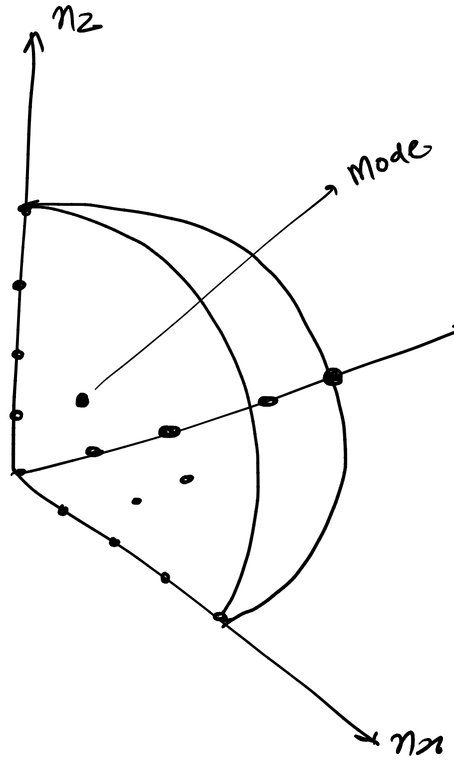
Lecture-20

3/10/2022.

Rayleigh's energy density distribution ⇒



$$n_x^2 + n_y^2 + n_z^2 = \left(\frac{2a\nu}{c}\right)^2$$

Each point corresponds to a set of integer of n_x, n_y, n_z & it represent a standing wave.

* Total no. of modes b/w 0 to ν.

$$n(\nu) \Rightarrow \frac{4}{3}\pi\left(\frac{2a\nu}{c}\right)^3 \times \frac{1}{8} \times 2$$

$$n(\nu) = \frac{8\pi a^3 \nu^3}{3c^3}.$$

$$n(\nu)d\nu = \frac{8\pi a^3}{\cancel{3}c^3} \cdot \cancel{3}\nu^2 d\nu$$

$$n(\nu)d\nu = \frac{8\pi\nu^2}{c^3} d\nu \; a^3.$$

∴ No. of modes (standing waves) of the radiation in the frequency interval ν to $\nu + d\nu$ per unit volume -

$$\boxed{N(\nu) = \frac{8\pi\nu^2}{c^3}} \text{ ——— (1)}$$

Acc. to the equipartition theorem of the classical thermodynamics. [it relates the tem. of the system to it's avergerage energy], all oscillators in the cavity have same mean energy.

$$\boxed{\langle E\rangle = \frac{\int_0^\infty E_i e^{-\beta E_i}\, dE}{\int_0^\infty e^{-\beta E_i}\, dE}} \quad \text{——— (a)} \quad \text{where, } \beta = \frac{1}{KT}$$

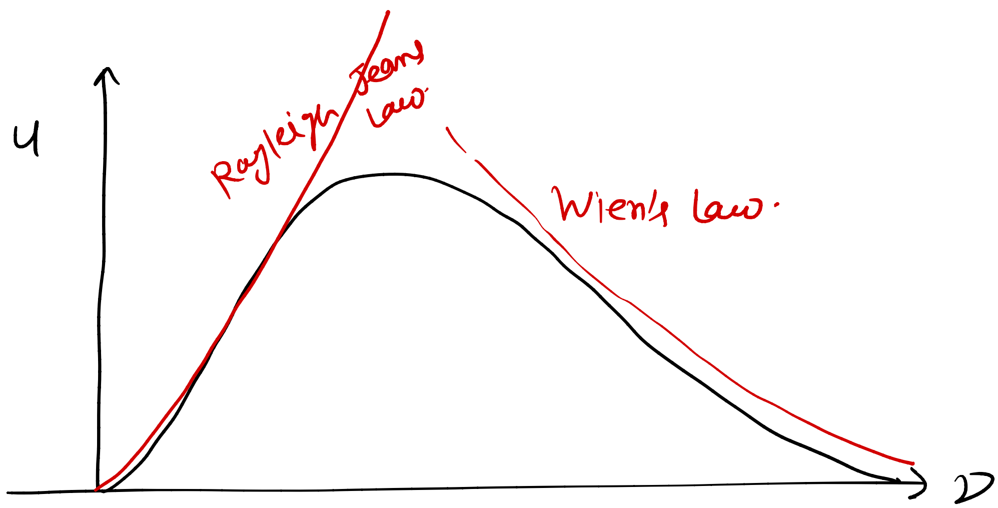
$$= \frac{\int_0^\infty E e^{-\beta E}\, dE}{\int_0^\infty e^{-\beta E}\, dE} = -\frac{\partial}{\partial\beta} \ln\Big[\underbrace{\int_0^\infty e^{-\beta E} dE}_{\left[\frac{e^{-\beta E}}{-\beta}\right]_0^\infty \Rightarrow \frac{1}{\beta}}\Big]$$

