Lecture-16

9/9/23.

Energy in Magnetic field:⇒

$$\boxed{W = \frac{1}{2}\int_V (A \cdot J)\, d\tau}$$

$$\boxed{W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\, d\tau}$$

Energy per unit volume $\frac{1}{2\mu_0} B^2$.

Energy of a continious charge distribution

$$W = \frac{1}{2}\int \rho V\, d\tau$$

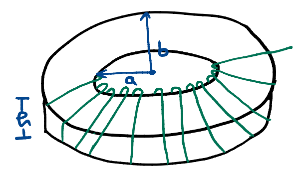
ρ → Volume charge density

V → Scalar Pot.

$$W = \frac{1}{2}\varepsilon_0 \int_{\text{all spac.}} E^2 d\tau$$

Energy per unit volume $= \frac{1}{2}\varepsilon_0 E^2$

Q.1:⇒ Calculate the energy stored in the toroidal coil.



A:⇒ $B = \frac{\mu_0 N I}{2\pi r}\hat{\phi}$ inside

$= 0$ outside

$$W = \frac{1}{2\mu_0}\int B^2 d\tau$$

$$= \frac{1}{2\mu_0}\int \frac{\mu_0^2 N^2 I^2}{4\pi^2 r^2} \cdot 2\pi r\, dr\, h$$

$$= \frac{\mu_0 N^2 I^2 h}{4\pi}\int_a^b \frac{1}{r}\, dr$$

$$= \frac{\mu_0 N^2 I^2 h^2}{4\pi}\ln b/a.$$

Homework

Q.2:⇒ A long coaxial cable carries current I (the current flows down the surface of the inner cylinder, radius a & back along the outer cylinder radius b) as shown. Find the magnetic energy stored in a section of length l.

Ans:⇒ $W = \frac{\mu_0 I^2 l}{4\pi}\ln b/a$

Electrodynamics before Maxwell :-

① $\nabla \cdot E = \rho/\varepsilon_0$ (Gauss's law)

② $\nabla \cdot B = 0$ (No name)

③ $\nabla \times E = -\frac{\partial B}{\partial t}$ (Faraday' law)

④ $\nabla \times B = \mu_0 J$ (Ampere's law).

Inconsistancy :-

eq^n (III) :- apply divergence.

$$\underbrace{\nabla \cdot (\nabla \times E)}_{0} = \nabla \cdot \left(-\frac{\partial B}{\partial t}\right)$$

$$= -\frac{\partial}{\partial t} \underbrace{(\nabla \cdot B)}_{0} \checkmark$$

eq^n (IV) apply divergence

$$\underbrace{\nabla \cdot (\nabla \times B)}_{0} = \mu_0 \underbrace{(\nabla \cdot J)}_{\neq \text{ not always zero.}}$$

* for steady current $\nabla \cdot J = 0$ but if we go beyond magnetostatic Ampere's law cannot be right

$$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0$$

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t}$$

$$= -\frac{\partial}{\partial t}(\varepsilon_0 \nabla \cdot E)$$

$$= -\nabla \cdot \left(\varepsilon_0 \frac{\partial E}{\partial t}\right)$$

If we combine $\varepsilon_0 \frac{\partial E}{\partial t}$ with J in Ampere's law it will kill off the extra divergence

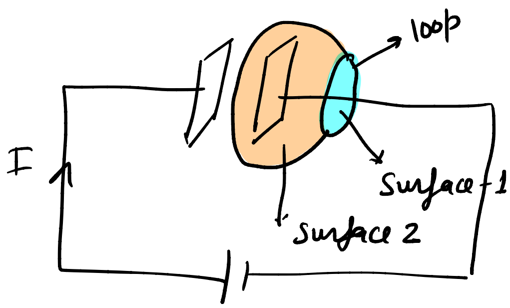
$$\boxed{\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}}$$

Ampere's law :- $\oint B \cdot dl = \mu_0 I_{enc}$.

I_{enc} ⇒ Total current passing through the loop.

~~or~~

It is the current piercing a surface that has the loop for it's boundary.



Surface 1 :- $I_{encl} = I$

surface 2 :- $I_{encl} = 0$.

