Orthonormal bases:

→ Let's say we have a set $B = \{\vec{v_1}, \vec{v_2}, -- \vec{v_k}\}$ and the special is all the vectors in B have length 1.

i.e $\|\vec{v_i}\| = 1$ for $i = 1, 2, --k$

$\|\vec{v_i}\|^2 = 1$

$\vec{v_i} \cdot \vec{v_i} = 1$ for $i = 1, 2, ---k$

we can say they have all been normalized.

→ And all of the vectors are orthogonal to each other.

$$\vec{v_i}^T \cdot \vec{v_j} = \begin{cases} 0, & \text{for } i \neq j \\ 1, & \text{for } i = j \end{cases}$$

So "B" is an <u>orthonormal set.</u>

→ We will prove 'B' is also linearly independent by contradiction:-

let's say $\vec{v_i}, \vec{v_j} \in B \quad i \neq j$

and $\vec{v_i} \cdot \vec{v_j} = 0 \rightarrow ⓐ$ because of orthogonality.

Proof:- assuming that v_i & v_j are linearly dependent.

i.e $v_i = c v_j \Rightarrow c \neq 0$ bcz v_i & v_j are non-zero vectors of length 1.

from ⓐ, $\vec{v_i} \cdot \vec{v_j} = 0$

$c\vec{v_j} \cdot \vec{v_j} = 0 = c(\vec{v_j} \cdot \vec{v_j}) = c\|v_j\|^2 = 0$

$c \neq 0$ so $\|v_j\|^2$ has to zero $\Rightarrow \|v_j\|^2 = 0$

but our set of vectors are orthonormal which is failing the contradiction, as length $\|v_j\|^2 = 0$ which is not true.

∴ B is the bases for subspace $V = \text{Span}\{\vec{v_1}, \vec{v_2}, --- \vec{v_k}\}$

↳ orthonormal bases.

Eg:- $B = \{ \vec{v_1}, \vec{v_2} \}$, $v_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$ $\vec{v_2} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

$$\|\vec{v_1}\|^2 = \vec{v_1} \cdot \vec{v_1} = \frac{1}{9} + \frac{4}{9} + \frac{4}{9}$$

$$\Rightarrow \|\vec{v_1}\| = 1$$

$$\|\vec{v_2}\|^2 = \vec{v_2} \cdot \vec{v_2} = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = \frac{9}{9} = 1$$

$$\Rightarrow \|\vec{v_2}\| = 1$$

$$\vec{v_1} \cdot \vec{v_2} = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$$

So, B is orthonormal set.

So we have a subspace $V = \text{span}\{\vec{v_1}, \vec{v_2}\}$ then B is orthonormal bases.

Note:- orthonormal basis indicates good Co-ordinate system.

Standard bases for $\mathbb{R}^n \Rightarrow$
$$\underbrace{\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \right\}}_{\text{length} = 1,\ \text{orthogonal}}$$

Basis:- a set of vectors in vector space that can be used to defines co-ordinate space and the vectors in basis are linearly independent.

Properties :- (Matrix A is orthogonal)

Orthogonal matrix:

1) $A^T A = I$; Where A ~~to be~~ is Orthogonal matrix

Let's say A spans $\{(\vec{v_1}), (\vec{v_2})\}$

$$\begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix}_{2\times 1} \begin{pmatrix} v_1 & v_2 \end{pmatrix}_{1\times 2} = \begin{bmatrix} v_1^T v & v_1^T v_2 \\ v_2^T v_1 & v_2^T v_2 \end{bmatrix}$$ *

from orthogonal vectors property, $V_i^T . V_j = 0$ $(i \neq j)$
unit vector, $V_i^T . V_i = 1$

$$\therefore A^T . A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \rightarrow ①$$

2) $\det(A) = \pm 1$

from 1st property $A^T . A = I$

applying det on both sides.

$$\det(A^T . A) = \det(I)$$

$$\det(A^T) \det(A) = 1$$

$$(\det(A))^2 = 1$$

$$\therefore \det(A) = \pm 1$$

$\left(\because \det(AB) = \det(A)\det(B) \right.$
$\left. \det(A) = \det(A^T) \right)$

3) $A^T A = I = A A^T$

from inverse property, $A . A^{-1} = I \rightarrow ②$

$$A^T A = I$$

$$\Rightarrow A^T = A^{-1} \rightarrow ③$$

applying ③ in ②,

$$\therefore A . A^T = I = A^T A$$

4) If A & B are orthgonal then product of A & B is also orthogonal.

