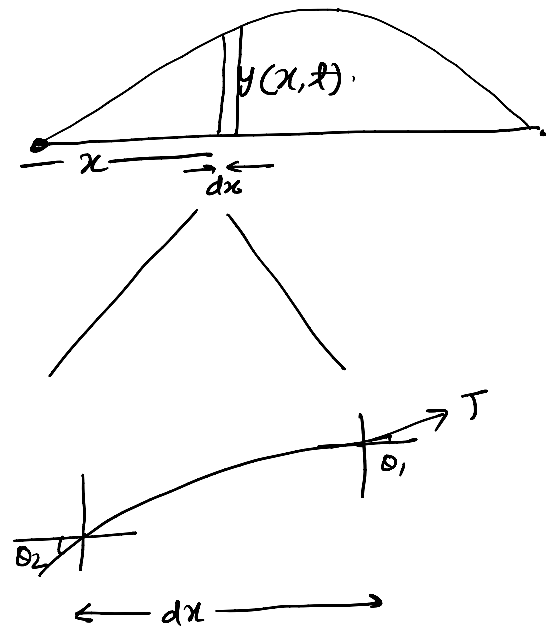
Lecture-17

26/09/2023.

Wave eq^n in One dimension ⇒

§ A long string tied b/w two fixed point. Uneder Tention T.



* If it is displaced from equilibrium the net transverse force on the segment ⇒

$$\text{force} = T\sin\theta_1 - T\sin\theta_2$$

For small θ ⇒

$$\text{force} = T(\tan\theta_1 - \tan\theta_2)$$

$$ma = T\left(\frac{dy}{dx}\Big|_{x+dx} - \frac{dy}{dx}\Big|_{x}\right)$$

$$\mu dx\, \frac{d^2y}{dt^2} = T\,\frac{d^2y}{dx^2}\,dx$$

$$\frac{d^2y}{dx^2} - \frac{\mu}{T}\frac{d^2y}{dt^2} = 0$$

If $v = \sqrt{\frac{T}{\mu}}$ Then.

$$\boxed{\frac{d^2y}{dx^2} - \frac{1}{v^2}\frac{d^2y}{dt^2} = 0}$$

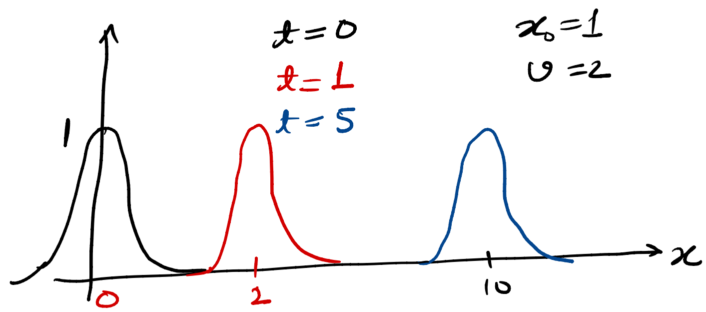
* Example of wave eq^n ⇒

$$y = f(x - vt)$$

for ex ⇒

$$y = e^{-\frac{(x-vt)^2}{x_0^2}} \quad ✓$$

$$y = e^{-\frac{x^2 - v^2t^2}{x_0^2}} \quad XX$$



General sol^n $\quad y = f(x - vt) + g(x + vt)$.

Maxwell eq^n -

$$\nabla \cdot E = \rho/\varepsilon_0$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

In Vaccum

$$\nabla \cdot E = 0 \quad \text{———} \quad (1)$$

$$\nabla \cdot B = 0 \quad \text{———} \quad (2)$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad \text{———} \quad (3)$$

$$\nabla \times B = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \quad \text{———} \quad (4)$$

Applying curl to eq^n (3)

$$\nabla \times (\nabla \times E) = -\frac{\partial}{\partial t}(\nabla \times B)$$

$$\Rightarrow \nabla(\cancelto{0}{\nabla \cdot E}) - \nabla^2 E = -\frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}\right)$$

$$\boxed{\nabla^2 E - \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} = 0}$$

$$\boxed{\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0}$$

Similarly

$$\boxed{\nabla^2 B - \mu_0 \varepsilon_0 \frac{\partial^2 B}{\partial t^2} = 0}$$

Compairing it with the wave eq^n -

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$$= 3 \times 10^8 \ m/sec = c.$$

§ In Vaccum E.M. wave propagate with the speed of ligh.

