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Q. Three isotropic sources, with spacing d between them, are placed along the z-axis. The excitation coefficient of each outside element is unity while that of the center element is 2. For a spacing of d = λ/4 between the elements, find the

(a) array factor

(b) angles (in degrees) where the nulls of the pattern occur ($0°$ ≤ θ ≤ $180°$)

(c) angles (in degrees) where the maxima of the pattern occur ($0°$ ≤ θ ≤ $180°$)
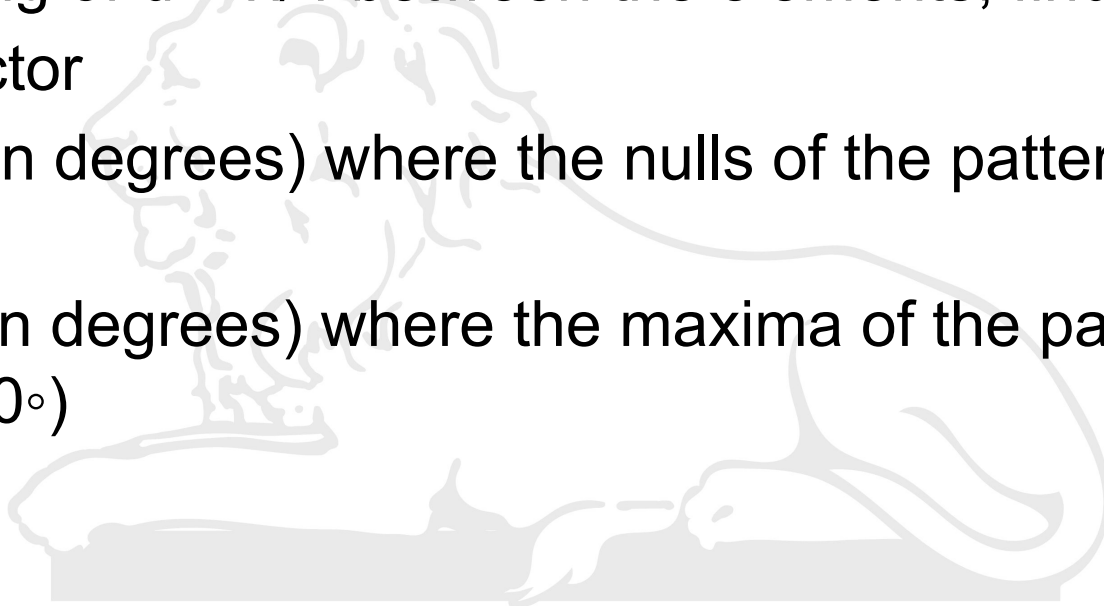
Contents

a. $E_* = E_1 + E_2 + E_3 = 2E_0 \dfrac{\bar{e}^{jkr}}{r} + E_0 \dfrac{\bar{e}^{jkr_1}}{r_1} + E_0 \dfrac{\bar{e}^{jkr_2}}{r_2}$

where the center element is placed at the origin. For far-field observations

$r_1 \simeq r - d\cos\theta$
$r_2 \simeq r + d\cos\theta$ $\Big\}$ for phase variations

$r_1 \simeq r_2 \simeq r$ for amplitude variations

and

$E_* \simeq E_0 \dfrac{\bar{e}^{jkr}}{r} \left\{ 2 + e^{jkd\cos\theta} + \bar{e}^{jkd\cos\theta} \right\}$

$\simeq E_0 \dfrac{\bar{e}^{jkr}}{r} \left\{ 2\left[1 + \tfrac{1}{2}\left(e^{jkd\cos\theta} + \bar{e}^{jkd\cos\theta}\right) \right] \right\}$

$= E_0 \dfrac{\bar{e}^{jkr}}{r} \left\{ 2\left[1 + \cos(kd\cos\theta) \right] \right\}$

Thus the array factor is equal to

$AF(\theta) = 2[1 + \cos(kd\cos\theta)] = 4\cos^2\left(\tfrac{kd}{2}\cos\theta\right)$

Computer Program
Directivity

$U = \cos^4\left(\tfrac{\pi}{4}\cos\theta\right)$

At. $d = \lambda/4$

$P_{rad} = 8.7119$

$D_0 = 1.44244$
    $= 1.5910\, dB$

$kd = \tfrac{2\pi}{\lambda} \cdot \tfrac{\lambda}{4} = \tfrac{\pi}{2}$

$AF(\theta) = 4\cos^2\left(\tfrac{\pi}{4}\cos\theta\right)$

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b. The nulls of the pattern can be found using either of the above forms for the array factor. For example.

