

Look at the other part:

Time-Independent Schrödinger equation (TISE)

$$\left[-\frac{\hbar^2}{2m} \vec{\nabla}^2 + \hat{V}(\vec{r}) \right] \Psi(\vec{r}) = E \Psi(\vec{r})$$

TISE is an eigenvalue problem

- Eigenvalue equations are of the type

$$\hat{A}\Psi_n = a_n \Psi_n$$

- \hat{A} is an operator and a_n is a number.
- A solution Ψ_n of such an equation is called an eigenfunction corresponding to the eigenvalue a_n of the operator \hat{A} .**
- Operator \hat{A} acting on certain function (the eigenfunction) will give back their function multiplied by constants a_n .

The total wavefunction :

$$\Psi(\vec{r}, t) = \Psi(\vec{r}) e^{-i\frac{Et}{\hbar}}$$

$$\hat{H}\Psi(\vec{r}) = E\Psi(\vec{r}) \quad (\text{TISE})$$

Probability density :

$$\begin{aligned} \rho(\vec{r}, t) &= \Psi^*(\vec{r}, t) \Psi(\vec{r}, t) \\ &= \Psi^*(\vec{r}) \Psi(\vec{r}) e^{-i\frac{(E-E^*)t}{\hbar}} \end{aligned}$$

$$\int \rho(\vec{r}, t) dV = 1 \quad \frac{\partial}{\partial t} \int \rho(\vec{r}, t) dV = 0$$

$$\Rightarrow -i \frac{(E - E^*)}{\hbar} e^{-i\frac{(E-E^*)t}{\hbar}} \int \Psi^*(\vec{r}) \Psi(\vec{r}) dV = 0$$

The integral $\int \Psi^*(\vec{r}) \Psi(\vec{r}) dV = 1$

Valid for all time.

$$E = E^* \Rightarrow E \text{ is real}$$

$$E = E^* \Rightarrow E \text{ is real}$$

So, what happens to the probability density?

$$\rho(\vec{r}, t) = \Psi^*(\vec{r}, t) \Psi(\vec{r}, t) = \Psi^*(\vec{r}) \Psi(\vec{r}) e^{-i\frac{(E-E^*)t}{\hbar}}$$

$$\rho(\vec{r}, t) = \rho(\vec{r}) = \Psi^*(\vec{r}) \Psi(\vec{r}) = |\Psi(\vec{r})|^2$$

As **E is real**, the **probability density is independent of time**

& then $\Psi(\vec{r}, t) = \Psi(\vec{r}) e^{-i\frac{Et}{\hbar}}$ is called **stationary state**

There is always a set of solutions for Equation $\hat{H}\psi(x) = E\psi(x)$

thus

$$\Psi_n(x, t) = \psi_n(x) \exp(-iE_n t / \hbar)$$

The probability density is

$$\begin{aligned} \Psi_n^*(x, t) \Psi_n(x, t) dx &= \psi_n^*(x) \exp(iE_n t / \hbar) \psi_n(x) \exp(-iE_n t / \hbar) dx \\ &= \psi_n^*(x) \psi_n(x) \end{aligned}$$

The average is

$$\langle \hat{H} \rangle = \frac{\int_{-\infty}^{\infty} \psi_n^*(x) \hat{H} \psi_n(x) dx}{\int_{-\infty}^{\infty} \psi_n^*(x) \psi_n(x) dx}$$

- Because the probability density and the average are independent of time, the $\psi_n(x)$ are known as stationary-state wave functions.
- The stationary energy states of an atom or molecule are obtained by the solving the Schrödinger equation in 1-, 2- and 3-dimensions.

Dirac Brackets

In Dirac notation, integrals are written as:

$$\int \psi^* \hat{Q} \psi \, d\tau = \langle \psi^* | \hat{Q} | \psi \rangle$$

$$\int \psi^* \psi \, d\tau = \langle \psi^* | \psi \rangle$$

$|\psi\rangle$ is called the **ket**, and denotes the state described by the function ψ .

$\langle \psi^* |$ is called the **bra**, and denotes the complex conjugate of the function ψ^* .

