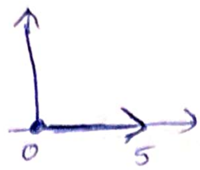
Vectors

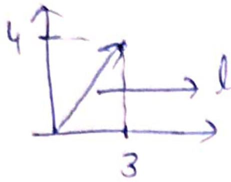
→ has magnitude & direction   Eg:- $\underbrace{\underbrace{5\text{ mph}}_{\text{speed} \to \text{scalar}} \text{ towards east}}_{\text{Velocity} \to \text{Vector}}$

→ represented by small letters.

$\vec{V} = (5,0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix} \longrightarrow$



$\vec{a} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \rightarrow$



length $= \sqrt{3^2+4^2} = \sqrt{9+16} = 5$

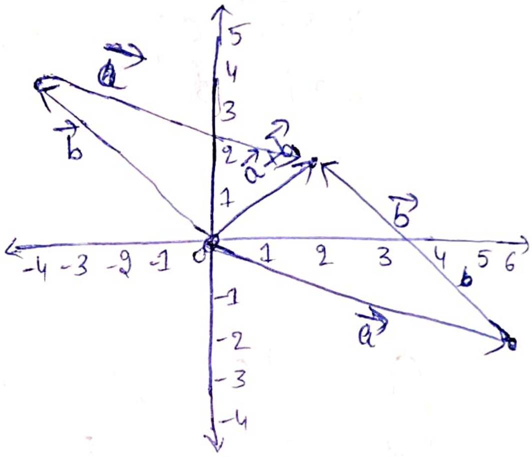
* Real Co-ordinate space :

$\mathbb{R}^n$ — n dimensional (real valued n-tuples)

i) Adding vectors :

$\vec{a} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$

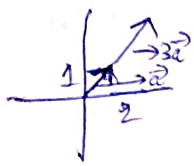
$\vec{a} + \vec{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$



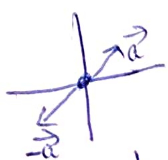
ii) Multiplying a vector by a scalar :

~~direction remains same~~, ~~but~~ changes magnitude & direction

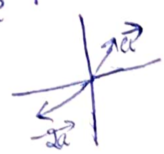
Eg:1) $\vec{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $3\vec{a} = 3\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$; only magnitude changed



$-1\vec{a} = -1\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$; only direction changed

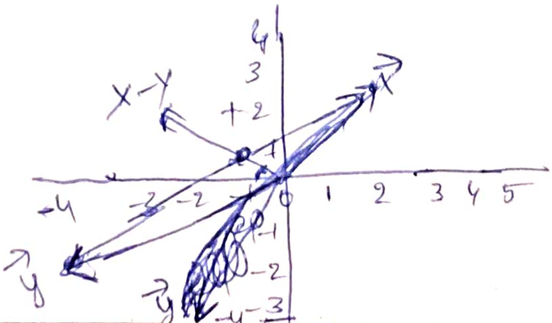


$-2\vec{a} = -2\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$; both magnitude & direction changes



Eg:2) $X = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $Y = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$

$X - Y = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$



→ Any 2-dim vector can be represented by the scaled version of unit vectors $\hat{i}$ & $\hat{j}$.

* Unit vector: $\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ → has magnitude 1

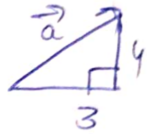
$$\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\hat{i} + 3\hat{j}$$

$$\vec{b} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} = -1\hat{i} + 4\hat{j}$$

$$\vec{v} + \vec{b} = \hat{i} + 7\hat{j} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

Eg:- $\vec{a} = (3,4)$

Magnitude of $\vec{a}$ = length of $\vec{a}$ = $\|\vec{a}\| = \sqrt{3^2+4^2}$



$\|\vec{a}\| = 5$ (this is not a unit vector bcz it doesn't has its magnitude as of 1).

∴ unit vector, $\hat{u}$ (unit) $= \left(\frac{3}{\|\vec{a}\|}, \frac{4}{\|\vec{a}\|} \right)$

$$\hat{u} = \left(\frac{3}{5}, \frac{4}{5} \right)$$

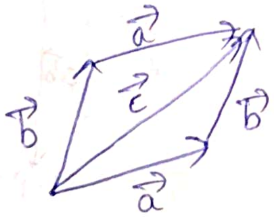
$$\|\hat{u}\| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = 1$$

↓

direction remain same as $\vec{a}$ but magnitude become 1.