One Form

$$AF(\theta) = 1 + \cos(kd\cos\theta_n) = 0$$

$$\cos(kd\cos\theta_n) = -1$$

$$kd\cos\theta_n = \cos^{-1}(-1) = n\pi, \quad n = \pm 1, \pm 3, \cdots$$

$$\theta_n = \cos^{-1}(n\lambda/2d), \quad n = \pm 1, \pm 3, \pm 5, \cdots$$

the other Form

$$2\cos^2\left(\frac{kd}{2}\cos\theta_n\right) = 0$$

$$\frac{kd}{2}\cos\theta_n = \cos^{-1}(0) = \frac{n\pi}{2}, \quad n = \pm 1, \pm 3.$$

$$\theta_n = \cos^{-1}(n\lambda(2d)), \quad n = \pm 1, \pm 3, \cdots$$

which are of identical form. Therefore both forms yield the same results. Thus for $d = \lambda/4$

$$\theta_n = \cos^{-1}\left(\frac{n\lambda}{2d}\right)_{d=\lambda/4} = \cos^{-1}(2n), \quad n = \pm 1, \pm 3, \cdots \Rightarrow \text{No nulls exist.}$$

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C. Similarly the maxima of the pattern can be found using either of the two forms for the array factor. For example

6-1(Contd) One Form

$$AF(\theta) = 1 + \cos(kd\cos\theta_m) = 2$$

$$\cos(kd\cos\theta_m) = 1$$

$$kd\cos\theta_m = \cos^{-1}(1) = 2m\pi, \ m = 0, \pm 1; \cdots,$$

$$\theta_m = \cos^{-1}\left(\frac{m\lambda}{d}\right), \ m = 0, \pm 1, \pm 2, \cdots$$

Other Form

$$AF(\theta) = 2\cos^2\left(\frac{kd}{2}\cos\theta_m\right) = 2$$

$$\cos\left(\frac{kd}{2}\cos\theta_m\right) = \pm 1$$

$$\frac{kd}{2}\cos\theta_m = \cos^{-1}(\pm 1) = m\pi, \ m = 0, \pm 1, \cdots$$

$$\theta_m = \cos^{-1}\left(\frac{m\lambda}{d}\right), \ m = 0, \pm 1, \pm 2, \cdots$$

which are of identical form. Therefore both yield the same results.
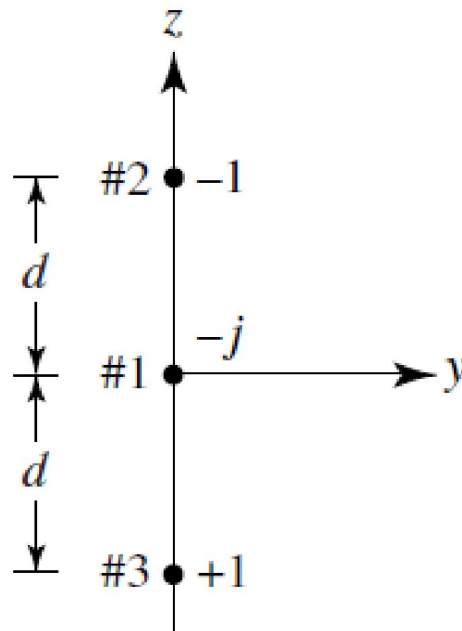Thus for $d = \lambda/4$.

$$\theta_m = \cos^{-1}(4m), \ m = 0, \pm 1, \pm 2, \rightarrow \begin{cases} m = 0: \ \theta_0 = \cos^{-1}(0) = 90° \\ m = \pm 1: \ \theta_1 = \cos^{-1}(4) \Rightarrow \text{Does not exist.} \end{cases}$$

Contents

Q. A three-element array of isotropic sources has the phase and magnitude relationships shown. The spacing between the elements is d = λ/2.

