UNIVERSITY OF JAFFNA

FACULTY OF ENGINEERING

END SEMESTER EXAMINATION– July 2023-SEMESTER III

MC 3010 – DIFFERENTIAL EQUATIONS AND NUMERICAL METHODS

Date:24/07/2023 *Duration: TWO Hours*

Instructions

1. This paper contains **THREE (3)** parts in six pages:
2. If you have any doubt as to the interpretation of the wording of a question, make your own decision, but clearly state in the script.
3. Scientific calculators are permitted.
4. The formula sheet is provided along with this question paper.

Part 1[Multivariable Calculus][32 marks]: You are advised to spend 40 minutes. This part have FIVE questions and best of FOUR will be considered for your Final Mark. All questions carry equal marks.

1. To divide 120 into three parts such that the sum of their products taken two at a time is maximized, we can formulate the function as follows: $f = xy + yz + xz$. In this function, x, y, and z represent the three parts into which 120 is divided. The objective is to find values for x, y, and z that maximize the sum of their products taken two at a time (xy, yz, and xz).

   (a) Write the function (f) to be maximized involving two variables, x and y only.

   (b) Differentiate f and find out

   i. $\dfrac{\partial f}{\partial x}$ ii. $\dfrac{\partial f}{\partial y}$ iii. $\dfrac{\partial^2 f}{\partial x^2}$ iv. $\dfrac{\partial^2 f}{\partial y^2}$ v. $\dfrac{\partial^2 f}{\partial x \partial y}$

   (c) Put $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$ and solve these equations for x and y.

   (d) Evaluate $r = \dfrac{\partial^2 f}{\partial x^2}$, $s = \dfrac{\partial^2 f}{\partial x \partial y}$ and $t = \dfrac{\partial^2 f}{\partial y^2}$ for the values obtained from (c) above.

   (e) Evaluate $rt - s^2$ and discuss the nature of points obtained from (c) above.

2. For the first two seconds of a race, a marathon runner follows a path given by:: $\mathbf{r}(t) = \left(\frac{t^2}{2} - 4t, \frac{8}{3}t^{\frac{3}{2}}\right)$, where t is the time (measured in seconds) since she left the starting point and $r(t)$ is a position vector describing her position (in metres) relative to the starting point of the race.

   (a) Find the velocity of the runner.

   (b) Show that the speed of the runner is given by $t + 4$. Is her path parameterised by arc length?

   (c) Find the distance travelled over the first two seconds of the race.

   (d) Find the acceleration of the runner.

   (e) Find the normal acceleration of the runner one second into the race.(Hint: Substitute time first!)

3. The period T of a pendulum of length l is given by $T = 2\pi\sqrt{\frac{l}{g}}$. Find the maximum error in T due to possible errors up to 1% in l and 2% in g according to the following steps.

   (a) $100\frac{\delta l}{l}$

   (b) $100\frac{\delta g}{g}$

   (c) $\frac{\delta T}{T}$

   (d) $100\frac{\delta T}{T}$

   (e) % of error in T

4. Consider the following integral: $Q = \int_0^8 \int_{\sqrt[3]{y}}^2 \exp x^4 dx dy$.

   (a) Sketch the region of integration and indicate on your sketch strips corresponding to the direction of the inner integral.

   (b) Find the value of the integral by changing the order of integration.

5. Find the Jacobian $J\left(\frac{u, v}{x, y}\right)$ for $u = e^x \sin y$ and $v = x \log(\sin y)$ according to the following steps:

   (a) $\frac{\partial u}{\partial x}$

   (b) $\frac{\partial v}{\partial x}$

   (c) $\frac{\partial u}{\partial y}$

   (d) $\frac{\partial v}{\partial y}$

   (e) $J\left(\frac{u, v}{x, y}\right)$

Part 2[Differential Equations][32 marks:] You are advised to spend 40 minutes. This part have FIVE questions and best of FOUR will be considered for your Final Mark. All questions carry equal marks.

1. (a) What are the main techniques and methods used to solve differential equations, and how do these solutions help in understanding and predicting the dynamics of various systems?

   (b) The solution to a differential equation (where t is time) is given by:

   $$x = \frac{e^{-2t} + 3}{1 - 2e^{-t}}.$$

   What is the steady state solution?

2. A tank initially contains 10 litres of pure water. Two pipes lead into the tank: in the first, salt water at 1 gram per litre is pumped in at a rate of 1 litre per minute; in the second, pure water is pumped in at 2 litres per minute. The salt solution leads out of the tank through two other pipes, the first at 1 litre per minute, the second at 3 litres per minute.

   (a) Write down a differential equation for the amount of salt in the tank at any given time.

   (b) Find the amount of salt in the tank at any given time.

3. Solve the differential equation $x\frac{dy}{dx} - y = x$, subject to the initial condition $y(1) = 2$, according to the following steps:

   (a) Put the differential equation in the standard form $\frac{dy}{dx} + p(x)y = r(x)$ and so determine $p(x)$ and $r(x)$.

   (b) Determine $h(x)$ using the formula: $h(x) = \int p(x)dx$

   (c) Find y by using the linear differential equation formula: $y = e^{-h(x)} \int e^{h(x)} r(x) dx$

   (d) Apply the initial condition to find the constant of integration, and hence write down the solution for y.

