Elementary Principles of Probability

The concept of probability originated from the analysis of the games of chance in the 17th Century. Now the subject has been developed to the extent that it is very difficult to imagine a discipline that can do without it.

The theory of probability is a study of statistical or random experiments. It is the backbone of statistical inference and decision theory that are essential tools of the analysis of most of the modern scientific and engineering problems.

To understand probability, we must know about some important terms which will be used to in defining probability.

(i) Random Experiment

If in each trial of an experiment conducted under identical conditions, the outcome is not unique, but may be any one of the possible outcomes, then such an experiment is called a *random experiment*.

Examples of random experiments are: - tossing a coin, throwing a die, selecting a card from a pack of playing card and so on. In all these cases, there are a number of possible results which can occur but there is an uncertainty as to which one of them will actually occur.

(ii) Outcome

The result of a random experiment is called an *outcome.*

(iii) Trial

Any particular performance of a random experiment is called a *trial*.

(iv) Sample Space

The set of all possible outcomes of a random experiment is defined as *sample space*, and is denoted by S, each member in the set is called a *sample point.*

Example 1:

A coin is tossed three times. List all the possible outcomes $(Sample\,Space)$ of the experiment.

Solution:

Since the random experiment of tossing a coin has two possible outcomes $(Head\ or\ Tail)$, and the experiment is repeated three times, then the number of the sample point we have can calculated as $2^3 = 8$.

Hence, the sample space is

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

(v) Event

An *event* is a subset of a sample space. Event is called simple if it corresponds to a single sample point of the sample space otherwise it is known as a compound or composite event.

Example 2:

From the above example, list the events such that

(i) Exactly 2 head appeared
(ii) At least 1 head obtained.

Solution:

(i) $S = \{HHT, HTH, THH\}$

(ii) $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH\}$

(vi) Equally Likely Outcomes

The outcomes of random experiment are said to be *equally likely* or *equally probable* if the occurrence of none of them is expected in preference to others. For example, if an unbiased coin is tossed once, the two possible outcomes, a head or a tail are *equally likely*.

Classical Definition of Probability

If n is the number of equally likely outcomes of a random experiment out of which m outcomes are favourable to the occurrence of an event A, then the probability that A occurs, denoted by $P(A)$, is given by:

$$P(A) = \frac{Number\ of\ outcomes\ favourable\ to\ A}{Total\ number\ of\ possible\ outcomes} = \frac{n}{m}$$

Types of Events

(1) Exhaustive Event

An event which is favourable by every outcome of a random experiment is called an *exhaustive event* and is denoted by symbol ς.

Thus, $P(\varsigma) = \frac{m+n}{m+n} = 1$

This is also called sure event.

(2) Impossible Event

An event which never happens, i.e., no outcome favours to this event, is called an impossible or a null event, denoted by ϕ.

Thus, $$P(\phi) = \frac{0}{m+n} = 0$$

(3) Compound Event

The events which are made up by the combination of two or more events are called *compound events*. For example, suppose we throw a single die, and A is the event of getting an even number on the die and B is the event of getting a number less than 3, then $A+B \, or \, A \cup B$ is the event of getting an even number or number less than 3. Thus, $A+B \, or \, A \cup B$ is termed as compound event.

(4) Complementary Event

If E is an event, then the event of not happening the event E is called *complementary event* of E, denoted by $E^c \, or \, E' \, or \, \bar{E}$. It is probability can be found by subtracting the $P(E)$ from 1. *i.e.*,

$$P(E^c) = 1 - P(E)$$

(5) Mutually Exclusive Events or Disjoint Events

Two events A, B are said to be *mutually exclusive or disjoint events* if A happens, then B cannot happen, and if B happens, then A cannot happen. Thus, out of A, B only one can happen at a time, i.e., both cannot happen simultaneously. In this case, $A \cap B = \phi$.

Hence, $P(A \cap B) \, or \, P(AB) = 0$

For example, from a pack playing cards, if R is the event of drawing a red card and B is the event of drawing a black card. Then this experiment of drawing a card, both R and B cannot happen simultaneously or we say when R happens, then B cannot happen and vice versa. So these are mutually exclusive events.

(6) Independent Events

Two events A, B are said to be independent if happening of one event does not affect the happening of other, i.e., when A happens, B may or may not occur. The occurrence of B is not linked with happening of A, i.e., B is independent of A.

For example, if we throw a coin and die simultaneously, then the result of coin is independent of result of die. So H and 5 are two independent events.

Axioms of Probability

In order to find the probability of any event of a sample space, the following rules, popularly known as axioms or postulates of probability, must satisfied:

Axiom I: For any event A in sample space S, we have

$$0 \le P(A) \le 1.$$

Axiom II: $P(S)=1$.

Axiom II: if $A_1, A_2, ..., A_k$ are k mutually exclusive event of the sample space S, then

$$P(A_1 \cup A_2 ... \cup A_k) = \sum_{i=1}^{k} P(A_i).$$

The first axiom implies that the probability of an event is non-negative number less than or equal to unity. The second axiom implies that the probability of an event that is certain to occur must be equal to unity. Axiom III gives a basic rule of addition of probabilities when events are mutually exclusive.

Theorems on Probability

Theorem 1.

$P(\phi)=0$, where ϕ is a null set.

Proof.

Let A be any event in the sample space S, We can write $S \cup \phi = S$.

Taking probability of both sides, we have $P(S \cup \phi) = P(S)$.

Since S and ϕ are mutually exclusive, using axiom III, we can write

$P(S)+P(\phi)=P(S)$. Hence, $P(\phi)=0$.

Theorem 2.

$P(\bar{A})=1-P(A)$, where $\bar{A}$ is compliment of A.

Proof.

Let A be any event in the sample space S. We can write

$A \cup \bar{A} = S$ or $(A \cup \bar{A}) = (S)$ Since A and $\bar{A}$ are mutually exclusive, we can write

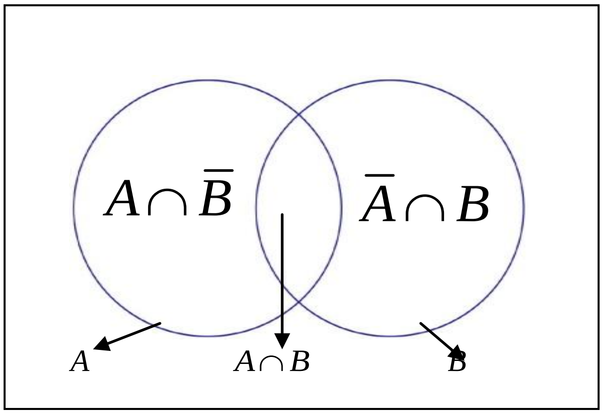
$P(A)+P(\bar{A})=P(S)=1$. Hence, $P(\bar{A})=1-P(A)$.

Theorem 3.

For any two events A and B in the sample space S

$P(\bar{A} \cap B) = P(B) - P(A \cap B)$

Proof.



From the Venn diagram above, we can write

$(B)=\left[\left(\bar{A}\cap B\right)\cup\left(A\cap B\right)\right]$ or $P(B)=P\left[\left(\bar{A}\cap B\right)\cup\left(A\cap B\right)\right]$

Since $\left(\bar{A}\cap B\right)$ and $\left(A\cap B\right)$ are mutually exclusive, we have

$$P(B)=P\left(\bar{A}\cap B\right)+P\left(A\cap B\right)$$

$$\therefore P\left(\bar{A}\cap B\right)=P(B)-P\left(A\cap B\right)$$

Similarly, it can be shown that

$$P\left(A\cap\bar{B}\right)=P(A)-P\left(A\cap B\right)$$

Theorem 4: (Addition Theorem of Probability)

If A and B are any two events of the same random experiment, then probability of the event of happening of at least one event out of these, is given by

$$P\left(A\cup B\right)=\left(A+B\right)=P(A)+P(B)-P\left(A\cap B\right)$$

Proof.

From the Venn diagram given above, we can write

$A\cup B=A\cup\left(\bar{A}\cap B\right)$ or $P\left(A\cup B\right)=P\left[A\cup\left(\bar{A}\cap B\right)\right]$

Since A and $\left(\bar{A}\cap B\right)$ are mutually exclusive, we can write

$$P\left(A\cup B\right)=P(A)+P\left(\bar{A}\cap B\right)$$

Substituting the value of $P(\bar{A}\cap B)$ from theorem 3, we get

$$P(A\cup B)=P(A)+P(B)-P(\bar{A}\cap B).$$

Note: The addition theorem can be generalized for than two events. If A, B and C are three events of a sample space S, the n the probability of occurrence of at least one of them is given by

$$P(A\cup B\cup C)=P[A\cup(B\cup C)]=P(A)+P(B\cup C)-P[A\cap(B\cup C)]$$

$$=P(A)+P(B\cup C)-P[(A\cap B)\cup(A\cap C)]$$

Applying theorem 4 on the second and third term, we get

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C).$$

Example 3:

The probability that a student passes a mathematics test is $2/3$ and the probability that he passes both mathematics test and statistics test is $14/45$. The probability that he passes at least one test is $4/5$. What is the probability that he passes the statistics test?

Solution:

Let us define the following events:

A: The student passes a mathematics test

B: The student passes a statistics test

Given, $P(A)=2/3,\ \ P(A\cap B)=14/45,\ \ P(A\cup B)=4/5\ \ and\ \ \ P(B)=?$

Applying addition theorem, we have

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$$\Rightarrow \qquad 4/5=2/3+P(B)-14/45$$

$$\therefore \qquad P(B)=\frac{4}{5}+\frac{14}{45}-\frac{2}{3}=\frac{36+14-30}{45}=\frac{4}{9}$$

Example 4:

If two dice are thrown together, find the probability that the sum is

(a) Greater than 8 (b) Neither 7 nor 11

Solution:

In a random toss of two dice, sample space S is given by $6^n = 6^2 = 36$

$$
\begin{matrix}
\{(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\
(6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6)\}
\end{matrix}
$$

(a) If S denotes the sum on the two dice, then we want $P(S>8)$.

The required events can happen in the following mutually exclusive ways:

$$P(S>8) = P(S=9) + P(S=10) + P(S=11) + P(S=12)$$

Thus,

$$S=9: \quad (3,6),(6,3),(4,5),(5,4) \;\Rightarrow\; P(S=9) = \frac{4}{36}$$

$$S=10: \quad (4,6),(6,4),(5,5) \;\Rightarrow\; P(S=10) = \frac{3}{36}$$

$$S=11: \quad (5,6),(6,5) \;\Rightarrow\; P(S=11) = \frac{2}{36}$$

$$S=12: \quad (6,6) \;\Rightarrow\; P(S=12) = \frac{1}{36}$$

$$\therefore \qquad P(S>8) = \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36} = \frac{5}{18}$$

(b) Let A denote the events of getting the sum of 7 and B denote the events of getting the sum of 11. We have

$$A=7: \quad (1,6),(6,1),(2,5),(5,2),(3,4),(4,3) \;\Rightarrow\; P(A) = P(A=7) = \frac{6}{36} = \frac{1}{6}$$

$$B=11: \quad (5,6),(6,5) \;\Rightarrow\; P(B) = P(B=11) = \frac{2}{36} = \frac{1}{18}$$

$$\therefore \quad \text{Required probability} = P\left(A^c \cap B^c\right) = 1 - P\left(A \cup B\right)$$
$$= 1 - \left[P\left(A\right) + P\left(B\right)\right]$$
$$= 1 - \frac{1}{6} - \frac{1}{18} = \frac{7}{9}$$

Example 5:

A man shoots at a target, the probability that he hits the target is $2/5$, if he shoots three times, what is the probability that

(i) He hits the target exactly once.
(ii) He fails to hit the target.

