Partition Functions, L-functions, and Point Counting: A Geometric Langlands Fairytale

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Contents

Partition Functions, L-functions, and Point Counting: A Geometric Langlands Fairytale

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Witten and Kapustin showed that the geometric Langlands program can be interpreted in terms of a generalized electric-magnetic duality for $\mathcal{N} = 4$ super Yang-Mills theories. This magical correspondence connects the seemingly disparate lands of physics and number theory, allowing for physical intuition to aid our understanding of what might seem a "ferocious beast" on the number theory side. We will explore these connections, especially in the context of arithmetic field theories defined over algebro-geometric or number-theoretic objects rather than actual spacetime manifolds. Surprisingly, analogies between number theory and physics give us many new tools. For example, we can understand primes as "knots" and compute their "linking numbers!" Our main result comes from studying partition functions on the B-side of the electric-magnetic duality. These give rise to L-functions such as the famous Riemann zeta function when the gauge group is a vector space. We then consider the case where our group is not simply connected through the use of arithmetic Chern-Simons theory for function fields, and prove partition functions in this setting count Frobenius fixed points of certain varieties.

Statement of original research

Chapters 1, 2, 3, and 4 primarily contain review of physics and math background material. However, in Chapter 3 we briefly mention an extension of work on the linking number in arithmetic field theories to the function field setting that appears to be novel. Chapter 5 describes original results, which are summarized in Chapter 6.

1 Introduction: An Invitation to Our Story

Once upon a time, in a land of powerful primes, a surprising link between number theory and physics was discovered. This is a tale with super(symmetric)-heroes, (topological) twists, and plenty of schemes. The main characters in this story are the seemingly disparate worlds of topological quantum field theories and number theory. Together these ideas will allow us to approach the – often mythical seeming – geometric Langlands correspondance![1] At the outset, the Langlands program may appear to be a scary firebreathing dragon. However, if you embark on this journey with us, we will develop the necessary tools to safely[2] approach this seemingly formidable creature, and we may find that the dragon is friendlier than we imagined.[3]

Before continuing we should meet the heroes of our story. Topology is the study of shapes, up to stretching and wiggling, so a *topological field theory* describes the physics coming from the topology of the space it lives in. The other character we must meet is *number theory*. A typical problem number theorists might study is finding integer or rational solutions to an equation. For example, integer solutions to $a^2 + b^2 = c^2$ give us Pythagorian triples. Even excluding multiples, this has infinitely many solutions. However, Fermat's last theorem, which tells us that there are no integer solutions to $a^n + b^n = c^n$ for any $n > 2$, took several hundred years to prove [5, 6]. There are a lot of cool structures that we can use to study number theory. For example, we can study groups of symmetries generated by polynomi-

[1] While an exaggeration, this theory has been referred to as the "mathematical theory of everything."

[2] For readability the text will often be imprecise, but we will direct you to appendices for details.

[3] Some dragons only hoard knowledge and are excited to share! See [1–4] for examples of good dragons.

als. This is called Galois theory, and we will tell you a bit about this and some of the math developed to understand Galois groups better. In particular, the Langlands program is an attempt to understand non-abelian Galois groups [7].

The connection between physics and the Langlands program was first explored in [8]. The big picture idea is to view the geometric Langlands program as a kind of generalized electric-magnetic duality. This is accomplished through considering dual "twists" of supersymmetric field theories to obtain topological quantum field theories. Strikingly, these topological theories contain the information that number theorists want to study! For example, partition functions give rise to L-functions – generalizations of the Riemann zeta function which play a core role in modern number theory.

1.1 Getting a Bit More Technical

Quantum physics and number theory have intersected in a surprising number of places. For example, the Hilbert-Pólya conjecture suggests that the non-trivial zeros of the Riemann zeta function (and L-functions in general) correspond to the spectrum of some observable in a quantum theory [9]. These ideas are supported by the form of the Selberg trace formula [10] and random matrix results [4] [12–15], and more recent efforts by Connes used a noncommutative geometry approach [16]. Another point of intersection between number theory and physics is found in Bost-Connes systems [17] which are C^*-dynamical systems – an operator algebraic approach to quantum statistical mechanics – that recover the Riemann zeta function and maximal cyclic field extensions of $\mathbb{Q}$ and generalize to other number fields [18]. In our setting, the connection between math and physics comes through dualities.

Dualities show up in many forms in physics. As a first motivating example we can consider the T duality of string theory [19, 20]. This duality comes from considering string theories on a spacetime with a compactified dimension. Explicitly, consider the simplest case of a bosonic theory with closed strings in 26 dimensions (indexed $0, 1, ..., 25$), and compatictify X^{25} to have radius R. Then T duality says this theory is equivalent to one on with $R \to \tilde{R} = \frac{\alpha'}{R}$ where α' determines the string tension. The intuition behind this, as explained in [21], is that the momentum in the compactified direction is quantized to $p^{25} = \frac{n}{R}$, $n \in \mathbb{Z}$. Then for large R the allowed momentum

[4] Hayes' article [11] provides an entertaining and accessible introduction to these ideas.

becomes a continuum, but if the string winds around the compactified dimension it will become too massive and drop out. Conversely, for small R the winding number can easily grow without the string becoming too massive, while the momentum drops out instead. Hence, the two effects are fully swapped. These ideas occur in a more complicated form to swap type IIA and IIB supersymmetric string theories as well as the two heterotic string theories. Further, this is a special case of mirror symmetry, which swaps theories on Calabi-Yau manifolds and their duals [22].

From a more mathematical side, we can understand this duality as coming from a special case of Pontryagin-Poincaré duality for an abelian group G. This duality says

$$H^i(M,G)^\vee \cong H^{n-i}(M,G^\vee) \tag{1}$$

where $G^\vee := \mathrm{Hom}(G,U(1))$ is the *Pontryagin dual* group of G.

These notions of duality will extend into a whole class of dualities in higher dimensions that are inspired by the electric-magnetic duality. However, if G is non-abelian then taking the Pontryagin dual gives a quantum group rather than an actual group [23]. Hence, we must either enter the realm of noncommutative geometry or change our approach. Langlands took the second route by proposing a Langlands dual group – which we will also denote $G^\vee$ but the context should be clear – defined in terms of the dual root data of G. Then the idea is that electric-magnetic duality gives a duality between theories defined with Langlands dual groups.

After giving background on these dualities, we will narrow our focus to consider L-functions – generalizations of the famous Riemann zeta function $\zeta(s) = \prod_{p=\text{prime}} \frac{1}{1-p^{-s}}$. These arise as certain operators for 3-dimensional topological boundary theories on the B-side of the duality. In particular, we will see that partition functions give rise to L-functions! Finally, in chapter 5 we will consider partition functions for an arithmetic version of Chern-Simons theory with a non-simply connected group and interpret it number theoretically. Our main result is the following theorem.

Theorem 1.1 (Theorem 5.1)**.** *The partition function for arithmetic Chern-Simons theory over curve $C/\mathbb{F}_q$ with gauge group $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^2$ and level $\begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}$ for $k = \ell^m$ is $Z = \#\left(\mathrm{Jac}(C)[k]^{\mathrm{Frob}_q}\right)$.*

2 The Quest Begins: Quantization & Topology

We begin by getting to know our characters through a first adventure through the land of physics. Here we will encounter various important ideas; starting with quantization, and then learning how to obtain topological quantum field theories through twisting.

2.1 Quantization

To begin, we consider what makes a theory quantum. For example, electric-magnetic duality requires a quantum theory of electromagnetism to discuss even the base case – actual electromagnetics. There are numerous approaches to quantization, including geometric quantization and deformation quantization. For our purposes geometric quantization is especially useful, so we begin by reviewing this. Then we sketch the more general and powerful BV formalism, which is necessary to discuss topologically twisting theories.

Geometric Quantization

Here we give a brief overview of the ideas of geometric quantization. More details can be found in numerous sources, for example [24]. The main idea is that we begin with a phase space $\mathcal{M}$ with symplectic structure ω. Then the goal is to output a Hilbert space $\mathfrak{H}$. To do so we first prequantize and then pick a polarization.

More specifically, *prequantization* is the choice of a Hermitian line bundle equipped with a Hermitian connection ∇ on $\mathcal{M}$ and such that the curvature of the connection F coincides with ω.[5]

Example 2.1. Consider $\mathcal{M} = T^*N$ for some manifold X with symplectic form $\omega = dp \wedge dq = d\alpha = d(pdq)$. Then we can consider the trivial complex line bundle $\mathcal{M} \times \mathbb{C}$ with the canonical 1-form as the prequantization data.

Given a prequantization, we then pick a *polarization*. This is a foliation $\mathcal{F}$ whose leaves are Lagrangian submanifolds. The Hilbert space will then be sections that are flat along the foliation:

$$\mathfrak{H} = \{s \in \Gamma(\mathcal{M}, L) | \nabla_X s = 0, \forall X \in \Gamma(\mathcal{M}, \mathcal{F})\}.$$

[5]The existence of a prequantum line bundle given a symplectic manifold is not guaranteed. Weil integrability conditions are sufficient, however.

Finally, a *metaplectic correction* and the introduction of square-roots of volume forms (known as "half-forms") must be introduced to obtain square integrable functions.[6] Once this is done we have *quantized* our theory!

Example 2.2. In the above example we can use the projection $\pi : T^*N \to N$ to view N as a Lagrangian submanifold. Taking $\mathcal{F} = \ker(d\pi : T(T^*N) \to TN)$ we get the desired foliation. The Hilbert space will be $\mathfrak{H} = \Gamma(N, N \times \mathbb{C}) = C^\infty(N, \mathbb{C})$ up to the metaplectic correction. However, there can be some subtleties. Suppose we have $\mathcal{M} = T^*S^1 \ni (p, q)$ where $q \in S^1$ so $q \sim q + 1$. Then $\mathfrak{H}$ is spanned by $\{e^{2\pi inq}\}$. However, we could instead choose to polarize so that our states are functions of p, and this quantization should somehow translate to this Hilbert space. To recover this, we find that p must be in the subset of $\mathbb{R}$ consisting of p where $\int_{S^1 \times p} \alpha \in \mathbb{Z}$.

Batalin-Vilkovisky Formalism

The Batalin-Vilkovisky (BV) formalism is a homological approach to path integrals that allows for a consistent treatment of gauge fixing and quantization [25]. The idea is to pass from the classical phase space of fields to a derived version which encodes infinitesimal neighborhoods of classical solutions and their symmetries. This is described by a cochain complex graded by "ghost number" such that the infinitesimal symmetries and higher relations appear as coboundaries. Formally, we describe this with a local L-infinity algebra. Our interest in this approach is due to its use in describing twisting to obtain topological field theories as described in section 2.3. Here we give a brief and relatively informal introduction.[7] See [26, 27] for nice exposition, and appendix A for a more mathematically technical introduction using derived algebraic geometry and shifted symplectic structures following [28].

Suppose we have a classical action $S_0(\phi)$ for a gauge theory that we want to quantize. In the BRST approach this is handled by introducing ghost fields and encoding gauge fixing via taking cohomology with respect to the BRST operator. However, this breaks down in supersymmetric theories where the symmetry is only on-shell. In these cases we must use the more general BV formalism. Let us denote the antifields corresponding to ϕ by ϕ^+. Then we extend our action to a functional $S = S_0 + \phi^+ S_1$ on the BV complex such that the classical part is recovered on fields of "ghost number" zero. Then

[6] These subtleties can be ignored if we only care about states on Kähler manifolds.

[7] This exposition is largely based on a verbal explanation given by Adrián López-Raven.

our path integrals get replaced with integrals over this extended space.

$$\int \mathcal{D}\phi^i e^{-\frac{1}{\hbar}S_0(\phi)}\mathcal{O} = \int \mathcal{D}\phi_i^+ \mathcal{D}\phi^i e^{-\frac{1}{\hbar}S(\phi,\phi^+)}\mathcal{O}\delta\left(\phi_i^+ = \frac{\partial\psi}{\partial\phi}\right). \tag{2}$$

Letting $\mathcal{A} = e^{-\frac{1}{\hbar}S(\phi,\phi^+)}\mathcal{O}$ and varying gives

$$\delta\int \mathcal{D}\phi\mathcal{A}\left(\phi, \frac{\partial\psi}{\partial\phi}\right) = \int \mathcal{D}\phi\frac{\partial\psi}{\partial\phi}\frac{\partial}{\partial\phi^+}\mathcal{A} = -\int \mathcal{D}\delta\psi\frac{\partial}{\partial\phi^i}\frac{\partial}{\partial\phi_i^+}\mathcal{A}. \tag{3}$$

Hence, for this to vanish $\mathcal{A}$ must be a cocycle under the fermionic BV Laplacian $\Delta = \frac{\partial^2}{\partial\phi_i^+\partial\phi^i}$. Applying this to our exponential, we get the quantum master equation (QME), $\frac{1}{2}\{S,S\} - \hbar\Delta S = 0$, which ensures the path integral is independent of gauge-fixing. More generally, physical observables must satisfy $\Delta\left(e^{-\frac{1}{\hbar}S}\mathcal{O}\right) = \text{QME} + \frac{1}{\hbar}\left(\{S,\mathcal{O}\} + \hbar\Delta\mathcal{O}\right) = 0$.

2.2 Topological field theories

In a topological field theory we take out all the dynamics and focus only on the aspects of a theory that depend on the topology of the manifold the theory lives within. This might seem like it could be boring, but actually there are many interesting effects that come from topology! The good news for us is that topological quantum field theories (TQFTs) are well founded mathematically.

To define an n-dimensional TQFT we first consider the category of oriented[8] cobordisms $\text{Bord}_{n-1,n}$. In this category our objects are closed oriented $n-1$-manifolds (including the empty $n-1$-manifold), and morphisms between two $n-1$-manifolds Σ_1 and Σ_2 are oriented n-manfolds M such that $\partial M = \Sigma_1 \sqcup \Sigma_2$. An example bordism for a 2D TQFT is shown in figure 1a.

This category has some additional structure: the disjoint union of $n-1$ manifolds $\Sigma_1 \sqcup \Sigma_2$, acts a lot like the tensor product. In fact, $(\text{Bord}_{n-1,n}, \sqcup)$ is a *monoidal category*[9].

Given this we follow Atiyah and Segal [29, 30] and define a (non-extended) *TQFT* to be a symmetric monoidal functor

$$(\text{Bord}_{n-1,n}, \sqcup) \xrightarrow{\mathcal{Z}} (\text{Vect}, \otimes).$$

[8] Or framed, although we will only discuss orientation.

[9] This section contains a bit of category theory. Definitions and references can be found in C.2

Unraveling this definition, we call $\mathcal{Z}$ the *partition function* because it sends

- Closed n-manifolds to numbers: $\mathcal{Z}(M^n) \in \mathbb{C}$
- Closed $n-1$-manifolds to vector spaces (the Hilbert space on a spatial slice)
- An n-manifold M with boundary $\partial M = \Sigma_1 \sqcup \Sigma_2$, oriented so we have an incoming boundary Σ_1 and outgoing boundary Σ_2, is sent to the linear map $\mathcal{Z}(M) : \mathcal{Z}(\Sigma_1) \to \mathcal{Z}(\Sigma_2)$ (think of this as time evolution).

Notice that the $\mathcal{Z}(\emptyset) = \mathbb{C}$ is the "trivial" vector space. Hence, a closed n-manifold can be thought of as a map between empty boundaries, i.e., a linear map from $\mathbb{C} \to \mathbb{C}$ which is just an element of $\mathbb{C}$ itself.

There is a lot that can be said about TQFTs, but instead of going into more depth on them here, we recommend other references such as [31]. In this work we are interested in *extended* TQFTs. Consider first extending to $n-2$-manifolds N. Objects in $\mathrm{Bord}_{n-2,n}$ are $n-2$ manifolds with $n-1$ manifolds being 1-morphism between them and n-manifolds being 2-morphisms. Hence, this is a 2-category. This idea can be extended to higher codimension manifolds as well by associating a symmetric monoidal $k-1$-category to codimension k manifolds as described quite nicely in [32].