Theorems of Quantum Mechanics

❖ **The eigenvalues of Hermitian operators are real (Can we prove it?)**

Suppose \hat{Q} is a **Hermitian operator** with a square-integrable eigenfunction $\psi(x)$:

$$\hat{Q}\psi = \alpha\psi$$

Expressing each side as a real and an imaginary part, keeping in mind that the real parts must be equal to each other and likewise the imaginary parts, gives:

$$\hat{Q}^* \psi^* = \alpha^* \psi^*$$

Multiplying ψ^* from the left of the Equation $\hat{Q}\psi = \alpha\psi$ followed by integration gives

$$\langle \psi^* | \hat{Q} | \psi \rangle = \alpha \langle \psi^* | \psi \rangle$$

Also, multiplying ψ from the left of the Equation $\hat{Q}^* \psi^* = \alpha^* \psi^*$ followed by integration gives:

$$\langle \psi | \hat{Q}^* | \psi^* \rangle = \alpha^* \langle \psi | \psi^* \rangle$$

Since the operator is Hermitian, the left-hand sides of the above two Equations are equal, and thus their right-hand sides are equal, and their difference is zero:

$$(\alpha - \alpha^*) \langle \psi | \psi^* \rangle = 0$$

Remember that Hermitian operator obey the following condition

$$\int_{-\infty}^{\infty} \psi_i^* \hat{Q} \psi_j \, d\tau = \int_{-\infty}^{\infty} \psi_i \hat{Q}^* \psi_j^* \, d\tau$$

Since ψ is square-integrable, the integral $\langle \psi | \psi^* \rangle \neq 0$
 & thus $(\alpha - \alpha^*) = 0 \rightarrow$ which requires that α be real.

❖ Orthogonality Theorem

Eigenfunctions corresponding to different eigenvalues for the same Hermitian operator are orthogonal.

Let ψ_i and ψ_j represent two eigenfunctions corresponding to two different eigenvalues, α_i and α_j , respectively, for the same Hermitian operator $\hat{\Omega}$:

$$\hat{\Omega} \psi_i = \alpha_i \psi_i$$

$$\hat{\Omega}^* \psi_j^* = \alpha_j \psi_j^*$$

$$\langle \psi_j^* | \hat{\Omega} | \psi_i \rangle = \alpha_i \langle \psi_j^* | \psi_i \rangle$$

$$\langle \psi_i | \hat{\Omega}^* | \psi_j^* \rangle = \alpha_j \langle \psi_i | \psi_j^* \rangle$$

Multiplying ψ_j^* from the left followed by integration

Multiplying ψ_i from the left followed by integration

Note: the LHSs are equal due the property of Hermitian operator

So, the difference of their RHSs gives:

So, the difference of their RHSs gives:

$$(\alpha_i - \alpha_j) \langle \psi_j^* | \psi_i \rangle = 0$$

The integral in the Equation above vanishes when $\alpha_i \neq \alpha_j \rightarrow$

$$\langle \psi_j^* | \psi_i \rangle = 0 \rightarrow \text{Hence, they are orthogonal}$$

Please refer to the following:

$$\int_{-\infty}^{\infty} \psi_i^* \psi_j d\tau = \delta_{ij} \text{ (Kronecker delta)}$$

$$\delta_{ij} = \begin{cases} 0 & \text{when } i \neq j \text{ (orthogonal)} \\ 1 & \text{when } i = j \text{ (normal)} \end{cases}$$

❖ **Commuting operators have simultaneous eigenfunctions**

Let \hat{A} and \hat{B} represent two different operators and f represents an arbitrary square-integrable function.

The operators are said to commute when

$$[\hat{A}, \hat{B}]f = \hat{A}\hat{B}f = \hat{B}\hat{A}f$$

$$(\hat{A}\hat{B} - \hat{B}\hat{A})f = 0 \quad \Rightarrow \quad \hat{A}\hat{B} - \hat{B}\hat{A} = \hat{0}$$

$\hat{0}$ is known as the null operator if $\hat{0}f = 0$.

The difference of the product of operators is called the commutator.

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

$[\hat{A}, \hat{B}] = 0 \quad \longrightarrow \quad$ The **operators commute** \rightarrow the corresponding observables can be measured simultaneously with arbitrary precision

$[\hat{A}, \hat{B}] \neq 0 \quad \longrightarrow \quad$ **Operators do not commute**
Operators are incompatible,
 \rightarrow leads to the **Heisenberg's uncertainty principle**.

