Lecture-31

28/10/2023.

$\langle \hat{x} p_x - p_x \hat{x} \rangle$ = commutation relation b/w x & p_x $\Rightarrow [x\ p_x]$

$$\langle \hat{x}\hat{p}_x - \hat{p}_x\hat{x} \rangle = \int \psi^* (x p_x - p_x x) \psi \, dx$$

$$= -i\hbar \int \psi^* \left[x \frac{\partial \psi}{\partial x} - \frac{\partial}{\partial x} \underset{\mathrm{I}\ \mathrm{II}}{x\psi} \right] dx$$

$$\boxed{\hat{p}_x = -i\hbar \frac{\partial}{\partial x}}$$

$$= -i\hbar \int \left(\cancel{\psi^* x \frac{\partial \psi}{\partial x}} - \cancel{\psi^* x \frac{\partial \psi}{\partial x}} - \psi^* \psi \frac{\partial x}{\partial x} \right) dx$$

$$= +i\hbar \int \psi^* \psi \, dx$$

$$= i\hbar$$

Conservation of Probability :⇒

* Total probability must be conserved.

$$\frac{\partial}{\partial t} \int \psi^* \psi \, d\tau = 0$$ [integration over all space].

Let say ρ is probability density then probility of finding the particle within a finite volume V -

$$\frac{\partial}{\partial t} \int \rho \, dV = \frac{\partial}{\partial t} \int (\psi^* \psi) \, dV$$

$$= \int \left(\psi^* \frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t} \psi \right) dV \quad \text{——— ①}$$

S.c. eqn :⇒

$$i\hbar \frac{\partial \psi}{\partial t} = \left(-\frac{\hbar^2 \nabla^2}{2m} + V \right) \psi \quad \text{——— ⓐ} \quad \times \psi^*$$

$$\therefore \quad -i\hbar \frac{\partial \psi^*}{\partial t} = \left(-\frac{\hbar^2 \nabla^2}{2m} + V \right) \psi^* \quad \text{——— ⓑ} \quad \times \psi$$

substract.

$$i\hbar\left[\psi^*\frac{\partial\psi}{\partial t}+\frac{\partial\psi^*}{\partial t}\psi\right]=-\frac{\hbar^2}{2m}\psi^*\nabla^2\psi+\cancel{\psi^*V\psi}+\frac{\hbar^2}{2m}\psi\nabla^2\psi^*-\cancel{\psi V\psi^*}$$

$$\psi^*\frac{\partial\psi}{\partial t}+\frac{\partial\psi^*}{\partial t}\psi=-\frac{\hbar}{2mi}\left[\psi^*\nabla^2\psi-\psi\nabla^2\psi^*\right]$$

Put in ①-

$$\frac{\partial}{\partial t}\int\rho\,dV=-\frac{\hbar}{2mi}\int\left(\psi^*\nabla^2\psi-\psi\nabla^2\psi^*\right)dV$$

$$\frac{\partial}{\partial t}\int\rho\,dV=-\frac{\hbar}{2mi}\int\nabla\cdot\left(\psi^*\nabla\psi-\psi\nabla\psi^*\right)dV$$

Let us define.

$$\boxed{J=\frac{\hbar}{2mi}\left(\psi^*\nabla\psi-\psi\nabla\psi^*\right)}$$

$$\frac{\partial}{\partial t}\int\rho\,dV=-\int\nabla\cdot J\,dV$$

$$\boxed{\nabla\cdot J+\frac{\partial\rho}{\partial t}=0}$$

$\rho\rightarrow$ Probability density.

$J\rightarrow$ Probability current density

EMT

$\nabla\cdot J+\frac{\partial\rho}{\partial t}=0$

$\rho\rightarrow$ Charge density

$J\rightarrow$ current density

$\Rightarrow$ Chare is conserved

$\Rightarrow$ Any increase or decrease in the probability for finding the particle in a given volume is compensated by a corresponding decrease or increase elsewhere.

$\Rightarrow$ Conservation of total probility implies that normalization is time independent.

For a microscopic particle in one-dimensional time-independent pot.

Sc. eq^n -

$$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x).$$

Sol^n $$\psi(x,t) = \psi(x)\, e^{-iEt/\hbar}$$

probability density does not depend on time.

To solve this differential eq^n we need to specify the pot. as well as boundary cond^n.

§ Boundary cond^n can be obtained from the physical requirement of the system

* Various one dimentional problem ⇒

1 → free particle.

2 → Particle in a box.

3 → Finite Potential well,

4 → Potential step ⇒ Tunneling effect.

Free particle ⇒ $V(x) = 0$

$$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

$$\Rightarrow \left(\frac{d^2}{dx^2} + \frac{2mE}{\hbar^2}\right)\psi(x) = 0$$

$$\Rightarrow \left(\frac{d^2}{dx^2} + k^2\right)\psi = 0$$

$\frac{2mE}{\hbar^2} = k^2.$ ↓ wave no.

Sol^n ⇒

$$\psi(x) = A_+ e^{ikx} + A_- e^{-ikx}.$$

$$\psi(x,t) = A_+ e^{i(kx-\omega t)} + A_- e^{-i(kx-\omega t)}.$$