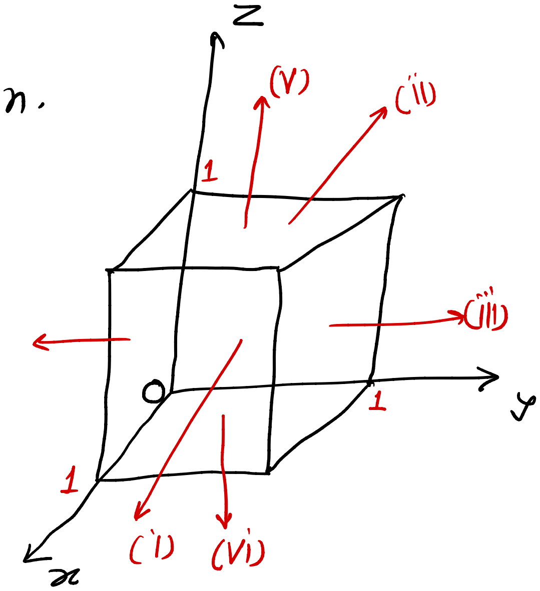
Lecture- 06

Example ⇒ Check the divergence theorem using the fun-

$$\bar{v} = y^2\hat{x} + (2xy+z^2)\hat{y} + (2yz)\hat{z}$$

& the unit cube situated at the origin.



Ans ⇒ $\int v \cdot ds = \int \nabla \cdot v \, d\tau$

$$\nabla \cdot v = \frac{\partial}{\partial x} v_x + \frac{\partial}{\partial y} v_y + \frac{\partial}{\partial z} v_z$$

$$= 0 + 2x + 2y$$

$$\int_V \nabla \cdot v \, d\tau = 2\int_0^1\int_0^1\int_0^1 (x+y)\, dx\, dy\, dz$$

$$= 2\int_0^1\int_0^1 \frac{x^2}{2} + yx \Big|_0^1 dy\, dz$$

$$= 2\int_0^1\int_0^1 \left(\frac{1}{2} + y\right) dy\, dz$$

$$= 2\int_0^1 \frac{1}{2}y + \frac{y^2}{2}\Big|_0^1 dz = 2\int_0^1 1\, dz = 2$$

Now Let us evaluate the surface integral ⇒

For side (1) ⇒ $\int v \cdot da = \int_0^1\int_0^1 y^2\, dy\, dz = \frac{1}{3}$

$\left\{ \begin{matrix} da = dy\, dz\, \hat{x} \\ x = 1 \end{matrix} \right\}$

For side (2) ⇒ $\int v \cdot da = -\int_0^1\int_0^1 y^2\, dy\, dz = -\frac{1}{3}$

$\left\{ \begin{matrix} da = -dy\, dz\, \hat{x} \\ x = 0 \end{matrix} \right\}$

For side (3) ⇒ $\int v \cdot da = \int_0^1\int_0^1 (2x+z^2)\, dx\, dz = 4/3$

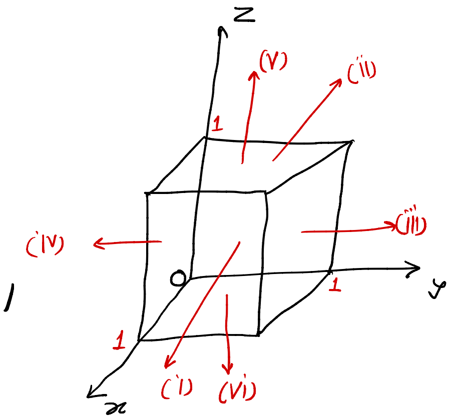
$\left\{ \begin{matrix} da = dx\, dz\, \hat{y} \\ y = 1 \end{matrix} \right\}$

For side (4) ⇒

$da = -dx\,dz\,\hat{y}$

$y = 0$

$$\int v \cdot da = -\int_0^1\int_0^1 z^2\,dx\,dz = -\frac{1}{3}$$



For side (5) ⇒ $da = dx\,dy\,\hat{z}$ $\therefore \int v \cdot da = \int_0^1\int_0^1 2y\,dx\,dy = 1$

$z = 1$

For side vi) ⇒ $da = -dx\,dy\,\hat{z}$ $\quad \int v \cdot da = -\int_0^1\int_0^1 0\,dx\,dy = 0$

$z = 0$

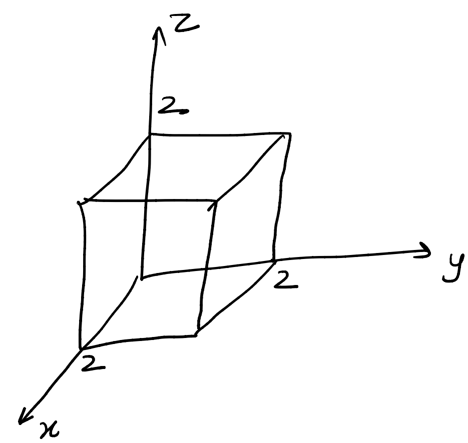
$$\text{Total flux} = \frac{1}{3} - \frac{1}{3} + \frac{4}{3} - \frac{1}{3} + 1 + 0 = 2$$ [As expected].