Let ψ_i be the eigenfunctions for \hat{B} so that: $\hat{B}\psi_i = b_i\psi_i$

where all the b_i are different, meaning that the eigenfunctions are nondegenerate.

When $[\hat{A}, \hat{B}] = \hat{0}$: $\hat{B}(\hat{A}\psi_i) = \hat{A}\hat{B}\psi_i = \hat{A}b_i\psi_i = b_i(\hat{A}\psi_i)$



The bracket is meant to stress that the function obtained by operating on ψ_i with \hat{A} is an eigenfunction of \hat{B} with eigenvalue b_i

However, the function can only be a constant multiplied by ψ_i itself and therefore for nondegenerate ψ_i :

$$\hat{A}\psi_i = c\psi_i$$

Commuting operators have simultaneous eigenfunctions, meaning that a set of eigenfunctions can be found for one of the operators that is also an eigenfunction set for the other operator.

Commutators and Uncertainty

There are important consequences for the existence of simultaneous eigenfunctions for various operators, and this is embodied in the uncertainty principle.

Consider a particle moving towards positive x .

Let the wave function $\psi(x)$ of the particle be $N e^{ikx}$ with N being the normalisation factor. Where can we find the particle?

To answer this question, calculate the probability density:

$$|\psi|^2 = (N e^{-ikx}) \times (N e^{ikx}) = N^2 (e^{-ikx})(e^{ikx}) = N^2$$

- ✓ Since the probability density is independent of x , there is an equal probability of finding the particle anywhere on the x -axis.
- ✓ In other words, the **position of the particle can not be predicted**.
- ✓ Put it in a different way: **knowing the linear momentum precisely makes it impossible to know anything about the position.**

Find $\langle \hat{x}\hat{p}_x - \hat{p}_x\hat{x} \rangle$

DO YOU REMEMBER THIS?

$$\begin{aligned} \langle \hat{x}\hat{p}_x - \hat{p}_x\hat{x} \rangle &= \int_{-\infty}^{+\infty} \Psi^*(\vec{r}, t) (\hat{x}\hat{p}_x - \hat{p}_x\hat{x}) \Psi(\vec{r}, t) \\ &= -i\hbar \int_{-\infty}^{+\infty} \Psi^*(\vec{r}, t) \left[\frac{x\partial\Psi}{\partial x} - \frac{\partial(x\Psi)}{\partial x} \right] dx \\ &= -i\hbar \int_{-\infty}^{+\infty} \Psi^*(\vec{r}, t) \left[\frac{x\partial}{\partial x} - \psi - x \frac{\partial(\Psi)}{\partial x} \right] dx \\ &= i\hbar \int_{-\infty}^{+\infty} \Psi^* \Psi dx \\ &= i\hbar \end{aligned}$$

$$[\hat{x}, \hat{p}_x] = i\hbar$$

Position and momentum are thus complementary observables

Time and energy are also complementary

$$[\hat{E}, \hat{t}] = ?$$

$$[\hat{t}, \hat{E}] = ?$$

$$[\hat{E}, \hat{t}]\psi(t) = (\hat{E}\hat{t} - \hat{t}\hat{E})\psi(t)$$

$$\hat{E}\hat{t}\psi(t) = i\hbar \frac{\partial}{\partial t} (t\psi(t)) = i\hbar(\psi(t) + t \frac{\partial \psi}{\partial t})$$

$$\hat{t}\hat{E}\psi(t) = t(i\hbar \frac{\partial \psi}{\partial t})$$

$$[\hat{E}, \hat{t}]\psi(t) = (\hat{E}\hat{t} - \hat{t}\hat{E})\psi(t) = i\hbar(\psi(t) + t \frac{\partial \psi}{\partial t}) - i\hbar(t \frac{\partial \psi}{\partial t}) = i\hbar\psi(t)$$

$$[\hat{E}, \hat{t}] = i\hbar$$

- A **non-zero constant**
- implies that the **operators do not commute**
- The corresponding observables (**energy and time**) **cannot be known with precision at the same time.**

This is the basis for Heisenberg's uncertainty principle!!

