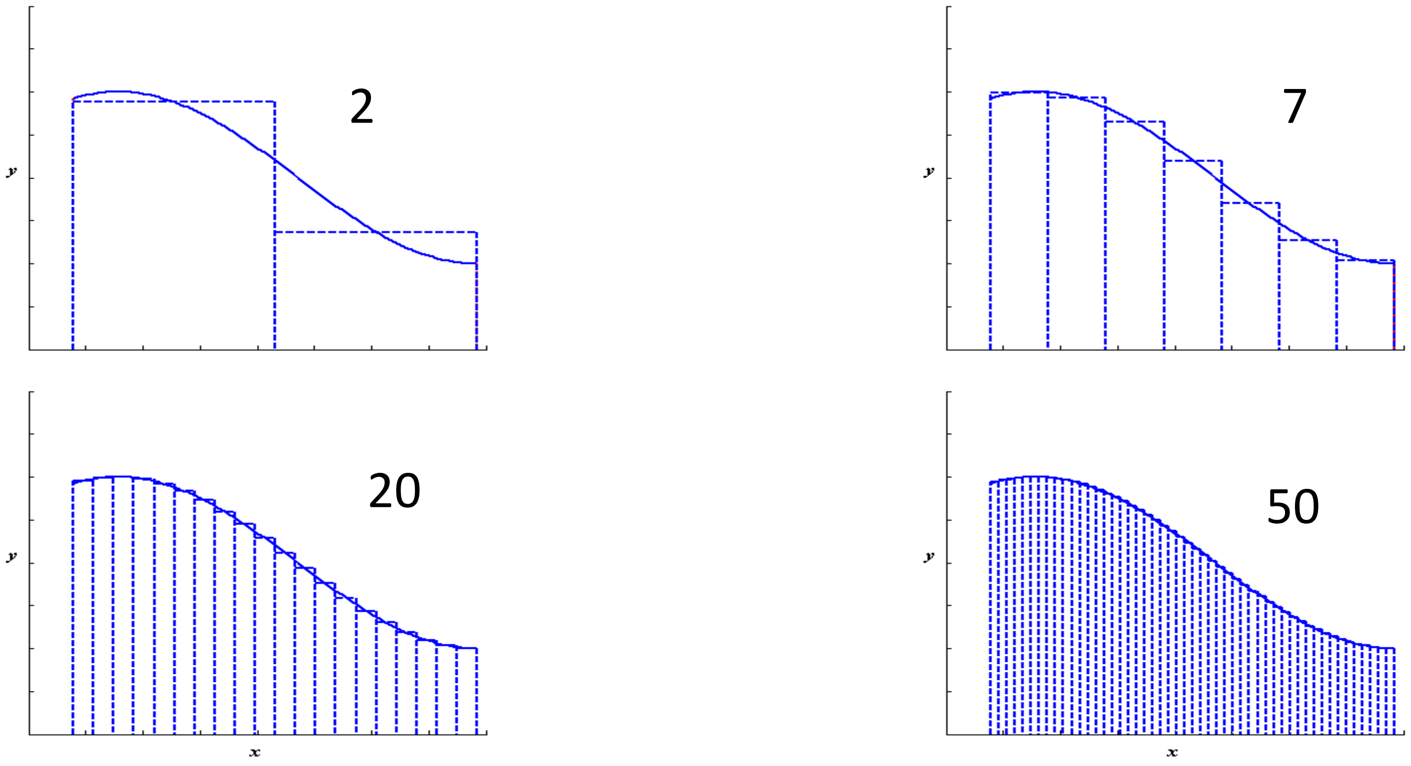
- Suppose we have a curve:



- We want to find the area under this curve between 2 points

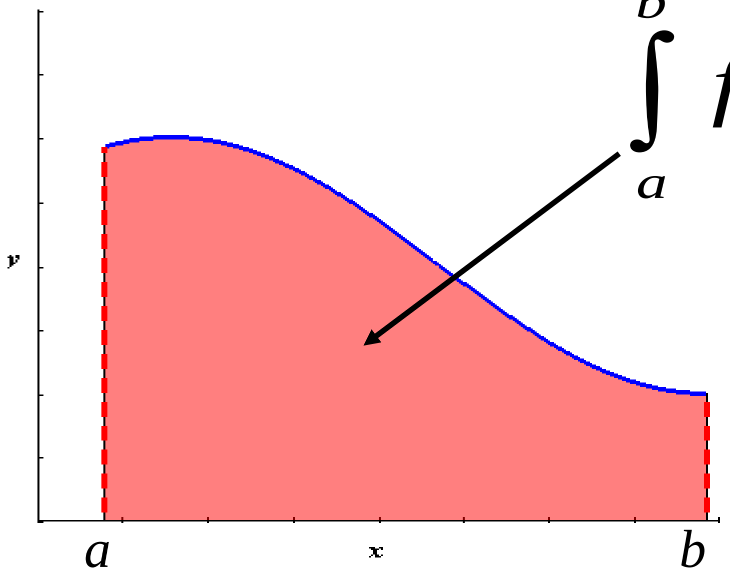
1-D Integrals

- We can approximate the area under the curve using rectangles:



1-D Integrals

- As the number of rectangles increases the approximation becomes more accurate.
- The integral is the area found by the limit as the number of rectangles approaches infinity.



$$\int_a^b f(x)dx$$

Multivariable integrals

- If we have a function of 2 (or more!) variables we can integrate it over a region.

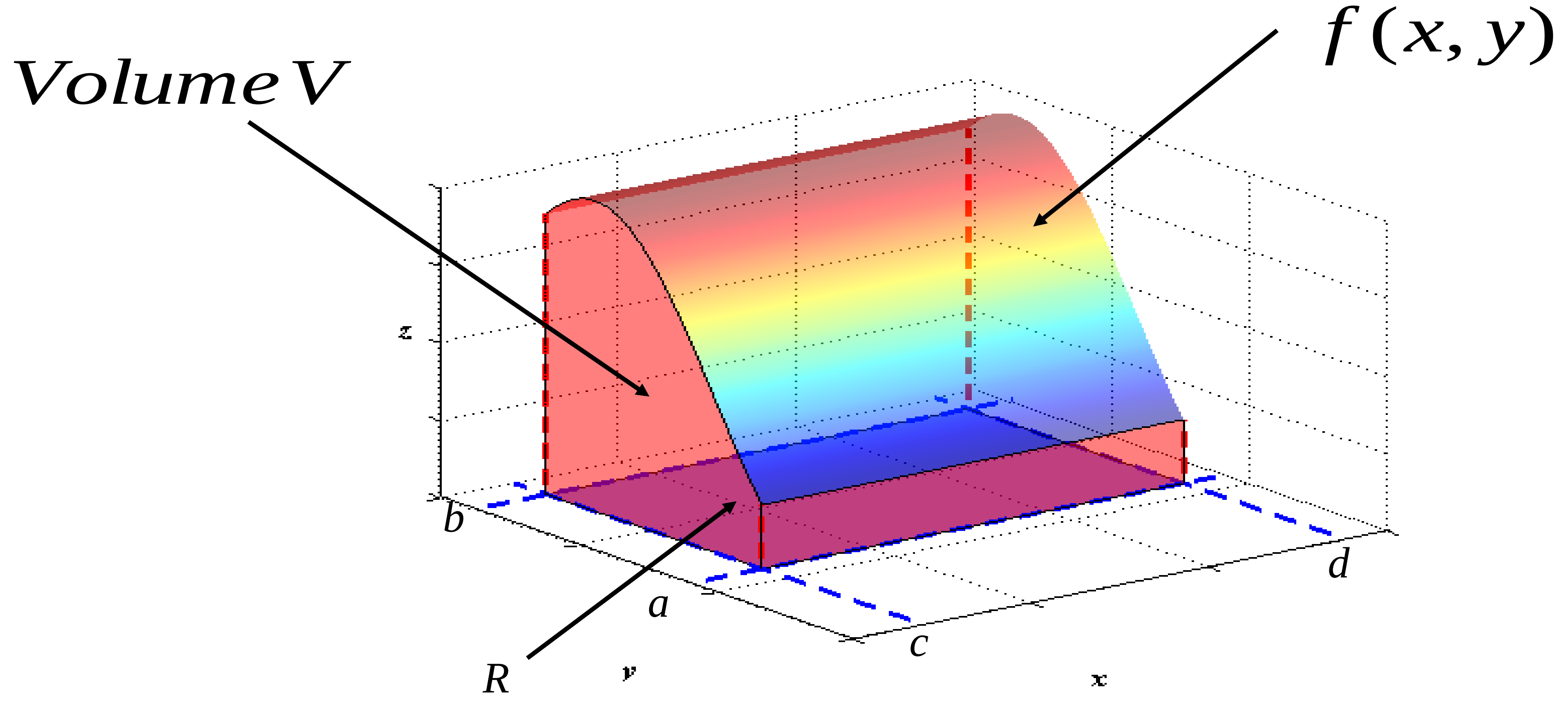
$$e.g.\ f(x, y)$$

$$y \text{ from } a \text{ to } b$$

$$x \text{ from } c \text{ to } d$$

- This will find the volume under the surface for that region.

