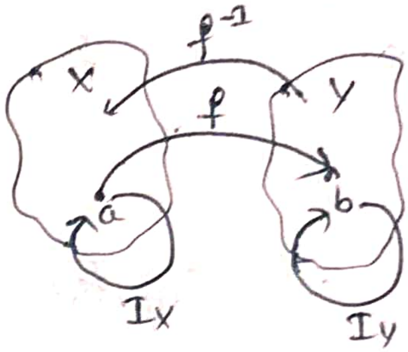
Introduction to the inverse of a function:

→ Let's say we have a function f that maps elements in X to elements in Y.

$$f : X \to Y$$
(domain) (codomain)

$a \in X$ & $b \in Y$ and $f(a) = b$



→ Identity function : $I_X : X \to X$
└ associates all points with themselves

$I_X(a) = a$

$I_Y(b) = b$

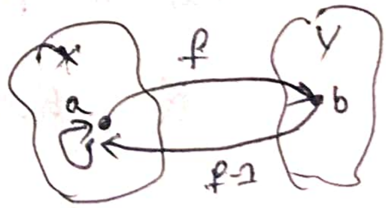
→ f is invertible iff there exists a function f^{-1} such that,

composition → (basically, a function is applied to the result of another function)

$\underbrace{f^{-1} \circ f}_{X \to X} = I_X$ and $\underbrace{f \circ f^{-1}}_{Y \to Y} = I_Y$

$f : X \to Y$ and $f^{-1} : Y \to X$

$(f^{-1} \circ f)(a) = I_X(a) = a$



$f^{-1}(f(a)) = a$

$f(f^{-1}(y)) = y$, $y \in Y$

→ Is f^{-1} is unique?

Let's assume it's not unique. And assume we have 2 functions $f : X \to Y$ and $g : Y \to X$, (g is mapping from $Y \to X$) then γ $g \circ f = I_X$
Composition of g ~~with~~ f,

(g is an inverse of f)

assuming 'g' is an inverse implies 2 things,
1) $g : Y \to X$
2) $g \circ f = I_X$
3) $f \circ g = I_Y$

1st inverse,

$g = I_X \circ g$
$= (h \circ f) \circ g$
$= h \circ (f \circ g) \Rightarrow h \circ I_Y = h$

2nd inverse, $h : Y \to X$ "h is another inverse of 'f', which implies

$h \circ f = I_X$
$f \circ h = I_Y$

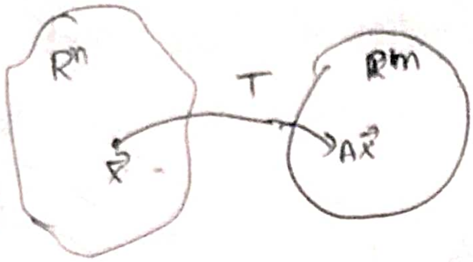
1) $h : Y \to X$
2) $h \circ f = I_X$
3) $f \circ h = I_Y$

$g = h \circ I_Y = h$

$\therefore$ implies we have unique inverse solution.

Let's say we have a linear transformation:

$T : \mathbb{R}^n \to \mathbb{R}^m$ where $T(\vec{x}) = \underset{m\times n}{A}\,\vec{x}_{n\times 1} = m\times 1$



$\Rightarrow$ for 'T' to be invertible,

i) T has to be 'onto' (surjective)

ii) T has to be 'one-to-one' (injective)

onto (surjective) — if we take any element in co-domain ($\mathbb{R}^m$), let say ~~any~~ vector $\vec{b}$ there always going to be some vector in domain ($\mathbb{R}^n$) (or) atleast one vector

$$A\vec{x} = \vec{b} \quad \text{where } \vec{x} \in \mathbb{R}^n.$$

$$T = A\vec{x} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots\ x_n\vec{a}_n$$

(linear combination of column vectors)

$\Rightarrow$ for T to be "onto", the span $(\vec{a}_1, \vec{a}_2 \ldots \vec{a}_n) = \mathbb{R}^m$ (co-domain) which implies column space; $C(A) = \mathbb{R}^m \Rightarrow$ rref (A) has a pivot entry in every row $\Rightarrow$ m pivot entries.

