Lecture-08

19/08/2023

Gauss's Law:

$$\boxed{\oint E\cdot ds = \frac{q_{encl.}}{\varepsilon_0}}$$ Integral form.

* Electric flux passing through any closed surface enclosing the charge is q/ε_0.

Differencial form:

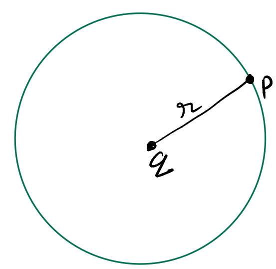
$$\oint E\cdot ds = \frac{q}{\varepsilon_0}$$

$$\Rightarrow \int_V (\nabla\cdot E)\, d\tau = \frac{1}{\varepsilon_0}\int_V \rho\, d\tau$$

$$\Rightarrow \boxed{\nabla\cdot E = \frac{\rho}{\varepsilon_0}}$$ Ist Maxwell's eqn.

Applications of Gauss's Law:

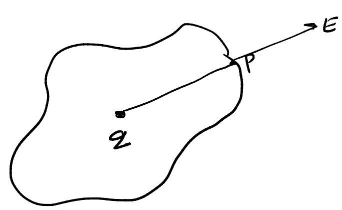
(a): Point charge:



$$\int E\cdot ds = \frac{q}{\varepsilon_0}$$

$$E\cdot 4\pi r^2 = \frac{q}{\varepsilon_0} \Rightarrow E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$$

Comment: * Gauss law is always true but not always useful.



$\Rightarrow$ * E & ds are in different direction & E is not const over the surface.

* Symmetry is crucial to the application of Gauss's law.

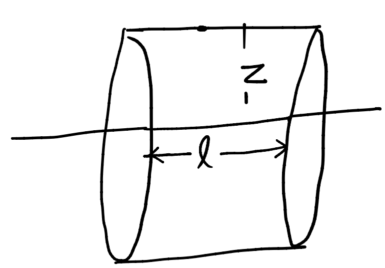
(ii) $\Rightarrow$ **Infinite Line charge:**

* Field at distance z from the wire,

$$\int E\cdot ds = \frac{q}{\varepsilon_0}$$

$$E\times 2\pi z l = \frac{\lambda l}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi\varepsilon_0 z}$$ (same as previously)



* line charge density $= \lambda$

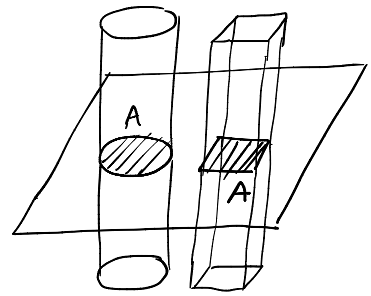
(iii) ⇒ Infinite Plane :⇒

Surface charge density $= \sigma$

$$\int E \cdot ds = \frac{q}{\varepsilon_0}$$

$$E \cdot 2A = \frac{\sigma A}{\varepsilon_0}$$

$$E = \frac{\sigma}{2\varepsilon_0}$$



(iv) :⇒ Uniformly charged sphere :⇒

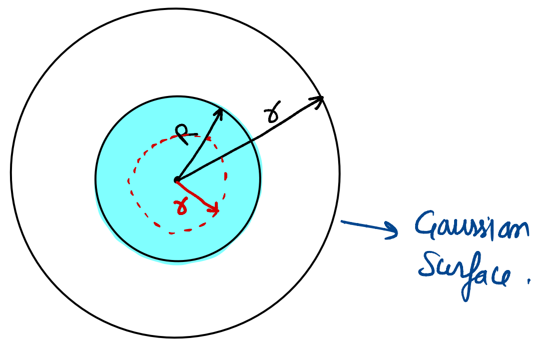
Radius R, Total charge $= q$

(a) :⇒ Outside :-

$$\int E \cdot ds = \frac{q}{\varepsilon_0}$$

$$E\, 4\pi r^2 = \frac{q}{\varepsilon_0}$$

$$E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2}$$



(b) :⇒ Inside :⇒

$$\int E \cdot ds = \frac{q}{\varepsilon_0}$$

$$E\, 4\pi r^2 = \frac{\frac{q}{\frac{4}{3}\pi R^3} \times \frac{4}{3}\pi r^3}{\varepsilon_0}$$

$$E = \frac{1}{4\pi\varepsilon_0} \frac{qr}{R^3}$$

Q :⇒ Suppose the electric field in some region is $\bar{E} = k r^3 \hat{r}$, in spherical coordinates.