Focusing on 2-dimensional TQFTs for the moment, the idea is as follows: we now want $\mathcal{Z}$ to be a functor to some symmetric monoidal 2-category $(\mathcal{C}, \otimes)$, with the condition that it should still recover the old notion of a TQFT when it makes sense. As a symmetric monoidal functor, $\mathcal{Z}$ should send the unit of Bord, the $\emptyset^{n-2}$ in $n-2$, to the unit of $1_{\mathcal{C}} \in \mathcal{C}$. However, morphisms $\{\emptyset^{n-2} \to \emptyset^{n-2}\} =: \mathrm{End}_{\mathrm{Bord}_{n-2,n}}(\emptyset^{n-2})$ are exactly oriented closed $n-1$-manifolds, and we already know $\mathcal{Z}$ should associate vector spaces to these. Hence, we require $\mathrm{End}_{\mathcal{C}}(1_{\mathcal{C}}) = \mathrm{Vect}$.

Example 2.3. A natural candidate to use for $\mathcal{C}$ is $\mathrm{Cat}_{\mathbb{C}}$. Objects in $\mathcal{C}$ are categories enriched [10] over $\mathrm{Vect}_{\mathbb{C}}$ – the category of vector spaces over $\mathbb{C}$. The 1-morphisms are linear functors between categories. Finally, the 2-morphisms are natural transformations.

We conclude with an example of untwisted Dijkgraaf-Witten theory.

[10] See C.9.

Example 2.4. Let G be a finite group. Classically we define a theory of G-local systems on 2-manifolds as the gauge theory of flat connections on 2-dimensional spacetime. Note that only non-contractible loops will result in non-trivial Wilson lines. Hence, these local systems are labeled by assigning elements of the group G to such loops in such a way that it agrees with loop concatenation/ deformation! More formally, and modding out by gauge as well, the phase space for 2-manifold M is[11]

$$\mathrm{Loc}_G M := \{\pi_1(M) \to G\}/G.$$

Then we define a 2-dimensional extended TQFT $\mathcal{Z} : \mathrm{Bord}_2 \to \mathrm{Cat}_{\mathbb{C}}$ via linearizing these G-local systems. Hence $\mathcal{Z}$ assigns the following:

- For M a closed 2-manifold we have $\mathcal{Z}(M) = \sum_{\rho \in \mathrm{Loc}_G M} \frac{1}{|\mathrm{Aut}(\rho)|}$
- For N a 1-manifold we have $\mathcal{Z}(N) = \mathbb{C}[\mathrm{Loc}_G N]$. Hence, $\mathcal{Z}(S^1) = \mathbb{C}[\mathrm{Loc}_G S^1] = \mathbb{C}[G/G]$ are class functions.
- For P a zero manifold then $\mathcal{Z}(P) = \mathrm{Vect}(\mathrm{Loc}_G P)$ is a category. Hence, $\mathcal{Z}(\mathrm{pt}) = \mathrm{Vect}(\mathrm{pt}/G) = \mathrm{Rep}(G)$.

Finally, notice that a bordism M from N_1 to N_2 gives a diagram like

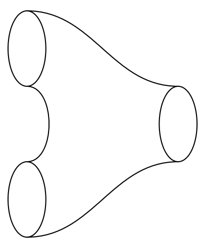
$$\begin{tikzcd} & \mathrm{Loc}_G M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathrm{Loc}_G N_1 & & \mathrm{Loc}_G N_2 \end{tikzcd}$$

from which we obtain a kind of pull-push map between the associated vector spaces $\pi_{2*}\pi_1^* : \mathbb{C}[\mathrm{Loc}_G N_1] \to \mathbb{C}[\mathrm{Loc}_G N_2]$ as $\mathcal{Z}(M)$.

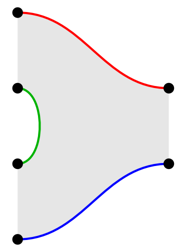
Boundaries and Interfaces in TQFTs

In a TQFT we often impose operators or defects. For example, we might consider a monopole charge inserted into our theory somewhere, or a line of

[11]Note that this is a groupoid with points given by local systems $\pi_1(M) \to G$ and morphisms the conjugations between them. Throughout we will abuse notation slightly to write $\cdot/G$ when modding out by conjugation. In particular, pt/G is (the stacky version of) the classifying space of G. As a groupoid this is a single object and automorphism group G. See the discussion below definition 3.5 for more details.



(a) The pair of pants is a morphism from $S^1 \times S^1 \to S^1$



(b) "Pair of chaps" bordism between interfaces. Boundary conditions are labeled by colors

Figure 1: Examples of bordisms in a TQFT (left), and with boundaries (right).

current. In our manifold these can be understood by removing a ball or tubular neighborhood around the defect and imposing some boundary conditions. The extremes of these boundary conditions are to require either trivializations or singularities along the defect, and the dualities we will discuss later on often exchange these two settings.

We need to describe how a theory can satisfy boundary conditions. Our first such notion may come as follows: Suppose we are studying fields on some spatial slice $\mathcal{F}(M)$, and that M has boundary $\partial M = N$. Then we get a map $\mathcal{F}(M) \xrightarrow{\pi_N} \mathcal{F}(N)$. We can describe imposing boundary conditions using sheaves, which we may think of as generalizations of fiber bundles where the fiber is no longer required to be constant.[12]

Definition 2.1 (Sketch of definition C.10.)**.** Let X be a topological space. A *sheaf* $\mathcal{F}$ of sets on X assigns a set $\mathcal{F}(U)$ to each open set in X such that restrictions compose well with each other. Further, sections satisfy a notion of gluing, and are defined locally.

We can use sheaves to impose boundary conditions "all at once" as Ben-Zvi put it: let $\mathcal{E}_N$ be any sheaf on N. Then we can pull back this sheaf to M to get $\pi_N^*\mathcal{E}_N$ as a sheaf on M. Then the vector space associated to M is the sections[13] $\mathfrak{H} = \Gamma(M, \pi_N^*\mathcal{E}_N)$.

More generally, we can consider interfaces between two (or more) TQFTs, say $\mathcal{Z}$ and $\mathcal{W}$, where pictorially we color manifolds with "red" and "blue" parts. Boundary conditions are equivalent to interfaces to the trivial theory.

[12]We can always get a sheaf of sections from a fiber bundle.

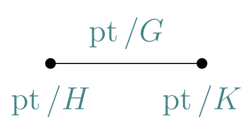
[13]More generally, for any map $\pi : X \to Y$ and $\mathcal{E} \in \mathrm{Shv}(Y)$ we get a modified vector space on X from boundary condition on Y.

Example 2.5. For 2-dimensional field theories, these interfaces correspond to maps $\mathcal{Z}(\mathrm{pt}) \to \mathcal{W}(\mathrm{pt})$. Hence, if $\mathcal{C} \ni C = \mathcal{Z}(\mathrm{pt})$ then trivial interfaces correspond to $\mathrm{Hom}(1, C)$, and being an object of Hom in a 2-category means these boundary theories form a category themselves. Physically, this is because there are interfaces between interfaces. In particular, given boundary conditions $\mathcal{B}, \mathcal{R}, \mathcal{G} \in \mathrm{Hom}(1, C)$, the collection of interfaces from $\mathcal{B}$ to $\mathcal{R}$ form a vector space, and the "pair of chaps" shown in Figure 1b gives a map $\mathrm{Hom}(\mathcal{B}, \mathcal{G}) \otimes \mathrm{Hom}(\mathcal{G}, \mathcal{R}) \to \mathrm{Hom}(\mathcal{B}, \mathcal{R})$.

A codimension k defect theory for $\mathcal{Z}$ means we extend $\mathcal{Z}$ from n-manifolds to n-manifolds with labeled codimension k submanifolds embedded. The same reasoning of interfaces having interfaces holds for larger n. In fact, an n-dimensional TQFT gives rise to an $(n-1)$-category of boundary theories with boundary conditions as the objects, 1-morphisms as interfaces between boundary conditions, and so on until $n-1$ [32].

Example 2.6. Recall $\mathcal{Z}(\mathrm{pt}) = \mathrm{Rep}(G)$. Hence, a representation V is a boundary condition, and we can extend $\mathcal{Z}$ to manifolds with boundaries marked by V. For example, a cylinder with one marked circle gives an element of $\mathcal{Z}(S^1) = \mathbb{C}[G/G]$ which is the character of V. In terms of fields, if we consider representations $\mathbb{C}[X]$ that come from some set with a G action $G \curvearrowright X$ then boundary conditions are specified by this G action!

Recall that a theory of G-bundles is a σ-model to pt/G, so specifying a boundary condition is giving some map $Y \to \mathrm{pt}/G$. Hence, on the boundary we will get a σ-model to some G-space Z such that $Y = Z/G$. As an example of maps between boundary conditions consider $Z_1 = G/H \implies Y_1 = \mathrm{pt}/H$ and $Z_2 = G/K \implies Y_2 = \mathrm{pt}/K$. Then an interface between these boundary conditions corresponds to the fiber product $H \setminus G/K$. Pictorially this is

$H\backslash G/K =$ 

. (4)

Composition of these interfaces, shown in figure 1b, gives us a *Hecke algebra* structure when all boundaries are reductions to H. An important case is $G \subset G \times G$ where $\mathcal{H}_{G \times G, G} = \mathbb{C}[G \setminus G \times G/G] = \mathbb{C}\left[\frac{G}{G}\right] = \mathcal{Z}(S^1)$, reducing $G \times G$ theory on the interval to G-theory on the circle! This is the commutative algebra of local operators in the G theory,[14] which acts

[14] Local operators are inserted at a point and are thus surrounded by S^{n-1} in an n-dimensional theory. Similarly line operators correspond to S^{n-2}.

on codimension 1 manifolds Σ via inserting S^1 into the "identity" bordism $\Sigma \times [0,1]$.[15]

Example 2.1. Later, we will be interested in another TQFT: Chern-Simons (CS) theory. This is a 3D gauge theory with classical phase space given by the moduli space of flat connections on the spacetime manifold, with action $S = \frac{k}{4\pi}\int_{M^3} \mathrm{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$ where A is the connection, and the constant k is called the *level* of the theory. For a compact, simple gauge group, the theory can be geometrically quantized, and the operators of interest are Wilson lines [34]. If G is finite this is 3D Dijkgraaf-Witten theory.

Of particular interest to us, the linking number of knots $\gamma_1, \gamma_2 \subset M$, $\mathrm{lk}(\gamma_1, \gamma_2) = \frac{1}{4\pi}\oint_{\gamma_1}\oint_{\gamma_2} \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} \cdot (d\vec{r}_1 \times d\vec{r}_2)$, arises in CS theory as follows. For simplicity take the gauge group $U(1)$, so $S = \frac{k}{4\pi}\int_M A \wedge dA$. Let $W_{\gamma_i} = \exp\left(i \int_{\gamma_i} A\right)$ be the Wilson line operators corresponding to the knots. Then

$$\langle W_{\gamma_1} W_{\gamma_2} \rangle = \exp\left(\frac{2\pi i}{k}\,\mathrm{lk}(\gamma_1, \gamma_2)\right). \tag{5}$$

2.3 A Twist in the Story: From SUSY to TQFTs

Now that we have a background on TQFTs, we need a way to obtain these theories. Witten introduced topological twisting as a mechanism for getting TQFTs from supersymmetric[16] (SUSY) theories [35]. Let us follow [36] and consider the simplest case of a SUSY theory: $\mathcal{N} = 1$ in $1D$. The Hilbert space is the super vector space of differential forms, and the symmetries form a super Lie group. Denote the fermionic supersymmetry operators by $Q := d$ and Q^*. Taking $H = \Delta$, the only non-trivial commutation relation is $[Q, Q^*] = H$. Thus, H acts on the Q cohomology by zero. We have "killed" time to get a TQFT! In higher dimensions there are more twists:

- **A-twist**: Let X be a Riemannian manifold with $\mathfrak{H} = \Omega^\bullet(X)$, $Q = d$, and $H = \Delta$. Then the twisted Hilbert space is $H^\bullet(\mathfrak{H}) = H^\bullet_{dR}(X)$.
- **B-twist**: For X a complex manifold, $\mathfrak{H} = \Omega^{0,\bullet}(X)$ with $Q = \bar{\partial}$ and $H = \Delta_{\bar{\partial}}$. Then the twisted Hilbert space becomes $H^\bullet(\mathfrak{H}) = H^{0,\bullet}(X)$.

[15] The giant tensor product action of $\mathcal{Z}(S^1)$ for every point in Σ factors through factorization homology [33].

[16] Theories where fields form a super space of even bosons and corresponding odd fermions.

In general, the philosophy described by Witten is to add fields until have an action of a big super-Lie group such that: (1) $H = [Q, -]$ is exact (killing time); and (2) the metric dependence is also exact (killing geometry) [37]. Formally, this is done using the BV formalism as described in appendix A.1.

Twisted Twins: Electric-Magnetic Duality

Consider Maxwell $S = \frac{g}{2\pi i}\int_{M\times\mathbb{R}} F \wedge \star F + \theta \int_{M\times\mathbb{R}} F \wedge F$. The space of fields is $U(1)$ bundles with connections modulo gauge transformations, $\mathcal{C}(M)$, so the Hilbert space is $\mathfrak{H} = L^2(\mathcal{C}(M^3))$. Notice $\mathcal{C}(M)$ is an abelian group that splits as $\Lambda \times T \times V$ where $\Lambda = H^2(M, \mathbb{Z})$ is the lattice of line bundles up to isomorphism,[17] $T = H^1(M, \mathbb{R})/H^1(M, \mathbb{Z})$ is the torus of flat connections, and we largely ignore vector space V of non-flat modes of the connection.

Following [36] we add matter fields to obtain an $\mathcal{N} = 4$ SUSY theory. Then we can pick some supercharge Q and take cohomology to get twists:[18]

- **A-twist:** We want cohomology equivariant with the gauge action, so
$$\mathfrak{H} = H^\bullet_{U(1)}(\mathcal{C}(M)) = H^\bullet(\mathrm{Pic}(M)) = H^\bullet(\Lambda \times T \times V \times B\mathrm{U}(1));$$
where $\mathrm{Pic}(M)$ is the underlying topological space of $U(1)$ line bundles with connection $\mathcal{C}(M)$. Notice $H^0(\mathrm{Pic}(M)) = \mathbb{C}[\pi_0(\mathcal{C}(M))] = \mathbb{C}\left[H^2(M, \mathbb{Z})\right]$ gives us the magnetic flux lattice that naturally couples to 't Hooft operators.

- **B-twist:** We define a $\mathbb{C}^\times$ connectn $\nabla + i\sigma = d + (A + i\sigma)$, where σ is an added Higgs field for the supersymmetry. The Hilbert space is
$$\mathfrak{H} = H^{0,\bullet}(\mathrm{Loc}_{\mathbb{C}^\times}(M)).$$
where $\mathrm{Loc}_{\mathbb{C}^\times}(M)$ is the moduli space of flat $\mathbb{C}^\times$-bundles, i.e., up to some derived factors, $\mathrm{Hom}(\pi_1(M), \mathbb{C}^\times) = \mathrm{Hom}(H_1(M), \mathbb{C}^\times) \cong T^\vee_{\mathbb{C}}$, the Pontryagin dual torus to the lattice Λ. This twist sees the algebraic geometry of $\mathrm{Loc}_{\mathbb{C}^\times}(M)$ and is the electric side of the theory. The corresponding operators of interest are Wilson lines, which are labeled by characters of $H_1(M)$.

Electric-magnetic duality exchanges these two twists. This exchanges Wilson line and 't Hooft operators with the line along a loop $\gamma \in H_1(M)$ being mapped to magnetic flux through $\Sigma \in H^2(M)$ via Poincare duality.

[17] These are labeled by Chern numbers.
[18] Linear combination of these also work, so we have a $\mathbb{CP}^1$ family of theories [32].

3 Powerful Primes: Math and Physics Meet

It is now time to meet number theory and its good friend algebraic geometry. The idea of arithmetic field theories is to use analogues to the concepts developed in TQFTs to study problems in number theory. In particular, while mathematicians understand abelian field extensions of number fields quite well due to the success of class field theory, non-abelian field extensions are much more difficult to study. Throughout this essay dualities play a key role, and this is no exception. The core idea of here is:

Theorem 3.1 (Artin-Verdier duality [38])**.** *For* $X = \operatorname{Spec} \mathcal{O}_K$ *the spectrum of the ring of integers of a number field*[19] K *and* $\mathcal{F}$ *a constructible étale abelian sheaf , there is a non-degenerate pairing*

$$H^r(X, \mathcal{F}) \times \operatorname{Ext}^{3-r}(\mathcal{F}, \mathbb{G}_m) \to H^3(X, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}. \tag{6}$$

Excitingly, this looks a lot like Poincaré duality for 3-manifolds! This should make us start thinking about CS theory, although we will need a bit of background to fully understand this statement and its implications.

