Curves and Integrals

May 2023

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Introduction

- Curves in two and three dimensions
- Position vectors
- Velocity and tangent vectors
- Arc length
- Acceleration
- Line integrals

Curves in Two and Three Dimensions

- We can picture a curve in space as the “path” or “trail” of a moving particle.
- This means that we are interested in the position of the particle as time changes.
- We use time as a parameter to determine the coordinates of the particle.

Curves in Two and Three Dimensions

- The position of the particle is given by the vector:

$$\mathbf{r}(t) = \big(x(t), y(t)\big) \quad \mathbf{r}(t) = \big(x(t), y(t), z(t)\big)$$

or, alternatively

$$\mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \qquad \mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

- The position vector can also be written as:

$$2-\text{dimensions} \qquad \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

$$3-\text{dimensions} \qquad \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

Example 1 – Straight Line

- The position of a particle moving along a straight line is given by:

$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} \qquad t \in [0,4]$$

- We can trace the curve by substituting values for t :

$$\mathbf{r}(t=0) = 0\mathbf{i} + 0\mathbf{j} \Rightarrow x = 0, y = 0$$

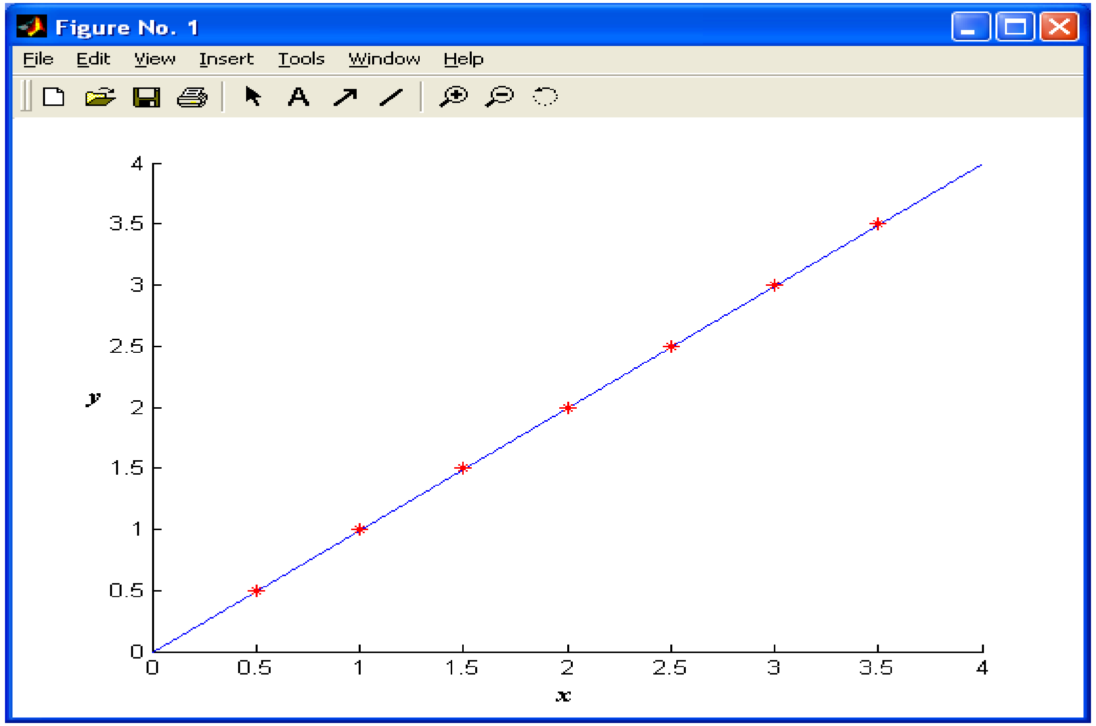
$$\mathbf{r}(t=2) = 2\mathbf{i} + 2\mathbf{j} \Rightarrow x = 2, y = 2$$