Heisenberg's uncertainty principle

The principle says it is impossible to specify simultaneously, with precision, both the momentum and the position of a particle.

$$\Delta x \Delta p \geq \frac{1}{2} \hbar$$

The **standard deviation** in these quantities, **position and momentum**, corresponding to the **operators**, represents the uncertainty in the observed values of the physical quantities:

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

standard deviation
in position

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

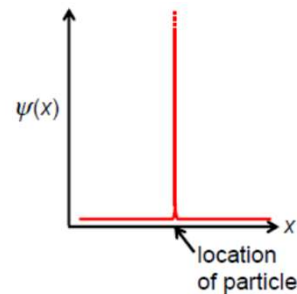
standard deviation
in momentum

Note that for any operator \hat{A} , the standard deviation ΔA of the measurement from the mean value is

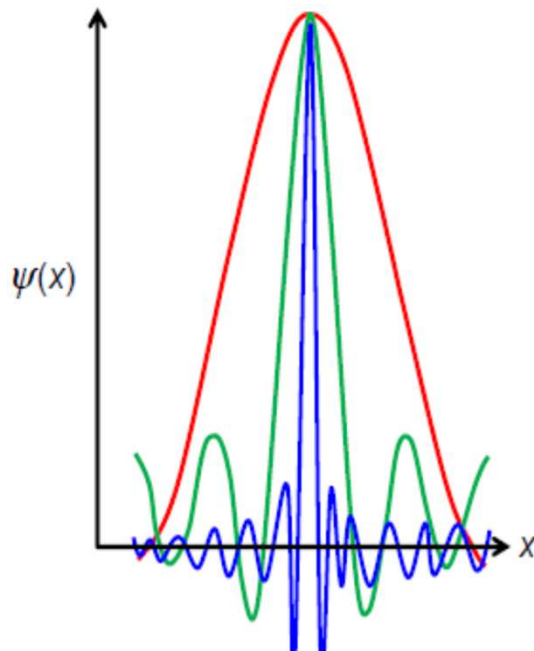
$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2}$$

where $\langle A^2 \rangle = \int \psi^* \hat{A}^2 \psi \, d\tau$ $\langle A \rangle^2 = \left(\int \psi^* \hat{A} \psi \, d\tau \right)^2$

If a particle is at a definite location, its wave function is large there and zero everywhere else. Such a wavefunction can be created by superimposing a large number of functions. Such a sharply localised wavefunction is known as a *wavepacket*.



wave function for a particle at a well-defined location



wavefunction for a particle with an ill-defined location

The superimposed wavefunctions interfere constructively in one place but destructively elsewhere

Applications of Quantum Mechanics

Particle in a Box - 1D

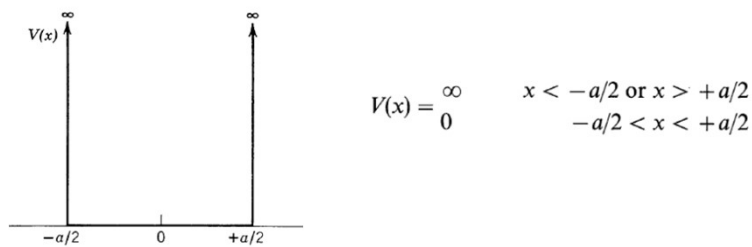
It tells us how boundary conditions and normalization determine wave functions

THE INFINITE SQUARE WELL POTENTIAL

A particle trapped in a box with infinitely hard walls.

Let us now tackle the problem in a more formal way.

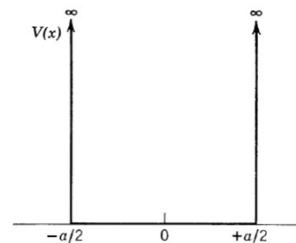
This will give us the wave function ψ_n that corresponds to each energy level.

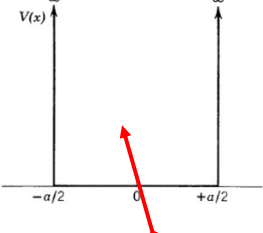


We specify the particle's motion

- ➔ It is restricted to traveling along the x-axis between $x = -a/2$ and $x = +a/2$ by infinitely hard walls.
- ➔ A particle does not lose energy when it collides with such walls, so that its total energy stays constant.
- ➔ The potential energy V of the particle is ∞ on both sides of the box
- ➔ while V is a constant — say 0 for convenience, inside
- ➔ Since the particle cannot have an infinite amount of energy, it cannot exist outside the box.
- ➔ Its wave function ψ is 0 for $x \leq -a/2$ and $x \geq +a/2$.