(a) :⇒ Find the charge density ρ

(b) :⇒ Find the total charge contained in a sphere of radius R.

Ans :⇒ (a) :⇒

$$\nabla \cdot E = \rho/\varepsilon_0$$

$$\nabla \cdot A = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \cdots\right]$$

$$\nabla \cdot E = \frac{1}{1 \cdot r \cdot r\sin\theta} \frac{\partial}{\partial r}\left(k r^3 \cdot r \cdot r\sin\theta\right) = \rho/\varepsilon_0$$

$$= \frac{1}{r^2 \sin\theta}\, k \sin\theta\, 5 r^4 = \rho/\varepsilon_0$$

$$\Rightarrow 5kr^2 = \rho/\varepsilon_0 \Rightarrow \rho = 5\varepsilon_0 k r^2$$

(b) ⇒ Total charge $Q = \int \rho \, d\tau$

$$\Rightarrow \int_0^R 5\varepsilon_0 k r^2 \, 4\pi r^2 \, dr$$

$$\Rightarrow 20\pi \varepsilon_0 k \int_0^R r^4 \, dr$$

$$\Rightarrow 4\pi \varepsilon_0 k R^5 \quad \underline{\underline{A}}$$

Homework

Q.1 ⇒ Find the electric field inside a sphere which carries a charge density $\rho = kr$.

Ans ⇒ $E = \frac{1}{4\pi\varepsilon_0} \pi k r^2 \, \hat{r}$.

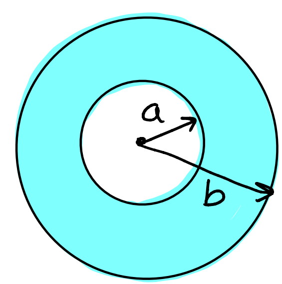
Homework.

Q.2 ⇒ A hollow spherical shell carries charge density $\rho = \frac{k}{r^2}$ in the region $a \leq r \leq b$. Find the electric field in three regions

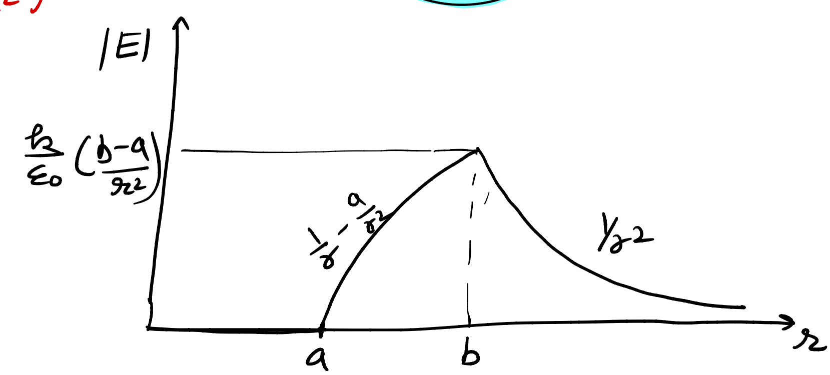
(i) ⇒ $r < a$ $\qquad E = 0$

(ii) ⇒ $a < r < b$ $\qquad E = \frac{k}{\varepsilon_0}\left(\frac{r-a}{r^2}\right)\hat{r}$

(iii) ⇒ $r > b$ $\qquad E = \frac{k}{\varepsilon_0}\left(\frac{b-a}{r^2}\right)\hat{r}$



& Plot $|E|$ vs r.



Electric Potential ⇒

* In electrostatic $\bar{E} = -y\hat{x} + x\hat{y}$

or

$\bar{E} = x\hat{y}$

is not possible with any set of charge, their size or position.

⇒ In electrostatic, the curl of Electric field is always zero.

Homework:

Q ⇒ One of these is an impossible electrostatic field. Which one?

(a) ⇒ $\bar{E}_1 = k(xy\,\hat{x} + 2yz\,\hat{y} + 3xz\,\hat{z})$, $\quad \nabla \times E_1 \neq 0$

(b) ⇒ $E_2 = k(y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\,\hat{z})$ $\quad \nabla \times E_2 = 0$

$\therefore$ E_2 is possible not E_1.