Say we are interested in the field of the following form

$$E = Re\left(E_0 e^{i(kz - \omega t)} \hat{x}\right)$$

$$E = [E_0 \cos(kz - \omega t) \quad 0 \quad 0]$$

Let us Check if this plane wave satisfy the wave eqn :→

$$* \quad \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}$$

$$E = E_0 \cos(kz - \omega t),\ 0,\ 0$$

$$-\cancel{E_0} k^2 \cancel{\cos(kz-\omega t)} = -\mu_0 \varepsilon_0 \omega^2 \cancel{E_0 \cos(kz-\omega t)}$$

$$k^2 = \mu_0 \varepsilon_0 \omega^2$$

$$\boxed{\omega = ck}$$

* What is the magnetic field ?

Easy choise is the farady law.

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

$$\Rightarrow \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_0 \cos(kz-\omega t) & 0 & 0 \end{vmatrix} = -\frac{\partial B}{\partial t}$$

$$\Rightarrow \hat{x} \times 0 - \hat{y}\left(E_0 k \sin(kz-\omega t)\right) + \hat{z} \cdot 0 = -\frac{\partial B}{\partial t}$$

$$\Rightarrow \cancel{-} E_0 k \sin(kz-\omega t)\, \hat{y} = \cancel{-}\frac{\partial B}{\partial t}$$

$$\bar{B} = \frac{k}{\omega} E_0 \cos(kz-\omega t)\, \hat{y}$$

$$\bar{B} = \frac{E_0}{c} \cos(kz-\omega t)\, \hat{y} \qquad \bar{E} = E_0 \cos(kz-\omega t)\, \hat{x}$$

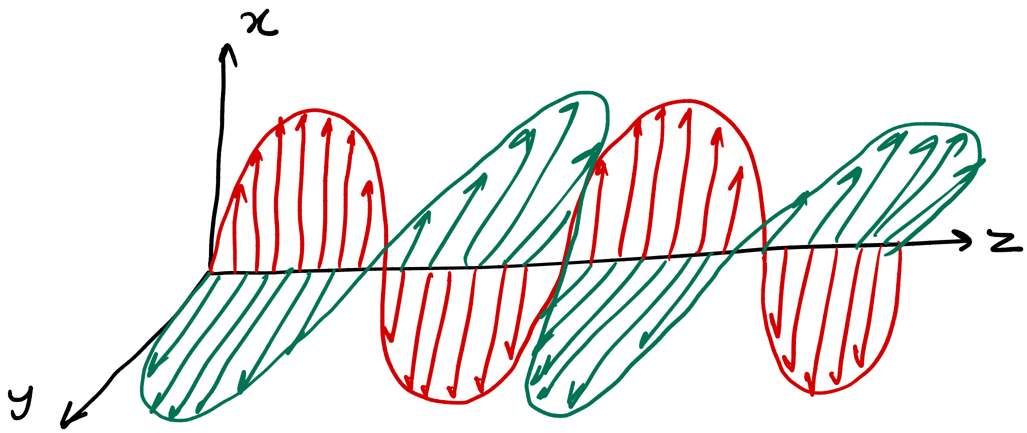
1- The electric field & magnetic field are $\perp^{ar}$ to the direction of propagation. i.e. EM wave is a Transverse wave.

Q- $\bar{E}$ & $\bar{B}$ are in phase & their amplitude are related by

$$B_0 = \frac{E_0}{c}.$$

$E = E_0 \cos(kz - \omega t)\,\hat{x}$ &

$B = B_0 \cos(kz - \omega t)\,\hat{y}$



3 ⇒ Energy density contained in the Electric & Magnetic field's

$$u_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0 E_0^2 \cos^2(kz - \omega t)$$ ✓

$$u_B = \frac{1}{2\mu_0} B^2 = \frac{B_0^2 \cos^2(kz - \omega t)}{2\mu_0}$$

$$= \frac{E_0^2}{2c^2\mu_0}\cos^2(kz - \omega t) = \frac{1}{2}\varepsilon_0 E_0^2 \cos^2(kz - \omega t)$$ ✓

* Equal amount of energy is contained in the Electric & magnetic fields.

Q.1 ⇒ An electric field in free space is given by $E = 50\cos(10^8 t + \beta x)\, a_y$ V/m

(a) ⇒ Find the direction of wave propagation.

(b) ⇒ Calculate β & the time it takes to travel a distance of $\lambda/2$.

(c) ⇒ Sketch the wave at $t = 0$, $T/4$ & $T/2$.