Our task is to find what ψ is within the box, between $x = -a/2$ and $x = +a/2$





$$V(x) = \begin{cases} \infty & x < -a/2 \text{ or } x > +a/2 \\ 0 & -a/2 < x < +a/2 \end{cases}$$

Time-Independent Schrödinger equation (TISE) for the particle of mass m

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + \hat{V}(\vec{r}) \right] \Psi(\vec{r}) = E \Psi(\vec{r})$$

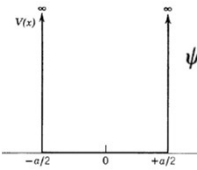
$$-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} = E \psi(x)$$

$$k = \frac{\sqrt{2mE}}{\hbar}$$

Constants

$$\psi(x) = A \sin kx + B \cos kx \quad \text{where } k = \frac{\sqrt{2mE}}{\hbar} \quad -a/2 \leq x \leq +a/2$$

Wavefunction inside the well



$$\psi(x) = A \sin kx + B \cos kx \quad \text{where } k = \frac{\sqrt{2mE}}{\hbar} \quad -a/2 \leq x \leq +a/2$$

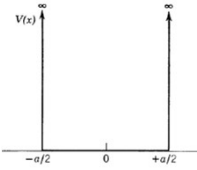
at the boundaries of the well $x = \pm a/2$ $\psi(x) = 0$

at $x = +a/2$, $A \sin \frac{ka}{2} + B \cos \frac{ka}{2} = 0$

At $x = -a/2$, $-A \sin \frac{ka}{2} + B \cos \frac{ka}{2} = 0$

Addition $2B \cos \frac{ka}{2} = 0$ Subtraction $2A \sin \frac{ka}{2} = 0$

Both must be satisfied



$\psi(x) = A \sin kx + B \cos kx$ where $k = \frac{\sqrt{2mE}}{\hbar}$ $-a/2 \leq x \leq +a/2$

at the boundaries of the well $x = \pm a/2$ $\psi(x) = 0$

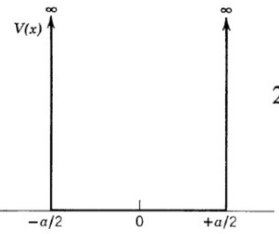
$$2B \cos \frac{ka}{2} = 0$$

$$2A \sin \frac{ka}{2} = 0$$

Both must be satisfied

There is no value of the parameter k for which both $\cos(ka/2)$ and $\sin(ka/2)$ are simultaneously zero. And we certainly do not want to satisfy the two equations by setting both A and B equal to zero, for then $\psi(x) = 0$ everywhere

$\psi(x) = A \sin kx + B \cos kx$ where $k = \frac{\sqrt{2mE}}{\hbar}$ $-a/2 \leq x \leq +a/2$



Boundary condition at $x = \pm a/2$ $\psi(x) = 0$

$$2B \cos \frac{ka}{2} = 0 \quad 2A \sin \frac{ka}{2} = 0$$

Both solutions must be satisfied

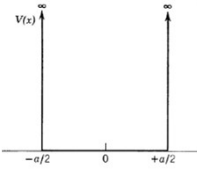
$A = 0 \quad \text{and} \quad \cos \frac{ka}{2} = 0$

$B = 0 \quad \text{and} \quad \sin \frac{ka}{2} = 0$

Thus there are two classes of solutions.

For the *first class*

$$\psi(x) = B \cos kx \quad \text{where} \quad \cos \frac{ka}{2} = 0$$



$\psi(x) = A \sin kx + B \cos kx$ where $k = \frac{\sqrt{2mE}}{\hbar}$ $-a/2 \leq x \leq +a/2$