* Parallelogram rule for vector addition:

$\vec{a} + \vec{b} = \vec{c}$



* Adding vectors: magnitude & direction to component

$\vec{v} = (5, 320°)$, $\vec{w} = (4, 250°)$

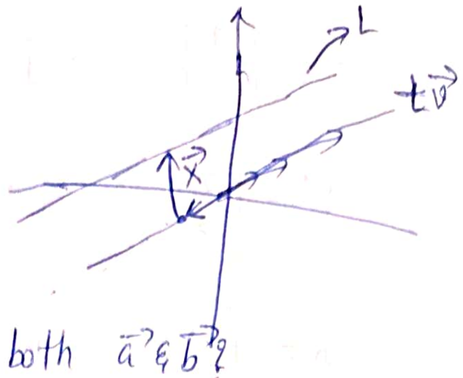
$$\vec{v} + \vec{w} = (v_x + w_x, v_y + w_y) = (5\cos 320° + 4\cos 250°), (5\sin 320° + 4\sin 250)$$

$$= (2.46, -6.97)$$

* Parametric representations of lines:

$S = \{ c\vec{v} \mid c \in \mathbb{R} \}$: set of Collinear vectors

$L = \{ \vec{x} + t\vec{v} \mid t \in \mathbb{R} \}$

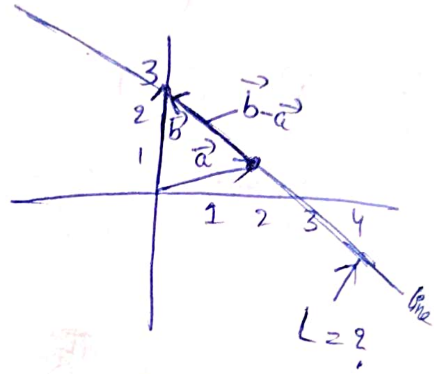


Eg:- Let $\vec{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

find out the line that has both $\vec{a}$ & $\vec{b}$?

$L = \{ t(\vec{b} - \vec{a}) + \vec{b} \mid t \in \mathbb{R} \}$

$L = \{ \vec{a} + t(\vec{b} - \vec{a}) \mid t \in \mathbb{R} \}$



$L = \left\{ \begin{pmatrix} 0 \\ 3 \end{pmatrix} + t \begin{bmatrix} -2 \\ 2 \end{bmatrix} \mid t \in \mathbb{R} \right\}$

$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ $L_1 \rightarrow$ x Co-ordinate $= x = 0 + -2t = -2t$

$L_2 \rightarrow$ y Co-ordinate $= y = 3 + 2t$

2. Linear Combinations and spans :-

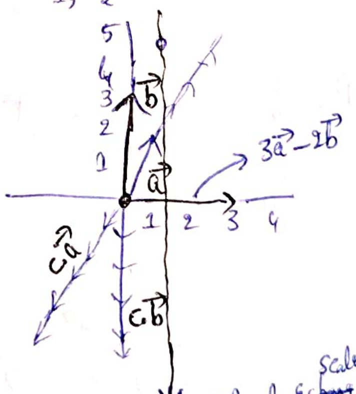
$c_1 V_1 + c_2 V_2 + \dots C_n V_n$ where $V_1, V_2 \dots V_n \in \mathbb{R}^n$

$c_1, c_2 \dots c_n \in \mathbb{R}$

Eg:- 1) $\vec{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\vec{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$0\vec{a} + 0\vec{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$3\vec{a} + -2\vec{b} = \begin{pmatrix} 3-0 \\ 6-6 \end{pmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

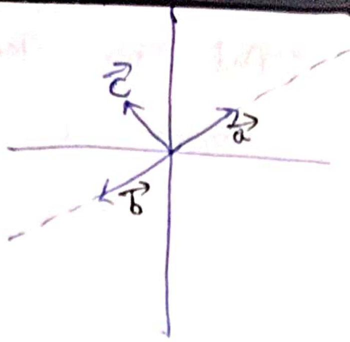


Keep $\vec{a}$ fixed & ~~changing~~ scaling $\vec{b}$ gives us a line & vice

→ The Span of vectors $(\vec{a}, \vec{b}) = \mathbb{R}^2$. This just means we can represent any vector in $\mathbb{R}^2$ with some linear combination of $\vec{a}$ & $\vec{b}$.

→ span of $\vec{a}$ = line = $c\vec{a}$

Eg. 2) $\vec{a} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$



Here whatever linear combination we apply on $\vec{a}$ & $\vec{b}$ it just scales up or down in the direction of $\vec{a}$ or $\vec{b}$ (line), and we can't get $\vec{c}$.

So, the span $(\vec{a}, \vec{b}) \neq \mathbb{R}^2$, but just a line.

