Physics - Assignment - 7.

Q1.

ID box

$0 \le x \le a$

$n = 1$

$x = a/4$ to $x = 3a/4$

$P = ? = |\psi^2|$

As per Schrodinger's Equation $\rightarrow$

$$\cancel{\frac{\partial^2\psi}{\partial x^2}} \quad \cancel{-\frac{\hbar^2}{2m}} \frac{2mE\psi}{\hbar^2} = \cancel{0}$$

$$\frac{\partial^2\psi}{\partial x^2} + \boxed{\frac{2mE}{\hbar^2}}\psi = 0. \qquad \frac{2mE}{\hbar^2} = k^2$$

$$\frac{\partial^2\psi}{\partial x^2} + k^2\psi = 0.$$

$$\psi = A\sin kx + B\cos kx.$$

Applying Boundary conditions $\rightarrow$

$\psi = 0$ when $x = 0$.

$\boxed{B = 0}$

$$\psi = A\sin kx.$$

$\psi = 0$ when $x = a$.

$\sin ka = 0$

$ka = n\pi$

$$k = \frac{n\pi}{a}.$$

So, $k = \sqrt{\frac{2mE}{\hbar^2}} = \frac{n\pi}{a}.$

$$\boxed{E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}}$$

$\psi = A\sin kx.$ where $k = \sqrt{\frac{2mE}{\hbar^2}}$

↳ Has to square integrable

$$\int_0^a \psi^*\psi\, dx = 1$$

$$\int_0^a A^2\sin^2 kx\, dx = 1$$

$$\frac{A^2}{2}\int_0^a 1 - \cos 2kx\, dx = 1$$

$$\frac{A^2}{2}\left[x - \frac{\sin 2kx}{2}\right]_0^a = 1$$

$$\frac{A^2}{2}\left[a - \frac{\sin 2\pi}{2}\right] = 1.$$

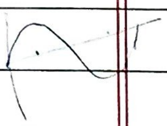
$$A = \sqrt{\frac{2}{a - \cancel{\sin a}^{\,0}}} = \sqrt{\frac{2}{a}}.$$

$$\boxed{\psi = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)}$$

For lowest energy state $n = 1$ $E = \frac{\pi^2\hbar^2}{2ma^2}$

$$P = |\psi^2| = \frac{2}{a}\sin^2\left(\frac{n\pi x}{a}\right)$$

$\frac{a}{4} \le x \le \frac{3a}{4}$ $\qquad P = \dfrac{\int_{a/4}^{3a/4} |\Psi|^2 \, dx}{\int_0^a |\Psi|^2 \, dx} = \dfrac{1}{a}\left[x - \dfrac{a\sin}{(n\pi)2}\left(\dfrac{2n\pi x}{a}\right)\right]_{a/4}^{3a/4}$



$$= \frac{1}{a}\left[\frac{a}{2} - \frac{a}{2\pi}\left[\sin\left(\frac{2\pi(3a)}{4a}\right) - \sin\left(\frac{2\pi \times a}{a \; 4/2}\right)\right]\right]$$

$$= \frac{1}{a}\left[\frac{a}{2} + \frac{a}{\pi}\right]$$

$$= \boxed{\frac{1}{2} + \frac{1}{\cancel{a}\pi}}$$

Q2. $\quad P = \int_x^{x+\Delta x} |\Psi|^2 \, dx = \lim_{n \to \infty} \dfrac{1}{a}\left[x - \dfrac{a\sin}{2\pi n}\left(\dfrac{2n\pi x}{a}\right)\right]_x^{x+\Delta x}$

So, $= \dfrac{[x]_x^{x+\Delta x}}{a}$

$$= \boxed{\frac{\Delta x}{a}}$$

Q3. $\quad$ mass $= m \qquad 0 \le x \le a \qquad \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$

For the n^{th} stationary state,

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}$$

$\sqrt{\frac{2}{a}} \frac{a}{n\pi} \cos\left(\frac{n\pi x}{a}\right)$

$$\Psi_n = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

$\frac{a\sqrt{2a}}{n^2\pi^2}\left(-\sin\left(\frac{n\pi x}{a}\right)\right)$

$$\Delta x = \int_0^a x \Psi^* \Psi \, dx = \int_0^a \frac{2x \sin^2}{a}\left(\frac{n\pi x}{a}\right) = \cancel{\frac{2}{a}\int_0^a x \sin^2\left(\frac{n\pi x}{a}\right)}$$

$$\frac{2}{a}\left[\cancel{\sin^2\left(\frac{n\pi x}{a}\right)\frac{x^2}{2} - \int \frac{x^2}{2}\sin\left(\frac{2n\pi x}{a}\right) dx}\right]$$

$$= \cancel{\frac{2}{a}\left[\frac{a^2 \sin^2}{2}\left(\frac{n\pi x}{a}\right) - \sin\left(\frac{2n\pi x}{a}\right)\right.}$$