Let $C = AB$

Prove :- $C^T C = I$ for 'C' to be orthogonal

$$\downarrow$$

$$(AB)^T (AB) = I$$

$$B^T \underbrace{A^T A}_{I} B = I$$

$$\underbrace{B^T I B}_{I} = I$$

$$I = I \Rightarrow \therefore C^T C = I$$

5) length of $||A\vec{x}||$

we have for any vector, $||\vec{V}|| = V^T.V$, using this

$$||A\vec{x}|| = (A\vec{x})^T (A\vec{x}) \Rightarrow \vec{x}^T \underbrace{A^T A}_{I} \vec{x}$$

$$\Rightarrow \vec{x}^T I \vec{x} \Rightarrow \vec{x}^T \vec{x} = ||\vec{x}||$$

6) If $\vec{V_1}$ & $\vec{V_2}$ have angle θ_1 then after transforming those 2 vectors with a orthogonal matrix, the resultant vectors will have same angle θ_1.

$$\theta_1 = \cos^{-1}\left(\frac{V_1^T V_2}{||V_1|| \, ||V_2||}\right) \quad \text{(before transformation)}$$

and $u_1 = AV_1$ and $u_2 = AV_2$ (after transformation)

$$\theta_2 = \cos^{-1}\left(\frac{u_1^T u_2}{||u_1|| \, ||u_2||}\right).$$

$$= \cos^{-1}\left(\frac{(AV_1)^T AV_2}{||AV_1|| \, ||AV_2||}\right)$$

$$= \cos^{-1}\left(\frac{V_1^T A^T A V_2}{||AV_1|| \, ||AV_2||}\right)$$

→ $A^T A = I$

→ from property 6 we have $||Ax_1|| = ||x_1||$

$$\therefore \theta_1 = \theta_2 = \cos^{-1}\left(\frac{V_1^T V_2}{||V_1|| \, ||V_2||}\right)$$

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Vectors to be orthogonal:

In a vector space if 2 vectors are orthogonal that means they are perpendicular to each other, the angle between them is $\theta = \pi/2$ radians or 90°, which makes the dot product between those 2 vectors = 0.

$$\vec{a}.\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta \xrightarrow{} \theta = \pi/2 \} \Rightarrow 0$$

Eg:-1) $\vec{a} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$

$$\vec{a}.\vec{b} = (a_1)(b_1) + (a_2)(b_2) + (a_3)(b_3)$$

$$= (4)(1) + (2)(-3) + (-1)(-2) = 0$$

$\therefore$ $\vec{a}$ & $\vec{b}$ are orthogonal

⇒ Another restriction we can add to the vectors is orthonormality, where

1) Vectors have length 1
2) satisfy orthogonal

Calculating vector length, $\|\vec{a}\| = \sqrt{\vec{a}.\vec{a}}$

unit vector of $\hat{a} \Rightarrow \dfrac{\vec{a}}{\|\vec{a}\|}$

Eg:- $\vec{a} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$

lengths,

$$\|\vec{a}\| = \sqrt{\vec{a}.\vec{a}} = \sqrt{16+4+1} = \sqrt{21}$$

$$\|\vec{b}\| = \sqrt{\vec{b}.\vec{b}} = \sqrt{1+9+4} = \sqrt{14}$$

unit vectors,

$$\hat{a} = \frac{\vec{a}}{\|a\|} = \frac{1}{\sqrt{21}}\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4/\sqrt{21} \\ 2/\sqrt{21} \\ -1/\sqrt{21} \end{bmatrix}$$

$$\hat{b} = \frac{\vec{b}}{\|b\|} = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} \\ -3/\sqrt{14} \\ -2/\sqrt{14} \end{bmatrix}$$

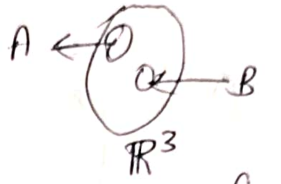
Note :- by doing normalization (unit vector) we are just changing the magnitude of vectors not their direction, so the vectors still follow orthogonality.