Solution:

Let H denotes the event the man hits the target, and F denotes the event he fails to hit the target, then

$$S = \{HHH, HHF, HFH, FHH, HFF, FHF, FFH, FFF\}$$

Given, $P\left(H\right) = 2/5 \quad \Rightarrow \quad P\left(F\right) = 1 - P\left(H\right) = 1 - 2/5 = 3/5$

$$(i) \quad P\left(He\, hits\, the\, \text{target exactly once}\right) = P\left(HFF\right) + P\left(FHF\right) + P\left(FFH\right)$$

$$= \left(\frac{2}{5} \times \frac{3}{5} \times \frac{3}{5}\right) + \left(\frac{3}{5} \times \frac{2}{5} \times \frac{3}{5}\right) + \left(\frac{3}{5} \times \frac{3}{5} \times \frac{2}{5}\right)$$

$$= \frac{18}{125} + \frac{18}{125} + \frac{18}{125} = \frac{54}{125}$$

$$(ii) \quad P\left(He\ fails\, to\, hit\, the\, \text{target}\right) = P\left(FFF\right)$$
$$= \left(\frac{3}{5} \times \frac{3}{5} \times \frac{3}{5}\right) = \frac{27}{125}$$

Example 6:

What is the probability of drawing a black card or a king from a pack of playing cards?

Solution:

There are 52 cards in a pack. Thus, $S = 52$

Let A be the event that the drawn card is black and B be the event that it is a king. We want find $P(A\cup B)$.

Since there are 26 black cards, 4 kings and 2 black kings in a pack, we have

$P(A)=26/52$, $P(B)=4/52$ and $P(A\cap B)=2/52$.

Thus,

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)=\frac{26}{52}+\frac{4}{52}-\frac{2}{52}=\frac{7}{13}$$

Theorem 5: (Multiplication Theorem of Probability)

If A and B are any two independent events, the probability of simultaneous happening of both the events is given by

$$P(A\cap B)=P(AB)=P(A).P(B)$$

Example 7:

Three balls are drawn from the box containing 6 white ball, 5 red balls and 4 blue balls. Find the probability that they are drawn in the order blue, red and white if each ball is

(a) Replaced (b) Not replaced

Solution:

(a) The probability of drawing a blue, a red and a white ball with replacement is

$$P(B\cap R\cap W)=P(B)\times(R)\times(W)$$

But $P(B)=\frac{4}{15}$, $P(R)=\frac{5}{15}$ *and* $P(W)=\frac{6}{15}$

$\therefore$ $P(B\cap R\cap W)=\frac{4}{15}\times\frac{5}{15}\times\frac{6}{15}=\frac{8}{225}$

(b) Probability of drawing a blue, a red and a white ball without replacement is

$$P(B\cap R\cap W)=P(B)\times(R)\times(W)$$

But $P(B)=\frac{4}{15}$, $P(R)=\frac{5}{14}$ *and* $P(W)=\frac{6}{13}$

$\therefore$ $P(B\cap R\cap W)=\frac{4}{15}\times\frac{5}{14}\times\frac{6}{13}=\frac{4}{91}$

Theorem 6: (Conditional Theorem of Probability)

Let A and B be two events, then the conditional probability of event A given that event B has already occurred is given by

$$P\left(A/B\right)=\frac{P\left(A\cap B\right)}{P\left(B\right)} \qquad ; P\left(B\right)\neq 0$$

Similarly, the conditional probability of event B given that event A has already occurred is given by

$$P\left(B/A\right)=\frac{P\left(A\cap B\right)}{P\left(A\right)} \qquad ; P\left(A\right)\neq 0$$

Example 8: The probabilities that a student will fail MTH1301 (M) and STA1301 (S) or both are: $P\left(M\right)=0.20, P\left(S\right)=0.15$ and $P\left(M\cap S\right)=0.03$. What is the probability that he/she will fail MTH1301 given that he/she will fail STA1301?

Solution:

We want $P\left(M/S\right)$ and by definition of conditional probability

$$P\left(M/S\right)=\frac{P\left(M\cap S\right)}{P\left(S\right)}=\frac{0.03}{0.15}=0.2$$

Example 9:

Find the probability that a single toss of a die results in a number less than 4 if it is given that the toss resulted in an odd number.

Solution:

Let A be the event (less than 4) and B be the event (odd number) then

$$P\left(A\right)=P\left(1\right)+P\left(2\right)+P\left(3\right)=\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{1}{2}$$

$$P\left(B\right)=P\left(1\right)+P\left(3\right)+P\left(5\right)=\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{1}{2}$$

Also, $P\left(A\cap B\right)=P\left(1\right)+P\left(3\right)=\frac{1}{6}+\frac{1}{6}=\frac{1}{3}$

$$\therefore \qquad P\left(A/B\right)=\frac{P\left(A\cap B\right)}{P\left(B\right)}=\frac{1/3}{1/2}=\frac{2}{3}$$

Theorem 7: (Bayes' Theorem)
If an event D can occur only in combination with any of the n mutually exclusive and

exhaustive events $A_1, A_2, ..., A_n$ and if, in an actual observation, D is found to have occurred, then the probability that it was preceded by a particular event A_k is given by

$$P\left(A_k/D\right) = \frac{P(A_k).P\left(D/A_k\right)}{\sum_{i=1}^{n} P(A_i).P\left(D/A_i\right)}$$

Example 10:

In a bolt factory, machine A, B and C manufacture 25%, 35% and 40% of the total respectively. If their output 5%, 4% and 2% are defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it was manufactured by machine B?

Solution:

A: bolt is manufactured by machine A

B: bolt is manufactured by machine B

C: bolt is manufactured by machine C

D: bolt is defective.

$P(A) = 0.25, P(B) = 0.35, P(C) = 0.40$

The probability of drawing a defective bolt manufactured by machine A, B and C is $P\left(D/A\right) = 0.05$, $P\left(D/B\right) = 0.04$ and $P\left(D/C\right) = 0.02$ respectively.

Thus, applying Bayes theorem, we have

$$P\left(B/D\right) = \frac{P(B)\left(D/B\right)}{P(A)P\left(D/A\right) + P(B)P\left(D/B\right) + P(C)P\left(D/C\right)}$$

$$= \frac{0.35 \times 0.04}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02} = 0.41$$

Counting Techniques

Counting techniques or combinatorial methods are often helpful in the enumeration of total number of outcomes of a random experiment and the number of cases favourable to the occurrence of an event.

Fundamental Principal of Counting

If the first operation can be performed in any one of the m ways and then a second operation can be performed in any one of the n ways, then both can be performed together in any one of the $m \times n$ ways.

This rule can be generalized. If first operation can be performed in any one of the n_1 ways, second operation in any one of the n_2 ways, ... Kth operation in any one of the n_k ways, then together these operations can be performed in any one of the $n_1 \times n_2 ... n_k$ ways.

Factorial Notation

The notation $n!$ is used to denote the product of all the positive whole numbers from 1 up to n. $i.e., n! =$ product of the first n natural numbers. For example,

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$
$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720 \quad etc.$$

In general, $n! = n(n-1)!$

Hence, by definition $0! = 1$.

Permutation

A permutation of a number of objects is their arrangement in some finite order. For example, given three letters a, b, c, we can permute them two at a time as "bc, cb, ca, ac, ab, ba" yielding 6 permutations.

Following are some rules regarding permutation:

(i) The number permutations of n different thing taken r at a time is

$n(n-1)(n-2)...(n-r+1)$

and is denoted by ${}^nP_r$.

where $${}^nP_r = \frac{n!}{(n-r)!}$$

Example 1:

Find the number of ways of arranging 7 students in 7 seats.

Solution:

$${}^nP_r = \frac{n!}{(n-r)!} \qquad ; \quad n = 7, \; r = 7$$

$$\Rightarrow \qquad {}^7P_7 = \frac{7!}{(7-7)!} = \frac{7!}{0!} = \frac{7!}{1}$$
$$= 7\times6\times5\times4\times3\times2\times1 = 5040\,ways$$

Example 2:

Find the number of ways of arranging 3 women in 5 seats.

Solution:

$${}^nP_r = \frac{n!}{(n-r)!} \qquad ; \quad n=5,\ r=3$$

$$\Rightarrow \qquad {}^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5\times4\times3\times2\times1}{2\times1}$$
$$= 5\times4\times3 = 60\,ways$$

(ii) Permutations with repetitions. The number of permutation of n object of which r_1 are alike, r_2 are alike and r_3 are alike is

$$\frac{n!}{r_1!r_2!r_3!}$$

Example 3:

How many distinct arrangements can be made from the word STATISTICS?

Solution:

$n=10,\ r_1=3\ for\ S,\ r_2=3\ \ for\ T, r_3=2\ \ for\ I$

Thus, the number of the distinct arrangements $= \dfrac{n!}{r_1!r_2!r_3!} = \dfrac{10!}{3!3!2!} = 50400\,ways$

(iii) The number of ways of arranging n objects in a circle is given by $(n-1)!$

Example 4:

Find the number of ways of arranging 6 people in a circle.

Solution:

$$n=6$$

$$\Rightarrow \qquad (n-1)! = (6-1)! = 5!$$
$$= 5\times4\times3\times2\times1 = 120\,ways.$$

Example 5:

In how many ways can 5 boys and 5 girls stand in a circle so that no two boys and no two girls stand together?

Solution:

By fixing the position of a boy, remaining four boys can stand in $4!=24\,ways$. Now the five girls can stand in $5!=120\,ways$. Thus, total number of required arrangements $=4!\times 5!=24\times 120=2880\,ways.$

(iv) The number of ways of arranging n objects in a circle which can be turned.

$$\frac{(n-1)!}{2}$$

Example 6:

Find the number of ways of arranging 5 objects in a ring that can be turned.

Solution:

$$n=5$$

$$\Rightarrow \qquad \frac{(n-1)!}{2}=\frac{(5-1)!}{2}=12\,ways.$$

Example 7:

$If\ {}^nP_4=12.{}^nP_2$, find n.

Solution:

$${}^nP_4=12.{}^nP_2$$

$$\Rightarrow \qquad \frac{n!}{(n-4)!}=12.\frac{n!}{(n-2)!}$$

$$\Rightarrow \qquad (n-2)!=12(n-4)!$$

$$\Rightarrow \qquad (n-2)(n-3)(n-4)!=12(n-4)!$$

$$\Rightarrow \qquad n^2-5n-6=0$$

$$\therefore \qquad n=6 \quad or \quad n=-1$$

Since n cannot be negative, hence, $n=6$.

Combinations

The number of combinations of n different objects taken r at a time is denoted by ${}^{n}C_{r}$.

Thus, $${}^{n}C_{r}=\frac{n!}{(n-r)!r!} \quad \text{or} \quad {}^{n}C_{r}=\frac{{}^{n}P_{r}}{r!}=\frac{Permutation}{r!}$$

Example 1:

Find the number of ways of forming a group of 4 people from 10 people.

Solution:

$$n=10,\quad r=4$$

$${}^{10}C_{4}=\frac{10!}{(10-4)!4!}=\frac{10!}{6!4!}=210\,ways.$$

Example 2:

Find the number of ways of forming a group of 5 people from 5 people.

Solution:

$$n=5,\quad r=5$$

$${}^{5}C_{5}=\frac{5!}{(5-5)!5!}=\frac{5!}{0!5!}=1\,way.$$

Example 3:

Find the number of ways of forming a club of 5 people from 5 women and 4 men, if

(a) Two women must be selected
(b) Five women must to be selected
(c) Two men and 3 women must to be selected
(d) Four men must to be selected
(e) The probability of selecting 3 men and 2 women
(f) The probability of selecting 5 women.

Solution:

$Women=5,\ Men=4,\ Total=4+5=9,\ No.of\ people\,in\,the$ club=5 people

(a) $${}^{5}C_{2}\times{}^{4}C_{3}=\left(\frac{5!}{(5-2)!2!}\right)\times\left(\frac{4!}{(4-3)!3!}\right)=\frac{5!}{3!2!}\times\frac{4!}{1!3!}=40\,ways.$$

(b) $${}^{5}C_{5}\times{}^{4}C_{0}=\frac{5!}{(5-5)!5!}\times\frac{4!}{(4-0)!0!}=\frac{5!}{0!5!}\times\frac{4!}{4!0!}=1\times1=1\,way$$

(c) $^4C_2 \times {}^5C_3 = \frac{4!}{(4-2)!2!} \times \frac{5!}{(5-3)!3!} = \frac{4!}{2!2!} \times \frac{5!}{2!3!} = 60\ ways.$

(d) $^4C_4 \times {}^5C_1 = \frac{4!}{(4-4)!4!} \times \frac{5!}{(5-1)!1!} = \frac{4!}{0!4!} \times \frac{5!}{4!1!} = 5\ ways.$

(e) The probability of selecting 3 men and 2 women

$$= \frac{^4C_3 \times {}^5C_2}{^9C_5} = \frac{\left(\frac{4!}{(4-3)!3!} \times \frac{5!}{(5-2)!2!}\right)}{\left(\frac{9!}{(9-5)!5!}\right)} = \frac{\left(\frac{4!}{1!3!} \times \frac{5!}{3!2!}\right)}{\left(\frac{9!}{4!5!}\right)} = \frac{10}{21}$$

(f) The probability of selecting 5 women $= \frac{^5C_5 \times {}^4C_0}{^9C_5} = \frac{\left(\frac{5!}{(5-5)!5!} \times \frac{4!}{(4-0)!0!}\right)}{\left(\frac{9!}{(9-5)!5!}\right)} = \frac{1}{126}$

Example 4:

If $^nC_2 + 2 = {}^{2n}P_1$. Find the possible value of n.