(a) Find the array factor. (b) Find all the nulls.

Contents

a. Derive the array factor;

$$AF = -e^{jkd\cos\theta} - j + e^{jkd\cos\theta} = -2j\sin(kd\cos\theta) - j$$

$$AF = 2\sin(kd\cos\theta) + 1$$

$$AF = 2\sin(\pi\cos\theta) + 1$$

b. $2\sin(\pi\cos\theta) = -1$

$$kd\cos\theta = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}, \ -\frac{5\pi}{6}, \ -\frac{13\pi}{6}, \dots, \ \frac{7\pi}{6}, \ \frac{11\pi}{6}, \ \frac{19\pi}{6}$$

$$\theta_n = \cos^{-1}\left(\frac{x}{\pi}\right)$$

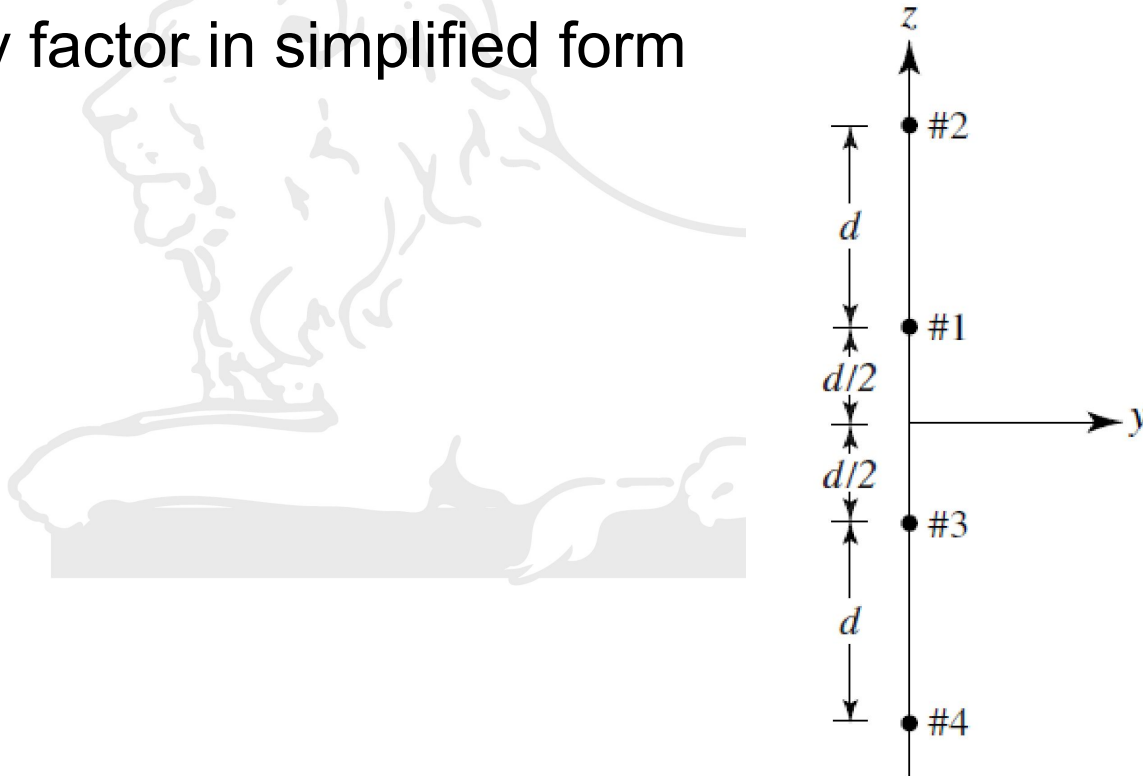$$-\frac{\pi}{6} \rightarrow \quad \theta_1 = 99.59°$$

$$-\frac{5\pi}{6} \rightarrow \quad \theta_2 = 146.44°$$

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Q. Four isotropic sources are placed along the z-axis as shown. Assuming that the amplitudes of elements #1 and #2 are +1 and the amplitudes of elements #3 and #4 are −1 (or 180 degrees out of phase with #1 and #2), find