$\boxed{\nabla \times E = 0}$

then $E = -\nabla V$ → Electric Pot.

if $\nabla \times \bar{A} = 0$
then there is always
a φ such that
$\bar{A} = -\nabla \varphi$

* If we know V, we can easily determine E.
* E is a vector quantity (3 components) & V is scalar [1 component]

§ How one funⁿ can contain the information of three independent fⁿ?

Ans: The three components are not independent.

$$\nabla \times E = 0$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = 0$$

$$\Rightarrow \frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x},$$

$$\frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z}$$

$$\frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}$$

* The advantage of pot. formaalism is that it reduces the vector problem to a scalar one.

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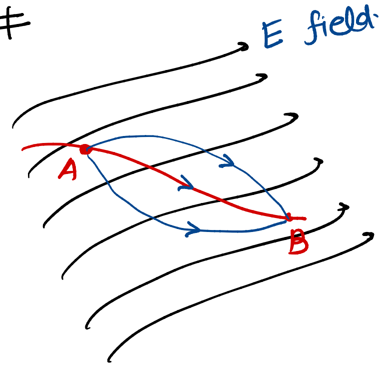


$$W = -\bar{F}\cdot d\bar{l}$$

$$W = -q\int_A^B E\cdot dl$$

* Pot. V is defined by Pot. Eenrgy per unit charge.

$$\boxed{V = \frac{W}{q}}$$

$$\therefore \quad \boxed{V = -\int_{O'}^{r} E\cdot d\bar{l}}$$

* $\nabla \times E = 0$

Stoke's theorem

$$\oint E\cdot dl = \int (\nabla \times E)\, ds$$

$$\Rightarrow \oint E\cdot dl = 0$$

* V_{AB} is independent of the path taken.

Ex:⇒ Find the pot. inside & outside a uniformly charged solid sphere whose radius is R & whose total charge is q. Use infinity as your reference point.

Ans:⇒ $V(r) = -\int_{\infty}^{r} E\cdot dl$

Outside	$\bar{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$
Inside	$\bar{E} = \frac{1}{4\pi\epsilon_0}\frac{qr}{R^3}\hat{r}$

① Outside ⇒ $r > R$

$$V(r) = -\int_{\infty}^{r} E_0\cdot dl = -\int_{\infty}^{r}\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,dr$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{r}\Big|_{\infty}^{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

② Inside ⇒ $r < R$

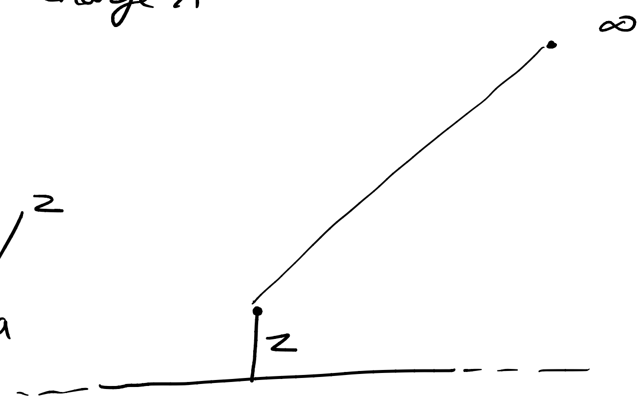
$$V(r) = -\int_{\infty}^{R} E_o \cdot dl - \int_{R}^{r} E_{in} \cdot dl$$

$$= -\int_{\infty}^{R} \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} dr - \int_{R}^{r} \frac{1}{4\pi\varepsilon_0} \frac{qr}{R^3} dr$$

$$= \frac{1}{4\pi\varepsilon_0} q \left[\frac{1}{R} - \frac{1}{R^3} \left(\frac{r^2 - R^2}{2} \right) \right] = \frac{1}{4\pi\varepsilon_0} \frac{q}{2R} \left(3 - \frac{r^2}{R^2} \right)$$

Ex ⇒ Find the pot. a distance z from an infinitely long straight wire that carries a uniform line charge λ.

Ans ⇒ $E = \frac{\lambda}{2\pi\varepsilon_0 z}$



$$V = -\int_{a}^{z} \frac{\lambda}{2\pi\varepsilon_0 z} dz = -\frac{\lambda}{2\pi\varepsilon_0} \ln z \Big|_{a}^{z}$$

$$V = -\frac{\lambda}{2\pi\varepsilon_0} \ln \frac{z}{a}$$