$$= -\frac{\partial}{\partial\beta} \ln\left(\frac{1}{\beta}\right)$$

$$= \frac{1}{\beta}$$

$$\boxed{\langle E\rangle = kT} \quad \text{———} \quad ②$$

$$\therefore \boxed{u(\nu,T) = \frac{8\pi\nu^2}{c^3} kT} \quad \text{———} \quad ③$$



* Except for low frequencies, this law is in complete disagreement with the experimental data. Moreover, if we integrate ③ over all frequencies the integral diverges. This implies that the cavity contains an infinite amount of energy. Historically, this was called the <u>ultraviolet catastrophe</u>.

Planck's energy density distribution ⇒

* Planck considered that the energy exchange b/w radiation & matter must be discrete. He postulated that the energy of the radiation emitted by the oscillating charge (from the walls of the cavity) must come only in integer multiples of $h\nu$.

$$\boxed{E = nh\nu \quad , \qquad n = 0, 1, 2, 3 \ldots}$$

here, h is universal const (Planck const)
& $h\nu$ is the energy of a quantum of radiation
ν represents the frequency of the oscillating charge in the cavity's wall as well as the frequency of the radiation emitted from the wall.

* Assuming energy of an oscillator is quantized, Planck replaced the integral in (a) with discrete summation.

$$\langle E \rangle = \frac{\sum_{n=0}^{\infty} nh\nu \, e^{-nh\nu/kT}}{\sum_{n=0}^{\infty} e^{-nh\nu/kT}} \quad \text{——— (b).}$$

Let say $x = e^{-h\nu/kT}$

$$\boxed{\begin{aligned} \sum x^n &= \frac{1}{1-x} \\ \sum n x^n &= \frac{x}{(1-x)^2} \end{aligned}}$$

$$= \frac{h\nu \sum_{n=0}^{\infty} n x^n}{\sum_{n=0}^{\infty} x^n}$$

$$= \frac{h\nu \, x}{(1-x)^{\cancel{2}}} \cdot \frac{\cancel{(1-x)}}{1}$$

$$= h\nu x / 1-x = \frac{h\nu \, e^{-h\nu/kT}}{1 - e^{-h\nu/kT}}$$

$$\boxed{\langle E \rangle = \frac{h\nu}{e^{h\nu/kT}-1}}$$

Hence, the energy density per unit frequency of the radiation emitted from the hole of the cavity

$$\boxed{u(\nu,T) = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/kT}-1}}$$

Homework.

$$u(\lambda,T) = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}$$

use $\nu = \frac{c}{\lambda}$

Planck's distribution in the limit of low frequency ⇒

$$u(\nu,T) = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/kT}-1} \qquad [h\nu \ll kT]$$

$$= \frac{8\pi\nu^2}{c^3}\frac{h\nu}{\cancel{1}+\frac{h\nu}{kT}\cancel{-1}}$$

$$= \frac{8\pi\nu^2}{c^3}\cdot kT$$

Rayleigh - Jeans law.

Planck's distribution in the limit of high frequency ⇒

$$u(\nu,T) = \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{h\nu/kT}\cancel{-1}}$$

$$= \frac{8\pi h\nu^3}{c^3}\cdot e^{-\frac{h}{k}\frac{\nu}{T}}$$

$$= A\nu^3 e^{-B\nu/T}$$

Wien's law.

Total energy density :-

$$\int_0^\infty u(\nu,T)\,d\nu = \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3}{e^{h\nu/kT}-1}\,d\nu$$

Let say, $\frac{h\nu}{kT} = x$

$\frac{h}{kT}\,d\nu = dx$

$$= \frac{8\pi k^4 T^4}{h^3 c^3}\underbrace{\int_0^\infty \frac{x^3}{e^x-1}\,dx}_{\frac{\pi^4}{15}}$$

$$= \frac{8\pi^5 k^4}{15 h^3 c^3}\,T^4$$

$$= \frac{4}{c}\sigma T^4$$

where, $\sigma = 5.67 \times 10^{-8}\ W m^{-2} K^{-4}$

Stefan's law.