3.1 The Who? What? and Why? of Number Theory

A core concept in number theory is fields and their extensions. For instance, we can understand solutions to polynomials by studying the extensions of $\mathbb{Q}$ given by adding their roots. We are then able to obtain many insights from the associated *Galois group*,[20] which describes automorphisms of the extended field that fix $\mathbb{Q}$. For example, adding the roots of the polynomial $x^2 + 1$ gives us $\mathbb{Q}(i) \subset \mathbb{C}$, and comes with a symmetry swapping i and $-i$. For a general field K, we denote the Galois group of field extensions L/K by $\operatorname{Gal}(L/K)$, and $\operatorname{Gal}(K)$ refers to the extension of K by its algebraic completion. See C.1 for more details.

Prime Time: Ramification

A natural first question is: what happens to primes in field extensions? For $\mathbb{Q}$, primes live in $\mathbb{Z}$. For a general number field K we consider prime ideals

[19] All number fields considered here will be totally imaginary.

[20] See definition C.1 for details.

in the *ring of integers,* $\mathcal{O}_K$, which contains all elements of K that are roots of monic polynomials in $\mathbb{Z}[x]$: $x^n + c_{n-1}x^{n-1} + ... + c_0$. For example, the ring of integers for $\mathbb{Q}(i)$ is $\mathbb{Z}[i]$ and the ring integers for $\mathbb{Q}(\sqrt{5})$ is $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$.

Now consider a degree[21] n extension L/K, with $\mathfrak{p} \subset \mathcal{O}_K$ a prime ideal. Is $\mathfrak{p} \cdot \mathcal{O}_L$ prime? In general, the answer is no. Instead primes in K split into a product of primes in L: $\mathfrak{p} \cdot \mathcal{O}_L = \mathfrak{p}_1^{e_1} \dots \mathfrak{p}_g^{e_g}$. We say $\mathfrak{p}$ *ramifies* if $e_i > 1$ for some i. Geometrically, we can imagine our extension corresponds to having a map from spaces $X_L \to X_K$ which is generically an n-fold cover, but has "singularities" where branches meet and ramification occurs.

For each j, the field $F = \mathcal{O}_K/\mathfrak{p}$[22] embeds into $F_j = \mathcal{O}_L/\mathfrak{p}_j$, giving *residue field extensions* F_j/F of degree f_j. Then we find $n = \sum_{j=1}^{g} e_j f_j$. This is useful because finite extensions of finite fields are of the form $\mathbb{F}_{q^m}/\mathbb{F}_q$ with $q = p^n$ a power of a prime, and $\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) = \langle \mathrm{Frob}_p^n \rangle$ where we define

Definition 3.1. The *Frobenius* automorphism for $\mathbb{F}_{q^m}/\mathbb{F}_q$ is $\mathrm{Frob}_q(x) := x^q$.

If we have a Galois extension then $\mathrm{Gal}(L/K)$ acts transitively on the $\mathfrak{p}_j$, and we find $\mathfrak{p} \cdot \mathcal{O}_L = \left(\prod_{j=1}^{g} \mathfrak{p}_j\right)^e$ and $f_j = f_i =: f$. Geometrically, we imagine this is similar to how the fundamental group acts on covers of a base space. The jth stabilizer subgroup D_j is known as the *decomposition subgroup* for $\mathfrak{p}_j$, and it contains another subgroup I_j called the *inertia subgroup* such that $D_j/I_j \cong \mathrm{Gal}(F_j/F) = \langle \mathrm{Frob}_{\mathfrak{p}_j} \rangle$. In the unramified case the inertia subgroups are trivial and we can identify $\mathrm{Frob}_{\mathfrak{p}_j} \in D_j \subset \mathrm{Gal}(L/K)$ giving a Frobenius element for the prime. In fact, $\mathrm{Frob}_{\mathfrak{p}_{j_1}}$ is congruent to $\mathrm{Frob}_{\mathfrak{p}_{j_2}}$, so in the abelian setting this is independent of j and we simply have $\mathrm{Frob}_{\mathfrak{p}}$!

What We Know and How We Know It

A major goal in number theory is to understand the Galois groups. This is well understood for abelian groups via class field theory (see [39] for a standard reference). However, the non-abelian setting is much more complicated, and is largely the point of the Langlands program! As we understand groups through their representations, we might say we understand 1-dimensional representations of Galois groups while the Langlands program seeks to understand higher dimensional representations. Class field theory comes in two

[21]The degree of L over K is the dimension of L as a vector space over K.

[22]If $K = \mathbb{Q}$ this would be $\mathbb{F}_p$.

flavors: global and local, where the global theory is often proven using the local case. Here *global* means number fields,[23] whereas *local* means the completion of a field with respect to some metric obtained from a prime. For $\mathbb{Q}$, this gives us the p-adic numbers.

Definition 3.2. For a prime $p \in \mathbb{Z}$ we define the p-adic absolute value via $|\frac{a}{b}p^k|_p = p^{-k}$ for a, b coprime to p. The *p-adic numbers*, denoted $\mathbb{Q}_p$, are the completion of $\mathbb{Q}$ with respect to the corresponding metric. This concept can be generalized to an arbitrary number field K with prime ideal $\mathfrak{p} \in \mathcal{O}_K$ to obtain the local field $K_\mathfrak{p}$. If we consider the "prime" ∞, then $|\cdot|_\infty = |\cdot|$ the usual absolute value, and the completion of $\mathbb{Q}$ is $\mathbb{R}$. For a general K we may have "primes" dividing ∞ and the completion is $\mathbb{R}$ or $\mathbb{C}$.

For $\mathfrak{p} \nmid \infty$ the *ring of integers* is $\mathcal{O}_{K_\mathfrak{p}} = \{a \in K_\mathfrak{p} : |a| \leq 1\}$. In particular, the p-adic integers are $\mathbb{Z}_p$. These are also the completion of $\mathbb{Z}$ in this metric.[24]

Hence, in the p-adic setting we say that if $p^n|(a-b)$ for large n then a and b are close to each other. Local fields K obtained via "finite primes" have a unique prime ideal, $\mathfrak{m}$, given by the open unit ball. This gives a finite residue field $k = \mathcal{O}_K/\mathfrak{m}$. Hence, local fields are often more tractable than global fields. While many of our motivating questions are in the global field setting, local fields are important to us because of the *Hasse principle*. This is the idea that one can often find an integer solution to an equation by piecing together solutions modulo powers of each different prime number. Hence, studying the solutions to a problem in all localizations of a field is often equivalent to solving it in the original number field.

The adeles and ideles allow us to consider all of the completions at once. Let K be a global field. The *adele ring* of K is $\mathbb{A}_K = \prod'_\nu K_\nu \ni (a_\nu)$ where $a_\nu \in K_\nu$ and the $'$ indicates that for all but finitely many ν we have $a_\nu \in \mathcal{O}_{K_\nu} \subset K_\nu$ making the adeles locally compact so we can do analysis. Similarly, the *idele group* of K is the topological group[25] of units $I_K := \mathbb{A}_K{}^\times$.

L-functions

Now that we have Cauchy completed our fields into local fields, we can use the tools of analysis! In particular, L-functions play an important role in

[23]Later this will also include function fields of a curve over a finite field.

[24]Notice that $\mathbb{Z}$ may have Cauchy sequences in this metric, unlike in the usual setting where $\mathbb{Z}$ is considered discrete.

[25]The topology is not the subspace topology from the adeles.

various aspects of number theory including the development of class field theory where they were used to prove the "second fundamental inequality," as well as in current work on the Langlands program. We recommend [40] as a nice review of their use. However, much about L-functions is currently conjectural. For example, the most famous L-function is the Riemann zeta function ζ, the subject of the Riemann hypothesis.

Our interest will be in L-functions constructed from the action of some endomorphism on a vector space: $F \circlearrowright V$. In particular, if V is a super vector space we can consider a supersymmetric version of the characteristic polynomial for F. Namely, for a cochain complex $V^\bullet$ we define

$$\mathrm{L}(V, F, t) = \prod_k \det\left((1 - tF)|_{H^k(V)}\right)^{(-1)^{k+1}} = \mathrm{Tr}_{\mathrm{gr}}\left(F, \mathrm{Sym}^\bullet V\right) \tag{7}$$

where the graded symmetric algebra $\mathrm{Sym}^\bullet$ acts as the symmetric algebra for even parts and as the exterior algebra $\wedge$ for odd parts of $V^\bullet$.

Remark 3.1. *The naturality of the product and inverse product structure can be seen in the 1-dimensional setting $F(x) = \lambda x$. Then $\mathrm{Sym}\,\mathbb{C} = \mathbb{C}[x] \implies \mathrm{Tr}_{\mathrm{Sym}\,\mathbb{C}} F = 1 + \lambda + \lambda^2 + \ldots = \frac{1}{1-\lambda}$, while $\wedge\mathbb{C} = \mathbb{C}_0 \oplus \mathbb{C}_1 \implies \mathrm{Tr}_{\wedge\mathbb{C}} F = 1 - \lambda$.*

We can use L-functions to study Galois representations. In particular, let us consider a Frobenius element $\mathrm{Frob}_p \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acting on a representation V. Then, given a global field K with $\mathrm{Gal}(K) \xrightarrow{\rho} G \xrightarrow{V} GL(V)$ for some group homomorphism ρ to group G, we obtain the global L-function as a product of the local ones.[26] $\mathrm{L}(\rho, V, t) = \prod_p \mathrm{L}_p\left(V \circlearrowleft \rho(\mathrm{Frob}_p), t\right)$

3.2 Schemes Behind the Scenes: Algebraic Geometry

To understand the intriguing world of number theory, we now introduce some of the main tools drawn from algebraic geometry. We will need them for two main purposes: first, it is often easiest to first study problems in the setting of function fields, and these are inherently geometric as their definitions come from algebraic varieties; and second, even in the number field setting the main ideas of this work are to use more topological notions to understand these problems. Hence, we must understand how to obtain geometric objects from number-theoretic data. Our exposition largely follows [41].

[26] If ramification occurs we must be a bit more careful. Namely, we restrict to invariants of the inertia .

We need a map to pass between algebraic and geometric data. This is similar to the duality stories from before. We first see this story as follows: An affine n space over a field k, $\mathbb{A}^n$, is all n-tuples of elements of k. Similarly, we may define projective space $\mathbb{P}^{n-1}$ to be n-tuples up to scaling.

- **Algebra to geometry:** For $T \subset k[x_1, ..., x_n]$ we define the *zero set* of T to be $Z(T) = \{P \in \mathbb{A}^n | f(P) = 0 \, \forall f \in T\}$.
- **Geometry to algebra:** Given a subset of (projective) affine space, Y, we define the *ideal of* Y, $I(Y) \subset k[x_1, ..., x_n]$, to be generated by all (homogeneous) polynomials that vanish on Y.

Hence, we can study a space through the functions on it, and understand a ring via the corresponding space. Note that the direction of maps is reversed when changing between the geometric and algebraic settings.

Example 3.1. You are actually already very used to working with zero sets!

- Let $k = \mathbb{R}$ and consider $f(x, y) = x^2 + y^2$. Then $Z(f)$ is the unit circle.
- Let $k = \mathbb{R}$ and consider $f(x, y) = y - x$ and $g(x, y) = x + y - 2$. Then $Z(\{f, g\})$ is the intersection of the two lines at $(1, 1)$.
- Let $k = \mathbb{F}_3$. Then $Z(f(x) = x^2 + 2) = \{1, 2\}$.
- Let $k = \mathbb{C}$. Then $\mathbb{P}^1 = \mathbb{CP}^1 = Z(0)$ is the Riemann sphere. Further if $f(x, y) = x - y$ then $Z(f)$ is a "circle" on the 2-manifold.

Definition 3.3. If Y is an algebraic set in $\mathbb{A}^n$ or $\mathbb{P}^{n-1}$, then $k[x_1, ...x_n]/I(Y)$ is called the *coordinate ring*.

Given this, the Zariski topology on affine space has a basis of open sets given by complements of sets $Z(T)$. A subset is called *irreducible* if it is nonempty and cannot be written as the union of two proper closed subsets. An *affine variety* is an irreducible closed subset of affine space. A *projective variety* is defined analogously, using the Zariski topology on projective space and homogeneous polynomials.

We will be interested in *curves*, i.e., projective varieties of dimension 1 over the relevant field. For example, Riemann surfaces like the torus are curves over $\mathbb{C}$. We denote a curve over k via C/k. Given such a curve, we define the *function field* for a field extension K/k to be $K(C)$, the field of fractions of the coordinate ring of any non-empty affine open subset of C.

Elliptic Curves

An elliptic curve is a nonsingular curve of genus 1 with a specified basepoint. Hence, when considered over $\mathbb{C}$ these are homeomorphic to a torus. However, we are interested in their algebraic structure in addition to topology, and our main use of elliptic curves is for curves over finite fields. Elliptic curves over any field have the equation $Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$. We denote by $O = [0, 1, 0]$ the point at infinity in $\mathbb{P}^2$. Elliptic curves are an example of an *abelian* variety. In particular, this means that the points $C(k)$ form an abelian group with identity O and group law described in Figure 2. See [42] for more details.

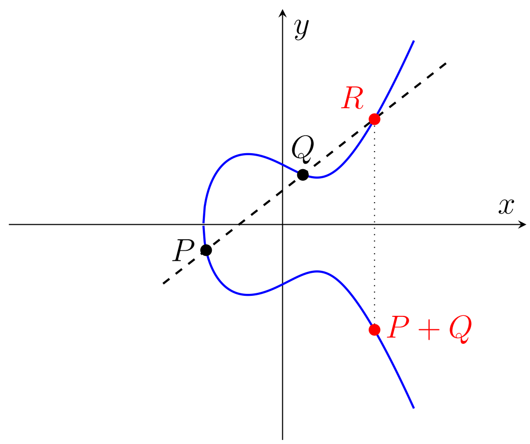


Figure 2: Group law on an elliptic curve. The axes are labeled with non-homogeneous coordinates $x = X/Z$ and $y = Y/Z$.

Since this is an abelian group, we will denote the group operation with $+$ and the inverse of a point P by $-P$. Then for $m \in \mathbb{Z}$ we define $[m]P = \overbrace{P + \cdots + P}^{m \text{ terms if } m>0}$, $[m]P = \overbrace{-P - \cdots - P}^{|m| \text{ terms if } m<0}$, $[0]P = O$. Over finite fields it is known that the m-torsion subgroups satisfy $C[m] \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$.

Curves have an action of the Frobenius endomorphism taking points $(x, y) \in C(\overline{\mathbb{F}}_q)$ to $\mathrm{Frob}_q : (x, y) \mapsto (x^q, y^q)$. The $\mathbb{F}_q$ rational points are precisely the fixed points of this automorphism.[27] This turns into an action of Frobenius on the (étale) ℓ-adic cohomology of $\overline{C}$. The action on the zeroth and second cohomology groups is always identity and multiplication by

[27] For higher genus curves more coordinates are required but the idea is the same.

q, while the action on the first cohomology group is given by a 2×2 matrix.[28] Given such an action of Frobenius, we can consider L-functions.

We define the Zeta function for the elliptic curve $C/\mathbb{F}_q$ to be

$$\zeta_C(T) = \exp\left(\sum_{n=1}^{\infty} \#C(\mathbb{F}_{q^n})\frac{T^n}{n}\right). \tag{8}$$

Then, Grothendieck's trace formula[29], tells us that $\#C(\mathbb{F}_q) = \sum(-1)^i \operatorname{Tr}_{H^i(\overline{C},\mathbb{Q}_\ell)} \operatorname{Frob}_q = 1 - \operatorname{Tr}\operatorname{Frob}_q + q$, Thus, if α and β are the eigenvalues of Frob_q then we have $\#C(\mathbb{F}_{q^n}) = 1 - \alpha^n - \beta^n + q^n$. In particular, we know $\beta = \bar{\alpha}$, and the Weil conjectures [43], proven by Deligne in [44, 45] and by Hasse [46] for the case of elliptic curves, give us $|\alpha| = \sqrt{q}$. Applying this to equation 8 gives

$$\zeta_C(T) = \frac{1 - \alpha T - \beta T + qT^2}{(1 - qT)(1 - T)}. \tag{9}$$

Higher genus curves do not have a group structure. Hence, we generalize these properties of elliptic curves with the following definition:

Definition 3.4. The *Jacobian variety* $\operatorname{Jac}(C)$ of a curve C/k is an abelian variety whose k-points are the degree-zero line bundles on C. See appendix C.6 for more details.