* $$\boxed{\nabla \times B = \mu_0 J + \underbrace{\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}}}$$

$$\boxed{\nabla \times B = \mu_0 [J + J_d]}$$

where $J_d = \frac{\partial D}{\partial t}$

& $D = \epsilon E$

J_d is called displacemt current density

$$\boxed{\int B \cdot dl = \mu_0 I_{encl} + \mu_0 \varepsilon_0 \int \left(\frac{\partial E}{\partial t}\right) \cdot ds}$$

* for flat surface ⇒ $E = 0$, $I_{encl} = I$

* For the capacitor plates :-

For $E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon A}$

$$\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0 A} \frac{\partial Q}{dt} = \frac{I}{\varepsilon_0 A}$$

∴ <u>for balloon-shaped surface</u> :- $I_{encl} = 0$ but $\int \frac{\partial E}{\partial t} \cdot ds$

$$\Rightarrow \frac{I}{\varepsilon_0}.$$

Q ⇒ A parallel-plate capacitor with plate area of 5 cm^2 & plate separation of 3 mm has a voltage $50 \sin 10^3 t$ V applied to it's plates. Calculate the displacement current assuming $\epsilon = 2\epsilon_0$.

A ⇒ $A = 5\,cm^2$, $d = 3$ mm $V = 50 \sin 10^3 t$.

displacement current density $J_d = \frac{\partial D}{\partial t} \Rightarrow \epsilon \frac{\partial E}{\partial t} = \frac{\epsilon}{d}\frac{\partial V}{\partial t}$

$$J_d = \frac{2\epsilon_0}{3\times10^{-3}} \times 50\times10^3 \cos 10^3 t$$

∴ Displacement current $I_D = \int J_d \cdot ds$

$$= \frac{2\epsilon_0}{3\times10^{-3}} \times 50\times10^3 \cos 10^3 t \times 5\times10^{-4}$$

$$= 147.4 \cos 10^3 t \ \ nA$$

Poynting theorem ⇒

* Energy stored in electric field $W_E = \frac{1}{2}\epsilon_0 \int E^2 d\tau$

" " " Magnetic field $W_B = \frac{1}{2\mu_0}\int B^2 d\tau$

Total energy stored in Electromagnetic field

$$U_{em} = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) d\tau$$

* Energy per unit time per unit area transported by the field

$$\boxed{S = E\times H} = \frac{1}{\mu_0} E\times B$$

= Energy flux density.

* $S \cdot da$ ⇒ Energy per unit time crossing the surface area da.
= energy flux.

Q.1 ⇒ In a nonmagnetic medium $\bar{E} = 4\sin(2\pi \times 10^7 t - .8x)\, a_z$ V/m

find

(a) ⇒ The time-average power carried by the wave.

(b) ⇒ The total power crossing 100 cm² of plane $2x+y=5$.

A ⇒

$$\bar{E} = 4\sin(2\pi \times 10^7 t - .8x)\, a_z$$

$$\boxed{E = E_0 \cos(kz - \omega t)\,\hat{x}}$$

$k = 0.8$

$\omega = 2\pi \times 10^7$

(a) ⇒ $\langle S \rangle = \langle E \times H \rangle = \frac{1}{\mu_0} \langle E \times B \rangle$

for B, $\nabla \times E = -\frac{\partial B}{\partial t}$

$$\nabla \times E = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & 4\sin(2\pi \times 10^7 t - .8x \end{vmatrix}$$

$\nabla \cdot E = \rho/\epsilon_0$

$\nabla \cdot B = 0$

$\nabla \times E = -\frac{\partial B}{\partial t}$

$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$

$$= 0 - \hat{j}\left(4\cos(2\pi \times 10^7 t - 0.8x)\right) \times -.8 + 0 = -\frac{\partial B}{\partial t}$$

$$\therefore B = \frac{-3.2\sin(2\pi \times 10^7 t - 0.8x)}{2\pi \times 10^7}\,\hat{j}$$

$$S = E \times H = \frac{1}{\mu_0}\, \frac{4 \times 3.2}{2\pi \times 10^7} \sin^2(2\pi \times 10^7 t - .08x)\,\hat{x}$$

$$\langle S \rangle = \frac{4 \times 3.2}{10^7 \times 4\pi \times 2\pi \times 10^7} \times \frac{1}{2} = 81\, a_x \text{ mW/m}^2$$ A