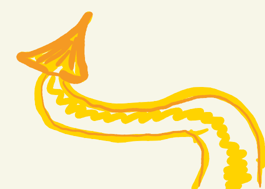
Partion functions, L-functions & Point Counting: A Geometric Langlands Fairytale

by Meenakshi McNamara
Supervised by Kevin Costello



Invitation to the story ⇝ Dualities

Physics

- electric-magnetic duality

- T-duality:

$\mathbb{R}^{25} \times S^1$

$$\boxed{\begin{matrix}\text{radius}\\ R\end{matrix}} \longleftrightarrow \boxed{\begin{matrix}\text{radius}\\ \alpha'/R\end{matrix}}$$

large R ⇝ continuum $p^{25} = \frac{n}{R}$
small R ⇝ continuum winding #

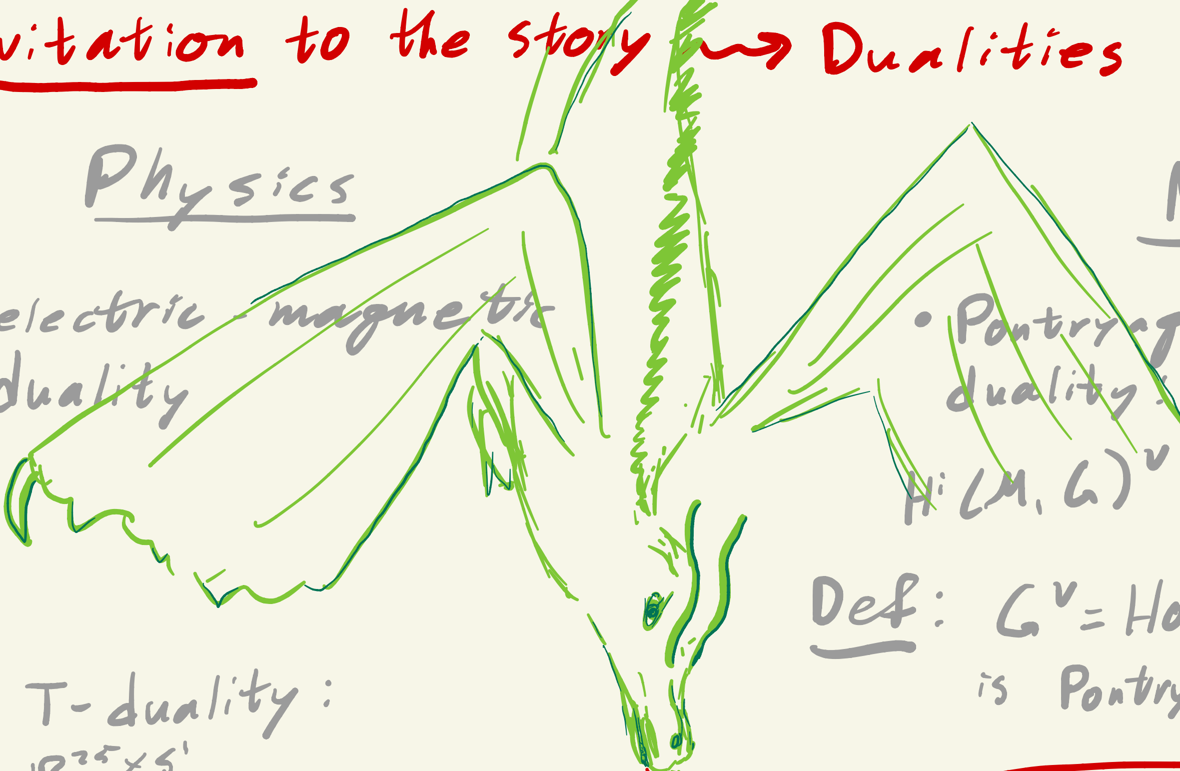
Math

- Pontryagin-Poincaré duality:

$$H^i(M, G)^\vee \cong H^{n-i}(M, G^\vee)$$

Def: $G^\vee = \mathrm{Hom}(G, U(1))$ is **Pontryagin dual group**

Invitation to the story ⇝ Dualities



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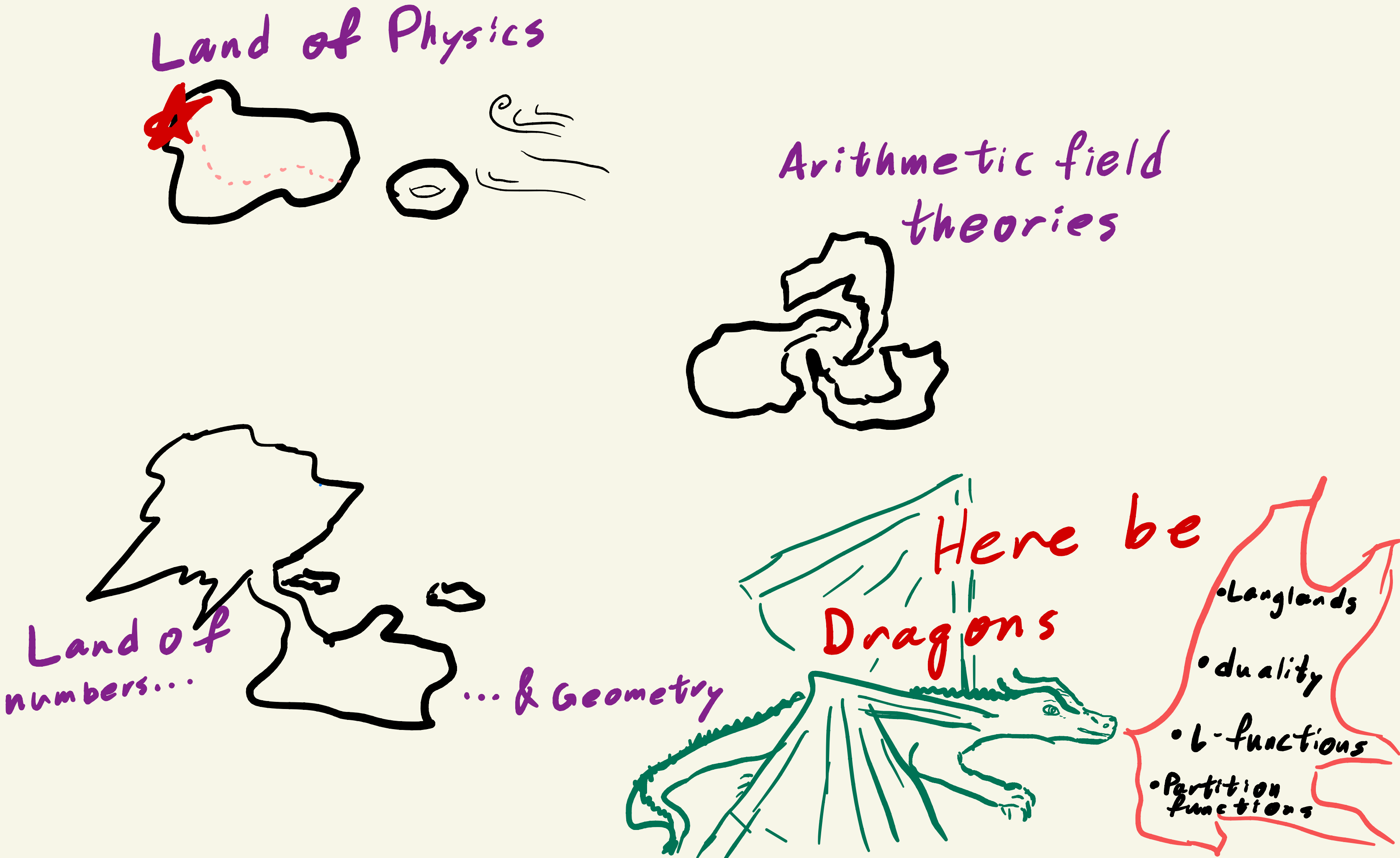
Math