Geometry from Number Fields

The theme of going between geometry and algebra can be greatly generalized using the concept of *schemes*. The story, told in [41], is as follows:

- **Algebra to geometry:** Given a commutative ring R, we obtain the *spectrum of the ring*, $\operatorname{Spec}(R)$, as the set of prime ideals. It has the Zariski topology with basis of open sets $\{D_f\}_{f \in R}$ where D_f is the set of prime ideals not containing f.

- **Geometry to algebra:** Given $X = \operatorname{Spec}(R)$ with Zariski topology, the *structure sheaf* $\mathscr{O}_X$ is defined on basis sets by $\Gamma(D_f, \mathscr{O}_X) = R_f = R[f^{-1}]$.[30] Note that $\mathscr{O}_X(X) = R$.

[28] For a general curve of genus g it will be a $2g \times 2g$ matrix.
[29] A general result for schemes .
[30] Notice if $R = k[x]$ then D_f is avoiding where f vanishes so f^{-1} is well defined.

The canonical example which will be explored later is R the ring of integers for some number field. Here is another example connecting back to TQFTs.

Example 3.2. Recall that the space of local operators in an n-dimensional TQFT $\mathcal{Z}$ is $\mathcal{Z}(S^{n-1})$. This has an algebra structure via (the n-dimensional analogue of) the "pair of pants bordism" in Figure 1a, which is commutative for $n \geq 2$. Hence, $\mathcal{M}^0_{\mathcal{Z}} := \mathrm{Spec}(\mathcal{Z}(S^{n-1}))$ is well defined. This is the *moduli space of vacua* of the theory via the operator-state correspondence.

The structure sheaf makes $\mathrm{Spec}\, R$ into a *locally ringed space*. We call any ringed space that is isomorphic to $\mathrm{Spec}\, R$ for some R an *affine scheme*. The story continues by gluing, similarly to how manifolds are built from $\mathbb{R}^n$.

Definition 3.5. A *scheme* is a locally ringed space X that has a covering by open sets U_i such that each U_i is an affine scheme.

We will find it convenient to take another viewpoint on schemes: the "functor of points" approach. We say a scheme X is *over* a scheme Y if there is a morphism [31] $X \to Y$. We say X is over a ring R if it has a morphism $X \to \mathrm{Spec}\, R$. Then the set of R-points $X(R)$ consists of sections of this morphism, i.e., morphisms $\mathrm{Spec}\, R \to X$. A scheme X over R is fully determined by its *functor of points* taking schemes $S \mapsto X(S) = \mathrm{Hom}(S, X) \in \mathit{Set}$.

Using this perspective, we define a *group scheme* to be an R-scheme G such that its functor of points is valued in groups. For example Artin-Verdier duality used the multiplicative group scheme $\mathbb{G}_m : S \mapsto S^\times$.

Finally, a *stack* generalizes the idea of a functor of points to be category rather than set valued. In particular, *algebraic stacks* take spaces to groupoids. This is useful when we want to keep track of automorphisms. For example, if X is a G-space for some group G, then the quotient stack $[X/G]$ remembers the orbits in addition to the stabilizers, enhancing the base topological space $|X/G|$.

Étale Cohomology

Now we have the tools to understand Theorem 6. Given some number field K, we considered the space $X = \mathrm{Spec}\, \mathcal{O}_K$. Then we appliead some cohomology using a sheaf $\mathcal{F}$. In particular, we make use of *étale* cohomology which we will only briefly describe here. A standard reference is [47].

[31] Note that morphisms $X \to Y$ take $\mathscr{O}_Y \to \mathscr{O}_X$.

The basic idea of sheaf cohomology is to understand the obstructions to local solutions extending to global solutions. Sheaf cohomology on a space X generalizes other cohomology theories. For example, if X is a "nice" space, then singular cohomology with coefficients A is given via the locally constant sheaf $\underline{A}$, and de Rham cohomology is given by the sheaf of differential forms $\Omega^\bullet$. More generally, sheaf cohomology is a functor from sheaves of abelian groups to abelian groups accounting for the lack of $\mathcal{F} \to \mathcal{F}(X) = \Gamma(X, \mathcal{F})$ being exact.

We can apply sheaf cohomology to a scheme X. However, it is often informative to use the *étale topology* rather than the coarser Zariski topology, as it sees more details. The intuition of this topology is that it plays well with morphisms of schemes. In particular the etale fundamental group is $\pi_1(X) = \varprojlim_Y \operatorname{Aut}(Y/X)$ for (nice[32]) covers Y of X. This matches with the fundamental group of a usual topological space acting on covering spaces, where $\pi_1(X)$ is the deck of transformations of the universal cover of X. Thus, since field extensions give us covers we get the following examples.

Example 3.3. (Examples taken from [48]).

- If X is a complex variety then the étale fundamental group is the profinite completion of the topological one.
- If $X = \operatorname{Spec} K$ then $\pi_1(X) \cong \operatorname{Gal}(K^{\text{sep}}/K)$.[33]
- For $X = \operatorname{Spec} \mathcal{O}_K$ then $\pi_1^{\text{ab}}(X)$ is the ideal class group of K. However, $\pi_1(X) \cong \varprojlim_L \operatorname{Gal}(L/K)$ for L unramified extensions.

3.3 Linking it all Together: Number Theory Meets Physics

Now that most of the main characters on both the number theory and physics side have been introduced, the plot begins to thicken. We can define "arithmetic field theories" where the usual manifolds of a TQFT are replaced by schemes and curves over finite fields. It is natural to consider analogues of 3D theories such as Chern-Simons because Artin-Verdier duality matches with Poincaré duality for 3-manifolds. In fact, we will soon see that this analogy allows us to treat prime numbers like knots, and we can even discuss "linking

[32] If X is connected Y should be a connected Galois finite étale cover of X.

[33] This contrasts the Zariski topology where $\operatorname{Spec} K$ is trivial.

numbers" of primes! Kim provides a good overview of arithmetic field theories in [49]. The main ideas of arithmetic Chern-Simons theory are outlined for both number fields and function fields below.

Number Fields

The scene of our story is $X = \operatorname{Spec}\mathcal{O}_K$, for some number field K. We wish to treat X as a "3-manifold" because of the analogy between Artin-Verdier duality and Poincaré duality [38]. The analogy is furthered by the observation that $\mathfrak{p}$ corresponds to the residue field $\mathcal{O}_K/\mathfrak{p}$, and $\pi_1(\operatorname{Spec}\mathbb{F}_q) \cong \hat{\mathbb{Z}}$ for any finite field.[34] Hence, primes "act like" S^1.

Now, given a set of primes S, we can make this "manifold" more interesting by removing "knots" to obtain the scheme $X_S = \operatorname{Spec}(\mathcal{O}_K[1/S]) = X \setminus \{\mathfrak{p}_\nu\}_{\nu \in S}$. Homotopically, we remove handle bodies surrounding primes; giving rise to "surface tori" at the boundary. The analogy is as follows:

$$\begin{array}{c} S^1 \times S^1 \cong \text{boundary} \hookrightarrow \text{handle body} \twoheadrightarrow \text{knot} \\ \updownarrow \\ \operatorname{Spec} K_{\mathfrak{p}} \hookrightarrow \operatorname{Spec}\mathcal{O}_{K_{\mathfrak{p}}} \twoheadrightarrow \operatorname{Spec}\left(\mathcal{O}_{K_{\mathfrak{p}}}/\mathfrak{p}\right). \end{array}$$

The kernel of the induced homomorphism $\pi_1(\operatorname{Spec} K_{\mathfrak{p}}) \to \pi_1(\operatorname{Spec}\mathcal{O}_{K_{\mathfrak{p}}})$ is the inertia group, which we will think of as bing the "meridian" around the torus.[35] The inverse image of $\operatorname{Frob} \in \pi_1(\operatorname{Spec}\left(\mathcal{O}_{K_{\mathfrak{p}}}/\mathfrak{p}\right))$ is the "longitude."

Hence, we can define an arithmetic version of CS theory, as introduced by Kim in [50, 51], and we wish to compute linking numbers as in example 2.1. Morishita gave a nice interpretation of $\mathbb{Z}/2\mathbb{Z}$ linking numbers of primes as Legendre symbols (definition C.3), which we sketch below:

Theorem 3.2 ([52] Section 2.)**.** *For primes $p, q \in K = \mathbb{Q}$ we have (see definition C.3)*

$$(-1)^{\mathrm{lk}_2(p,q)} = \left(\frac{p}{q}\right). \tag{10}$$

[34] Recall the fundamental group is the Galois group of the finite field $\mathcal{O}_{K_p}/p$.

[35] The maximal tame quotient of the inertia group is topologically generated by a monodromy τ such that $\pi_1^{\mathrm{tame}}(\operatorname{Spec} K_{\mathfrak{p}})$ is topologically generated by τ and $\operatorname{Frob}_{\mathfrak{p}}$.

Sketch. First, $\mathbb{Q}(\sqrt{p})$ is unramified outside of p if and only if $p \equiv 1 \mod 4$. So, for simplicity assume we are in the setting that $p, q \equiv 1 \mod 4$.[36] Now, take the unique étale double covering Y_p of X_p.[37] Then by definition,

$$\pi_1(X_p) \to \mathrm{Gal}(Y_p/X_p) = \mathbb{Z}/2\mathbb{Z}, \qquad [\mathrm{Frob}_q] \mapsto \mathrm{lk}_2(p, q).$$

Now consider $\mathrm{Frob}_q|_{Y_p}$. This will be $id_{Y_p} \iff \mathrm{Frob}_q(\sqrt{p}) = \sqrt{p} \iff p$ is a quadratic residue $\mod q$.

Kim's arithmetic CS theory with gauge group $G = \mathbb{Z}/n\mathbb{Z}$ generalizes this construction to a $\mathbb{Z}/n\mathbb{Z}$ "linking number" of primes [53]. Intuitively, the $\mathbb{Z}/2\mathbb{Z}$ linking number comes from quadratic reciprocity, and $\mathbb{Z}/n\mathbb{Z}$ comes from the more general Artin reciprocity. See appendix B for arithmetic CS theories.

Function Fields

The above theory can be generalized to the function field setting. Let $C/\mathbb{F}_r$ be a curve for $r = p^n$ some power of a prime. In this setting C plays the role of the three manifold, and closed points in $x \in C$ look like "knots."

There are now two ways to obtain "surfaces." First, we can take the tubular neighborhood around a "knot" as in the number field case, and we will find the completed field looks like $K(C)_x \cong K((t))$.[38] In this setting we were able to define a function field version of Kim's arithmetic CS and verify an analogous statement about linking numbers of closed points in C.

Second, instead of completing with respect to a "prime" to obtain a surface, we instead take the algebraic closure of the finite field $\mathbb{F}_r$ to $\overline{\mathbb{F}}_r$. This produces $\overline{C}$ with an action of the Frobenius for $\mathbb{F}_r$ on it. While these "surfaces" do not have a notion of linking numbers, they allow us view our "3-manifold" C as globally fibered over "S^1," something we will make use in chapter 5. In both cases the idea is to obtain a surface via replacing a "circle" with its "universal cover," a homotopically trivial space.

[36]Another definition in terms of "Seifert surfaces" using cup products works more generally.

[37]Y_p is the spectrum of elements of $\mathbb{Q}(\sqrt{p})$ integral over $\mathbb{Z}\left[\frac{1}{p}\right]$.

[38]The analogue of the p-adics in the function field setting.

4 Here be Dragons: The Geometric Langlands Program via Physics

Now all the main characters have been introduced and they even had the chance to get some practice working together on a side quest through computing linking numbers of primes in arithmetic field theory. Hence, it is time for the ideas from math and physics to join forces to explore the exciting world of the geometric Langlands program through electric-magnetic duality. As with most journeys one can only start at the beginning. In our case this is understanding $\mathcal{N} = 4$ supersymmetric Yang Mills theory. From there, our heroes must traverse the treacherous terrain of topological twists, reducing to boundary theories, and understanding deep dualities. This is a story that began with [8], and the intuition from such approaches influenced further work on the geometric Langlands program and culminated in a proof [54–58]. The possibility of success in our journey requires the combined strengths of our various characters. In particular, it relies upon techniques for finding the "commutative piece" of theories using Hecke algebras, developed for TQFTs, applied to number theoretic problems when Galois groups are no longer abelian. This chapter largely follows a combination of the expositions found in [32, 36, 59]. However, we will focus specifically on the function field setting where our "3-manifolds" are curves $X/\mathbb{F}_q$.[39] If you do not wish to look for the dragon in full generality you may skip to section 4.3 to see the story in the specific case of partition functions as L-functions.

4.1 Finding the Dragon: What is Geometric Langlands?

To set the scene, we consider 4-dimensional $\mathcal{N} = 4$ super Yang-Mills theory for some group G. In particular, we do not assume this group is abelian, unlike for Maxwell theory. Just as in the Maxwell setting, we can consider A- and B-twists of this theory. We then hope to recover something analogous to electric-magnetic duality. The main point is this: the abelian theory described in Section 2.3 had the A and B sides corresponding to Pic and Loc. Everything worked well because we have many dualities for abelian groups, and Pic has a group structure via tensor products. When we now consider a non-abelian gauge group, the local systems picture survives but Pic breaks

[39] Replace $\mathbb{F}_q$ with $\mathbb{C}$ to recover actual physical theories, or with number fields for the arithmetic theory.

down. Instead we end up building the spaces Bun_G as a stack of principal G bundles on a "3-manifold" This has a Hecke algebra type structure:

$$|\,\mathrm{Bun}_G(X)(k)| = G(K) \setminus G(\mathbb{A}_X)/G(\mathcal{O}_{\mathbb{A}_X}) \tag{11}$$

where $K = k(X)$, and we think of $G(\mathbb{A}_X) = \prod'_{x\in X} G(K_x)$ as an analogous product to defining the adeles. The idea is the bundle is given by assigning elements of G to every point and noticing that we can trivialize everywhere except finitely many points, and then we quotient out on the left by the choice of trivializations and on the right by the choice of "rational" section. Then our quantum theory studies functions on $\mathrm{Bun}_G(X)$.

Now consider this from the math perspective. Number theorists are also interested in understanding $\mathrm{Bun}_G(X)$[40]. We study such spaces via the functions defined on them. The initial "functions" that we might consider are (sections of) the category $\mathrm{QCoh}_{\,\mathrm{Bun}_G(X)}$ of quasicoherent (think "nice") sheaves on $\mathrm{Bun}_G(X)$. However, we want to consider a derived version of everything, so instead we might consider $\mathrm{DMod}\,\mathrm{Bun}_G(X)$, the D-modules on X, which we can think of as a sheaf with connection. Hence our goal is more specifically to understand $\mathrm{DMod}\,\mathrm{Bun}_G(X)$ via a categorified version of spectral decomposition: finding some space Y such that $\mathrm{DMod}\,\mathrm{Bun}_G(X)\text{“}=\text{”}\,\mathrm{QCoh}(Y)$.

Geometric Satake

In order to understand these "functions" we consider symmetries on them. In particular, for $x \in X$[41] we have point modifications given by doubling this point and adding x' to obtain $X' = X \cup \{x'\}$ so bundles on X' are $\mathrm{Bun}_G(X') = \mathrm{Bun}_G(X) \times_{\mathrm{Bun}_G X\setminus x} \mathrm{Bun}_G(X)$. This space now has a monoidal structure via convolution.[42]

We can understand this more easily by looking at only the part local to x. Namely the formal disk D_x. We call the formal disk with a doubled center, D'_x, a "ravioli." Notice that ravioli look topologically like $S^2 = S^{4-2}$, and thus correspond to line operators in our 4D theories. Hence, we now define:

Definition 4.1. $\mathcal{H}_x := \mathrm{DMod}\,\mathrm{Bun}_G(D'_x)$ is called the *Hecke category.*

[40] This corresponds to studying automorphic forms

[41] Recall points correspond to "knots" and hence line operators here.

[42] Think of how functions on $\{1,...,n\} \times \{1,...,n\}$ with convolution are the same as matrix multiplication.

In example 2.6 we viewed Hecke algebras as encoding actions on $H \leq G$-invariants for all G representations. Hence, they are characterized by the double-coset structure $\mathrm{Fun}(H \setminus G/H)$. In our setting we find:

$$\mathcal{H}_x = \mathrm{DMod}\left(G\left(\mathcal{O}_x\right) \backslash G\left(K_x\right)/G\left(\mathcal{O}_x\right)\right).$$

Then the action of $\mathcal{H}_x$ on $\mathrm{DMod}\,\mathrm{Bun}_G(X)$ is the categorified version of the Hecke algebra action on H invariants of G representations because

$$\mathrm{DModBun}_G(X) = \mathrm{DMod}\left(G(K)\backslash \prod_{y\neq x}{}' G\left(K_y\right)/G\left(\mathcal{O}_y\right) \times G\left(K_x\right)\right)^{G(\mathcal{O}_x)}$$

and this is $G(\mathcal{O}_x)$ invariants of a $G(K_x)$ representation.