$A = 0 \quad \text{and} \quad \cos \frac{ka}{2} = 0$

$B = 0 \quad \text{and} \quad \sin \frac{ka}{2} = 0$

For the *first class*

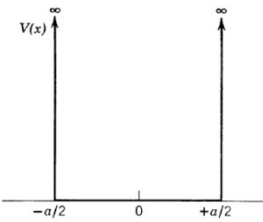
$\psi(x) = B \cos kx$ where $\cos \frac{ka}{2} = 0$

$$\frac{ka}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$

$$k_n = \frac{n\pi}{a} \quad n = 1, 3, 5, \dots$$

For the *second class*

$\psi(x) = A \sin kx$ where $\sin \frac{ka}{2} = 0$



For the *second class*

$\psi(x) = A \sin kx$ where $\sin \frac{ka}{2} = 0$

$$\frac{ka}{2} = \pi, 2\pi, 3\pi, \dots$$

$$k_n = \frac{n\pi}{a} \quad n = 2, 4, 6, \dots$$

Knowing the allowed values of k , we can then obtain the solutions to the time-independent Schroedinger equation for the infinite square well :

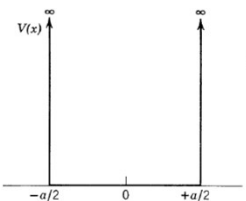
$$\psi_n(x) = B_n \cos k_n x \quad \text{where } k_n = \frac{n\pi}{a} \quad n = 1, 3, 5, \dots$$

$$\psi_n(x) = A_n \sin k_n x \quad \text{where } k_n = \frac{n\pi}{a} \quad n = 2, 4, 6, \dots$$

The solution corresponding to $n = 0$ is $\psi_0(x) = A \sin 0 = 0$; it is ruled out because it does not describe a particle in a box.

A_n and B_n : from normalization of the wavefunction (Exercise)

Remember:



$$\psi(x) = A \sin kx + B \cos kx \quad \text{where } k = \frac{\sqrt{2mE}}{\hbar} \quad -a/2 \leq x \leq +a/2$$

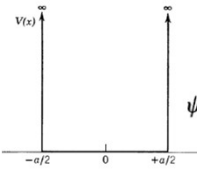
Energy Eigenvalues

for the allowed values of k , we find

$$E_n = \frac{\hbar^2 k_n^2}{2m}$$

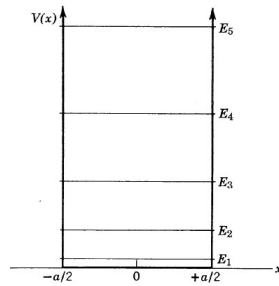
$$= \frac{\pi^2 \hbar^2 n^2}{2ma^2} \quad n = 1, 2, 3, 4, 5, \dots$$

only certain values of the total energy E are allowed. The total energy of the particle in the box is *quantized*.



$$\psi(x) = A \sin kx + B \cos kx \quad \text{where } k = \frac{\sqrt{2mE}}{\hbar} \quad -a/2 \leq x \leq +a/2$$

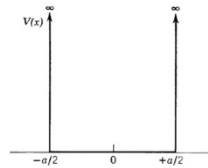
Energy Eigenvalues

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\pi^2 \hbar^2 n^2}{2ma^2} \quad n = 1, 2, 3, 4, 5, \dots$$


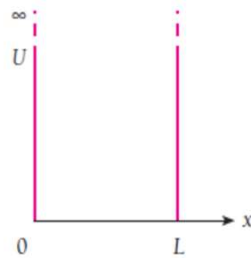
$$E_1 = \frac{\pi^2 \hbar^2}{2ma^2} \quad (\text{non zero})$$

The first few eigenvalues of an infinite square well potential.

THE INFINITE SQUARE WELL POTENTIAL (Particle in a box)



$$V(x) = \begin{cases} \infty & x < -a/2 \text{ or } x > +a/2 \\ 0 & -a/2 < x < +a/2 \end{cases}$$



$$U(x) = \begin{cases} 0, & 0 < x < L \\ \infty, & \text{elsewhere} \end{cases}$$

wave function ψ is 0 for $x \leq 0$ and $x \geq L$.

Our task is to find what ψ is within the box, between $x = 0$ and $x = L$



$$U(x) = \begin{cases} 0, & 0 < x < L \\ \infty, & \text{elsewhere} \end{cases}$$

wave function ψ is 0 for $x \leq 0$ and $x \geq L$.