- Pontryagin - Poincaré duality:

$$H^i(M, G)^\vee \cong H^{n-i}(M, G^\vee)$$

<u>Def</u>: $G^\vee = \mathrm{Hom}(G, U(1))$ is Pontryagin dual group

Challenge: Non-abelian


Land of Physics
Arithmetic field theories
Land of numbers...
...& Geometry
Here be Dragons
•Langlands
•duality
•L-functions
•Partition functions

The Quest Begins: Quantization

Begin with Phase space (M, ω)

T^*N

Prequantize: A Hermitian line bundle L w/ ∇ so $F = \omega$

$\omega = dp \wedge dq = d\alpha$ use $L = N \times \mathbb{C}$ with α

Polarize: Pick Foliation with Lagrangian leaves

$$\mathcal{H} = \{ s \in \Gamma(M, L) \mid \nabla_X s = 0 \ \forall X \in \Gamma(M, F) \}$$

$L^2(N, \mathbb{C})$

Land of Physics

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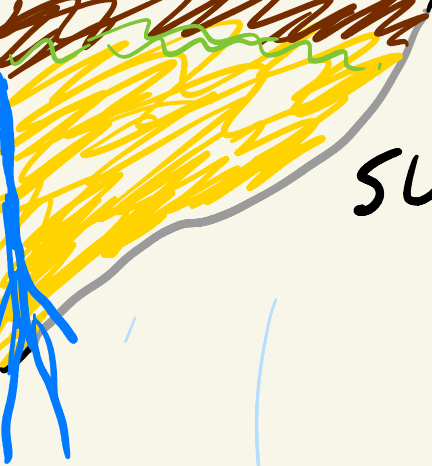
A TWIST in the story

SUSY theories have super operator Q

↳ they exchange particles with super partners

↳ $\boxed{Q^2 = 0}$ ⇝ cohomological twisted topological field theory



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(super) Maxwell theory

$\mathcal{H} = L^2(\mathcal{C}(M))$ for $U(1)$ + fields

A-twist ⟷ B-twist

$\mathcal{H} = H^\bullet_{U(1)}(\mathrm{Pic}(M))$ ⟷ $\mathcal{H} = H^{\bullet,\bullet}(\mathrm{Loc}_{\mathbb{C}^\times}(M))$

t'Hooft ops ⟷ Wilson lines

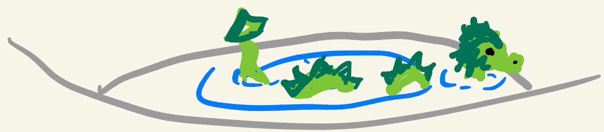
...

Island of TQFTs

$$\mathrm{Bord}_n \xrightarrow{Z} \mathrm{Vec}$$

Def:

- $(n-1)$  $\longmapsto$ V. S. (Hilbert space)
- n  $\longmapsto$ linear operator



(Dijkgraaf-Witten)

Ex: $(n=2)$ Phase space $\mathcal{M} = \mathrm{Loc}_G M^2 := \boxed{\{\pi_1(M^2) \to G\}/\mathrm{conj}}$

- $Z($  $) = \mathbb{C}[G/G]$
-  $\longrightarrow$ $\mathrm{Loc}_G N_1 \leftarrow \mathrm{Loc}_G M \rightarrow \mathrm{Loc}_G N_2$

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$$\mathrm{Bord}_n \xrightarrow{Z} \mathrm{Vec}$$

Def:

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- n  $\longmapsto$ linear operator

Extended

- $(n-k)$ $\bullet$ $\longmapsto$ category (higher)

Kapustin 2010

(Dijkgraaf-Witten)

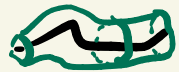
Ex $(n=2)$ Phase space $\mathcal{M} = \mathrm{Loc}_G M^2 := \{\pi_1(M^2) \to G\}/\mathrm{conj}$

- $Z(\bigcirc) = \mathbb{C}[G/G]$
-  $\rightsquigarrow$

$$\begin{array}{ccc} & \mathrm{Loc}_G M & \\ \swarrow & & \searrow \\ \mathrm{Loc}_G N_1 & & \mathrm{Loc}_G N_2 \end{array}$$

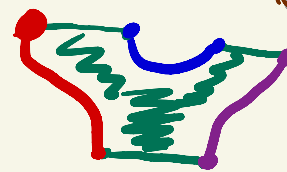
- $Z(\bullet) = \mathrm{Vect}(pt/G) \simeq \mathrm{Rep}\, G$

Defects & Boundaries

- local operator  → S^{n-1}
- line operator  → S^{n-2}

⋮

→ Boundary conditions & interfaces



Defects & Boundaries

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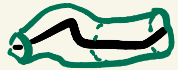
- line operator  $\to S^{n-2}$

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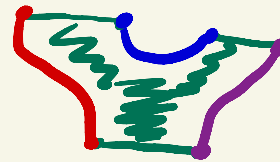
$\rightsquigarrow$ Boundary conditions & interfaces