$\cos 2x = 1 - 2\sin^2 x$

$\frac{1-\cos 2x}{2}$ $1 -$

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$$= \frac{2}{a}\left[\frac{x}{2} - \frac{\sin}{4}\left(\frac{2n\pi x}{a}\right) - \int \frac{x}{2} - \frac{\sin}{4}\left(\frac{2n\pi x}{a}\right) dx\right]$$

$$= \frac{2}{a}\left[x\left(\frac{x}{2} - \frac{a\sin}{4n\pi}\left(\frac{2n\pi x}{a}\right)\right) - \frac{x^2}{4} - \frac{a\cos}{4(2n\pi)}\left(\frac{2n\pi x}{a}\right)\right]_0^a$$

$$= \frac{2}{a}\left[\frac{a^2}{2} - \frac{a^2}{4} - \frac{a^0}{8n\pi}\right] = \frac{a^2}{4_2}\left(\frac{2}{a}\right) = \boxed{\frac{a}{2}}$$

$$\langle x \rangle = \int_0^a x|\psi^2|dx = \frac{2}{a}\int^a x \sin^2\left(\frac{n\pi x}{a}\right) dx.$$

$$= \frac{2}{a}\left[x\left[\frac{x}{2} - \frac{\sin}{4}\left(\frac{2n\pi x}{a}\right)\right]^a_0 - \int \frac{x}{2} - \frac{\sin}{4}\left(\frac{2n\pi x}{a}\right) dx\right]$$

$$= \frac{2}{a}\left[\frac{a^2}{2} - \frac{a^2}{4}\right]$$

$$= \boxed{\frac{a}{2}}$$

$$\langle p \rangle = ?$$

using operator $p = \frac{\hbar}{i}\frac{\partial}{\partial x}$.

$$\langle p \rangle = \int_0^a \psi^* \frac{\hbar}{i}\frac{\partial \psi}{\partial x} dx = \frac{\hbar \cancel{a}}{i n\pi}\int_0^a \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\sqrt{\frac{2}{a}}\cos\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{\hbar}{n\pi i}\int_0^a \sin\left(\frac{2n\pi x}{a}\right) dx$$

$$= \frac{\hbar}{n\pi i}\left[-\frac{a\cos}{2n\pi}\left(\frac{2n\pi x}{a}\right)\right]_0^a$$

$$= \frac{\hbar}{n\pi i}\left(\frac{-a}{2n\pi}\right)\left[\cos(2n\pi) - \cos 0\right]$$

$$= \boxed{0}$$

$$\langle x^2 \rangle = \int_0^a x^2 |\psi|^2 \, dx$$

$$= \int x^2 \left(\frac{2}{a}\right) \sin^2\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{2}{a} \int x^2 \sin^2\left(\frac{n\pi x}{a}\right) dx$$

$$= \frac{2}{a} \left[x^2 \left(\frac{x}{2} - \frac{a\sin}{2n\pi}\left(\frac{2n\pi x}{a}\right)\right) - \int 2x \left(\frac{x}{2} - \frac{a}{2n\pi} \sin\left(\frac{2n\pi x}{a}\right)\right) dx \right]$$

$$= \frac{2}{a} \left[x^2 \left(\frac{x}{2} - \frac{a}{2n\pi} \sin\left(\frac{2n\pi x}{a}\right)\right) - \frac{x^3}{3} + \frac{a}{2n\pi} \int x \sin\left(\frac{2n\pi x}{a}\right) dx \right]$$

$$= \frac{2}{a} \left[\frac{x^3}{2} - \frac{x^2 a}{2n\pi} \sin\left(\frac{2n\pi x}{a}\right) - \frac{x^3}{3} + \frac{a}{2n\pi} \left[\frac{a\cos}{2n\pi}\left(\frac{2n\pi x}{a}\right) x - \frac{a^2 \sin}{4n^2\pi^2}\left(\frac{2n\pi x}{a}\right) \right] \right]$$

$$= \frac{2}{a} \left[\frac{x^3}{6} - \frac{x^2 a}{2n\pi} \sin\left(\frac{2n\pi x}{a}\right) + \frac{a^2 x}{4n^2\pi^2} \cos\left(\frac{2n\pi x}{a}\right) - \frac{a^3}{8n^3\pi^3} \sin\left(\frac{2n\pi x}{a}\right) \right]_0^a$$

$$= \frac{\cancel{2}}{\cancel{a}} \left[\frac{a^{\cancel{3}2}}{\cancel{6}\,3} - \cancel{\frac{a^3}{2n\pi}} 0 + \frac{a^{\cancel{3}2}}{{}_2\cancel{4}n^2\pi^2} \right]$$

$$= \boxed{\frac{a^2}{3} + \frac{a^2}{2n^2\pi^2}}$$

$$\langle p^2 \rangle = \int_0^a \psi^* \left(-\hbar^2 \frac{\partial^2 \psi}{\partial x^2}\right) dx \qquad p = \frac{\hbar}{i} \frac{\partial}{\partial x} \qquad p^2 = -\hbar^2 \frac{\partial^2}{\partial x^2}$$

$$= \int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \left(-\hbar^2 \frac{\partial^2 \psi}{\partial x^2}\right) dx$$

$$= \int_0^a \sqrt{\frac{2}{\cancel{a}}} \sin\left(\frac{n\pi x}{a}\right) (\hbar^2) \left(\frac{a\sqrt{2/a}}{n^2\pi^2}\right) \left[\sin\left(\frac{n\pi x}{a}\right)\right] dx$$

$$= \frac{2a\hbar^2}{n^2\pi^2} \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx = \frac{a\hbar^2}{n^2\pi^2} \int_0^a 1 - \cos\left(\frac{2n\pi x}{a}\right) dx$$

$$= \frac{a\hbar^2}{n^2\pi^2} \left[x - \frac{a}{2n\pi} \sin\left(\frac{2n\pi x}{a}\right) \right]_0^a$$

$$= \boxed{\frac{a^2\hbar^2}{n^2\pi^2}}$$

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{a^2}{3} - \frac{a^2}{2n^2\pi^2} - \frac{a^2}{4}}$$