$$\mathbf{r}(t=4) = 4\mathbf{i} + 4\mathbf{j} \Rightarrow x = 4, y = 4$$

Example 1 – Straight Line

- Plotting these points gives the straight line

$$x = y$$



Example 2

- Find a parametric representation of the 2D curve given by

$$(x+2)^2+(y-2)^2=4$$

- This is a circle with radius 2 centered at

$$x=-2, y=+2$$

Example 2

- For a circle

$$X^2 + Y^2 = R^2$$

we have
$$X = R\cos t$$
$$Y = R\sin t$$

- We can satisfy the equation of the curve by letting

$$(x+2) = 2\cos t \Rightarrow x = -2 + 2\cos t$$
$$(y-2) = 2\sin t \Rightarrow y = 2 + 2\sin t$$

Example 2

We are given:-

$$(x+2)^2+(y-2)^2=4$$

By making the substitution on the previous slide in above equation
we get:-

$$(2\cos t)^2+(2\sin t)^2=4\cos^2 t+4\sin^2 t$$

$$=4(\cos^2 t+\sin^2 t)=4=\text{Right hand side}$$

More Examples …

- See Section 9.2 in Coursebook

Velocity

- We can find the *velocity* of a moving particle by differentiating the position vector with respect to time.

- Given the position vector:

$$\mathbf{r}(t) \Rightarrow \mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$$

Velocity

- We can see that this is:

$$\text{2-dimensions} \quad \mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \Rightarrow \mathbf{v}(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$$

$$\text{3-dimensions} \quad \mathbf{r}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} \Rightarrow \mathbf{v}(t) = \begin{bmatrix} x'(t) \\ y'(t) \\ z'(t) \end{bmatrix}$$

Example

- Find the velocity of a particle moving with position:

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$$

  - Differentiate each component of r(t) with respect to time

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \frac{d(\cos(t))}{dt}\mathbf{i} + \frac{d(\sin(t))}{dt}\mathbf{j} = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j}$$

Further example ...

- See Section 9.3 in Coursebook

Speed

- The *speed* of the particle is the length of it's velocity vector:

$$|\mathbf{v}(t)|$$

$$\Rightarrow 2\text{-dimensions} \quad |\mathbf{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\Rightarrow 3\text{-dimensions} \quad |\mathbf{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Speed

- However, the speed is also the rate of change of the distance travelled along the particle path
- Let the distance travelled by the particle along its path, also referred to as the *arc length*, be denoted by *s*
- Hence

$$\frac{ds}{dt} = |\mathbf{v}(t)| = \sqrt{\mathbf{v}(t) \bullet \mathbf{v}(t)}$$

Arc Length

- Therefore, once the speed has been found the distance travelled at any given time, or arc length, can be found

- The curve’s length L can then be found by integrating between the start time (*t=a*) and end time (*t=b*)

$$s = \int \frac{ds}{dt}\, dt = \int \left|\mathbf{v}(t)\right| dt$$

$$L = \int_{t=a}^{t=b} \left|\mathbf{v}(t)\right| dt$$

Example – Arc Length

- Find the length of the semicircle defined by:

$$\mathbf{r}(t) = \left(2\cos(t), 2\sin(t), 0\right) \quad t \in [0, \pi]$$

- First we find the velocity vector:

$$\mathbf{v}(t) = \mathbf{r}'(t)$$

$$\Rightarrow \mathbf{v}(t) = \left(-2\sin(t), 2\cos(t), 0\right)$$

Example – Arc Length

- This gives the speed:

$$|\mathbf{v}(t)| = \sqrt{(-2\sin(t))^2 + (2\cos(t))^2 + 0^2}$$

$$= \sqrt{4\sin^2(t) + 4\cos^2(t)}$$

$$= \sqrt{4} = 2$$

Example – Arc Length

- Put the speed into the arc length formula:

$$L = \int_{t=a}^{t=b} |\mathbf{v}(t)| dt$$

$$\Rightarrow L = \int_{0}^{\pi} 2dt$$

$$\Rightarrow L = [2t]_{0}^{\pi} = 2\pi$$

Arc Length as a Parameter

- Time is typically the parameter we use when tracing a curve of the movement of a particle.