Dijkgraaf-Witten

G-bundles theory = σ-model to pt/G

$\rightsquigarrow$ bdry cond is map $Y \to pt/G$ so $Y = X/G$

$X \circlearrowleft G$

Land of physics

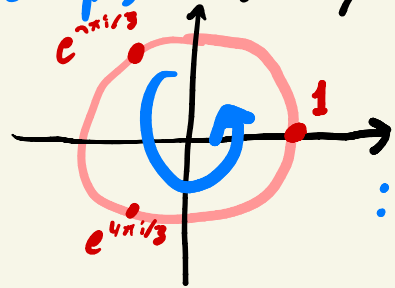
Arithmetic field theories

Land of numbers... ...& Geometry

Land of Numbers

Galois groups study symmetries of polynomials

Ex: $x^3 - 1 \rightsquigarrow$



$\cong \mathbb{Z}_3$

$:= \mathrm{Gal}(\mathbb{Q}(e^{2\pi i/3})/\mathbb{Q})$

Land of Numbers

Galois groups study symmetries of polynomials

Ex: $x^3 - 1 \rightsquigarrow$

$\cong \mathbb{Z}_3 := \mathrm{Gal}(\mathbb{Q}(e^{2\pi i/3})/\mathbb{Q})$

Tool: Want analysis: $\mathbb{Q} \to \mathbb{R}$ by completing with $|\cdot|$.

↳ or Def: $\mathbb{Q}_p$ by completing via

$$\left|\tfrac{a}{b}p^k\right|_p = p^{-k}$$

"local" fields combine to study global

Land of Numbers

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$\hookrightarrow$ or Def: $\mathbb{Q}_p$ by completing via

$$\left|\tfrac{a}{b} p^k\right|_p = p^{-k}$$

Tool: L-functions: $L(V^{\bullet} \circlearrowleft F, t) = \prod_k \det(1 - tF)_{H^k/V}^{(-1)^{k+1}}$

$= Tr_{gr}(F, Sym^{\bullet} V)$

Ex: $L(s) = \zeta(s) = \prod \frac{1}{1 - p^{-s}}$

Schemes Behind the Scenes

Algebra ⟷ Geometry

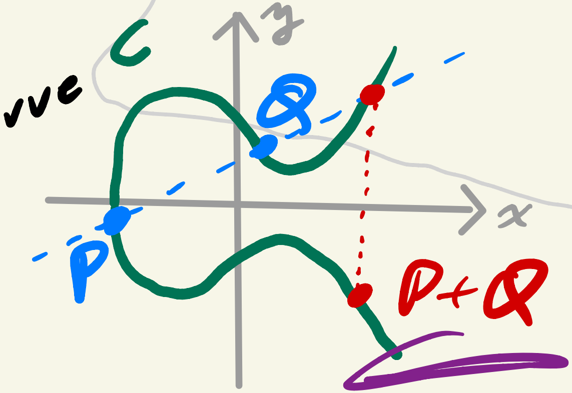
Algebra

- Function field
  $K(C)$
  $\hookrightarrow \mathbb{F}_3(t),\ \dfrac{\mathbb{F}_5(s,t)}{(s^2-t)}$

- $\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) = \langle \mathrm{Frob}_q \rangle$
  $\mathrm{Frob}_q(x) = x^q$