Multivariable integrals



Examples of Multivariable Integrals

- Volume

$$V = \iint_R f(x,y)\,dxdy$$

- Area

$$A = \iint_R dxdy$$

- Averages

$$M = \iint_R \rho(x,y)\,dxdy$$

$$\overline{\rho} = \frac{1}{A}\iint_R \rho(x,y)\,dxdy$$

- Centre of mass

$$\overline{x} = \frac{1}{M}\iint_R x\rho(x,y)\,dxdy$$

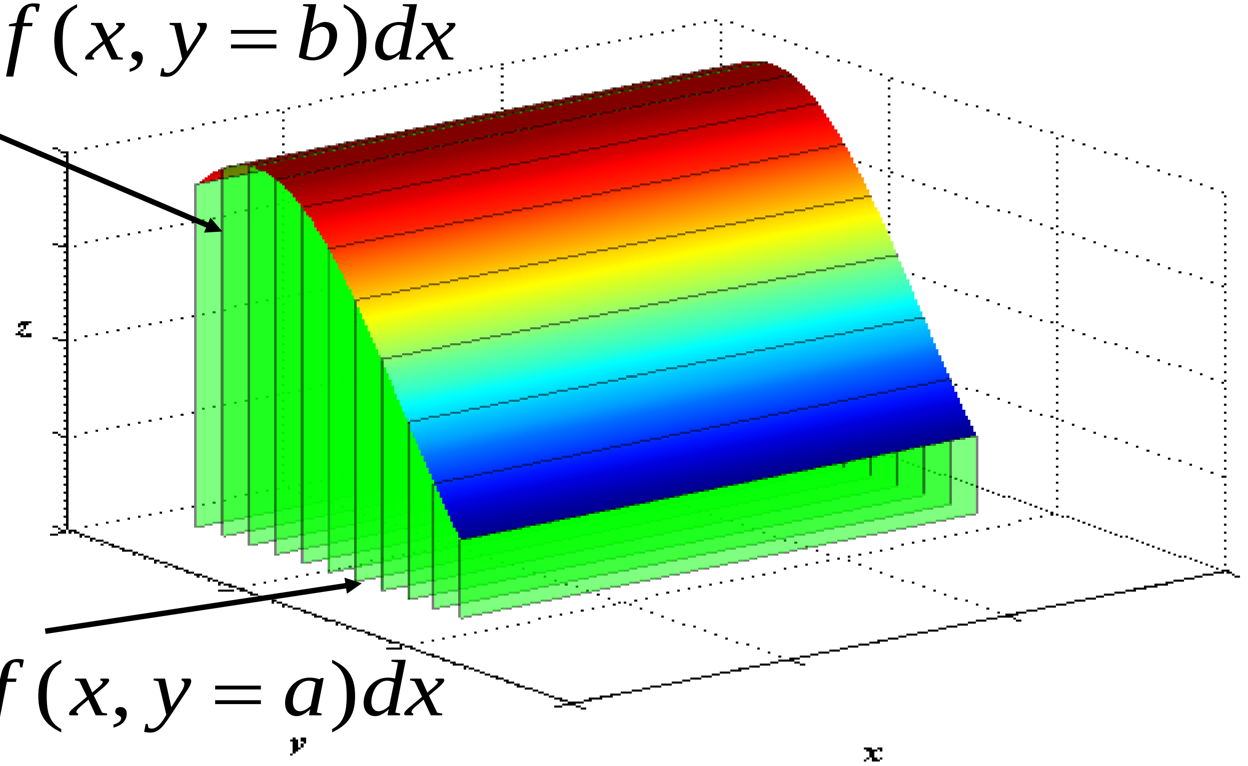
$$\overline{y} = \frac{1}{M}\iint_R y\rho(x,y)\,dxdy$$

Multivariable integrals

- How do we calculate the multiple integral i.e. the volume under a surface?
- We carry out the integration by working out two 1D integrals
- We form “slices” of the volume, by holding one variable constant
- This is called the inner integral
- The total volume is found by adding these slices together.

Multivariable integrals

$$A(y=b)=\int_{c}^{d} f(x,y=b)\,dx$$



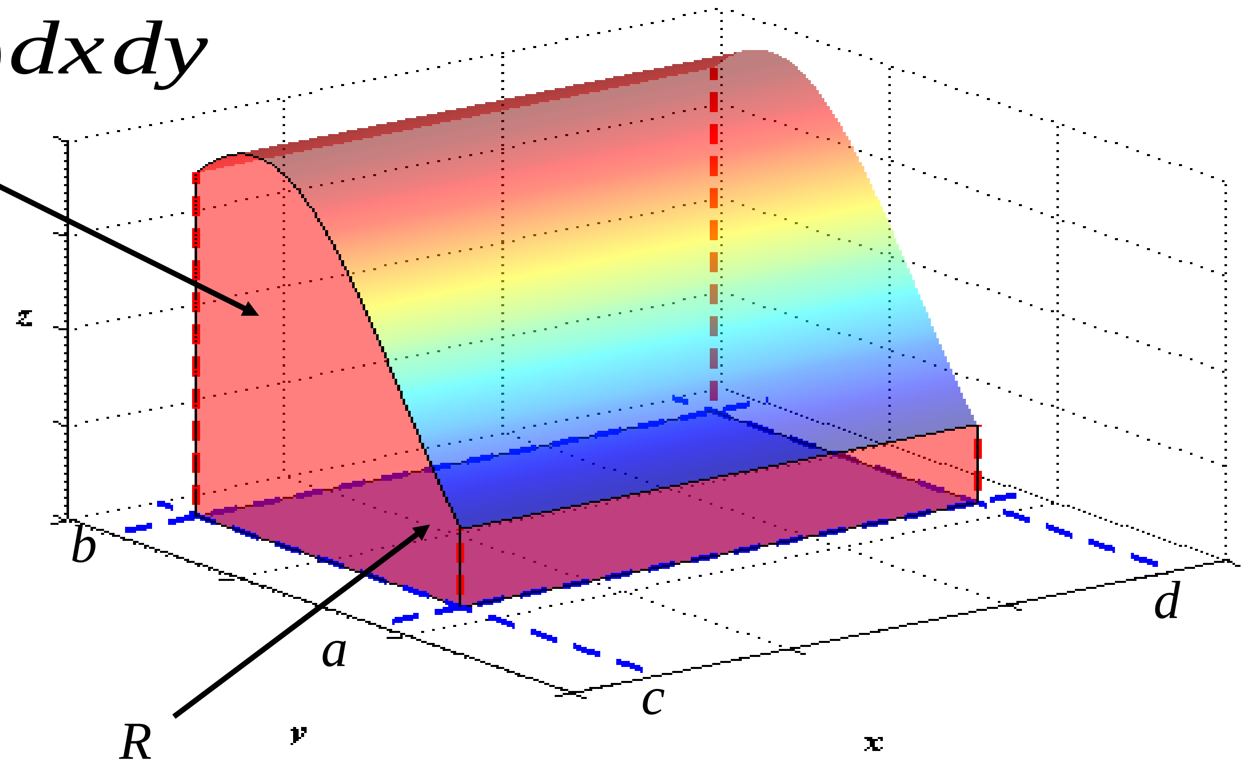
$$A(y=a)=\int_{c}^{d} f(x,y=a)\,dx$$

- The slices form the volume under the surface that covers the region of integration.

Multivariable integrals

- There is a slice at each value of *y*
- Find the volume by integrating the area of these slices over *y*

$$\int_a^b \int_c^d f(x,y)\,dx\,dy$$



Summary of method:

- We evaluate the inner integral first (holding the variable in the outer integral constant).
- We then evaluate the outer integral to give the volume.

Example – Multivariable integrals

- Evaluate:

$$\int_3^4 \int_0^1 x^2 y\,dx\,dy$$

- Find the inner integral (hold *y* constant):

$$\int_{y=3}^{y=4} \int_{x=0}^{x=1} x^2 y\,dx\,dy = \int_{y=3}^{y=4} \left[\int_{x=0}^{x=1} x^2 y\,dx \right] dy$$

$$= \int_{y=3}^{y=4} \left[\frac{x^3}{3} y \right]_{x=0}^{x=1} dy = \int_{y=3}^{y=4} \left(\frac{1}{3} - \frac{0}{3} \right) y\,dy = \int_{y=3}^{y=4} \frac{1}{3}\, y\,dy$$

Example – Multivariable integrals

- Now evaluate the outer integral (it has just a single variable!):

$$\int_{y=3}^{y=4} \frac{1}{3} y dy = \left[\frac{1}{3} \times \frac{y^2}{2} \right]_{y=3}^{y=4} = \frac{16}{6} - \frac{9}{6} = \frac{7}{6}$$

- So the answer is:

$$\int_{3}^{4} \int_{0}^{1} x^2 y dx dy = \frac{7}{6}$$

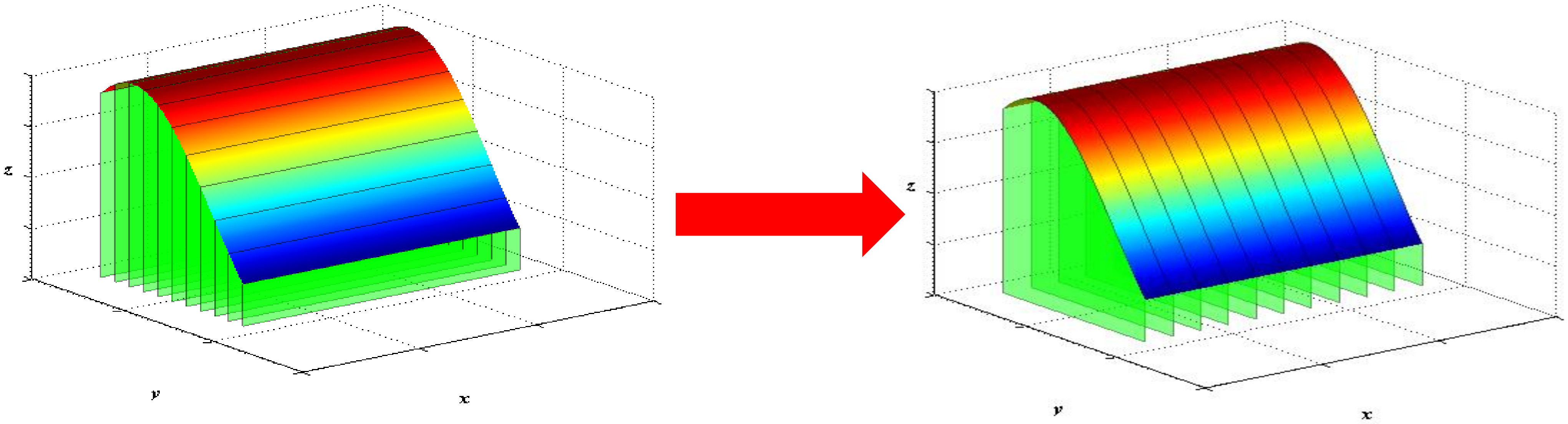
Example – Changing the order of integration