4. Solve the differential equation $\left(D^2 - 3D + 2\right) y = e^x$ according to the following steps:

   (a) Find the complementary function

   (b) Find a particular integral

   (c) Find the solution.

5. The vertical motion of a particle is modelled by:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 9;\ y(0) = 0,\ \frac{dy}{dx}(0) = 0$$

   Use Laplace transforms to solve according to the following steps:

   (a) Taking the Laplace transform of both sides of the differential equation by applying the formulae for the Laplace transforms.

   (b) Put in the given initial conditions $y(0)$ and $\frac{dy}{dx}(0)$

   (c) Re-arrange the equation to make $\mathcal{L}(y)$ the subject.

   (d) Determine y by using, where necessary, partial fractions, and taking the inverse of each term.

Part 3[Numerical Methods][36 marks] You are advised to spend 40 minutes. This part have SIX questions and best of FOUR will be considered for your Final Mark. All questions carry equal marks.

1. (a) Complete the following computation and state what type of error is present in this situation.

$$\int_0^{0.5} e^{x^2} dx \simeq \int_0^{0.5} (1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!})dx$$

If true value $p = 0.544987104$, find the absolute error.

(b) Given the Taylor polynomial Expansions $tan^{-1}(h) = h - \frac{h^3}{3} + \frac{h^5}{5} + O(h^7)$ and $ln(h+1) = h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} + O(h^5)$ determine the order of approximation for their product.

2. (a) Let $f(x) = x + \frac{2}{x}$. Use quadratic Lagrange interpolation based on the nodes $x_0 = 1, x_1 = 2, x_2 = 2.5$ to approximate $f(1.5)$

(b) Apply Newton-Raphson method to find an approximate solution of the equation $e^x - 3x = 0$ correct up-to three decimal figures (assume $x_0 = 0.4$)

3. In a vibration experiment, the position (x) of a block of mass is given as a function of time(t). The recorded data for the first 2 seconds are given in the table below.

t(s)	0	0.5	1	1.5	2
x(mm)	200	123	27	-56	-100

The velocity of the block is the derivative of the position w.r.t time. Use Forward divided difference, backward divided difference or central divided difference approximation method to find the velocity at time $t = 1.5s$

4. Find the number $2m$(number of sub intervals) and the step size h so that the error $E_S = -\frac{b-a}{180}h^4 f^{(4)}(\hat{t})$ for the Simpson rule is less than 5×10^{-9} for the approximation $\int_2^7 \frac{dx}{x}$, where $\hat{t}$ is suitable value between a and b.
The maximum value of taken over [2,7] occurs at the end point $x = 2$.

5. Show that two integrals are equivalent and calculate $G_2(f)$ (Two-point Gauss-Legendre)

$$\frac{1}{\pi}\int_0^{\pi} \cos(0.6\sin(t))dt = 0.5\int_{-1}^{1} \cos(0.6\sin((x+1)\frac{\pi}{2}))dx$$

6. Use Runge-Kutta method of order $N = 4$ to solve the initial value problem $y' = \frac{t-y}{2}$ on $[0,3]$ with $y(0) = 1$.

Formula sheet

1. Quadratic Lagrange Polynomial $P_2(x) = l_0 f_0 + l_1 f_1 + l_2 f_2$ where, $l_0 = \dfrac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$, $l_1 = \dfrac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$, $l_2 = \dfrac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$

2. Forward Difference $f'(x_0) \approx \frac{f(x_0+h)-f(x_0)}{h}$

3. Backward Difference $f'(x_0) \approx \frac{f(x_0)-f(x_0-h)}{h}$

4. Central Difference $f'(x_0) \approx \frac{f(x_0+h)-f(x_0-h)}{2h}$

5. Gauss Quadrature Formula-2 Point $\int_{-1}^{1} f(x)\,dx \approx f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$

   To approximate the integral: $\displaystyle\int_a^b f(x)dx \approx \frac{b-a}{2}\sum_{k=1}^{N} w_{N,k} f(t_{N,k})$ by sampling $f(x)$ at the N unequally spaced points $\{t_{N,k}\}_{k=t}^{N}$, the changes of variable $t = \dfrac{a+b}{2} + \dfrac{b-a}{2}x$ and $dt = \dfrac{b-a}{2}dx$ are used.

6. Newton-Raphson Method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

7. Laplace Transforms

   (a) $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$
   (b) $\mathcal{L}\{1\} = \frac{1}{s}$
   (c) $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$
   (d) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$
   (e) $\mathcal{L}\{t\} = \frac{1}{s^2}$
   (f) $\mathcal{L}\{F'(t)\} = sF(s) - f(0)$
   (g) $\mathcal{L}\{F''(t)\} = s^2F(s) - sf(0) - f'(0)$

8. Runge-Kutta method of order $N = 4$: $y_{k+1} = y_k + \dfrac{h(f_1 + 2f_2 + 2f_3 + f_4)}{6}$, where $f_1 = f(t_k, y_k)$, $f_2 = f\left(t_k + \dfrac{h}{2}, y_k + \dfrac{h}{2}f_1\right)$, $f_3 = f\left(t_k + \dfrac{h}{2}, y_k + \dfrac{h}{2}f_2\right)$, $f_4 = f(t_k + h, y_k + hf_3)$.

—— *End of Examination* ——