Eg: 3) $\text{Span}(\vec{0}) = 0$

Def of Span: $\text{span}(v_1, v_2, \ldots v_n) = \{ c_1 v_1 + c_2 v_2 + \ldots c_n v_n \mid c_i \in \mathbb{R} \text{ for } 1 \leq i \leq n \}$

Proof: Given 2 vectors (not colinear) and a arbitrary vector we should find c's such that combination of 2 vectors lead to arbitrary vector.

$$c_1 \vec{a} + c_2 \vec{b} = \vec{x}$$

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$1c_1 + 0c_2 = x_1 \Rightarrow c_1 = x_1$$

$$2c_1 + 3c_2 = x_2 \Rightarrow c_2 = \frac{1}{3}(x_2 - x_1 2)$$

let's say $\vec{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$,

$$c_1 = 2, \quad c_2 = \frac{1}{3}(2 - 2 \times 2) = \frac{2}{3}$$

$$\therefore 2\vec{a} - \frac{2}{3}\vec{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

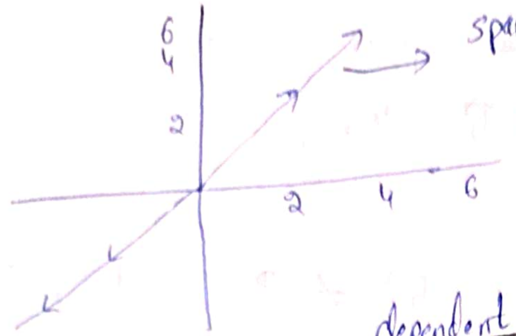
3. Linear independence and dependence.

The span of $\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix} \right\}$ is $c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$$\Rightarrow c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \cdot 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} \Rightarrow \underbrace{(c_1 + 2c_2)}_{} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow c_3 \begin{bmatrix} 2 \\ 3 \end{bmatrix} \rightarrow$$ it became scalar combination of one vector

∴ And we know in $\mathbb{R}^2$ scalar combination of one vector becomes a line along that vector.



$\longrightarrow \text{span}\left(\left\{\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}4\\6\end{pmatrix}\right\}\right)$

Co-linear vectors.

∴ These vectors are called as linear dependent which means one of the vectors in the set can be represented by some combinations of other vectors in the set.

Egs:-1) In $\mathbb{R}^2$, check these vector are linear dependent?

$$\begin{pmatrix}2\\3\end{pmatrix} \quad \begin{bmatrix}7\\2\end{bmatrix} \quad \begin{bmatrix}9\\5\end{bmatrix}$$

span$(\vec{v_1}, \vec{v_2}) = \mathbb{R}^2$ where $\vec{v_3} \in \mathbb{R}^2$.

$$\vec{v_1} + \vec{v_2} = \vec{v_3}$$

Since it's a 2-dimension for sure the ~~at~~ third vector well be linear dependen~~t~~. And span$(v_1, v_2, v_3) \doteq \mathbb{R}^2$.

Eg:-2) $\left\{\begin{bmatrix}7\\0\end{bmatrix}, \begin{bmatrix}0\\-1\end{bmatrix}\right\}$ and $\left\{\begin{pmatrix}2\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\7\end{pmatrix}\right\}$

↓ linearly endependent ↓ linearly independent.

Note: The set, $S = \{V_1, V_2 .. V_n\}$ is linearly dependent iff $c_1V_1 + c_2V_2 + .. c_nV_n = 0$ and atleast one c_i is non-zero.

proof:- 1) let, $V_1 = a_2V_2 + -- a_nV_n$

$$0 = -1V_1 + a_2V_2 + -- a_nV_n \quad \text{(proved)}$$

2) Assume $c_1 \neq 0$ and deviding linear combination by c_1,

$$\frac{c_1}{c_1}V_1 + \frac{c_2}{c_1}V_2 + -- \frac{c_n}{c_1}V_n = 0$$

$$\frac{c_2}{c_1}V_2 + -- \frac{c_n}{c_1}V_n = -V_1 \Rightarrow V_1 = -\frac{c_2}{c_1}V_2 \cdots -\frac{c_n}{c_1}V_n$$

(proved)

Eg:3) i) $\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{pmatrix}3\\2\end{pmatrix}\right\}$ → linearly dependent?

$c_1\begin{pmatrix}2\\1\end{pmatrix} + c_2\begin{pmatrix}3\\2\end{pmatrix} = 0$ $\Big\}$ if c_1 or c_2 non zero → dependent
c_1 & c_2 both zero ⇒ independent.

$2c_1 + 3c_2 = 0$

$(-)\times 2 \quad c_1 + 2c_2 = 0$

$c_2 = 0$
$c_1 = 0$ $\Big\}$ lineary independent set

ii) $\left\{\begin{pmatrix}2\\1\end{pmatrix} \begin{pmatrix}3\\2\end{pmatrix} \begin{pmatrix}1\\2\end{pmatrix}\right\}$

$c_1\begin{pmatrix}2\\1\end{pmatrix} + c_2\begin{pmatrix}3\\2\end{pmatrix} + c_3\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$$\left.\begin{aligned} 2c_1 + 3c_2 + c_3 &= 0 \\ c_1 + 2c_2 + 2c_3 &= 0 \end{aligned}\right\} \xrightarrow{\text{if } c_3 = -1} \begin{aligned} 2c_1 + 3c_2 - 1 &= 0 \\ (-)\times 2 \quad 2c_1 + 2c_2 - 2 &= 0 \\ \hline 0 + (-c_2) + 3 &= 0 \\ c_2 &= 3 \\ \Rightarrow c_1 &= -4 \end{aligned}$$