- Let’s change the order of the integrals:

$$\int_{y=3}^{y=4}\int_{x=0}^{x=1} x^2 y\,dx\,dy = \int_{x=0}^{x=1}\int_{y=3}^{y=4} x^2 y\,dy\,dx$$

Changing the order of integration

- Changing the order of integration means that we will form the slices of the volume in the other direction:



Example – Changing the order of integration

- Now find the integral:

$$\int_{x=0}^{x=1}\int_{y=3}^{y=4} x^2 y\,dy\,dx = \int_{x=0}^{x=1}\left[\frac{x^2y^2}{2}\right]_{y=3}^{y=4} dx$$

$$= \int_{x=0}^{x=1}\frac{(16-9)x^2}{2}\,dx = \int_{x=0}^{x=1}\frac{7x^2}{2}\,dx$$

$$= \left[\frac{7x^3}{6}\right]_{x=0}^{x=1} = \frac{7}{6}$$

- So the answer is the same as before.

Non-rectangular integration regions

- Lets try changing the order for a more complicated example

$$\int_0^2 \int_0^{2x} y\,dy\,dx$$

- Find the inner integral (hold x constant):

$$\int_{x=0}^{x=2} \int_{y=0}^{y=2x} y\,dy\,dx = \int_{x=0}^{x=2} \left[\frac{y^2}{2} \right]_{y=0}^{y=2x} dx = \int_{x=0}^{x=2} 2x^2\,dx$$

- Now find the outer integral:

$$\int_{x=0}^{x=2} 2x^2\,dx = \left[\frac{2}{3} x^3 \right]_{x=0}^{x=2} = \frac{16}{3}$$

Non-rectangular integration regions

- Changing the order of integration and finding the inner integral:

$$\int_{x=0}^{x=2}\int_{y=0}^{y=2x} y\,dy\,dx = \int_{y=0}^{y=2x}\int_{x=0}^{x=2} y\,dx\,dy$$

$$= \int_{y=0}^{y=2x}\left[xy\right]_{x=0}^{x=2} dy = \int_{y=0}^{y=2x} 2y\,dy$$

Non-rectangular integration regions

- Now find the outer integral:

$$\int_{y=0}^{\boxed{y=2x}} 2y\,dy = \left[y^2\right]_{y=0}^{y=2x}$$

$$= 4x^2$$

- This answer does not make sense – it still contains variables.
- What has gone wrong

Non-rectangular integration regions

- Notice in the second example the limits of the inner integral involved the variable of the outer integral.
- This means that we are integrating over a NON RECTANGULAR region.

Non-rectangular integration regions

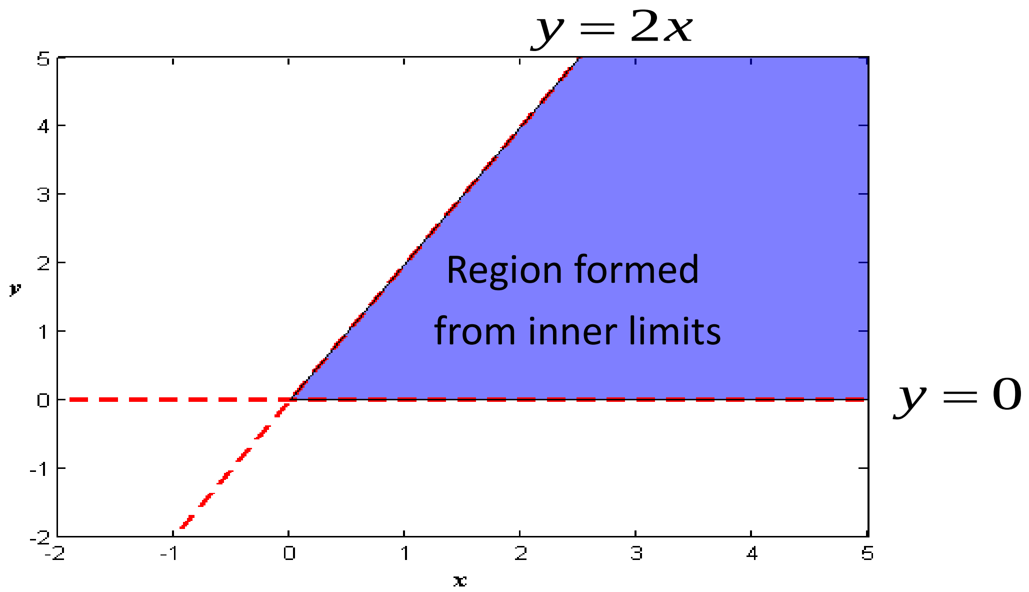
- We have to make sure that the region we are integrating over stays the same.
- The first thing we do is draw the region of integration.
- Let’s try this with the example from above:

Non-rectangular integration regions

- The integral is:

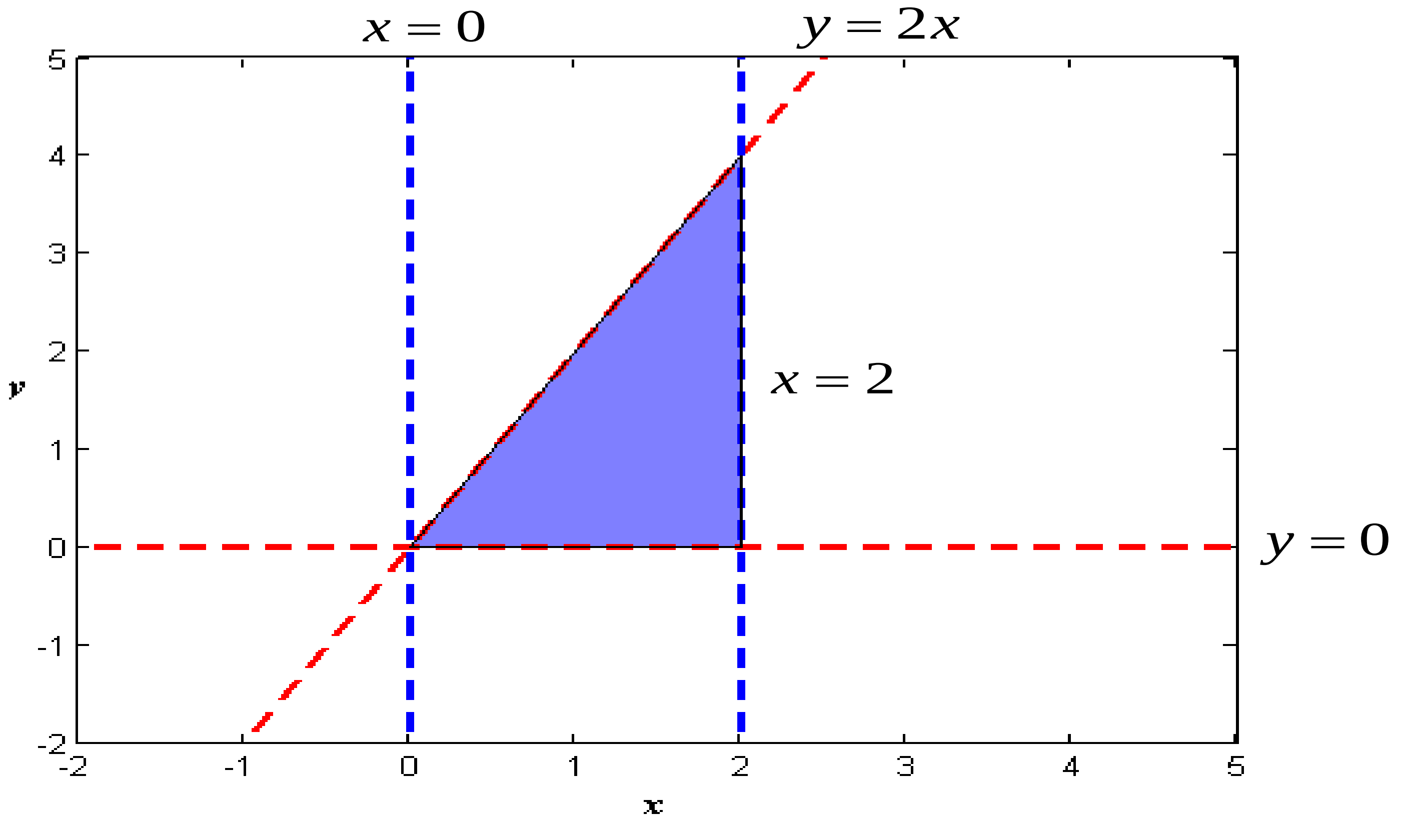
$$\int_0^2 \int_0^{2x} y\,dy\,dx$$

- If we plot the inner limits:



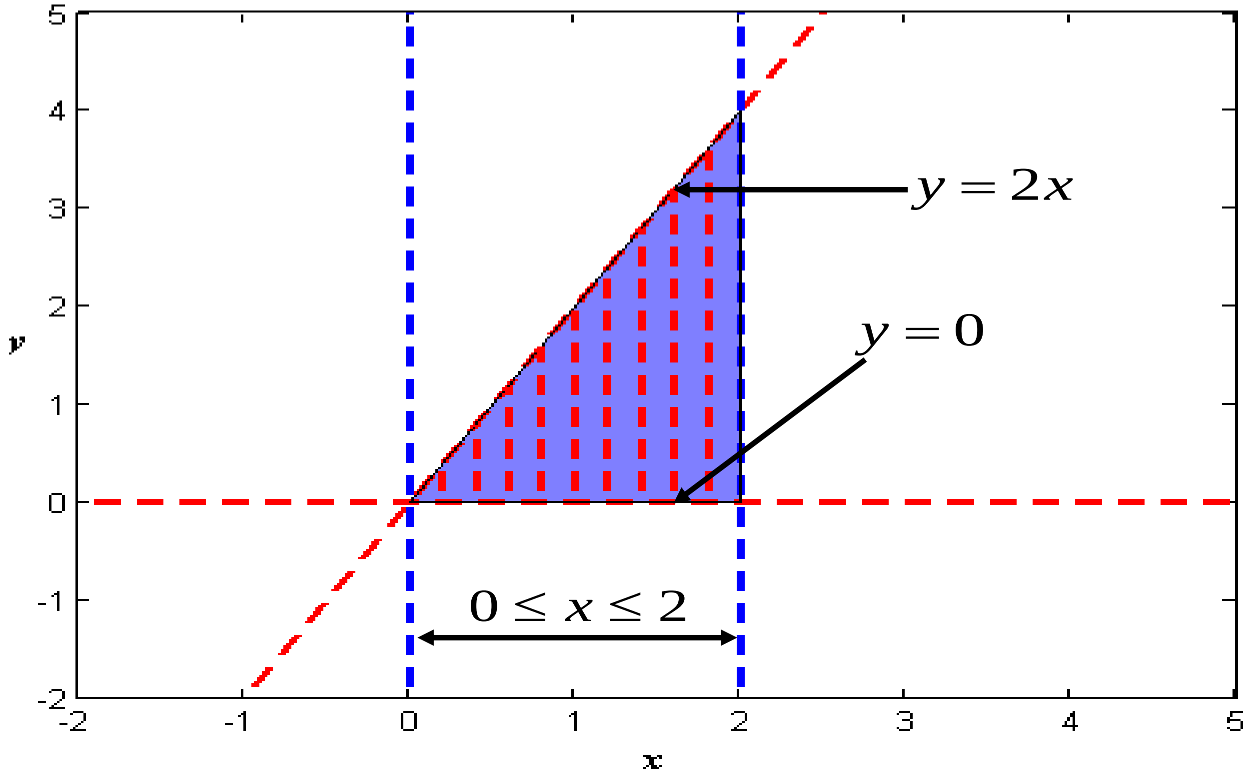
Non-rectangular integration regions

- Now lets look at the outer limits:



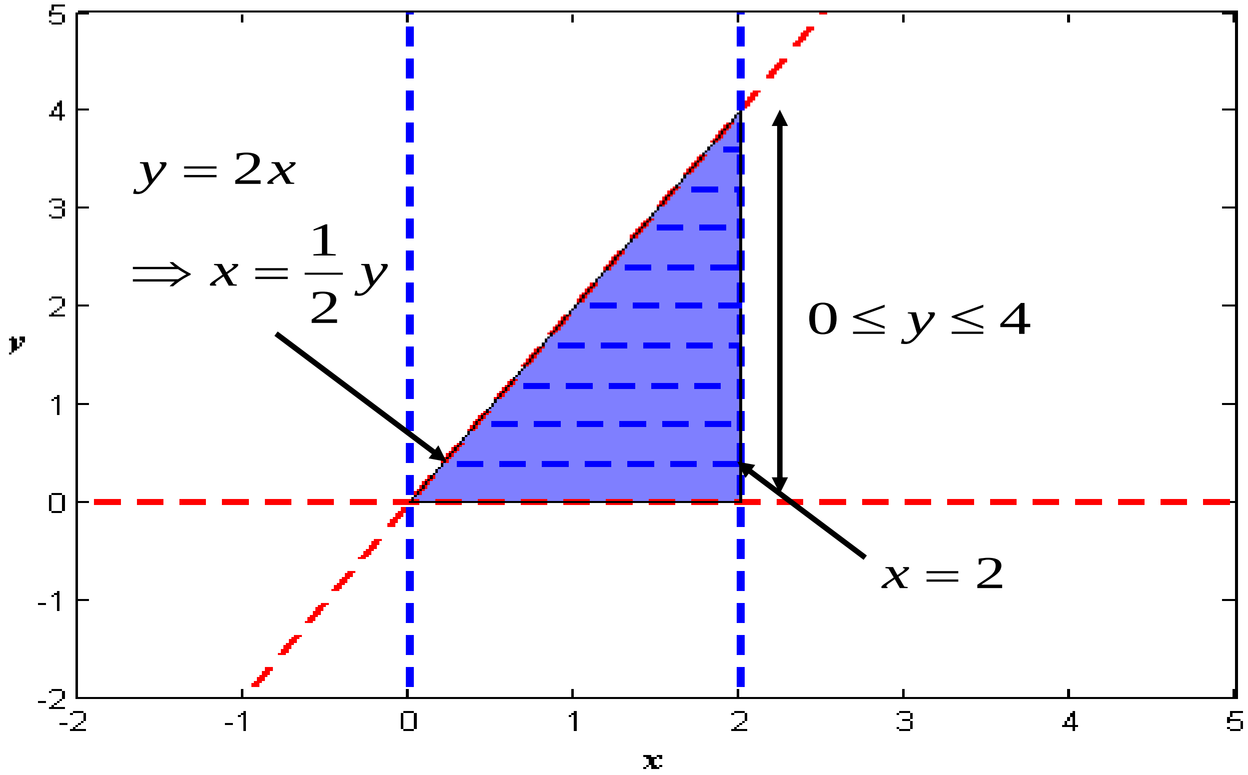
Non-rectangular integration regions

- We can think of the region being formed out of strips in the direction of the inner integral (y direction):



Non-rectangular integration regions

- To change the order of integration we must consider the region to be formed out of strips in the x direction:



Non-rectangular integration regions

- So we can put these new limits into the integral:

$$\int_{x=0}^{x=2}\int_{y=0}^{y=2x} y\,dy\,dx \quad = \int_{y=0}^{y=4}\int_{x=\frac{1}{2}y}^{x=2} y\,dx\,dy$$

$$= \int_{y=0}^{y=4} [xy]_{x=\frac{1}{2}y}^{x=2}\, dy \quad = \int_{y=0}^{y=4}\left(2y-\frac{1}{2}y^2\right)dy$$

$$= \left[y^2-\frac{1}{6}y^3\right]_{y=0}^{y=4} = 16-\frac{64}{6} \;= \frac{16}{3}$$

- Which is the same answer as before.

Example 2 – Non-rectangular integration regions

- We want to evaluate the integral:

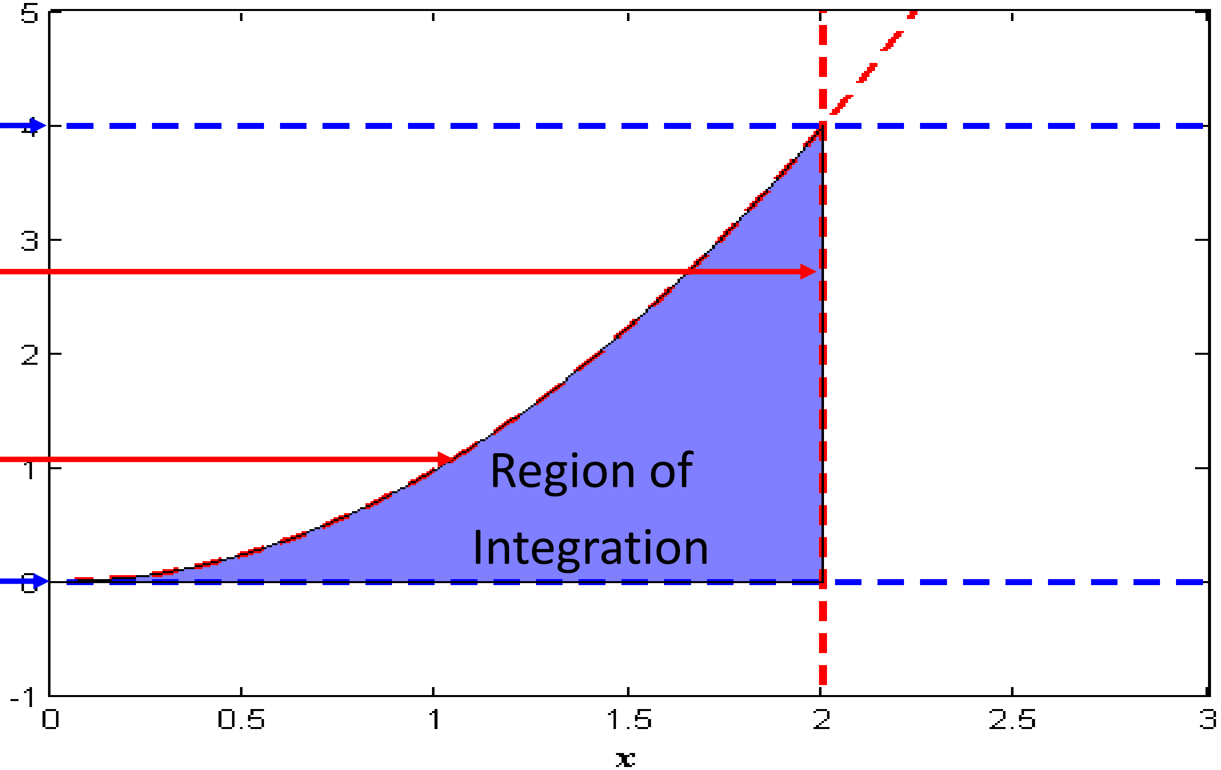
$$\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\, dx\, dy$$

- The inner integral here is difficult – perhaps impossible?
- We will try changing the order of integration.
- We must sketch the region of integration.