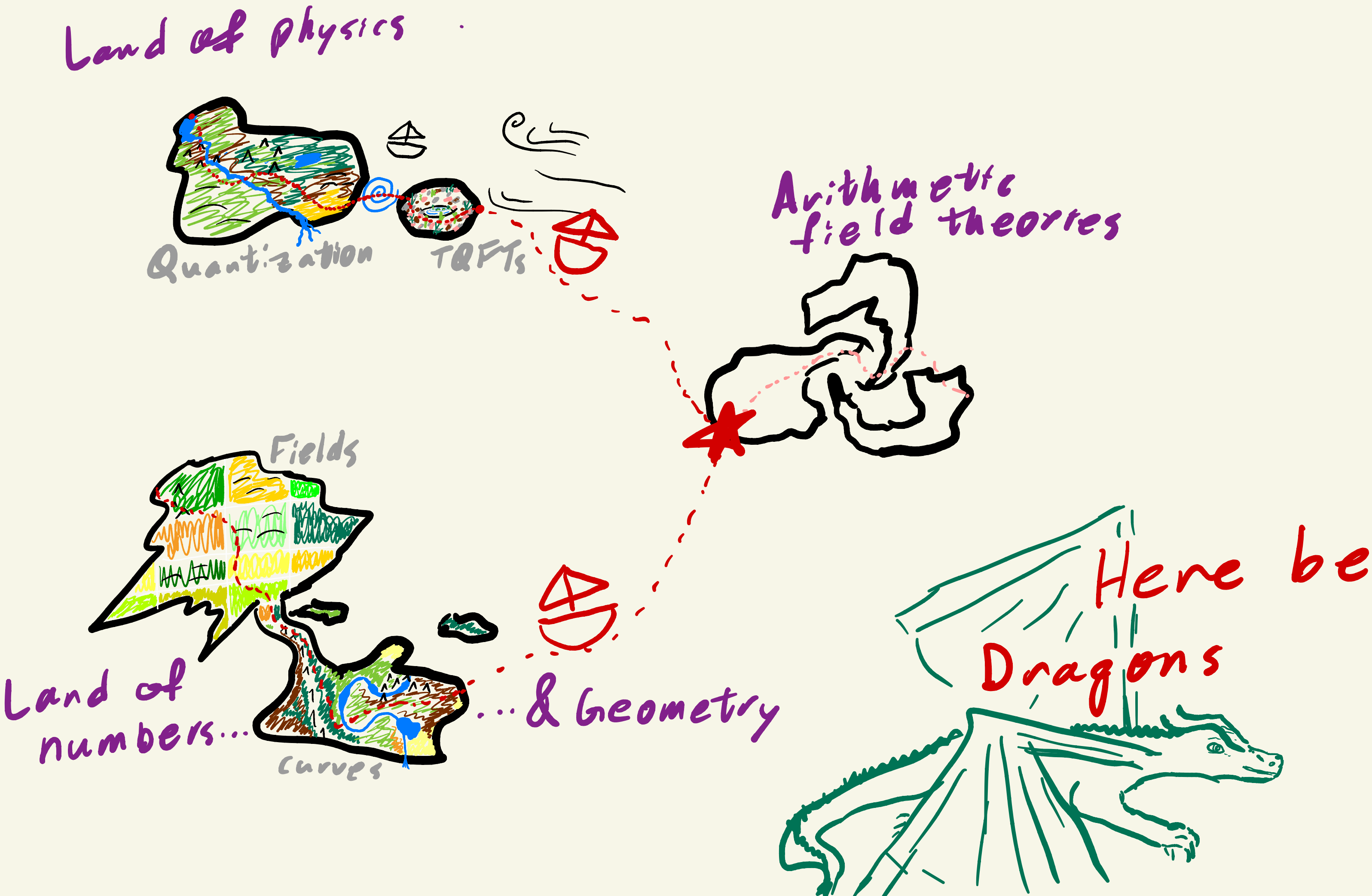
Geometry

- Curve



Elliptic curves are abelian varieties

- Frob endomorphism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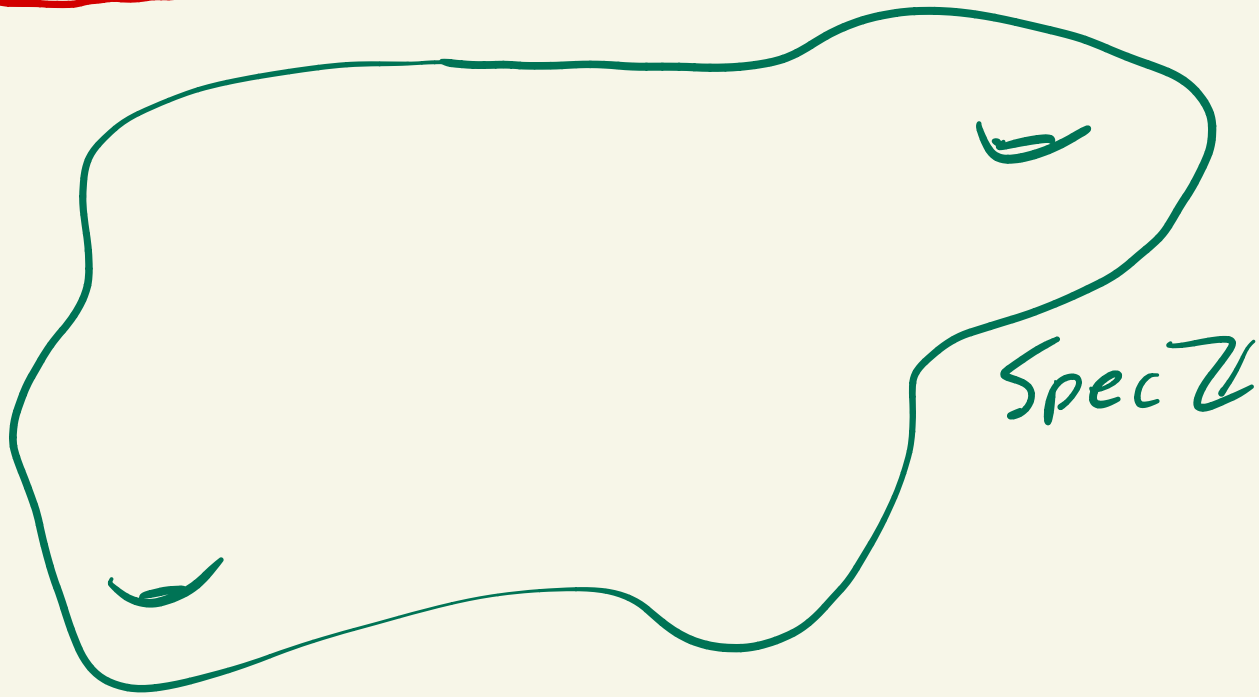
Arithmetic field theories

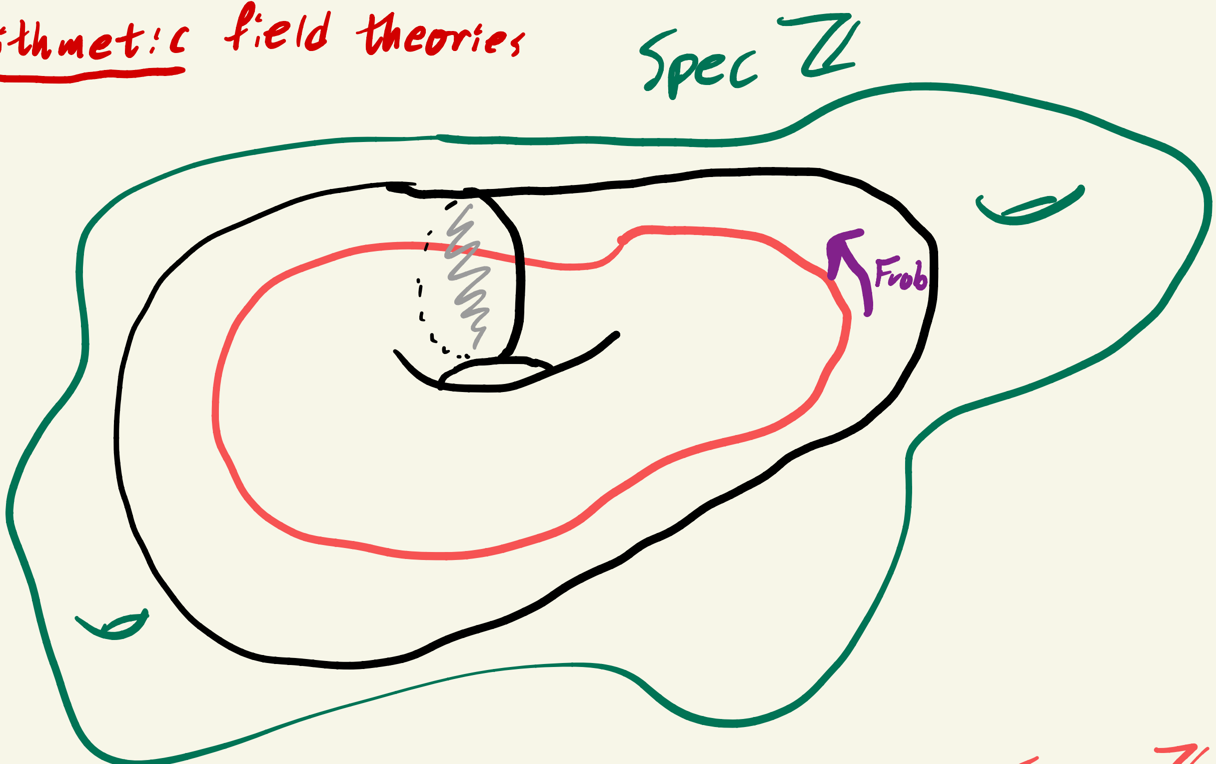


Thm (Artin-Verdier)

$$H^i(\operatorname{Spec}\mathbb{Z}, \mathcal{F}) \times \operatorname{Ext}^{3-i}(\mathcal{F}, \mathbb{G}_m) \to H^3(X, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$$

