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STA 1311: PROBABILITY 1 (Part A) Lecture Notes

Note: These notes are based upon a work by ***Mallam Hassan Adamu Shitu*** of Department of Mathematical Sciences, Bayero University, Kano.

Introduction:

The subject of probability theory began on the 19th century due to the increasing demands on Mathematicians such as Fermat, Pascal, Leibniz, and James Bernoulli, by gamblers to come up with optimum strategies for various games of chances. Moreover, with the advent of probability theories, it was realized that statistics could be used in drawing valid conclusions and making reasonable decisions on the basis of analysis of data, such as sampling theory and predictions or forecasting. The subject of statistics originated much earlier than probability and dealt mainly with the collection, organization and presentation of data in tables and charts.

The world “probability” is often used in day to day activities to describe ***uncertainty situations***. For instance, we may say “*Probably*, I will pass this course”, “Nigeria will *probably* win the A.C Nations”, etc. In all the cases, one is not certain about the possible outcomes; however, supportive arguments/reasons of making the statements are available. Example passing exams: hard work, brilliance, attending lectures, etc. These reasons are what are called data. The recorded *information* in its original collected form is called *data*.

Definition 1 (Probability):
Probability can be defined as the mathematical study of *uncertainty*.

It has wide application in areas such as Sciences, Agric, traffic-studies, Engineering, etc. The study of set theory is very essential in both probability and statistics.

Set Theory

A fundamental concept in all branches of mathematics in general, is the concept of set.

Definition 2 (Set):
A set can be defined as a well-defined collection of objects.

Examples;

- The set of students from Katsina,
- the set of local governments in Nigeria,
- set of fruits,
- set of odd numbers less than 30,
- multiples of 6, etc.

Each object in a set is called an ***element*** or a ***member*** of the set. A set is denoted by a capital letter such as *A, B, X, Y…*, etc and an element by a lower case letters such as *a, b, x, y…*, etc. If an element *x* belongs to a set *X* we write $x \in X$ otherwise we use $x \notin X$.

<u>Notation and Describing Sets</u>

A set can be described in two ways:

1. First, by direct enumeration of members in a curly bracket separated by a comma: Examples
   a. $B = \{2, 4, 6, 7\}$,
   b. $H = \{\text{Niger,Ghana,Cameroun}\}$, etc

2. Second, a set can b e described by a statement or rule, popularly called *set-builder notation*. Examples:
   a. $B = \{$the set of all Nigerian military head of states$\}$
   b. $H = \{$The set of all natural numbers less than 50$\}$
   c. $A = \{$The set of all states in Nigeria that implement Sharia law$\}$
   d. $X = \{$Thes set of all numbers between -2 and 5$\}$, etc.

   by rule, the above sets can be described as follows
   e. $B = \{x \mid x$ is a Nigerian military head of states$\}$
   f. $H = \{y : y \in$,less than 50$\}$
   g. $A = \{a : a$ is a state in Nigeria that implement Sharia law$\}$
   h. $X = \{x : -2 \le x \le 5\}$

Type of Sets:

1. **Subset**: Suppose every element of a set *A* also belongs to a set *B*, that is suppose $a \in A$ implies $a \in B$. Then *A* is called a *subset* of *B*, or *A* is said to be *contained* in *B*, written as $A \subseteq B$ or $B \supseteq A$. If *A* is not a subset of *B* it is denoted by $A \not\subseteq B$ .If $A \subseteq B$ and $A \neq B$, then we say that *A* is a *proper subset* of *A.*
2. **Equal Sets:** Two sets are equal if both have the same elements or, equivalently, if each is contained in the other. That is $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
3. **Universal Set:** All sets under investigating in any application of set theory are assumed to be contained in some large fixed set called the *universal set* usually denoted by **U.**
4. **Null or Empty Set:** A set with no elements is called the empty set or null set, and is denoted by $\varnothing$. ***Note*** that the null set is a subset of every set and every set is a subset of itself; i.e. for any set *A* we have $\varnothing \subseteq A$ and $A \subseteq A$
5. **Complement of a Set:** Recall that all sets under consideration at a particular time are subsets of a fixed universal set **U.** The *complement* of a set *A*, denoted by A^c, is the set of elements which belong to **U** but which do not belong to *A*, that is $A^c = \{x : x \in \mathbf{U}, x \notin A\}$

Special Symbols

Some sets occur very often in mathematics, and so we use special symbols for them. Some such symbols follow:

- $=$ the *natural numbers* or positive integers: $\{1, 2, 3, \quad\}$
- $=$ all integers; positive, negative and zero: $\{\quad, -2, -1, 0, 1, 2, \quad\}$
- $=$ the real numbers.

Thus we have $\subseteq \quad \subseteq$.

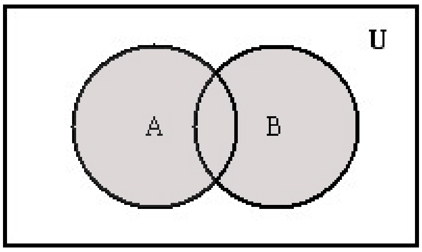
Set Operations

Venn Diagram:

These are the pictorial representations of sets in which a rectangle is used to represent a universal set and a circle is used to represent a sunset.

(a) Union:

The union of two or more sets is the set of all elements which belong to either of the sets or both. It is denoted by $\cup$ i.e., $A \cup B = \{x : x \in A$ or $x \in B\}$

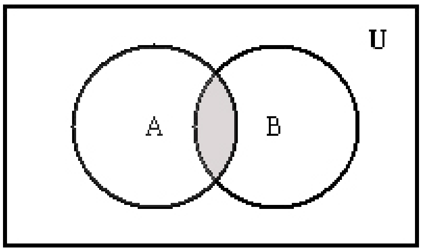


$A \cup B$ is shaded

Example:
1) If $A = \{x : -2 < x < 0\}$, $B = \{y : 4 \geq y \geq -1\}$ then $A \cup B = \{z : -2 < z \leq 4\}$

(b) Intersection:

The intersection of sets is a set of all elements which belong to both sets. It is denoted by $\cap$ i.e., $A \cap B = \{y : y \in A \text{ and } y \in B\}$



$A \cap B$ is shaded

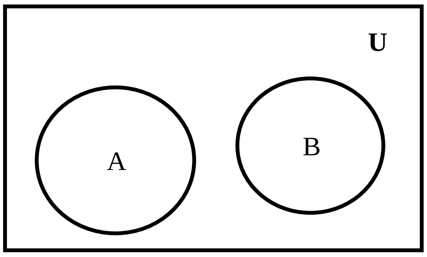
Example:
2) Let $H = \{x : x \text{ is an Engineering student}\}$

$J = \{y : y \text{ is a student from Jigawa State}\}$, then

$H \cap J = \{i : i \text{ is an Engineering student from Jigawa State}\}$

(c) Disjoint:

If $A \cap B = \emptyset$, then A and B are said to be *disjoint sets*.



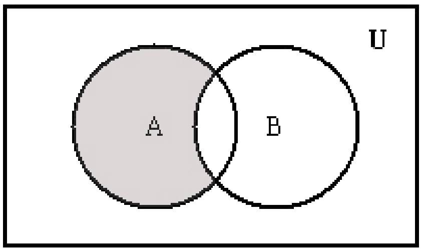
Example:
3) Let $P = \{p : p \text{ is natural number greater than } 20\}$

$Q = \{q : q \text{ is counting number less than } 19\}$, then

$P \cap Q = \emptyset = \{\}$

(d) Difference:

The set containing all elements in A which denote belong to B is called the difference of A and B denoted by $A - B$ or $A \setminus B$. Note that $A \setminus B \equiv A \cap B^c \equiv A \cap B'$



$A \setminus B$ is shaded

Set Theorems

Let *A*, *B*, and *C* be sets, then the following statements hold for sets.

Idempotent Laws	
$1a.\ A \cup A = A$	$1b.\ A \cap A = A$
Associative Laws	
$2a.\ (A \cup B) \cup C = A \cup (B \cup C)$	$2b.\ (A \cap B) \cap C = A \cap (B \cap C)$
Commutative Laws	
$3a.\ A \cup B = B \cup A$	$3b.\ A \cap B = B \cap A$
Distributive Laws	
$4a.\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$4b.\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity Laws	
$5a.\ A \cup \varnothing = A$	$5b.\ A \cap \mathbf{U} = A$
$6a.\ A \cup \mathrm{U} = \mathbf{U}$	$6a.\ A \cap \varnothing = \varnothing$
Involution Law	
$7.\ (A^c)^c = A$	
Complement Laws	
$8a.\ A \cup A^c = \mathbf{U}$	$8b.\ A \cap A^c = \varnothing$
$9a.\ \mathbf{U}^c = \varnothing$	$9a.\ \varnothing^c = \mathbf{U}$
DeMorgan's Laws	
$10a.\ (A \cup B)^c = A^c \cap B^c$	$10b.\ (A \cap B)^c = A^c \cup B^c$

Also not that if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$, also the following statements hold:

(a) $A \cap B = A$ **(b)** $A \cup B = B$

Partitions:

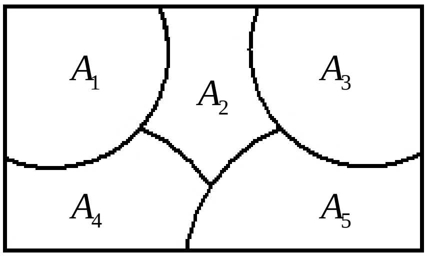
Definition 3 (Partition):
Let *S* be a nonempty set. A *partition* of *S* is a *subdivision* of *S* into nonoverlapping, nonempty subsets. Precisely, a *partition* of *S* is a collection $\{A_i\}$ of nonempty subsets of *S* such that

(i) Each a in *S* belong to one of the A_i

(ii) The sets of $\{A_i\}$ are mutually disjoint; that is, if

$$A_i \neq A_j, \text{ then } A_i \cap A_j = \varnothing$$

The subsets in a partition are called *cells*. Below is Venn diagram of a partition of the rectangular set *S* of points into five cells, A_1, A_2, A_3, A_4, A_5.



Example:

Consider the following collection of subsets of $S = \{1, 2, 3, \quad, 8, 9\}$:

(i) $[\{1,3,5\},\{2,6\},\{4,8,9\}]$

(ii) $[\{1,3,5\},\{2,4,6,8\},\{5,7,9\}]$

(iii) $[\{1,3,5\},\{2,4,6,8\},\{7,9\}]$

Solution:

(i) is not a partition of S since 7 in S does not belong to any of the subsets.

(ii) is not a partition of S since $\{1,3,5\}$ and $\{5,7,9\}$ are not disjoint.

(iii) is a partition of S

Exercise 1

(1) Let $\mathbf{U} = \{\frac{1}{3}, 0, 5, -\sqrt{3}, -4\}$ be the universal set of $A = \{-\sqrt{3}, 0, 5\}$, $B = \{\frac{1}{3}, -4, -\sqrt{3}\}$ and $C = \{\frac{1}{3}, -4\}$. Find

(a) $A \cap B$ (b) $B \cup C$ (c) $(A \cup B) \cap C$ (d) $B' \cup C^c$ (e) $A \setminus B$ (f) $(B \cap C)'$

(2) Let $F = \{x : x \text{ is an odd integer}\}$, $B = \{x : x^2 - 8x + 15 = 0\}$. Show that $B \subset A$.