Example 2 – Non-rectangular integration regions

- Looking at the limits:

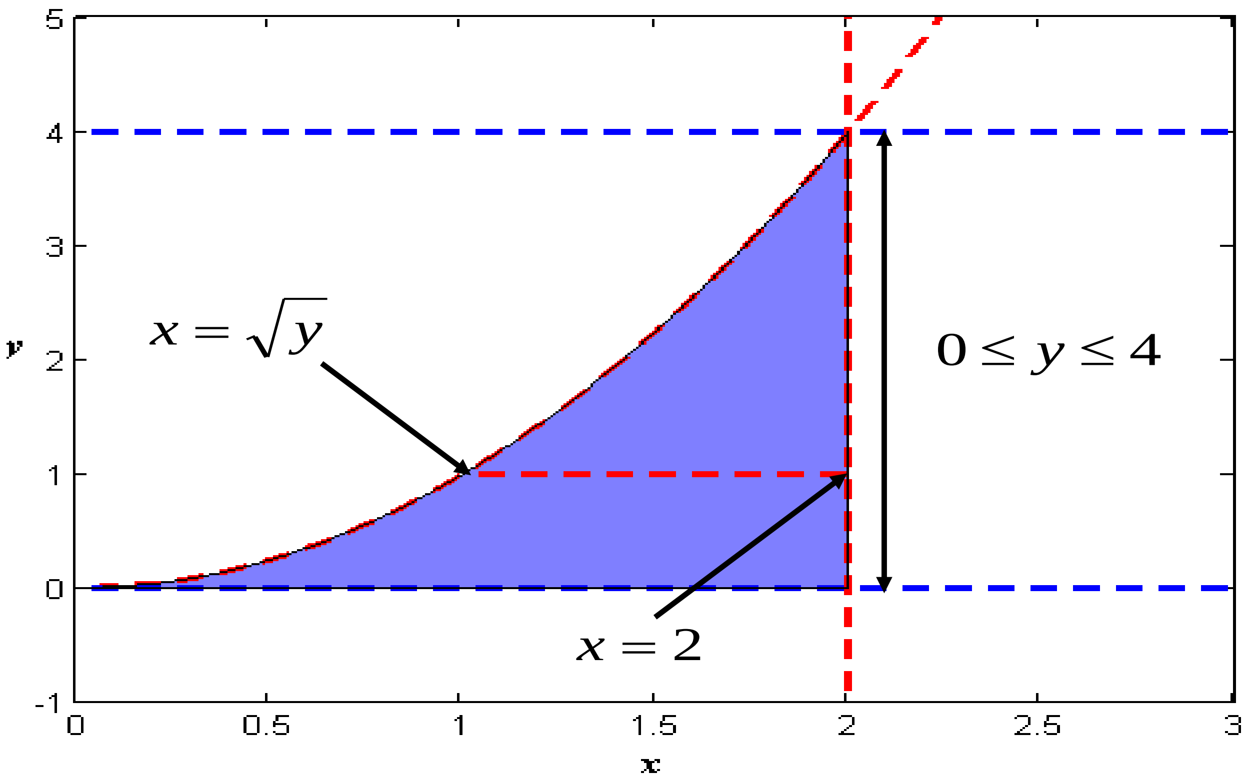
$$\int_{\boxed{y=0}}^{\boxed{y=4}} \int_{\boxed{x=\sqrt{y}}}^{\boxed{x=2}} e^{x^3}\, dx\, dy$$



Example 2 – Non-rectangular integration regions

- Draw a strip in the x direction (the inner integral):

$$\int_{y=0}^{y=4} \int_{x=\sqrt{y}}^{x=2} e^{x^3} \, dx \, dy$$

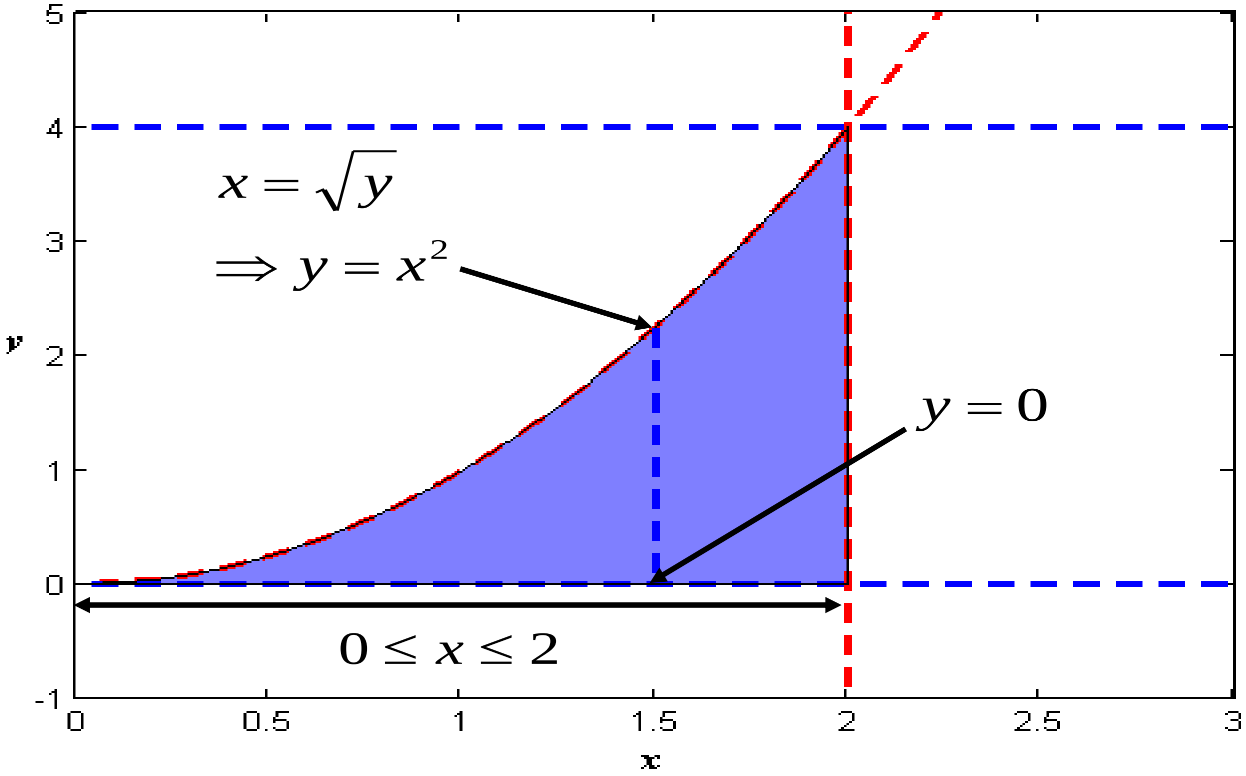


Example 2 – Non-rectangular integration regions

- To change the order of integration we now have to draw a strip in the y direction, and evaluate the new limits:

$$\int_{y=0}^{y=4} \int_{x=\sqrt{y}}^{x=2} e^{x^3}\, dx\, dy$$

$$= \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} e^{x^3}\, dy\, dx$$



Example 2 – Non-rectangular integration regions

- We have found the new limits for integral with the change order of integration.
- We can now find the integral:

$$\int_{y=0}^{y=4}\int_{x=\sqrt{y}}^{x=2} e^{x^3}\,dx\,dy = \int_{x=0}^{x=2}\int_{y=0}^{y=x^2} e^{x^3}\,dy\,dx$$

$$= \int_{x=0}^{x=2}\left[ye^{x^3}\right]_{y=0}^{y=x^2} dx = \int_{x=0}^{x=2} x^2 e^{x^3}\,dx$$

$$= \left[\frac{e^{x^3}}{3}\right]_{x=0}^{x=2} = \frac{e^8-1}{3}$$

Questions for Practice

Q1: Find the integral of $f(x,y) = xy^2$ for $0 \le x \le 1$ and $2 \le y \le 3$

Q2: Find $\int_0^1 \int_0^{x^2} y\,dy\,dx$. Also find this integral by changing the order of integration.

Q3: Find $\int_0^2 \int_y^2 e^{x^2}\,dx\,dy$

Questions for Practice

- :

We want to find the integral:

$$\int_0^1 \int_2^3 xy^2 \, dy dx$$

First evaluate the inner integral, holding x as a constant:

$$\int_0^1 \int_2^3 xy^2 \, dy dx = \int_0^1 \left[x \frac{y^3}{3} \right]_{y=2}^{y=3} dx$$

$$= \int_0^1 \left(9x - \frac{8}{3} x \right) dx$$

Questions for Practice

Now we will perform a normal 1-D integral:-

$$\int_0^1 \left(9x - \frac{8}{3}x \right) dx$$

$$= \left(\frac{9x^2}{2} - \frac{8x^2}{6} \right) \Bigg|_0^1 = \frac{9}{2} - \frac{8}{6} = \frac{27-8}{6} = \frac{19}{6}$$