The geometric Satake correspondence categorifies the idea that there is some *Langlands dual group* $G^\vee$ such that $\{G(\mathcal{O}_x)\text{-orbits}\} \leftrightarrow \{\text{irreps of } G^\vee\}$:

Theorem 4.1 (Geometric Satake [60]). *There is a monoidal equivalence*

$$\mathcal{H}_x = \mathrm{Rep}\, G^\vee. \tag{12}$$

From a physics point of view this gives an identification between line operators in these 4-dimensional A and B theories denoted $\mathcal{A}_G$ and $\mathcal{B}_{G^\vee}$.

$$\begin{gathered}\mathcal{A}_G\left(S^2\right) = \left(\mathrm{DMod}_{G(\mathcal{O}_x)}\left(G(K_x)/G(\mathcal{O}_x)\right), *\right) = \text{'t Hooft line operators} \\ \updownarrow \\ \mathcal{B}_{G^\vee}\left(S^2\right) = \mathrm{QCoh}\left(\mathrm{Loc}_{G^\vee} S^2\right) = (\mathrm{Rep}(G^\vee), \otimes) = \text{Wilson line operators.}\end{gathered} \tag{13}$$

The idea behind this is that generalizations of the "pair of pants" bordism define a product $*$ for our line operators via colliding points in X. Further, the B-theory is a kind of "affinization" of the A theory. Finally, recall $\mathrm{Loc}_{G^\vee}(X) = \{\mathrm{Gal}(K) \to G^\vee\}/G^\vee$ so this is also studying representations of the Galois group.

Langlands at Last!

Now we want to spectrally decompose $\mathrm{DMod}\,\mathrm{Bun}_G(X)$ as something like $\mathrm{QCoh}(Y)$ so Y is some space of "eigenvectors". In the A-theory, the action of inserting line operators at $x \in X$ is $\mathcal{H} = \bigotimes_{x\in X} \mathcal{H}_x$,[43] in a manner similar to

[43]Or actually, the factorization homology .

the action of local operators described in the footnote at the end of example 2.6.

Notice that for every $x \in X$ we have a map

$$\operatorname{Loc}_{G^\vee}(X) \to \frac{G^\vee}{G^\vee},\ \rho \mapsto \rho(\operatorname{Frob}_x) \tag{14}$$

for the Frobenius of x, Frob_x. Further, we recall from Example 2.6 that $\mathcal{O}\left(\frac{G^\vee}{G^\vee}\right) = |\operatorname{Rep} G^\vee|$. Hence, we obtain an action of $\mathcal{H}$ on $\mathcal{O}\left(\operatorname{Loc}_{G^\vee}(X)\right)$, and we might hope that $Y = \operatorname{Loc}_{G^\vee}(X)$.

Conjecture 4.1 (Naive Geometric Langlands Correspondence)**.** *There is an equivalence commuting with the action of the Hecke operators:*

$$\operatorname{DMod}\operatorname{Bun}_G(X) \simeq \operatorname{QCoh}\operatorname{Loc}_{G^\vee}(X) \tag{15}$$

The left-hand-side is the "automorphic side" since functions on Bun_G correspond to automorphic forms, while the right-hand-side is the "Galois side" since $\operatorname{Loc}_{G^\vee}$ corresponds to Galois representations. Physically this is a correspondence between the A and B-theories for G and $G^\vee$ via a generalization of electric-magnetic duality!

Remark 4.1. *As given the conjecture is false. We should take everything derived and address singularities properly. See C.1 for a corrected version.*

4.2 Catching the Dragon: Boundary Theories

Now that we have glimpsed the mythical beast known as the geometric Langlands program, we wish to capture it for closer study. We will do this by considering boundary theories from codimension one defects in our A and B theories [61]. Let us follow Ben-Zvi's notes [36] for this.

Our motivation is as follows: Langlands functoriality asks us to understand what happens on the A-side of this diagram:

$$\begin{tikzcd}
\mathcal{B}_{H^\vee} \arrow[r] \arrow[d, leftrightsquigarrow] & \mathcal{B}_{G^\vee} \arrow[d, leftrightsquigarrow] \\
\mathcal{A}_H \arrow[r, dashed, "?"] & \mathcal{A}_G
\end{tikzcd}$$

In particular, a group homomorphism $H^\vee \to G^\vee$ does not necessarily give $H \to G$, but electric-magnetic duality leads us to want some operation going from automorphic forms for H to automorphic forms for G.

Recall that in Example 2.4 we saw bordisms between boundaries give a kind of "pull-push" structure inducing maps on the associated theories. A similar concept can be applied here for a domain wall between theories Z and W. Recall further, as discussed at the beginning of Section 2.2, that defects correspond to interfaces or boundary conditions. Hence, instead of the domain wall, we will consider a boundary condition for the single theory $Z \otimes W$. Combining all of this, we find a correspondence from the $\mathcal{B}_{G^\vee}$ theory to the $\mathcal{A}_G$ theory using the "pull-push" map.[44]

$$\begin{tikzcd}
& B^\vee \arrow[dl, hook'] \arrow[dr, two heads] & & & \mathrm{pt}/B \arrow[dl] \arrow[dr] & \\
G^\vee & & T^\vee \arrow[r, Rightarrow, dashed] & \mathrm{pt}/G & & \mathrm{pt}/T
\end{tikzcd}$$

The upshot of this is that, as in example 2.6, boundary theories for $\mathcal{B}_{G^\vee}$ give rise to a (shifted) Hamiltonian $G^\vee$ space. In particular, our boundary theory will be a σ-model with this $G^\vee$ space the target. Given a boundary condition $G \circlearrowright M$ for the $\mathcal{A}_G$ theory we can build a Hamiltonian $G^\vee$-space $M^\vee$ from the spectrum of the local operators of the theory, so we might hope that this corresponds to the dual boundary condition for the $\mathcal{B}_{G^\vee}$ theory.

4.3 Befriending the Dragon: RW Theory & L-functions

The boundary theories for $\mathcal{B}_{G^\vee}$ coming from Hamiltonian $G^\vee$ spaces are known as Rozansky-Witten (RW) theories. Surprisingly, these theories give us a way of organizing L-functions for Galois representations, and the duality should give us corresponding periods on the $\mathcal{A}_G$ side [62]. We tend to think of the A-side as asking questions and the B-side providing answers, so in this essay we will focus on the B-side.

Given a linear representation $G^\vee \circlearrowright W$ we obtain a symplectic vector space $V = T^*W = W \oplus W^*$. Suppose that this is the target space of our σ-model. Then we can use the representation to study Galois groups via $\rho : \mathrm{Gal}(C()) = \pi_1(C) \to G^\vee \to GL(W)$, and obtain L-functions. (We will

[44] More specifically, parabolic induction. See C.22

assume we are unramified everywhere). This connects back with physics by noting that $\mathrm{Loc}_{G^\vee}(C) = \{\pi_1(C) \to G^\vee\}/G^\vee$. Thus, we have

$$\mathrm{L}(\rho, V, T=1) = \mathrm{L}(V^{\pi_1^{\text{geom}}} \circlearrowleft \mathrm{Frob}_q) = \mathrm{L}\left(H^\bullet(C, V_\rho) \circlearrowleft \mathrm{Frob}_q\right) \tag{16}$$

where V_ρ is the induced sheaf for the local system ρ similarly to how flat connections correspond to vector bundles. Hence, we get a function $\rho \mapsto \mathrm{L}(\rho, V)$ for each representation $G^\vee \circlearrowright V$.

We view these L-functions as "observables" as follows: Recall that L-functions come from the trace of Frob_q on the giant tensor product[45] $\mathrm{Sym}\, H^\bullet(C, V_\rho)$ which we will view as functions on some (derived) scheme . In particular, the structure sheaf $\mathscr{O}$ (locally constant sections of $V_\rho \to C$). Then the L-functions are the trace of these functions.

Partition functions

Before continuing with the function field setting, let us briefly discuss what happens in an actual TQFT. Let M be a three manifold of the form $M = \Sigma \times [0,1]/\sim$ where $(x,0) \sim (\sigma(x),1)$ for some σ generating the Hamiltonian, and Σ some Riemann surface. Then the Hilbert space for this theory will be

$$\mathfrak{H}_{\mathrm{RW}} = \text{geometric quantization of } \Pi H^\bullet(\Sigma) \otimes V \tag{17}$$

$$= \wedge^\bullet H^1(\Sigma, W) \otimes \mathrm{Sym}^\bullet\left(H^0(\Sigma, W) \oplus H^2(\Sigma, W)\right). \tag{18}$$

where Π indicates the fermionic grading of the super vector space.

Now, recalling the relationship between graded traces and products of determinants discussed in section 3.1, the partition function should be

$$Z(M) = \mathrm{Tr}_{\mathcal{H}}\, \sigma = \frac{\det_{H^{\text{even}}}(1-\sigma)}{\det_{H^{\text{odd}}}(1-\sigma)} = \mathrm{L}\left(H^\bullet(\Sigma, W) \circlearrowleft \sigma\right). \tag{19}$$

In the function field setting, we replace M with a curve $C/\mathbb{F}_q$, Σ with $\overline{C}$, and σ with the Frobenius, Frob_q.

Example 4.1 (Elliptic curve)**.** Consider C an elliptic curve over $\mathbb{F}_q$ and take $W = \mathbb{Q}_\ell$. Then equation 9 gives us the partition function:

$$\zeta_C(T) = \frac{1 - \alpha T - \bar{\alpha} T + qT^2}{(1-T)(1-qT)}. \tag{20}$$

[45] Thinking factorization homology again.

This is entirely determined by the trace of Frobenius on the first ℓ-adic étale cohomology. Hence, we obtained an L-function giving us number theoretic data. This dragon is friendly and willing to share its treasures! In chapter 5 we will discuss generalizing this idea using arithmetic CS.

5 Riding Dragons: Counting Points via Partition Functions

Having befriended the dragon of the geometric Langlands program, it may offer to take us on a ride and explore new lands. Thus far, we were particularly amazed to learn how partition functions in RW theory give rise to L-functions in section 4.3. We can connect this back to our previous study of arithmetic CS by noticing that we can consider CS theory as the square root of this RW theory in the following sense.

Including all the trivial derived parts, the Hilbert space of CS theory with gauge group $\mathbb{R}$ is $\mathfrak{H}_{\text{CS}} = S^\bullet H^1(\Sigma) \otimes \wedge^\bullet (H^0(\Sigma) \oplus H^2(\Sigma))$. This looks very much like the Hilbert space for RW theory in equation 17. In fact if we take $V = \mathbb{R}^2$ then, the only changes are: (1) opposite fermionic grading, and (2) "doubling" the theory from $\mathbb{R}$ to $\mathbb{R}^2$ (note that only W survives polarization in equation 18). Thus, we get $Z_{RW}(M)$ " $=$ " $Z_{CS}(M)^{-2}$, so partition functions in CS theory also correspond to L-functions.

This invites the question of what happens when we consider other gauge groups for the CS theory? As a first case we might consider the circle group and hope that taking the radius to infinity recovers the L-functions present for $\mathbb{R}$. Thus, our goal is to determine what number theoretic objects the partition functions give us in this setting.

5.1 A First Pass: Quantizing the Actual TQFT

To make the study of this easier, let us assume the symplectic structure comes from the gauge group $G = \mathbb{R}^2/\mathbb{Z}^2$ with level $\begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}$ which defines an "inner product," and hence gives a "radius" of k, for the gauge group. Note we must have $k \in \mathbb{Z}$ to be well defined over the quotient. The symplectic structure can be seen better after a change of basis so that the level is $\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$.

We will consider the situation described in section 4.3, so the 3 manifold is given by $M = \Sigma \times [0,1]/\sim$ the Hamiltonian is $H = \log \sigma$ acting on the curve, and $(x, 0) \sim (\sigma(x), 1)$. Hence, M is a fibration over S^1. For definiteness let $\Sigma = T^2$ be a torus. Then the classical phase space will be $\mathcal{P} = H^1(T^2, \mathbb{R}) \oplus H^1(T^2, \mathbb{R}) \cong \mathbb{R}^4$ up to an action of $\mathbb{Z}^2 = H^1(T^2, \mathbb{Z})$ on $x_1, x_2, y^1, y^2 \in \mathcal{P}$ via $x_1 \to x_1 + a$ and $x_2 \to x_2 + b$ (and a similar action for the y^i).[46] Finally, the symplectic form is $\omega = k\,(dy^1 dx_1 + dy^2 dx_2)$.

Applying geometric quantization , the Hilbert space will naively be functions of x_1 and x_2. However, everything should be invariant under the action of $H^1(\Sigma, \mathbb{Z}) \oplus H^1(\Sigma, \mathbb{Z})$, so we obtain[47]

$$\mathfrak{H} = \left\{ \sum_{\ell_i \in \mathbb{Z}} f(\ell_1, \ell_2) e^{2\pi i (x_1 \ell_1 + x_2 \ell_2)} \,|\, f(\ell_1 + k, \ell_2) = f(\ell_1, \ell_2) = f(\ell_1, \ell_2 + k) \right\}. \tag{21}$$

This is a k^2-dimensional Hilbert space isomorphic to $\mathbb{C}[\mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}]$. It has basis vectors $\lambda_{\binom{\ell_1}{\ell_2}}(x, y) = \delta_{x=\ell_1, y=\ell_2}$, and σ then acts on them via

$$\sigma \cdot \lambda_{\binom{\ell_1}{\ell_2}} = \lambda_{\sigma\binom{\ell_1}{\ell_2}}. \tag{22}$$

Now that the theory is quantized, we consider the partition function as a graded trace of σ as in example 4.1, although we focus on the first cohomology as the other terms are trivial. To compute this we use the following:

Lemma 5.1. *For A acting on $\mathbb{C}[\mathbb{Z}/k \times \mathbb{Z}/k]$ as σ does in equation 22:*

$$\mathrm{Tr}(A) = \#\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in (\mathbb{Z}/k\mathbb{Z})^2 \,|\, A \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \right\}. \tag{23}$$

Proof.

$$\begin{aligned} \mathrm{Tr}\, A &= \sum_{\ell_1, \ell_2 \in \mathbb{Z}/k\mathbb{Z}} \lambda_{\binom{\ell_1}{\ell_2}} \cdot \left(A \cdot \lambda_{\binom{\ell_1}{\ell_2}} \right) \\ &= \sum_{\ell_1, \ell_2 \in \mathbb{Z}/k\mathbb{Z}} \sum_{a, b \in \mathbb{Z}/k\mathbb{Z}} \lambda_{\binom{\ell_1}{\ell_2}}^{\;*}(a, b) \lambda_{A\binom{\ell_1}{\ell_2}}(a, b) \\ &= \#\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in (\mathbb{Z}/k\mathbb{Z})^2 \,|\, A \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \right\}. \end{aligned}$$

[46] $H^1(\mathbb{R}/\mathbb{Z}) \cong H^1(\mathbb{R})/H^1(\mathbb{Z})$ because of the cohomological LES from $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$ applied to the torus.

[47] The factor of k in ω leads to momentum translation by multiples of k.

5.2 The Full Flight: Function Fields

The "three manifold" is now a curve C over a finite field $\mathbb{F}_q$ where $q = p^n$ is a power of a prime. To find a similar construction to the above, we need our 3-manifold to look something like $\Sigma \times [0,1]/\sim$. We want our time evolution to be generated by the Frobenius , so we should have $H = \log \mathrm{Frob}_q$. Now we need a notion of the surface Σ. As discussed in section 3.3, there are two ways to obtain surfaces in the function field setting, and the one that produces the "3-manifold" structure of interest is $\Sigma = \overline{C}$.

In this algebraic setting we take the gauge group to be $G = \mathbb{Q}_\ell/\mathbb{Z}_\ell \oplus \mathbb{Q}_\ell/\mathbb{Z}_\ell$ (ℓ coprime to p) with level $\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$ for $k = p^m$ giving the symplectic structure on the phase space $\mathcal{P} = H^1(\Sigma, \mathbb{Q}_\ell) \oplus H^1(\Sigma, \mathbb{Q}_\ell)$ up to a quotient by $H^1(\Sigma, \mathbb{Z}_\ell) \oplus H^1(\Sigma, \mathbb{Z}_\ell)$.