- However, the arc length (s) is a very convenient parameter to use in some situations.

Questions for Practice

Q1:Find the velocity of a particle moving with position:

$$r(t) = (\cos t, \sin t, 1)$$

Q2:Find the *speed* of a particle moving with velocity: as a function of time

$$v(t) = (-\sin t, \cos t, 0)$$

Q3:Find the length of the circular helix defined by:

$$\mathbf{r}(t) = (\cos t, \sin t, t) \quad t \in [0, 2\pi]$$

Questions

Q1:Find the velocity of a particle moving with position:

$$r(t) = (\cos t, \sin t, 1)$$

Ans1:

$$v(t) = r'(t) = (-\sin t, \cos t, 0)$$

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Q2:Find the *speed* of a particle moving with velocity: as a function of time.

$$v(t) = (-\sin t, \cos t, 0)$$

Ans2:The speed is the magnitude or length of the velocity and is given by:

$$|\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 0^2} = \sqrt{\sin^2 t + \cos^2 t + 0} = \sqrt{1} = 1$$

Questions

Q3:Find the length of the circular helix defined by:

$$\mathbf{r}(t) = (\cos t, \sin t, t) \quad t \in [0, 2\pi]$$

Ans: Find the velocity vector:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}$$

Now find the speed:

$$\sqrt{\mathbf{v}(t) \bullet \mathbf{v}(t)} = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$$

Questions

Put the speed into the length formula:

$$L = \int_a^b \sqrt{\mathbf{v}(t) \bullet \mathbf{v}(t)}\,dt$$

$$\Rightarrow L = \int_0^{2\pi} \sqrt{2}\,dt = \left[\sqrt{2} \times t\right]_0^{2\pi} = 2\sqrt{2}\pi$$

Tangent Vectors

- If we are using the particle's path to trace a curve then the velocity vector is a tangent to the curve.
- We can calculate a *unit tangent* vector by dividing the velocity vector by its length:

$$\tau(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

Example – Tangent Vector

- Find the *unit* tangent vector to the curve:

$$\mathbf{r}(t) = \left(\cos^2(t), -2\sin(2t), t^2\right), \quad t \in [0, \pi]$$

at the point:

$$P = \left(0, 0, \tfrac{1}{4}\pi^2\right)$$

Example – Tangent Vector

- First differentiate the position vector with respect to time to find the velocity vector:

$$\mathbf{r}(t) = \left(\boxed{\cos^2(t)}, \boxed{-2\sin(2t)}, \boxed{t^2}\right)$$

$$\frac{d}{dt}\left(\cos^2(t)\right) = -2\cos(t)\sin(t)$$

$$\frac{d}{dt}\left(-2\sin(2t)\right) = -4\cos(2t)$$

$$\frac{d}{dt}\left(t^2\right) = 2t$$

$$\Rightarrow \mathbf{v}(t) = \left(-2\cos(t)\sin(t), -4\cos(2t), 2t\right)$$

Example – Tangent Vector

- Given the position vector and the point:

$$\mathbf{r}(t) = \left(\cos^2(t), -2\sin(2t), \boxed{t^2}\right)$$

$$P = \left(0, 0, \boxed{\tfrac{1}{4}\pi^2}\right)$$

$$\Rightarrow \boxed{t^2} = \boxed{\tfrac{1}{4}\pi^2}$$

$$\Rightarrow t = \tfrac{\pi}{2}$$

$$\left(check: \quad \cos^2(\tfrac{\pi}{2}) = 0 \quad and \quad -2\sin(2\tfrac{\pi}{2}) = 0\right)$$

Example – Tangent Vector

- Substitute in t to find the velocity vector at P:

$$\mathbf{v}(t) = \left(-2\cos(t)\sin(t), -4\cos(2t), 2t\right)$$

$$\Rightarrow \mathbf{v}(t = \tfrac{\pi}{2}) = \left(-2\cos(\tfrac{\pi}{2})\sin(\tfrac{\pi}{2}), -4\cos(\pi), \pi\right)$$

$$\Rightarrow \mathbf{v}(t = \tfrac{\pi}{2}) = \left(0, 4, \pi\right)$$

Example – Tangent Vector

- Now we divide the velocity vector by its length to make it a unit vector:

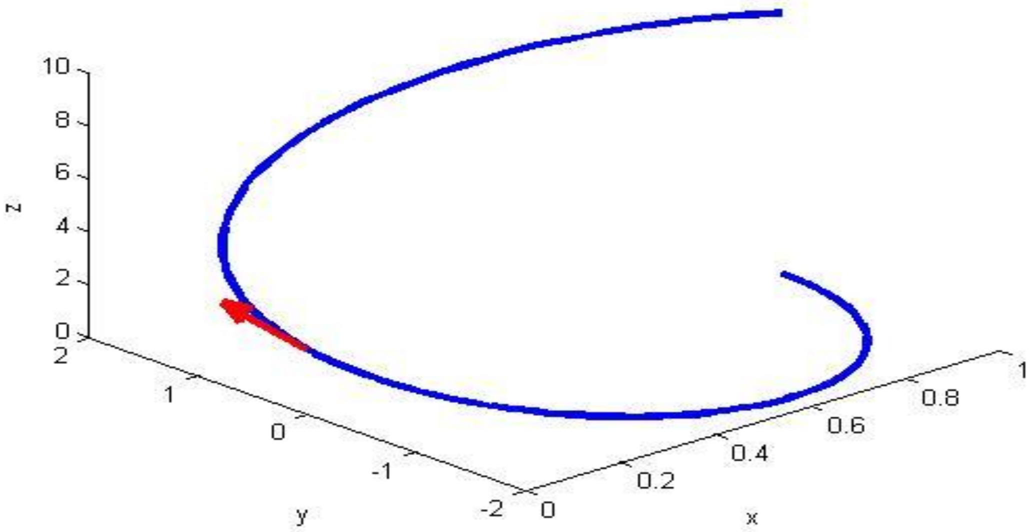
$$\mathbf{v}(t=\tfrac{\pi}{2})=(0,4,\pi)$$

$$\Rightarrow \left|\mathbf{v}(t=\tfrac{\pi}{2})\right|=\sqrt{0^2+4^2+\pi^2}$$

$$\Rightarrow \left|\mathbf{v}(t=\tfrac{\pi}{2})\right|=\sqrt{16+\pi^2}$$

$$\Rightarrow \hat{\mathbf{v}}=\frac{1}{\sqrt{16+\pi^2}}(0,4,\pi)$$

Example – Tangent Vector



Arc Length as a Parameter

- Suppose that the position vector is parameterized using arc length s

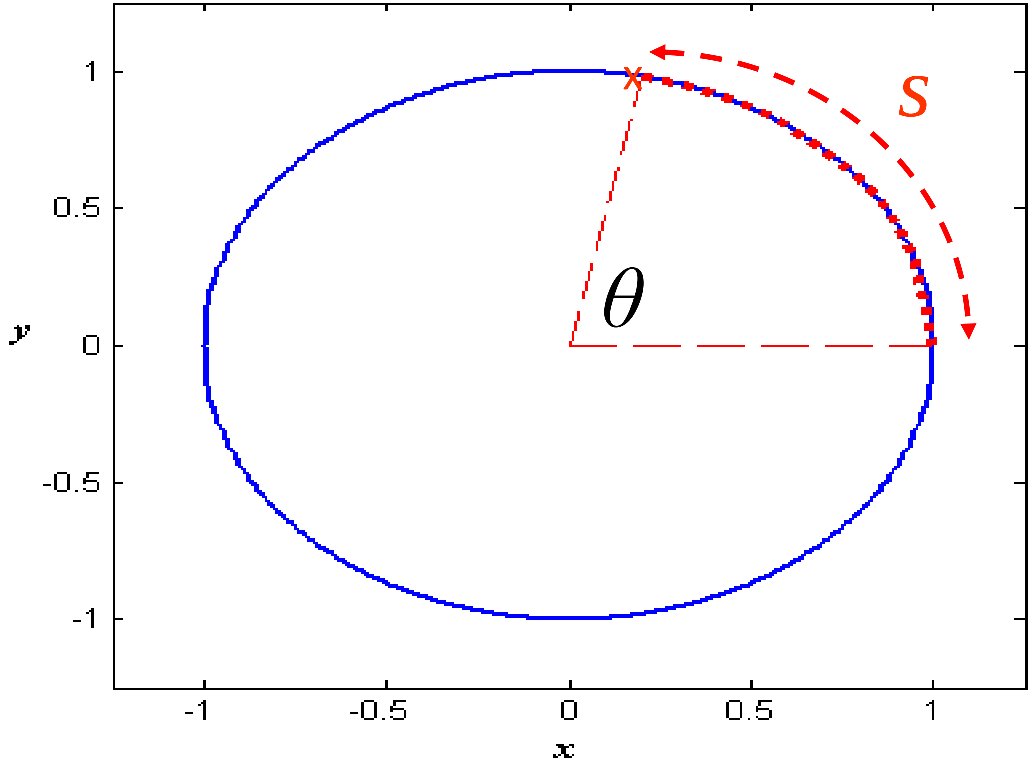
$$\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k}$$

