Plane Waves

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Plane Waves in Free Space

Plane Waves in Free Space

n a <u>source free region</u>, the homogeneous vector wave quation and corresponding Helmholtz's equation:

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \mathbf{H}) = -\mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$$

$$\nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0 \qquad k = \omega\sqrt{\mu\epsilon}$$

Plane Waves in Free Space

In <u>free space</u>:

$$\nabla^2 \mathbf{E} + k_0^2 \mathbf{E} = 0 \qquad k_0 = \omega\sqrt{\mu_0 \epsilon_0} = \frac{\omega}{c} = \frac{2\pi}{\lambda_0}$$

It is a vector wave equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_0^2\right) E_x = 0$$

one each in the components E_x, E_y, and E_z

Plane Waves in Free Space

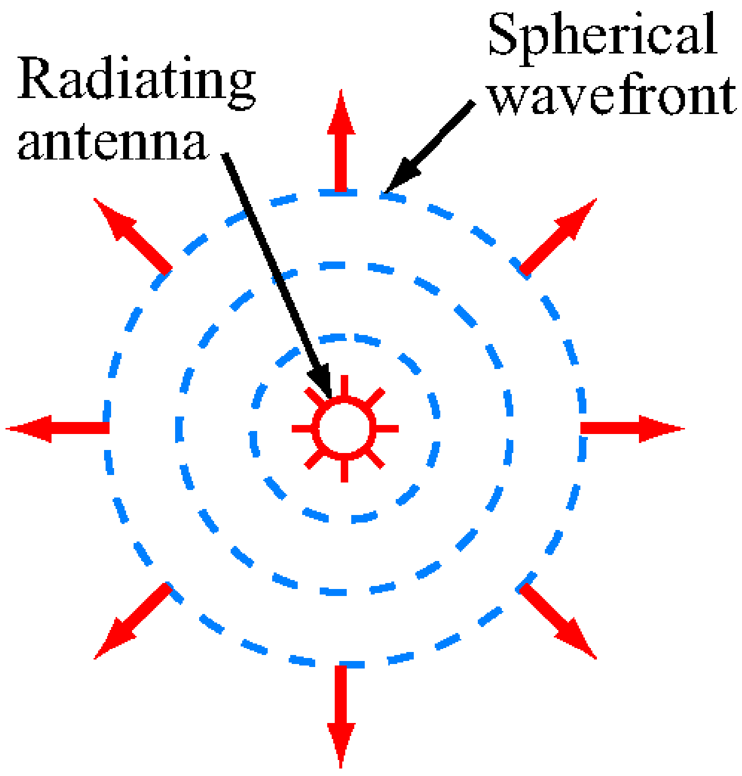
In <u>free space</u>:

$$\nabla^2 \mathbf{E} + k_0^2 \mathbf{E} = 0 \qquad k_0 = \omega\sqrt{\mu_0 \epsilon_0} = \frac{\omega}{c} = \frac{2\pi}{\lambda_0}$$

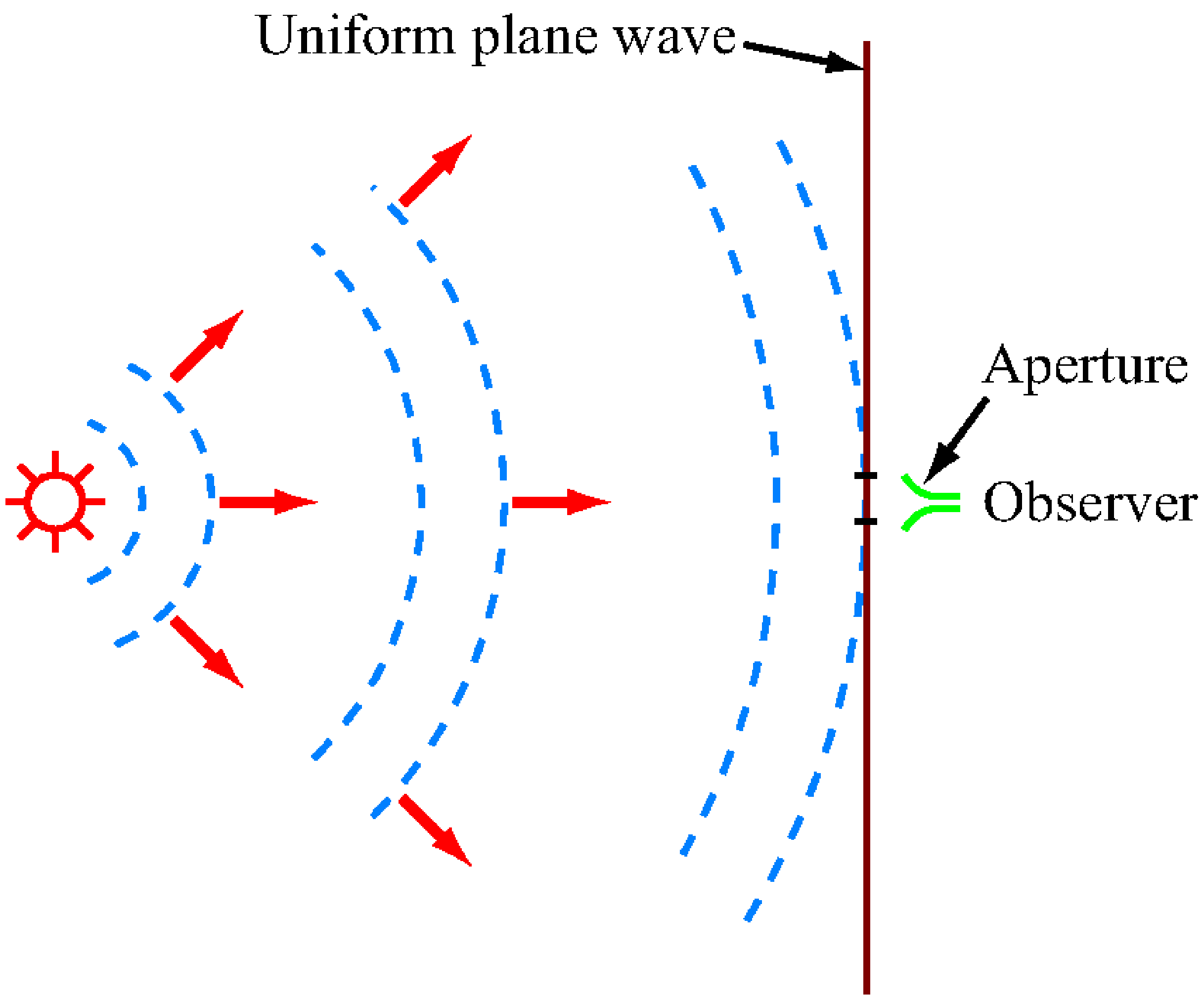
<u>Uniform Plane Wave Travelling in Z direction</u>