$\therefore \underbrace{-4\begin{pmatrix}2\\1\end{pmatrix} + 3\begin{pmatrix}3\\2\end{pmatrix} - 1\begin{pmatrix}1\\2\end{pmatrix}}_{\text{linearly dependent}} = \begin{pmatrix}0\\0\end{pmatrix}$

Eg:4) $S = \left\{\begin{pmatrix}1\\-1\\2\end{pmatrix}, \begin{pmatrix}2\\1\\3\end{pmatrix}, \begin{pmatrix}-1\\0\\2\end{pmatrix}\right\}$ span(S) = $\mathbb{R}^3$?
linearly Independent?

Span, $c_1\begin{pmatrix}1\\-1\\2\end{pmatrix} + c_2\begin{pmatrix}2\\1\\3\end{pmatrix} + c_3\begin{pmatrix}-1\\0\\2\end{pmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$

① — $c_1 + 2c_2 - c_3 = a$ $\qquad$ $c_1 + 2c_2 - c_3 = a$

② — $-c_1 + c_2 = b$ $\xrightarrow{②+①}$ $3c_2 - c_3 = b + a$

③ — $2c_1 + 3c_2 + 2c_3 = c$ $\xrightarrow{③+(-2)①}$ $-c_2 + 4c_3 = c - 2a$

do row elemenation procedure we get

$c_3 = \frac{1}{11}(3c - 5a + b)$

span(S) = $\mathbb{R}^3$

$c_2 = \frac{1}{3}(b + a + c_3)$

$c_1 = a - 2c_2 + c_3$

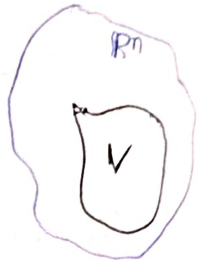
To prove independence,

let $a = b = c = 0$, then

$c_3 = 0, \; c_2 = 0, \; c_1 = 0$

$\therefore$ 'S' is linearly independent.

<u>Linear subspaces:-</u>



- Let V be a subset of R^n.

$$\text{Subspace of } R^n \leftarrow \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, x_i \in \mathbb{R}, \; 1 \le i \le n \right\}$$

Definition of subspace: If V is subspace of R^n then it implies 3 things:

1) V contains $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

2) If $\vec{x}$ in V then $c\vec{x}$ should be in V (Closure under scalar multiplication)

3) If $\vec{a}$ in V & $\vec{b}$ in $\vec{V}$ then $\vec{a} + \vec{b}$ should be in V. (Closure under addition)

Eg:-) $V = \{0\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$, is V a subspace of R^3?

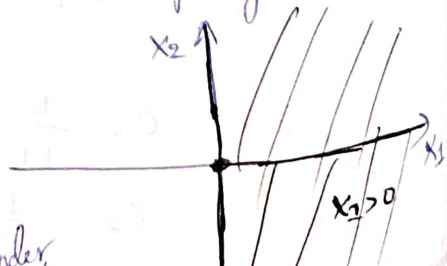
i) Zero vector ✓

ii) $c \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, closed under scalar multiplication ✓

iii) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, closed under addition ✓

$\therefore$ V is a subspace of $\mathbb{R}^3$.

Eg:-) $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; x_1 \ge 0 \right\}$, is S subspace of R^2?



i) Contain $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ✓

ii) $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$ with $a \ge 0$, $c \ge 0$, $a+c \ge 0$ ✓ Closed under addition

iii) $-1\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a \\ -b \end{bmatrix} \rightarrow \neq \geq 0$ (not closed under scalar multiplication)

$\therefore$ S is not a subspace of $\mathbb{R}^2$.

Eg:-) $U = \text{span}(v_1, v_2, v_3)$. Valid subspace of $\mathbb{R}^n$?

i) $0\vec{v_1} + 0\vec{v_2} + 0\vec{v_3} = \vec{0}$ ✓

ii) let $\vec{x} = c_1\vec{v_1} + c_2 v_2 + c_3\vec{v_3}$

$$a.\vec{x} = a c_1\vec{v_1} + a c_2\vec{v_2} + a c_3\vec{v_3}$$

$$= b_1\vec{v_1} + b_2\vec{v_2} + b_3\vec{v_3}$$ (one of the linear combination included in span)

└→ closed under mul ✓

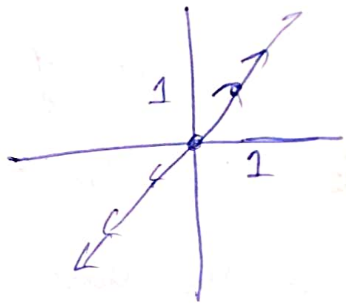
iii) $\vec{y} = d_1\vec{v_1} + d_2\vec{v_2} + d_3\vec{v_3}$

$$\vec{x} + \vec{y} = (c_1+d_1)\vec{v_1} + (c_2+d_2)\vec{v_2} + (c_3+d_3)\vec{v_3}$$ (linear combination)

closed under addition ✓

U is valid subspace.