- The derivative automatically gives a unit tangent vector:

$$\left|\boldsymbol{\tau}(s)\right| = \left|\frac{d\mathbf{r}}{ds}\right| = 1$$

Example – Arc Length Parameter

- Consider a point on the unit circle:



Example – Arc Length Parameter

- We know that the arc length is:

$$s = \theta, \ \ 0 \leq \theta \leq 2\pi$$

- So we have the position vector:

$$\Rightarrow \mathbf{r}(s) = \cos(s)\mathbf{i} + \sin(s)\mathbf{j}$$

$$\Rightarrow \boldsymbol{\tau}(s) = \frac{d\mathbf{r}(s)}{ds}$$

$$= -\sin(s)\mathbf{i} + \cos(s)\mathbf{j}$$

Acceleration

- We can use the position vector to find the acceleration of a moving particle.

- By differentiating the velocity vector we obtain the acceleration vector:

$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \frac{d^2\mathbf{r}(t)}{dt^2}$$

Example – Acceleration

- Find the acceleration at time t of a particle moving in the trajectory:

$$\mathbf{r}(t) = \left(2\cos(t), 2\sin(t), t\right)$$

$$\Rightarrow \mathbf{v}(t) = \mathbf{r}'(t) = \left(-2\sin(t), 2\cos(t), 1\right)$$

$$\Rightarrow \mathbf{a}(t) = \mathbf{r}''(t) = \left(-2\cos(t), -2\sin(t), 0\right)$$

Components of Acceleration

- The acceleration of an object is made up of two components.
- The *tangential acceleration* is the acceleration in the direction of travel.
- The *normal acceleration* is perpendicular to the direction of travel.

Components of Acceleration

- Given the unit tangent vector:

$$\boldsymbol{\tau}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

we can find the magnitude of the tangential component by taking the dot product of the acceleration with unit vector $\boldsymbol{\tau}$:

$$|\mathbf{a}_{\tan}(t)| = \mathbf{a}(t) \bullet \boldsymbol{\tau}(t) = \mathbf{a}(t) \bullet \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

Components of Acceleration

- We can find the tangential acceleration vector by multiplying this magnitude by the unit tangent vector:

$$\mathbf{a}_{\tan}(t) = \left(\mathbf{a}(t) \bullet \boldsymbol{\tau}(t)\right)\boldsymbol{\tau}(t)$$

$$\mathbf{a}_{\tan}(t) = \left(\mathbf{a}(t) \bullet \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}\right)\frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

Components of Acceleration

- We can then use the tangential acceleration vector to find the *normal acceleration* vector:

$$\mathbf{a}(t) = \mathbf{a}_{\tan}(t) + \mathbf{a}_{\text{norm}}(t)$$

$$\Rightarrow \mathbf{a}_{\text{norm}}(t) = \mathbf{a}(t) - \mathbf{a}_{\tan}(t)$$

