Lecture-28

20/10/2023.

Schrodinger eq^n :

Schrodinger eq^n cannot be derived from other basic principle of physics.

* For a particle moving in x direction with a precise value of linear moment & energy, the wave fu^n can be a sinusoidal fu^n.

$$\psi(x,t) = \sin(kx - \omega t).$$

$$\frac{\partial \psi}{\partial x} = k \cos(kx - \omega t)$$

$$\frac{\partial^2 \psi}{\partial x^2} = -k^2 \sin(kx - \omega t)$$ ⟹ Second space derivative introduce a factor of $-k^2$.

$$\frac{\partial \psi}{\partial t} = -\omega \cos(kx - \omega t)$$ ⟹ I^{st} time derivative introduce a factor of $-\omega$.

$$\frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \sin(kx - \omega t).$$

The wave eq^n should satisfy the following properties

① It must be consistant with de-Broglie & Einstein postulates.

$$\lambda = \frac{h}{p} \quad \& \quad E = h\nu.$$

② It must be consistant with

$$E = \frac{p^2}{2m} + V$$ [total energy = K.E + P.E]

③ It must be linear in ψ
(Wave fu^n exhibit interference effect).

④ generally $V \longrightarrow V(x,t)$.

But if $V(x,t) \longrightarrow V_0$

then $F = -\frac{\partial V}{\partial x} = 0.$

⟹ linear momentum p & energy E will be constant.

In this situation $\lambda = \frac{h}{p}$ & $\nu = \frac{E}{h}$ $\Rightarrow$ const.

Therefore, in this case, the desired differential eqⁿ should have travelling wave sinusoidal solⁿ of const λ & ν.

* $\frac{p^2}{2m} + V(x,t) = E$

$\frac{\hbar^2}{2m\lambda^2} + V(x,t) = \hbar\omega$ $\quad\quad\quad \lambda = \frac{h}{p}$

$\frac{\hbar^2 k^2}{2m} + V(x,t) = \hbar\omega$ ——— (A)

For linearity, every term in wave eqⁿ must be linear & the wave eqⁿ must be consistent with eqⁿ (A)

* k^2 term in (A) suggest $\rightarrow$ eqⁿ contain IInd space derivative
* ω " " " $\rightarrow$ " " Ist time ".
& $V(x,t)$ " " $\rightarrow$ " " ψ.

Let us try —

$$\alpha \frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x,t)\,\psi(x,t) = \beta \frac{\partial \psi(x,t)}{\partial t} \quad\text{———}\quad (B)$$

here α & β are const.

If $V(x,t) = V_0$
$\psi = \sin(kx - \omega t)$ $\Big\}$ put above

$$-\alpha \sin(kx-\omega t)\, k^2 + \sin(kx-\omega t)\, V_0 = -\beta \cos(kx-\omega t)\,\omega.$$

For free particle $\Rightarrow$

$\psi(x,t) = \cos(kx-\omega t) + \gamma \sin(kx-\omega t)$. $\quad\quad$ γ is constant.

$$\frac{\partial \psi}{\partial x} = -k \sin(kx-\omega t) + k\nu \cos(kx-\omega t)$$

$$\frac{\partial^2 \psi}{\partial x^2} = -k^2 \cos(kx-\omega t) - k^2 \nu \sin(kx-\omega t)$$

$$\frac{\partial \psi}{\partial t} = \omega \sin(kx-\omega t) - \omega\nu \cos(kx-\omega t).$$

put it in eq^n B

$$-\alpha k^2 \cos(kx-\omega t) - \alpha k^2 \nu \sin(kx-\omega t) + V_0 \cos(kx-\omega t) + V_0 \nu \sin(kx-\omega t)$$
$$= \beta\omega \sin(kx-\omega t) - \beta\omega\nu \cos(kx-\omega t).$$

$$\cos(kx-\omega t)\left[-\alpha k^2 + V_0 + \beta\omega\nu\right] + \sin(kx-\omega t)\left[-\alpha k^2\nu + V_0\nu - \beta\omega\right] = 0$$

$$\Rightarrow \quad \left.\begin{aligned} -\alpha k^2 + V_0 &= -\beta\omega\nu \\ -\alpha k^2 + V_0 &= \beta\omega/\nu \end{aligned}\right\} \text{———} \quad (a)$$

Substract

$$0 = -\beta\nu\omega - \beta\omega/\nu$$

$$\nu^2 = -1$$

$$\nu = \pm i \qquad \text{put in (a).}$$

$$-\alpha k^2 + V_0 = \mp i\beta\omega \quad \text{———} \quad C$$

Compare with A $\quad \frac{\hbar^2}{2m} k^2 + V_0 = \hbar\omega$

$$\Rightarrow \alpha = -\frac{\hbar^2}{2m} \quad \& \quad \mp i\beta = \hbar$$

Thus
$$\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + V\psi = i\hbar \frac{\partial \psi}{\partial t}}$$

This is Schrodinger eq^n.