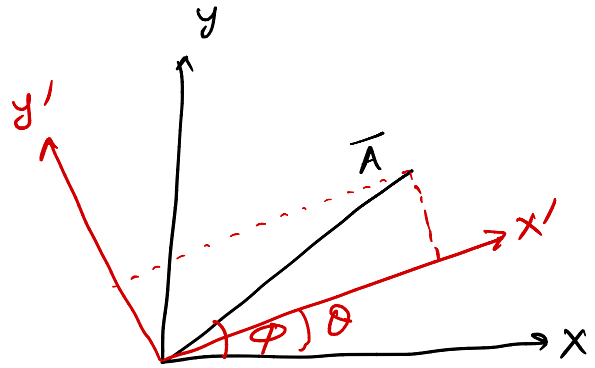
Lecture-2

4/8/2023.

How Vectors transform ?



$$A_{x'} = A\cos(\varphi-\theta)$$
$$= A(\cos\varphi\cos\theta + \sin\varphi\sin\theta)$$
$$A_{x'} = A_x\cos\theta + A_y\sin\theta \qquad \text{—— (1)}$$

$$A_{y'} = A\sin(\varphi-\theta)$$
$$= A(\sin\varphi\cos\theta - \cos\varphi\sin\theta)$$
$$= A_y\cos\theta - A_x\sin\theta \qquad \text{—— (2)}$$

In Matrix notation –

$$\begin{pmatrix} A_{x'} \\ A_{y'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \quad A_y(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

* Is $N = N_x\hat{x} + N_y\hat{y} + N_z\hat{z}$ a vector ?

where,
$N_x \rightarrow$ pears
$N_y \rightarrow$ apples
$N_z \rightarrow$ oranges.

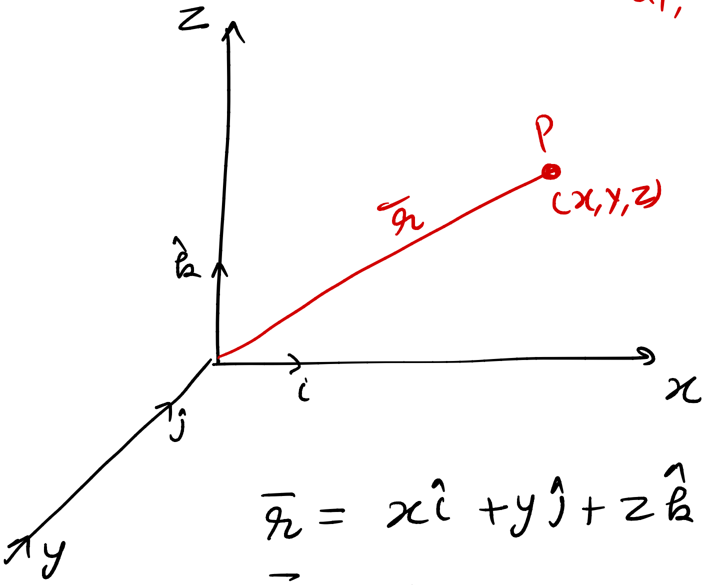
Ans ⇒ No.
Because N does not transform properly when we change the coordinates.

Coordinate Systems & Transformation ⇒

* In order to describe the special variation of the quantities, we must be able to describe all points uniquely in the space.

(1) ⇒ Cartesian coordinates (x, y, z) ⇒

u_1, u_2, u_3



$\hat{i}$ or $\hat{x}$

$\hat{j}$ $\hat{y}$

$\hat{k}$ $\hat{z}$

unit vectors

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\frac{\partial \bar{r}}{\partial x} = \hat{i} \quad , \quad \left|\frac{\partial \bar{r}}{\partial x}\right| = 1 = h_1 \text{ (Scale factor)}$$

$$\frac{\partial r}{\partial y} = \hat{j} \quad , \quad \left|\frac{\partial \bar{r}}{\partial y}\right| = 1 = h_2$$

$$\frac{\partial r}{\partial z} = \hat{k} \quad , \quad \left|\frac{\partial r}{\partial z}\right| = 1 = h_3.$$