Bases for column space (A)

$$A \xrightarrow{\text{rref}} R$$

$$(\vec{a}_1)\ \vec{a}_2 \cdots (\vec{a}_n) \longrightarrow \begin{bmatrix} 1 & 2 & 3 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & & & & \\ \vdots & & \cdots & & 1 \end{bmatrix}$$

$\to$ pivot ~~entries~~ colums are correspond to bases.

Rank(A) = dim (C(A))
= # of bases vectors of C(A)

Rank (A) = m (invertible)

Eg-1) $S : \mathbb{R}^2 \to \mathbb{R}^3$

S applied to some vector $\vec{x}$, $S(\vec{x}) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \vec{x}$, is S onto ?

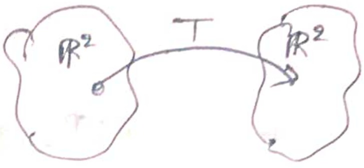
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \xrightarrow[R_3 = R_3 - 5R_1]{R_2 = R_2 - 3R_1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 0 & -4 \end{bmatrix} \xrightarrow[R_3 = R_3 - 2R_2]{R_1 = R_1 + R_2,\ R_2 = -R_2/2} \begin{bmatrix} \underline{1} & 0 \\ 0 & \underline{1} \\ 0 & 0 \end{bmatrix} = 2 \text{ pivot entries}$$

So $\text{Rank}(S) = 2 \neq 3$ (Codomain $\mathbb{R}^3$)

$\therefore$ S is not onto $\Rightarrow$ not invertible

Eg - 2) $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$T(\vec{x}) = \underbrace{\begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix}}_{A} \vec{x}$$



$\vec{b} \in \mathbb{R}^2$

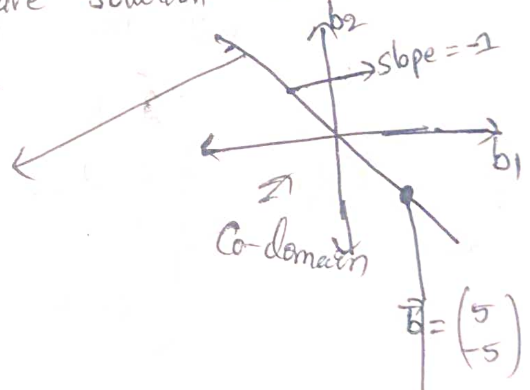
$$A\vec{x} = \vec{b} = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Now we try to find all possible b's,

Augmented matrix, $\left[\begin{array}{cc|c} 1 & -3 & b_1 \\ -1 & 3 & b_2 \end{array}\right] \xrightarrow{R_2 = R_2 + R_1} \left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 0 & 0 & b_1 + b_2 \end{array}\right]$

Only members $\vec{b} \in \mathbb{R}^m$ that have solution are the ones that $b_1 + b_2 = 0$, $b_2 = -b_1$

all possible b's on this line have a solution



$x_1 - 3x_2 = b_1$

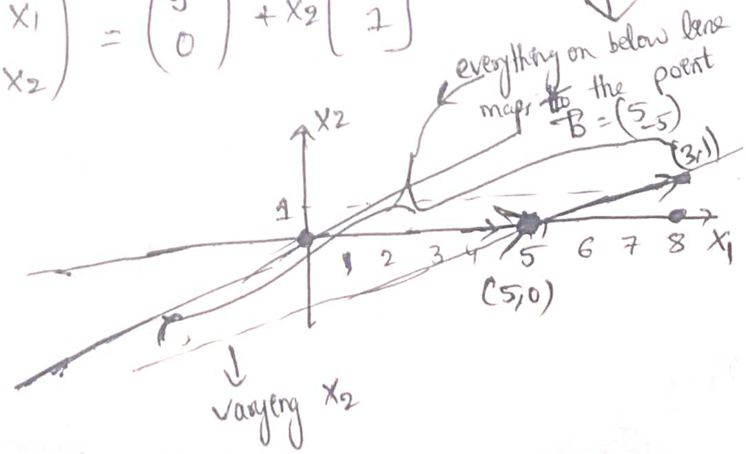
$x_1 = b_1 + 3x_2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

i) if $\vec{b} = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$ then $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

ii) if $\vec{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ then

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$



⇒ Matrix condition for one-to-one transformation :-

* $A\vec{x} = \vec{0}$: Null space of A, $N(A)$ ⇒ all $\vec{x}$'s that satisfy $A\vec{x} = \vec{0}$