Proposition 5.1. *The action of* Frob_q *on the quotient of* $\mathcal{P}$ *is well-defined.*

Proof. Étale cohomology is built from torsion sheaves . Hence, we define the $\mathbb{Z}_\ell$ cohomology as an inverse limit of $\mathbb{Z}/\ell^s\mathbb{Z}$ cohomologies. Each of these groups will have an action of the Galois group, and hence Frobenius, on them induced by the action on $\overline{C}$. The $\mathbb{Q}_\ell$ cohomology is defined as $H^i(\overline{C}, \mathbb{Q}_\ell) = H^i(\overline{C}, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Hence, the action of Frob_q on the ℓ-adic cohomology is already defined so that it factors over $\mathbb{Z}_\ell$ cohomology.

Now let us specialize to the case that C is an elliptic curve.

Proposition 5.2. *For* C *an elliptic curve, the Hilbert space is*

$$\begin{aligned}\mathfrak{H} = &\left\{ \sum_{\ell_i \in \mathbb{Z}_\ell} f(\ell_1, \ell_2) e^{2\pi i (x_1\ell_1 + x_2\ell_2)} \,|\, f(\ell_1 + ak, \ell_2) = f(\ell_1, \ell_2) = f(\ell_1, \ell_2 + bk),\ a, b \in \mathbb{Z}_\ell \right\} \\ \cong &\,\mathbb{C}\left[\mathbb{Z}_\ell/k\mathbb{Z}_\ell \times \mathbb{Z}_\ell/k\mathbb{Z}_\ell\right] \cong \mathbb{C}\left[\mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}\right].\end{aligned} \tag{24}$$

Proof. We want to apply the same arguments as in the TQFT case so that geometric quantization should give us the Hilbert space for an elliptic curve. We just need to check if using ℓ-adics this changes anything.

Since $\mathbb{Z}_\ell$ is the Cauchy completion of $\mathbb{Z}$, functions constant on $\mathbb{Z}$ are constant on $\mathbb{Z}_\ell$ as well. Thus the only remaining check is if these functions are invariant under the action of $\exp\left(\frac{1}{k}\lambda\partial_{x_i}\right)$ where the $1/k$ comes from the symplectic form being the inverse of the Poisson bracket, and $\lambda \in k\mathbb{Z}_\ell$. This works if k is a power of ℓ since then $\lim_{n\to\infty}\exp\left(2\pi i\ell^n \frac{1}{k}\right) = 1$.

Now that we have "quantized" the theory in the function field setting we can come to our main objective: computing the partition functions. As described in chapter 4, the usual setting with G a vector space and an actual TQFT should result in partition functions that are L-functions. We would like to determine what the interpretation of partition functions are in this new setting. Thus, using the basis defined above and lemma 5.1, we wish to compute the partition function

$$Z = \#\left\{\begin{pmatrix} a \\ b \end{pmatrix} \in (\mathbb{Z}/k\mathbb{Z})^2 \,|\, \mathrm{Frob}_q \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}\right\}. \tag{25}$$

Before diving into our actual problem, consider a simpler example.

Example 5.1. Suppose we have a 1×1 matrix A acting on $\mathbb{Z}/k\mathbb{Z}$ where $k = \ell^m$ and $A-1 = a\ell^d$ for a coprime to ℓ. We can compute $\#\{x \in \mathbb{Z}/k\mathbb{Z} | Ax = x\}$ as the nullspace of $A-1$. It can be quickly checked that this set is $\{0, \ell^{m-d}, 2\ell^{m-d}, ..., (\ell^d - 1)\ell^{m-d}\}$ which gives us $\ell^d = \frac{1}{|A-1|_\ell}$ elements.

Thus, we might interpret the partition function as generalizing the notion of (one over) the ℓ-adic norm to matrices (minus the identity). However, we can actually find a more satisfying interpretation in terms of counting certain torsion points of elliptic curves.

Theorem 5.1. *The partition function for a curve C is given by*

$$Z = \#\left(\mathrm{Jac}(C)[k]^{\mathrm{Frob}_q}\right) \tag{26}$$

the number of Frobenius fixed points of the Jacobian variety of the curve.

Proof. For a curve of genus g, the Hilbert space defined in proposition 5.2 becomes $\mathbb{C}\left[(\mathbb{Z}/\ell\mathbb{Z})^{2g}\right]$, and Frob_q is a $2g \times 2g$ matrix. In particular, this is precisely $H^1(C, \mathbb{Z}/k\mathbb{Z})$. Thus, equation 25 becomes

$$Z = \#\left(H^1(\overline{C}, \mathbb{Z}/k\mathbb{Z})\right)^{\mathrm{Frob}_q}. \tag{27}$$

Let $\mu_k \cong \mathbb{Z}/k\mathbb{Z}$ denote the multiplicative group of kth roots of unity. The short exact sequence $0 \to \mu_k \to \mathbb{G}_m \xrightarrow{(\cdot)^k} \mathbb{G}_m \to 0$ gives us the following cohomological long exact sequence

$$...H^0(\overline{C}, \mathbb{G}_m) \to H^0(\overline{C}, \mathbb{G}_m) \to H^1(\overline{C}, \mu_k) \to H^1(\overline{C}, \mathbb{G}_m) \to H^1(\overline{C}, \mathbb{G}_m) \to ... \tag{28}$$

and we should notice that $H^1(\overline{C}, \mathbb{G}_m) = \mathrm{Pic}(C)$, the group of line bundles on C with tensor product. Further, $H^0(\overline{C}, \mathbb{G}_m) = \mathscr{O}_C(C)^\times$ so the map to itself is invertible. Hence, we find

$$H^1(\overline{C}, \mu_k) \cong \ker\left(\mathrm{Pic}(C) \xrightarrow{(\cdot)^k} \mathrm{Pic}(C)\right) := \mathrm{Pic}(C)[k] := \mathrm{Jac}(C)[k] \tag{29}$$

where the last equality is because if L is a line bundle of degree d then $L^{\otimes n}$ will have degree nd, so only degree zero line bundles can have finite order.

Corollary 5.1. *For C an elliptic curve, the partition function is*

$$Z = \#\left(C[k]^{\mathrm{Frob}_q}\right). \tag{30}$$

Proof. For an elliptic curve $C \cong \mathrm{Jac}(C)$ via the Abel-Jacobi map C.3.

We conclude by noting that the field of n-torsion points of the elliptic curve over number fields played an important role in the proof of Fermat's Last Theorem. In the function field setting such field extensions are entirely determined by their degree over $\mathbb{F}_q$, and several algorithms for computing such degrees have been given [63]. It is interesting to note that the Frobenius fixed points are the points already in $\mathbb{F}_q$, indicating that a large partition function in this theory corresponds to fewer possible points being added to the field extension and thus bounding the possible degree.

6 Conclusion: The Moral of the Story

Today we have tried to share a beautiful story that brings together many concepts across math and physics. We began by meeting the characters;

quantization and TQFTs on the physics side, and number theory and algebraic geometry on the math side. Then built intuition by following the heroes of our story on a side quest to uncover the meaning of linking numbers of primes! There we learned that we can think of primes as knots, and do an arithmetic version of TQFTs. Using this intuition we delved into the geometric Langlands program where we saw how our understanding of physics informed our approach. In particular, a generalized electric-magnetic duality took us across the Langlands correspondence. Finally, we quantized an arithmetic version of Chern-Simons theory, and considered the partition functions of this theory with gauge group an analogue of $U(1)$. Upon letting the radius go to infinity we expected to recover L-functions from prior results, but we were interested in determining the meaning of the finite radius partition function. Our story concluded with a rather satisfying interpretation of these partition functions, although there is certainly much more to explore!

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There's a lot of cool stuff to explore!

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A Details of BV formalism

Here we give a brief introduction to this formalism, largely following the exposition in [28]. For more details, see [25]. Further, more accessible exposition and motivation for the BV formalism is developed in [27] and [26], and another couple of very useful references with good exposition are [64, 65].

First, we introduce the classical BV formalism. The idea here is that fields on our spacetime M are considered to be sections of a local L_∞ algebra L on M.

Definition A.1. An L_∞-algebra is a $\mathbb{Z}$ graded vector space $\mathfrak{g}$ as well as, for each $n \in \mathbb{N}_{\geq 1}$, an n-ary bracket taking in n copies of $\mathfrak{g}$:

$$l_n(...) := [-, -, ..., -]_n : \mathfrak{g} \otimes ... \otimes \mathfrak{g} \to \mathfrak{g}$$

and such that

1. Each l_n is graded antisymmetric
2. There is a strong homotopy Jacobi identity

The *bundle of BV fields* is $E = L[1]$, and we denote sections of E with $\mathcal{E}$. Then the idea is the equations of motion of fields and ghosts correspond to the ℓ_n brackets in the L_∞ structure. In particular, the l_1 bracket acts as a differential, which we denote Q_{BV}. This differential gives us a complex of fields where, similarly to BRST, physical observables are cohomology classes of the BV differential. Then we have some action S, defined in terms of a shifted symplectic pairing ω and an interaction term I, that should satisfy the classical master equation $\{S, S\} = X_S(S) = \omega \circ d_{dR}(S)(S) = 0$.

From another point of view, [65] tells us that L_∞ algebras are equivalent to formal moduli problems. Note that through the functor of points viewpoint on schemes which will be described in 3.2 we can view moduli spaces as both spaces and "nice" functors sending connective Artinian algebras to simplicial sets. We denote this formal moduli problem corresponding to $\mathfrak{g}$ by $\mathcal{M} = B\mathfrak{g} = \mathfrak{g}/\mathfrak{g}$ which we think of as our derived version of the phase space[48]. Then by definition $X \in B\mathfrak{g}(A)$ satisfies the Maurer-Cartan equation $\sum_{n=1}^{\infty} \frac{1}{n!}\ell_n\left(X^{\otimes n}\right) = 0$. This extends to the local versions of both formal moduli problems and L_∞ algebras defined on the spacetime manifold M[49].

A *free BV theory* on a manifold M is the following data:

- a finite rank vector bundle $E \to M$ with sections $\mathcal{E}$ that has an even differential operator $Q_{BV} : \mathcal{E} \to \mathcal{E}[1]$ of cohomological degree $+1$[50]
- a map of bundles $\omega : E \otimes E \to \mathrm{Dens}_M[-1]$ that is
  1. fiberwise nondegenerate
  2. graded antisymmetric
  3. satisfies $\int_M \omega(e_0, Q_{BV}e_1) = (-1)^{|e_0|} \int_M \omega(Q_{BV}e_0, e_1)$ for e_i compactly supported sections of E.

To obtain a general *classical BV theory* we must include interactions as well. Hence, we we additionally have an even functional $I \in \mathcal{O}^+_{loc}(\mathcal{E})$ of degree

[48]Notice this is a stack. In general a nice exposition on formal moduli problems is [66].
[49]A local formal moduli problem on M is a presheaf of formal moduli problems on M
[50]We also require $Q_{BV}^2 = 0$ and $(\mathcal{E}, Q_{BV})$ is an elliptic complex for quantization.

zero that satisfies the Maurer-Cartan equation:

$$Q_{BV} I + \frac{1}{2}\{I, I\} = 0. \tag{31}$$

Then the action will be

$$S = \frac{1}{2}\int_M \omega(e, Q_{BV} e) + I \in \mathcal{O}_{loc}(E).$$

Our main example of BV theories will be Chern-Simons theory. However, the BV formalism is actually a generalization of BRST theories, as the next example illustrates. This generalization is useful when the theory has symmetries that are only preserved on-shell. For example, supersymmetry is not conserved off-shell so these theories are better described using BV formalism.

Example A.1. A classical BRST theory on M consists of a local L_∞ algebra L with $\mathcal{M} = BL$ the associated formal moduli problem, and an action $S_{BRST} \in \mathcal{O}_{\text{loc}}(\mathcal{M})$. We require these satisfy $Q_{BRST} S_{BRST} = 0$ where Q_{BRST} is the Chevalley-Eilenberg differential defined by the local L_∞ structure on L.

Then the associated classical BV theory is given by the bundle $T^*[-1]\mathcal{M}$. This has a natural (-1)-shifted symplectic structure. The BV action functional will then be $S_{BRST} + S_{anti}$ for S_{anti} for $S_{anti} = \frac{1}{2}\int_M \omega\left(e, Q_{\text{BV}} e\right)$

We call $L[1]$ the space of *BRST fields*. Typically this is made up of the physical fields ($\mathbb{Z}$ degree 0) and ghost fields (degree -1) which generate the gauge symmetry.

Finally, we can *quantize* this theory. Following the point of view of [67] axiomatized in [26], the idea is to let $\langle f\rangle_\mu = \frac{\int_M f\mu}{\int_M \mu}$ for $f \in C^\infty_M$ a field and μ a top form that plays the analogue of $e^{-S/\hbar}D\phi$ the path integral measure. Using μ we are able to define a BV Laplacian Δ_μ giving a quantum version of the BV complex with this differential. Then if we let $[\cdot]_{BV}$ denote cohomology classes in the zeroeth cohomology of the complex we can define $\langle f\rangle_\mu = [f]_{BV}/[1]_{BV}$. To phrase this more closely to the classical formalism, BV quantization finds an action functional $S_q = S + I_q$ such that

- $S_q \mod \hbar = S$
- S_q satisfies the quantum master equation $\{S_q, S_q\} - \hbar\Delta S_q = 0$.

Example A.1. Chern-Simons can be described as a specialized case of generalized Chern-Simons theories using the BV formalism as follows [28]:

Let M be a 3-manifold, and $\mathfrak{g}$ be an ordinary Lie algebra. Then, as a local formal moduli problem, CS is the classical field theory $\mathrm{Map}(M_{\mathrm{dR}}, B\mathfrak{g})$ with underlying local L_∞ algebra $\Omega^\bullet_M \otimes \mathfrak{g}$. It has BV action

$$S = \frac{k}{4\pi} \int_M \mathrm{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

where A is a (not necessarily homogeneous) differential form on M. However, the only non-trivial cohomology of a spatial slice is H^1, and we obtain the usual CS for $A \in \Omega^1 \otimes \mathfrak{g}$ by recognizing this as the connection for a gauge theory with group G.

A.1 SUSY and twisting

To describe symmetries of a Lie algebra $\mathfrak{g}$ on a BV theory we use the Chevalley-Eilenberg complex with differential d_{CE} to add $\mathfrak{g}$ valued background fields. We define this action in terms of a Noether current $S_\mathfrak{g}$ below, although it also gives $\mathcal{E}$ an L_∞ structure for $\mathfrak{g}$.

Definition A.2. An *action* of a super Lie algebra $\mathfrak{g}$ on a classical field theory (E, S, ω) is a cohomological degree zero element $S_\mathfrak{g} = \sum_{k \geq 0} S_\mathfrak{g}^{(k)} \in C^\bullet(\mathfrak{g}, \mathcal{O}_{\mathrm{loc}}(\mathcal{E}))$ where $S_\mathfrak{g}^{(k)} : \mathfrak{g}^{\otimes k} \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$ are such that $S_\mathfrak{g}^{(0)} = S$, for $K \geq 1$ the $S_\mathfrak{g}^{(k)}$ are at least quadratic in fields, and the $S_\mathfrak{g}$ satisfy the Maurer-Cartan equation $\mathrm{d}_{\mathrm{CE}} S_\mathfrak{g} + \frac{1}{2}\{S_\mathfrak{g}, S_\mathfrak{g}\} = 0$.

An *action* of a super Lie group G on (E, S, ω) acting on G is (up to some natural details) an action of G on $\mathcal{E}$ compatible with the action on M, and an action $S_\mathfrak{g}$ for its super Lie algebra with $S_\mathfrak{g}^{(k)} = 0$ for $k \geq 2$.

Given this, we can now describe supersymmetry. Let our spacetime be a real vector space $V_\mathbb{R}$ for simplicity, and denote its complexifications as V.

Definition A.3. A classical field theory (E, S, ω) is a *supersymmetric classical field theory* if $E \to M$ is a $(\mathrm{Spin}(V_\mathbb{R}) \ltimes V_\mathbb{R}) \times G_R$-equivariant vector bundle, and the infinitesimal strict action of the translation Lie algebra V on the theory is extended to a $\mathrm{Spin}(V_\mathbb{R}) \times G_R$-equivarient L_∞ action of the supertranslation Lie algebra $\mathfrak{A}$.