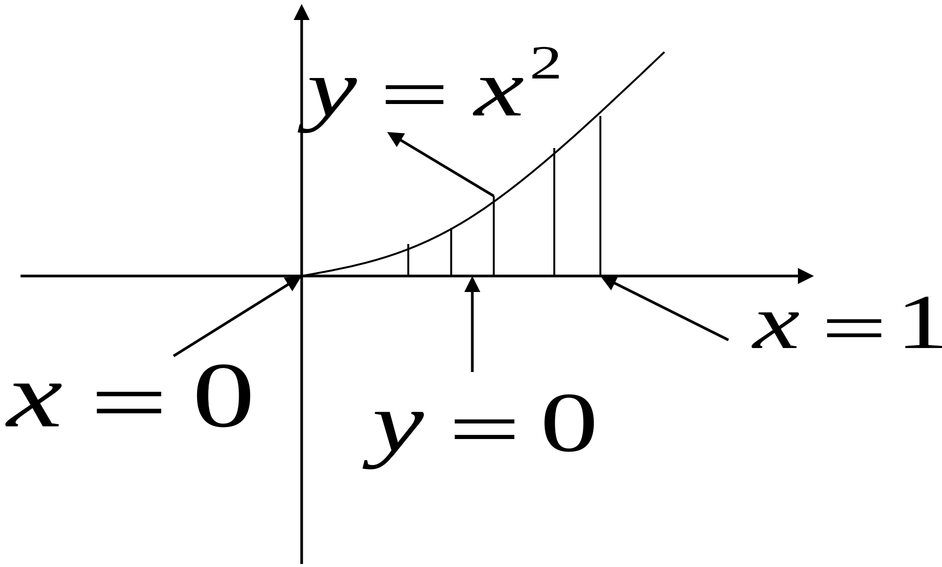
Questions for Practice

Ans2: We do the following:-

$$\int_0^1 \int_0^{x^2} y\,dy\,dx$$

$$= \int_0^1 \left(\frac{y^2}{2} \right)\Bigg|_0^{x^2} dx$$

$$= \int_0^1 \left(\frac{x^4}{2} \right) dx = \frac{x^5}{10}\Bigg|_0^1 = \frac{1}{10}$$



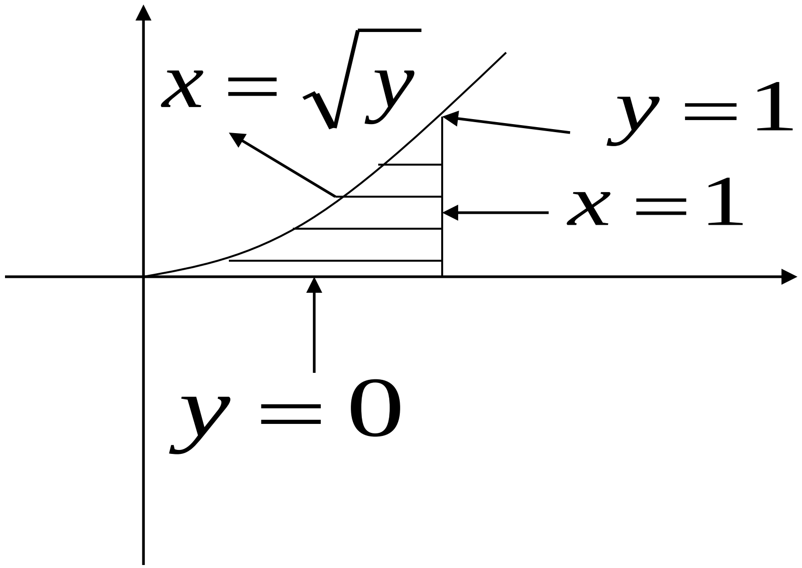
Questions for Practice

Now if we reverse the order of integration we get:-

$$\int_0^1 \int_{\sqrt{y}}^1 y\,dx\,dy$$

$$= \int_0^1 y\,x\Big|_{\sqrt{y}}^1 dy$$

$$= \int_0^1 y\left(1 - \sqrt{y}\right)dy$$



Questions for Practice

$$= \left(\frac{y^2}{2} \right) \Bigg|_0^1 - \left(\frac{2y^{\frac{5}{2}}}{5} \right) \Bigg|_0^1$$

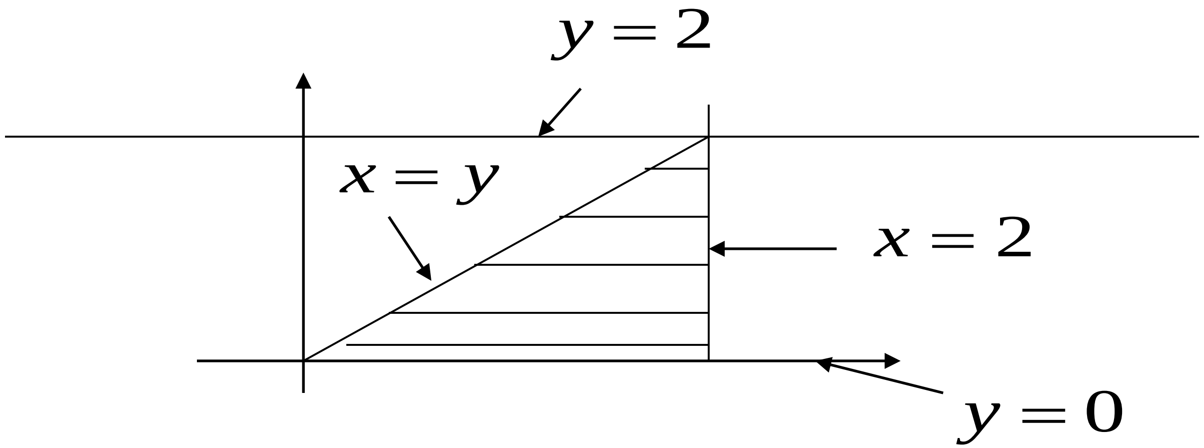
$$= \frac{1}{2} - \frac{2}{5} = \frac{1}{10}$$

Questions for Practice

Ans3:The following integral cannot be evaluated

$$\int_0^2 \int_y^2 e^{x^2}\, dx\, dy$$

The area of integration of this integral is given as:-



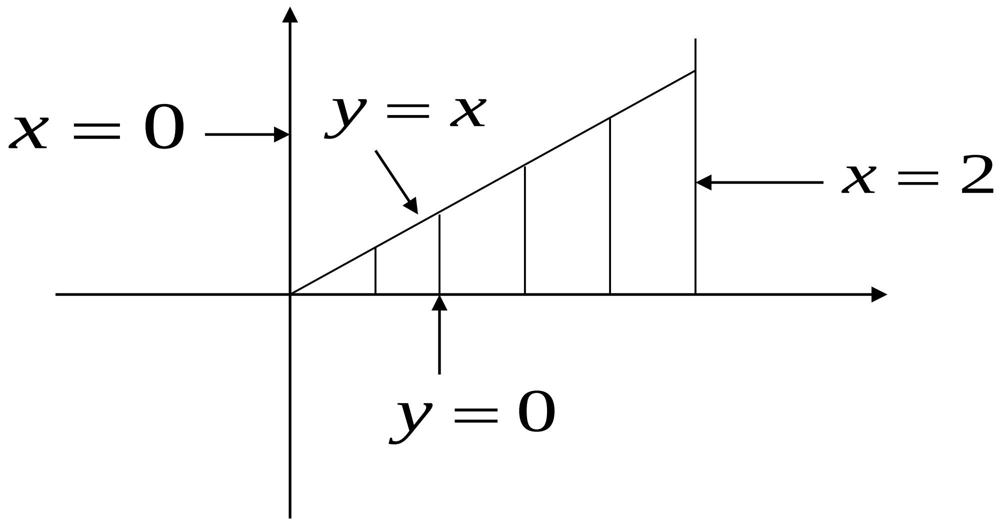
Questions for Practice

Therefore we reverse the area of integration to get:-

$$\int_0^2 \int_0^x e^{x^2}\, dy\, dx$$

$$= \int_0^2 e^{x^2}\, y\Big|_0^x \, dx$$

$$= \int_0^2 e^{x^2}\, x\, dx$$



Questions for Practice

Now we put $x^2 = t$ and evaluate the limits accordingly to get:-

$$\int_0^4 \frac{1}{2} e^t dt$$

$$= \frac{1}{2} e^t \Big|_0^4$$

$$= \frac{1}{2}\left(e^4 - 1\right)$$

Example 3 – Non-rectangular integration regions

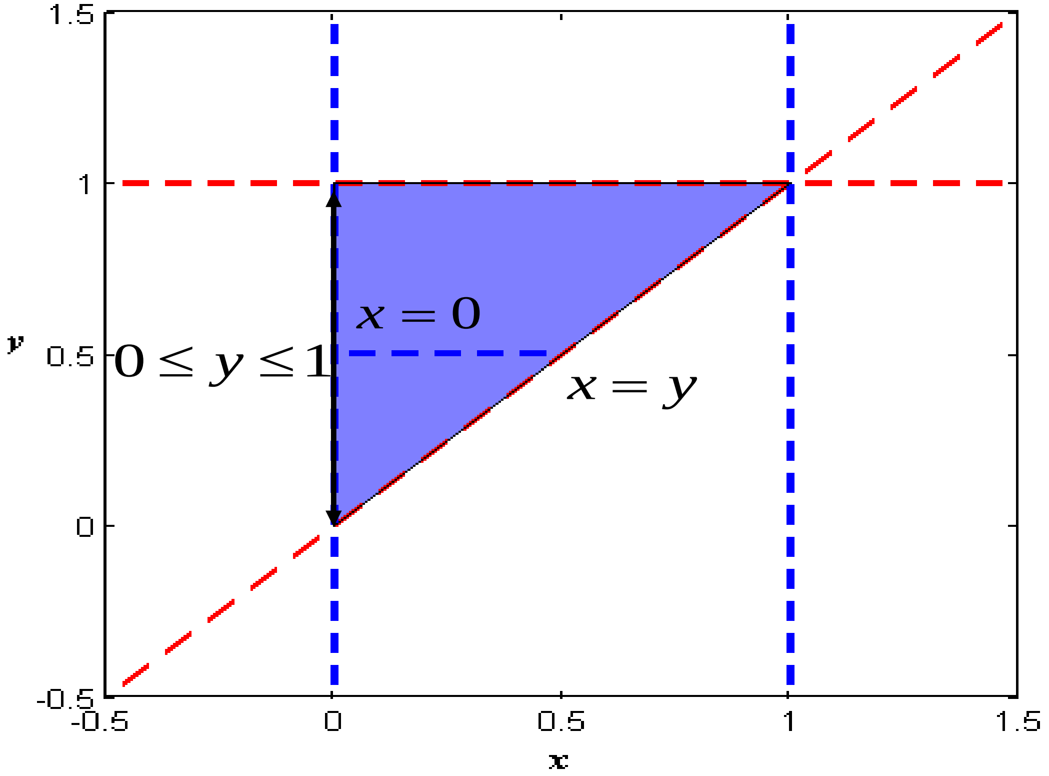
- Evaluate:

$$\int_0^1 \int_x^1 \sqrt{1-y^2}\,dydx$$

- Try changing the order of integration.
- We have to draw the region of integration.