Time Independent Schrödinger Equation
(for particle in the box)

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}E\psi = 0$$

$$\psi = A \sin \frac{\sqrt{2mE}}{\hbar} x + B \cos \frac{\sqrt{2mE}}{\hbar} x$$

Boundary condition : $\psi = 0$ for $x = 0$ \rightarrow $B = 0$

$$\psi = A \sin \frac{\sqrt{2mE}}{\hbar} x$$



$U(x) = 0, \quad 0 < x < L$
 $= \infty, \quad \text{elsewhere}$

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} E\psi = 0$$

$\Rightarrow \psi = A \sin \frac{\sqrt{2mE}}{\hbar} x$

Boundary condition : ψ will be 0 at $x = L$

$\Rightarrow \frac{\sqrt{2mE}}{\hbar} L = n\pi \quad n = 1, 2, 3, \dots$

Energy Eigenvalues : $E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2} \quad n = 1, 2, 3, \dots$

$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2} \quad n = 1, 2, 3, \dots$

wave functions of a particle in a box whose energies are E_n

$$\psi_n = A \sin \frac{\sqrt{2mE_n}}{\hbar} x$$

$$\psi_n = A \sin \frac{n\pi x}{L}$$

for each quantum number n , ψ_n is a finite, single-valued function of x ,
 ψ_n and $\partial\psi_n/\partial x$ are continuous (except at the ends of the box)

Find A

$$\int_{-\infty}^{\infty} |\psi_n|^2 dx = \int_0^L |\psi_n|^2 dx = A^2 \int_0^L \sin^2 \left(\frac{n\pi x}{L} \right) dx$$

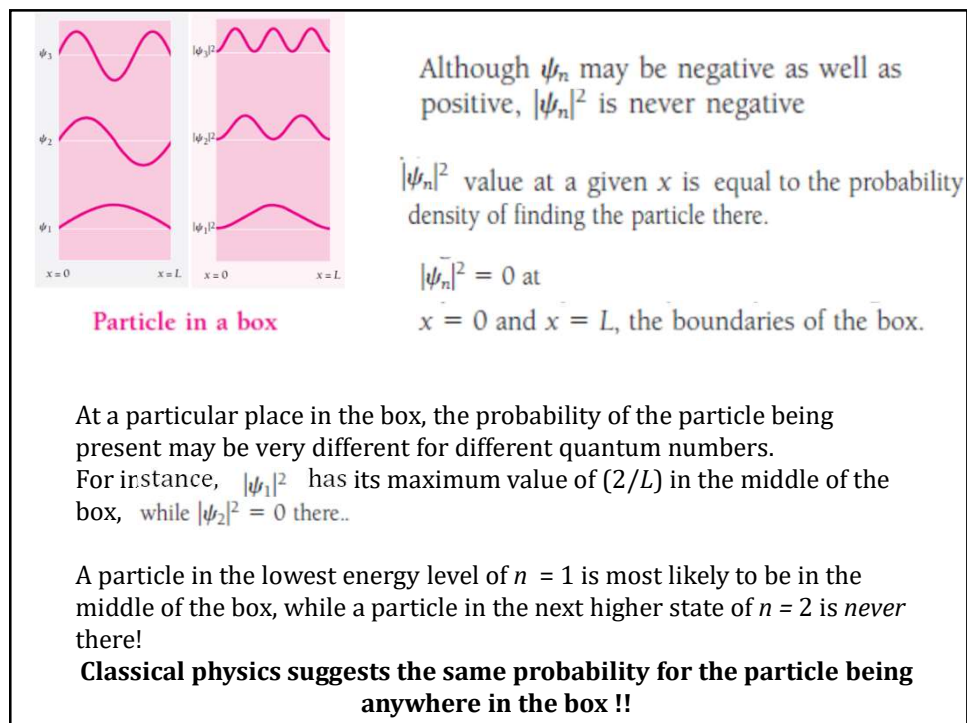
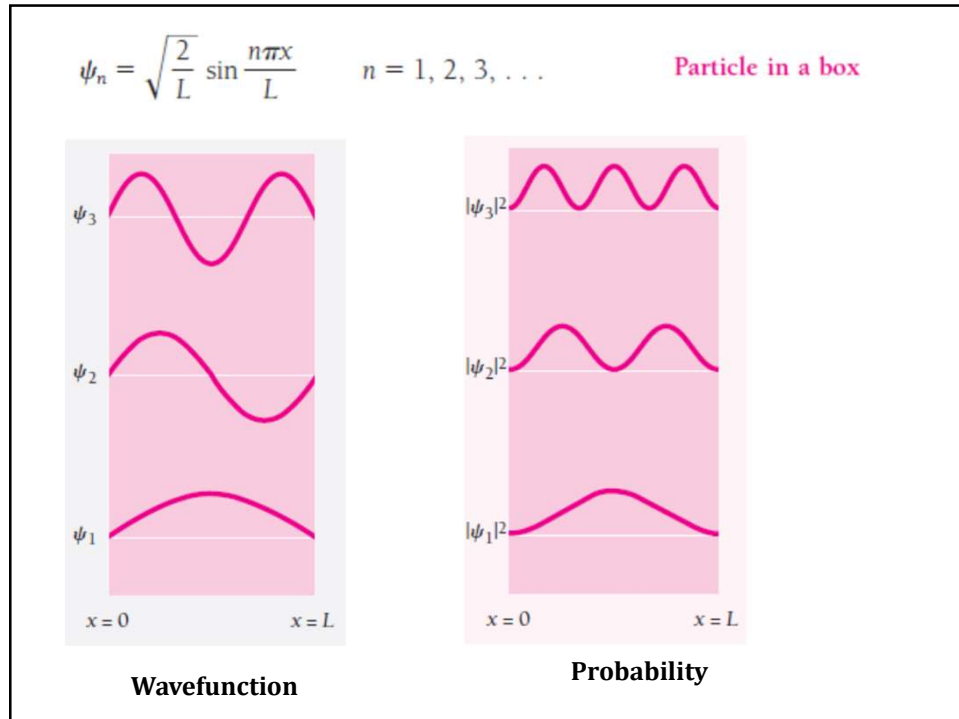
$$= \frac{A^2}{2} \left[\int_0^L dx - \int_0^L \cos \left(\frac{2n\pi x}{L} \right) dx \right]$$