This definition requires a bit of unraveling. First, the *supertranslation Lie algebra* $\mathfrak{A}$ can be thought of as the fermionic part of the Poincare group. Then the Lie group G_R is the group of *R-symmetries* which acts on spinorial representations of $\mathfrak{so}(V)$ in a nice way. This encodes the internal symmetries of rotating between supercharges. Finally, we will call $(\mathfrak{so}(V) \oplus \mathfrak{g}_R) \times \mathfrak{A}$ the *supersymmetry algebra* of the theory.

We can now begin with some supersymmetric theory (E, S, ω) and twist it using a supercharge Q. The action of the action of $\mathfrak{A}$ gives us a Maurer-Cartan element $S_{\mathfrak{A}} = S + \sum_{k\geq 1} S_{\mathfrak{A}}^{(k)} \in C^\bullet(\mathfrak{A}, \mathcal{O}_{\text{loc}}(\mathcal{E}))$, and the classical theory has an action of the R-symmetry.

Definition A.4. Suppose Q is a square-zero supercharge on the theory. The *Q-twisted classical field theory* is the $\mathbb{Z}/2\mathbb{Z}$-graded classical field theory with the same bundle of fields and ω, but with the modified action

$$S^Q = S + \sum_{k\geq 1} S_{\mathfrak{A}}^{(k)}(Q, ..., Q).$$

Then we say a homomorphism $\alpha : U(1) \to G_R$ is *compatible* with Q if Q has weight 1 and the α-weight $\mod 2$ is the fermionic grading. Thus, we obtain a new grading on E via the sum of the original cohomological grading and the α grading. Then we find S has total degree zero, so (E, S^Q, ω) is a $\mathbb{Z}$ graded classical field theory.

Finally, this new theory carries a G action if there is a twisting homomorphism:

Definition A.5. A *twisting homomorphism* is a homomorphism $\phi : G \to G_R$ for some group G with fixed homomorphism $\iota : G \to \text{Spin}(V_{\mathbb{R}})$ such that Q is preserved under $(\iota, \phi) : G \to \text{Spin}(V_{\mathbb{R}}) \times G_R$.

B Arithmetic Chern-Simons

Here we describe some of the ideas of Kim's arithmetic CS in more detail, continuing off of section 3.3.

If no primes are removed, the "phase space" for this theory with (finite) gauge group G is our analog of the space of flat connections: $\mathcal{M}(G) := \text{Hom}_{cont}(\pi_1(X), G)/G \ni [\rho]$. Then, fixing some $c \in H^3(G, \mathbb{Z}/n\mathbb{Z})$ we can define the simplest case of the arithmetic CS functional

$$\begin{aligned}\mathrm{CS}_c : \mathcal{M}(G) \to & \frac{1}{n}\mathbb{Z}/\mathbb{Z} \\ [\rho] \mapsto & \operatorname{inv}(\rho^*(c))\end{aligned}$$

where the idea is to use the *invariant map* induced by Artin-Verdier duality $\operatorname{inv} : H^3(X, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$ and analogous statements we also call the invariant map such as $\operatorname{inv} : H^3(X, \mathbb{Z}/n\mathbb{Z}) \cong H^3(X, \mu_n) \cong \frac{1}{n}\mathbb{Z}/\mathbb{Z}$.

To extend to X_S, Kim defined $\mathcal{M}_S(G)$ to be the groupoid with objects $Y_S(G) = \mathrm{Hom}_{cont}(\pi_1(X_S), G)$ and morphisms via conjugation by G. He also defined local versions of these using products $\prod_{\nu\in S}$ for the localized fields via embeddings $i_\nu : K \to K_\nu$ which combine into a total embedding i_S. Then, letting $\Sigma : \prod_{\nu\in S} \frac{1}{n}\mathbb{Z}/\mathbb{Z} \to \frac{1}{n}\mathbb{Z}/\mathbb{Z}$ via summing, Kim defined a more interesting CS functional:

$$CS_c([\rho]) = \Sigma\left([i_S^*(\beta)]\right)$$

where $d\beta = c \circ \rho$ is a boundary.[51] Kim also defined this for p-adics, although we will not discuss this here.

Additionally, Kim's arithmetic CS theory with abelian group $G = \mathbb{Z}/n\mathbb{Z}$ generalizes Morishita's construction of linking numbers to a $\mathbb{Z}/n\mathbb{Z}$ "linking number" of primes [53]. In particular, the duality pairing $\langle\cdot,\cdot\rangle : H^1(X, \mathbb{Z}/n\mathbb{Z}) \times \mathrm{Ext}^3_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m) \to \frac{1}{n}\mathbb{Z}/\mathbb{Z}$ allows us to set

$$\mathrm{lk}_n(\mathfrak{p}, \mathfrak{q}) := \langle d^{-1}\mathfrak{p}, \mathfrak{q}\rangle \tag{32}$$

where $d : H^1(X, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{Ext}^2_X(\mathbb{Z}/n\mathbb{Z}, \mathbb{G}_m)$.

This also connects more directly to the CS theory itself since the abelian CS functional of $A \in H^1(X, \mathbb{Z}/n\mathbb{Z}) = \mathrm{Hom}_{cont}(\pi_1(X), \mathbb{Z}/n\mathbb{Z})$ is $\langle A, dA\rangle$.

C A Glossary of Quick Definitions

Here we will restate many definitions and results that were given rather imprecisely in the text. This is mainly meant to be a glossary of corrected theorems and definitions (although in some cases we still do not give full details). Rather than including extensive exposition most explanations are left to references. An effort has been made to make as many of the terms and notation used in the text linked to their relevant definitions here.

[51] Using $H^3\left(X_S, \frac{1}{n}\mathbb{Z}/\mathbb{Z}\right) = 0$.

C.1 Miscellaneous Number Theory

Definition C.1. Suppose L is a field extension of the field K. An automorphism of L/K is defined to be an isomorphism $\alpha : L \to L$ such that $\alpha(x) = x$ for $x \in K$. Then the set of all such automorphisms forms a group.

We say L/K is a *Galois extension* if it is normal (any irreducible polynomial in K with a root in L splits into linear factors in L) and separable (any irreducible polynomial in K does not have repeated roots in any extension of L).

Given a Galois extension we defined the *Galois group* $\mathrm{Gal}(L/K)$ to be the automorphism group. If L/K is not Galois then we can let E be the Galois closure of L and define $\mathrm{Gal}(L/K) := \mathrm{Gal}(E/K)$. Finally, we define $\mathrm{Gal}(K) := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ for K^{sep} the maximal separable extension of K.

In nice Galois extensions, the degree of the extension $[L : K]$ is exactly the order $|\mathrm{Gal}(L/K)|$.

In an integral domain R we can define fractional ideals to be R-submodules, I, in the field of fractions of R such that there exists some nonzero $r \in R$ such that $Ir \subseteq R$. For example, in $\mathbb{Z}$ we would say $\frac{1}{3}\mathbb{Z}$ is a fractional ideal with $r = 3$.

Definition C.2. If the ring R is an integral domain then we can define an equivalence relation between nonzero fractional ideals by $I \sim J$ whenever there are nonzero $a, b \in R$ such that $(a)I = (b)J$. Then $[I][J] = [IJ]$ is well defined and commutative, and the class of principal ideals (generated by a single element) form the identity element. Inverses might not always exist, but if R is a Dedekind domain (such as the ring of integers of a number field) then these form an abelian group known as the *ideal class group*.

This is closely related to the Picard group in algebraic geometry.

Definition C.3. Let p be an odd prime. Then $a \in \mathbb{Z}$ is a quadratic residue modulo p if there exists $x \in \mathbb{Z}$ such that $a \equiv x^2 \mod p$. Then we define the Legendre symbol via

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue mod } p \text{ and } a \not\equiv 0, \\ -1 & \text{if } a \text{ is not a quadratic residue,} \\ 0 & \text{if } a \equiv 0 \mod p \end{cases} . \tag{33}$$

These are multiplicative in their top argument and satisfy quadratic reciprocity

$$\left(\frac{q}{p}\right)\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}. \tag{34}$$

Definition C.4. A real number α is a *period* if it can be expressed in integral form: $\alpha = \int_{P(x_1,\ldots,x_n)\geq 0} Q\left(x_1,\ldots,x_n\right) \mathrm{d}x_1 \ldots \mathrm{d}x_n$ where P is a polynomial, and Q is a rational function on $\mathbb{R}^n$ with rational coefficients. We say a rational number is a period if both the real and imaginary parts are. These include algebraic numbers and many constants such as π, and they form a ring that lies between the algebraic numbers $\overline{\mathbb{Q}}$ and $\mathbb{C}$.

C.2 (Higher) Categories

A good introductory reference to category theory is Riehl's book [68].

Definition C.5. A (small) *category* C consists of

- a set $\mathrm{ob}(C)$ of objects
- a set of morphisms between objects. For a, b objects in C we denote the morphisms between them as $\mathrm{Hom}_C(a,b)$
- For every three objects a, b, c we can compose morphisms $\circ : \mathrm{Hom}(a,b)\times \mathrm{Hom}(b,c) \to \mathrm{Hom}(a,c)$

and satisfying

- If $f \in \mathrm{Hom}(a,b)$, $g \in H(b,c)$ and $h \in \mathrm{Hom}(c,d)$ then $h\circ(g\circ f) = (h\circ g)\circ f$
- For every object x there exists $1_x \in \mathrm{Hom}(x,x)$ called the identity morphism satisfying for any $f : a \to x$ and $g : x \to b$ then $1_x \circ f = f$ and $g\circ 1_x = g$.

We map between categories with functors.

Definition C.6. Let C and D be categories. Then a *functor* F from C to D is a mapping that

- takes $c \in C$ to $F(c) \in D$

- takes a morphism $f : c \to d$ in C to $F(f) : F(c) \to F(d)$ in D

and satisfies

- $F(1_c) = 1_{F(c)}$ for every object c in C
- $F(g \circ f) = F(f) \circ F(g)$ for any morphisms $g : a \to b$ and $f : b \to c$ in C.

We map between functors with natural transformations.

Definition C.7. If F and G are each a functor from category C to category D then a *natural transformation* η from F to G is a family of morphisms that

- associate to every object c in C a morphism $\eta_c : F(c) \to G(c)$ in D
- satisfy that, for every morphism $f : c \to d$ in C, we have

$$\eta_d \circ F(f) = G(f) \circ \eta_c. \tag{35}$$

We can make "higher" versions of categories. A fairly comprehensive reference for this is Lurie's Higher Topos Theory [69]. A *higher category* generalizes the idea of a category where there are not only morphisms between objects but k-morphisms between $k-1$-morphisms.

Categories can have additional structure. For example, monoidal/ tensor categories. A good reference for these is [70].

Definition C.8. A *monoidal category* is a category C equipped with a monoidal structure that consists of:

- a bifunctor $\otimes : C \times C \to C$ called the *monoidal product* or *tensor product*,
- An object I called the monoidal unit,
- the following three natural isomorphisms:
  - the associator $\alpha_{a,b,c} : a \otimes (b \otimes c) \cong (a \otimes b) \otimes c$
  - the left and right unitors $\lambda_a : I \otimes a \cong a$ and $\rho_a : a \otimes I \cong a$

that satisfy the natural coherence relations you would want for a tensor product.

A functor between monoidal categories that preserves the monoidal structure is called a *monoidal functor*.

Another type of structure we can add to a category is "enriching" it.

Definition C.9. A (small) category C *enriched* over a monoidal category K is

- a set of objects in C
- for each ordered pair of objects (x, y) in C, a *hom object* $\mathrm{Hom}_C(x, y)$ in K that plays the role of the set of morphisms from x to y
- for each ordered triple (a, b, c) of objects in C, a morphism $\circ : \mathrm{Hom}(b, c) \otimes \mathrm{Hom}(a, b) \to \mathrm{Hom}(a, c)$ in K called the composition morphism
- for each object a in C, a morphism $j_a : I \to \mathrm{Hom}(a, a)$ called the *identity element*

and such that the diagrams we would want to commute do commute.

C.3 Algebraic Geometry

See [41] for a standard introduction to these topics.

Definition C.10. Consider a topological space X. Then a *sheaf* $\mathcal{F}$ on X consists of

- For each open set $U \subseteq X$ a set $\mathcal{F}(U) = \Gamma(U, \mathcal{F})$ of sections over U. Sections over X are called global sections
- Fear each inclusion of open sets $V \subseteq U$, a function $\mathrm{res}^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ called *restriction morphisms*. For a section $s \in \mathcal{F}(U)$ we denote $s|_V := \mathrm{res}^U_V(s)$

such that the restriction morphisms satisfy

- For every open $U \subseteq X$ then res^U_U is the identity morphism on $\mathcal{F}(U)$
- If we have three open sets $W \subseteq V \subseteq U$ then $\mathrm{res}^V_W \circ \mathrm{res}^U_V = \mathrm{res}^U_W$

and further the sheaf should satisfy the locality and gluing axioms:

- (Locality) Say U is open and $\{U_i\}_{i\in I}$ is an open cover of U with $U_i \subseteq U$ for all $i \in I$. If $s, t \in \mathcal{F}(U)$ are sections such that $s|_{U_i} = t_{U_i}$ for all i, then $s = t$
- (Gluing) Again suppose U is open and $\{U_i\}_{i\in I}$ is an open cover with $U_i \subseteq U$ for all $i \in I$. Also let $\{s_i \in \mathcal{F}(U_i)\}$ be a family of sections. Then if for all $i, j \in I$ we have $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ then there exists a "glued" section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

There are a couple important types of sheaves that we will care about in particular.

Definition C.11. A *quasicoherent sheaf* on a ringed space $(X, \mathscr{O}_X)$ is a sheaf $\mathcal{F}$ of $\mathscr{O}_X$-modules such that for every point $x \in X$ there exists an open neighborhood $U \ni x$ in which the following exact sequence holds

$$\mathscr{O}_X^{\oplus I}|_U \to \mathscr{O}_X^{\oplus J}|_U \to \mathcal{F}_U \to 0 \tag{36}$$

for some possibly infinite sets I and J.

These are a generalization of coherent sheaves:

Definition C.12. A *coherent sheaf* on ringed space $(X, \mathscr{O}_X)$ is a sheaf $\mathcal{F}$ of $\mathscr{O}_X$-modules satisfying:

- For every $x \in X$ there exists an open neighborhood $U \ni x$ such that there is a surjective morphism $\mathscr{O}_X^n|U \to \mathcal{F}|_U$ for some $n \in \mathbb{N}$. We call this "being of finite type."
- For any open set $U \subseteq X$, natural number $n \in \mathbb{N}$, and morphism $\phi : \mathscr{O}_X^n|U \to \mathcal{F}|_U$ of $\mathscr{O}_X$-modules, the kernel of ϕ is of finite type.

We also used the following notion:

Definition C.13. A *D-module* is a sheaf of modules over the sheaf D_X of regular differential operators on a variety (or scheme, manifold,...) X which is quasicoherent as an $\mathscr{O}_X$-module. (Note that $\mathscr{O}_X$ is a subsheaf of D_X so every D_X module is an $\mathscr{O}_X$ module).

Finally, to properly state the geometric Langlands program we need [71]:

Definition C.14. We let IndCoh(X) denote the homotopy category of (possibly unbounded) injective complexes on X. Note that we do not localize with respect to quasi-isomorphisms.

Note that there is a canonical completion functor taking IndCoh$(X) \to$ QCoh(X).

C.4 Representation theory

Recall that a representation of a Lie algebra is $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ so that $\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$. To continue, we first need the idea of the universal enveloping algebra. The idea is to embed $\mathfrak{g}$ into an associative algebra $\mathcal{A}$ with identity such that the commutator is the Lie bracket. The *universal enveloping algebra* is the largest such algebra.

Definition C.15. Define the tensor algebra $T(\mathfrak{g}) = K \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \ldots$. Then the *universal enveloping algebra* is defined as $U(\mathfrak{g}) = T(\mathfrak{g})/\sim$ where $a \otimes b - b \otimes a \sim [a,b]$.

We can now define

Definition C.16. A *Cartan subalgebra* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a nilpotent Lie subalgebra such that if $[X,Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$ then $Y \in \mathfrak{h}$.

If this is a finite dimensional semisimple Lie algebraic over an algebraic closed characteristic 0 field then the above definition this is equivalent to being a *maximal* abelian subalgebra of semisimple elements x (meaning $ad(x) : \mathfrak{g} \to \mathfrak{g}$ via $ad(x)(y) = [x,y]$ is diagonalizable).