(a) the array factor in simplified form

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(a) $E = \dfrac{e^{-jkr_2}}{r_2} + \dfrac{e^{-jkr_1}}{r_1} - \dfrac{e^{-jkr_3}}{r_3} - \dfrac{e^{-jkr_4}}{r_4}$

$= \dfrac{e^{-jkr}}{r}\left[e^{jk\frac{3d}{2}\cos\theta} + e^{jk\frac{d}{2}\cos\theta} - e^{-jk\frac{d}{2}\cos\theta} - e^{-jk\frac{3d}{2}\cos\theta} \right]$

$r_1 = r - \dfrac{d}{2}\cos\theta, \quad r_2 = r - \dfrac{3d}{2}\cos\theta, \quad r_3 = r + \dfrac{d}{2}\cos\theta, \quad r_4 = r + \dfrac{3d}{2}\cos\theta$
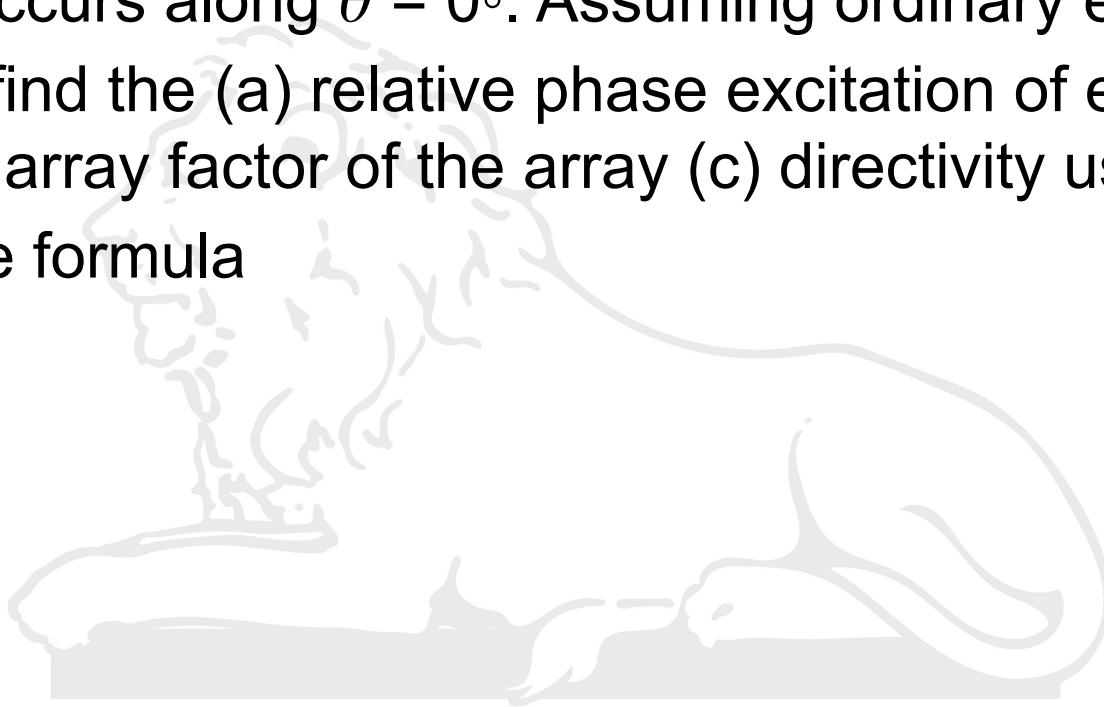
$AF = 2j\left[\sin\left(\dfrac{3kd}{2}\cos\theta\right) + \sin\left(\dfrac{kd}{2}\cos\theta\right) \right]$

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Q. Design a two-element uniform array of isotropic sources, positioned along the z-axis a distance λ/4 apart, so that its only maximum occurs along $\theta = 0°$. Assuming ordinary end-fire conditions, find the (a) relative phase excitation of each element (b) array factor of the array (c) directivity using Kraus' approximate formula