(3) Which of these sets are equal: $\{r, s, t\}$, $\{t, s, r\}$, $\{s, r, t\}$, $\{t, r, s\}$?

(4) Determine which of the following sets are equal: $\varnothing$, $\{0\}$, $\{\varnothing\}$.

(5) Consider the following sets where $\mathbf{U} = \{1, 2, 3, \quad, 8, 9\} : \varnothing$, $A = \{1\}$, $B = \{1, 3\}$, $C = \{1, 5, 9\}$, $D = \{1, 2, 3, 4, 5\}$, $E = \{1, 3, 5, 7, 9\}$. Insert the correct symbol $\subseteq$ or $\not\subseteq$ between each pair of sets: $(a)\ \varnothing, A$ $(c)\ A, B$ $e)\ C, D$ $g)\ D, E$

$(b)\ B, C$ $(d)\ B, E$ $f)\ C, E$ $(h)\ D, \mathbf{U}$

Solution 1

A(1) $\mathbf{U} = \{\frac{1}{3}, 0, 5, -\sqrt{3}, -4\}$

(a) $A \cap B = \{-\sqrt{3}, 0, 5\} \cap \{\frac{1}{3}, -4, -\sqrt{3}\} = \{-\sqrt{3}\}$

(b) $B \cup C = \{\frac{1}{3}, -4, -\sqrt{3}\} \cup \{\frac{1}{3}, -4\} = \{\frac{1}{3}, -4, -\sqrt{3}\} = B$

(c) $(A \cup B) \cap C = (\{-\sqrt{3}, 0, 5\} \cup \{\frac{1}{3}, -4, -\sqrt{3}\}) \cap \{\frac{1}{3}, -4\}$

$$= \{-\sqrt{3}, 0, 5, \tfrac{1}{3}, -4\} \cap \{\tfrac{1}{3}, -4\} = \{\tfrac{1}{3}, -4\} = C$$

(d) $B' \cap C^c = \{\frac{1}{3}, -4, -\sqrt{3}\}' \cup \{\frac{1}{3}, -4\}^c$

$$= \{0, 5\} \cup \{0, 5, -\sqrt{3}\} = \{0, 5, -\sqrt{3}\}$$

(e) $A \setminus B = \{-\sqrt{3}, 0, 5\} \setminus \{\frac{1}{3}, -4, -\sqrt{3}\} = \{0, 5\}$

(f) $(B \cap C)' = (\{\frac{1}{3}, -4, -\sqrt{3}\} \cap \{\{\frac{1}{3}, -4\})' = \{\frac{1}{3}, -4\}' = \{0, 5, -\sqrt{3}\}$

A(2) $F = \{x : x \text{ is an odd integer}\} = \{\ \ldots, -5, -3, -1, 1, 3, 5, \ldots\ \}$

$B = \{x^2 - 8x + 15 = 0\} = \{x : x^2 - 3x - 8x + 15 = 0\} = \{x : x(x-3) - 5(x-3) = 0\}$

$= \{x : (x-5)(x-3) = 0\} = \{x : x = 5 \text{ or } x = 3\} = \{5, 3\}$

Therefore B contains 5 and 3 which are odd integers, hence $B \subset F$

A(3) The sets are all equal; order does not change a set.

A(4) Each of the three sets is different from the other.
The set $\varnothing$ contains no element; it is the empty set.
The set $\{0\}$ contains one element, the number zero.
The set $\{\varnothing\}$ also contains one element, the null set

A(5)

(a) $\varnothing \subseteq A$ (b) $B \not\subseteq C$ (c) $A \subseteq B$ (d) $B \subseteq E$
(e) $C \not\subseteq D$ (f) $C \subseteq E$ (g) $D \not\subseteq E$ (h) $D \subseteq \mathbf{U}$

Answers 1

Q(1)

(a) $\{-\sqrt{3}\}$ (b) $\{\frac{1}{3}, -4, -\sqrt{3}\}$ (c) $\{\frac{1}{3}, -4\}$ (d) $\{0, 5, -\sqrt{3}\}$ (e) $\{0, 5\}$ (f) $\{0, 5, -\sqrt{3}\}$

Probability Theory

Random Experiments

The scientist performs experiments to produce observations or measurements that will assist him in drawing a valid conclusion. For instance, a chemist in determining the liter value of a particular chemical carrying out titrations several times, Biologist in determining different species of bacteria colony, agriculturalist in determining the effect of fertilizer on sorghum species, etc.

Definition 4 (Random Experiment):
In probability, a random experiment is any process of observations or measurement that generates raw data. The results one obtains from an experiment are called *outcomes.*

The result of each performance depends on "*chance*" and therefore cannot be predicted with certainty.

Examples:

1) tossing a coin or die
2) drawing a card or two from a shuffled deck of playing cards
3) random selection/inspection of 20 light bulbs from 100 light bulbs produced by a company.
4) Sampling the opinion of Nigerians about the constitution of Nigeria, etc.

Definition 5 (Sample Space):
The set which consists of all the possible outcomes of a random experiment is called sample space, and it is usually denoted by *S*. Each outcome in a sample space is called a *sample point.*

Examples:

1) An experiment of rolling a die has $S = \{1, 2, 3, 4, 5, 6\}$
2) Experiment of tossing a coin twice has $S = \{HH, HT, TH, TT\}$
3) Observation of industrial production whether it is defective or not has $S = \{\text{defective, nondefective}\}$.
4) Experiment involving sampling of opinions of Nigerians about the review of Nigerian constitution has $S = \{\text{yes, no}\}$, etc.

Definition 6 (Event):
An event is a subset of a sample space, i.e. it is the set of possible outcomes.

The empty set $\varnothing$ and S are the subsets of S and hence they are events; $\varnothing$ is sometimes called the *impossible* or *null* event, and S is sometimes called the *certain* or *sure* event

Example:

1) If a coin is toss twice, the event that only one tail comes up is given by $A = \{HT, TH\}$.

Types of Events:

1) **Equally likely Events:** Events are said to be equally likely if they have equal chances of occurring.

2) **<u>Mutually Exclusive Events:</u>** Two or more events are said to be mutually exclusive if they cannot occur simultaneously. Example in tossing a coin heads and tails cannot occur at the same time.
3) **<u>Independent Events:</u>** Events are said to be independent whenever the occurrence of one does not in anyway affects the occurrence of the other. Example obtaining 2 and 3 in tossing a die twice.

<u>Combining Events:</u>
Events can be combined to form new events using the various set operations:
(i) $A \cup B$ is the event occurs iff *A* occurs ***<u>or</u>*** *B* occurs (or both).
(ii) $A \cap B$ is the event occurs iff *A* occurs ***<u>and</u>*** *B* occurs.
(iii) A^c, the compliment of *A*, is the event that occurs iff *A* does not occur.

<u>Probability</u>

The probability of an event *A* in an experiment is supposed to measure how frequently *A* is about to occur if we make many experiments/trials.

Definition 7 (Probability 2):
If an experiment can result in any one of *N* different equally likely outcomes, and if *n* of these outcomes together constitute event *A*, then the probability *P(A)* of an event *A* is

$$P(A) = \frac{\text{Number of outcomes in } A}{\text{Number of outcomes in } S} = \frac{n(A)}{n(S)} = \frac{n}{N}$$

Thus, in particular, $P(S) = 1$ as follows from (definition 7) above.

<u>Example:</u>
In rolling a fair die, what is the:
a. probability *P(A)* of *A* of obtaining at least 5
b. probability *P(B)* of *B* of obtaining an "even number".

<u>Solution</u>
There six outcomes in rolling a die, i.e. $S = \{1,2,3,4,5,6\}$ thus $N = 6$.
a. $A = \{5,6\}$, thus $n(A) = 2$ therefore $P(A) = {}^2\!/_6 = {}^1\!/_3$
b. $B = \{2,4,6\}$, thus $n(B) = 3$ therefore $P(A) = {}^3\!/_6 = {}^1\!/_2$

<u>*Axioms of Probability*</u>

Given a sample space *S*, with each event *A* of *S* (subset of *S*) there is associated a number *P(A)*, called the probability of *A*, such that the following ***axioms of probability*** are satisfied.

[P1] For every *A* in *S*, $\boxed{0 \le P(A) \le 1}$

[P2] The entire sample space *S* has the probability $\boxed{P(S) = 1}$

[P3] For mutually exclusive events *A* and *B* (i.e., $A \cap B = \varnothing$) $\boxed{P(A \cup B) = P(A) + P(B)}$

[P3]' More generally, for mutually exclusive events $A_1, A_2, A_3,$.

$$P(A_1 \cup A_2 \cup A_3 \cup ...) = P(A_1) + P(A_2) + P(A_3)....$$

Basic Theorems on Probability

Theorem 1: The impossible event or, in other words, the empty set $\varnothing$ has probability zero, that is, $P(\varnothing)=0$.

Proof: For any event *A*, we have $A\cup\varnothing = A$, where $A\cap\varnothing=\varnothing$ (i.e. *A* and $\varnothing$ are disjoint). By axiom **[P3]**,

$$P(A)=P(A\cup\varnothing)=P(A)+P(\varnothing)$$

$$\Rightarrow P(A)=P(A)+P(\varnothing)$$

subtract $P(A)$ from both sides

$$\begin{aligned} P(A)-P(A) &= P(A)+P(\varnothing)-P(A) \\ &= P(A)-P(A)+P(\varnothing) \\ &= P(\varnothing) \end{aligned}$$

thus $P(\varnothing)=0$.

Theorem 2 (Complement Rule): For any event *A* we have and its compliment A^c in the sample space *S*, we have

$$P(A^c)=1-P(A)$$

Proof: $S=A\cup A^c$ where *A* and A^c are disjoint. By axiom **P[2],** $P(S)=1$. Thus, by **P[3],**

$$1=P(S)=P(A\cup A^c)=P(A)+P(A^c)$$

$$\Rightarrow 1=P(A)+P(A^c)$$

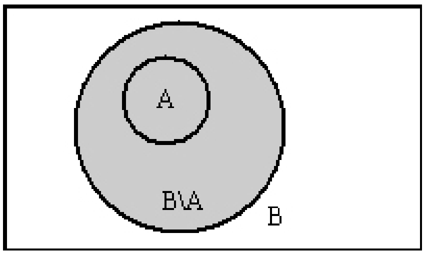
subtract $P(A)$ from both sides

$$\begin{aligned} 1-P(A) &= P(A)+P(A^c)-P(A) \\ &= P(A^c) \end{aligned}$$

thus $P(A^c)=1-P(A)$.

Theorem 3: If $A\subseteq B$, then $P(A)\le P(B)$.

Proof: If $A\subseteq B$, then, as indicated by the following Venn diagram



B is shaded

$B=A\cup(B\setminus A)$ where *A* and $B\setminus A$ are disjoint. Hence

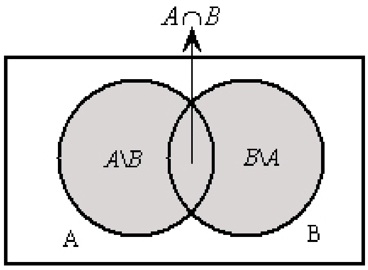
$$P(B)=P(A)+P(B\setminus A)$$

By axiom **P[1]**, we have $P(B\setminus A)\ge 0$, hence $P(A)\le P(B)$.