$p = \hbar$

$$= a\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{a^2\hbar^2}{n^2\pi^2} - 0} = \frac{a\hbar}{n\pi}$$

$$\Delta x \Delta p = \frac{a^2\hbar}{n\pi}\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}$$

$$= \frac{a^2\hbar}{2\pi^2 n^2}\sqrt{\frac{n^2\pi^2 - 2}{3}}$$

$$= \frac{\hbar a}{2n^2\pi^2}\sqrt{\frac{(n\pi)^2 - 2}{3}}$$

Q4. $\psi_1 = Ae^{ikx}$

$\psi_2 = Ae^{qx}$

$\psi_1 = Ae^{ikx}$

(ψ_1) Characteristics = This type of wave function is a sinosoidal curve. It is usually observed in cases where energy of wave is greater than the potential barrier.

It is continuous, single-valued, square-integrable.

(ψ_2) Characteristics = This type of wave function is an exponential function. ~~It is usually observed in cases where energy is less than potential barrier.~~

It is continuous, single valued, ~~square integrable~~.

It is not an admissible wave function as it is not normalisable. It approaches infinity as x approaches infinity.

Q5. $\psi(x) = Ae^{ikx} \rightarrow$ ~~[illegible]~~ can be written as $\rightarrow$

$= A[i\sin kx + \cos kx]$

- It is continuous, single-valued and normalisable.

~~Hence it is an a~~

- Its derivative is also continuous, single valued & normalisable

so it is an acceptable eigen function.

For $-a \le x \le a$.

$$\int_{-a}^{a} \psi^* \psi = 1$$

$$\int_{-a}^{a} (A e^{ikx})(A e^{-ikx})\, dx = 1$$

$$A^2 \int_{-a}^{a} dx = 1$$

$$A^2 (2a) = 1$$

$$\boxed{A = \frac{1}{\sqrt{2a}}}$$

$$\boxed{\psi = \frac{e^{ikx}}{\sqrt{2a}}}$$

Q6. $V(x) \rightarrow 0 \quad -a/2 \le x \le a/2$

$\quad\quad\ \rightarrow \infty \quad$ otherwise.

Since $V(x) = \infty$ otherwise, $\psi = 0$ outside the boundary.

For within the boundary, as per Schrodinger Eq^n →

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{2m(E - V_0)\psi}{\hbar^2} = 0.$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{2mE\psi}{\hbar^2} = 0.$$

let $\frac{2mE}{\hbar^2} = k^2$

so, $$\frac{\partial^2 \psi}{\partial x^2} + k^2 \psi = 0.$$

So, solution : $\psi = A\sin kx + B\cos kx$.

Applying boundary conditions →

$\psi = 0$ when $x = -a/2$

$$0 = -A\sin\left(\frac{ka}{2}\right) + B\cos\left(\frac{ka}{2}\right) \quad \text{--- (1)}$$

$\psi = 0$ when $x = a/2$

$$0 = A\sin\left(\frac{ka}{2}\right) + B\cos\left(\frac{ka}{2}\right) \quad \text{--- (2)}.$$

Adding (1) & (2)

$$0 = 2B\cos\left(\frac{ka}{2}\right)$$

Subtracting (1) & (2)

$$0 = 2A\sin\left(\frac{ka}{2}\right)$$

$\cos\left(\frac{ka}{2}\right)$ and $\sin\left(\frac{ka}{2}\right)$ simultaneously can't be 0.

so, 2 classes of solutions are possible :→

$\boxed{A=0 \text{ and } \cos\left(\frac{ka}{2}\right) = 0}$ OR $B=0$ and $\sin\left(\frac{ka}{2}\right) = 0$

$\frac{ka}{2} = \frac{n\pi}{2}$ for $n = 1, 3, 5, 7 \ldots$ $\qquad$ $\frac{ka}{2} = n\pi$ for $n = 1, 2, 3, 4 \ldots$

$k_n = n\pi/a$ for $n = 1, 3, 5, 7 \ldots$ $\qquad$ $k_n = \frac{2n\pi}{a}$ for $n = 1, 2, 3, 4 \ldots$

$\sqrt{\frac{2mE_n}{\hbar^2}} = \frac{n\pi}{a}$ $\qquad$ $= \frac{n\pi}{a}$ for $n = 2, 4, 6, 8 \ldots$

$$\boxed{E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}} \text{ for } n = 1, 2, 3 \ldots$$

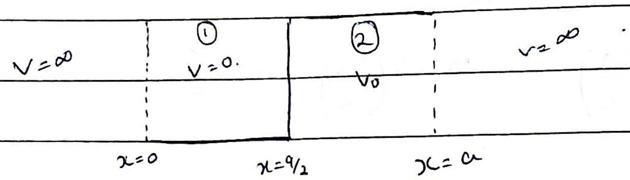
~~$\psi = B\cos($~~

Yes the results are consistent with a well centred at $x = a/2$ of width a.

No, we cannot obtain this wave function by using that substitution.

Q7.

$$V(x) \rightarrow \begin{cases} \infty & x \leq 0 \\ 0 & 0 < x < a/2 \\ V_0 & a/2 < x < a \\ \infty & x \geq a \end{cases}$$



In the regions where $V = \infty$, $\psi = 0$ as wave function cannot exist.