Example – Circular Helix

- Find both components of acceleration of a particle travelling along the curve:

$$\mathbf{r}(t) = \left(2\cos(t), 2\sin(t), t\right)$$

$$\Rightarrow \mathbf{v}(t) = \mathbf{r}'(t) = \left(-2\sin(t), 2\cos(t), 1\right)$$

$$\Rightarrow \mathbf{a}(t) = \mathbf{r}''(t) = \left(-2\cos(t), -2\sin(t), 0\right)$$

Example – Circular Helix

- Calculate the unit velocity vector (*i.e.* unit tangent vector):

$$\boldsymbol{\tau}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

$$\Rightarrow \boldsymbol{\tau}(t) = \frac{\left(-2\sin(t), 2\cos(t), 1\right)}{\sqrt{\left(-2\sin(t)\right)^2 + \left(2\cos(t)\right)^2 + 1^2}}$$

$$\Rightarrow \boldsymbol{\tau}(t) = \frac{1}{\sqrt{5}}\left(-2\sin(t), 2\cos(t), 1\right)$$

Example – Circular Helix

- Now we can find the tangential acceleration:

$$\mathbf{a}_{\tan}(t) = \left(\mathbf{a}(t) \bullet \boldsymbol{\tau}(t)\right)\boldsymbol{\tau}(t)$$

$$\Rightarrow \mathbf{a}_{\tan}(t) = \left(\left(-2\cos(t), -2\sin(t), 0\right) \bullet \frac{1}{\sqrt{5}}\left(-2\sin(t), 2\cos(t), 1\right)\right)\boldsymbol{\tau}(t)$$

$$\Rightarrow \mathbf{a}_{\tan}(t) = \frac{1}{\sqrt{5}}\left(4\cos(t)\sin(t) - 4\sin(t)\cos(t) + 0\right)\boldsymbol{\tau}(t)$$

$$\Rightarrow \mathbf{a}_{\tan}(t) = (0, 0, 0)$$

i.e. no tangential acceleration

Example – Circular Helix

- Which gives the normal acceleration:

$$\mathbf{a}_{\text{norm}}(t) = \mathbf{a}(t) - \mathbf{a}_{\tan}(t)$$

$$\Rightarrow \mathbf{a}_{\text{norm}}(t) = \left(-2\cos(t), -2\sin(t), 0\right) - \left(0, 0, 0\right)$$

$$\Rightarrow \mathbf{a}_{\text{norm}}(t) = \left(-2\cos(t), -2\sin(t), 0\right)$$

Line Integrals

- Consider a particle subject to a force field, $\mathbf{F}(x, y, z)$.

- The force causes the particle to accelerate along the trajectory:

$$\mathbf{r}(t), \quad t \in [a, b]$$

Line Integrals

- The total work done on the particle over this interval is:

$$W = \int_{t=a}^{t=b} \mathbf{F}(x(t), y(t), z(t)) \bullet \mathrm{d}\mathbf{r}(t)$$

$$\Rightarrow W = \int_{a}^{b} \mathbf{F}(t) \bullet \frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t}\,\mathrm{d}t$$

$$\Rightarrow W = \int_{a}^{b} \mathbf{F}(t) \bullet \mathbf{v}(t)\,\mathrm{d}t$$

Example – Line Integral

- Find the work done by the force:

$$\mathbf{F}(t) = \left(y(t)^2, -x(t)^2, 0\right)$$

on the particle moving along the straight line segment from $\left(0,0,0\right)$

to $\left(1,2,0\right)$

Example – Line Integral

- We define the path using:

$$\mathbf{r}(t) = (t, 2t, 0), \quad t \in [0,1]$$

- So the velocity vector is:

$$\mathbf{v}(t) = \mathbf{r}'(t) = (1, 2, 0)$$

Example – Line Integral

- So we can now use the work formula:

$$W = \int_a^b \mathbf{F}(t) \bullet \mathbf{v}(t)\,\mathrm{d}t$$

$$\Rightarrow W = \int_0^1 \left(y(t)^2, -x(t)^2, 0\right) \bullet (1, 2, 0)\,\mathrm{d}t$$

$$\Rightarrow W = \int_0^1 \left((2t)^2, -(t)^2, 0\right) \bullet (1, 2, 0)\,\mathrm{d}t$$