Example 3 – Non-rectangular integration regions

- The region of integration is:
- Draw a strip in the direction of the new inner integral (*x* direction)



Example 3 – Non-rectangular integration regions

- Change order – use new limits:

$$\int_0^1 \int_x^1 \sqrt{1-y^2}\,dydx = \int_0^1 \int_0^y \sqrt{1-y^2}\,dxdy$$

$$= \int_0^1 \left[x\sqrt{1-y^2}\right]_0^y dy = \int_0^1 y\sqrt{1-y^2}\,dy$$

$$= \left[-\frac{1}{3}\left(1-y^2\right)^{\frac{3}{2}}\right]_0^1 = \frac{1}{3}$$

Change of variable in double integrals

- Sometimes it is easier to evaluate an integral using variables other than x and y.
- This could be to make the region of integration easier to describe.
- It may also make the integral easier to evaluate.

Recall: Change of variable in 1-D integrals

- If we describe our variable as a function of some parameter, i.e.:

$$x = x(u)$$

- Then we can express the integral:

$$\int_a^b f(x)dx = \int_\alpha^\beta f[x(u)]\frac{dx}{du}du$$

- where:

$$x(u = \alpha) = a$$

$$x(u = \beta) = b$$

Example – Change of variable in 1-D integrals

- Evaluate the integral:

$$\int_0^1 \sqrt{1-x^2}\,dx$$

- Put: $x = \cos(\theta)$

$$\Rightarrow \frac{dx}{d\theta} = -\sin(\theta)$$

- Find the new limits:

$$x = 0 \;\Rightarrow\; \cos\theta = 0 \;\Rightarrow\; \theta = \frac{\pi}{2}$$

$$x = 1 \;\Rightarrow\; \cos\theta = 1 \;\Rightarrow\; \theta = 0$$

Example – Change of variable in 1-D integrals

- So the integral becomes:

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_{\frac{\pi}{2}}^{0} \sqrt{1-\cos^2(\theta)}(-\sin(\theta))d\theta$$

$$= \int_{\frac{\pi}{2}}^{0} (\sin(\theta)\times -\sin(\theta))d\theta \quad = \int_0^{\frac{\pi}{2}} \sin^2(\theta)d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos(2\theta))d\theta = \frac{1}{2}\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

Change of variable in double integrals

- For the double integral:

$$\iint_R f(x,y)\,dx\,dy$$

- we use the substitution:

$$x = x(u,v),\, y = y(u,v)$$

- What do we use instead of $\dfrac{dx}{du}$

  which we used in the 1-D example?

Change of variable in double integrals

- We use a term called the *Jacobian*, which is defined as:

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

- The integral becomes:

$$\iint_R f(x,y)\,dx\,dy = \iint_R f\big(x(u,v), y(u,v)\big)\boxed{|J|}\,du\,dv$$

Make sure you remember that this is the absolute value of the Jacobian!

Example 1 – Change of variable in double integrals

- Find the integral:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} 1\,dy\,dx$$

- We use the substitution:

$$x = r\cos(\theta),\ y = r\sin(\theta)$$

- Finding the Jacobian:

$$J = \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} = \cos(\theta)\times r\cos(\theta) + r\sin(\theta)\times \sin(\theta)$$

$$= r\cos^2(\theta) + r\sin^2(\theta) \ = r$$

Example 1 – Change of variable in double integrals

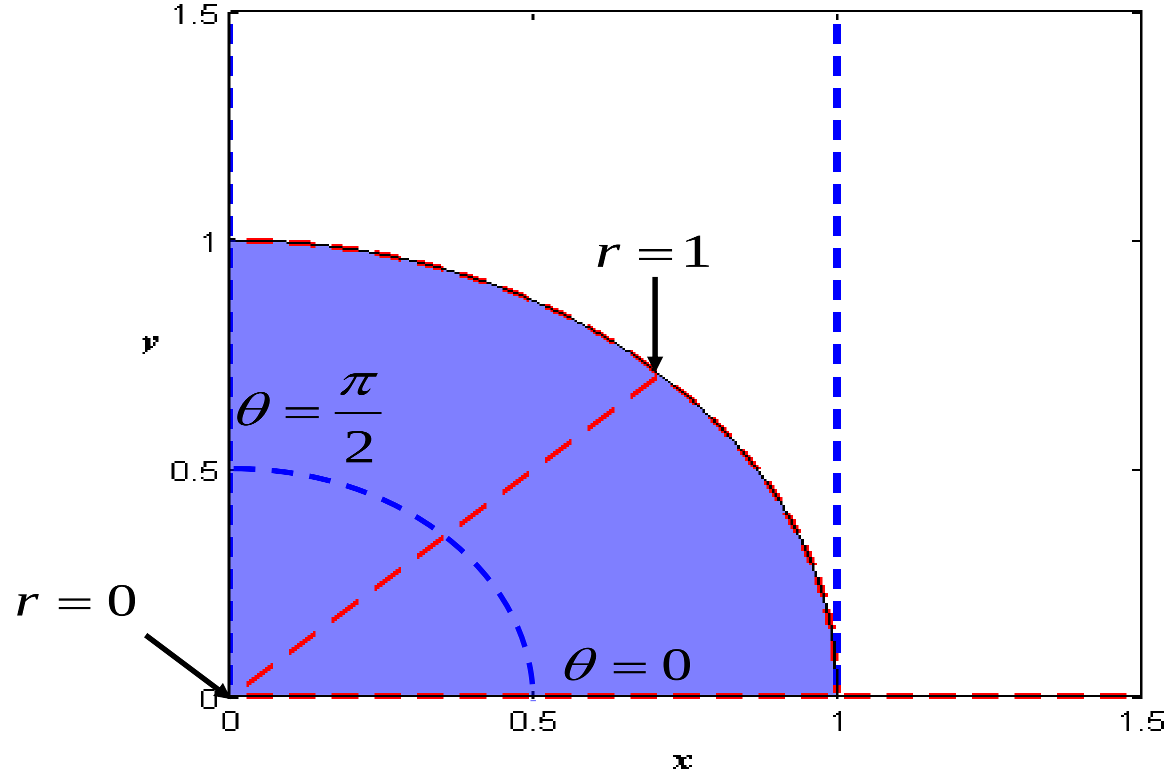
- This give:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} 1 dy dx = \int_?^? \int_?^? r dr d\theta$$

- We need to find the limits for the new variables.
- We have to draw the region of integration and determine how to form it with the new variables.

Example 1 – Change of variable in double integrals

- Sketch the region of integration.
- Draw a strip in the direction of the inner integral (*r* is the radius in polar coordinates).



Example 1 – Change of variable in double integrals

- So the integral becomes:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} 1\,dy\,dx = \int_0^{\frac{\pi}{2}} \int_0^1 r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_0^1 d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2} d\theta$$

$$= \left[\frac{1}{2}\theta\right]_0^{\frac{\pi}{2}} \qquad = \frac{\pi}{4}$$

Example 2 – Change of variable in double integrals

- Find the integral:

$$\int_0^1 \int_0^{\sqrt{1-y^2}} e^{-x^2} e^{-y^2} \, dy dx$$

- We use the substitution:

$$x = r\cos(\theta), y = r\sin(\theta)$$

- Finding the Jacobian:

$$J = \frac{\partial x}{\partial r}\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\frac{\partial y}{\partial r} = \cos(\theta) \times r\cos(\theta) + r\sin(\theta) \times \sin(\theta)$$

$$= r\cos^2(\theta) + r\sin^2(\theta) \; = r$$

Example 2 – Change of variable in double integrals

- This give
s:

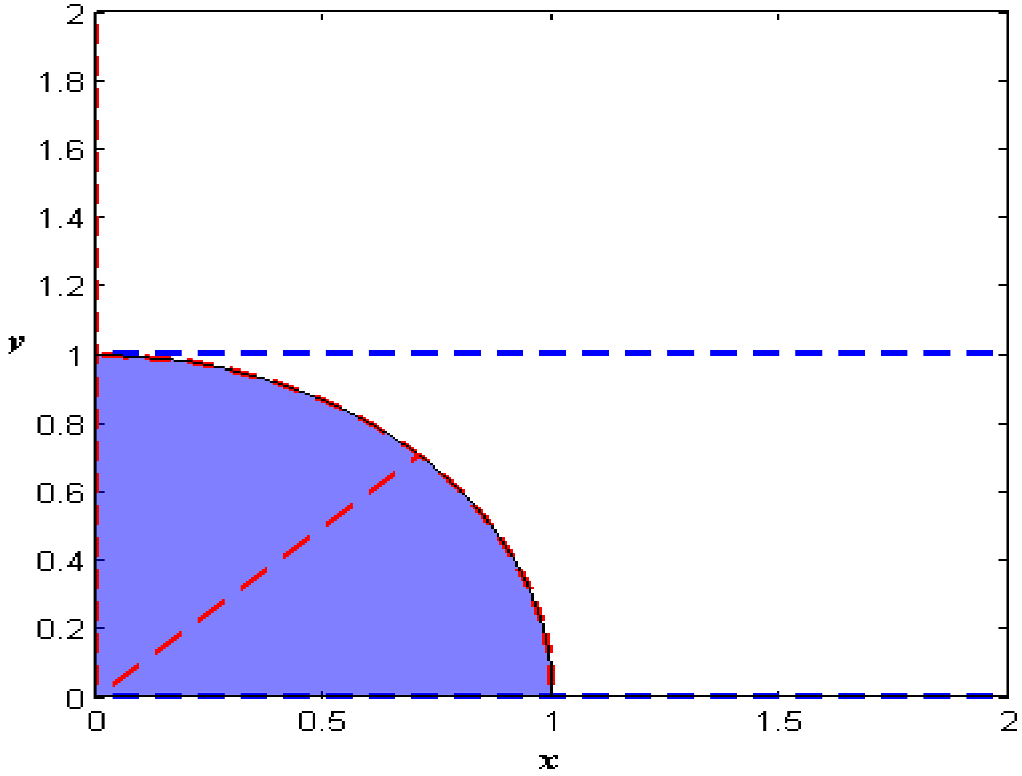
$$\int_0^1 \int_0^{\sqrt{1-y^2}} e^{-x^2} e^{-y^2}\, dydx = \int_?^? \int_?^? e^{-(r\sin(\theta))^2} e^{-(r\cos(\theta))^2}\, rdrd\theta$$

$$= \int_?^? \int_?^? e^{-r^2\left(\cos^2(\theta)+\sin^2(\theta)\right)} rdrd\theta \quad = \int_?^? \int_?^? re^{-r^2}\, drd\theta$$

- We now draw the region of integration to find the new limits.