Solution:

$$^nC_2 + 2 = {}^{2n}P_1$$

$$\Rightarrow \qquad \frac{n!}{(n-2)!2!} + 2 = \frac{(2n)!}{(2n-1)!}$$

$$\Rightarrow \qquad \frac{n(n-1)(n-2)!}{(n-2)!2\times1} + 2 = \frac{(2n)(2n-1)!}{(2n-1)!}$$

$$\Rightarrow \qquad \frac{n(n-1)}{2} + 2 = 2n$$

$$\Rightarrow \qquad n^2 - 5n + 4 = 0$$

$$\therefore \qquad n = 1 \quad or \quad n = 4$$

Random Variable (R.V.)

A real valued function defined on the sample space of a random experiment is called a *random variable*. Thus, the values of random variables are real numbers connected with the outcomes of an experiment.

A random variable is generally denoted by capital letters, i.e., X, Y, Z *etc.* and the values of these random variables are denoted by corresponding small letters i.e., x, y, z *etc.* For example, in an experiment of tossing a coin twice, if we define X to be the numbers of head, then the values of the random variable X are $0,1,1,2$ corresponding to outcomes HH, HT, TH, TT respectively.

If in a random experiment, the event corresponding to a number "a" occurs, then the corresponding random variable X is said to assume the value "a" and the probability of the event is denoted by $P(X = a)$. Similarly, the probability of the event X assuming any value in the interval $a < X < b$ is denoted by $P(a < X < b)$. The probability of the event $X \le c$ is written as $P(X \le c)$.

Types of Random Variable

(1) Discrete Random Variable

A variable which can assume only a countable number of real values is called a *discrete random variable*. Examples of discrete random variables are: number of academic staff in KUST, Wudil, number of accidents per month, number of telephone calls per unit time and so on.

Probability Mass Function (PMF)

If we tabulate all probabilities corresponding to all possible values of the random variable X, then the table of values of probabilities is called *probability distribution, distribution function, discrete density function or probability mass function* of X.

The distribution can be represented in a tabular form as

X	x_1	x_2	……	x_i	…….	x_k
$P(X = x)$	p_1	p_2	……	p_i	…….	p_k

Where $P(X = x_i) = \sum_{i=1}^{k} p_i = 1$

Note:

(i) $P(X \le x_i) = p(X = x_1) + p(X = x_2) + \ldots + p(X = x_i)$

(ii) $P(X \ge x_i) = p(X = x_i) + p(X = x_{i+1}) + \ldots + p(X = x_k)$

Definition: if X is a discrete random variable with distinct values $x_1, x_2, \ldots, x_n, \ldots,$ then the function $f(x)$ is defined as :

$$f(x)=\begin{cases} P(X=x_i) & ;\ if\ x=x_i \\ 0 & ;\ if\ x\neq x_i \end{cases} \quad for\ i=1,2,\ldots$$

Is called the *probability mass function or discrete density function* of random variable X .

Properties of PMF

(i) $f(x_i)\geq 0 \quad \forall_i$

(ii) $\sum_{i=1}^{\infty} f(x_i)=1$

Example 1:

Find the probability distribution of the random variable (number of tail) when two coins are tossed.

Solution:

Let S be the sample space and X be the discrete random variable “number of tails). Then $S=\{HH,HT,TH,TT\}$ and the possible values of X are $0,1,2$.

Thus, $p(X=0)=p(HH)=\frac{1}{4}$

$$p(X=1)=p(HT,TH)=\frac{2}{4}=\frac{1}{2}$$

$$p(X=2)=p(TT)=\frac{1}{4}$$

The required probability distribution is as follows:

X	0	1	2
$P(X=x)$	$1/4$	$1/2$	$1/4$

Example 2:

A random variable X has the following probability distribution:

X	-2	-1	0	1	2	3
$P(X=x)$	0.1	k	0.2	$3k$	$2k$	0.3

(i) Find k

(ii) Obtain (a) $p(X<2)$ (b) $p(X\geq 2)$ (c) $p(-2<X<2)$

Solution:

(i) $\sum P(X=x)=1 \quad (by\,definition)$

Thus, $0.1+k+0.2+3k+2k+0.3=1$

$\Rightarrow \quad 6k=0.4$

$\therefore \quad k=0.07$

$$(i)\ (a)\quad p(X<2)=p(X=-2)+p(X=-1)+p(X=0)+p(X=1)$$
$$=0.1+k+0.2+3k$$
$$=0.1+0.07+0.2+3(0.07)$$
$$=0.58$$

$$(b)\quad p(X\geq 2)=p(X=2)+p(X=3)$$
$$=2k+0.3=2(0.07)+0.3=0.44$$

$$(c)\quad p(-2<X<2)=p(X=-1)+p(X=0)+p(X=1)$$
$$=k+0.2+3k$$
$$=4(0.07)+0.2=0.48.$$

Example 3:

The random variable has the following distribution:

X	0	1	2	3	4	5
$F(X)$	0.1	0.23	0.28	0.11	0.19	0.09

Calculate

$(i)\quad P(X\leq 2)\quad (ii)\quad P(X\geq 3)\quad (iii)\quad The\ probability\ that\ X\ is\ an\ odd\ number$

Solution:

(i) $\sum_{x=0}^{2} f(x)=0.1+0.23+0.28=0.61$

(ii) $\sum_{x=3}^{5} f(x)=0.11+0.19+0.09=0.39$

(iii) $\sum_{x=1,3,5} f(x)=0.23+0.11+0.09=0.43$

Distribution Function or Cumulative Distribution Function (CDF)

Let X be a discrete random variable, then the distribution function of X is defined as :

$F(x) = P(X \le x) = \sum f(x).$

The value $F(x)$ is also known as the cumulative distribution function of X. It gives the probability that the random variable X takes on a value less than or equal to a given value of X.

The distribution function for a discrete random variable must satisfy the following conditions:

1. $F(-\infty) = 0$
2. $F(\infty) = 1$
3. If $a < b$, then $F(a) < F(b)$ for any real numbers a and b.

Example 1:

Given the following probability distribution:

X	0	1	2	3
$F(X)$	1/8	3/8	3/8	1/8

Find the distribution function of X.

Solution:

We can obtain the distribution function of X as follows:

$F(0) = f(0) = 1/8$

$F(1) = f(0) + f(1) = 1/8 + 3/8 = 1/2$

$F(2) = f(0) + f(1) + f(2) = 1/8 + 3/8 + 3/8 = 7/8$

$F(3) = F(X) = f(0) + f(1) + f(2) + f(3) = 1/8 + 3/8 + 3/8 + 1/8 = 8/8 = 1$

Hence, the distribution function of X is given by

$$F(X) = \begin{cases} 0 & ; -\infty \le x < 0 \\ 1/8 & ; \quad 0 \le x < 1 \\ 1/2 & ; \quad 1 \le x < 2 \\ 7/8 & ; \quad 2 \le x < 3 \\ 1 & ; \quad 3 \le x < \infty \end{cases}$$

Example 2:

Let X be the number of boys in a family of 3. Find the distribution of X and the CDF of X.

Solution:

Let b denote boys and g denote girls.

Then,

$$S = \{bbb, bbg, bgb, gbb, bgg, gbg, ggb, ggg\}$$

Now,

$$p(X=0) = p(ggg) = 1/8$$
$$p(X=1) = p(bgg, gbg, ggb) = 3/8$$
$$p(X=2) = p(bbg, bgb, gbb) = 3/8$$
$$p(X=3) = p(bbb) = 1/8$$

The values of X are $0,1,2, and\, 3$ and hence, the distribution of X is given by

X	0	1	2	3
$f(x)$	1/8	3/8	3/8	1/8

The CDF is obtained as follows:

$$F(X) = \begin{cases} 0 & ; \quad x < 0 \\ 1/8 & ; \quad 0 \le x < 1 \\ 4/8 & ; \quad 1 \le x < 2 \\ 7/8 & ; \quad 2 \le x < 3 \\ 1 & ; \quad \ge 3 \end{cases}$$

(2) Continuous Random Variable

A random variable X is continuous if it has no jumps, i.e., $X \in (-\infty, \infty)$. Continuous random variables represent measured data such as height, weight, temperature, time, blood pressure and so on.

For a continuous random variable X and for any real number a and b with $a \le b$, we can obtain the value of the probability distribution of X at a given point x within an interval $[a,b]$ by subtracting the value of the distribution function of X at the point (a) from the distribution function of X at the point (b). Also we can obtain the density function by differentiating the distribution function with respect to x

Definition: Let X be a continuous random variable, then probability distribution function of X is denoted by $f(x)$ and defined as

$$f(x) = \frac{dF_x(X)}{dx}$$

Where, d is the derivative of probability distribution function of X.

Properties of PDF for a Continuous random Variable

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(iii) $P(a < X < b) = \int_a^b f(x)\,dx = F(b) - F(a)$

Example 1:

The random variable X has probability density function $f(x) = \begin{cases} ax & ; \; 0 < x < 1 \\ 0 & elsewhere \end{cases}$

Obtain the value of a and hence calculate the following probabilities:

$(i) \quad p(X \leq 0.8) \quad (ii) \quad p(X \geq 0.2)$

Solution:

To obtain a constant a, we use

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \qquad \int_0^1 axdx = 1 \qquad = a\int_0^1 xdx = 1$$

$$\Rightarrow \qquad a\left[\frac{x^2}{2}\right]_0^1 = 1$$

$$\therefore \qquad a = 2$$

Thus, $$f(x) = \begin{cases} 2x & ; \; 0 < x < 1 \\ 0 & elsewhere \end{cases}$$

(i) $p(X \leq 0.8) = \int_0^{0.8} 2xdx = 2\left[\frac{x^2}{2}\right]_0^{0.8} = 0.64$

(ii) $p(X \geq 0.2) = \int_{0.2}^{1} 2xdx = 2\left[\frac{x^2}{2}\right]_{0.2}^{1} = 0.96$

Example 2:

(a) Find the constant c such that the function

$$f(x)=\begin{cases} cx^2 & ; \quad 0<x<3 \\ 0 & \quad otherwise \end{cases}$$

is a probability density function.

(b) Compute $P(1<X<2)$.

Solution:

(a) Since $$\int_{-\infty}^{\infty} f(x)\,dx=1$$

$\Rightarrow$ $$\int_0^3 cx^2\,dx=1$$

$\Rightarrow$ $$c\left[\frac{x^3}{3}\right]_0^3=1$$

$\therefore$ $$c=1/9$$

(b) $$P(1<X<2)=\frac{1}{9}\int_1^2 x^2\,dx=\frac{1}{9}\left[\frac{x^3}{3}\right]_1^2=\frac{7}{28}$$

Distribution Function or Cumulative Distribution Function

Definition: Let X be a continuous random variable, then the distribution function of X is given by

$$F(X)=P(X\le x)=\int_{-\infty}^{x} f(t)\,dt \quad ; \qquad (-\infty<x<\infty)$$

Thus, $F(X)$ is a cumulative distribution function since it gives the distribution of the values of the continuous random variable X in cumulative form.

Properties of CDF

(i) $F(-\infty)=0$

(ii) $F(\infty)=1$

(iii) If $a\le b$, then $F(a)\le F(b)$ for any real number a and b i.e., $P(a<X<b)=\int_a^b f(x)\,dx$

(iv) $f(x)\ge 0$

(v) $\int_{-\infty}^{\infty} f(x)\,dx=1$

Example 1:

A continuous random variable X has the probability density function.

$$f(x)=\begin{cases}\frac{1}{3}x^2, & -1<x<2,\\ 0, & elsewhere.\end{cases}$$

Determine the distribution function of X .