Arithmetic field theories

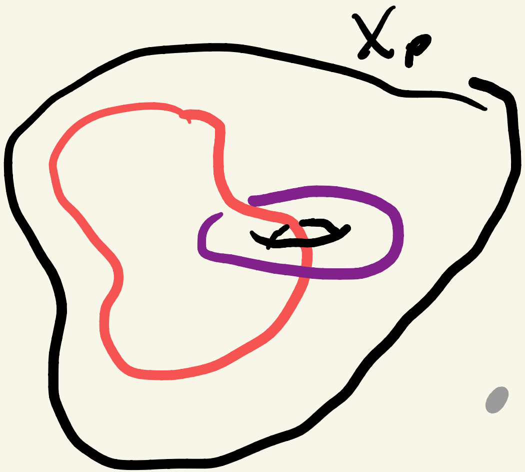
Morishita



$$\begin{array}{ccccc}
\mathrm{Spec}\,\mathbb{Q}_p & \hookrightarrow & \mathrm{Spec}\,\mathbb{Z}_p & \twoheadrightarrow & \mathrm{Spec}\,\mathbb{Z}_p/p = \langle \mathrm{Frob}_p \rangle \\
\pi_1 \downarrow & & \downarrow \pi_1 & & \downarrow \pi_1 \\
\mathrm{Gal}(\mathbb{Q}_p) & \hookrightarrow & \varprojlim_L \mathrm{Gal}^{\mathrm{unr}}(L/\mathbb{Q}) & \twoheadrightarrow & \hat{\mathbb{Z}}
\end{array}$$

Linking it all together

Kim, Morishita



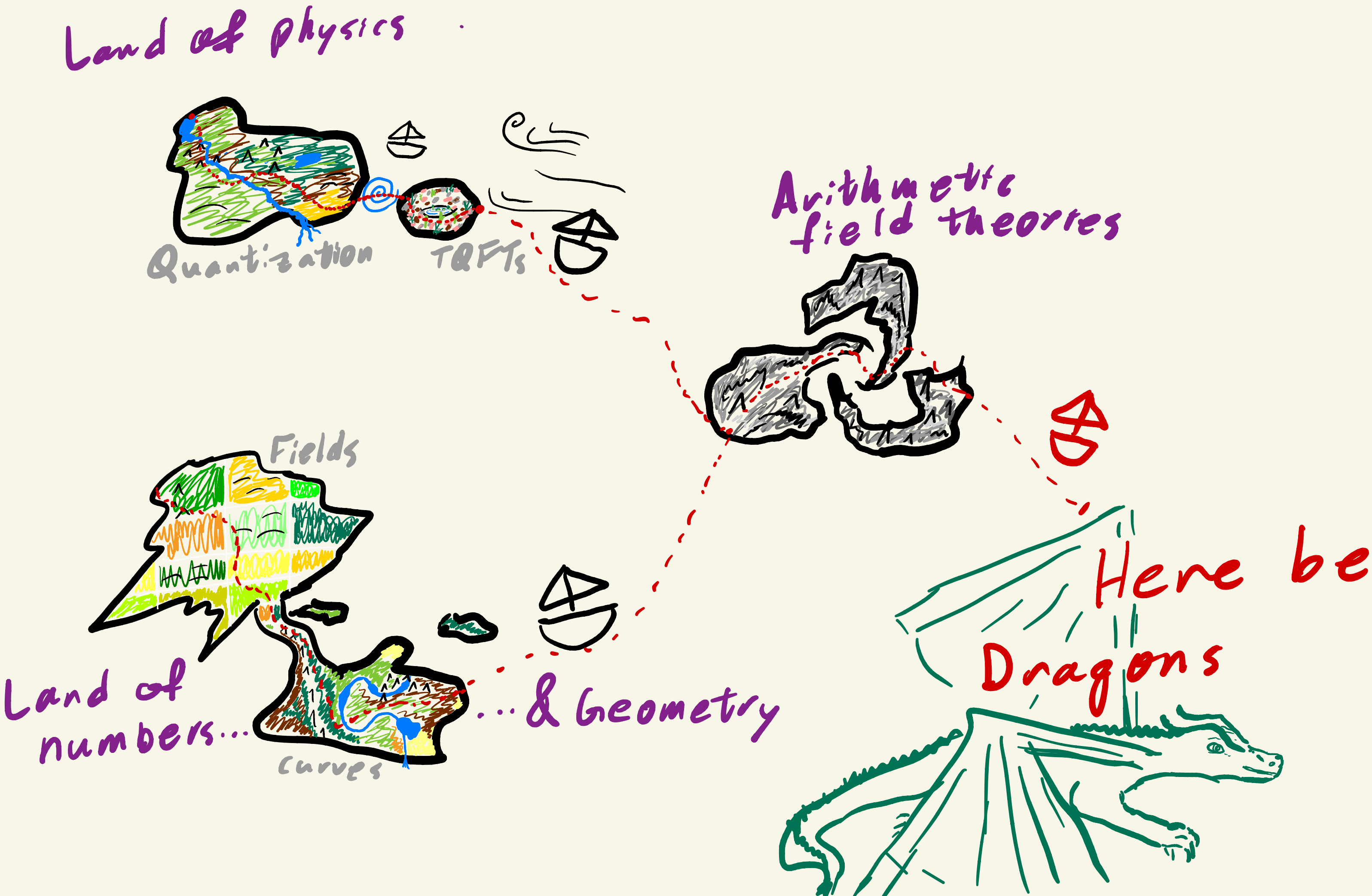
Thm: For primes p, q we have

$$(-1)^{lk_2(p,q)} = \left(\frac{p}{q}\right)$$

- Double cover X_p w/ Y_p
- $\pi_1(X_p) \to Gal(Y_p/X_p) = \mathbb{Z}/2\mathbb{Z}$
  $Frob_q \mapsto lk(p,q)$
- $Frob_q/Y_p = id$ iff $Frob_q(\sqrt{p}) = \sqrt{p}$
  iff $\left(\frac{p}{q}\right) = 1$