Theorem 4 (Addition Rule): For any two events *A* and *B*.

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

Proof: By the Venn diagram below (recall that $A\setminus B=A\cap B^c$ and $B\setminus A=B\cap A^c$)



$A\cup B$ is shaded

$$A = (A\cap B^c)\cup(A\cap B)$$

$$\therefore P(A) = P(A\cap B^c) + P(A\cap B)$$

$$\text{but } A\cup B = B\cup(A\cap B^c)$$

$$\Rightarrow P(A\cup B) = P(B) + P(A\cap B^c)$$

$$\text{but } P(A\cap B^c) = P(A\cup B) - P(B)$$

$$\Rightarrow P(A) = P(A\cup B) - P(B) + P(A\cap B)$$

$$\therefore P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

H/W: Show that

$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(A\cap C) - P(B\cap C) + P(A\cap B\cap C)$

Exercise 2

(1) Write a sample space S with equally likely outcomes for each of the following experiments
- **(a)** A two-headed coin is tossed once.
- **(b)** Two ordinary coins are tossed.
- **(c)** Three ordinary coins are tossed.
- **(d)** Five slips of paper marked with the numbers 1, 2, 3, 4, and 5 are placed in a box. After mixing well, two slips are drawn.
- **(e)** An unprepared student takes a three-question true/false quiz in which he guesses the answer to all the three questions.
- **(f)** A die is rolled and then a coin is tossed.

(2) A ball is drawn from a box containing 6 red balls, 3 white balls and 4 green balls. Find the probability that the ball is:

(a) green
(b)white or red
(c) not white
(d) white and green

(3) A card is drawn at random from an ordinary deck of 52 playing cards. Find the probability that it is:

(a) a club
(b) a queen of hearts
(c) a queen or a heart
(d) a 7 of spade or a 5 of diamonds
(e) any suit except hearts
(f) neither a ten nor a spade
(g) a red king or a black ace

(4) Two dice (green and blue) are thrown together. Find:

(a) the sample space
(b) the probability of getting a prime number with green die and an even number with the blue die.
(c) the probability that the sum of the faces that turn up is an even number greater than 7
(d) the probability that the sum is divisible by 3

(5) In one of their shooting exercises, an army in Bukavu Barracks is allowed to shoot a target three times. If we label the outcome of each shoot 0 for a miss and 1 for a hit. Find:

(a) the sample space of the experiment
(b) the probability of hitting the target once and missing it twice
(c) two consecutive misses
(d) missing the target alternately

(6) Suppose A and B are events with $P(A) = 0.6$, $P(B) = 0.3$, and $P(A \cap B) = 0.2$. Find the probability that:

(a) A does not occur
(b) B does not occur
(c) A or B occurs
(d) neither A nor B occurs

(7) A sample space S consists of four elements, that is, $S = \{a_1, a_2, a_3, a_4\}$. Under which of the following functions P does S become a probability space?

(a) $P(a_1) = 0.4, \quad P(a_2) = 0.3 \quad P(a_3) = 0.2 \quad P(a_4) = 0.3$
(b) $P(a_1) = 0.4, \quad P(a_2) = -0.2 \quad P(a_3) = 0.7 \quad P(a_4) = 0.1$
(c) $P(a_1) = 0.4, \quad P(a_2) = 0.2 \quad P(a_3) = 0.1 \quad P(a_4) = 0.3$
(d) $P(a_1) = 0.4, \quad P(a_2) = 0 \quad P(a_3) = 0.5 \quad P(a_4) = 0.1$

(8) A coin is weighted so that heads is three times as likely to appear as tails. Find $P(T)$ and $P(H)$ where T = tails and H = heads.

(9) Consider the following table:

Outcome	1	2	3	4	5	6
Probability	0.1	0.3	0.1	0.2	0.2	0.1

Consider the following events:

$$A = \{\text{even number}\}, \quad B = \{2, 3, 4, 5\}, \quad C = \{1, 2\}$$

Find: (a) $P(A)$, (b) $P(B)$, (c) $P(C)$, (d) $P(\varnothing)$, (e) $P(S)$

(10) From the events *A*, *B*, *C* in Problem 9 above, find:
(a) $P(A \cap B)$, (b) $P(A \cup C)$, (c) $P(B \cap C)$, (d) $P(A^c)$, (e) $P(B \cap C^c)$

(11) Let P be a probability function on $S = \{a_1, a_2, a_3\}$. Find $P(a_1)$ if

(a) $P(a_2) = 0.3,$ and $P(a_3) = 0.5;$
(b) $P(a_1) = 2P(a_2)$ and $P(a_3) = 0.7;$
(c) $P(\{a_2, a_3\}) = 2P(a_1);$
(d) $P(a_3) = 2P(a_2)$ and $P(a_2) = 3P(a_1).$

(12) A die is loaded in such away that each odd number is twice as likely to occur as each even number. If E is the event that a number greater than 3 occurs on a single toss of the die, find $P(E)$.

<u>Solution 2</u>

A(1)

(a) Let H = heads and T = tails, then $S = \{H\}$

(b) $S = \{HH, HT, TH, TT\}$, where H = heads and T = tails, and the first letter indicates the outcome of the first coin and the second letter indicates the outcome of the second coin, thus, HT indicates the first coin turns up a head and the second coin turns up a tail.

(c) $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

(d) $S = \{1, 2, 3, 4, 5\}$

(e) Let C = correct and W = wrong, then
$S = \{CCC, CCW, CWC, CWW, WCC, WCW, WWC, WWW\}$

(f) $S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$

A(2) Let G, W, R be the events that green, white and red ball is drawn respectively,

(a) $P(G) = \dfrac{n(G)}{n(S)} = \dfrac{4}{15}$

(b) $P(W) = \dfrac{n(W)}{n(S)} = \dfrac{3}{13}$, $P(R) = \dfrac{n(R)}{n(S)} = \dfrac{6}{13}$

$\therefore\ P(\text{white or red}) = P(W \cup R)$

$\Rightarrow P((W \cup R) = P(W) + P(R) - P(W \cap R)$

$$= \frac{3}{13} + \frac{6}{13} - 0 = \frac{9}{13}$$

(c) $P(\text{not white}) = P(W') = 1 - P(W)$, but $P(W) = \frac{3}{13}$

$\therefore P(W') = 1 - \dfrac{3}{13} = \dfrac{13-3}{13} = \dfrac{10}{13}$

(d) It is not possible to get a white and green ball in a single draw, hence G and W are mutually exclusive events, i.e., $G \cap W = \varnothing$; therefore $P(G \cap W) = P(\varnothing) = 0$.

A(3) Let the sample space be S, then S consists of the four suits, club (*C*), diamonds (*D*), hearts (*H*) and spades (*S*), where each suit contains 13 cards which are numbered 2 to 10, and a jack (*J*), queen (*Q*), king (*K*), and ace (*A*). The hearts (*H*) and diamonds (*D*) are red cards, and the spades (*S*) and clubs (*C*) are black cards. Below is a tabular view of the sample space

	A	***K***	***Q***	***J***	**10**	**9**	**8**	**7**	**6**	**5**	**4**	**3**	**2**
♥***H***	*	*	*	*	*	*	*	*	*	*	*	*	*
♦***D***	*	*	*	*	*	*	*	*	*	*	*	*	*
♠***S***	*	*	*	*	*	*	*	*	*	*	*	*	*
♣***C***	*	*	*	*	*	*	*	*	*	*	*	*	*

Let H = heart, D = diamond, S = spade, and C = club. Then

$H = \{A_H, K_H, Q_H, J_H, 10_H, 9_H, 8_H, 7_H, 6_H, 5_H, 4_H, 3_H, 2_H\}$, $n(H) = 13$

$D = \{A_D, K_D, Q_D, J_D, 10_D, 9_D, 8_D, 7_D, 6_D, 5_D, 4_D, 3_D, 2_D\}$, $n(D) = 13$

$S = \{A_S, K_S, Q_S, J_S, 10_S, 9_S, 8_S, 7_S, 6_S, 5_S, 4_S, 3_S, 2_S\}$, $n(S) = 13$

$C = \{A_C, K_C, Q_C, J_C, 10_C, 9_C, 8_C, 7_C, 6_C, 5_C, 4_C, 3_C, 2_C\}$, $n(C) = 13$

$S = \{H, D, S, C\}$, $n(S) = 52$

(a) The event of a club is $C = \{A_C, K_C, Q_C, J_C, 10_C, 9_C, 8_C, 7_C, 6_C, 5_C, 4_C, 3_C, 2_C\}$

$\therefore P(\text{of a club}) = P(C) = \dfrac{n(C)}{n(S)} = \dfrac{13}{52} = \dfrac{1}{4}$

(b) The event of queen of hearts is $\{Q_H\}$, $\quad n(\{Q_H\}) = 1$

$\therefore P(\{Q_H\}) = \dfrac{n(\{Q_H\})}{n(S)} = \underline{\underline{\dfrac{1}{52}}}$

(c) The event of a queen is $Q = \{Q_H, Q_D, Q_S, Q_C\}$, $\quad n(Q) = 4$ and
the event of a heart is $H = \{A_H, K_H, Q_H, J_H, 10_H, 9_H, 8_H, 7_H, 6_H, 5_H, 4_H, 3_H, 2_H\}$
the event of a queen or a heart is $Q \cup H$ and by the addition rule

$P(Q \cup H) = P(Q) + P(H) - P(Q \cap H)$

$P(Q) = \dfrac{n(Q)}{n(S)} = \dfrac{4}{52} = \dfrac{1}{13}$

$P(H) = \dfrac{n(H)}{n(S)} = \dfrac{13}{52} = \dfrac{1}{4}$

$Q \cap H = \{Q_H\}$

$P(Q \cap H) = P(\{Q_H\}) = \dfrac{1}{52}$

$P(Q \cup H) = \dfrac{1}{13} + \dfrac{1}{4} - \dfrac{1}{52} = \dfrac{4+13-1}{52} = \dfrac{16}{52} = \underline{\underline{\dfrac{4}{13}}}$

(d) The event of 7 of spades is $\{7_S\}$ and the event of 5 of diamonds is $\{5_D\}$. Thus the event of 7 of spades or 5 of diamonds is $\{7_S\} \cup \{5_D\}$. But $\{7_S\} \cap \{5_D\} = \varnothing$, thus,

$P(\{7_S\} \cup \{5_D\}) = P(\{7_S\}) + P(\{5_D\}) - P(\{7_S\} \cap \{5_D\})$

$= \dfrac{1}{52} + \dfrac{1}{52} - 0 = \dfrac{2}{52} = \underline{\underline{\dfrac{1}{26}}}$

(e) The event of a heart is *H*. The probability of a heart is $P(H) = \frac{13}{52}$. So by the complement rule, the probability of drawing a card that is not a suit of heart is

$$P(H') = 1 - P(H) = 1 - \frac{13}{52} = 1 - \frac{1}{4} = \underline{\underline{\frac{3}{4}}}$$