For region-1

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{2mE\psi}{\hbar^2} = 0.$$

$$\frac{\partial^2 \psi}{\partial x^2} + k_1^2 \psi = 0.$$

~~$\psi = Ae^{ikx}$~~

~~$E = \frac{n^2 \pi^2 \hbar^2}{2m}$~~

$$\psi = A \sin k_1 x + B \cos k_1 x$$

Applying boundary conditions :

$\psi = 0$ when $x = 0$.

So, $\boxed{B = 0}$.

~~$\psi = 0$ when $x = a/2$~~

$$\psi_1 = A \sin k_1 x$$

~~$\psi = Ae^{ikx} + Be^{-ikx}$~~

$$\psi = A\left[e^{ik_1 x} - e^{-ik_1 x}\right]$$

For region-2

For $E > V_0$ (E_2)

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{2m(E - V_0)\psi}{\hbar^2} = 0$$

$$\frac{\partial^2 \psi}{\partial x^2} + k_2^2 \psi = 0.$$

$$\boxed{\psi_2 = C \sin k_2 x + D \cos k_2 x}$$

$\psi = 0$ when $x = a$

$$0 = C \sin k_2 a + D \cos k_2 a$$

$\psi_1 = \psi_2$ at $x = a/2$

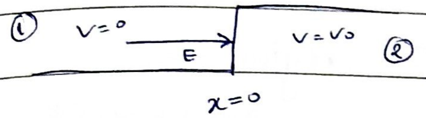
$$\boxed{A \sin \frac{k_1 a}{2} = C \sin \frac{k_2 a}{2} + D \cos \frac{k_2 a}{2}}$$

$$\left.\frac{\partial \psi_1}{\partial x}\right|_{x = a/2} = \left.\frac{\partial \psi_2}{\partial x}\right|_{x = a/2}$$

$$\boxed{k_1 A \cos \frac{k_1 a}{2} = k_2 \left[C \cos \frac{k_2 a}{2} - D \sin \frac{k_2 a}{2}\right]}$$

Q. $V(x) = 0 \quad x \le 0 \qquad E > V_0.$

$\quad\;\; = V_0 \quad x > 0.$ Transmission Coefficient $T = ?$



For region-1

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{2mE\psi}{\hbar^2} = 0.$$

let $K_1^2 = \frac{2mE}{\hbar^2}$

So, $$\frac{\partial^2 \psi}{\partial x^2} + K_1^2 \psi = 0.$$

$$\psi_1 = \underbrace{Ae^{iK_1x}}_{\text{incident}} + \underbrace{Be^{-iK_1x}}_{\text{reflected}}$$

For region-2

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{2m(E-V_0)\psi}{\hbar^2} = 0.$$

Let $K_2^2 = \frac{2m(E-V_0)}{\hbar^2}$

$$\frac{\partial^2 \psi}{\partial x^2} + K_2^2 \psi = 0.$$

$$\psi_2 = Ce^{iK_2x} + De^{-iK_2x}$$

But $D=0$ as there is no barrier to reflect back

So, $\psi_2 = \underbrace{Ce^{iK_2x}}_{\text{Transmitted}}$.

Applying Boundary conditions →

$\psi_1 = \psi_2$ at $x=0$.

$$A + B = C. \quad -①$$

$$\left.\frac{\partial \psi_1}{\partial x}\right]_{x=0} = \left.\frac{\partial \psi_2}{\partial x}\right]_{x=0}.$$

$$iK_1\left[Ae^{iK_1x} - Be^{iK_1x}\right] = iK_2\, Ce^{iK_2x}.$$

$$K_1(A-B) = K_2C.$$

$$A - B = K_2/K_1\, C. \quad -②$$

From ① & ②

① + ②.

$$2A = C\left(1 + \frac{K_2}{K_1}\right).$$

$$\frac{C}{A} = \frac{2K_1}{K_1+K_2}.$$

$$T = \frac{K_2|C|^2}{K_1|A|^2} = \frac{4K_1K_2}{(K_1+K_2)^2} = \frac{4\left(\sqrt{\frac{2mE}{\hbar^2}}\right)\sqrt{2m(E-V_0)}}{\frac{1}{\hbar^2}\left(\sqrt{2mE} + \sqrt{2m(E-V_0)}\right)^2} = \frac{\overset{4}{\cancel{8m}}E}{2mE + 2m(E-V_0) + 2(2m)\sqrt{E(E-V_0)}}$$

$$= \frac{4\sqrt{E(E-V_0)}}{2E - V_0 + 2\sqrt{E(E-V_0)}} = \boxed{\frac{4\sqrt{E(E-V_0)}}{\left(\sqrt{E} + \sqrt{E-V_0}\right)^2}}$$

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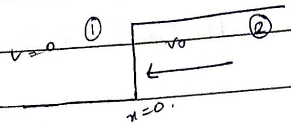
If the wave is incident from right $\rightarrow$



For region-1

$$\frac{\partial^2 \psi}{\partial x^2} + \underbrace{\boxed{\frac{2mE}{\hbar^2}}}_{k_1^2}\psi = 0.$$

For region-2

$$\frac{\partial^2 \psi}{\partial x^2} + \underbrace{\boxed{\frac{2m(E-V_0)}{\hbar^2}}}_{k_2^2}\psi = 0.$$

$$\psi_1 = \cancel{A\sin}\ Ae^{ik_1x} \qquad \psi_2 = Be^{ik_2x} + Ce^{-ik_2x}$$

Applying boundary conditions -

$$A = B + C. \quad -①$$

$$\cancel{i}k_1 A = \cancel{i}k_2 [B - C]$$

$$A = \frac{k_2}{k_1}(B - C) \quad -②.$$

$$J = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right)$$

$$T = \frac{J_T}{J_I} = \frac{\cancel{|B|^2}}{\cancel{|A|}} \frac{k_1 |A^2|}{k_2 |B^2|}$$

$$=$$

From ① & ② $\rightarrow$ $2B = A\left(1 + \frac{k_1}{k_2}\right)$

$$\frac{A}{B} = \frac{2k_2}{k_1 + k_2}$$

$$T = \frac{k_1}{\cancel{k_2}}\left(\frac{4k_2^{\cancel{2}}}{(k_1+k_2)^2}\right) = \boxed{\frac{4k_1k_2}{(k_1+k_2)^2}}$$

= Same as previous case.