* Any solution to the inhomogeneous system $A\vec{x} = \vec{b}$ will take the forms x_p (some particular soln) + x_{Null} (some member of null space)

* $T: \mathbb{R}^n \to \mathbb{R}^m$, Is T invertible? $T(\vec{x}) = A\vec{x}$

Soln Invertible :
1) onto ⇐ $Rank(A) = m$
2) one-to-one ⇐ $Rank(A) = n$

for satisfying both 1) & 2) conditions, A has to be square matrix, $Rank(A) = m = n$, which implies all the column vectors are linearly independent.

for linear independence, the reduced row echelon form will have pivot element in every column.

$$A_{n\times n} \xrightarrow{RRef} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = I_n$$

* Finding inverses and determinants :

→ When doing the reduced row echelon form, the operations are equivalent to the linear transformations on the column vectors of A.

Eg:- $$A = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \to S_1 A$$

Step 1

$$T_1 \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \longrightarrow \begin{bmatrix} a_1 \\ a_2 + a_1 \\ a_3 - a_1 \end{bmatrix} \Rightarrow T_1(\vec{x}) = S_1 \vec{x}$$

Column vectors

let $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{T_1(\vec{x})} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = S_1$

$$S_1\vec{x} \Rightarrow \left[S\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad S\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \quad S\begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} \right]$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{S_1} \underbrace{\begin{bmatrix} 1 & -1 & -1 \\ -1 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix}$$

$\therefore$ The row operations can be represented by matrix multiplication

Step 2:

$$\overbrace{\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix}}^{S_1 A} \xrightarrow[\underbrace{R_3 \to R_3 - 2R_2}]{R_1 \to R_1 + R_2} \overbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}^{S_2 S_1 A} \quad \cancel{S_2 S_1 A}$$

$$T_2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_2 \\ x_3 - 2x_2 \end{bmatrix} \Rightarrow T_2(\vec{x}) = S_2\vec{x}$$

Step 3:-

$$\overbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}^{S_2 S_1 A} \xrightarrow[\underbrace{R_2 \to R_2 - 2R_3}]{R_1 \to R_1 - R_3} \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{S_3 S_2 S_1 A}$$

$$T_3(\vec{x}) = S_3\vec{x}$$

We got an identity matrix which implies the matrix A is Invertible.

A	I
$S_1 A$	$S_1 I$
$S_2 S_1 A$	$S_2 S_1 I$
$A^{-1} \leftarrow \boxed{S_3 S_2 S_1} A = I$	$\boxed{S_3 S_2 S_1} I \; A^{-1}$

One solution to find A^{-1} is using rref of an Augmented matrix,

$$[A \mid I] \to [I \mid A^{-1}]$$

Let's find the inverse of matrix A,

$$[A\,|\,I] = \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \longrightarrow$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_2]{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -3 & -2 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - 2R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 3 & -1 \\ 0 & 1 & 0 & 7 & 5 & -2 \\ 0 & 0 & 1 & -3 & -2 & 1 \end{array}\right] \rightarrow A^{-1}$$

Formula for finding Inverse:

→ let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$[A\,|\,I]$$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\underbrace{\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad-bc & -c & a \end{array}\right]}_{\text{so}}$$

① $T_1\left(\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right) = \begin{bmatrix} c_1 \\ ac_2 - cc_1 \end{bmatrix}$ → To make 1st element of 2nd row $=0$

$$T_1\left(\begin{bmatrix} a \\ c \end{bmatrix}\right) = \begin{bmatrix} a \\ a\times c - c\times a \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

$$T_1\left(\begin{bmatrix} b \\ d \end{bmatrix}\right) = \begin{bmatrix} b \\ a\times d - c\times b \end{bmatrix} = \begin{bmatrix} b \\ ad-bc \end{bmatrix}$$

$$T_1\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ a\times 0 - c\times 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -c \end{bmatrix}$$

$$T_1\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ a\times 1 - c\times 0 \end{bmatrix} = \begin{bmatrix} 0 \\ a \end{bmatrix}$$