Homework.

Q ⇒ Test the divergence theorem for the fun

$\bar{v} = (xy)\hat{x} + (2yz)\hat{y} + (3zx)\hat{z}$.

Take the cuble with sides of length 2.

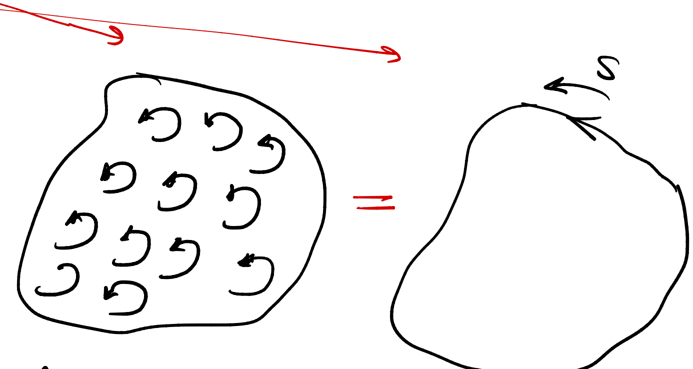


Stoke's Theorem ⇒

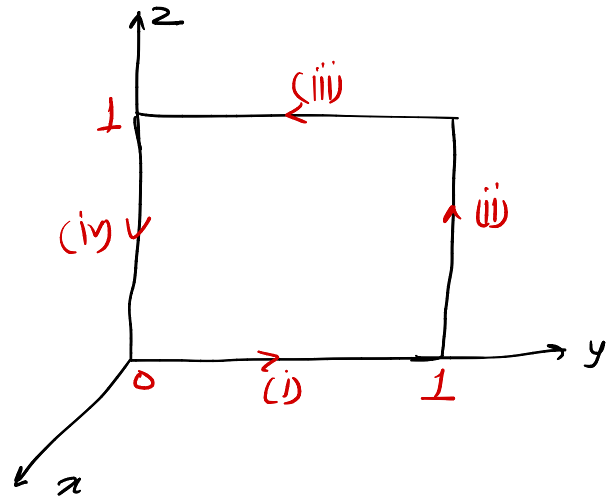
$$\text{Curl } \bar{A} = \nabla \times A = \lim_{ds \to 0} \frac{\oint A \cdot dl}{dS}\,\hat{a}_n$$

$$\boxed{\oint_C \bar{A} \cdot dl = \int_S (\nabla \times A) \cdot ds}$$

* Line integral of a vector field over a loop is equal to the surface integral of the curl of the field enclosed by the loop.



Ex1 ⇒ Suppose $\bar{v} = (2xz + 3y^2)\hat{y} + (4yz^2)\hat{z}$. Check Stoke's theorem for the following square surface.



Ans $\Rightarrow$ $\oint A \cdot dl = \iint_S (\nabla \times A) \cdot ds$

$$\nabla \times A = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 2xz+3y^2 & 4yz^2 \end{vmatrix}$$

$$= \hat{x}(4z^2 - 2x) + \hat{y} \cdot 0 + \hat{z}(2z)$$

$$= (4z^2 - 2x)\hat{x} + 2z\hat{z}$$

$ds = dy\,dz\,\hat{x}$, & $x = 0$

$$\therefore \int (\nabla \times A) \cdot ds = \int_0^1 \int_0^1 (4z^2)\,dy\,dz = 4/3$$

Now the line integral $\Rightarrow$

Segment 1 $\Rightarrow$ $x=0$, $z=0$, $dl = dy\,\hat{y}$

$$\therefore \int v \cdot dl = \int_0^1 3y^2\,dy = 1$$

segment 2 $\Rightarrow$ $x=0$, $y=1$, $dl = dz\,\hat{z}$

$$\therefore \int v \cdot dl = \int_0^1 4z^2\,dz = 4/3$$

segment 3 $\Rightarrow$ $x=0$, $z=1$, $dl = -dy\,\hat{y}$

$$\therefore \int v \cdot dl = -\int_0^1 3y^2\,dy = -1$$

segment 4 $\Rightarrow$ $x=0$, $y=0$, $dl = -dz\,\hat{z}$

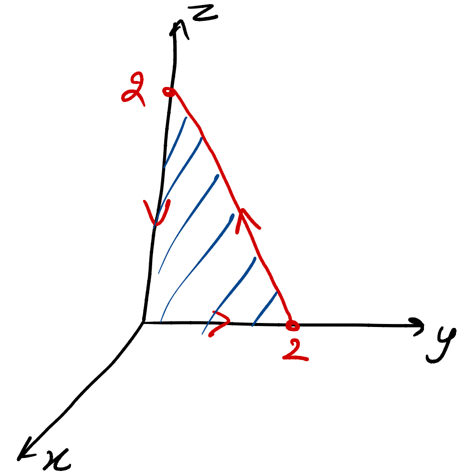
$$\therefore \int v \cdot dl = -\int 0\,dz = 0$$