Eg:-) $U = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$



i) $0\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ✓

ii) $c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ✓

iii) $c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = (c_1 + c_2)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ✓

U is a valid subspace of $\mathbb{R}^2$

Note:- $V = \text{span}(\vec{v_1}, \vec{v_2}, \ldots \vec{v_n})$ is always a valid subspace.

$\{\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}\}$ are linearly independent, implies

$$c_1\vec{v_1} + c_2\vec{v_2} + \ldots c_n\vec{v_n} = 0$$

$$c_1 = c_2 = \ldots c_n = 0$$

If both linear independence & subspace is valid then those set of vectors are called as "bases".

$S = \{v_1, v_2, \ldots \vec{v_n}\} \Rightarrow S$ is a bases for V.

Bases $\rightarrow$ Minimum set of vectors that spans the subspace.

Eg:- $S = \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ 0 \end{pmatrix} \right\}$, span(s) = ?

$$c_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 7 \\ 0 \end{pmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$2c_1 + 7c_2 = x_1$$

$$3c_1 + 0.c_2 = x_2 \Rightarrow c_1 = \frac{x_2}{3}$$

$$\frac{2}{3}x_2 + 7c_2 = x_1 \Rightarrow 7c_2 = x_1 - \frac{2}{3}x_2$$

$$c_2 = \frac{x_1}{7} - \frac{2}{21}x_2$$

$$\text{Span}(s) = \mathbb{R}^2$$

linear independence

$$c_1 \begin{pmatrix} 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 7 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c_1 = 0, \quad c_2 = 0$$

The set of vectors 'S' is a basis for $\mathbb{R}^2$.

Note:- We can have 'many' bases for a subspace.

→ If we have bases for some subspace, any member of subspace can be uniquely determined by a unique combination of these vectors.

let $\{v_1, \ldots v_n\}$ = bases for U (subspace)

$\vec{a} \in U$ then $\vec{a} = c_1\vec{v_1} + c_2\vec{v_2} \ldots + c_n\vec{v_n}$

$$(-)\ \vec{a} = d_1\vec{v_1} + d_2\vec{v_2} + \ldots + d_n\vec{v_n}$$

$$\vec{0} = \underbrace{(c_1-d_1)}_{0}\vec{v_1} + \underbrace{(c_2-d_2)}_{0}\vec{v_2} + \ldots \underbrace{(c_n-v_n)}_{0}\vec{v_n}$$

$$\Rightarrow c_1 = d_1; \ c_2 = d_2 \ldots c_n = v_n$$

unique solution.

Vector dot product and vector length:

Addition: $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix}$

Scalar Multiplication: $c\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix}$

- Dot product:

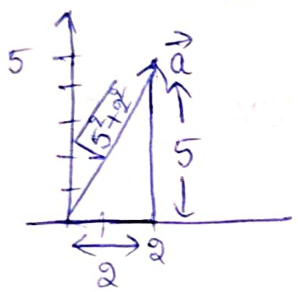
$$\vec{a}\cdot\vec{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \underbrace{a_1 b_1 + a_2 b_2 + \cdots\cdots a_n b_n}_{\text{Scalar}}$$

Eg:- $\begin{bmatrix} 2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 1 \end{bmatrix} = 2.7 + 5.1 = 14 + 5 = 19$

- Length of vector:

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

Eg:- $\vec{a} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$



from pythagoras theorem,

$$\|\vec{a}\| = \sqrt{5^2 + 2^2} = \sqrt{25+4} = \sqrt{29}$$

- Relation btw dot product and length of vector.

$$\vec{a}\cdot\vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = a_1^2 + a_2^2 + \cdots a_n^2$$

$$\|a\| = \sqrt{\vec{a}\cdot\vec{a}} \quad (\text{or}) \quad \|a\|^2 = \vec{a}\cdot\vec{a}$$

<u>Properties of dot product:</u>

1) $\vec{V}\cdot\vec{W} = \vec{W}\cdot\vec{V}$ (Commutative) ✓ -- order doesnot matter

$\vec{V} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}$ $\vec{W} = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_n \end{bmatrix}$

$\vec{V}\cdot\vec{W} = V_1 W_1 + V_2 W_2 + \cdots V_n W_n$

$\vec{W}\cdot\vec{V} = W_1 V_1 + W_2 V_2 + \cdots W_n V_n$

$\therefore V_1 W_1 = W_1 V_1$ (Normal multiplication Commutative property)