② To make 2nd element of 1st row $=0$

$$T_2\left(\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right) = \begin{bmatrix} \cancel{bc_1 - dc_2}\ (ad-bc)c_1 - b(c_2) \\ c_2 \end{bmatrix}$$

$$T_2\left(\begin{bmatrix} a \\ 0 \end{bmatrix}\right) = \begin{bmatrix} (ad-bc)a - b(0) \\ 0 \end{bmatrix} = \begin{bmatrix} a(ad-bc) \\ 0 \end{bmatrix}$$

$$T_2\left(\begin{bmatrix} b \\ ad-bc \end{bmatrix}\right) = \begin{bmatrix} \cancel{(ad-bc)}b - b\cancel{(ad-bc)} \\ ad-bc \end{bmatrix} = \begin{bmatrix} 0 \\ ad-bc \end{bmatrix}$$

$$T_2\left(\begin{bmatrix} 1 \\ -c \end{bmatrix}\right) = \begin{bmatrix} \cancel{(ad-bc)}1 - b(-c) \\ -c \end{bmatrix} = \begin{bmatrix} ad \\ -c \end{bmatrix}$$

$$T_2\left(\begin{bmatrix} 0 \\ a \end{bmatrix}\right) = \begin{bmatrix} \cancel{(ad-bc)(0)} - b(a) \\ a \end{bmatrix} = \begin{bmatrix} -ab \\ a \end{bmatrix}$$

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad-bc & -c & a \end{array}\right) \xrightarrow{T_2\left(\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right)} \left(\begin{array}{cc|cc} a(ad-bc) & 0 & ad & -ba \\ 0 & ad-bc & -c & a \end{array}\right)$$

$$\Big\downarrow T_3\left(\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right)$$

③ Removing scaling factors.

$$T_3\left(\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{(ad-bc)a} \cdot c_1 \\ \frac{1}{(ad-bc)} c_2 \end{bmatrix}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{\cancel{a}d}{\cancel{a}(ad-bc)} & \frac{-b\cancel{a}}{\cancel{a}(ad-bc)} \\ 0 & 1 & \frac{-c}{(ad-bc)} & \frac{a}{(ad-bc)} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{d}{(ad-bc)} & \frac{-b}{(ad-bc)} \\ \frac{-c}{(ad-bc)} & \frac{a}{(ad-bc)} \end{pmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Given any 2x2 matrix we can find the inverse using above formula and it is not defined $\boxed{\text{when } (ad-bc)=0,}$ which we call as "determinant".

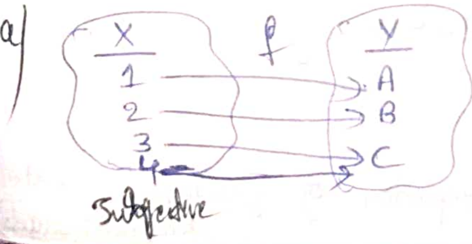
$$\boxed{\text{determinant}(A) = ad-bc \neq 0 \Leftrightarrow A \text{ is invertible}}$$

$$\text{Det}(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$$

Surjective (or) onto function:

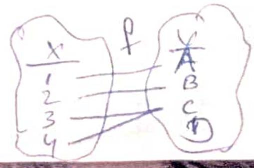
every $y \in Y$ there exist ($\exists$) atleast one $x \in X$ such that $f(x) = y$

a)



Surjective

b)

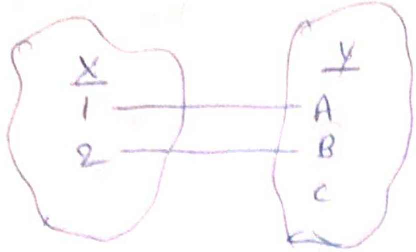


Not surjective

2) Injective (or) one-to-one function:

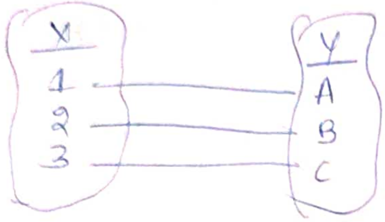
for ~~every~~ any $y \in Y$ there should be atmost 1 'x' such that $f(x) = y$.