$\therefore$ Hence proved.

Q9. $E = 10\,eV$ $\quad V_0 = 13.8\,eV$

Here $E < V_0$.

For region-2 So, $$\frac{\partial^2 \psi}{\partial x^2} - \frac{2m(V_0 - E)\psi}{\hbar^2} = 0.$$

$$\frac{2m(V_0 - E)}{\hbar^2} = k_2^2 =$$

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$$\frac{\partial^2 \psi}{\partial x^2} - K_2^2 \psi = 0.$$

Solution : $\psi_2 = C e^{-K_2 x}$

From prev. Qs we know $\psi_1 = A e^{iK_1 x} + B e^{-iK_1 x}$.

$\psi_1 = \psi_2$ for $x = 0$.

$$\boxed{A + B = C} \quad - ①$$

$$\frac{\partial \psi_1}{\partial x} = \frac{\partial \psi_2}{\partial x} \quad \text{for } x = 0.$$

$$iK_1 (A - B) = -K_2 C.$$

$$\boxed{A - B = \frac{i K_2 C}{K_1}} \quad - ②.$$

$$A = \frac{C}{2}\left(1 + \frac{iK_2}{K_1}\right)$$

$$B = \frac{C}{2}\left(1 - \frac{iK_2}{K_1}\right)$$

$$\text{Probability} = \int_0^x |\psi^2|\, dx$$

At $x = 0$. $P = |\psi_2|^2 = C^2 e^{-2K_2 x(0)} = C^2$

After penetrating Δx, $P = \dfrac{C^2}{100}.$

$$\frac{\cancel{C^2}}{100} = \cancel{C} e^{-2K_2(\Delta x)}$$

$$100 = e^{2K_2(\Delta x)}$$

$$\ln 100 = 2 K_2 \Delta x$$

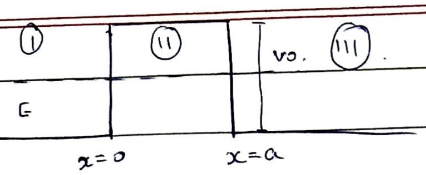
$$\Delta x = \frac{\ln 100}{2K_2} = \frac{4.605\ (6.6 \times 10^{-34})}{2 \times 2\pi \sqrt{2 \times 9.1 \times 10^{-31} \times 3.8 \times 1.6 \times 10^{-19}}}$$

$$= \frac{2.428 \times 10^{-34}}{10.519 \times 10^{-25}}$$

$$= 0.230 \times 10^{-9}\ m$$

$$= \boxed{0.23\ nm}$$

Q10.(a) $E = V_0$.



For $0 < x < a$.

Schrodinger $Eq^n \rightarrow \dfrac{\partial^2\psi}{\partial x^2} + \boxed{\dfrac{2m(E-V_0)}{\hbar^2}}\psi = 0.$ $\quad = K_2^2$

$\dfrac{\partial^2\psi}{\partial x^2} + \cancel{K_2^2}\psi = 0.$ $\qquad \dfrac{\partial^2\psi}{\partial x^2} = 0.$

$\cancel{\psi_2 = Ae^{iK_2x} + Be^{iK_2x}}$ $\qquad \psi_2 = Ax + B$

~~since $E = V_0$ so, $K_2 = 0$.~~

~~so, ψ_2 = constant = $\boxed{A+B}$~~

(b) Region - 1 $\qquad$ Region - 2

$\dfrac{\partial^2\psi}{\partial x^2} + \boxed{\dfrac{2mE}{\hbar^2}}\psi = 0$ $\quad = K_1^2$ $\qquad \psi_2 = Ax + B.$

~~Region 3~~

So, $\psi_1 = \underbrace{Ce^{iK_1x}}_{\text{incident}} + \underbrace{De^{-iK_1x}}_{\text{Reflected}}$.

Region - 3

$\dfrac{\partial^2\psi}{\partial x^2} + \dfrac{2mE\psi}{\hbar^2} = 0.$

$\psi_3 = \underbrace{Fe^{iK_1x}}_{\text{Transmitted}}.$

$\psi_1 = \psi_2$ at $x = 0$. $\qquad (\psi_2 = \psi_3)_{x=a}$.