2) $(\vec{V}+\vec{W})\cdot\vec{X} = V\cdot\vec{X} + \vec{W}\cdot\vec{X}$. (distributive) ✓ ← order doesn't matter

<u>L.H.S</u> $(\vec{V}+\vec{W})\cdot\vec{X} = \begin{bmatrix} V_1 + W_1 \\ \vdots \\ V_n + W_n \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$

$(\vec{v}+\vec{w})\cdot\vec{x} = (v_1+w_1)x_1 + \dots + (v_n+w_n)x_n$

R.H.S $\vec{v}\cdot\vec{x} = v_1x_1 + v_2x_2 + \dots v_nx_n$

$\vec{w}\cdot\vec{x} = w_1x_1 + w_2x_2 + \dots w_nx_n$

$\vec{v}\cdot\vec{x} + \vec{w}\cdot\vec{x} = (v_1x_1 + w_1x_1) + (v_2x_2 + w_2x_2) + \dots (v_nx_n + w_nx_n)$

$= (v_1+w_1)x_1 + (v_2+w_2)x_2 + \dots (v_n+w_n)x_n$

$\therefore$ L.H.S = R.H.S

3) $(c\vec{v})\cdot\vec{w} = c(\vec{v}\cdot\vec{w})$ (associative)

L.H.S $\begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}\cdot\vec{w} = cv_1w_1 + \dots cv_nw_n$

R.H.S $c(\vec{v}\cdot\vec{w}) = c(v_1w_1 + \dots + v_nw_n) = cv_1w_1 + \dots cv_nw_n$

Proof of the Cauchy-Schwarz inequality:

Non-zero vectors, $\vec{x}, \vec{y} \in R^n$,

$|\vec{x}\cdot\vec{y}| \le \|\vec{x}\|\,\|\vec{y}\|$ and

$|\vec{x}\cdot\vec{y}| = \|\vec{x}\|\,\|\vec{y}\| \Leftrightarrow \vec{x} = c\vec{y}$ (Co-linear)

Proof: Let's say a function of a scalar,

$p(t) = \|t\vec{y} - \vec{x}\|^2 \ge 0$ → any real-valued vector length,

$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \dots v_n^2} \ge 0$

$p(t) = (t\vec{y} - \vec{x})\cdot(t\vec{y} - \vec{x})$ ← $\boxed{\|\vec{v}\|^2 = \vec{v}\cdot\vec{v}}$

$= t\vec{y}\cdot t\vec{y} - \vec{x}\cdot t\vec{y} - t\vec{y}\cdot\vec{x} + \vec{x}\cdot\vec{x} \ge 0$

$= (\vec{y}\cdot\vec{y})t^2 - 2(\vec{x}\cdot\vec{y})t + \vec{x}\cdot\vec{x} \ge 0$

$p(t) = at^2 - bt + c \ge 0$

for any real 't', $t = \frac{b}{2a}$ then,

$P\left(\frac{b}{2a}\right) = a\cdot\frac{b^2}{4a^2} - b\cdot\frac{b}{2a} + c \ge 0$

$$P\left(\frac{b}{2a}\right) = \frac{b^2}{4a} - \frac{b^2}{2a} + c \geq 0$$

$$= \frac{b^2}{4a} - \frac{2}{2}\cdot\frac{b^2}{2a} + c \geq 0$$

$$= \frac{b^2}{4a} - \frac{2b^2}{4a} + c \geq 0 \Rightarrow -\frac{b^2}{4a} + c \geq 0$$

$$\Rightarrow c \geq \frac{b^2}{4a} \Rightarrow 4ac \geq b^2$$

Substituting a & b, in c,

$$4(\underbrace{\|\vec{y}\|^2}_{a}\ \underbrace{\|\vec{x}\|^2}_{c}) \geq (\underbrace{2(\vec{x}\cdot\vec{y})}_{b})^2$$

$$\cancel{4}\|\vec{y}\|^2\|\vec{x}\|^2 \geq \cancel{4}(\vec{x}\cdot\vec{y})^2$$

apply square root on both sides,

$$\boxed{\|\vec{y}\|\ \|\vec{x}\| \geq |\vec{x}\cdot\vec{y}|}$$

∴ Hence proved Cauchy-Schwarz inequality.

if $\vec{x} = c\vec{y}$,

$$|\vec{x}\cdot\vec{y}| = |c\vec{y}\cdot\vec{y}| = |c|\ |\vec{y}\cdot\vec{y}| = |c|\ \|y\|^2$$

$$= |c|\ \|y\|\|y\|$$

$$= \|c\vec{y}\|\ \|\vec{y}\|$$

$$\boxed{|\vec{x}\cdot\vec{y}| = \|\vec{x}\|\ \|\vec{y}\|}$$

Vector ~~quart~~triangle inequality:

$$\|\vec{x}+\vec{y}\|^2 = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y})$$