* <u>Subspaces to be orthogonal :-</u>

Every vector in A is orthogonal to every vector in B subspace.

$\Rightarrow$ for every $\vec{a} \in A$, every $\vec{b} \in B$

$$\vec{a} \cdot \vec{b} = 0$$



Eg:- Subspace have the vectors of the form, $A = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix}$ and subspace B of the form, $B = \begin{bmatrix} 0 \\ b_1 \\ b_2 \end{bmatrix}$. There could be infinite no of values that can satisfy above condition replacing $a_1, b_1, b_2 \in \mathbb{R}$

$\Rightarrow$ The dot product of any possible vector in subspace A & b gives us,

$$\vec{a} \cdot \vec{b} = (a)(0) + (0)(b_2) + (0)(b_3) = 0$$

$\therefore$ subspace A & B are orthogonal.

* <u>Square matrices to be orthogonal :-</u>

If columns of square matrices make an orthonormal set of vectors then the matrix considered to be orthogonal.

Eg:- $\begin{bmatrix} \frac{1}{\sqrt{2}} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

dot product $= \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{-1}{\sqrt{2}} = \frac{1}{2} - \frac{1}{2} = 0$ //

lengths, $\sqrt{(1/\sqrt{2})^2 + (1/\sqrt{2})^2} = 1$ & $\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + (-1/\sqrt{2})^2} = 1$

The defenetion for orthonormal vectors,

$$q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

orthogonal matrex = Q and orthogonal bases $q_1 \cdots q_n$.

$$Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} \quad \text{then} \quad Q^T Q \Rightarrow \begin{bmatrix} \text{—} q_1^T \text{—} \\ \vdots \\ \text{—} q_n^T \text{—} \end{bmatrix} \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$$

$$\therefore Q^T Q = \begin{bmatrix} q_1^T q_1 & \cdots & q_1^T q_n \\ q_2^T q_1 & q_2^T q_2 \cdots & \vdots \\ \vdots & \vdots & \vdots \\ q_n^T q_1 & \cdots & q_n^T q_n \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ 0 & 1 \cdots & 0 \\ \vdots & 1 & \\ & & 1 \end{bmatrix}$$

first row times the first column

Note:- If Q is a square matrex then $Q^T Q = I$ tells us $\boxed{Q^T = Q^{-1}}$

Eg:- if $Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ ⇒ it is orthogonal to make it athonormal $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

* Gram-Schmedth – to convert given matrex to orthogonal matrex

Note:- If 'Q' has orthonormal columns, then projection onto its column space,

$$P = Q(Q^T Q)^{-1} Q^T = QQ^T = \begin{cases} I, \text{ if Q is squa} \\ \end{cases}$$

Property: 7) $(QQ^T)(QQ^T) = Q(Q^TQ)Q^T = QQ^T$

Property: 8) $AX = b$

Multiplying both sides by A^T,

$$A^TA\vec{X} = A^T\vec{b}$$

If A is orthogonal matrix, $A = Q$

$$\underbrace{Q^TQ}_{I}\vec{x} = Q^T\vec{b}$$

$$\vec{X}_{\$} = Q^T\vec{b}$$

$$\vec{x_i} = q_i^T b$$

Property 9): eigen values of orthogonal matrix will always be equal to ± 1.

$$A\vec{x} = \lambda\vec{x}$$

$$\|A\vec{x}\| = \|\lambda\vec{x}\|$$

$$(Ax)^T(Ax) = (\lambda x)^T(\lambda x)$$

$$x^T\underbrace{A^TA}_{I}x = x^T\underbrace{\lambda^T\lambda}_{\lambda^2}x$$

$$x^Tx = \lambda^2x^Tx$$

$$\Rightarrow \boxed{\lambda = \pm 1}$$