a)



→ Injective but not surjective

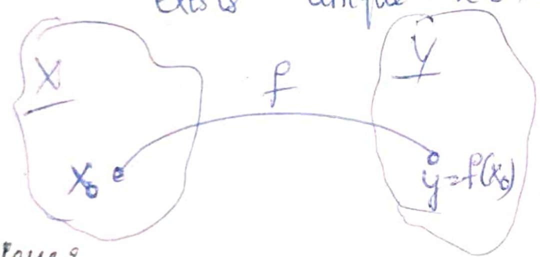
b)



→ both injective & surjective

3) Invertible:

$f: X \to Y$, f is invertible iff $\overbrace{\text{for every } y \in Y}^{\text{surjective}}$ there exists $\underbrace{\text{unique}}^{\text{injective}}$ $x \in X$ such that $f(x) = y$



Is f^{-1} unique?

Let's say there exists 2 inverses g & h for function 'f',

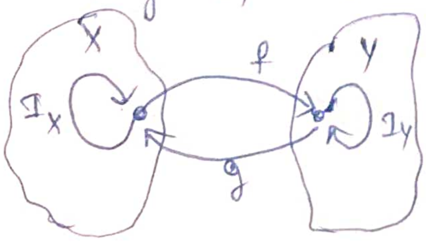
which implies, if $f: X \to Y$

1st:

$g: Y \to X$

$g \circ f = I_X$

$f \circ g = I_Y \rightarrow$ eq ②

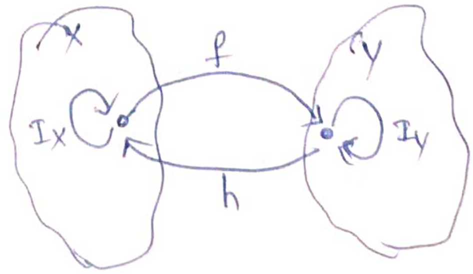


2nd:

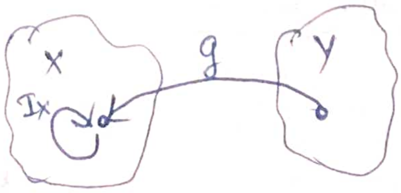
$h: Y \to X$

$h \circ f = I_X \rightarrow$ eq ①

$f \circ h = I_Y$



Now, $g = I_X \circ g \Rightarrow$



substituting eq-①,

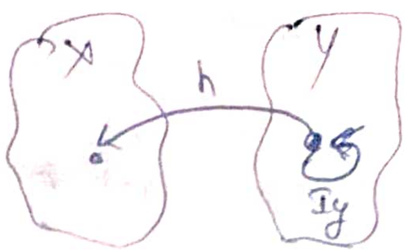
$g = (h \circ f) \circ g$ ($\because$ the composition of function is associative $a \circ (b \circ c) = (a \circ b) \circ c$

$g \Rightarrow (h \circ f) \circ g = h \circ (f \circ g)$ from eq-②

$= h \circ (I_y)$

$\boxed{\therefore g \Rightarrow h}$

$h: Y \to X$



So, any function has unique inverses.

$\Rightarrow$ for every $y \in Y$, is there a unique solution $x \in X$ such that it satisfies $f(x) = y$? $\rightarrow$

if f is invertible $\exists$ (there exists) $f^{-1}: Y \to X$ such that $f^{-1} \circ f = I_X$ and $f \circ f^{-1} = I_y$.

let $f(x) = y$

$\underbrace{f^{-1} f(x)} = f^{-1}(y)$ (applying inverse on both sides)

$I_X(x) = f^{-1}(y)$

↳ only 1 inverse

$x = f^{-1}(y)$

$\therefore$ if f is invertible, then $f(x) = y$. $\forall\, y \in Y$ has a unique solution.

$\Rightarrow$ redefining invertibility:- The $f: X \to Y$ is invertible if and only if f is both surjective (onto) and injective (one-to-one).

$A_{m \times n}$ matrix:

Onto	One-to-one	Rank
✓	X	Rank(A) = m & < n (tall matrix)
X	✓	Rank(A) = n & < m (wide matrix)
✓	✓	Rank(A) = m = n
X	X	Rank(A) < m & < n