$\boxed{C + D = B.}$ $\qquad Aa + B = Fe^{iK_1a}.$

$\left\{\dfrac{\partial\psi_1}{\partial x} = \dfrac{\partial\psi_2}{\partial x}\right\}_{x=0}.$ $\qquad \cancel{2C} = Fe^{iK_1a}.$

$\boxed{\cancel{C = \dfrac{Fe^{iK_1a}}{2}}}$

$iK_1(C-D) = A.$

$\boxed{\cancel{C = D}}$ $\qquad \left[\dfrac{\partial\psi_2}{\partial x} = \dfrac{\partial\psi_3}{\partial x}\right]_{x=a}$

$C - D = \dfrac{A}{iK_1}$ $\qquad A = iK_1Fe^{iK_1a}.$

$2C =$ $\qquad$ ~~$Aa + B =$~~

So, $C + D = B$. $\qquad Aa + B = Fe^{ik_1a}$

$C - D = \frac{A}{ik_1} \qquad A = ik_1 Fe^{ik_1a}$.

$C - D = Fe^{ik_1a} \qquad ik_1a Fe^{ik_1a} + B = Fe^{ik_1a}$

$$B = Fe^{ik_1a}(1 - ik_1a).$$

$$C + D = Fe^{ik_1a}(1 - ik_1a).$$

$$2C = Fe^{ik_1a}(2 - ik_1a).$$

$$\frac{F}{C} = \frac{2}{e^{ik_1a}(2 - ik_1a)} = \frac{2e^{-ik_1a}}{2 - ik_1a}$$

(c) Transmission coefficient $T = \cancel{\frac{|F|^2}{|C|^2}}$

$$= \cancel{\frac{4}{e^{2ik_1a}}}$$

Probability $= |\psi^2|$

Probability of transmission $= \cancel{\frac{|\psi_2|^2}{|\psi_1|^2}} = \cancel{\frac{|F^2|}{|C^2|}}$

$$= \cancel{\frac{(A+B)^2}{|C|^2}}$$

Probability of Transmission $= \frac{|F|^2}{|C|^2} = \frac{4}{4 + k_1^2a^2}$

$$= \frac{4}{4 + \frac{2mEa^2}{\hbar^2}} = \boxed{\frac{1}{1 + \frac{ma^2V_0}{2\hbar^2}}}$$

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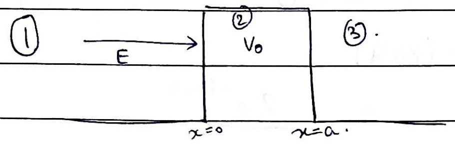
Q11. $V_0 = 6eV$

width $= a = 0.2nm$.

Energy $= ?$

Probability of transmission $= 1\%$



Let us consider $E < V_0$.

Region-1

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{2mE\psi}{\hbar^2} = 0$$

$$\psi_1 = Ae^{ik_1x} + Be^{-ik_1x}.$$

Region-2

$$\frac{\partial^2 \psi}{\partial x^2} - \underbrace{\boxed{\frac{2m(V_0-E)}{\hbar^2}}}_{=k_2^2}\psi = 0.$$

$$\psi_2 = \cancel{Ce^{-k_2x} + De^{k_2x}}$$

$$= Ce^{k_2x} + De^{-k_2x}$$

Region-3

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{2mE\psi}{\hbar^2} = 0.$$

$$\psi_3 = Fe^{ik_1x}$$

Applying Boundary conditions -

At $x=0$

$$A + B = C + D$$

$$ik_1(A-B) = k_2(C-D).$$

$$A - B = \frac{k_2}{ik_1}(C-D)$$

At $x=a$

$$Ce^{k_2a} + De^{-k_2a} = Fe^{ik_1a}.$$

$$k_2(Ce^{k_2a} - De^{-k_2a}) = ik_1Fe^{ik_1a}$$

$$2A = C\left(1 + \frac{k_2}{ik_1}\right) + D\left(1 - \frac{k_2}{ik_1}\right)$$

$$T = \frac{1}{100} = \frac{|F^2|}{|A|^2}$$

$$C = \frac{Fe^{ik_1a-k_2a}}{2}\left[1 + \frac{ik_1}{k_2}\right]$$

$$D = \frac{Fe^{ik_1a+k_2a}}{2}\left[1 - \frac{ik_1}{k_2}\right]$$

$$4A = Fe^{ik_1a-k_2a}\left(1 + \frac{ik_1}{k_2}\right)\left(1 + \frac{k_2}{ik_1}\right) + Fe^{ik_1a+k_2a}\left(1 - \frac{ik_1}{k_2}\right)\left(1 - \frac{k_2}{ik_1}\right)$$

e

$$\frac{4A}{Fe^{ik_1a}} = e^{-k_2a}\left(2 + \frac{k_2}{ik_1} + \frac{ik_1}{k_2}\right) + e^{k_2a}\left(2 - \frac{k_2}{ik_1} - \frac{ik_1}{k_2}\right) \quad \frac{k_2}{ik_1}\left(e^{-k_2a} - e^{k_2a}\right)$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\frac{4A}{Fe^{ik_1a}} = 4\cosh(k_2a) + \cancel{2\sinh\left(\frac{k_2}{ik_1} - \frac{ik_1}{k_2}\right)}\ \frac{k_2}{ik_1}2\sinh(-k_2a) + \frac{ik_1}{k_2}2\sinh(-k_2a)$$

$$\frac{4Ae^{-ik_1a}}{F} = 4\cosh(k_2a) - \frac{i2k_2}{k_1}\sinh(-k_2a) + \frac{2ik_1}{k_2}\sinh(-k_2a)$$

$$\cancel{\frac{4Ae^{ik_1a}}{F}} = \cancel{4\cosh(k_2a)} +$$

$$= 4\cosh(k_2a) + 2i\left[\frac{k_1}{k_2} - \frac{k_2}{k_1}\right]\sinh(-k_2a)$$

