Solving a system of Linear equations

→ n linear equations, n unknowns, below are the ways to understand the equations

i) Row picture – picture of 1 equation at a time, we can visualize where the lines meet

~~ii) Co~~

ii) Column picture – picture a column at a time

iii) Matrix form – algebraic way

Eg:- Let's say,
$$\left.\begin{aligned}2x - y &= 0\\ -x + 2y &= 3\end{aligned}\right\}$$
2 equations with 2 unknowns

i) Row picture :

we'll look into the all points that satisfies both the equation. first we see whether a equation goes through origin,

We have to choose x & y that solve the equation,

eq-①, $x=0, y=0$, eq-① $= 0$

$x=1, y=2$, eq-① $= 0$

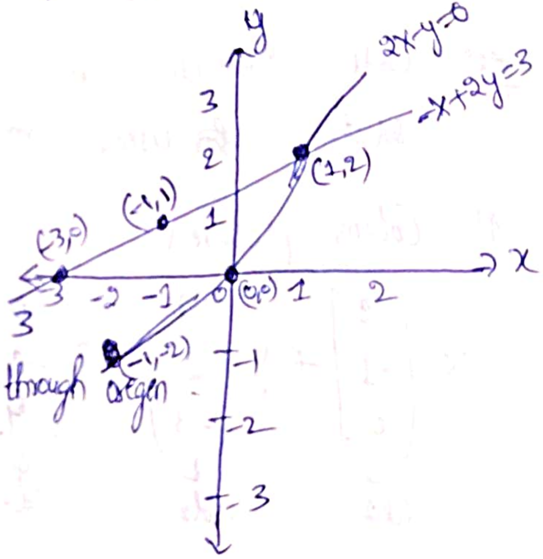
$x=-1, y=-2$, eq-① $= 0$

eq-②, when x & y $= 0$, we don't get 3 that mean eq-② is not going through origin.

if $y=0$, eq-② ⇒ $x=-3$

if $x=-1, y=1$, eq-② ⇒ 3

if $x=1, y=2$, eq-② ⇒ 3

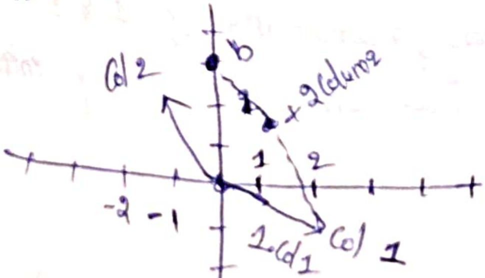


$x=1, y=2$ solves both equations (intersection points)

ii) Column picture :

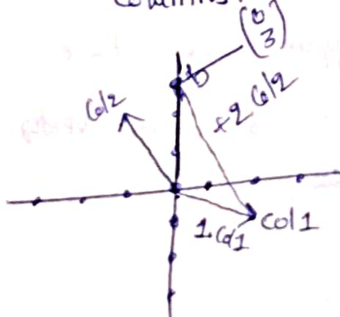
$$x\underset{\text{col1}}{\begin{bmatrix}2\\-1\end{bmatrix}} + y\underset{\text{col2}}{\begin{bmatrix}-1\\2\end{bmatrix}} = \underset{\text{'b'}}{\begin{bmatrix}0\\3\end{bmatrix}}$$

→ add right amount of x & y to get 'b' : linear combination of columns.



$x=1$

$y=2$



→ $x=1$, $y=2$

Note: for all the x & y combination it'll fill the whole plane.

Eg: 3 equations, 3 unknowns.

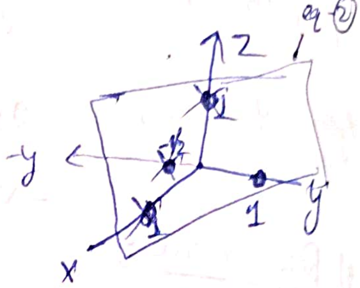
$$\left.\begin{aligned} 2x-y &= 0 \\ -x+2y-z &= -1 \\ -3y+4z &= 4 \end{aligned}\right\} \underset{A}{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{b}{\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}}$$

i) Row picture:

eq-②, if $x=1,\ y=0,\ z=0$, eq-② $=-1$

$x=0,\ y=0,\ z=1$, eq-② $=-1$

$x=0,\ y=-\frac{1}{2},\ z=0$, eq-② $=-1$



all the (x,y,z) that solves eq-② will become a plane and all planes (all 3 eqs) will meet at a point that's a solution.