Placing one element at the origin and the other at d distance above it, the array factor is equal to

$$AF(\theta) = 1 + e^{j(kd\cos\theta + \beta)} = 2e^{j\frac{1}{2}(kd\cos\theta + \beta)} \left[\frac{e^{-j\frac{1}{2}(kd\cos\theta + \beta)} + e^{+j\frac{1}{2}(kd\cos\theta + \beta)}}{2} \right]$$

$$AF(\theta) = 2e^{j\frac{1}{2}(kd\cos\theta + \beta)} \cos\left[\frac{1}{2}(kd\cos\theta + \beta)\right]$$

which in normalized form can be written as

$$(AF)_n = \cos\left[\frac{1}{2}(kd\cos\theta + \beta)\right]$$

a. $\beta = -kd = -\frac{2\pi}{\lambda}\left(\frac{\lambda}{4}\right) = -\frac{\pi}{2}$.

b. For $d = \lambda/4$, $\quad (AF)_n = \cos\left[\frac{\pi}{4}(\cos\theta - 1)\right]$

c. $(AF)_n\big|_{max} = 1 = \cos\left[\frac{\pi}{4}(\cos\theta_m - 1)\right] \Rightarrow \theta_m = 0°$

$(AF)_n = 0.707 = \cos\left[\frac{\pi}{4}(\cos\theta_h - 1)\right] \Rightarrow \frac{\pi}{4}(\cos\theta_h - 1) = \cos^{-1}(0.707) = \begin{cases} +\frac{\pi}{4} \\ -\frac{\pi}{4} \end{cases}$

For $+\pi/4 \Rightarrow \cos\theta_h - 1 = 1 \Rightarrow \cos\theta_h = 2 \Rightarrow \theta_h = \cos^{-1}(2) \Rightarrow$ Does not exist

For $-\pi/4 \Rightarrow \cos\theta_h - 1 = -1 \Rightarrow \cos\theta_h = 0 = \theta_h = \cos^{-1}(0) = 90° = \frac{\pi}{2}$ radians

Therefore $\theta_{1r} = \theta_{2r} = 2\left(\frac{\pi}{2} - \theta\right) = \pi$

and $D_0 \simeq \frac{4\pi}{\theta_{1r}\theta_{2r}} = \frac{4\pi}{(\pi)^2} = \frac{4}{\pi} = 1.273 = 1.049 \text{dB}$

Q. Repeat the design of above problem so that its only maximum occurs along θ = 180 degrees.

a. $\quad \beta = + kd = + \dfrac{\pi}{2}$

b. $\quad (AF)_n = \cos\left[\dfrac{\pi}{4}(\cos\theta + 1)\right]$

$(AF)_n\big|_{max} = 1 = \cos\left[\dfrac{\pi}{4}(\cos\theta_m + 1)\right] \Rightarrow \theta_m = 180° = \pi \text{ radians}$

$(AF)_n = 0.707 = \cos\left(\dfrac{\pi}{4}(\cos\theta_h + 1)\right) \Rightarrow \theta_h = 90° = \dfrac{\pi}{2} \text{ radians}$

$\Theta_{1r} = \Theta_{2r} = 2\left(\pi - \dfrac{\pi}{2}\right) = \pi$

and

$D_0 \simeq \dfrac{4\pi}{\pi^2} = \dfrac{4}{\pi} = 1.273 = 1.049 \, dB$

Q. Design a four-element ordinary end-fire array with the elements placed along the z-axis a distance d apart with a maximum array factor directed towards θ = 0 deg. For a spacing of d = λ/2 between the elements find the

(a) progressive phase excitation between the elements to accomplish this

(b) angles (in degrees) where the nulls of the array factor occur

(c) angles (in degrees) where the maximum of the array factor occur

(d) beamwidth (in degrees) between the first nulls of the array factor

(e) directivity (in dB) of the array factor

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a. $\beta = -kd = -\frac{2\pi}{\lambda}\left(\frac{\lambda}{2}\right) = -\pi = -180°$

b. $\theta_n = \cos^{-1}\left[1 - \frac{n\lambda}{Nd}\right] = \cos^{-1}\left(1 - \frac{n\lambda}{4\,\lambda/2}\right) = \cos^{-1}\left(1 - \frac{n}{2}\right)$, $n = 1, 2, 3, \cdots, n \neq 4, 8, \cdots$