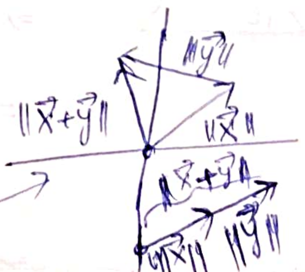
$$= \vec{x}(\vec{x}+\vec{y}) + \vec{y}\cdot(\vec{x}+\vec{y})$$

$$= \vec{x}\vec{x} + \vec{x}\vec{y} + \vec{y}\vec{x} + \vec{y}\cdot\vec{y}$$

$$= \|\vec{x}\|^2 + 2(\vec{x}\cdot\vec{y}) + \|y\|^2 \qquad \because \vec{x}\cdot\vec{y} \leq \|\vec{x}\|\|\vec{y}\|$$

$$\|\vec{x}+\vec{y}\|^2 \leq \|x\|^2 + 2\|\vec{x}\|\|\vec{y}\| + \|y\|^2$$

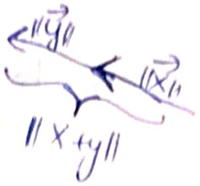
$$\|\vec{x}+\vec{y}\|^{\cancel{2}} \leq (\|\vec{x}\| + \|\vec{y}\|)^{\cancel{2}}$$



$$\boxed{\therefore \|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|}$$

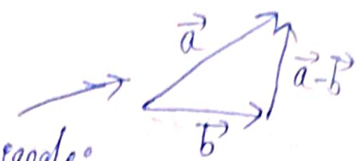
$\|\vec{x}+\vec{y}\| = \|\vec{x}\| + \|\vec{y}\|$ when $\vec{x} = c\vec{y}\,;\ c>0$

↳ Co-linear



* Angle between vectors:

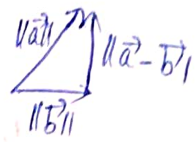
Let $\vec{a}, \vec{b} \in \mathbb{R}^n$ & non-zero.



In few conditions we cannot construct a triangle:

↓ length

1) $\|\vec{b}\| > \|\vec{a}\| + \|\vec{a}-\vec{b}\|$



2) $\|\vec{a}\| > \|\vec{a}-\vec{b}\| + \|\vec{b}\|$

3) $\|\vec{a}-\vec{b}\| > \|\vec{a}\| + \|\vec{b}\|$

In general the above 3 can never happen we will prove using triangular inequality,

$\|\vec{x}+\vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$

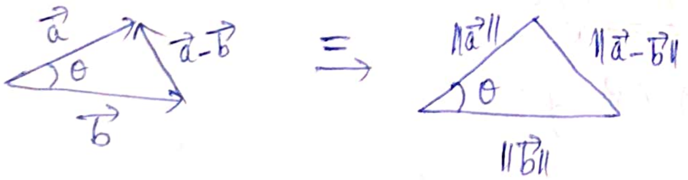
$\Rightarrow \|\vec{a}\| = \|(\vec{b})+(\vec{a}-\vec{b})\| \le \|\vec{b}\| + \|\vec{a}-\vec{b}\|$ → above 2) is not correct from this left side eqn

$\|\vec{b}\| = \|\vec{a} + (\vec{b}-\vec{a})\| \le \|\vec{a}\| + \|\vec{b}-\vec{a}\|$ → above 1) not correct

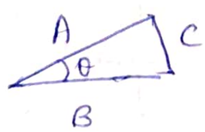
$\|\vec{b}-\vec{a}\| = \|(-1)(\vec{a}-\vec{b})\|$

$\|\vec{a}-\vec{b}\| = \|\vec{a}+(-\vec{b})\| \le \|\vec{a}\| + \|(-\vec{b})\| \le \|\vec{a}\|+\|\vec{b}\|$ → above 3)

Angle:

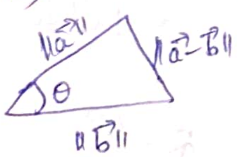


law of cosines tells,



$\Rightarrow C^2 = A^2+B^2-2AB\cos\theta$

so,



$\Rightarrow \|\vec{a}-\vec{b}\|^2 = \|\vec{b}\|^2 + \|\vec{a}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta$

$\Rightarrow (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = \|\vec{b}\|^2 + \|\vec{a}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta$

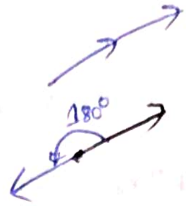
$\Rightarrow \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b} =$ " " "

$\Rightarrow \cancel{\|\vec{a}\|^2} - \cancel{2}(\vec{a}\cdot\vec{b}) + \cancel{\|\vec{b}\|^2} = \cancel{\|\vec{b}\|^2} + \cancel{\|\vec{a}\|^2} - \cancel{2}\|\vec{a}\|\|\vec{b}\|\cos\theta$

$\boxed{(\vec{a}\cdot\vec{b}) = \|\vec{a}\|\,\|\vec{b}\|\cos\theta}$ → angle between the vectors $\vec{a}, \vec{b}$

if $\vec{a} = c\vec{b}$, $c > 0 \Rightarrow \theta = 0$

$c < 0 \Rightarrow \theta = 180°$



Note:- if there is zero vector, the above formula is not valid.