Alternatively,
the event of any suit except hearts is $H' = \{D, S, C\}$ and $n(H') = 39$. Hence,

$$P(H') = \frac{n(H')}{n(S)} = \frac{39}{52} = \frac{3}{4}$$

(f) The event of neither a ten nor a spade is an event of not a ten and not a spade. Let the event of a ten be $T = \{10_H, 10_D, 10_S, 10_C\}$, the event of a spade is *S*. So the event of not a ten and not a spade is $T' \cap S'$, but by DeMorgan's law $T' \cap S' = (T \cup S)'$. Therefore,

$P(T' \cap S') = P\big((T \cup S)'\big) = 1 - P(T \cup S)$

$P(T \cup S) = P(T) + P(S) - P(T \cap S)$

$P(T) = \dfrac{4}{52}, \quad P(S) = \dfrac{13}{52}, \quad P(T \cap S) = P(\{10_S\}) = \dfrac{1}{52}$

$\Rightarrow P(T \cup S) = \dfrac{4}{52} + \dfrac{13}{52} - \dfrac{1}{52} = \dfrac{4+13-1}{52} = \dfrac{16}{52}$

$\therefore P(T' \cap S') = 1 - P(T \cup S)$

$$= 1 - \frac{16}{52} = \frac{52-16}{52}$$

$$= \frac{36}{52} = \underline{\underline{\frac{9}{13}}}$$

(g) Let $R_K = \{K_H, K_D\}$ be the event of a red king, and $B_A = \{A_S, A_D\}$ be the event of a black ace. Hence the event of a red king or a black ace is $R_K \cup B_A$. Thus

$P(R_K \cup B_A) = P(R_K) + P(B_A) - P(R_K \cap B_A)$

$$= \frac{2}{52} + \frac{2}{52} - 0$$

$$= \frac{4}{52} = \underline{\underline{\frac{1}{13}}}$$

A(4)

(a) The sample space is given below:

		Blue Die					
		1	2	3	4	5	6
Green Die	1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
	3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
	4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

(b) Let the event of a prime number with the green die and an even number with the blue die be denoted by $B = \{(1,2),(1,4),(1,6),(2,2),(2,4),(2,6),(3,2),(3,4),(3,6),(5,2),(5,4),(5,6)\}$ thus $n(B) = 12$; there are total of 36 possible outcomes. Therefore,

$$P(B) = \frac{12}{36} = \underline{\underline{\frac{1}{3}}}$$

(c) Let C be the event "Sum of faces is an even number greater than 7", thus $C = \{(2,6),(3,5),(4,4),(4,6),(5,3),(5,5),(6,2),(6,4),(6,6)\}$, thus $n(C) = 9$. Therefore,

$$P(C) = \frac{9}{36} = \underline{\underline{\frac{1}{4}}}$$

(d) Let D be the event "sum is divisible by 3", then $D = \{(1,2),(1,5),(2,1),(2,4),(3,3),(3,6),(4,2),(4,5),(5,1),(5,4),(6,3),(6,6)\}$, thus $n(D) = 12$. Therefore,

$$P(D) = \frac{12}{36} = \underline{\underline{\frac{1}{3}}}$$

A(5)

(a) The sample space is, $S = \{(1,1,1),(1,1,0),(1,0,1),(1,0,0),(0,1,1),(0,1,0),(0,0,1),(0,0,0),\}$ $n(S) = 8$

(b) Let H be the event of "hitting once and missing twice". Thus,

$H = \{(1,0,0),(0,1,0),(0,0,1)\}$, $n(H) = 3$. Therefore, $P(H) = \underline{\underline{\frac{3}{8}}}$

(c) Let K be the event "Two consecutive misses". Thus $K = \{(1,0,0),(0,0,1)\}$, $n(K) = 2$. Therefore, $P(K) = \frac{2}{8} = \underline{\underline{\frac{1}{4}}}$

(d) Let T be the event "missing the target alternatively". Thus $T = \{(0,1,0)\}$, $n(T) = 1$. Therefore $P(T) = \frac{1}{8}$

A(6)

(a) By the complement rule, $P(\text{not } A) = P(A^c) = 1 - P(A) = 1 - 0.6 = \underline{\underline{0.4}}$

(b) By the complement rule, $P(\text{not } B) = P(B^c) = 1 - P(B) = 1 - 0.3 = \underline{\underline{0.7}}$

(c) By the addition rule, $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= 0.6 + 0.3 - 0.2 = \underline{\underline{0.7}}$$

(d) "Neither A nor B" is the same as "not A and not B". That is $A^c \cap B^c = (A \cup B)^c$. Therefore $P(\text{neither } A \text{ nor } B) = P\left((A \cup B)^c\right) = 1 - P(A \cup B) = 1 - 0.7 = \underline{\underline{0.3}}$

A(7)

(a) The sum of the points in S exceed one hence P does not define S to be a probability space.

(b) $P(a_2) < 0$, P does not define S to be a probability space.

(c) Each value is nonnegative and their sum does equal one, hence P does define S to be a probability space.

(d) P define S to be a probability space.

A(8) Let $P(T) = p$, then $P(H) = 3p$ and $S = \{H, T\}$

$\Rightarrow P(T) + P(H) = 1$

$\Rightarrow p + 3p = 1$

$\Rightarrow 4p = 1, \qquad \Rightarrow p = \frac{1}{4}$

$\Rightarrow P(T) = \frac{1}{4}$

$\Rightarrow P(H) = 3p = 3 \cdot \frac{1}{4} = \underline{\underline{\frac{3}{4}}}$

A(9) For any event E, find $P(E)$ by summing the probabilities of elements of E.

(a) $A = \{2,4,6\}$, so $P(A) = 0.3 + 0.2 + 0.1 = \underline{\underline{0.6}}$

(b) $P(B) = 0.3 + 0.1 + 0.2 + 0.2 = \underline{\underline{0.8}}$

(c) $P(C) = 0.1 + 0.3 = \underline{\underline{0.4}}$

(d) $P(\varnothing) = 0$

(e) $S = \{1,2,3,4,5,6\}$, so $P(S) = 0.1 + 0.3 + 0.1 + 0.2 + 0.2 + 0.1 = \underline{\underline{1}}$

A(10)

(a) $A \cap B = \{2,4\}$, $\quad P(A \cap B) = 0.3 + 0.2 = \underline{\underline{0.5}}$

(b) $A \cup C = \{1,2,4,6\}$, $\quad P(A \cup C) = 0.1 + 0.3 + 0.2 + 0.1 = \underline{\underline{0.7}}$

(c) $B \cap C = \{2\}$, $\quad P(B \cap C) = \underline{\underline{0.3}}$

(d) $A^c = \{1,3,5\}$, $P(A^c) = 0.1 + 0.1 + 0.2 = \underline{\underline{0.4}}$ or

$P(A^c) = 1 - P(A) = 1 - 0.6 = \underline{\underline{0.4}}$

(e) $B \cap C^c = \{2,3,4,5\} \cap \{1,2\}^c = \{2,3,4,5\} \cap \{3,4,5,6\} = \{3,4,5\}$

$P(B \cap C^c) = 0.1 + 0.2 + 0.2 = \underline{\underline{0.5}}$

A(11) We use the fact that $P(a_1) + P(a_2) + P(a_3) = 1$

(a) $P(a_1) + P(a_2) + P(a_3) = 1$

$P(a_1) + 0.3 + 0.5 = 1$

$\Rightarrow P(a_1) + 0.8 = 1$

$\Rightarrow P(a_1) = 1 - 0.8 = \underline{\underline{0.2}}$

(b) $P(a_1) + P(a_2) + P(a_3) = 1$

$2P(a_2) + P(a_2) + P(a_3) = 1$

$3P(a_2) + 0.7 = 1$

$\Rightarrow 3P(a_2) = 1 - 0.7 = 0.3$

$\Rightarrow P(a_2) = \frac{0.3}{3} = 0.1$

but $P(a_1) = 2P(a_2) = 2(0.1) = \underline{\underline{0.2}}$

(c) $P(a_1) + \left(P(a_2) + P(a_3)\right) = 1$

$P(a_1) + 2P(a_1) = 1$

$\Rightarrow 3P(a_1) = 1$

$\Rightarrow P(a_1) = \underline{\underline{\frac{1}{3}}}$

A(12) The sample space of rolling a die is $S = \{1,2,3,4,5,6\}$, let E be the event of rolling a number greater than 3, thus $E = \{4,5,6\}$. Let the probability of even numbers be p, then the probability of odd numbers is $2p$. $P(1) = P(3) = P(5) = 2p$, $P(2) = P(4) = P(6) = p$, and $P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$

$\Rightarrow 2p + p + 2p + p + 2p + p = 1$

$\Rightarrow 9p = 1$

$\Rightarrow p = \frac{1}{9}$

$\Rightarrow P(1) = P(3) + P(5) = \frac{2}{9}$, and

$P(2) = P(4) + P(6) = \frac{1}{9}$

$P(E) = P(4) + P(5) + P(6)$

$= \frac{1}{9} + \frac{2}{9} + \frac{1}{9} = \underline{\underline{\frac{4}{9}}}$

Answers 2

Q(2)

(a) $4/13$ (b) $9/13$ (c) $10/13$ (c) 0

Q(3)

(a) $1/4$ (b) $1/52$ (c) $1/52$ (d) $1/26$ (e) $3/4$ (f) $9/13$ (g) $1/13$

Q(4)

(b) $1/3$ (c) $1/4$ (d) $1/3$

Q(5)

(b) $3/8$ (c) $1/4$ (d) $1/8$

Q(6)

(a) 0.4 (b) 0.7 (c) 0.7 (d) 0.3

Q(9)

(a) 0.6 (b) 0.8 (c) 0.4 (d) 0 (e) 1

Q(10)

(a) 0.5 (b) 0.7 (c) 0.3 (d) 0.4 (e) 0.5

Q(11)

(a) $P(a_1) = 0.2$ (b) $P(a_1) = 0.2$ (c) $P(a_1) = 1/3$ (d) $P(a_1) = 0.1$

Q(12)

$P(E) = 4/9$

Conditional Probability

Probabilities are sometimes affected when specification or condition is attached on the entire sample space or the event in question.

Example 1:

Consider an experiment in which a box containing 10 defective and 8 non-defective items. Two items are taking at random. Find the probability that both items are defective: (a) with replacement (b) without replacement.

Solution:

Let $A = \{\text{the first item is defective}\}$ and $B = \{\text{the second item is defective}\}$

(a) $P(A) = P(B) = \dfrac{10}{18} = \dfrac{5}{9}$

(b) $P(A) = \dfrac{10}{18} = \dfrac{5}{9}$, when the second item is drawn the number of defective reduced from 10 to 9; thus $P(B) = \dfrac{9}{17}$.