Comment:

CS $\mathbb{Z}/n$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Finding the Dragon (Kapustin-Witten, 2007)

$\mathcal{N}=4$ SYM w/ G non-abelian $K = k(X)$

Replace Pic with $\mathrm{Bun}_G(X)(k) = G(K)\backslash G(\mathbb{A}_k)/G(\mathcal{O}_{\mathbb{A}_k})$

Quantum theory: functions on $\mathrm{Bun}_G(X)$

Finding the Dragon

$\mathcal{N}=4$ SYM w/ G non-abelian $\qquad K=k(X)$

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Thm (Geometric Satake)

$$\mathcal{H}_x \simeq \mathrm{Rep}\, G^\vee$$
$$\|$$
$$\mathrm{DMod}(G(\mathcal{O}_x)\backslash G(K_x)/G(\mathcal{O}_x))$$
$$\mathrm{DMod}\,\mathrm{Bun}_G(X)$$

via inserting line op at x

Physically

- $\mathcal{A}_G(☺) = (\mathrm{DMod}_{G(\mathcal{O}_x)}(G(K_x)/G(\mathcal{O}_x)), *)$

  't Hooft lines

$\updownarrow$

- $\mathcal{B}_{G^\vee}(☺) = \mathrm{QCoh}(\mathrm{Loc}_{G^\vee} S^2) \simeq (\mathrm{Rep}\, G^\vee, \otimes)$

  $\{\mathrm{Gal}(K) \to G^\vee\}/G^\vee$

  Wilson lines

... want to "spectrally decompose" $\mathrm{DMod}\,\mathrm{Bun}_G(X)$ as "$\mathrm{QCoh}(Y)$" for some Y.

Think $Y \simeq \mathrm{spec}\,\mathcal{H}$

Geometric Langlands

"$\mathrm{DMod}\,\mathrm{Bun}_G(X) \simeq \mathrm{QCoh}\,\mathrm{Loc}_{G^\vee}(X)$"

duality

automorphic side

Galois side

(this is false, but close)

Catching the Dragon

Currently

$$\begin{tikzcd} & \mathcal{N}=4 \text{ SYM in 4D} \arrow[dl, "\text{twist}"'] \arrow[dr, "\text{twist}"] & \\ \text{A-twist} \arrow[rr, leftrightarrow] & & \text{B-twist} \end{tikzcd}$$

TFTs

Catching the Dragon

Currently $\mathcal{N}=4$ SYM in 4D

A-twist $\longleftrightarrow$ B-twist   TFTs

use "pull-push" maps

Boundary theory $\overset{?}{\longleftrightarrow}$ Rozansky-Witten theory

Hamiltonian $G^\vee$-space gives boundary theory

Topological σ-model to $G^\vee$-space.

Befriending the Dragon

Focus on symplectic V.S. $V = T^*W \otimes G^\vee$

"3 mnfd" $= C/\mathbb{F}_q = M$ "2 mnf" $= \bar{C} = \Sigma$

$\simeq \bar{C} \times [0,1]/\sim$ where $(x,0) \sim (\mathrm{Frob}\cdot x, 1)$

$\Rightarrow \boxed{H = \log \mathrm{Frob}}$, $\mathcal{H}$ = funcs on $\Pi H^\bullet(\bar{C}) \otimes W$

$\cong \Lambda^\bullet H^1 \otimes \mathrm{Sym}^\bullet(H^0 \oplus H^2)$

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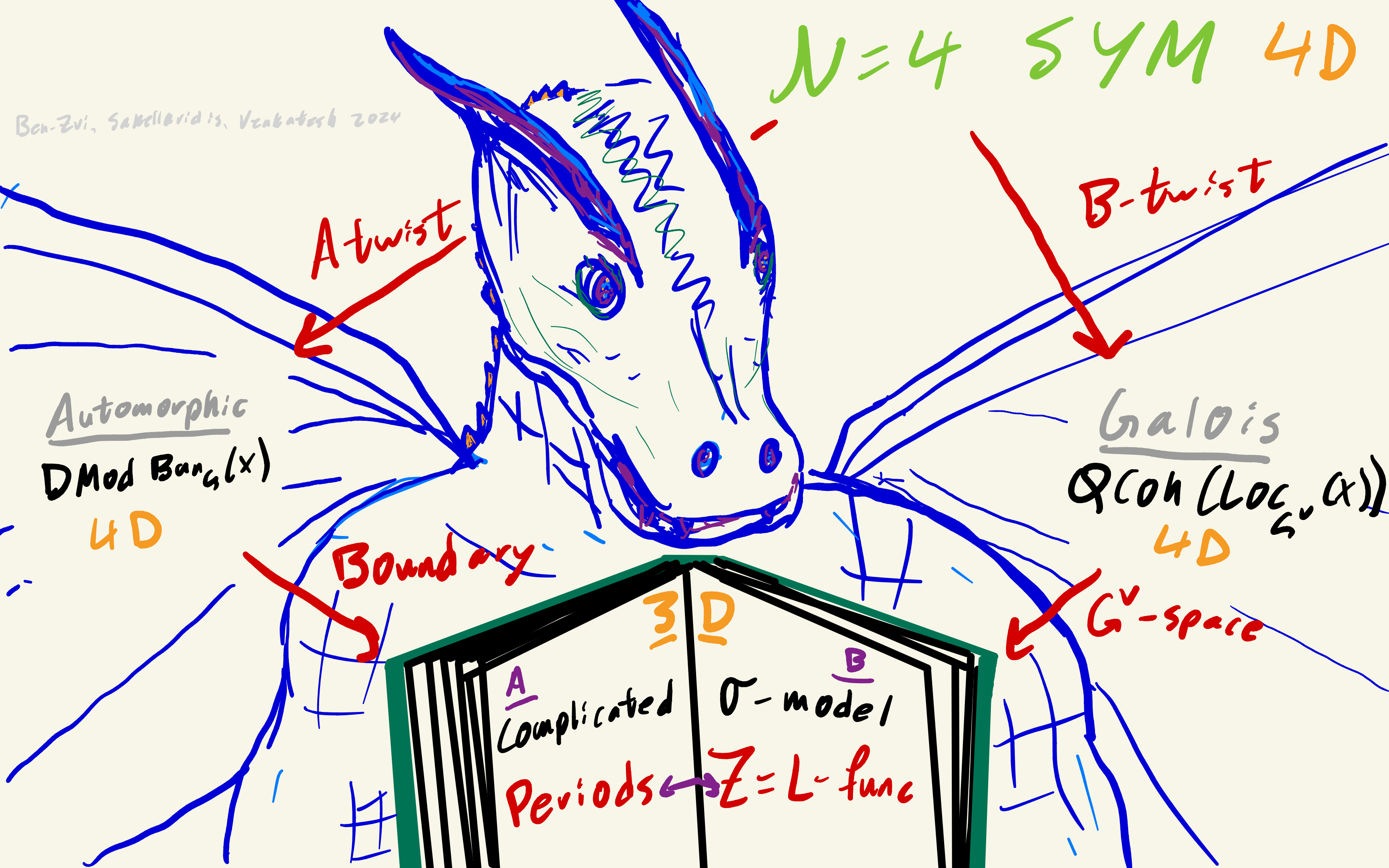
$$\boxed{Z = \mathrm{Tr}\, e^H = \mathrm{Tr}_{\mathcal{H}}\, \mathrm{Frob} = L(H^\bullet(\bar{C}) \circlearrowleft \mathrm{Frob})}$$