$$\therefore \oint v \cdot dl = 1 + 4/3 - 1 + 0 = 4/3$$

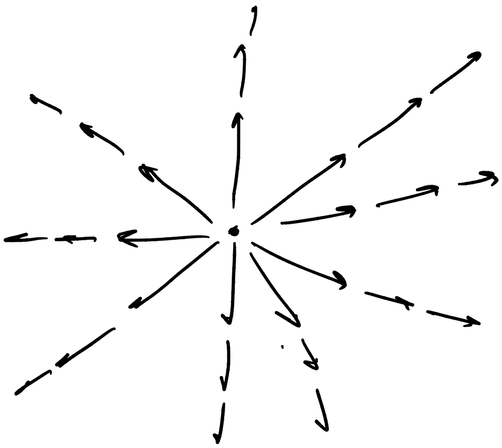
Ans.

Homework

Q $\Rightarrow$ Test Stoke's theorem for the funⁿ $v = (xy)\hat{x} + (2yz)\hat{y} + 3zx\hat{z}$, using the following triangular shaded area.



Consider a vector field $\bar{v} = \frac{1}{r^2}\hat{r}$



Gauss div. theorem $\Rightarrow$

$$\int v \cdot ds = \int \nabla \cdot v \, d\tau$$

LHS $\Rightarrow$ $\int v \cdot ds = \int \frac{1}{\cancel{r^2}} \hat{r} \cdot \cancel{r^2} \sin\theta \, d\theta \, d\phi \, \hat{r}$

$$= \int_0^{\pi} \int_0^{2\pi} \sin\theta \, d\theta \, d\phi$$

$$= 4\pi$$

RHS $\Rightarrow$ $\nabla \cdot v = \frac{1}{r^2} \frac{\partial}{\partial r} \left(\cancel{r^2} \frac{1}{\cancel{r^2}} \right)$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} (1).$$

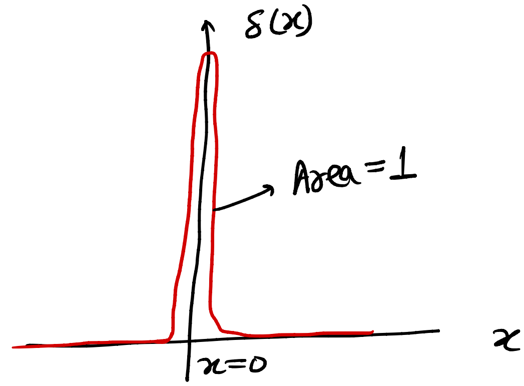
$$= 0$$

Dirac Delta funⁿ ⇒ The One-D, Dirac Delta funⁿ ($\delta(x)$) can be pictured as an infinitely high, infinitesimally narrow spike with area 1.

$$\delta(x) = \begin{cases} 0 & , \text{ if } x \neq 0 \\ \infty & , \text{ if } x = 0 \end{cases}$$

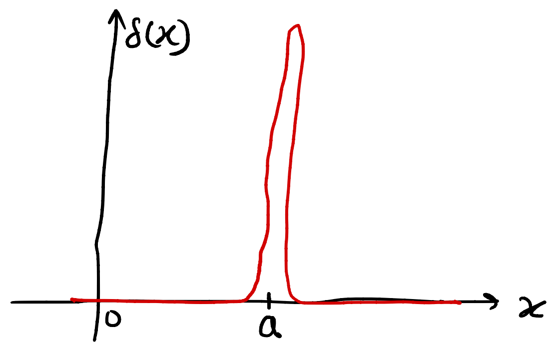
&

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$



* If we shift the spike from $x=0$ to some other point say $x=a$.

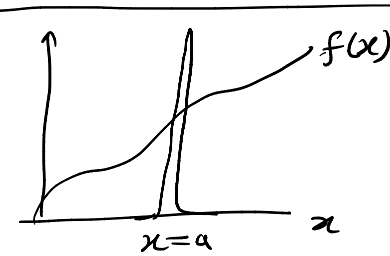
$$\boxed{\delta(x-a) = \begin{cases} 0 & \text{if } x \neq a \\ \infty & \text{if } x = a \end{cases}}$$



&

$$\boxed{\int_{-\infty}^{\infty} \delta(x-a) = 1}$$

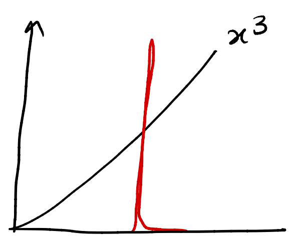
$$f(x)\,\delta(x-a) = f(a)\,\delta(x-a)$$



* $$\boxed{\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\, dx = f(a)}$$

Ex ⇒ $\int_0^3 x^3 \delta(x-2)\, dx$

$= 2^3 = 8$ Ans.



* $\int_0^1 x^3 \delta(x-2)\, dx = 0$ Ans.