$$a + ib$$
$$a - ib.$$

Taking conjugate :

$$\frac{4Ae^{ik_1a}}{F} = 4\cosh(k_2a) - 2i\left[\frac{k_1}{k_2} - \frac{k_2}{k_1}\right]\sinh(-k_2a)$$

$$\frac{16|A^2|}{|F^2|} = 16\cosh^2(k_2a) + 4\sinh^2(-k_2a)\left[\frac{k_1^2 - k_2^2}{k_1k_2}\right]^2$$

$$\frac{|A^2|}{|F^2|} = \cosh^2(k_2a) + \frac{\sinh^2(-k_2a)}{4}\left[\frac{k_1^2 - k_2^2}{k_1k_2}\right]^2$$

$$100 = 1 + \frac{(k_1^2 + k_2^2)^2}{4k_1^2k_2^2}\sinh^2(k_2a)$$

$$99 = \frac{(k_1^2 + k_2^2)^2\sinh^2(k_2a)}{4k_1^2k_2^2}$$

$$= \left[\frac{\left(\frac{2mE}{\hbar^2} + \frac{2m(E-V_0)}{\hbar^2}\right)^2}{4\left(\frac{4m^2}{\hbar^4}\right)E(E-V_0)}\right]\sinh^2\left(\sqrt{\frac{2m(E-V_0)}{\hbar^2}}\,a\right) \rightarrow \lambda$$

$$= \frac{[E + (E-V_0)]^2}{4E(E-V_0)}\ \cancel{\sinh^2}(\quad)\cdot\left[\frac{e^{2\lambda} - e^{-2\lambda}}{4}\right]$$

$$=$$

Since Transmission is very less, we assume $\epsilon << V_0$.

So, $T = 16\,\epsilon(1-\epsilon)\,e^{-2\lambda\sqrt{1-\epsilon}} = \frac{1}{100}$

$$\lambda = a\sqrt{\frac{2mV_0}{\hbar^2}} = 0.2\times10^{-9}\times 2\pi\sqrt{\frac{2\times9.1\times10^{-31}\times6\times1.6\times10^{-19}}{(6.626\times10^{-34})^2}}$$

$$= \frac{0.4\pi\times\cancel{10^{-9}}\times13.218\times\cancel{10^{-25}}}{6.626\times\cancel{10^{-34}}}$$

$$= 2.506$$

$$\epsilon(1-\epsilon)\,e^{-5.013\sqrt{1-\epsilon}} = \frac{1}{1600}.$$

Solving this for ϵ.

$$\epsilon \simeq 0.0708 = \frac{E}{V_0}.$$

$$E = 6\,(0.0708)$$

$$\boxed{= 0.4248\ eV}$$

$-ye^{-y} - \int e^{-y}\,dy$

$e^{-y}(-y-1)$

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Q12. For a linear harmonic oscillator →

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \boxed{\frac{1}{2}kx^2}_{=V_0}\psi = E\psi$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{2m(E-V_0)}{\hbar^2}\psi = 0. \quad \text{where} \quad V_0 = \frac{1}{2}kx^2.$$

$$\psi_n = \left(\frac{2mv}{\hbar}\right)^{1/4}(2^n\,\lfloor n)^{-1/2}\,H_n(y)\,e^{-y^2/2}$$

$$E_n = (n+\tfrac{1}{2})\hbar\omega.$$

$$\langle x\rangle = \int_{-A}^{A} x\,|\psi^2|\,dx. \quad \text{for } n=0$$

$$\sqrt{\frac{2mv}{\hbar^2}}\;\cancel{}(0) \qquad A = \text{Amplitude}.$$

$$= \boxed{0}$$

for $n=1$

$$\langle x\rangle = \int_{-A}^{A} x\,|\psi|^2\,dx \qquad \psi_1 = \left(\frac{2mv}{\hbar}\right)^{1/4}\frac{1}{\sqrt{2}}\cdot 2\beta x\,e^{-\beta^2x^2/2}$$

$$= 4\sqrt{\frac{mv}{2\hbar}}\int x\,\beta^2x^2\,e^{-\beta^2x^2}\,dx.$$

$$\text{let} \quad \beta^2x^2 = y.$$

$$\beta^2(2x\,dx) = dy.$$

$$4\sqrt{\frac{mv}{2\hbar}}\int \frac{y\,e^{-y}\,dy}{2\beta^2} \qquad 2\sqrt{\frac{mv}{2\hbar}}\left[e^{-\beta^2x^2}(-\beta^2x^2-1)\right]_{-A}^{A}$$

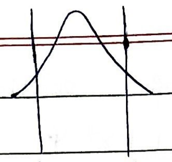
$$= \boxed{0}.$$

Q13. For a one dimensional harmonic oscillator :-

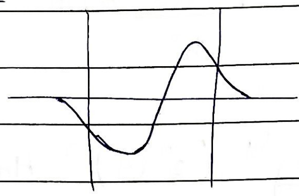
$$\psi_g = \text{ground state} = \left(\frac{2mv}{\hbar}\right)^{1/4} e^{-y^2/2}$$

$$\psi_1 = \left(\frac{2mv}{\hbar}\right)^{1/4}\frac{1}{\sqrt{2}}\,2y\,e^{-y^2/2}$$

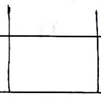
Graphically $\quad \Psi_0 =$



$\Psi_1 =$

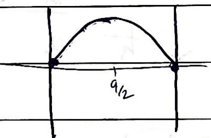


For a particle trapped in a 1-D box :-

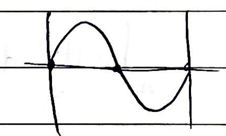


$$\Psi = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

Ψ_g = Ground state =



1st excited state Ψ_1 =



The wave are different as for a particle trapped in 1-D box, it cannot go beyond the box in classically forbidden areas but a harmonic oscillator can go there.