N=4 SYM 4D
Ben-Zvi, Sakellaridis, Venkatesh 2024
A twist
B-twist
Automorphic
DMod Bun$_G$(X)
4D
Galois
QCoh(Loc$_{G^\vee}$(X))
4D
Boundary
$G^\vee$-space
3D
A
Complicated
B
σ-model
Periods ↔ Z=L-func

Riding the Dragon

Question: Instead of V.S. try circle group?

Idea: CS "=" $(RW)^{-2}$ $\rightsquigarrow$ Use CS theory!

Riding the Dragon

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Idea: CS "=" $(RW)^{-2}$ $\rightsquigarrow$ use CS theory!

Quantizing

$G = \mathbb{Q}_\ell^2/\mathbb{Z}_\ell^2$ with level $\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$

$\rightsquigarrow \mathcal{P} = H^1(\bar{C}, \mathbb{Q}_\ell) \oplus H^1(\bar{C}, \mathbb{Q}_\ell) / H^1(\bar{C}, \mathbb{Z}_\ell)^2$

$\underset{x_1, x_2}{\Downarrow}$ $\underset{y^1, y^2}{\Downarrow}$ $\rightsquigarrow \omega = k(dy^1 \wedge dx_1 + dy^2 \wedge dx_2)$

$\Rightarrow \mathcal{H} \cong \mathbb{C}[\mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/k\mathbb{Z}]$

Treasure!

$$Z = \operatorname{Tr} \mathrm{Frob}_q = \#\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in (\mathbb{Z}/k\mathbb{Z})^2 \,\middle|\, \mathrm{Frob}_q \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \right\}$$

Thm For C an elliptic curve,

$$Z = \#\left\{ (C[k])^{\mathrm{Frob}_q} \right\}$$

21

Treasure!

$$Z = \operatorname{Tr} \mathrm{Frob}_q = \#\left(\begin{pmatrix} a \\ b \end{pmatrix} \in (\mathbb{Z}/k\mathbb{Z})^2 \,\middle|\, \mathrm{Frob}_q \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \right\}$$

Thm For C an ~~elliptic~~ curve,

$$Z = \#\left\{ \left(\cancel{C[k]} \right)^{\mathrm{Frob}_q} \right\}$$

$\mathrm{Jac}(C)[k]$

$\mathrm{Jac}(C)$ = Degree zero line bundles on C

Summary


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Pf details

Curve genus $g \rightsquigarrow h = \mathbb{C}[(\mathbb{Z}/k\mathbb{Z})^{2g}]$ so Frob is $2g \times 2g$

$$\cong H^1(\bar{C}, \mathbb{Z}/k\mathbb{Z})$$

so $$Z = \#\left(H^1(\bar{C}, \mathbb{Z}/k\mathbb{Z})\right)^{\mathrm{Frob}_q}$$

$$0 \to \mu_k \to \mathbb{G}_m \xrightarrow{(\cdot)^k} \mathbb{G}_m \to 0$$

$$\Downarrow$$

$$\cdots \to H^0(\bar{C}, \mathbb{G}_m) \xrightarrow{\text{invert}} H^0(\bar{C}, \mathbb{G}_m) \xrightarrow{0} H^1(\bar{C}, \mu_k) \hookrightarrow H^1(\bar{C}, \mathbb{G}_m) \to H^1(\bar{C}, \mathbb{G}_m) \to \cdots$$

$$H^0(\bar{C}, \mathbb{G}_m) = \mathcal{O}_C(C)^\times$$

$$H^1(\bar{C}, \mu_k) \cong \ker\left(\mathrm{Pic} \xrightarrow{(\cdot)^k} \mathrm{Pic}\right) = \mathrm{Pic}(C)[k] = \mathrm{Jac}(C)[k]$$

$$H^1(\bar{C}, \mathbb{G}_m) = \mathrm{Pic}(C)$$

$\blacksquare$

Twist

Ex $N=1$ in 1D. HS is super space of diff forms

$$Q = d,\ Q^*,\ H = \Delta \implies [Q, Q^+] = H$$

$\rightsquigarrow$ H acts on Q cohomology by zero!
"killed time"

If higher dim:

- A: $HS = H^\bullet_{dR}(X)$

- B: $HS = H^{0,\bullet}(X)$

Langlands

"Def": $G^\vee$ is gp s.t.

$$\left\{ G(\mathcal{O}_x) \text{ orbits on } G(K_x)/G(\mathcal{O}_x) \right\} \xleftrightarrow{\text{Geom Satake}} \left\{ \text{irreps of } G^\vee \right\}$$

Corrected geometric Langlands:

$$\mathrm{DMod}(\mathrm{Bun}_G(X)) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{Loc}_{G^\vee}(X))$$

- Deals with singularities
- Derived properly

More Langlands intuition

Want Y so $\text{DMod}\,\text{Bun}_G(X)$ "$\simeq$" $\text{QCoh}(Y)$"

Use $\forall x \in X$ $\quad \text{Loc}_{G^\vee}(X) \to \dfrac{G^\vee}{G^\vee}$

$\rho \mapsto \rho(\text{Frob}_x)$

Now

$\mathcal{O}\left(\frac{G^\vee}{G^\vee}\right) = |\text{Rep}\, G^\vee|$

$\mathcal{H} = \bigotimes_x \mathcal{H}_x \curvearrowright \mathcal{O}(\underbrace{\text{Loc}_{G^\vee}(X)}_{\text{hope } Y \text{ is this}})$