Solution:

If $x\le -1$, then

$$F(X)=\int_{-\infty}^{x} f(t)dt=\int_{-\infty}^{x} 0dt=0$$

If $-1<x\le 2,$ then

$$F(X)=\int_{-\infty}^{x} f(t)dt=\int_{-\infty}^{-1} 0dt+\int_{-1}^{x}\tfrac{1}{3}t^2dt=\frac{1}{9}\left[t^3\right]_{-1}^{x}=\frac{1}{9}\left(x^3+1\right)$$

If $x\ge 2$, then

$$\begin{aligned}F(X)=\int_{-\infty}^{x} f(t)dt&=\int_{-\infty}^{-1} f(t)dt+\int_{-1}^{2} f(t)dt+\int_{2}^{x} f(t)dt\\ &=\int_{-\infty}^{-1} 0dt+\int_{-1}^{2}\tfrac{1}{3}t^2dt+\int_{2}^{x} 0dt\\ &=0+\left[\tfrac{1}{9}t^3\right]_{-1}^{2}+0=\frac{1}{9}(8+1)=1.\end{aligned}$$

Therefore,

$$F(X)=\begin{cases}0, & x\le 0,\\ \dfrac{1}{9}(x^3+1), & -1\le x\le 2,\\ 1, & x\ge 2.\end{cases}$$

Mathematical Expectation and Variance of Random Variables

Mathematical Expectation

If X is a discrete random variable and $f\left(x\right)$ is the value of its probability distribution function at x, then the expectation or expected value of X , denoted by $E\left[X\right]$ or μ is defined by

$$E[X]=\sum xf(x)$$

Similarly, if X is a continuous random variable and $f(x)$ of its probability density function at x, then the expectation or expected value of X, denoted by $E[X]$ or μ is defined by

$$E[X]=\int_{-\infty}^{\infty} xf(x)dx \qquad ; \qquad -\infty < x < \infty$$

Properties of Expectation

(1) $E[c]=c$, where c is constant.

(2) $E[aX]=aE[X]$

(3) $E[aX+b]=aE[X]+b$

(4) $E[X \pm Y]=E[X] \pm E[Y]$

(5) $E[XY]=E[X]E[Y]$, if X and Y are independent.

Variance

The variance of a random variable X with a probability distribution function $f(x)$, is denoted by $Var(X)$ or σ^2 and is defined as

$Var(X)=E[X-\mu]^2$ or $\sigma^2=\sum(x-\mu)^2 f(x)$ or $E[X^2]-\mu^2$ $(for\ discrete\ case)$ $Var(X)=E[X-\mu]^2$ or $\sigma^2=\int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx$ $(for\ continuous\ case)$

The quantity $\sigma=\sqrt{Var(X)}$ is called the standard deviation of X.

Properties of Variance

(1) $Var(c)=0$, where c is constant.

(2) $Var(aX+b)=a^2Var(X)$

(3) $Var(aX)=a^2Var(X)$

Example 1:

Find the expectation and variance of the number of heads in three tosses of a coin.

Solution:

Let X denotes the number of heads. Then X takes the values $0,1,2,3$. Their respective probabilities are calculated below:

$$p(X=0)=p(TTT)=1/8$$

$$p(X=1)=p(HTT,THT,TTH)=3/8$$

$$p(X=2)=p(HHT,THH,HTH)=3/8$$

$$p(X=3)=p(HHH)=1/8$$

For finding expectation and variance, we form the following table:

$X=x_i$	$p(X=x)=f(x)$	$xf(x)$	$x^2 f(x)$
0 1 2 3	1/8 3/8 3/8 1/8	0 3/8 6/8 3/8	0 3/8 12/8 9/8
Total	1	$\sum xf(x)=3/2$	$\sum x^2 f(x)=3$

$$\therefore E(X)=\sum xf(x)=3/2$$

$$\sigma^2=\sum(x-\mu)^2 f(x)=E\left[X^2\right]-\mu^2=\sum x^2 f(x)-\mu^2$$

$$=3-(3/2)^2=3-\frac{9}{4}=\frac{3}{4}$$

Example 2:

(a) Find the expectation of the number on a die when thrown.
(b) Two unbiased dice are thrown. Find the expected values of the sum of numbers of points on them.
(a) Let X be random variable representing the number on a die when thrown. Then X can take any one of the values 1, 2, 3, 4, 5 and 6 each with equal probability $\frac{1}{6}$.

Hence,

$$E[X]=\sum xf(x)=1\times\frac{1}{6}+2\times\frac{1}{6}+3\times\frac{1}{6}+4\times\frac{1}{6}+5\times\frac{1}{6}+6\times\frac{1}{6}$$

$$=\frac{1}{6}(1+2+3+4+5+6)=\frac{7}{2}$$

(b) The probability function of X (the sum of numbers obtained on two dice)

$X:x$	2	3	4	5	6	7	8	9	10	11	12
$f(x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

$$E\left[X\right]=\sum xf(x)$$

$$=2\times\frac{1}{36}+3\times\frac{2}{36}+4\times\frac{3}{36}+5\times\frac{4}{36}+6\times\frac{5}{36}+7\times\frac{6}{6}+8\times\frac{5}{36}+9\times\frac{4}{36}+10\times\frac{3}{36}+11\times\frac{2}{36}+12\times\frac{1}{36}$$

$$=\frac{1}{36}\left(2+6+12+20+30+42+40+36+30+22+21\right)=7$$

Example 3:

Let X be a random variable with the following probability distribution:

X	-3	6	9
$p\left(X=x\right)$	$1/6$	$1/2$	$1/3$

Find

(a) $E\left(X\right)$ and $E\left(X^2\right)$

(b) Variance of X .

Solution:

(a) $E\left[X\right]=\sum xf(x)=(-3)\times\frac{1}{6}+6\times\frac{1}{2}+9\times\frac{1}{3}=\frac{11}{2}$

$$E\left[X^2\right]=\sum x^2 f(x)=9\times\frac{1}{6}+36\times\frac{1}{2}+81\times\frac{1}{3}=\frac{93}{2}$$

(b)

$$Var\left(X\right)=\sum\left(X-\mu\right)^2=E\left[X^2\right]-\mu^2$$

$$=\frac{93}{2}-\left(\frac{11}{2}\right)^2=\frac{65}{4}$$

Example 4:

The diameter of an electric cable is assumed to be continuous random variable with probability density function: $6x\left(1-x\right),\qquad 0\le x\le 1$. Find the mean variance.

Solution:

$$Mean = \int_{-\infty}^{\infty} x.f(x)dx = \int_{0}^{1} x.6x(1-x)dx$$

$$= \int_{0}^{1}(6x^2 - 6x^3)dx = \left[2x^3 - \tfrac{3}{2}x^4\right]_0^1 = \frac{1}{2}$$

$$Variance = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx$$

$$= \int_{0}^{1}(x-\tfrac{1}{2})^2.6x(1-x)dx$$

$$= \int_{0}^{1}(x^2 - x + \tfrac{1}{4})^2.(6x - 6x^2)dx$$

$$= \int_{0}^{1}(12x^3 - 6x^4 - \tfrac{15}{2}x^2 + \tfrac{3}{2}x)dx$$

$$= \left(2x^4 - \tfrac{6}{5}x^5 - \tfrac{5}{2}x^3 + \tfrac{3}{4}x^2\right)_0^1 = \frac{1}{20}$$

Example 5:

If $f(x) = \begin{cases} \tfrac{1}{2}(x+1), & -1 < x < 1 \\ 0, & elsewhere \end{cases}$

represents the density of a random variable X, find $E(X)$ and $Var(X)$.

Solution:

$$E(X) = \int_{-\infty}^{\infty} x.f(x)dx$$

$$= \int_{-1}^{1} x.\tfrac{1}{2}(x+1)dx = \frac{1}{2}\int_{-1}^{1}(x^2 + x)dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{1} = \frac{1}{3}$$

We find variance as follows:

$$E(X^2) = \int_{-\infty}^{\infty} x^2.f(x)dx$$

$$= \int_{-1}^{1} x^2.\tfrac{1}{2}(x+1)dx = \frac{1}{2}\int_{-1}^{1} x^3 + x^2 dx$$

$$= \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3}$$

Thus,

$$Var(X) = E\left(X^2\right) - \mu^2$$

$$= \frac{1}{3} - \left(\frac{1}{3}\right)^2 = \frac{1}{3} - \frac{1}{9} = \frac{2}{9}$$

Some Special Probability Distributions

After discussing probability distribution of a random variable, we shall now discuss about some special probability distributions as follows:

(1) Discrete Probability Distributions

(i) Bernoulli Distribution

A Bernoulli trial is a random experiment in which there are precisely two possible outcomes, statistically known as 'success' (p) and 'failure' (q) in any single trial.

A random variable X is defined as (1), if a Bernoulli results in success and (0), if same trial results in failure. Thus, X have Bernoulli distribution with parameter (p) (i.e., the probability of success) and (q) (i.e., the probability of failure).

Definition: A random variable X is define to have a Bernoulli distribution if the probability mass function of X is of the form

$$f(x) = P(X = x) = \begin{cases} p^x (1-p)^{1-x}, & x = 0,1 \\ 0, & otherwise \end{cases}$$

where p is the probability of success and $q = (1-p)$ which is the probability of failure.

Theorem 1: if X is a Bernoulli random variable with parameter p, then the mean and variance are respectively given by

$$\mu = p$$
$$\sigma^2 = p(1-p)$$

Proof: The mean of the Bernoulli random variable is

$$\begin{aligned}\mu &= \sum_{x=0}^{1} xf(x) \\ &= \sum_{x=0}^{1} xp^x(1-p)^{1-x} \\ &= p.\end{aligned}$$

Similarly, the variance of X is given by

$$\sigma^2 = E(X-\mu)^2 = E(X^2) - \mu^2$$

Thus, $$E(X^2) = \sum_{x=0}^{1} x^2 p^x (1-p)^{1-x} = p$$

$$\Rightarrow \qquad \sigma^2 = p - p^2$$
$$= p(1-p)$$

Example 1:

Find the probability of getting a head in a single toss of a coin. Hence find the mean and variance.

Solution:

The two possible outcomes are $S=\{H,T\}$

Now, $p=\frac{1}{2}$ and $q=1-p=1-\frac{1}{2}=\frac{1}{2}$.

Thus, this is a Bernoulli trial with

$P(X=0)=P(failure)=\frac{1}{2}$ and $P(X=1)=P(success)=\frac{1}{2}$

Hence, the probability of getting a head in a single toss of a coin is $\frac{1}{2}$.

$$\begin{aligned}\mu &= \sum_{x=0}^{1} xf(x)=\sum_{x=0}^{1} xp^x(1-p)^{1-x}\\ &= \sum_{x=0}^{1} x(\tfrac{1}{2})^x(1-\tfrac{1}{2})^{1-x}=\frac{1}{2}\end{aligned}$$

$$\sigma^2=E(X-\mu)^2=E(X^2)-\mu^2$$

$$E(X^2)=\sum_{x=0}^{1} x^2p^x(1-p)^{1-x}=\sum_{x=0}^{1} x^2(\tfrac{1}{2})^x(1-\tfrac{1}{2})^{1-x}=\frac{1}{2}$$

$$\Rightarrow \qquad \sigma^2=p-p^2=\frac{1}{2}-\left(\frac{1}{2}\right)^2=\frac{1}{4}$$

(ii) Binomial Distribution

An experiment that consists of finite repeated independent Bernoulli trials is called a Binomial experiment. In Binomial distribution the two possible outcomes are called 'success' and 'failure'. The probability of success is denoted by $'p'$ and that of failure denoted by $'q'$, such that $p+q=1$.

Definition: A random variable X is defined to have a Binomial distribution, if the probability mass function of X is given by

$$f(x)=\binom{n}{x}p^xq^{n-x}, \qquad x=0,1,2,\ldots,n$$

where $\binom{n}{x}$ is the Binomial coefficient.