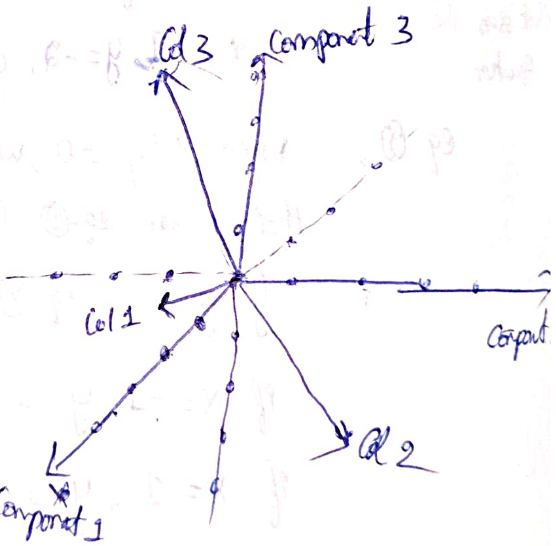
Note: with increase in dimension its becomes harder to solve the solutions ~~for~~ using row-picture.

ii) Column picture:

$$x\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

$\downarrow$ Col 1 $\quad \downarrow$ Col 2 $\quad \downarrow$ Col 3

if $x=0, y=0, z=1$



- Do the linear combinations of the columns fill 3-D ~~plane~~ space (d)
Can we solve $Ax=b$ for every b?

Not for all A's, if the matrix A is non-singular & invertible then we can solve for any b and its fills entire 3-D space.

↳ Suppose we have 3 column vectors, & combination of 1 & 2 col vectors give 3rd column vector then that's $\overset{\text{mean}}{\text{i}}$ we are not covering entire 3-D space.

- Two ways of Matrix multiplication

$Ax = b$ — A times of x. is a combination of columns of A.

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \overbrace{1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 3 \end{bmatrix}} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$; 1) Column multiplication

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{bmatrix} 2\times1 + 5\times2 \\ 1\times1 + 3\times2 \end{bmatrix} = \begin{pmatrix} 12 \\ 7 \end{pmatrix}$$; 2) row-wise operation

Lecture-2

- Systematic way to find solution if there is one, to a system of any size using elimination and back-substitution.

1) success scenario of elimination

Eg:-

$$x + 2y + z = 2$$
$$3x + 8y + z = 12$$
$$4y - 1z = 2$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{array}\right] \xrightarrow{R_2 \to R_2 - 3R_1}$$

A b

~~trying to make~~ elimination

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & 1 & 2 \end{array}\right] \xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 2 & +1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & \underline{-10} \end{array}\right]$$

→ upper triangular matrix

→ c

↓ back-substitution

$$\left.\begin{aligned} x + 2y + z &= 2 \\ 2y - 2z &= 6 \\ 5z &= -10 \end{aligned}\right\} \quad \begin{aligned} x &= 2 \\ y &= 1 \\ z &= -2 \end{aligned}$$

Rules,

we can interchange rows if pivot element is zero accordingly and in 3x3 → 3 pivot elements $\neq 0$. (pivot - diagonal elements of a matrix)

- Whole purpose of elimination is to get upper triangular matrix U from A.
- determinant = multiplying the pivot elements = $1 \times 2 \times 5 = 10$

2) Failure scenario of elimination.

$$\left|\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & -4 & 2 \end{array}\right. \xrightarrow{R_2 \to R_2 - 3R_1} \left|\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 4 & -4 & 2 \end{array}\right. \xrightarrow{R_3 \to R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 0 & -10 \end{array}\right)$$

pivot element is zero hence elimination failed.

→ Adding 'b' to 'A' matrix is called as Augmented matrix.

$[A|b]$

* Elimination Matrices:

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = 3 \times col1 + 4 \times col2 + 5 \times col3$$

(weighted combination of column vectors of a matrices)

- Matrix times a column is a column.
- Matrix times a row is a row.

$$\begin{bmatrix} 1 & 2 & 7 \end{bmatrix}\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix} = 1 \times row1 + 2 row2 + 7 row3$$

(weighted combination of row vectors of a matrices)

- Previous example using elimination matrices:

$$\left.\begin{array}{ccc} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{array}\right| \begin{array}{c} 2 \\ 12 \\ 2 \end{array}$$

Step 1: Subtract 3×row1 from row 2

$$E_{21} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

Matrix needed to fix 21 position → E_{21}

Step 2: Subtract 2×row2 from row 3

$$E_{32} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

$$E_{32}(E_{21}A) = u$$

$$(E_{32}E_{21})A = u$$

(from associate law) - changing the order of performing multiplication but not the order of elements (~~complete~~ commutative law fails)

- Permutation matrix - exchanging rows 1 and 2

Eg1- $\hookrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$

- Exchanging columns of a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

Note: To do column operation, the multiplier should be on right and to do row operation the multiplier should be on left.

- $(E_{32} E_{21}) A = u$

Instead of thinking what matrix to be multiplied by A to get u. Instead of thinking how to get A from u, we thing of

Instead of thinking what matrix to be multiplied by A to get u, we think of how to get A from u. This were inverse comes into picture.