$0 = 0 \cdot \cos\theta \Rightarrow \cos\theta = \frac{0}{0}$ (undefined)

⇒ if $\vec{a}, \vec{b}$ are perpendicular ⇒ $\vec{a}\cdot\vec{b} = 0$ ($\vec{a}, \vec{b}$ are non-zero)

$\theta = 90°$, $\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\,\cancel{\cos 90}^{\,0}$

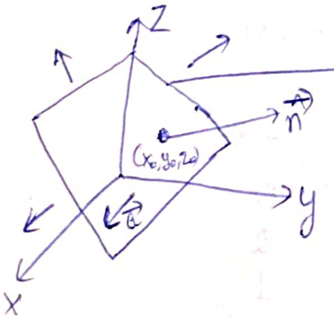
$\boxed{\vec{a}\cdot\vec{b} = 0}$

⇒ all perpendicular vectors are "orthogonal" ⇒ $\vec{a}\cdot\vec{b} = 0$

Note:- zero vector is orthogonal to every other vector.

Defining a plane in $\mathbb{R}^3$ with a point and normal vector.

→ Equation of a plane in $\mathbb{R}^3$:



$Ax + By + Cz = D$ (every point (x, y, z) satisfies this eq^n)

$\vec{n}$ ⇒ "normal" perpendicular to everything on the plane.

vector $\vec{a}$ is perpendicular to $\vec{n}$ then $\vec{n}\cdot\vec{a} = 0$

let's say we have 2 points on the plane, $\vec{x}_0$ & $\vec{x}$

$\vec{x}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\vec{n}(\vec{x} - \vec{x}_0) = 0$

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$$

$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$ (of the form $Ax + By + Cz = 0$)

Matrices for solving systems by elimination

3 equations 4 unknowns, → infinite solutions with some constraint

Eg:-
$$x_1 + 2x_2 + x_3 + x_4 = 7$$
$$x_1 + 2x_2 + 2x_3 - x_4 = 12$$
$$2x_1 + 4x_2 \qquad + 6x_4 = 4$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 7 \\ 1 & 2 & 2 & -1 & 12 \\ 2 & 4 & 0 & 6 & 4 \end{array}\right]$$

Using reduced row echelon form, finding the soln for this system of eq

$$\xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 7 \\ 0 & 0 & +1 & -2 & 5 \\ 0 & 0 & -2 & 4 & -10 \end{array}\right] \xrightarrow[R_1 \to R_1 - R_2]{R_3 \to R_3 + 2R_2} \left[\begin{array}{cccc|c} \underline{1} & 2 & 0 & 3 & 2 \\ 0 & 0 & \underline{1} & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

non-zero leading entries in a row are called 'pivot',

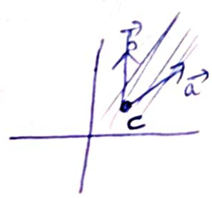
$\cancel{1 \quad 2}$ $\quad x_1 + 2x_2 + 3x_4 = 2 \Rightarrow x_1 = 2 - 2x_2 - 3x_4$

$$x_3 - 2x_4 = 5 \Rightarrow x_3 = 5 + 2x_4$$

in vector form,
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 5 \\ 0 \end{bmatrix}}_{c} + x_2 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{a}} + x_4 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 1 \end{bmatrix}}_{\vec{b}}$$

x_1, x_3 → pivot variables and x_2, x_4 → free variables.

Solution for A is a plane in R^4.



Matrix-vector products :-

$$A = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix} \begin{array}{l} \to \vec{C_1}^T \\ \to \vec{C_2}^T \\ \to \vec{C_3}^T \end{array} \qquad \vec{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ \vec{V_1} & \vec{V_2} & \vec{V_3} \end{array}$$

$$A\vec{X} = x_1\vec{V_1} + x_2\vec{V_2} + x_3\vec{V_3} \quad \to \quad 1 \text{ form } \left\{ \begin{array}{l} \text{linear combination} \\ \text{of column vectors of} \\ \text{'A'.} \end{array} \right.$$

2^{nd} form,

$$\begin{pmatrix} C_1^T \\ C_2^T \\ C_3^T \end{pmatrix} \vec{X} = \begin{pmatrix} C_1 \cdot \vec{X} \\ C_2 \cdot \vec{X} \\ C_3 \cdot \vec{X} \end{pmatrix}$$

where $C_1 = \begin{pmatrix} a \\ d \\ g \end{pmatrix}$

$$C_1 \cdot \vec{X} = ax_1 + dx_2 + gx_3$$