Lecture-05

11/08/2023.

Let us see the dot & Cross product

$\underbrace{\bar{A}\cdot\nabla}$

does not mean anything because it is still an operator.

$\nabla\cdot A = ?$

Some Scalar field.

$\nabla\cdot \underset{\substack{\downarrow \\ \text{scalar} \\ \text{[Tem field]}}}{a} \longrightarrow$ meaningless.

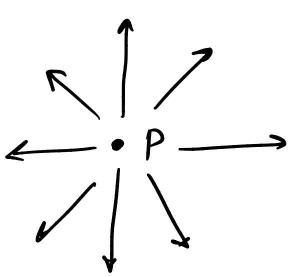
$$\nabla\cdot\bar{A} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}\right)$$

$$\boxed{\nabla\cdot\bar{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$

Interpretation:→ The divergence of $\bar{A}$ at a given point P is the outward flux per unit volume as the volume shrinks about P.

$$\boxed{\text{div}\,\bar{A} = \nabla\cdot\bar{A} = \lim_{d\tau\to 0}\frac{\oint A\cdot d\bar{s}}{d\tau}}$$

$d\tau$ is volume enclosed by the closed surface $d\bar{s}$ in which point P is located.



+ ve. divergence

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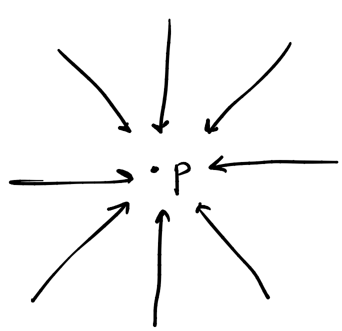

− ve. divergence

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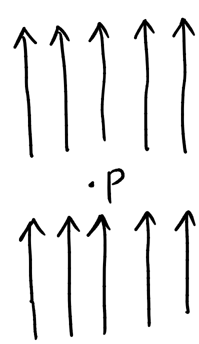


zero divergence

Ex 1 $\Rightarrow$ $\bar{v}_a = x\hat{x} + y\hat{y} + z\hat{z}$

$\bar{v}_b = z\hat{z}$

$\bar{v}_c = \hat{z}$

Calculate it's divergence.

A $\Rightarrow$ (a) $\Rightarrow$ $\nabla \cdot \bar{v}_a = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$ +ve.

(b) $\Rightarrow$ ✓

(c) $\Rightarrow$ ✓

What about the cross product?

$\nabla \times \bar{A}$ = vector.

$$\boxed{\nabla \times A = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}}$$

$\nabla \times a \longrightarrow$ meaningless.

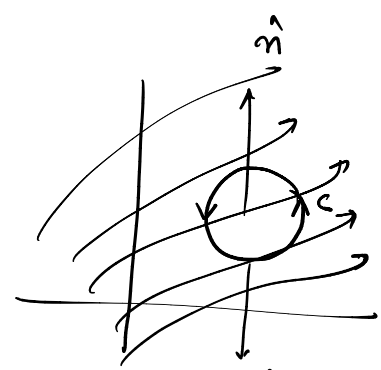
$\downarrow$

Scalar

$$\text{Curl } \bar{A} = \nabla \times \bar{A} = \left(\lim_{ds \to 0} \frac{\oint A \cdot dl}{ds} \right) \hat{a}_n$$

= $\dfrac{\text{circulation of } A \text{ around } C}{\text{area contained by } C.}$

as the area tends to zero.



surface ds
length dl.

Curvilinear coordinate $\Rightarrow$ $A = A_1\hat{u}_1 + A_2\hat{u}_2 + A_3\hat{u}_3$

	u_1	u_2	u_3	h_1	h_2	h_3
Cartesian	x	y	z	1	1	1
Spherical	r	θ	φ	1	r	$r\sin\theta$
Cylindrical	ρ	φ	z	1	ρ	1

Gradient :→

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1}\hat{u}_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2}\hat{u}_2 + \frac{1}{h_3}\frac{\partial T}{\partial u_3}\hat{u}_3$$

Ex :→

Cartesian $\nabla T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$

Spherical $\nabla T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \varphi}\hat{\varphi}$

Cylindrical $\nabla T = \frac{\partial T}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial T}{\partial \varphi}\hat{\varphi} + \frac{\partial T}{\partial z}\hat{z}$

Divergence :→

$$\nabla\cdot A = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_3 h_1) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2)\right]$$

Curl :→

$$\nabla\times A = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \hat{u}_1 h_1 & \hat{u}_2 h_2 & \hat{u}_3 h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

Second derivative of vector field :→

(a) :→ $\nabla\cdot(\nabla T)$ ⟶ Laplacian

(b) :→ $\nabla\times(\nabla T)$ ⟶ $\bar{A}\times(\bar{A}T) = (A\times A)T = 0$

(c) :→ $\nabla(\nabla\cdot\bar{A})$ ⟶ This is some vector field

(d) :→ $\nabla\cdot(\nabla\times A)$ ⟶ $B\cdot(B\times A) = 0$

(e) :→ $\nabla\times(\nabla\times A)$ ⟶ $\nabla(\nabla\cdot A) - \nabla^2 A$

(a) $\Rightarrow$ $\nabla \cdot (\nabla T) \Rightarrow \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right)$

$$\boxed{\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

This is a new operator. This scalar operator operator is called Laplacian of T.

Theorem 1 $\Rightarrow$

If $\nabla \times \bar{A} = 0$

then, there is always a ϕ

such that $\bar{A} = \nabla \phi$

Theorem 2 $\Rightarrow$

If $\nabla \cdot \bar{A} = 0$

then, there is always a $\bar{F}$

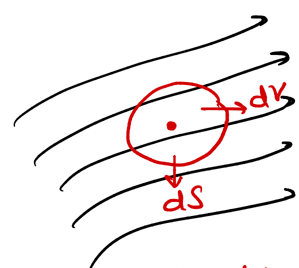
$$\bar{A} = \nabla \times \bar{F}$$

Gauss divergence Theorem $\Rightarrow$

$$\text{div } \bar{A} = \nabla \cdot \bar{A} = \lim_{d\tau \to 0} \frac{\oint A \cdot ds}{d\tau} = \text{Flux per unit volume.}$$

$$\boxed{\oint_S A \cdot ds = \int_V \nabla \cdot A \, d\tau}$$

* Surface integral of a vector field $\bar{A}$ over a closed surface is equal to the divergence of that field over the volume enclosed by the closed surface.



dV Volume enclosed by the closed surface ds.

Example :- Check the divergence theorem using the fu^n-

$$\bar{v} = y^2\hat{x} + (2xy+z^2)\hat{y} + (2yz)\hat{z}$$

& the unit cube situated at the origin.