For a Binomial distribution, following points are to be noted:

(i) Each trial has only two possible outcomes (i.e., success and failure)
(ii) The repeated trials are independent
(iii) The probability of success in each trial remains constant.

Theorem 2: if X is Binomial random variable with parameter p and n, then the mean and variance are respectively given by

$$\mu = np$$

$$\sigma^2 = np(1-p).$$

Proof:

$$\mu = \sum_{x=0}^{n} xf(x)$$

$$= \sum_{x=0}^{n} x\binom{n}{x} p^x q^{n-x}, \qquad x = 0,1,2,\ldots,n$$

$$= \sum_{x=0}^{n} x \tfrac{n!}{(n-x)!x!} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} x \tfrac{n!}{(n-x)!x(x-1)!} p^x q^{n-x}$$

$$= np\sum_{x=1}^{n} \tfrac{(n-1)!}{(n-x)!(x-1)!} p^{x-1} q^{n-x}$$

Letting $x-1 = y \ \ and \ \ n-1 = m$,

$$\Rightarrow \qquad \mu = np\sum_{y=0}^{m} \tfrac{m!}{y!(m-y)!} p^y q^{m-y}$$

But $\sum_{y=0}^{m} \tfrac{m!}{y!(m-y)!} p^y q^{m-y}$ is the complete summation of Binomial distribution and therefore, is equal to unity.

Therefore, $\mu = np$

Now consider evaluation of the variance of X .

$$\sigma^2 = E\left(X-\mu\right)^2 = E\left(X^2\right) - \mu^2$$

$$E\left(X^2\right) = \sum_{x=0}^{n} x^2 \binom{n}{x} p^x q^{n-x}$$
$$= \sum_{x=0}^{n} \left(x(x-1)+x\right) \tfrac{n!}{(n-x)!x!} p^x q^{n-x}$$
$$= \sum_{x=0}^{n} x(x-1) \tfrac{n!}{(n-x)!x!} p^x q^{n-x} + \sum_{x=0}^{n} x \tfrac{n!}{(n-x)!x!} p^x q^{n-x}$$

Since the second summation is the expected value of Binomial distribution. Thus, we have

$$E\left(X^2\right) = \sum_{x=0}^{n} x(x-1) \tfrac{n(n-1)(n-2)!}{(n-x)!x(x-1)(x-2)!} p^x q^{n-x} + np$$

$$= \sum_{x=2}^{n} \tfrac{n(n-1)(n-2)!}{(n-x)!(x-2)!} p^x q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \tfrac{(n-2)!}{(n-x)!(x-2)!} p^{x-2} q^{n-x} + np$$

Letting $x-2=y \quad and \quad n-2=m$, we have

$$E\left(X^2\right) = n(n-1)p^2 \sum_{y=0}^{m} \tfrac{m!}{(m-y)!y!} p^y q^{m-y} + np$$
$$= n(n-1)p^2(1) + np$$

Thus,

$$\sigma^2 = n(n-1)p^2 + np - (np)^2$$
$$= n^2p^2 - np^2 + np - n^2p^2$$
$$= np(1-p).$$

Example 1:

If a machine produces 10% defective items, what is the probability of getting 2 defective items out of 5 items produced by the machine?

Solution:

Here $\quad n=5, \quad p=10\%=0.1, \quad q=1-0.1=0.9, \quad x=2$

Applying Binomial distribution, we have

$$f(x)=\binom{n}{x}p^x q^{n-x}, \qquad x=1,2,3,4,5$$

$$f(x)=\binom{5}{2}(0.1)^2(0.9)^3=0.0729$$

Example 2:

A die is thrown 20 times. Getting a number greater than 4 is considered a success. Find the mean and variance of the number of success.

Solution:

We shall fit the Binomial distribution for $n=20$. Here $p = probability\,of\ success = p(getting\,5\,or\,6)=\frac{1}{3}$ and $q=\frac{2}{3}$.

Thus,

$$Mean = np = 20\times\frac{1}{3}=\frac{20}{3}$$

$$Variance = npq = 20\times\frac{1}{3}\times\frac{2}{3}=\frac{40}{9}$$

(iii) Poisson Distribution

A random variable X is defined to have a Poisson distribution if its probability mass function is given by

$$P(X=x)=f(x)=\frac{\ell^{-\lambda}\lambda^x}{x!}, \qquad x=0,1,2,\ldots,\infty$$

where λ is a positive Poisson parameter i.e., $\lambda>0$.

Poisson distribution is the limiting form of the Binomial distribution when the number of trials n becomes very large and the probability of success, p in a trial very small $(i.e.,\ np=\lambda)$.

Theorem 3: if X is Poisson random variable with parameter λ, then the mean and variance are respectively given by

$$E(X)=\lambda$$

$$Var(X)=\lambda$$

Proof:

If λ is the average n umber of occurrence of an event at a given time interval and the random variable x assumes the infinite set of values $0,1,2,3,...,\infty$, then the mean of Poisson distribution is given by

$$\begin{aligned} E\left(X\right) &= \sum_{x=0}^{\infty} xf\left(x\right) \\ &= \sum_{x=0}^{\infty} x\frac{\ell^{-\lambda}\lambda^{x}}{x!} = \sum_{x=0}^{\infty} x\frac{\lambda\ell^{-\lambda}\lambda^{x-1}}{x(x-1)!} \\ &= \lambda\sum_{x=1}^{\infty}\frac{\ell^{-\lambda}\lambda^{x-1}}{(x-1)!} \end{aligned}$$

Letting $x-1=y$, we have

$$\begin{aligned} E\left(X\right) &= \lambda\sum_{y=0}^{\infty}\frac{\ell^{-\lambda}\lambda^{y}}{y!} \\ &= \lambda(1) = \lambda \end{aligned}$$

Now consider evaluation of the variance of X .

$$\sigma^2 = E\left(X-\mu\right)^2 = E\left(X^2\right) - \mu^2$$

$$\begin{aligned} E\left(X^2\right) &= \sum_{x=0}^{\infty} x^2\frac{\ell^{-\lambda}\lambda^{x}}{x!} \\ &= \sum_{x=0}^{\infty}\left(x(x-1)+x\right)\frac{\ell^{-\lambda}\lambda^{x}}{x!} \\ &= \sum_{x=0}^{\infty} x(x-1)\frac{\ell^{-\lambda}\lambda^{x}}{x!} + \sum_{x=0}^{\infty} x\frac{\ell^{-\lambda}\lambda^{x}}{x!} \\ &= \lambda^2\sum_{x=2}^{\infty}\frac{\ell^{-\lambda}\lambda^{x-2}}{(x-2)!} + \lambda \end{aligned}$$

Letting $x-2=y$, we have

$$\begin{aligned} E\left(X^2\right) &= \lambda^2\sum_{y=0}^{\infty}\frac{\ell^{-\lambda}\lambda^{y}}{y!} + \lambda \\ &= \lambda^2 + \lambda \end{aligned}$$

Thus,

$$\begin{aligned} Var\left(X\right) &= \lambda^2 + \lambda - (\lambda)^2 \\ &= \lambda \end{aligned}$$

Some Example of Poisson Distribution

1. The number of cars passing through KUST gate in a time t.
2. The number of deaths in a city in one year by a rare disease.
3. The number of accidents in Kano in one year.
4. The number of telephone calls received in some unit of time.
5. The number of customers arriving in a certain bank in a given time.

Example 1:

The number of accidents in a junction in a week time follows a Poisson distribution with mean 3. Find the probability that:

(i) There are no accident in a week
(ii) There are at most 3 accidents in a week
(iii) There are more than 3 accidents in a week

Solution:

Let X denote the number of accidents occurring in a junction during the week and $\lambda = 3$.

Applying $P(X = x) = f(x) = \dfrac{\ell^{-\lambda}\lambda^{x}}{x!}$; $\quad x = 0, 1, 2, ..., \infty$

We have

(i) $P(X = 0) = \dfrac{\ell^{-3}(3)^{0}}{0!} = \ell^{-3} = 0.050$

(ii)

$$
\begin{aligned}
P(X \le 3) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) \\
&= \frac{\ell^{-3}(3)^{0}}{0!} + \frac{\ell^{-3}(3)^{1}}{1!} + \frac{\ell^{-3}(3)^{2}}{2!} + \frac{\ell^{-3}(3)^{3}}{3!} \\
&= \ell^{-3}\left(1 + 3 + \frac{9}{2} + \frac{27}{6}\right) = 0.647
\end{aligned}
$$

(iii)

$$
\begin{aligned}
P(X > 3) &= 1 - P(X \le 3) \\
&= 1 - \left[P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)\right] \\
&= 1 - 0.647 = 0.353
\end{aligned}
$$

Example 2:

If 2% of electric bulbs manufactured by a certain company are defective, find the probability that in a sample of 200 bulbs

(i) Less than 2 bulbs are defective

(ii) More than 3 bulbs are defective. $\left[Given\ \ell^{-4} = 0.0183\right]$

Solution:

Here $p = 0.02, \quad n = 200$, then $\lambda = np = 200\times0.02 = 4$

(i)

$$\begin{aligned} P(X<2) &= P(X=0)+P(X=1) \\ &= \frac{\ell^{-4}(4)^0}{0!}+\frac{\ell^{-4}(4)^1}{1!} \\ &= \ell^{-4}(5) = 0.0183\times5 = 0.0915 \end{aligned}$$

(ii)

$$\begin{aligned} P(X>3) &= 1-P(X\le3) \\ &= 1-\left[P(X=0)+P(X=1)+P(X=2)+P(X=3)\right] \\ &= 1-\left[0.0915+\frac{\ell^{-4}(4)^2}{2!}+\frac{\ell^{-4}(4)^3}{3!}\right] = 0.5668 \end{aligned}$$

(iii) Hypergeometric Distribution

The Hypergeometric distribution describes the number of successes in a sequence of n draws without replacement from a population of N that contained M total successes. Suppose that a sample of n objects is drawn without replacement from the population of N objects, and the defective objects are counted. Then, if X denotes the number of defective objects in the sample, then the probability of exactly x defectives objects in a sample of size n is said to follow Hypergeometric distribution.

Definition: A random variable X is defined to have a Hypergeometric distribution if its probability mass function is given by

$$f(x) = P(X=x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} \quad for \quad x = 1, 2, \ldots, n$$

where M, N, n are parameters of hypergeometric distribution.

Theorem 4: if X is hypergeometric random variable with parameters M, N, n, then the mean and variance are respectively given by

$$E(X) = \frac{nM}{N}$$

$$Var(X) = \frac{nM}{N^2}\left[\frac{(N-n)(N-M)}{(N-1)}\right].$$

Proof

We derive the mean as follows:

$$\begin{aligned}
E[X] &= \sum_{x=0}^{n} xf(x) = \sum_{x=0}^{n} x.\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} \\
&= 0.\frac{\binom{M}{0}\binom{N-M}{n-0}}{\binom{N}{n}} + \sum_{x=1}^{n} x.\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} \\
&= \sum_{x=1}^{n} x.\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} = \sum_{x=1}^{n} x.\frac{M!\times(N-n)!n!\binom{N-M}{n-x}}{(M-x)!x!N!} \\
&= \sum_{x=1}^{n} x.\frac{M(M-1)!\times(N-n)!n(n-1)!\binom{N-M}{n-x}}{(M-x)!x(x-1)!N(N-1)!} = \frac{nM}{N}\sum_{x=1}^{n}\frac{\binom{M-1}{x-1}\binom{N-M}{n-x}}{\binom{N-1}{n-1}}
\end{aligned}$$

Now, we define the variables inside the sum as $M' = M-1,\quad x' = x-1,\quad N' = N-1,\ n' = n-1.$

$$E[X] = \frac{nM}{N}\sum_{x'=0}^{n'}\frac{\binom{M'}{x'}\binom{N'-M'}{n'-x'}}{\binom{N'}{n'}}$$

Now, we can see that $\sum_{x'=0}^{n'}\frac{\binom{M'}{x'}\binom{N'-M'}{n'-x'}}{\binom{N'}{n'}} = 1$

Hence, $E[X] = \frac{nM}{N}$

To derive variance, we first determine $E[X^2]$.

$$\begin{aligned}
E[X^2] &= \sum_{x=0}^{n} x^2 f(x) = \sum_{x=0}^{n} x^2.\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} \\
&= 0^2.\frac{\binom{M}{0}\binom{N-M}{n-0}}{\binom{N}{n}} + \sum_{x=1}^{n} x^2.\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} \\
&= \sum_{x=1}^{n} x.\frac{M\binom{M-1}{x-1}\binom{N-M}{n-x}}{\frac{N}{n}\binom{N-1}{n-1}} = \frac{nM}{N}\sum_{x=1}^{n} x.\frac{\binom{M-1}{x-1}\binom{N-M}{n-x}}{\binom{N-1}{n-1}}
\end{aligned}$$

Now, we define the variables inside the sum as $M' = M-1,\quad x' = x-1,\quad N' = N-1,\ n' = n-1.$

$$E\left[X^2\right] = \frac{nM}{N}\sum_{x'=0}^{n'}(x'+1).\,\frac{\binom{M'}{x'}\binom{N'-M'}{n'-x'}}{\binom{N'}{n'}}$$
$$= \frac{nM}{N}\left[\sum_{x'=0}^{n'}x'.\,\frac{\binom{M'}{x'}\binom{N'-M'}{n'-x'}}{\binom{N'}{n'}} + \sum_{x'=0}^{n'}\frac{\binom{M'}{x'}\binom{N'-M'}{n'-x'}}{\binom{N'}{n'}}\right]$$

We can observe that, the first sum is the expected value of a hypergeometric distribution with parameters n', M', N'. The second sum is the total sum of that distribution which is equal to unity.