Chern-Simons

$$S = \frac{k}{4\pi}\int_{M^3} \mathrm{tr}\left(A\wedge dA + \frac{2}{3} A\wedge A\wedge A\right),$$ Gauge group G

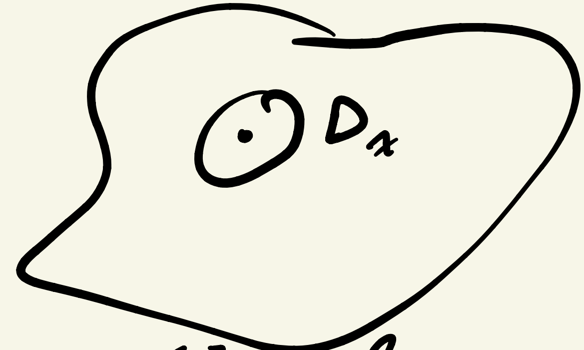
↳ Phase space = flat connections

↳ observables Wilson lines $W_\gamma = P\exp\left(i\int_\gamma A\right)$

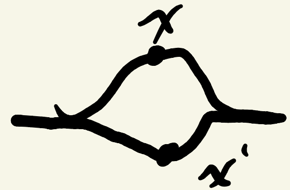
$$\boxed{\langle W_{\gamma_1}, W_{\gamma_2}\rangle = \exp\left(\frac{2\pi i}{k}\, lk(\gamma_1, \gamma_2)\right)}$$

Hecke Category

At $x \in X \rightsquigarrow$ formal disk D_x



$\downarrow$

D'_x 

$\simeq S^2$ = line ops

$$\mathcal{H}_x = \mathrm{DMod}\,\mathrm{Bun}_G(D'_x)$$

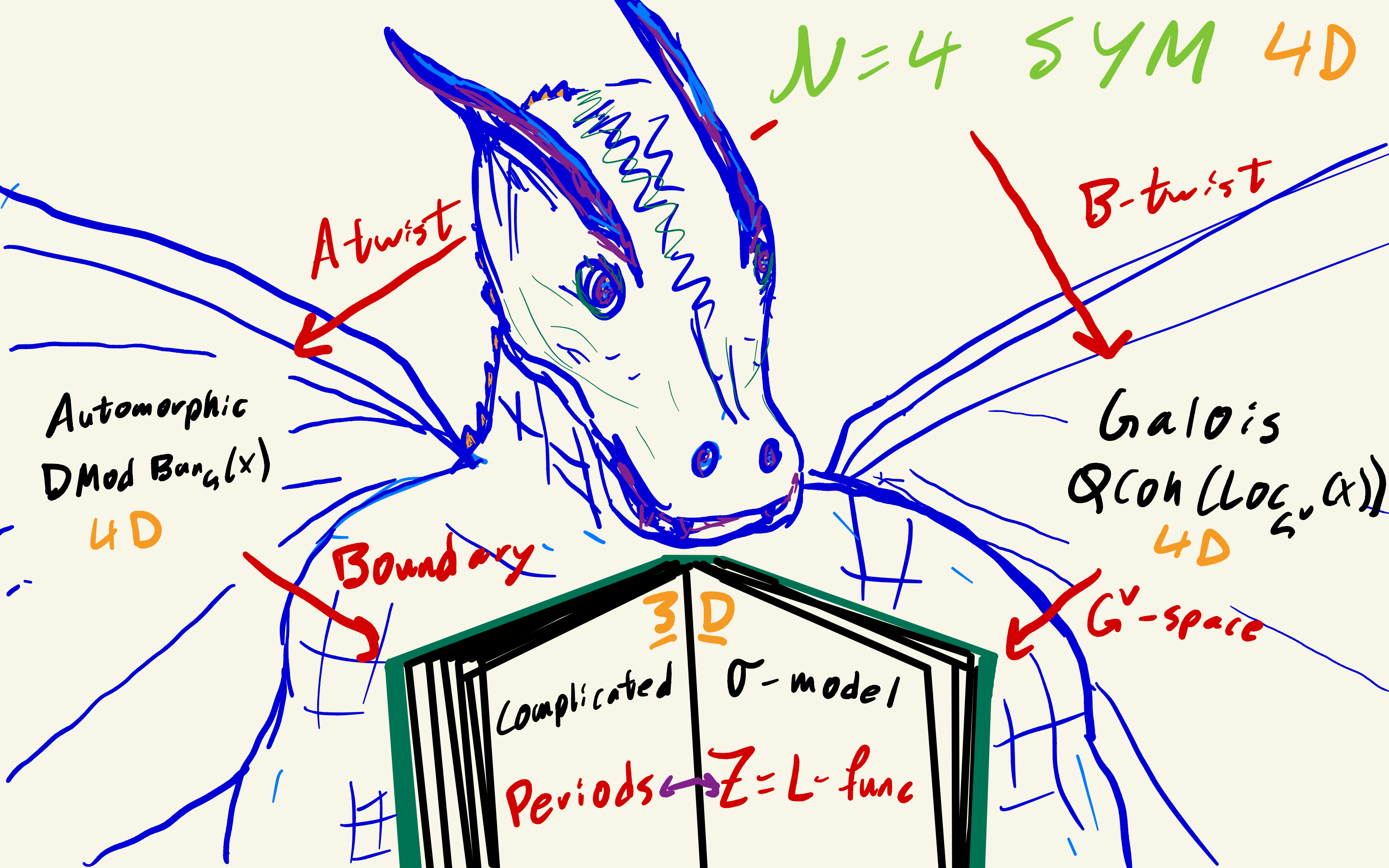
Also, $\mathrm{Fun}(H\backslash G/H)$ encodes actions on H-inv for all G-reps

Here $\mathcal{H}_x \simeq \mathrm{DMod}(G(\mathcal{O}_x)\backslash G(K_x)/G(\mathcal{O}_x))$

$\circlearrowleft \mathrm{DMod}\,\mathrm{Bun}_G(X)$ as $G(\mathcal{O}_x)$-inv of $\underset{\text{rep}}{G(K_x)}$

$$BG = EG/G$$

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & EG \arrow[d] \\ X \arrow[r, "f"] & BG \end{tikzcd}$$



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