Example 2 – Change of variable in double integrals

- The region is:
- This has the same limits as the last problem.



Example 2 – Change of variable in double integrals

- So the integral is:

$$\int_0^1 \int_0^{\sqrt{1-y^2}} e^{-x^2} e^{-y^2} \, dy dx = \int_0^{\frac{\pi}{2}} \int_0^1 r e^{-r^2} \, dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[-\frac{1}{2} e^{-r^2} \right]_0^1 d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2} e^{-1} \right) d\theta$$

$$= \left[\frac{1}{2}\theta - \frac{1}{2} e^{-1} \theta \right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}\left(1 - e^{-1}\right)$$

Change of variable in double integrals – Polar Coordinates

- Changing from Cartesian (x,y) coordinates to polar coordinates (r,θ) is one of the most common transformations.
- It is worth remembering the Jacobian and the transformation, i.e. for:

$$x = r\cos(\theta),\ y = r\sin(\theta)$$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = r$$

$$\iint_R f(x,y)\,dx\,dy = \iint_R f\left(r\cos(\theta), r\sin(\theta)\right) r\,dr\,d\theta$$

Change of variables and geometry

- It can be helpful to consider the change of variables geometrically
- A multivariable integral can be expressed without reference to a coordinate system:

$$V = \iint_R f \, dA$$

- This integral is interpreted as the volume under the surface f and above the region R

Change of variables and geometry

- The region is made up of elemental areas dA
- This means that the volume can be made up by summing lots of elemental volumes:

$$dV = f\,dA$$

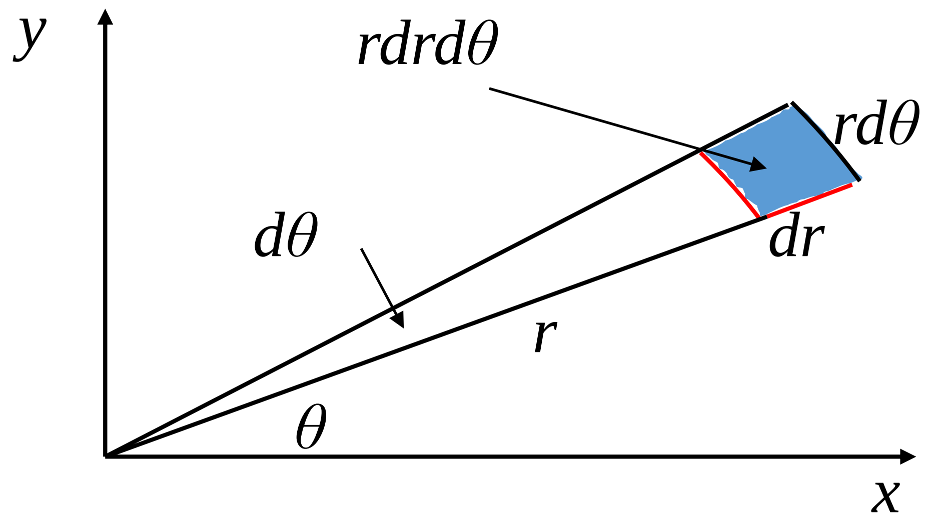
- In the xy-coordinate system we write the integral as:

$$V = \iint_R f\,dA = \iint_R f(x, y)dxdy$$

$$\Rightarrow dV = f\,dA = f(x, y)dxdy$$

Change of variables and geometry

- In some cases (e.g. Polar coordinates) it is possible to find dA using geometry.



- From the sketch:

$$dA = rdrd\theta$$

Change of variables and geometry

- So the geometric interpretation gives the integral:

$$V = \iint\limits_R f \, dA = \iint\limits_R f \, r dr d\theta$$

which is the same as the integral found by using the Jacobian.

Changing the variable in double integrals - Notes

- Find the Jacobian for your new variables.
- Substitute the new variables into the integral.
- DRAW the region of integration.
- Use your drawing to find the limits of your new variables.

Summary

- Revised 1-D integrals.
- How to evaluate multivariable integrals.
- Changing the order of integration in multivariable integrals.
- Changing variables in multivariable integrals.

Questions for Practice

Q1:Evaluate the following by converting into polar coordinates

$$I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dydx$$

Q2:A washer has inner radius r_1 and outer radius r_2. The thickness of the washer is given by

$$f(x,y) = ae^{-b\left(x^2+y^2\right)}$$

What is the average thickness of the washer?

Questions for Practice

Ans1:

$$I = \int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2}\, dydx$$

$$x = r\cos\theta, y = r\sin\theta$$

Put to get:-

$$I = \int_?^? \int_?^? \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\; |J| dr d\theta$$

Questions for Practice

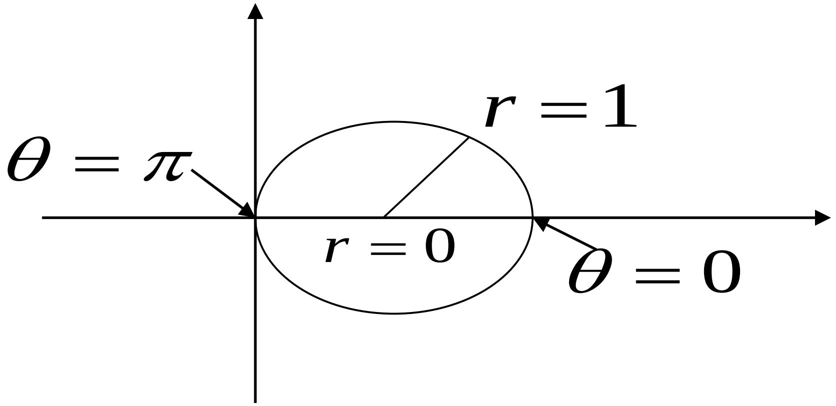
So $$I = \int_{?}^{?}\int_{?}^{?} \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\;|J|\,dr\,d\theta$$

For $$x = r\cos(\theta),\ y = r\sin(\theta)$$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = r$$

Questions for Practice

Now to calculate the limits we consider the following diagram:-



$$which \ is \ obtained \ using \ y^2 = 2x - x^2$$

$$\Rightarrow y^2 + x^2 - 2x + 1 - 1 = 0$$

$$\Rightarrow y^2 + (x-1)^2 = 1$$

$$which \ is \ equation \ of \ a \ circle \ with \ radius \ 1$$

$$and \ centre (1,0)$$

Questions for Practice

Therefore we have:-

$$I = \int_0^{\pi}\int_0^{1} \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\; r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_0^{1} \sqrt{r^2}\; r\,dr\,d\theta \quad as\ \cos^2\theta + \sin^2\theta = 1$$

$$= \int_0^{\pi}\int_0^{1} r^2\,dr\,d\theta$$

Questions for Practice

$$= \int_0^{\pi} \left. \frac{r^3}{3} \right|_0^1 d\theta$$

$$= \int_0^{\pi} \frac{1}{3} d\theta$$

$$= \frac{\pi}{3}$$

Questions for Practice

Ans2: The area of the washer (A) = $\pi(r_2{}^2 - r_1{}^2)$

Here the region of integration is a difference of area of two circles.

For $f(x, y) = ae^{-b\left(x^2+y^2\right)}$ the average thickness is given by:-

$$Averagethickness = \frac{1}{A}\iint_R f(x, y)dxdy$$

$$= \frac{1}{\pi(r_2^2 - r_1^2)}\int_0^{2\pi}\int_{r_1}^{r_2} ae^{-br^2} rdrd\theta$$

Questions for Practice

$$= -\frac{1}{\pi(r_2^{\,2} - r_1^{\,2})} \int_0^{2\pi} \left(\frac{e^{-br^2}}{b} \times \frac{a}{2} \right) \Bigg|_{r_1}^{r_2} d\theta$$

$$= -\frac{1}{\pi(r_2^{\,2} - r_1^{\,2})} \times \frac{a}{2b} \times \int_0^{2\pi} \left(e^{-b(r_2)^2} - e^{-b(r_1)^2} \right) d\theta$$

$$= -\frac{1}{\pi(r_2^{\,2} - r_1^{\,2})} \times \frac{a}{2b} \times 2\pi \times \left(e^{-b(r_2)^2} - e^{-b(r_1)^2} \right)$$

$$= \frac{1}{(r_2^{\,2} - r_1^{\,2})} \times \frac{a}{b} \times \left(e^{-b(r_1)^2} - e^{-b(r_2)^2} \right)$$