$n = 1 : \theta_1 = \cos^{-1}(1/2) = 60°$

$n = 2 : \theta_2 = \cos^{-1}(0) = 90°$

$n = 3 : \theta_3 = \cos^{-1}(-1/2) = 120°$

c. $\theta_m = \cos^{-1}(1 - m\lambda/d) = \cos^{-1}\left(1 - m\lambda/\lambda/2\right) = \cos^{-1}(1 - 2m)$, $m = 0, 1, 2, \cdots$

$m = 0 : \theta_0 = \cos^{-1}(1) = 0°$

$m = 1 : \theta_1 = \cos^{-1}(-1) = 180°$

d. $\Theta_0 = 2\cos^{-1}\left(1 - \frac{\lambda}{Nd}\right) = 2\cos^{-1}\left(1 - \frac{\lambda}{4\,\lambda/2}\right) = 2\cos^{-1}\left(1 - \frac{1}{2}\right) = 2\cos^{-1}\left(\frac{1}{2}\right) = 2(60°)$

$\Theta_0 = 120°$

e. $D_0 = 4N\left(\frac{d}{\lambda}\right) = 4(4)\left(\frac{\lambda/2}{\lambda}\right) = 8 = 9.03\,dB$

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Q. Design an ordinary end-fire uniform linear array with only one maximum so that its directivity is 20 dB (above isotropic). The spacing between the elements is λ/4, and its length is much greater than the spacing. Determine the

(a) number of elements

(b) overall length of the array (in wavelengths)

(c) approximate half-power beamwidth (in degrees)

(d) amplitude level (compared to the maximum of the major lobe) of the first minor lobe (in dB)

(e) progressive phase shift between the elements (in degrees).

Contents

a. $D_0 = 4N\left(\frac{d}{\lambda}\right)$

   $20 = 10 \log_{10} D_0$ (dimensionless) $\Rightarrow D_0$ (dimensionless) $= 10^2 = 100$

   $100 = 4N\left(\frac{\lambda}{4\lambda}\right) = N \Rightarrow N = 100$

b. $L = 99\left(\frac{\lambda}{4}\right) = \frac{99}{4}\lambda = 24.75\lambda$

c. $\Theta_{3dB} = \Theta_h = 2\cos^{-1}\left(1 - \frac{1.391\lambda}{Nd\pi}\right)\Big|_{n=100} = 2\cos^{-1}\left(1 - \frac{1.391\lambda}{\pi\left(\frac{\lambda}{4}\right)100}\right)$

   $= 2\cos^{-1}\left(1 - \frac{1.391(4)}{100\pi}\right) = 2\cos^{-1}(1 - 0.01771) = 2\cos^{-1}(0.98228)$

   $\Theta_h = 2(10.799°) = 21.598° \simeq 21.6°$

d. Sidelobe (dB) $\simeq -13.5$ dB

e. $\beta = \pm kd = \pm\frac{2\pi}{\lambda}\left(\frac{\lambda}{4}\right) = \pm\frac{\pi}{2} = \pm 90°$

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Q. Redesign the end-fire uniform array of above problem in order to increase its directivity while maintaining the same, as in the problem, uniformity, number of elements, spacing between them, and end-fire radiation.

(a) What different from the design of the problem are you going to do to achieve this? Be very specific, and give values.

(b) By how many decibels (maximum) can you increase the directivity, compared to the design of the problem?

(c) Are you expecting the half-power beamwidth to increase or decrease? Why increase or decrease and by how much?

(d) What antenna figure of merit will be degraded by this design? Be very specific in naming it, and why is it degraded?