> **Definition 8 (Conditional Probability):**
> Often it is required to find the probability of an event B under the condition that an event A occurs. This probability is called *conditional probability of B given A* and is denoted by $P(A|B)$. In this case A serves as a new (reduced) sample space, and that probability is the

fraction of $P(A)$ which corresponds to $A \cap B$, provided $P(A) \neq 0$. Thus

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \qquad [P(A) \neq 0].$$

Similarly, the *conditional probability of A given B* is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad [P(B) \neq 0].$$

Example 2:

Two fair dice are tossed, the outcome being recorded as (x_1, x_2), where x_i is the outcome of the i^{th} die, $i = 1, 2$. Let event $A = \{(x_1, x_2) : x_1 + x_2 = 9\}$ and $A = \{(x_1, x_2) : x_1 > x_2\}$. Find (a) $P(A \cap B)$ (b) $P(B \mid A)$ (b) $P(A \mid B)$

Solution:

The sample space S has 36 points $A = \{(3,6), (4,5), (5,4), (6,3)\}$

$B = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3), (5,1), (5,2), (5,3), (5,4), (6,1), (6,2), (6,3), (6,4), (6,5)\}$

$A \cap B = \{(5,4), (6,3)\}$, $n(S) = 36$, $n(A) = 4$, $n(B)$, and $n(A \cap B) = 2$; thus

$P(A) = \frac{n(A)}{n(S)} = \frac{4}{36} = \frac{1}{9}$, $P(B) = \frac{n(B)}{n(S)} = \frac{15}{36} = \frac{5}{12}$

(a) $P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{2}{36} = \frac{1}{18}$

(b) $P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{1/18}{1/9} = \frac{9}{18} = \frac{1}{2}$

(c) $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/18}{5/12} = \frac{1}{18} * \frac{12}{5} = \frac{1}{3} * \frac{2}{5} = \frac{2}{3 * 5} = \frac{2}{15}$

Example 3:

A fair die is tossed twice. Find the probability of getting a 5, 4, or 6 on the first toss and 2, 3, 4, 5 on the second toss.

Solution:

Let H be the event "5, 4, or 6 on the 1st toss", and
K be the event "2, 3, 4, 5 on the second toss"

$\therefore P(H) = \frac{3}{6} = \frac{1}{2}$, $P(K) = \frac{4}{6} = \frac{2}{3}$

$$P(K \mid H) = \frac{P(H \cap K)}{P(H)}$$

$$\therefore P(H \cap K) = P(K \mid H) * P(H)$$
$$= \frac{2}{3} * \frac{1}{2} = \frac{1}{3}$$

Theorem 5 (Multiplication Rule/Theorem): If A and B are events in a sample space S and $P(A) \neq 0$, $P(B) \neq 0$, then

$$P(A \cap B) = P(A) * P(B \mid A) = P(B) * P(A \mid B)$$

Proof:

Solve as an exercise.

Independent Events

In example 3 above it can be noted that $P(K \mid H) = P(K) = 2/3$, this is due to *independency*. Events A and B in a probability space S are said to be *independent* if the occurrence of one of them does not influence the occurrence of the other. More specifically, B is independent of A if $P(B)$ is the same as $P(B \mid A)$.Now substitute $P(B)$ for $P(B \mid A)$ in the Theorem 5 above (multiplication theorem) that $P(A \cap B) = P(A) * P(B \mid A)$. This yields $P(A \cap B) = P(A) * P(B)$.

> **Definition 9 (Independent Events):**
> Events A and B are *independent* if $P(A \cap B) = P(A) * P(B)$;
> otherwise they are *dependent.*

Example:

Two men A and B fire at target. Suppose $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{5}$ denote their probabilities of hitting the target. (We assume that the events A and B are independent.) Find the probability that:

a) A does not hit the target **b)** Both hit the target
c) One of them hits the target **d)** Neither hits the targetr

Solution:

a) By the complement rule,

$$P(\text{not } A) = P(A^c) = 1 - P(A) = 1 - \frac{1}{3} = \frac{2}{3}.$$

b) Since the events A and B are independent (Definition 9 above),

$$P(A \text{ and } B) = P(A \cap B) = P(A) \cdot P(B) = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}$$

c) By the addition rule (Theorem 4),

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{3} + \frac{1}{5} - \frac{1}{15} = \frac{7}{15}$$

d) By DeMorgan's Law, "neither A nor B" is the compliment of $A \cup B$. Hence,

$$P(\text{neither } A \text{ nor } B) = P\left((A \cup B)^c\right) = 1 - P(A \cup B) = 1 - \frac{7}{15} = \frac{8}{15}$$

Theorem 6:

Suppose A and B are independent events. Then A^c and B^c are independent events.

Proof:

<u>Goal</u>: $P(A^c \cap B^c) = P(A^c) \cdot P(B^c)$

Let $P(A) = x$ and $P(B) = y$. Then $P(A^c) = 1 - x$ and $P(B^c) = 1 - y$. Since A and B are independent events $P(A \cap B) = P(A) \cdot P(B) = xy$. Thus by Theorem 4 (addition rule)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = x + y - xy$$

By DeMorgan's law, $(A \cup B)^c = A^c \cap B^c$; hence

$$P(A^c \cap B^c) = P\left((A \cup B)^c\right)$$
$$= 1 - P(A \cup B) = 1 - (x + y - xy)$$
$$= 1 - x - y + xy$$

On the other hand,

$$P(A^c) \cdot P(B^c) = (1-x) \cdot (1-y)$$
$$= 1 - x - y + xy$$

Thus, $P(A^c \cap B^c) = P(A^c) \cdot P(B^c)$, and so A^c and B^c are independent events.

Exercise:

Corollary: Suppose *A* and *B* are independent events. Show that

(a) A and B^c are independent events.

(b) A^c and B are independent events.

Example:

Let *A* be the event that a man will live 10 years, and let *B* be the event that his wife lives 10 more years. Suppose $P(A) = \frac{1}{4}$ and $P(B) = \frac{1}{3}$. Assuming *A* and *B* are independent events; find the probability that, in 10 years

a) Both will be alive. **b)** At least one will be alive

c) Neither will be alive **d)** Only the wife will be alive

Solution:

a) We seek $P(A \cap B)$. Since A and *B* are independent events,

$$P(A \cap B) = P(A) \cdot P(B) = \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}.$$

b) We seek $P(A \cup B)$. By the addition rule,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{4} + \frac{1}{3} - \frac{1}{12} = \frac{1}{2}$$

c) By DeMorgan's Law, "neither *A* nor *B*" is the compliment of $A \cup B$.

$$P(A^c \cap B^c) = P\left((A \cup B)^c\right) = 1 - P(A \cup B) = 1 - \frac{1}{2} = \frac{1}{2}$$

Alternately, we have $P(A^c) = \frac{3}{4}$ and $P(B^c) = \frac{2}{3}$; and, since A^c and B^c are independent,

$$P(A^c \cap B^c) = P(A^c) \cdot P(B^c) = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$$

d) We seek $P(A^c \cap B)$. Since A^c and *B* are also independent,

$$P(A^c \cap B) = \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}$$

The multiplicative rule can be extended to three *A*, *B*, and *C* independent events as follows

$$P(A \cap B \cap C) = P(A) \cdot P(B \mid A) \cdot P\left(C \mid (A \cap B)\right)$$

In general, for $A_1, A_2, \quad, A_n$ pair wise mutually independent then

$$P(A_1 \cap A_2 \cap \quad \cap A_n) = P(A_1) \cdot P(A_2) \cdot \quad \cdot P(A_n)$$

or

$$P(A_1 \cap A_2 \cap \quad \cap A_n) = P(A_1) \cdot P(A_2 \mid A_1) \cdot P(A_3 \mid A_1 \cap A_2) \cdot P(A_3 \mid A_1 \cap A_2 \cap A_3)$$
$$P(A_n \mid A_1 \cap A_2 \cap \quad \cap A_{n-1})$$

Independent Repeated Trials

Here we are concerned with probability spaces which were associated with an experiment repeated a finite number of times such as the tossing of a coin three times.

Example:

A fair coin is tossed three times. Find the probability that there will appear:
(a) three heads, (b) exactly two heads, (c) exactly one head, (d) no heads.

Solution:

Let H denote a head and T a tail on any toss. The sample space S of the tosses is

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

However, since the result of any one toss does not depend on the result of any other toss, the three tosses are three independent trials in which $P(H) = \frac{1}{2}$ and $P(T) = \frac{1}{2}$ on any trial. Then,

(a) $P(\text{three heads}) = P(HHH) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

(b) $P(\text{exactly two heads}) = P(HHH \text{ or } HTH \text{ or } THH)$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{8}$$

(c) As in (b), $P(\text{exactly one head}) = P(\text{exactly two tails}) = \frac{3}{8}$

(d) As in (a), $P(\text{no heads}) = P(TTT) = \frac{1}{8}$

Example:

A certain football team wins (*W*) with probability 0.6, loses (*L*) with probability 0.3, and ties (*T*) with probability 0.1. The team plays three games over the weekend.

(a) Determine the elements of the event *A* that the team wins at least twice and does not lose; and find *P*(*A*).

(b) Determine the elements of the event *B* that the team wins, loses, and ties in some order; and find *P*(*B*).

Solution:

(a) *A* consists of all ordered triples with at least two *W*'s and no *L*'s. Thus

$$A = \{WWW, WWT, WTW, TWW\}$$

Since these events are mutually exclusive,

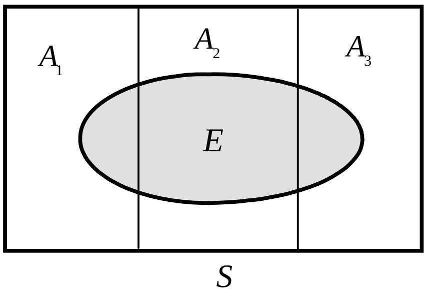
$$\begin{aligned} P(A) &= P(WWW) + P(WWT) + P(WTW) + P(TWW) \\ &= (0.6)(0.6)(0.6) + (0.6)(0.6)(0.1) + (0.6)(0.1)(0.6) + (0.1)(0.6)(0.6) \\ &= 0.216 + 0.36 + 0.36 + 0.36 = 0.324 = 32.4\% \end{aligned}$$

(b) Here $B = \{WLT, WTL, LWT, LTW, TWL, TLW\}$. Each element in *B* has the probability $(0.6)(0.3)(0.1) = 0.018$. Hence

$$P(B) = 6(0.018) = 0.108 = 10.8\%.$$

Theorem on Total Probability

Suppose a set *S* is the union of mutually disjoint subsets $A_1, A_2, \ldots, A_n$, that is, suppose, the sets $A_1, A_2, \ldots, A_n$ form a partition of the set *S*. Furthermore, suppose *E* is any subset of *S*. Then, as illustrated below for the case $n = 3$,



$$E = E \cap S = E \cap (A_1 \cup A_2 \cup \cdots \cup A_n)$$
$$= (E \cap A_1) \cup (E \cap A_2) \cup \cdots \cup (E \cap A_n)$$

Moreover, the n subsets on the right in the above equation, are also mutually disjoint, that is form a partition of E.

Theorem 7 (Total Probability):

Let E be an event in a sample space S and let $A_1, A_2, \cdots, A_n$, be mutually disjoint events whose union is S. Then

$$P(E) = P(A_1)P(E \mid A_1) + P(A_2)P(E \mid A_2) + \cdots + P(A_n)P(E \mid A_n)$$

The equation in theorem 7 above is called the *law of total probability*. It must be emphasize that the sets $A_1, A_2, \cdots, A_n$ are pairwise disjoint and their union is all if S, that is, the A's form a *partition* of S.