Thus,

$$E\left[X^2\right] = \frac{nM}{N}\left[\frac{n'M'}{N'}+1\right]$$
$$= \frac{nM}{N}\left[\frac{(n-1)(M-1)}{(N-1)}+1\right] = \frac{nM}{N}\left[\frac{(n-1)(M-1)+(N-1)}{(N-1)}\right]$$

Since $Var\left(X\right) = E\left[X^2\right] - \left(E\left[X\right]\right)^2$, we have

$$Var\left(X\right) = \frac{nM}{N}\left[\frac{(n-1)(M-1)+(N-1)}{(N-1)}\right] - \left(\frac{nM}{N}\right)^2$$
$$= \frac{nMN}{N^2}\left[\frac{(n-1)(M-1)+(N-1)}{(N-1)}\right] - \frac{(N-1)n^2M^2}{(N-1)N^2}$$
$$= \frac{nM}{N^2}\left[\frac{(N-n)(N-M)}{(N-1)}\right]$$

Example 1:

A bag contains 10 items out of which 4 are defectives. Samples of 4 items are selected, what is the probability that there are:

(i) Two defectives
(ii) Less than 2 defectives in the sample.

Solution:

$N = 10,\quad M = 4\quad and\quad n = 4$

(i) $$P(X=x)=\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

$\Rightarrow$ $$P(X=2)=\frac{\binom{4}{2}\binom{10-4}{4-2}}{\binom{10}{4}}$$

$$=\frac{{}^{4}C_{2}\times{}^{6}C_{2}}{{}^{10}C_{4}}=0.4285$$

(ii)

$$P(X<2)=P(X=0)+P(X=1)$$

$$=\frac{\binom{4}{0}\binom{10-4}{4-0}}{\binom{10}{4}}+\frac{\binom{4}{1}\binom{10-4}{4-1}}{\binom{10}{4}}$$

$$=\frac{{}^{4}C_{0}\times{}^{6}C_{4}}{{}^{10}C_{4}}+\frac{{}^{4}C_{1}\times{}^{6}C_{3}}{{}^{10}C_{4}}$$

$$=0.0714+0.3810=0.4524$$

Example 2:

Among 120 applicants for a job, only 80 are actually qualified; if 5 of these applicants are randomly selected for an interview, find the probability that only 2 of the five will be qualified for the job.

Solution:

Let x be the number of qualified applicants, then

$N=120,\ M=80,\ n=5\ \ and\ \ x=2$, we have

$$P(X=x)=f(x)=\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{n}{x}}$$

$$P(X=2)=\frac{{}^{80}C_2\times{}^{(120-80)}C_{(5-2)}}{{}^{180}C_5}=0.0210$$

(2) Normal Distribution

The normal distribution is the most widely used continuous distribution. This distribution serves as an excellent approximation to a large class of distributions which have great practical importance. One reason why the normal distribution is so important is that a number of natural phenomena are normally distributed. Phenomena such as heights and weights of individuals and sores on various tests all have normal distribution.

Definition: The random variable X has a normal distribution with mean μ and variance σ^2 if its probability density function is given by

$$f(x)=P(X=x)=\frac{1}{\sqrt{2\pi\sigma^2}}.\ell^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad -\infty<x<\infty$$

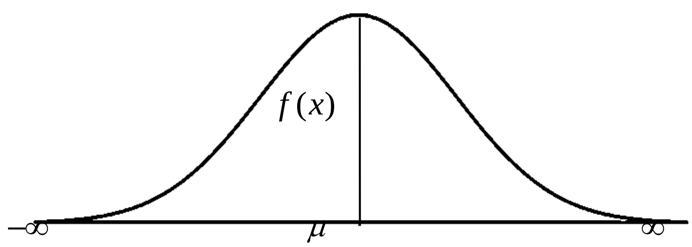


Figure 1: The graph of the normal distribution

Standard Normal Distribution

The random variable X is said to have a standardized normal distribution with mean $(\mu=0)$ and variance $(\sigma^2=1)$ if its probability density function is given by

$$f(x)=\frac{1}{\sqrt{2\pi}}\ell^{-\frac{x^2}{2}}, \qquad -\infty<x<\infty$$

Thus, if X is a normal variate with $(\mu=0)$ and $(\sigma^2=1)$, we can form a new variate Z by the formula

$$f(z)=\frac{1}{\sqrt{2\pi}}\ell^{-\frac{z^2}{2}}, \qquad -\infty<z<\infty$$

where $Z=\dfrac{X-\mu}{\sigma}$

For finding the area under the normal curve, it is necessary to convert every normal distribution to the standard normal distribution (Z- transformation) due to the fact that Z variate is tabulated.

Properties of Normal Distribution Curve

i. The graph is symmetrical about $X = \mu$
ii. The frequencies on the either side of the mean, median or mode are equal.
iii. There are two parameters of the normal distribution which are μ and σ. If these parameters are known the complete normal distribution is known.
iv. There is a perfect balance between the right half and the left half of the curve. This is why, it is also called 'bell shaped' curve.
v. The height of the normal curve is at its maximum at the mean.
vi. The total area under the normal curve is unity.
vii. On the basis of mean, μ and standard deviation, σ the area of normal curve is distributed as:
(a) $\mu \pm 1\sigma$ covers 68.27% area
(b) $\mu \pm 2\sigma$ covers 95.45% area
(c) $\mu \pm 3\sigma$ covers 99.73% area

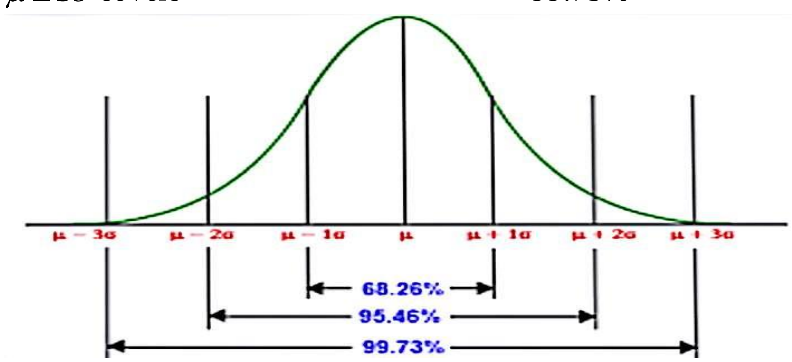


Figure 2: Showing areas under the normal curve.

(d) $\mu \pm 0.67\sigma$ covers 50% area
(e) $\mu \pm 1.96\sigma$ covers 95% area
(f) $\mu \pm 2.58\sigma$ covers 99% area

Area under Normal Curve

The area or probability under a normal probability distribution or curve bounded by two ordinate at $X = a$ and $X = b$ is writing as $P(a \le X \le b)$.

This probability is the probability that the normal variate X lies between two specified value a and b which can be represented by shaded area below:

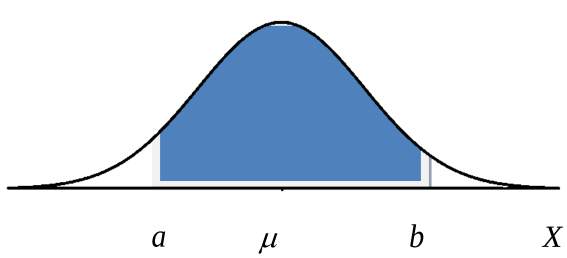


Figure 3: A normal curve showing variate between two ordinate

Thus, the area between the two ordinate $X = a$ and $X = b$ is transformed to

$$z_1 = \frac{a-\mu}{\sigma} \text{ and } z_2 = \frac{a-\mu}{\sigma}.$$

Symbolically, we can write

$$P\left[a \le X \le b\right] = P\left[\frac{a-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right] = P\left[z_1 \le Z \le z_2\right]$$

Then we use the standard normal curve table given at the end of this note to compute the needed probability or area.

Theorems 5: if X is a random variable with parameters μ and σ, then the mean and variance are respectively given by

$$E\left[X\right] = \mu$$

$$Var\left[X\right] = \sigma^2$$

Proof

There is need to know the following identity in proving means and variance of normal distribution:

1. $\int_{-\infty}^{\infty} \ell^{-\frac{1}{2}z^2} dz = \sqrt{2\pi}$
2. $\int_{-\infty}^{\infty} z.\ell^{-\frac{1}{2}z^2} dz = 0$
3. $\int_{-\infty}^{\infty} z^2.\ell^{-\frac{1}{2}z^2} dz = \sqrt{2\pi}$

$$E\left[X\right] = \int_{-\infty}^{\infty} xf(x)dx$$

$$E\left[X\right] = \int_{-\infty}^{\infty} x.\frac{1}{\sqrt{2\pi\sigma^2}}.\ell^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Let $z = \dfrac{x-\mu}{\sigma}$ $\Rightarrow x = \mu + \sigma z$, $dx = \sigma dz$

Thus,

$$E\left[X\right] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma(\mu + \sigma z).\ \ell^{-\frac{1}{2}z^2} dz$$

$$= \frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \ell^{-\frac{1}{2}z^2} dz + \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z \,.\, \ell^{-\frac{1}{2}z^2} dz$$

It follows that, the first integral is a even function, which is found to be $\sqrt{2\pi}$. Similarly, the second integral is an odd function, which is equal to 0.

Hence,

$$E[X] = \frac{\mu}{\sqrt{2\pi}} . \sqrt{2\pi} + \frac{\sigma}{\sqrt{2\pi}} . 0 = \mu$$

We derive variance as follows:

$$Var(X) = E\left[(X-\mu)^2\right]$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 \,.\, \ell^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$ By substituting $z = \frac{x-\mu}{\sigma}$, we have

$$Var(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma((\mu + \sigma z) - \mu)^2 \,.\, \ell^{-\frac{1}{2}(z)^2} dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 \,.\, \ell^{-\frac{1}{2}z^2} dz$$

Consider $\int_{-\infty}^{\infty} z^2 \,.\, \ell^{-\frac{1}{2}z^2} dz$ as an even function which is equal to $\sqrt{2\pi}$.

Thus,

$$Var(X) = \frac{\sigma^2}{\sqrt{2\pi}} . \sqrt{2\pi} = \sigma^2 .$$

The following important points should be kept in mind while computing area or probability under a standard normal curve:

1. The total area under standard normal curve is 1.
2. The mean of the distribution is zero. Thus, the negative and positive values of Z will be on the left and right of mean respectively.
3. The ordinate at mean, ie., at $Z = 0$ divides the area under the standard normal curve into two equal parts . Thus, the area on the right and left of the ordinate at $Z = 0$ is 0.5. Symbolically,

$$P[-\infty < Z < 0] = P[0 \le Z \le \infty] = 0.5$$

4. Since the curve is symmetrical, the area between 0 to a (say) is equal to the area between $-a$ to 0. In other words,

$$P\left[-a \le Z \le 0\right] = P\left[0 \le Z \le a\right]$$

Example 1:

For a standard normal random variable Z, find the probability that

(a) Z is between 0 and 1.50,
(b) Z is between -2.33 and 2.33,
(c) Z is less than 1.69,
(d) Z is more than or equal to -1.25.

