Properties of Determinant

1) Row multiplied by scalar.

i) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} = \quad |A| = ad - bc$

$$\begin{vmatrix} a & b \\ Kc & Kd \end{vmatrix} = Kad - Kbc = K(ad-bc) = K|A|$$

$$\begin{vmatrix} Ka & Kb \\ Kc & Kd \end{vmatrix} = K^2ad - K^2bc = K^2(ad-bc) = K^2|A|$$

ii) $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow |A| = -d\begin{vmatrix} b & c \\ h & i \end{vmatrix} + e\begin{vmatrix} a & c \\ g & i \end{vmatrix} - f\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$A' = \begin{pmatrix} a & b & c \\ Kd & Ke & Kf \\ g & h & i \end{pmatrix} \Rightarrow |A'| = -Kd\begin{vmatrix} b & c \\ h & i \end{vmatrix} + Ke\begin{vmatrix} a & c \\ g & i \end{vmatrix} - Kf\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

$$= K|A|$$

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} A_{ij}.$$

$$\det(A') = K\sum_{j=1}^{n} (-1)^{i+j} a_{ij} A_{ij}$$

$\det(KA) = K^n \det(A) \longrightarrow$ 'n' : no. of rows K is multiplied in matrix A.

2) When row is added.

let $X = \begin{bmatrix} a & b \\ x_1 & x_2 \end{bmatrix} \quad Y = \begin{bmatrix} a & b \\ y_1 & y_2 \end{bmatrix} \quad Z = \begin{bmatrix} a & b \\ x_1+y_1 & x_2+y_2 \end{bmatrix}$

$|X| = ax_2 - bx_1$; $|Y| = ay_2 - by_1$; $|Z| = a(x_2+y_2) - b(x_1+y_1)$

$$= ax_2 + ay_2 - bx_1 - by_1$$

$$= \underbrace{ax_2 - bx_1}_{|X|} + \underbrace{ay_2 - by_1}_{|Y|}$$

3) Duplicate row

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = (ab - ba) = 0$$

$\therefore$ If matrix is invertible then rref $= I_n$

duplicate rows never get rref to be $I_n \Rightarrow$ not invertible.

$\Rightarrow det = 0$

4) Swapping of row

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -(ad - bc) = -|A|$$

5) After row operations

$$A = \begin{bmatrix} \vec{r_1} \\ \vec{r_2} \\ \vec{r_3} \end{bmatrix}, \quad B = \begin{bmatrix} \vec{r_1} \\ \vec{r_2} - c\vec{r_1} \end{bmatrix}$$

$$\Downarrow$$

$$\det(B) = \left| \begin{pmatrix} \vec{r_1} \\ \vec{r_2} \end{pmatrix} \right| + \left| \begin{pmatrix} \vec{r_1} \\ -c\vec{r_1} \end{pmatrix} \right|$$

$$= |A| + (-c) \left| \begin{pmatrix} \vec{r_1} \\ \vec{r_1} \end{pmatrix} \right| \rightarrow \text{duplicate row (prop 3)} = 0$$

$$\det(B) = \det(A)$$

6) Upper-triangular and lower-triangular

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \Rightarrow \det(A) = ad \text{ (diagonal product)}$$

Note: Whenever we have larger $n \times n$ matrix, first do rref to get in the form of upper triangular matrix then take the product of diagonal elements to get the determinant.

$$A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \Rightarrow \det(A) = ad$$

Properties of matrix Inverses

1) I is invertible and $I^{-1} = I$

since $I \cdot I = I$

2) If A is invertible then A^{-1} is also invertible and

$(A^{-1})^{-1} = A$. since $AA^{-1} = A^{-1}A = I$

3) If A and B are invertible then AB is also invertible,

and $(AB)^{-1} = B^{-1}A^{-1}$

since $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$

4) If A is invertible then λA is also invertible $\forall \lambda \in \mathbb{R}|\{0\}$,

and $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$

since $\left(\frac{1}{\lambda} A^{-1}\right)(\lambda A) = \left(\frac{1}{\lambda} \cdot \lambda\right) A^{-1}A = 1 \cdot I = I$

and $(\lambda A)\left(\frac{1}{\lambda} A^{-1}\right) = \left(\lambda \cdot \frac{1}{\lambda}\right) AA^{-1} = 1 \cdot I = I$

5) If A is invertible, the A^T is also invertible and

$(A^T)^{-1} = (A^{-1})^T$

$(A \cdot A^{-1})^T = (I)^T$

$(A^{-1})^T (A)^T = I^T$

since $(A^T)(A^{-1})^T = (A^{-1}A)^T = I^T = I$

and $(A^{-1})^T(A^T) = (AA^{-1})^T = I^T = I$

6) If A is invertible then A^K is also invertible $\forall K \in \mathbb{N}$,

and $(A^K)^{-1} = (A^{-1})^K$

$(A^{n+1})^{-1} = (A^n A)^{-1} = A^{-1}(A^n)^{-1}$

$(A^{-1})^{n+1} \Leftarrow A^{-1} \cdot (A^{-1})^n$

Proof: $(AB^T)_{pq} = (AB)_{qp} = \sum_K (A)_{qK} (B)_{Kp} = \sum_K (A^T)_{Kq} (B^T)_{pK}$

$= \sum_K (B^T)_{pK} (A^T)_{Kq} = (B^T A^T)_{pq} \Rightarrow (AB)^T = B^T A^T$