Note that a *toral* algebra is one consisting of only semisimples. In $\mathbb{C}$ this means abelian so Cartan = maximal toral.

In the algebraically closed and characteristic 0 setting we now have $ad(\mathfrak{h})$ is abelian and the adjoint representation $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ only has diagonalizable operators. Hence, we can simultaneously diagonalize to obtain

$$\mathfrak{g} = \bigoplus_{\lambda} \mathfrak{g}_\lambda$$

for $\lambda \in \mathfrak{h}^*$w a *weight* (so $x \in \mathfrak{g}_\lambda \implies ad(h)(x) = \lambda(h)x$).

Now we can define

Definition C.17. Let $\Phi = \{\lambda \in \mathfrak{h}^* \setminus \{0\} | \mathfrak{g}_\lambda \neq \{0\}\}$. Then Φ is a *root system*

Say we have a representation $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$. There is a decomposition

$$V_\lambda = \{v \in V | \pi(h)(v) = \lambda(h)v,\, h \in \mathfrak{h}\} \tag{37}$$

called the *weight space* for weight λ. Then $V = \oplus_\lambda V_\lambda$.

A useful result is the following: If ρ is a representation, v is weight vector for λ, and $X \in \mathfrak{g}$ is a root vector with root α so that $[h, X] = \alpha(h)X$ for all $h \in \mathfrak{h}$, then

$$\rho(h)(\rho(X)(v)) = [(\lambda + \alpha)(h)](\rho(X)v). \tag{38}$$

Hence, X action maps V_λ to $V_{\lambda+\alpha}$.

More generally, we have

Definition C.18. A *root system* Φ in E a finite dimensional Euclidean vector space with standard inner product is a finite set of non-zero vectors (*roots*) such that

- They span E
- For $\alpha \in \Phi$ only other multiple is $-\alpha$
- For every root α let h_α be the hyperplane orthogonal. Then reflecting Φ over h_α is still Φ. (so for all $\alpha, \beta \in \Phi$ also $\beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in \Phi$. Call this *reflection* s_α
- $\langle \beta, \alpha \rangle = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$

Given such a root system we have

Definition C.19. The *Weyl group* W of Φ is the subgroup of the orthogonal group $O(E)$ generated by the s_α's.

Now if Φ is from a Lie algebra with Cartan subalgebra setting as above, pick $\Phi^+ \subset \Phi$ such that either for all $\alpha \in \Phi\alpha$ or $-\alpha$ is contained and closed under addition. Then say $\lambda \in \mathfrak{h}^*$ is integral is dominant integral if $(\lambda, \alpha) \geq 0\, \forall \alpha \in \Phi^+$ and $2\frac{(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}\, \forall \alpha \in \Phi$.

Now also have a result for Weyl groups:

Theorem C.1. *Let K be a connected maximal Lie group and T a maximal torus. The normalizer is $N(T) = \{x \in K | xtx^{-1} \in T\}$ and centralizer is $Z(T) = \{x | xtx^{-1} = t\}$ (actually, $Z(T) = T$). Then the Weyl group W of K is $W = N(T)/T$*

Bruhat decomposition

Definition C.20. A *linear* algebraic group over a field k is a smooth closed subgroup scheme of $GL(n)$ over k (i.e., a smooth affine group scheme over k).

A connected linear algebraic group is *semisimple* if every smooth connected solvable normal subgroup is trivial (can be constructed from abelian groups using extensions).

A connected linear algebraic group over an algebraically closed field is *reductive* if the largest smooth connected unipotent (isomorphic to a closed subgroup scheme of U_n) normal subgroup is trivial. We call this subgroup $R_u(G)$. Over an arbitrary field k we say G is reductive if the base change $G_{\overline{k}}$ is reductive.

Definition C.21. A *Borel subgroup* of an algebraic group G is a maximal Zariski closed and connected solvable algebraic subgroup.

Example C.1. For GL_n, the invertible upper triangular matrices form a Borel subgroup.

Finally, let W be a Weyl group of G corresponding to a maximal torus of B. Then we have a decomposition

$$G = BWB = \bigsqcup_{w \in W} BwB \tag{39}$$

as a disjoint union of double cosets of B parametrized by W (even though W is not a subgroup of G, we have coset wB well defined because the maximal torus is contained in B).

Note that Bruhat decomposition underlies much of the structure of group actions and orbits playing a role in geometric Satake and the geometric Langlands correspondance discussed above.

Parabolic induction

Given a subgroup $H \subset G$ and a representation (π, V) of H, the *induced representation* of G, denoted $\mathrm{Ind}_H^G \pi$, is the representation on the space

$$\mathrm{Ind}_H^G \pi = \left\{ f : G \to V \,\middle|\, f(gh) = \pi(h^{-1}) f(g) \text{ for all } h \in H, g \in G, \text{ and } f \in L^2(G/H) \right\}, \tag{40}$$

with G acting by left translation: $(g \cdot f)(x) = f(g^{-1}x)$. This construction extends a representation of a subgroup to one of the whole group.

More algebraically, given H a subgroup of G with representation (π, V), then the induced representation of G is in a sense the most general possible representation we can get from the representation of H. Namely, given $g_1 ... g_n$ representatives of the (left) cosets of H we have for any $g \in G$ and i then $g \cdot g_i = g_{j(i)} h_i$ for some $h_i \in H$ and $j(i) \in \{1, ..., n\}$. Thus, we obtain a representation on $\bigoplus_i g_i V$ (so a copy of V for each g_i) with group action given by

$$g \cdot \sum_i g_i v_i = \sum_i g_{j(i)} \pi(h_i) v_i \tag{41}$$

Definition C.22. If H is a *parabolic subgroup* of a reductive algebraic group G, then $\operatorname{Ind}_H^G \pi$ is referred to as *parabolic induction.*

The standard inner product on class functions $\alpha, \beta : G \to \mathbb{C}$ is given by

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g \in G} \alpha(g) \overline{\beta(g)}. \tag{42}$$

Theorem C.2 (Frobenius Reciprocity)**.** *Let ψ be a class function on H and ϕ a class function on G. Then*

$$\langle \operatorname{Ind}_H^G \psi, \phi \rangle_G = \langle \psi, \operatorname{Res}_H^G \phi \rangle_H, \tag{43}$$

so induction and restriction are adjoint functors with respect to the class function inner product.

C.5 Langlands

To define the Langlands dual group we use a generalization of root systems to *root datum.* In particular, if G is a reductive algebraic group over an algebraically closed field K with a split maximal torus T then it has root datum $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ where

- X^* is the lattice of characters of the torus
- X_* is the dual lattice
- Φ is the set of roots

- $\Phi^\vee$ is the set of coroots.

Definition C.23. Let G be a connected reductive algebraic group over an algebraically closed field with root datum $(X^*(T), \Phi, X_*(T), \Phi^\vee)$, where T is a maximal torus. The *Langlands dual group* $G^\vee$ is the complex connected reductive group with dual root datum:

$$(X_*(T), \Phi^\vee, X^*(T), \Phi). \tag{44}$$

In particular, roots and coroots are swapped, as are characters and cocharacters.

Over a general field k that may not be algebraically closed, let K be its algebraic closure. Then over K we obtain $\left(\bar{G}^\vee\right)^o$ the connected component of the Langlands dual as above, and we define $G^\vee = \left(\bar{G}^\vee\right)^o \times \mathrm{Gal}(K/k)$.

The naive version of the geometric Langlands program is 4.1 because QCoh does not correctly capture singularities on the derived stack $\mathrm{Loc}_{G^\vee}(X)$, while DMod is sensitive to singular support.

Conjecture C.1 (Geometric Langlands program (no longer a conjecture))**.**

$$\mathrm{DMod}\left(\mathrm{Bun}_G(X)\right) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}\left(\mathrm{Loc}_{G^\vee}(X)\right) \tag{45}$$

C.6 More on Jacobian and Picard groups

Let X be a ringed space.

Definition C.24. The *Picard group* is $\mathrm{Pic}(X)$ is the group of isomorphism classes of invertible sheaves (line bundles) on X with group operation via tensor product $(L \otimes_{\mathcal{O}_X} M)$.

Alternatively, $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^\times)$ from sheaf cohomology.

We should think of this as a global version of ideal class group.

For complex manifolds the Picard group can be studied given by the exponential sheaf sequence

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^\times \to 0$$

where the maps are exponentials. This then leads to

$$\ldots \to H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^\times) \to H^2(X, \mathbb{Z})$$

where the second map sends a line bundle to its first Chern class.

Also, recall *divisors*:

- On a *Riemann surface* have divisors are linear combinations of points with integer coefficients. The *degree* is sum of coefficients. For f meromorphic the a principal divisor is of the form $(f) = \sum_p ord_p(f)p$ a finite sum. If it is a compact Riemann surface then the degree of a principal divisor is zero! For a divisor D we have the *line bundle* given by sections being the vector space of meromorphic functions with poles at most given by D, called $\mathcal{O}(D)$.

- *Weil divisors* are formal sums over integral closed subschemes of codimension 1 of form $\sum_Z n_Z Z$. (On a curve this is a sum of closed points. For a divisor on Spec $\mathbb{Z}$ use prime numbers and this corresponds to a nonzero fractional ideal in $\mathbb{Q}$).

- The *canonical divisor* is important too. Say we have a perfect field, and X be a normal (so local rings are integrally closed domains) variety over it of dimension n. Then the canonical sheaf is $\omega_X = i_*\Omega^n_U$ where i is inclusion of the smooth locus U (an open subset with codimension at least 2) and Ω^n_U is the sheaf of differential forms on U of degree n. When char=0 instead of normal then instead of i use a resolution of singularities Now the canonical divisor K_X denotes the divisor class so that $\mathcal{O}_X(K_X) = \omega_X$

Note if line bundles are given by a divisor $\sum_x n_x x$ then this maps to the degree of the line bundle $\sum n_x$ so we get

$$0 \to \text{Jac} \to \text{Pic} \to \mathbb{Z} \to$$

a short exact sequence where the kernel of the degree map is:

Definition C.25. The *Jacobian variety* of a non-singular algebraic curve C of genus g is the moduli space of degree 0 line bundles.

$$\text{Jac}(C) = H^0(C, \Omega^1_C)^*/H_1(C, \Omega)$$

with H_1 embedded in H^0 via $[\gamma] : \omega \to \int_\gamma \omega$.

This makes it the connected component of the identity in the Picard group and hence an abelian variety (abelian, connected, projective algebraic group).

An important property comes from the Abel-Jacobi map $u : C \to \text{Jac}(C)$ via $u(p) = \left(\int_{p_0}^p \omega_1, ..., \int_{p_0}^p \omega_g\right) \mod \Lambda$ where p_0 is a base point and the ω_i are linearly independent forms chosen for the torus.

Theorem C.3 (Abel-Jacobi). *Suppose $D = \sum_i n_i p_i$ is a divisor, and defined $u(D) = \sum n_i u(p_i)$ where u is the Abel-Jacobi map. Then $u(D) = u(E)$ iff D is linearly equivalent to E (differ by principal divisor). Hence, u induces an injective map of abelian groups from the divisor classes of degree zero to the Jacobian*

It can also be proved this is surjective so the groups are isomorphic.

Another important thing is: Riemann Abel and Jacobi showed that $\mathrm{Jac}(C) = (\mathrm{Jac}(C))^*$ where for A an abelian variety then A^* is the dual torus. However, also it happens to then be the collection of multiplicative line bundles on A namely $A^\vee[1]$.

Then if we have $\mathrm{Pic}(C)$ the stack of line bundles so $\mathrm{Pic}(C) = |\mathrm{Pic}(C)| \times B\mathbb{G}_m$ we have $\mathrm{Pic}(C) = \mathbb{Z} \times \mathrm{Jac}(C) \times B\mathbb{G}_m$ i.e the degree $\times$ the degree 0 line bundles $\times$ the automorphisms. Hence, $(\mathrm{Pic}(C))^* = \mathrm{Pic}(C)^\vee[1] \cong B\mathbb{G}_m \times \mathrm{Jac} \times \mathbb{Z}$ so Pic is also self dual via swapping automorphisms with degrees.

C.7 Formal schemes and thickening

A formal scheme includes data about its surroundings. I.e., infinitesimal data beyond what is in a usual scheme. These are useful in deformation theory. To define this we first have

Definition C.26. Let A be a Noetherian, commutative topological ring. A is *linearly topologized* if the topology admits a fundamental system of neighborhoods of 0 that consist of ideals of A.

Definition C.27. An *ideal of definition* J for such a ring is an open ideal s.t. $\forall 0 \in V \subset A$ open nbhd, there exists a positive integer n s.t. $J^n \subset V$.

Note that this necessarily makes $\{J^n\}$ into a neighborhood basis of 0. Hence, a general basis of A is cosets of J^n, so open sets of unions of cosets of J^n.

We say a linearly topologized is ring is *preadmissible* if it admits an ideal of definition, and is *admissible* if also complete. (Recall, a $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ at p).

Now assume that A is admissible, and let J be the ideal of definition. Then a prime ideal is open iff it contains J (definitely requires contains J. Also, since $p \supset J$ means $p \supset J^n, \forall n$ this means p is made up entirely of cosets of J^n is is thus open.)

Definition C.28. The set of open prime ideals in A, or equivalently the prime ideals of A/J, is the underlying topological space of the *formal spectrum* of A, $\mathrm{Spf}(A)$

- $\mathrm{Spf}\, A$ has a structure sheaf from using the spectrum of the ring
- Let J_λ be a neighborhood basis of 0 consisting of ideals of def.
- All the spectra of $A/\mathbb{J}_\lambda$ have the same underlying top space but a different structure sheaf.

Definition C.29. The structure sheaf of $\mathrm{Spf}\, A$ is the inverse limit $\lim_{\leftarrow\lambda} \mathcal{O}_{\mathrm{Spec}\, A/J_\lambda}$

If $f \in A$ and D_f the set of all open primes not containing f then $\mathcal{O}_{\mathrm{Spf}\, A}(D_f) = \hat{A}_f$ the completion of the localization of A at f.

Finally, we get,

Definition C.30. A *locally Noetherian formal scheme* is a topologically ringed space $(X, \mathcal{O}_X)$ (so the sheaf of rings is a sheaf of top rings) such that each point of X admits an open nbhd isomorphic (as top ringed spaces) to the formal spectrum of a Noetherian ring.

Example C.2.

- For any ideal I get I-adic topology with basis $a + I^n$. This is admissible and preadmissible if A is I-adically complete. Then $\mathrm{Spf}\, A$ is the top space $\mathrm{Spec}\, A/I$ with sheaf of rings $\lim_n \mathcal{O}_{\mathrm{Spec}\, A/I^n}$.
- Say $A = k[[t]]$ and $I = (t)$. Then $A/I = k$, so $\mathrm{Spf}\, A = \{(t)\} = pt$ and the structure sheaf takes the value $k[[t]]$.
- In comparison, $\mathrm{Spec}\, A/I$ has structure sheaf takes value k at this point. This is idea that $\mathrm{Spf}\, A$ is a *formal thickening* of $\mathrm{Spec}\, A$ around I.
- Consider the closed subscheme X of the affine plane over k given by ideal $I = (y^2 - x^3)$. Then $A_0 = k[x, y]$ is not I-adically complete. Write A as the completion. Then $\mathrm{Spf}\, A = X$ in this case as spaces, and we also have its structure sheaf is $\lim_n k[x, \widetilde{y}]/I^n$. The global sections are A as opposed to X which has global sections A/I.

Definition C.31. A *formal disk* in the differential setting is: for any space X and point $x : * \to X$ then the formal disk in X at x is the infinitesimally thickened point that is the formal neighborhood of the point. If dimension makes sense it has the same dimension as X.

Another concept that is important is the formal neighborhood of the diagonal which basically is pairs of points that are infinitesimally close. More formally we have

Definition C.32. The *formal neighborhood of the diagonal* is the formal completion of the map $\Delta : X \to X \times X$.

If X is a scheme then it is the formal scheme around X.

The opposite of being a formal scheme is

Definition C.33. A scheme is *reduced* if it has no "purely infinitesimal directions". More specifically, if all stalks of the structure sheaf are local rings without nonzero nilpotent elements.

Equivalently, X is reduced iff it can be covered by a family of open sets $\{U_\alpha\}$ s.t. $\mathcal{O}_X(U_\alpha)$ has no nonzero nilp elements. i.e., for every open $U \subset X$ need $\mathcal{O}_X(U)$ has no nonzero nilp.