$$= \frac{A^2}{2} \left[x - \left(\frac{L}{2n\pi} \right) \sin \frac{2n\pi x}{L} \right]_0^L = A^2 \left(\frac{L}{2} \right)$$

$$\int_{-\infty}^{\infty} |\psi_n|^2 dx = 1$$

$$A = \sqrt{\frac{2}{L}}$$

$$\psi_n = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \quad n = 1, 2, 3, \dots$$



Example 5.4

Find the probability that a particle trapped in a box L wide can be found between $0.45L$ and $0.55L$ for the ground and first excited states.

Solution

This part of the box is one-tenth of the box's width and is centered on the middle of the box.

Classically, one would expect the particle to be in this region 10% of the time. Quantum mechanics gives quite different predictions that depend on the quantum number of the particle's state.

The probability of finding the particle between x_1 and x_2 when it is in the n^{th} state is

$$P_{x_1, x_2} = \int_{x_1}^{x_2} |\psi_n|^2 dx = \frac{2}{L} \int_{x_1}^{x_2} \sin^2 \frac{n\pi x}{L} dx$$

$$= \left[\frac{x}{L} - \frac{1}{2n\pi} \sin \frac{2n\pi x}{L} \right]_{x_1}^{x_2}$$

Here $x_1 = 0.45L$ and $x_2 = 0.55L$.

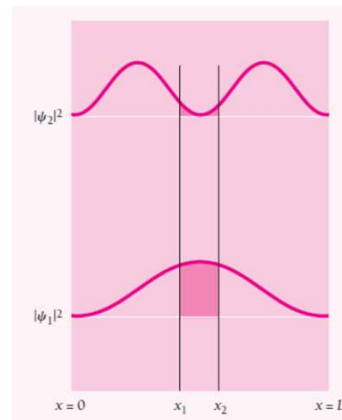
For the ground state, which corresponds to $n = 1$,

$$P_{x_1, x_2} = 0.198 = 19.8 \text{ percent}$$

This is about twice the classical probability.

For the first excited state, which corresponds to $n = 2$,

$$P_{x_1, x_2} = 0.0065 = 0.65 \text{ percent}$$



This low figure is consistent with the probability density of $|\psi_n|^2 = 0$ at $x = 0.5L$.

For more fun of Quantum Mechanics

Google for [Schrödinger's cat] → A thought experiment