Q15.

Q14. $\Psi_1 = \left(\frac{8m\nu}{\hbar}\right)^{1/4} y e^{-y^2/2} \qquad y = \sqrt{\frac{2\pi m\nu}{\hbar}}\, x = \beta x.$

$$P = \int_0^\infty |\Psi_1|^2 dx = \int_0^\infty \sqrt{\frac{8m\nu}{\hbar}}\, \beta^2 x^2 e^{-\beta^2 x^2/2} dx$$

let $\beta^2 x^2 = K$

$\beta^2 (2x\,dx) = dK$

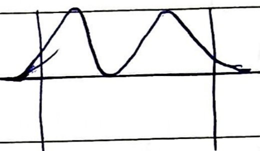
$$\cancel{\beta} \sqrt{\frac{2\cancel{8}m\nu}{\hbar}} \int_0^\infty \frac{e^{-K/2}\, dK}{\cancel{2}\beta^{\cancel{2}}} = \frac{1}{\beta}\sqrt{\frac{2m\nu}{\hbar}} \left(-2e^{-K/2}\right)_0^\infty$$

$$= \frac{2}{\beta}\sqrt{\frac{2m\nu}{\hbar}}$$

$$= 2\sqrt{\frac{\cancel{\hbar}}{2\pi \cancel{m\nu}} \cdot \frac{\cancel{2m\nu}}{\cancel{\hbar}}}$$

$$= \boxed{\frac{2}{\sqrt{\pi}}}$$

$|\psi_1|^2 =$



Q15. $E_n = \left(n+\frac{1}{2}\right)\hbar\omega$

$$V(x) \rightarrow \begin{cases} \frac{1}{2} m\omega^2 x^2 & \text{for } x > 0 \\ \infty & x < 0 . \end{cases}$$

Eigen value equation =

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \frac{1}{2}kx^2\psi = E\psi$$

let $\alpha = \frac{2E}{\hbar\omega} = \frac{2E}{h\nu}$

let $y = \beta x$

$$\frac{d^2\psi}{dy^2} + (\alpha - y^2)\psi = 0 .$$

So basically ψ for all negative values should become 0 in this case so only the states where ~~energy~~ ψ has an odd parity should be allowed. So $n = 1, 3, 5, 7 \ldots$ will be allowed states & $E_n = \left(n+\frac{1}{2}\right)\hbar\omega$.

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Q16. mass = m

energy = $\frac{1}{2}\hbar\omega$

$V = \frac{1}{2} m\omega^2 x^2$

- This is the case of a harmonic oscillator in its ground state.

wave function $\psi = \left(\frac{2m\nu}{\hbar}\right)^{1/4} e^{-y^2/2}$ $y = \sqrt{\frac{2\pi m\nu}{\hbar}}\, x$

Probability $= |\psi^2|$

$= \sqrt{\frac{2m\nu}{\hbar}}\, e^{-y^2}$

For mean position

probability $= \sqrt{\frac{2m\nu}{\hbar}}$

Q17.

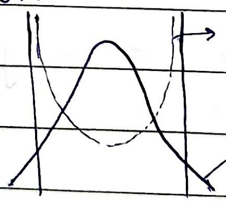
For extreme position,

probability $= \sqrt{\frac{2m\nu}{\hbar}}\, e^{-\frac{2\pi m\nu}{\hbar}A}$

Relative probability $= e^{-\frac{2\pi m\nu}{\hbar}A}$

Probability distribution →

Q18.



$\langle x \rangle = \int_{-A}^{A} x|\psi^2|\, dx = \int_{-A}^{A} x \sqrt{\frac{2m\nu}{\hbar}}\, e^{-\beta^2 x^2}\, dx.$

$\sqrt{\frac{2m\nu}{\hbar}} \left[\frac{e^{-\beta^2 x^2}}{2\beta^2}\right]_{-A}^{A}$

$= \boxed{0}$

$\langle V \rangle = \int_{-A}^{A} \frac{1}{2} m\omega^2 x^2 |\psi^2|\, dx$

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$$= \frac{1}{2} m\omega^2 \int_{-A}^{A} x^2 \sqrt{\frac{2m\upsilon}{\hbar}}\, e^{-\beta^2 x^2}\, dx$$

$$= \frac{1}{2} m\omega^2 \sqrt{\frac{2m\upsilon}{\hbar}} \int x^2 e^{-\beta^2 x^2} = \frac{1}{2} m\omega^2 \langle x^2 \rangle_n .$$

$$\langle x^2 \rangle_n = \frac{\hbar}{2m\omega}(2n+1)$$

$$\langle V \rangle_n = \frac{\hbar\omega}{4}(2n+1) = \boxed{\frac{E_n}{2}}$$

Q17. $\Psi(x,0) = A\left[\psi_1(x) + \psi_2(x)\right]$

For a particle trapped inside a box $\rightarrow$

$V \Rightarrow V_0 \quad 0 \leq x \leq a$

$\hookrightarrow \infty \quad$ otherwise.

$$\psi = A\sin kx + B\cos kx .$$

$$\psi = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

Q18. Same as Q13. Already solved before.