Solution:

For determining the area under standard normal curve, we use the table given at the end of this lecture note.

(a) From the table, it can be seen that the tabulated value at $Z = 1.50$ is $0.4322 = 43.22\%$.

$\therefore \; P(0 \le Z \le 1.50) = 0.4322 = 43.22\%$, which is actually the area between $Z = 0$ and $Z = 1.50$

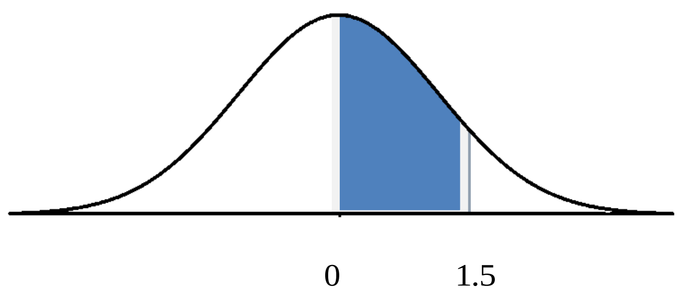


Figure 4: Areas under a standard normal curve between 0 and 1.5

$(b) \qquad P(-2.33 \le Z \le 2.33) = 0.9802 = 98.02\%$

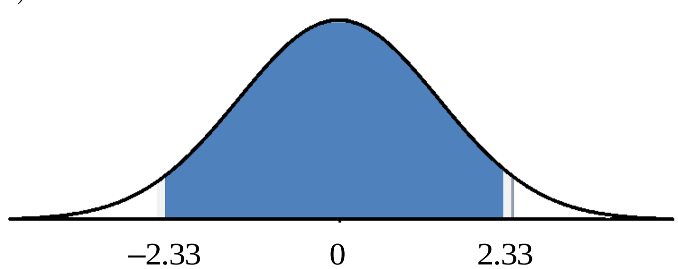


Figure 5: Areas under a standard normal curve between -2.33 and 2.33.

$(c) \quad P(Z \le 1.69) = 1 - P(Z \ge 1.69) = 1 - 0.0455 = 0.9545 = 95.45\%$

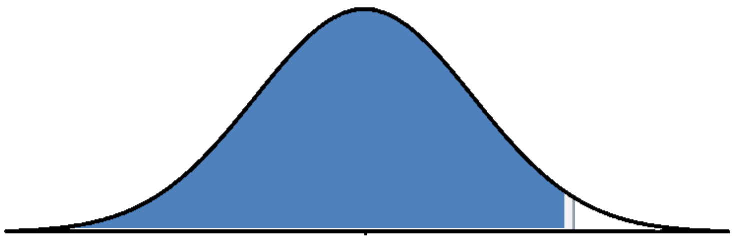


Figure 6: Areas under a standard normal curve between $-\infty$ to 1.96

$(d)\quad P(Z \geq -1.25) = 1 - P(Z \leq -1.25) = 1 - P(Z \geq 1.25) = 1 - 0.1056 = 0.8944 = 89.44\%$

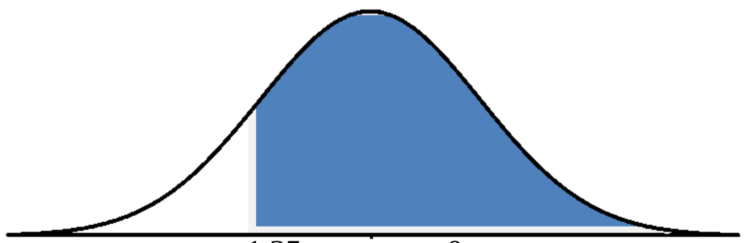


Figure 7: Areas under a standard normal curve between -1.25 and ∞

Example 2:

Suppose that in a certain pediatric population, systolic blood pressure is normally distributed with $\mu = 115\, mm\, of\ Hg$ and $\sigma^2 = 225$. Find the probability that a child randomly selected from the population will have:

(a) Systolic blood pressure less than $140\, mm$.
(b) A pressure greater than $100\, mm$.
(c) A pressure between $110\, mm$ and $120\, mm$.
(d) A pressure between $120\, mm$ and $140\, mm$.

Solution:

We are given that $\mu = 115 \quad and \quad \sigma^2 = 225$.

(a) First we find the value of Z by converting the data to standard normal distribution.

$$Z = \frac{X - \mu}{\sigma} = \frac{140 - 115}{15} = \frac{25}{15} = 1.67$$

Thus, $P(Z<1.67)$ is the area under the curve for $Z=0$ and $Z=1.67$ as shown below:

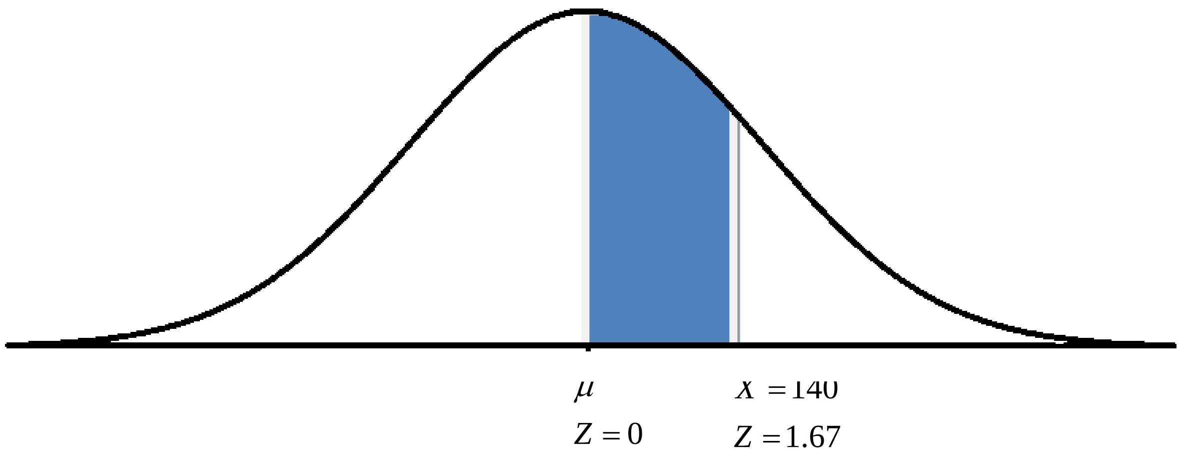


Figure 8: Area under curve for $Z=0$ and $Z=1.67$.

Therefore, the probability that the child will have systolic pressure of $140\,mm$ is $0.4525+0.5000=0.9525$

(b) Here $Z=\dfrac{100-115}{15}=-1$. Thus, $P(Z>-1)$ is the shaded area shown below*:*

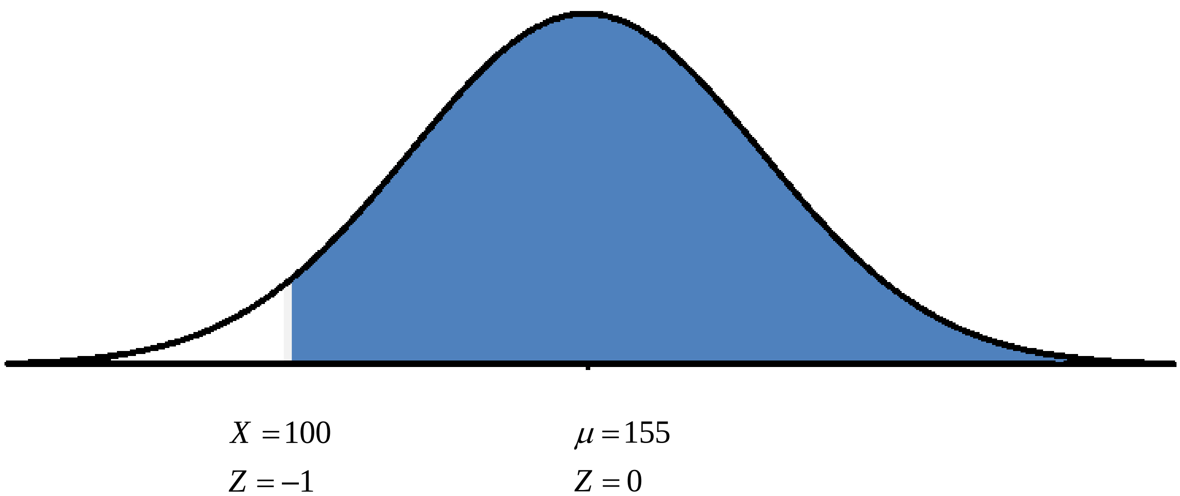


Figure 9: Area under the curve for Z= -1

$$\therefore \qquad P(Z>-1)=P(-1<Z<0)+P(0<Z<\infty)=0.3413+0.5=0.8413.$$

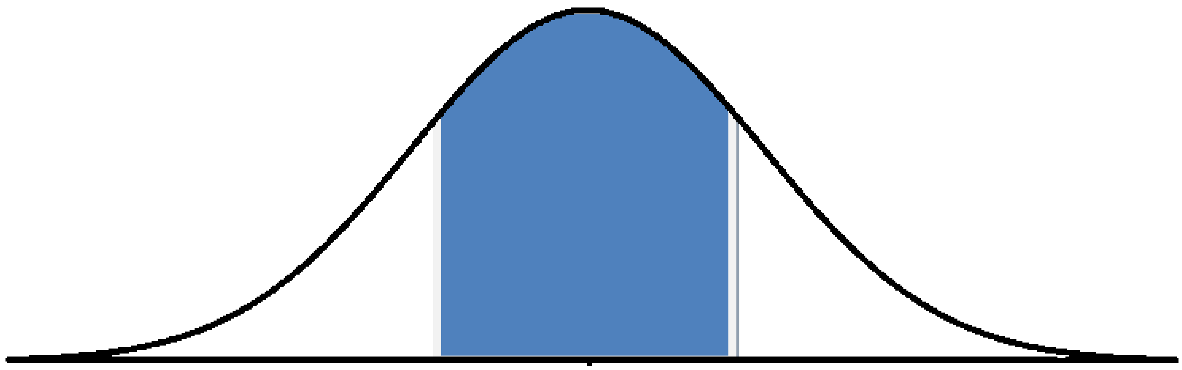
Thus, the probability that the child will have systolic pressure greater than 100 is 0.8413.

(c) Here, $X_1=110$ *and* $X_2=120$

For X_1, $\quad Z_1=\dfrac{110-115}{15}=-0.33$

For X_2, $Z_2 = \dfrac{120-115}{15} = 0.33$

Thus, we find that $P[Z_1 \le Z \le Z_2] = [-0.33 \le Z \le 0.33]$. This area is shown below:



$X_1 = 110$ μ $X_2 = 120$

$Z_1 = -0.33$ $Z = 0$ $Z_2 = 0.33$

Figure 10: Area under the curve for Z= -0.33 and Z= 0.33.

From the table we find that the area between $Z=0$ and $Z=0.33$ is 0.1293. Similarly, area between $Z=0$ and $Z=-0.33$ is 0.1293. Thus, total area from $Z_1=-0.33$ to $Z_2=0.33$ is $0.1293+0.1293=0.2586$.

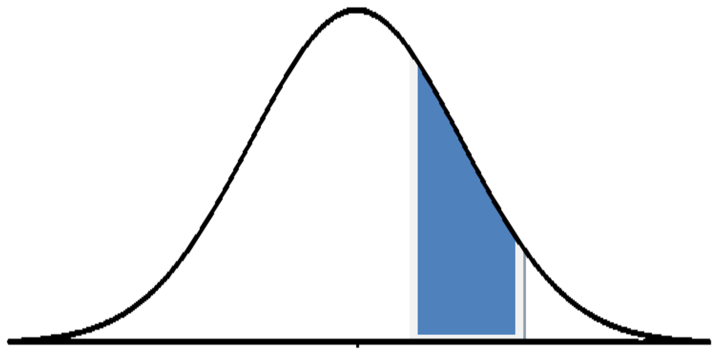
Thus, the probability that a child picked at random will have systolic pressure between 10 and 120 is 0.2586.