Example:

A factory uses three machines X, Y, Z to produce certain items. Suppose:

(1) Machine X produces 50% of the items of which 3% are defective
(2) Machine Y produces 30% of the items of which 4% are defective
(3) Machine Z produces 20% of the items of which 5% are defective

Find the probability p that a randomly selected item is defective

Solution:

Let D denote the event that an item is defective. Then by the law of total probability,

$$P(D) = P(X)P(D \mid X) + P(Y)P(D \mid Y) + P(Z)P(D \mid Z)$$
$$= (0.50)(0.03) + (0.30)(0.04) + (0.20)(0.05)$$
$$= 0.037 = 3.7\%$$

Bayes' Formula (theorem)

Let $A_1, A_2, \cdots A_n$ be a partition of the sample space S and let E be an event associated with S. Supposed we seek for

$$P(A_k \mid E) = \frac{P(A_k \cap E)}{P(E)} = \frac{P(A_k)P(E \mid A_k)}{P(E)}$$

$$= \frac{P(A_k)P(E \mid A_k)}{P(A_1)P(E \mid A_1) + P(A_2)P(E \mid A_2) + \cdots + P(A_n)P(E \mid A_n)}$$

$$= \frac{P(A_k)P(E \mid A_k))}{\sum_{i=1}^{n} P(A_i)P(E \mid A_i)}$$

The above rule is called *Bayes' rule* or *Bayes' formula*, after an English mathematician Thomas Bayes (1702-1761). If we think of the events $A_1, A_2, \cdots A_n$ as possible causes of the event

E, then Bayes' formula enables us to determine the probability that a particular one of the *A*'s occurred, given that *E* occurred.

Example:

Consider the factory in the example above. Suppose a defective item is found among the output. Find the probability that it comes from each of the machines, that is, find

(1) $P(X \mid D)$, i.e. the defective is from machine *X*

(2) $P(Y \mid D)$, i.e. the defective is from machine *Y*

(3) $P(Z \mid D)$, i.e. the defective is from machine *Z*

Solution:

Recall that $P(D) = P(X)P(D \mid X) + P(Y)P(D \mid Y) + P(Z)P(D \mid Z) = 0.037$. Therefore, by Bayes' formula,

(1) $P(X \mid D) = \dfrac{P(X)P(D \mid X)}{P(D)} = \dfrac{(0.50)(0.03)}{0.037} = \dfrac{15}{37} = 40.5\%$

(2) $P(Y \mid D) = \dfrac{P(Y)P(D \mid Y)}{P(D)} = \dfrac{(0.30)(0.04)}{0.037} = \dfrac{12}{37} = 32.5\%$

(3) $P(Z \mid D) = \dfrac{P(Z)P(D \mid Z)}{P(D)} = \dfrac{(0.20)(0.05)}{0.037} = \dfrac{10}{37} = 27.0\%$

Exercise 3

(1) A box contains 10 screws, 3 of which are defective. Two screws are drawn at random. Find the probability of the event that

(a) neither of the 2 screws is defective,

(b) the second item is defective give that the first is replaced,

(c) one defective and one non defective.

(2) *Urn I* contains 1 white and 3 black balls, *Urn II* contains 3 white and 2 black balls, and *Urn III* contains 4 white and 1 black balls. An urn is selected at random and a ball is drawn form it. Given that the ball drawn is black, what is the probability that *Urn I* was chosen.

(3) Three balls are drawn successively from a box containing 6 red balls, 4 white balls and 5 blue balls. Find the sample space *S*, hence or otherwise find the probability that

(a) They are drawn in the order white, red and blue,

(b) are of different colours,

(c) at least 2 white balls are drawn.

(4) In a certain university, 25 percent of the students failed mathematics, 15 percent failed chemistry, and 10 percent failed both mathematics and chemistry. A student is selected at random

(a) If the student failed chemistry, what is the probability that he or she failed mathematics?

(b) If the student failed mathematics, what is the probability that he or she failed chemistry?

(c) What is the probability that the student failed mathematics or chemistry?

(d) What is the probability that the student failed neither mathematics nor chemistry?

(5) Let A and B be events with $P(A) = 0.3$, $P(A \cup B) = 0.5$, and $P(B) = p$. Find p if:
- (a) A and B are disjoint,
- (b) A and B are independent,
- (c) A is a subset of B.

(6) A coin is loaded so that heads are four times as likely as tails. If the coin is tossed three times, determine the sample space, hence find the probability of getting:
- (a) all heads
- (b) two tails and a head (not necessary in that order)

(7) A city is partition into wards A, B, C having 20 percent, 40 percent, and 40 percent of the registered voters, respectively. The registered voters listed as Democrats are 50 percent in A, 25 percent in B, and 75 percent in C. A registered voter is chosen randomly in the city.
- (a) Find the probability that the voter is a listed Democrat.
- (b) If the registered voter is a listed Democrat, find the probability that the voter comes from ward B.

(8) Refer to ***(Q7)*** above. Suppose a ward is chosen at random, and then a registered voter is randomly chosen from the ward.
- (a) Find the probability that the voter is a listed Democrat.
- (b) If the voter is a listed Democrat, what is the probability that the voter came from ward A?

(9) In City University, students from departments of Biology, Chemistry, Mathematics, and Physics consist of 60 percent, 40 percent, 40 percent, and 45 percent women respectively. The Faculty of Science population is 30 percent from Biology, 25 percent from Chemistry, 25 percent from Mathematics, and 20 percent from Physics. A student from Faculty of Science is chosen at random.
- (a) Find the probability that the student is a woman.
- (b) If the student is a woman, what is the probability that she is from chemistry department?

(10) Refer to ***(Q9)*** above. Suppose one of the four departments is chosen at random, and then a student is randomly chosen from the department.
- (a) Find the probability that the student is a woman.
- (b) If the student is a woman, what is the probability that she is from chemistry department?

(11) A company produces light bulbs in three factories A, B, C.
- (a) Factory A produces 40% of the total number of bulbs, of which 2% are defective
- (b) Factory B produces 35% of the total number of bulbs, of which 4% are defective
- (c) Factory C produces 25% of the total number of bulbs, of which 3% are defective

A defective bulb is found among the total output. Find the probability that it came from
- (a) Factory A,
- (b) Factory B,
- (c) Factory C.

(12) Refer to ***(Q11)*** above. Suppose a factory is chosen at random, and one of its bulbs is randomly selected. If the bulb is defective, find the probability that it came from
- (a) Factory A,
- (b) Factory B,

(c) Factory *C*.

Solution 3

Q(1)
(a) $7/15$ (b) 0.30 (c) $7/15$

Q(2)
$5/9$

Q(3)
(a) $4/91$ (b) $421/455$ (c) $2/13$

Q(4)
(a) $2/3$ (b) $2/5$ (c) 0.30 (d) 0.70

Q(5)
(a) 0.2 (b) $2/7$ (c) 0.5

Q(6)
(a) $64/125$ (b) $12/125$

Q(7)
(a) 0.5 (b) 0.2

Q(8)
(a) 0.5 (b) $1/3$

Q(9)
(a) 0.47 (b) 0.213

Q(10)
(a) 0.4625 (b) 0.216

Q(11)
(a) 0.271 (b) 0.475 (c) 0.254

Q(12)
(a) $2/9$ (b) $4/9$ (c) $1/3$

Combinatorial Analysis

Definition 10 (Combinatorial Analysis):
Combinatorial analysis is a method for determining, without direct enumeration, the number of possible outcomes of a particular experiment or event or the number of elements ina particular set

In many random experiments where the sample space is not large, counting the sample points is not difficult. However, problems/difficulties arise where direct enumeration or counting becomes practically impossible or one becomes tedious when dealing with experiments with large sample points, e.g. tossing a coin 8 times etc. This task can be facilitated by the use of some basic notions and results from elementary combinatorial analysis, which could be called a *counting technique* (*sophisticated* way of counting).

Product Rule Principle (fundamental principle of counting): Suppose an event *E* can occur in *m* ways and, independent of this event, an event *F* can occur in *n* ways. Then combinations of events *E* and *F* can occur in $m \cdot n$ ways.

Cleary this principle can also be extended to three or more events. That is, suppose E_1 can occur n_1, then a second event E_2 can occur in n_2 ways, then a third event E_3 can occur in n_3 ways and so on. Then all of the events can occur in $n_1 \cdot n_2 \cdot n_3 \cdot$ ways.

Example:

Suppose a password consists of 4 characters, the first 2 being letters in the (English) alphabet and the last 2 being digits. Find the number *n* of:

(a) Passwords
(b) Passwords beginning with a vowel

Solution:

(a) There are 26 ways to choose each of the first 2 characters and 10 ways to choose each of the last 2 characters. Thus by the product rule

$$n = 26 \cdot 26 \cdot 10 \cdot 10 = 67,600 .$$

(b) Here there are only 5 ways to choose the first character. Hence

$$n = 5 \cdot 26 \cdot 10 \cdot 10 = 13,000 .$$

Example:

A student has 5 different books that he wishes to arrange side by side. How many different arrangements are possible?

Solution:

There are 5 decisions (events) to be made, one for each space which will hold a book. To select a book for the first place, the student has 5 choices, for the second place, 4 choices (one book has already been put in the first place), for the third place, 3 choices, and so on. By product rule, we see that the number of different arrangements is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

Example:

There are 5 bus lines from city A to City B and 4 bus lines from city *B* to city *C*. Find the number *n* of ways a person can travel by bus:

(a) from *A* to *C* by way of *B*,
(b) round-trip from *A* to *C* by way of *B*,
(c) round-trip from *A* to *C* by way of *B* without using a bus line more than once.

Solution:

(a) There are 5 ways to go fro *A* to *B* and 4 ways to go from *B* to *C*; hence, by the product rule, $n = 5 * 4 = 24$.

(b) There are 20 ways to go from *A* to *C* by way of *B* and 20 ways to return, Thus, by product rule, $n = 20 * 20 = 400$.

(c) The person will travel from *A* to *B* to *C* to *B* to *A*. Enter these letters with connecting arrows as follows:

$$A \longrightarrow B \longrightarrow C \longrightarrow B \longrightarrow A$$

There are 5 ways to go from *A* to *B* and 4 ways to go from *B* to *C*. Since a bus line is not to be used more than once, there are only 3 ways to go from *C* back to *B* and only 4 ways to go from *B* back to *A*. Enter these numbers above the corresponding arrows as follows:

$$A \xrightarrow{\quad 5 \quad} B \xrightarrow{\quad 4 \quad} C \xrightarrow{\quad 3 \quad} B \xrightarrow{\quad 4 \quad} A$$

Thus, by the product rule, $n = 5 * 4 * 3 * 4 = 240$

Definition 11 (Factorial):

The product of the positive integers 1 to *n* inclusive occurs very often in mathematics and hence it is denoted by the special symbol *n*! read "*n* factorial". That is,

$$n! = 1 \cdot 2 \cdot 3 \quad (n-2)(n-1)n$$
$$= n(n-1)(n-2) \quad 3 \cdot 2 \cdot 1$$

In other words, *n*! may be defined by

$$1! = 1 \quad \text{and} \quad n! = n \cdot (n-1)!$$

It is also convenient to define $0! = 1$.