ⓐ differential length ⇒

$$\boxed{dl = h_1 du_1 \hat{u}_1 + h_2 du_2 \hat{u}_2 + h_3 du_3 \hat{u}_3}$$

$$dl = 1 \cdot dx\,\hat{x} + 1 \cdot dy\,\hat{y} + 1\,dz\,\hat{z}$$

ⓑ differential area ⇒

$$\boxed{ds = h_1 du_1\, h_2 du_2} \cdot \hat{u}_3$$

$$ds = dx\,dy\,\hat{k}.$$

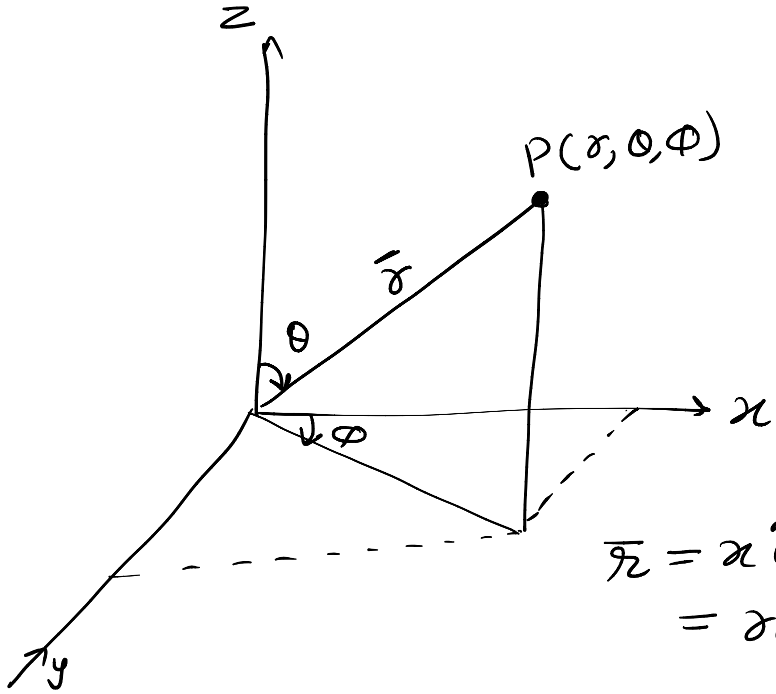
ⓒ differential volume ⇒

$$\boxed{dV = h_1 h_2 h_3\, du_1 du_2 du_3}$$

$$dV = 1 \cdot 1 \cdot 1 \cdot dx\,dy\,dz$$

Spherical Polar Coordinates $(r, \theta, \phi) \rightarrow$

$u_1 \quad u_2 \quad u_3$



Transformation eq^n -

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$= r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$$

$$\frac{\partial \bar{r}}{\partial r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\left|\frac{\partial \bar{r}}{\partial r}\right| = \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}$$
$$= 1 = h_1$$

$$\therefore \hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\frac{\partial \bar{r}}{\partial \theta} = r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}$$
$$= r\left[\cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}\right]$$

$$\left|\frac{\partial \bar{r}}{\partial \theta}\right| = r = h_2$$

$$\therefore \hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

$$\frac{\partial \bar{r}}{\partial \phi} = -r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j}$$
$$= r\sin\theta\left[-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right)$$

$$\left|\frac{\partial \bar{r}}{\partial \phi}\right| = r\sin\theta = h_3$$

$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$d\ell = h_1 du_1 \hat{u}_1 + h_2 du_2 \hat{u}_2 + h_3 du_3 \hat{u}_3$$
$$= 1 \cdot dr \hat{r} + r\, d\theta\, \hat{\theta} + r \sin\theta\, d\varphi\, \hat{\varphi}$$

$$dS = h_1 du_1 h_2 du_2 \quad \hat{u}_3$$
$$= r\, dr\, d\theta \quad \hat{\varphi}$$

$$dV = h_1 h_2 h_3\, du_1 du_2 du_3$$
$$= 1 \cdot r \cdot r \sin\theta \quad dr\, d\theta\, d\varphi$$

* <u>In Matrix form</u> $\Rightarrow$

$$\bar{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$$
$$\bar{A} = A_r \hat{r} + A_\theta \hat{\theta} + A_\varphi \hat{\varphi}$$

$(\hat{i}, \hat{j}, \hat{k}) \longrightarrow (\hat{r}, \hat{\theta}, \hat{\varphi})$

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\varphi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

$(\hat{r}, \hat{\theta}, \hat{\varphi}) \longrightarrow (\hat{i}, \hat{j}, \hat{k})$

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\varphi} \end{bmatrix}$$

\# Suppose we want to transform vector $\bar{A}$ from (A_x, A_y, A_z) to $(A_r, A_\theta, A_\varphi)$ $\Rightarrow$

$$\boxed{\begin{aligned} \begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix} &= \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \\ \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} &= \begin{bmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\varphi \end{bmatrix} \end{aligned}}$$