Contents

a. Choose different phase excitation. That is

$$\beta = \pm \left(kd + \frac{2.94}{N} \right) \simeq \pm \left(kd + \frac{\pi}{N} \right)$$

$$\beta = \pm \left(\frac{2\pi}{\lambda} \frac{\lambda}{4} + \frac{2.94}{100} \right) = \pm \left(\frac{\pi}{2} + 0.0294 \right) = \pm (1.570796 + 0.0294) = \pm (1.6) = \pm 91.684°$$

b. Directivity increase by 1.789 factor = 2.526 dB

c. The HPBW will decrease because sidelobe level will increase.

$$\theta_h = 2 \cos^{-1}\left(1 - \frac{0.1398\lambda}{Nd} \right) = 2 \cos^{-1}\left(1 - \frac{0.1398\lambda}{100 \, \lambda/4} \right) = 2 \cos^{-1}\left(1 - \frac{0.1398 \cdot (4)}{100} \right)$$

$$= 2 \cos^{-1}(1 - 0.005592) = 2 \cos^{-1}(0.9944) = 2 \, (6.066°) = 12.13°$$

   decreased by 9.47°
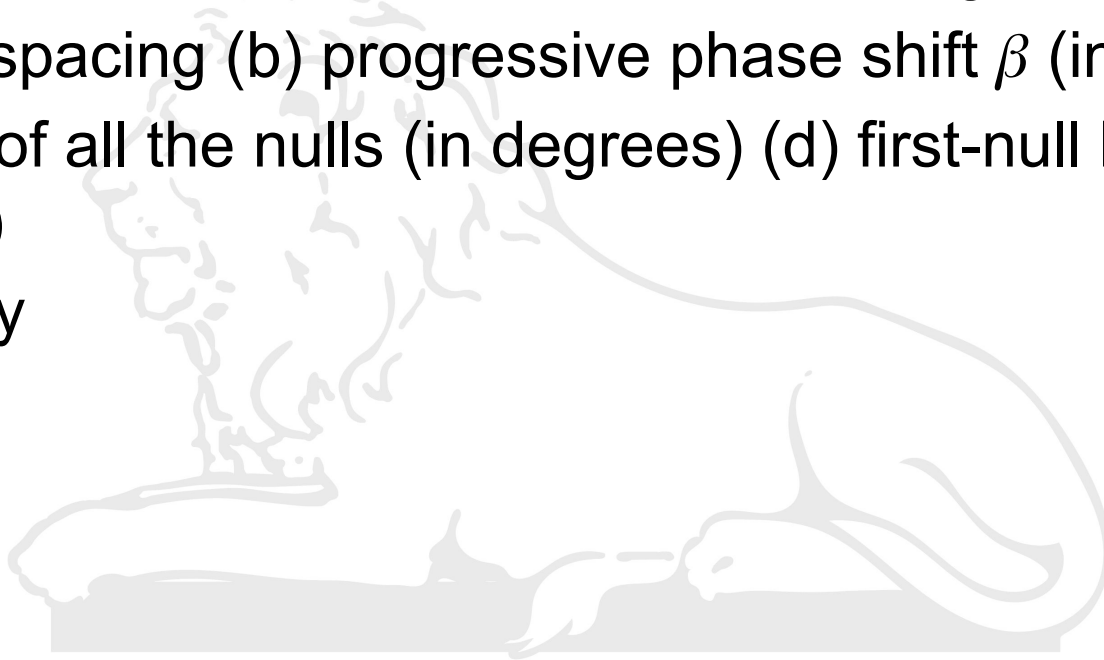
d. Sidelobe level will increase. It will be higher than -13.5 dB

Q. Ten isotropic elements are placed along the z-axis. Design a Hansen-Woodyard end-fire array

with the maximum directed toward θ = 180 degrees. Find the:

(a) desired spacing (b) progressive phase shift β (in radians)

(c) location of all the nulls (in degrees) (d) first-null beamwidth (in degrees)