Examples:

(a) $2! = 2 \cdot 1 = 2$; $3! = 3 \cdot 2 \cdot 1 = 6$; $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$; $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

(b) $\dfrac{8!}{6!}=\dfrac{8\cdot 7\cdot 6!}{6!}=8\cdot 7=56;\quad \dfrac{8!}{6!}=\dfrac{8\cdot 7\cdot 6!}{6!}=8\cdot 7=56;\quad 12\cdot 11\cdot 10=\dfrac{12\cdot 11\cdot 10\cdot 9!}{9!}=\dfrac{12!}{9!}$

(c) $\dfrac{12\cdot 11\cdot 10}{3\cdot 2\cdot 1}=12\cdot 11\cdot 10\cdot\dfrac{1}{3!}=\dfrac{12!}{3!9!}$

(d) $n(n-1)\quad (n-r+1)=\dfrac{n(n-1)\quad (n-r+1)(n-r)(n-r-1)\quad 3\cdot 2\cdot 1}{(n-r)(n-r-1)\quad 3\cdot 2\cdot 1}=\dfrac{n!}{(n-r)!}$

(e) $\dfrac{n(n-1)\quad (n-r+1)}{r(r-1)\quad 3\cdot 2\cdot 1}=n(n-1)\quad (n-r+1)\cdot\dfrac{1}{r!}=\dfrac{n!}{(n-r)!}\cdot\dfrac{1}{r!}=\dfrac{n!}{r!(n-r)!}$

Exercise 4

(1) Compute: $(a)\dfrac{16!}{14!},\quad (b)\dfrac{14!}{11!},\quad (c)\dfrac{8!}{10!},\quad (d)\dfrac{10!}{13!}$

(2) Simplify: $(a)\dfrac{(n+1)!}{n!},\quad (b)\dfrac{n!}{(n-2)!},\quad (c)\dfrac{(n-1)!}{(n+2)!},\quad (d)\dfrac{(n-r+1)!}{(n-r-1)!}$

(3) Suppose a password consists of 4 characters where the first character must be a letter of the (English) alphabet, but each of the other characters may be a letter or a digit. Find the number of :
(a) passwords (b) passwords beginning with one of the 5 vowels.

(4) Suppose a code consists of 2 letters followed by 3 digits. Find the number of:
(a) codes (b) codes with distinct letters (c) codes with the same letters

(5) There are 6 roads between *A* and *B* and 4 roads between *B* and *C*. Find the number *n* of ways a person can drive:
(a) from *A* to *C* by way of *B*,
(b) round-trip from *A* to *C* by way of *B*,
(c) round-trip from *A* to *C* by way of *B* without using the same road more than once.

Solution 4

Q(1)
(a) 240 (b) 2184 (c) $1/90$ (d) $1/1716$

Q(3)
(a) 1,210,056 (b) 233,280

Q(4)
(a) 676,000 (b) 650,000 (c) 26,000

Q(5)
(a) 24 (b) 576 (c) 360

Permutations

Definition 11 (Permutations):
Any arrangement of a set of *n* objects in a given order is called a *permutation* of the objects (taken all at a time). Any arrangement of any $r \le n$ of these objects in a given order is called an *r permutation* or a *permutation of the n objects taking r at a time*

Consider, for example, the set of letters *a*, *b*, *c*, there are 6 permutations of the 3 letters namely, *abc, bac, cab, acb, bca, cba.*

The number of permutations of *n* objects taking *r* at a time will be denoted by

$$P(n,r) \quad \text{or} \quad {}^{n}P_{r}$$

Theorem 8 (Permutation Formula):

The number of permutations of *n* objects taking *r* at a time

(a) without repetition is $P(n,r)=\dfrac{n!}{(n-r)!}$

(b) and with repetitions is n^r

Proof:

(a) The first element of an *r* permutation (without replacement) of *n* objects can be chosen in *n* different ways; following this, the second element in the permutations can be chosen in $n-1$ ways; ways; and, following this the third element in the permutation can be chosen in $n-2$ ways. Continuing this manner, we have the *r*th (last) element in the *r* permutation can be chosen in $n-(r-1)=n-r+1$ ways. Thus, by the product rule principle, we have

$$P(n,r)=n(n-1)(n-2) \quad (n-r+1)$$

and

$$n(n-1)(n-2) \quad (n-r+1)=\frac{n(n-1)(n-2) \quad (n-r+1)(n-r)!}{(n-r)!}=\frac{n!}{(n-r)!}$$

Thus,

$$P(n,r)=\frac{n!}{(n-r)!}$$

(b) The first element of an *r* permutation (with replacement) of *n* objects can be chosen in *n* different ways; the object is then replaced before the next object is chosen; following this, the second element in the permutations can also be chosen in *n* ways and then replaced; and, following this the third element in the permutation can be chosen in *n* ways and then replaced. Continuing this manner, we have the *r*th (last) element in the *r* permutation can be chosen in *n* ways. Thus, by the product rule principle, we have

$$\overbrace{n\cdot n\cdot n \quad n}^{r\ \text{times}}=n^r$$

Example:

Suppose 8 people enter an event in the recent Olympics. Assuming there are no ties, in how many ways could the gold, silver, and bronze medals be awarded.

Solution:

Using the formula for $P(n,r)$, with $n=8$ and $r=3$

$$\begin{aligned} P(8,3)&=\frac{8!}{(8-3)!} \\ &=\frac{8!}{5!}=\frac{8\cdot 7\cdot 6\cdot 5!}{5!} \\ &=8\cdot 7\cdot 6=336 \end{aligned}$$

Example:

Three cards are chosen in succession from a deck with 52 cards.3 Find the number of ways this can be done

(a) with replacement, (b) without replacement.

Solution:

(a) Since each card is replaced before the next card is chosen, each card can be chosen in 52 ways. Thus, there is

$$52(52)(52) = 52^3 = 140,608$$

is the number of different ways of choosing the cards with replacement.

(b) Since there is no replacement, the first card can be chosen in 52 ways, the second in 51 ways, and the last card in 50 ways. Thus,

$$P(52,3) = 52(51)(50) = 132,600$$

is the number of different ways of choosing the cards without replacement.

Corollary: The number of permutations of *n* different objects taken all at a time is equal to *n*!

Proof: Solve as an exercise.

Example:

In how many ways can a team manager introduce the 11 players of a football team to the public.

Solution:

In $11! = 39,916,800$ different ways

Theorem 9(Permutations with repetitions):

The number of permutations of *n* objects of which n_1 are alike, n_2 are alike, …, n_r are alike is

$$P(n; n_1, n_2, \quad , n_r) = \frac{n!}{n_1! n_2! \quad n_r!}$$

Example:

Find the number *m* of seven letter words that can be formed using the letters of the word "*BENZENE*".

Solution:

We seek the number of permutations of seven objects of which three are alike, the three *E*'s, and two are alike, the two *N*'s. By theorem 9 above,

$$m = P(7;3,2) = \frac{7!}{3!2!} = \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1\cdot 2\cdot 1} = 420$$

Theorem 10 (Circular Permutation)

The number of permutations of *n* distinct objects arranged in a circle is $(n-1)!$.

Example:

Find the number *n* of ways that 7 people can arrange themselves:

(a) in a row of 7 chairs,

(b) around a circular table.

Solution:

(a) The 7 people can arrange themselves in a row in $n = 7*6*5*4*3*2*1 = 7!$ ways.

(b) One person can sit at any place at the circular table. The other 6 people can then arrange themselves in $n = 6*5*4*3*2*1 = 6!$ ways around the table.

Example:

Suppose repetitions are not allowed.

(a) Find the number n of three-digit numbers that can be formed from the six digits: 2, 3, 5, 6, 7, 9.
(b) How many of them are even?
(c) How many of them exceed 400?

Solution:

There are 6 digits, and the three-digit number can be pictured by

$$\_\_\_, \ \_\_\_, \ \_\_\_.$$

in each case, write down the number of ways that one can fill each of the positions.

(a) There are 6 ways to fill the first position, 5 ways for the second position. And 3 ways for the third position. This may be pictured by: $\underline{\ 6\ }, \underline{\ 5\ }, \underline{\ 4\ }$. Thus, $n = 6 * 5 * 4 = 120$.

Alternately, n is the permutations o 6 things taken 3 at a time, and so

$$n = {}^{6}P_{3} = P(6,3) = 6 * 5 * 4 = 120$$

(b) Since the number must be even, the last digit must be either 2 or 4. Thus, the third position is filled first and it can be done in 2 ways. Then there are 5 ways to fill the middle position and 4 ways to fill the first position. This may be pictured by: $\underline{\ 4\ }, \underline{\ 5\ }, \underline{\ 2\ }$. Thus, $n = 4 * 5 * 2 = 40$ of the numbers are even.

(c) Since the numbers must exceed 400, they must begin with 5, 6, 7, or 9. Thus, we first fill the first position and it can be done in 4 ways. Then there are 5 ways to fill the second position and 4 ways to fill the third position. This may be pictured by: $\underline{\ 4\ }, \underline{\ 5\ }, \underline{\ 4\ }$. Thus, $n = 4 * 5 * 4 = 80$ of the numbers exceed 400.

Exercise 5

(1) A multiple-choice test consists of 10 questions each permitting a choice of four alternatives. In how many different ways can a student check off his/her answers to these questions?

(2) A televised debate will include 4 women and 3 men as finalists.
(a) In how many ways can the finalists be seated in a row of 7 chairs?
(b) In how many ways can the finalists be seated if the men and women are to be alternated?

(3) Find the number n of permutations that can be formed from all the letters of each word:
(a) *QUEUE*
(b) *COMMITTEE*
(c) *PROPOSITION*
(d) *STATISTICS*

(4) Six boys and two girls are to sit together on a bench. In how many ways can they arrange themselves if:
(a) The girls do not sit next to each other?
(b) The girls sit next to each other?

(5) Eight women and two men are to sit at a round table. Find the number of ways if
(a) no restriction is imposed
(b) two men must sit next to each other

(6) For many years, the state of Tokunbo used three letters followed by three digits on its car license plate.
(a) How many different license plates are possible with this arrangement?
(b) When the state ran out of new plates, the order was reversed to three digits followed by three letters. How many additional plates were then possible?

(c) Several years ago, the plates prescribed in *(b)* were used up. The state then issued plates with one letter followed by three digits and then three letters. How many plates does this scheme provide?

(7) How many ways can 5 boys and 4 girls sit in a row if the boys and the girls must alternate

(8) In how many ways can 9 books be arranged on a shelf if
(a) any arrangement is possible,
(b) four particular books must always stand together,
(c) three particular books must occupy the ends.

(9) $*$ Show that ${}^{n}P_{r} = {}^{n-1}P_{r} + {}^{n-1}P_{r-1} * r$

(10) For what value of n is ${}^{n}P_{2} = 20$

Solution 5

Q(1)
4^{10} = 1,048,576

Q(2)
(a) 5040 (b) 144

Q(3)
(a) 30 (b) 45,360 (c) 1,663,200 (d) 50,400

Q(4)
(a) 30,240 (b) 5040

Q(5)
(a) 9! = 362,880 (b) 8! = 40,320

Q(6)
(a) $26^3(10^3)$ = 17,576,000 (b) $10^3(26^3)$ = 17,576,000 (c) $26^4(10^3)$ = 456,376,000

Q(7)
2880

Q(8)
(a) 9! = 362,880 (b) 17,280 (c) 4320

Q(10)
$n = 5$

Combinations

Definition 12 (Combinations):
Suppose we have a collection of *n* objects. A *combination* of these *n* objects taking *r* at a time is any selection of *r* of the objects where order does not count. In other words, an *r combination* of a set of *n* objects is any subset of *r* elements.