(d) Here, $X_1 = 120$ *and* $X_2 = 140$

For X_1, $Z_1 = \dfrac{120-115}{15} = 0.33$

For X_2, $Z_2 = \dfrac{140-115}{15} = 1.66$

Thus, we find the $P[Z_1 \le Z \le Z_2] = P[0.33 \le Z \le 1.66]$. This area is shown below:



$\mu \quad X_1 = 120 \quad X_2 = 140$

$Z = 0 \quad Z_1 = 0.33 \quad Z_2 = 1.66$

Figure 11: Area under the curve for z= 0.33 and z=1.66.

From the table the area for $Z = 0$ and $Z = 0.33$ is 0.1293 and for $Z = 1.66$ is 0.4415. The required area is $0.4415 - 0.1293 = 0.3122$. Thus, the probability that the child will have systolic pressure between 120 and 140 is 0.3122.

Exercises of interest

1. If two dice are thrown; what is the probability that the sum is
   (i) Greater than 8? (ii) Neither 7 nor 12?
2. A bag contains 4 white and 8 black balls. A ball is drawn at random. Find the probability of it being black.
3. A card is drawn at random from a pack of 52 cards. Find the chance that
   (i) It is red (ii) It is a king (iii) It is either red or a king.
4. In a family there are three children. What is the probability that the family has
   (i) No male child.
   (ii) At least one male child. (Assuming that the chances of a child being a male or female are equal).
5. What is the probability of randomly drawing either a heart or a club or a seven in a single draw from a pack of 52 cards?

6. A bag contains 3 red, 6 white and 7 blue balls. What is the probability that two balls drawn are white and blue?
7. Two monitors are to be selected in a class of 15 boys and 13 girls. What is the probability that the selected are one girl and one boy?
8. A ball is drawn at random from a box containing 6 red balls, 4 white balls, and 5 blue balls. Determine the probability that it is
   (a) Red (b) White (c) Blue (d) Not red (e) Red or white (d) Red, blue or white.
9. Two dice are thrown.
   (i) Find the probability of getting an odd number on the one and a multiple of three on the other.
   (ii) What is the probability that the sum of the numbers on the two dice is greater than 8?
10. If we toss a coin twice, let A be the event 'at least one head occurs' and B the event 'the second toss results in tail'. Find
    (i) $A \cup B$ (ii) $A \cap B$ (iii) A' (iv) $A - B$
11. In how many ways can 10 people be seated on a bench if only 4 seats are available?
12. Define Random Variable. Give two examples each of discrete and continuous random variable.
13. A random variable X has the following probability distribution:

X:	-2	-1	0	1	2
Pr:	$1/6$	p	$1/4$	p	$1/6$

    (i) Find the value of p.
    (ii) Calculate $E(X+2)$ and $E(2X^2+3X+5)$
14. Find the probability of drawing an ace or spade or both from a deck of cards.
15. Given $P(A)=1/4$, $P(B)=1/3$ and $P(A\cup B)=1/2$, evaluate $P\left(A/B\right)$, $P\left(B/A\right)$, $P(A\cap B')$ and $P\left(A/B'\right)$
16. (a) Find the constant c such that the function $f(x)=\begin{cases} cx^2, & 0<x<3 \\ 0, & otherwise \end{cases}$ is a density function.
    (b) Compute $P(1<X<2)$
    (c) Find the distribution function of X.
17. For an events A, B and C, show that
18. When does a Binomial distribution tends to a Poisson distribution?
19. Explain the clearly Binomial distribution and find out its mean, variance and standard deviation.
20. $P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$
21. If ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3:5$ find the value of n.
22. If ${}^{8}C_r - {}^{7}C_3 = {}^{7}C_2$, find r.
23. In a factory, machines A, B and C manufacture 25%, 35% and 40% of the total bolts produced. Of the total of their output, 5%, 4% and 2% are defective bolts. A bolt is drawn at random and found to be defective. What are the probabilities that it was manufactured by Machine A or B or C?

24. The probability that a KUST applicant will get admission in statistics is $2/3$, and the probability that he will not get admission in geography is $5/9$. If the probability of getting at least one admission is $4/5$, what is the probability that he will get both?
25. Eight coins are thrown simultaneously. Find the probability of getting at least six heads.
26. There are 3 different books of Mathematics, 4 different books of Engineering and 5 different books of Statistics. In how many ways these books can be arranged on KUST library's shelf when
(a) All the books are arranged at random.
(b) Books of each subject are arranged together.
27. The average number of customers who appear at a counter of a certain bank per minute is 2. Find the probability that during a given minute.
(i) No customer appears
(ii) Three or more customers appear . $\left[Given\ \ell^{-2} = 0.1353\right]$.
28. A man and a woman appear in an interview for two vacancies. The probability of man's selection is $1/4$ and that of woman is $1/3$. Find the probability of the event
    (i) Both of them will be selected,
    (ii) None of them will be selected,
    (iii) At least one of them will be selected.
29. What is a hypergeometric distribution? Find the mean and variance of this distribution. How is this distribution related to the binomial?
30. The time (t in minute) for a machine to be repaired has the following probability density function:

$$f(t) = \begin{cases} 10ct^2, & 0 \le t \le 40 \\ 9c(1-t), & 40 \le t \le 60 \\ 0, & otherwise \end{cases}$$

    (a) Using the fact that the probability $0 \le t \le 60$ is 1, evaluate c.
    (b) Determine the expected value of t.
31. Let X be a continuous random variables with probability distribution function given by

$$f(x) = \begin{cases} kx & ; \quad 0 \le x < 1 \\ k & ; \quad 1 \le x < 2 \\ -kx + 3k & ; \quad 2 \le x < 3 \\ 0 & ; \quad elsewhere \end{cases}$$

    (a) Determine the constant k
    (b) Determine $F(x)$.
32. If mortality rate for certain disease is 0.10 and 10 people are with that disease, what is the probability that none will survive?
33. The probability that a particular insecticide is successful in killing an insect is $1/20$. if 20 insects are subjects to that insecticide, what is the probability that all the insects will be killed?

34. A hospital records the weights of every new born child at the hospital. The distribution of weights is normally shaped with a mean, $\mu = 2.9$ kg., and standard deviation, $\sigma = 0.45$. Determine the following:
    (i) The percentage of new born babies who weighted under 2.1 kg.
    (ii) The percentage of new born babies who weighted between 1.8 kg. and 4 kg.
    (iii) If 500 babies have been born at the hospital, how many weighted less than 2.5 kg.?
35. Write a short note on Normal distribution and hence prove its mean and variance.

Table of the standard normal distribution values ($z \leq 0$)

$-z$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.49601	0.49202	0.48803	0.48405	0.48006	0.47608	0.47210	0.46812	0.46414
0.1	0.46017	0.45621	0.45224	0.44828	0.44433	0.44038	0.43644	0.43251	0.42858	0.42466
0.2	0.42074	0.41683	0.41294	0.40905	0.40517	0.40129	0.39743	0.39358	0.38974	0.38591
0.3	0.38209	0.37828	0.37448	0.37070	0.36693	0.36317	0.35942	0.35569	0.35197	0.34827
0.4	0.34458	0.34090	0.33724	0.33360	0.32997	0.32636	0.32276	0.31918	0.31561	0.31207
0.5	0.30854	0.30503	0.30153	0.29806	0.29460	0.29116	0.28774	0.28434	0.28096	0.27760
0.6	0.27425	0.27093	0.26763	0.26435	0.26109	0.25785	0.25463	0.25143	0.24825	0.24510

0.7	0.24196	0.23885	0.23576	0.23270	0.22965	0.22663	0.22363	0.22065	0.21770	0.21476
0.8	0.21186	0.20897	0.20611	0.20327	0.20045	0.19766	0.19489	0.19215	0.18943	0.18673
0.9	0.18406	0.18141	0.17879	0.17619	0.17361	0.17106	0.16853	0.16602	0.16354	0.16109
1.0	0.15866	0.15625	0.15386	0.15151	0.14917	0.14686	0.14457	0.14231	0.14007	0.13786
1.1	0.13567	0.13350	0.13136	0.12924	0.12714	0.12507	0.12302	0.12100	0.11900	0.11702
1.2	0.11507	0.11314	0.11123	0.10935	0.10749	0.10565	0.10384	0.10204	0.10027	0.09853
1.3	0.09680	0.09510	0.09342	0.09176	0.09012	0.08851	0.08692	0.08534	0.08379	0.08226
1.4	0.08076	0.07927	0.07780	0.07636	0.07493	0.07353	0.07215	0.07078	0.06944	0.06811
1.5	0.06681	0.06552	0.06426	0.06301	0.06178	0.06057	0.05938	0.05821	0.05705	0.05592
1.6	0.05480	0.05370	0.05262	0.05155	0.05050	0.04947	0.04846	0.04746	0.04648	0.04551
1.7	0.04457	0.04363	0.04272	0.04182	0.04093	0.04006	0.03920	0.03836	0.03754	0.03673
1.8	0.03593	0.03515	0.03438	0.03363	0.03288	0.03216	0.03144	0.03074	0.03005	0.02938
1.9	0.02872	0.02807	0.02743	0.02680	0.02619	0.02559	0.02500	0.02442	0.02385	0.02330
2.0	0.02275	0.02222	0.02169	0.02118	0.02068	0.02018	0.01970	0.01923	0.01876	0.01831
2.1	0.01786	0.01743	0.01700	0.01659	0.01618	0.01578	0.01539	0.01500	0.01463	0.01426
2.2	0.01390	0.01355	0.01321	0.01287	0.01255	0.01222	0.01191	0.01160	0.01130	0.01101
2.3	0.01072	0.01044	0.01017	0.00990	0.00964	0.00939	0.00914	0.00889	0.00866	0.00842
2.4	0.00820	0.00798	0.00776	0.00755	0.00734	0.00714	0.00695	0.00676	0.00657	0.00639
2.5	0.00621	0.00604	0.00587	0.00570	0.00554	0.00539	0.00523	0.00509	0.00494	0.00480
2.6	0.00466	0.00453	0.00440	0.00427	0.00415	0.00403	0.00391	0.00379	0.00368	0.00357
2.7	0.00347	0.00336	0.00326	0.00317	0.00307	0.00298	0.00289	0.00280	0.00272	0.00264
2.8	0.00256	0.00248	0.00240	0.00233	0.00226	0.00219	0.00212	0.00205	0.00199	0.00193
2.9	0.00187	0.00181	0.00175	0.00170	0.00164	0.00159	0.00154	0.00149	0.00144	0.00140
3.0	0.00135	0.00131	0.00126	0.00122	0.00118	0.00114	0.00111	0.00107	0.00104	0.00100
3.1	0.00097	0.00094	0.00090	0.00087	0.00085	0.00082	0.00079	0.00076	0.00074	0.00071
3.2	0.00069	0.00066	0.00064	0.00062	0.00060	0.00058	0.00056	0.00054	0.00052	0.00050
3.3	0.00048	0.00047	0.00045	0.00043	0.00042	0.00040	0.00039	0.00038	0.00036	0.00035
3.4	0.00034	0.00033	0.00031	0.00030	0.00029	0.00028	0.00027	0.00026	0.00025	0.00024
3.5	0.00023	0.00022	0.00022	0.00021	0.00020	0.00019	0.00019	0.00018	0.00017	0.00017

Table of the standard normal distribution values ($z \geq 0$)

z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524

0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91308	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983

Recommended Books

1. D.R., Agarwal, ‘Business Statistics’, Vrinda, (2008).
2. Feller, W., An Introduction to Probability Theory and Its Applications, Vol. 1, John Wiley (1965).
3. J.N., Kapur, H.C., Saxena, ‘Mathematical Statistics’, S. Chand, (2011).
4. K.S., Negi, ‘Methods in Biostatistics’, AITBS, (2012).
5. Murray, R.S., John S., R.A. Srinivasan, ‘Probability and Statistics’, Schaum’s Outlines, (2011).
6. Nabendu, P., Sahadeb, S., ‘Statistics Concept and Application’, Prentice – Hall, (2008).
7. R.S., Bhardwaj, ‘Business Statistics’, Excel Books, (2006).
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9. Sheldon, M.R., ‘Probability for Engineers and Scientists’, Academic Press, (2009).

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