Inverses:

previous example of E_{21}, the inverse of E_{21} is adding 3 x row 1 from row 2,

$$\underset{E^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Matrix Multiplication:

1) Dot product

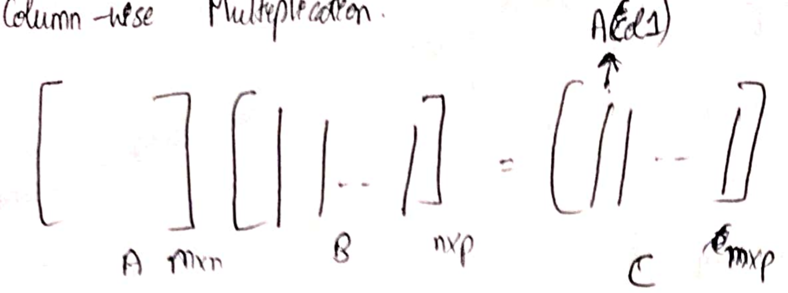
$$\begin{array}{cc} \text{row 3} & \begin{bmatrix} & & \\ a_{31} & a_{32} & \text{---} \\ & & \end{bmatrix}_{m\times n} \end{array} \begin{array}{c} \text{column 4} \\ \begin{bmatrix} b_{14} \\ b_{24} \\ | \end{bmatrix}_{n\times p} \end{array} = \begin{bmatrix} c_{34} \searrow \\ \\ \end{bmatrix}_{m\times p}$$

Matrix A Matrix B $C = AB$

$$c_{34} = (\text{row 3 of } A) \cdot (\text{column 4 of } B)$$

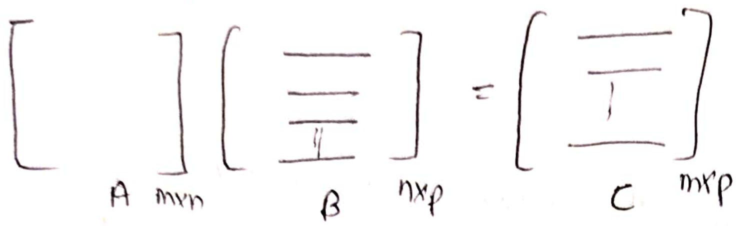
$$= a_{31} b_{14} + a_{32} b_{24} + \cdots = \sum_{k=1}^{n} a_{3k} b_{k4}$$

2) Column-wise Multiplication.



Columns of C are combinations of columns of A.

3) row-wise Multiplication



rows of C are combinations of rows of B.

4) Column-times rows.

Column of A × row of B = m×p

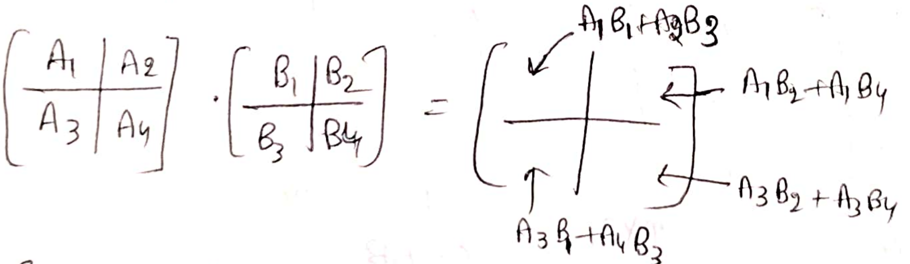
m×1 1×p

$$\begin{bmatrix}2\\3\\1\end{bmatrix}_{3\times1}\begin{bmatrix}1 & 6\end{bmatrix}_{1\times2}=\begin{bmatrix}2 & 12\\3 & 18\\4 & 24\end{bmatrix}_{3\times2}$$

AB = Sum of ((columns of A) × rows of B]

$$\begin{bmatrix}2 & 7\\3 & 8\\4 & 9\end{bmatrix}\begin{bmatrix}1 & 6\\0 & 0\end{bmatrix}=\begin{bmatrix}2\\3\\4\end{bmatrix}(1\ \ 6)+\begin{bmatrix}7\\8\\9\end{bmatrix}\begin{bmatrix}0 & 0\end{bmatrix}$$

5) Block Multiplication:



→ Square Matrices Inverses:

- If A^{-1} exists for matrix A then, $A^{-1}A = I = AA^{-1}$
  these are called as invertible, non-singular matrices.

- ~~non~~ No inverse, Singular case
  - ⇒ det = 0
  - ⇒ find a vector x such that $Ax=0$ $(x\neq0)$

Eg:- $Ax = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ → Non-invertible matrix

Eg:- Inverteble matrex

$$\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A \quad A^{-1} \quad = I$$

$A \times$ Column of A^{-1} = columns of I

Solving 2 equations at once (Gauss-Jordan)

$$\left.\begin{aligned} \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix} \begin{bmatrix} c \\ d \end{bmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}\right\} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right)$$

$$\underset{I \qquad A^{-1}}{\left(\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right)} \xleftarrow{R_1 \to R_1 - 3R_2}$$

$$\begin{pmatrix} 1 & 3 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} 7 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$E[A] = I$ tell us $E = A^{-1}$

$$\underset{\swarrow \text{ elemenation matrex}}{E}[A \ \ I] = [I \ \ A^{-1}]$$