(e) directivity

Contents

a. $d = \left(\frac{N-1}{N}\right)\frac{\lambda}{4} = 0.225\lambda$

b. $\beta = kd + \frac{2.94}{10} = 2\pi(0.225) + 0.294 = 1.7077$ rad

c. $\theta_n = \cos^{-1}\left(1 + (1-2n)\frac{\lambda}{2dN}\right)$

$\theta_n = \cos^{-1}\left(1 + (1-2n)\frac{1}{4.5}\right)$

$\theta_1 = \cos^{-1}(0.777) = 38.9°$, $\theta_2 = \cos^{-1}(0.333) = 70.53°$,

$\theta_3 = \cos^{-1}(-0.111) = 96.38°$, $\theta_4 = \cos^{-1}(-0.555) = 123.7°$

d. First null Beamwidth

$\theta_n = 2\cos^{-1}\left(1 - \frac{\lambda}{2dN}\right) = 2\cos^{-1}\left(1 - \frac{1}{2(0.225)\cdot 10}\right) = 77.88°$

e. $D_0 = 1.789\left[4N\cdot\left(\frac{d}{\lambda}\right)\right] = 1.789\left[4\cdot 10\cdot(0.225)\right] = 16.101 = 12,068\,dB$

Q. A uniform array of 20 isotropic elements is placed along the z-axis a distance λ/4 apart with a progressive phase shift of β rad. Calculate β (give the answer in radians) for the following array designs:

(a)   broadside

(b)   end-fire with maximum at $\theta = 0°$

(c) end-fire with maximum at $\theta = 180°$

(d) phased array with maximum aimed at $\theta = 30°$

(e) Hansen-Woodyard with maximum at $\theta = 0°$

(f) Hansen-Woodyard with maximum at $\theta = 180°$

Contents

$$kd = \frac{2\pi}{\lambda} \frac{\lambda}{4} = \frac{\pi}{2}$$

a. $\beta = 0$ radians

b. $\beta = -\pi/2$

c. $\beta = +\pi/2$

d. $\beta = -1.36 = -\frac{\sqrt{3}}{4}\pi = -0.433\,\pi$

e. $\beta = -\left(\frac{\pi}{2} + 0.147\right)$ or $-\left(\frac{\pi}{2} + 0.157\right) = -\frac{11}{20}\pi = -1.72$

f. $\beta = +\left(\frac{\pi}{2} + 0.147\right)$ or $+\left(\frac{\pi}{2} + 0.157\right) = \frac{11}{20}\pi = 1.72$

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Q. For a uniform broadside linear array of 10 isotropic elements, determine the approximate directivity (in dB) when the spacing between the elements is

(a) λ/4 (b) λ/2 (c) 3λ/4 (d) λ

$$D_0 \simeq 2N(d/\lambda)$$

a. $d = \frac{\lambda}{4}$, $D_0 \simeq 2 \cdot 10 \cdot \frac{1}{4} = 5 = 6.99\text{dB}$

   Computer Program : $D_0 = 7.132\text{dB}$

b. $d = \frac{\lambda}{2}$, $D_0 \simeq 2 \cdot 10 \cdot \frac{1}{2} = 10 = 10\text{dB}$

   Computer Program : $D_0 = 10.00\text{dB}$

c. $d = \frac{3\lambda}{4}$, $D_0 \simeq 2 \cdot 10 \cdot (0.75) = 15 = 11.76\text{dB}$

   Computer Program : $D_0 = 11.624\text{dB}$

d. $d = \lambda$, $D_0 \simeq 2 \cdot 10 \cdot (1) = 20 = 13.0\text{dB}$

   Computer Program : $D_0 = 10.011\text{dB}$

Q. The maximum distance d between the elements in a linear scanning array to suppress grating lobes is

$$d_{\max} = \frac{\lambda}{1 + |\cos(\theta_0)|}$$

where θ is the direction of the pattern maximum. What is the maximum distance between the elements, without introducing grating lobes, when the array is designed to scan to maximum angles of

(a) $\theta = 30°$ (b) $\theta = 45°$ (c) $\theta = 60°$

Contents

The recommended element spacing is

$$d = \frac{1}{1+\cos\theta},$$ where θ is the scan angle in degrees

a. $\theta_0 = 30°$

$$d = \frac{1}{1+\cos 30°} = 0.5359 \text{ wavelength}$$

b. $\theta_0 = 45°$

$$d = \frac{1}{1+\cos 45°} = \frac{1}{1+0.7071} = 0.58578 \text{ wavelength}$$

c. $\theta_0 = 60°$

$$d = \frac{1}{1+\cos 60°} = \frac{1}{1+0.5} = 0.6667 \text{ wavelength}.$$

Thank You

Questions?