Example – Line Integral

- So we can now use the work formula:

$$\Rightarrow W = \int_0^1 \left(4t^2, -t^2, 0\right) \bullet \left(1, 2, 0\right) \mathrm{d}t$$

$$\Rightarrow W = \int_0^1 \left(4t^2 - 2t^2\right) \mathrm{d}t$$

$$\Rightarrow W = \int_0^1 2t^2 \,\mathrm{d}t = \left[\frac{2}{3}t^3\right]_0^1 = \frac{2}{3}$$

Additional Examples ...

- See section 9.7 in Coursebook
- See Module 9 Exercises for different examples of Line Integrals

Questions for Practice

Q1:Find the unit tangent vector for

a curve given by:

$$a)r(t) = 3\cos ti + 4\sin tj + 4k$$

$$b)r(s) = \sin(2s)i + \cos(3s)j$$

Q2:Using the answer for part (a) of question 1 find the total, tangential and normal components of acceleration.

Q3:Find the work done by a force

along a path given by

$$x(t)i + y(t)j$$

$$r(t) = (3t, t, 0)\ ,\ \ t \in (0,1)$$

Questions for Practice

Ans1:a)

$$v(t) = r'(t) = -3\sin ti + 4\cos tj$$

$$Unit \quad \tan gent \quad vector = \frac{v(t)}{|v(t)|}$$

$$= \frac{-3\sin ti + 4\cos tj}{\sqrt{4^2 \cos^2 t + 3^2 \sin^2 t}}$$

$$= \frac{-3\sin ti + 4\cos tj}{\sqrt{16\cos^2 t + 9\sin^2 t}}$$

Questions for Practice

b) $$Unit\tan gentvector = \frac{dr}{ds} = 2\cos 2si - 3\sin 3sj$$

Ans2:

$$Total\ \ acceleration = a(t) = -3\cos ti - 4\sin tj$$

$$Tangential\ \ acceleration = a_{\tan}(t)$$

$$= \left(a(t).\frac{v(t)}{|v(t)|} \right) \frac{v(t)}{|v(t)|}$$

$$= \left(\frac{9\sin t\cos t - 16\sin t\cos t}{\sqrt{16\cos^2 t + 9\sin^2 t}} \right) \frac{v(t)}{|v(t)|}$$

Questions for Practice

$$= \left(-\frac{7\sin t\cos t}{\sqrt{16\cos^2 t + 9\sin^2 t}}\right)\left(\frac{-3\sin t i + 4\cos t j}{\sqrt{16\cos^2 t + 9\sin^2 t}}\right)$$

$$= \frac{21\sin^2 t\cos t i - 28\sin t\cos^2 t j}{16\cos^2 t + 9\sin^2 t}$$

$$a_{norm} = a(t) - a_{\tan}(t)$$

$$= (-3\cos t i - 4\sin t j) + \left(\frac{28\sin t\cos^2 t j - 21\sin^2 t\cos t i}{16\cos^2 t + 9\sin^2 t}\right)$$

$$= \left(-3\cos t - \frac{21\sin^2 t\cos t}{16\cos^2 t + 9\sin^2 t}\right)i + \left(-4\sin t + \frac{28\sin t\cos^2 t}{16\cos^2 t + 9\sin^2 t}\right)j$$

Questions for Practice

Ans3: $\Rightarrow Workdone = \int_a^b \mathbf{F}(t) \bullet \mathbf{v}(t)\, \mathrm{d}t$

$$= \int_0^1 (3ti + tj).(3,1,0)dt$$

$$= \int_0^1 (9t + t + 0)dt$$

$$= \int_0^1 10tdt == 5t^2\Big|_0^1 \quad = 5$$