In permutation, the order in which the objects are arranged is very essential. Thus *abc* is a different permutation from *bac* or *bca*. But in combination, the order in which the objects are arranged does not count. Thus *abc, bac,* and *bca* are the same combination.

The number of combinations of *n* objects taken *r* at a time is denoted by

$$C(n,r) \quad \text{or} \quad {}^{n}C_{r} \quad \text{or} \quad \binom{n}{r}$$

Example:

Find the number of combinations of four objects, *a, b, c, d,* taking three at a time.

Solution:

Each combination consisting of three objects determines $3! = 6$ permutations of the objects in the combination as in the table below.

Combinations	Permutations
abc	abc, acb, bac, bca, cab, cba
abd	abd, adb, bad, bda, dab,dba
acd	acd, adc, cad, cda, dac, dca
bcd	bcd, bdc, cbd, cdb, dbc, dcb

Thus, the number of combinations multiplied by 3! equals the number of permutations. That is,

$$C(4,3)\cdot 3! = P(4,3) \quad \text{or} \quad C(4,3) = \frac{P(4,3)}{3!}$$

But $P(4,3) = 4\cdot 3\cdot 2 = 24$ and $3! = 6.$ Thus, $C(4,3) = 4$, which is noted in the table above.

Theorem 11 (Combination Formula):

The number of combinations of n objects taking r at a time is without repetition is

$$\binom{n}{r} = {}^{n}C_{r} = C(n,r) = \frac{P(n,r)}{r!} = \frac{{}^{n}P_{r}}{r!} = \frac{n!}{r!(n-r)!}$$

and the number of combinations with repetitions is

$$\binom{n+r-1}{r} = {}^{n+r-1}C_{r} = C(n+r-1,r) = \frac{(n+r-1)!}{r!(n+r-1-r)!} = \frac{(n+r-1)!}{r!(n-1)!}$$

Example:

How many different committees of 3 people can be chosen from a group of 8 people?

Solution:

Since the order in which the members of the committee are chosen does not effect the result, use combination to get

$$\binom{8}{3} = \frac{8!}{5!\,3!} = \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{(5\cdot 4\cdot 3\cdot 2\cdot 1)\cdot(3\cdot 2\cdot 1)} = 56\,.$$

Example:

From a group of 30 employees, 3 are to be selected to work on a special project.

(a) In how many different ways can the employees be selected?

(b) In how many different ways can the group of 3 be selected if it has already been decided that a certain employee must work on the project

(c) In how many ways can a (nonempty) group of *at most* 3 employees be selected from the group of 30?

Solution:

(a) The number of 3-element combinations from a set of 30 elements must be found. Using the formula gives

$$\binom{30}{3} = \frac{30!}{27!\;3!} = 4060.$$

There are4060 ways to select the project group.

(b) Since one employee has already been selected to work on the project, the problem is reduced to selecting 2 more employees from the 29 employees that are left:

$$\binom{29}{2} = \frac{29!}{27!\;2!} = 406.$$

In this case, the project group can be selected in 406 different ways.

(c) Here "at most 3" means "exactly 1 or exactly 2 or exactly 3." We shall find the number of ways to select employees for each case.

Case	Number of Ways
1	$\binom{30}{1} = \frac{30!}{29!\ 1!} = \frac{30 \cdot 29!}{1 \cdot 29!} = 30$
2	$\binom{30}{2} = \frac{30 \cdot 29 \cdot 28!}{1 \cdot 2 \cdot 28!} = 435$
3	$\binom{30}{3} = \frac{30 \cdot 29 \cdot 28 \cdot 27!}{1 \cdot 2 \cdot 3 \cdot 27!} = 4060$

The total number of ways to select at most 3 employees will be the sum 30 + 435 4060 = 4525.

Example:

Out of 6 economists, 7 sociologists and 5 medical doctors, a committee consisting of 3 economists, 4 sociologists and 2 medical doctors is to be formed. How many different committees can be formed if

(a) any economist, sociologist and medical doctor can included,
(b) 1 particular economist and 2 particular sociologists must be in the committee.
(c) 2 particular economists and 1 particular medical doctor cannot be in the committee.

Solution:

(a) Ways of selecting an economists is ${}^4C_3$

Ways of selecting sociologists is ${}^7C_4$

Ways of selecting a medical doctor is ${}^5C_2$

Therefore total number of ways $= {}^6C_3 \times {}^7C_4 \times {}^5C_2$

$= 20 \times 35 \times 10$

$= 7000$

(b) 2 economists can be selected out of 5 in ${}^5C_2$

2 sociologists out of 5 can be selected in ${}^5C_2$

2 medical doctors out of 5 can be selected in ${}^5C_2$

Therefore total number of possible selections $= {}^5C_2 \times {}^5C_2 \times {}^5C_2$

$= 3 \times {}^5C_2 = 3 \times 10$

$= 30$ ways

(c) 3 economists can be selected out of 4 in ${}^4C_3$

4 sociologists out of 7 can be selected in ${}^7C_4$

2 medical doctors out of 4 can be selected in ${}^4C_2$

Therefore total number of possible selections $= {}^4C_3 \times {}^7C_4 \times {}^4C_2$

$= 4 \times 35 \times 6$

$= 840$ ways

Exercise 6

(1) A store has 8 different mystery books. Find the number of ways a customer can buy 3 of the books.

(2) A box contains 6 blue socks and 4 white socks. Find the number of ways two socks can be drawn from the box where:
(a) there are no restrictions,
(b) they are different colors,
(c) they are to be the same color.

(3) A bag contains 5 purple, 4 green, and 9 black marbles, how many samples of 3 can be drawn in which:
(a) all the marbles are black
(b) exactly 2 marbles are black

(4) A class contains 9 boys and 3 girls. Find the number of ways a teacher can select a committee of 4.

(5) Repeat ***(Q4)*** above, but where
(a) there are to be 2 boys and 2 girls,
(b) there is to be exactly 1 girl,
(c) there is to be at least 1 girl.

(6) A woman has 11 close friends. Find the number of ways she can invite 5 of them to dinner.

(7) Repeat ***(Q6)*** above, but where 2 of the friends are married and will not attend separately.

(8) Repeat ***(Q6)*** above, but where 2 of the friends are not on speaking terms and will not attend together.

(9) For what value of *n* is $3 \cdot {}^{n+1}C_3 = 7 \cdot {}^{n}C_2$?

(10) *******Prove the following properties on combinations
(a) ${}^{n}C_n = {}^{n}C_0 = 1$
(b) ${}^{n}C_{n-r} = {}^{n}C_r$
(c) ${}^{n}C_r + {}^{n}C_{r+1} = {}^{n+1}C_{r+1}$

(11) Padlocks with digit dials are often referred to as "combination locks." According to the mathematical definition of combination, is this an accurate description? Why or why not?

Solution 6

Q(1) 56

Q(2) (a) 56 (b) 24 (c) 21

Q(3) (a) 84 (b) 324

Q(4) 495

Q(5) (a) 108 (b) 252 (c) 369

Q(6) 462

Q(7) 240

Q(8) 252

Q(9) $n = 6$

Probability using Combinatorial Analysis

Distinguishing b/w Permutations & Combinations:

The formulas for permutations and combinations are very useful in solving probability problems. Any difficulty in using these formulas usually comes from being unable to differentiate between them. Both permutations and combinations give the number of ways to choose r objects from set of n objects. The differences between permutations and combinations are outlined below.

Permutations	Combinations
Different orderings or arrangements of the r objects are different permutations	Each choice or subset of r objects gives 1 combination. Order within the r objects does not matter
$P(n,r)=\dfrac{n!}{(n-r)!}$	$\dbinom{n}{r}=\dfrac{n!}{(n-r)!r!}$
Clue words: Arrangement, Schedule, Order	*Clue words*: Group, Committee, Sample

Many probability problems involve numbers that are too large to determine the number of outcomes easily, even with a tree diagram. In such cases, we can use combinations. **For example**, if 3 engines are tested from a shipping container packed with 12 engines, 1 of which is defective, what is $P(E)$, the probability that the defective engine will be found?

(a) How many ways are there to choose the sample of 3 from the 12 engines?
(b) How many ways are there to choose a sample of 3 with 1 defective and 2 good engines?
(c) What is $n(E)$ in this experiment if E is the event "The defective engine is in the sample"?
(d) What is $n(S)$ in this experiment?
(e) Find $P(E)$.

Solution:

(a) The number of 3-element combinations from a set of 12 elements must be found. Using the formula gives

$$\binom{12}{3}=\frac{12!}{(12-3)!3!}=\frac{12!}{9!3!}=\frac{12\cdot 11\cdot 10\cdot 9!}{9!\cdot 3\cdot 2\cdot 1}=4\cdot 11\cdot 5=220\,.$$

There are 220 ways to select the 3 engines.

(b) There is 1 way to choose 1 defective engine and there are ${}^{11}C_2=55$ ways to choose 2 good engines from the remaining 11 engines. Thus by the product rule

$${}^{1}C_1\cdot {}^{11}C_2=1\cdot 55=55\,.$$

There are 55 ways to choose 1defective and 2 good engines.

(c) If E is the event "The defective engine is in the sample" then $n(E)$ is the number of different combinations of sample that contain the defective engine, which is just the number calculated in ***(b)*** above. Thus,

$$n(E)=55$$

(d) The sample space S is the number of all possible combinations of choosing 3 engines from 12 engines, which is just the number calculate in ***(a)*** above. Thus,

$$n(S)=220$$

(e) By the definition of probability,

$$P(E) = \frac{n(E)}{n(S)} = \frac{55}{220} = \frac{1}{4} = 0.25 = 25\%$$

The above example illustrates how combinations is used to some problems in probability.

Exercise 7

(1) From a shelf containing 8 Biology, 4 Chemistry, 7 Mathematics and 6 Physics books, 5 books are picked at random. What is the probability that:
   (a) 2 Mathematics, 3 Physics books are selected?
   (b) no Chemistry book is selected?

(2) A box contains 9 white, 6 blue and 5 green balls. 4 balls are drawn at random without replacement. Find the probability that:
   (a) all the balls drawn are white,
   (b) 3 are blue and 1 is green,
   (c) at least one is green.

(3) Five cards are drawn at random from a well-shuffled deck of 52 playing cards. Find the probability that:
   (a) 4 are kings
   (b) 3 are twos and 2 are queens
   (c) 2 are of one suit 3 are of another suit
   (d) at least one diamond is obtained.

Solution 7

Q(1)		**Q(2)**
(a) $2/253$ (b) $969/2530$		(a) $42/1615$ (b) $20/969$ (c) $232/323$

Q(3)
(a) $4/54145$ (b) $1/108290$ (c) $429/4165$ (d) $7411/9520$