A *uniform plane wave* is a wave (i.e., a solution to the wave equation) in which the electric and magnetic field intensities are directed in fixed directions in space and are *constant in magnitude and phase on planes perpendicular to the direction of propagation.*

- infinite planes → infinite source → not possible (strictly)
- approximation : small antenna and larger distance of trans-reception



(a) Spherical wave



(b) Plane-wave approximation

Plane Waves in Free Space

In free space:

$$\nabla^2 \mathbf{E} + k_0^2 \mathbf{E} = 0 \qquad\qquad k_0 = \omega\sqrt{\mu_0\epsilon_0} = \frac{\omega}{c} = \frac{2\pi}{\lambda_0}$$

Uniform Plane Wave Travelling in Z direction

Consider a uniform plane wave characterized by a uniform E_x (uniform magnitude and constant phase) over plane surfaces perpendicular to z; that is,

$$\mathbf{E} = \hat{\mathbf{x}} E_x(z) \qquad E_y = E_z = 0 \quad \text{and} \quad \frac{\partial E_*}{\partial x} = \frac{\partial E_*}{\partial y} = 0$$

where $*$ denotes any component of $\mathbf{E}$

Plane Waves in Free Space

In free space:

$$\nabla^2 \mathbf{E} + k_0^2 \mathbf{E} = 0 \qquad k_0 = \omega\sqrt{\mu_0 \epsilon_0} = \frac{\omega}{c} = \frac{2\pi}{\lambda_0}$$

Uniform Plane Wave Travelling in Z direction

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_0^2\right) E_x = 0$$

$$\partial^2 E_x / \partial x^2 = 0 \quad \text{and} \quad \partial^2 E_x / \partial y^2 = 0$$

$$\frac{d^2 E_x}{dz^2} + k_0^2 E_x = 0$$

Plane Waves in Free Space

In free space:

Uniform Plane Wave Travelling in Z direction

$$\frac{d^2E_x}{dz^2} + k_0^2E_x = 0$$

$$\omega t - k_0 z = \text{A constant phase}$$

$$E_x(z) = E_x^+(z) + E_x^-(z) = E_0^+ e^{-jk_0z} + E_0^- e^{jk_0z}$$

$$u_p = \frac{dz}{dt} = \frac{\omega}{k_0} = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c$$

$$E_x^+(z, t) = \mathcal{R}e\left[E_x^+(z)e^{j\omega t}\right] = \mathcal{R}e\left[E_0^+ e^{j(\omega t - k_0 z)}\right] = E_0^+ \cos(\omega t - k_0 z) \quad \textit{traveling wave}$$

Plane Waves in Free Space

The associated magnetic field H

$$\nabla \times \mathbf{E} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ E_x^+(z) & 0 & 0 \end{vmatrix} = -j\omega\mu_0(\mathbf{a}_x H_x^+ + \mathbf{a}_y H_y^+ + \mathbf{a}_z H_z^+)$$

$$\frac{\partial E_x^+(z)}{\partial z} = \frac{\partial}{\partial z}(E_0^+ e^{-jk_0 z}) = -jk_0 E_x^+(z),$$

$$H_x^+ = 0,$$

$$H_y^+ = \frac{1}{-j\omega\mu_0}\frac{\partial E_x^+(z)}{\partial z}$$

$$H_z^+ = 0.$$

$$H_y^+(z) = \frac{k_0}{\omega\mu_0} E_x^+(z) = \frac{1}{\eta_0} E_x^+(z)$$

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \cong 120\pi \cong 377$$

intrinsic impedance of the free space

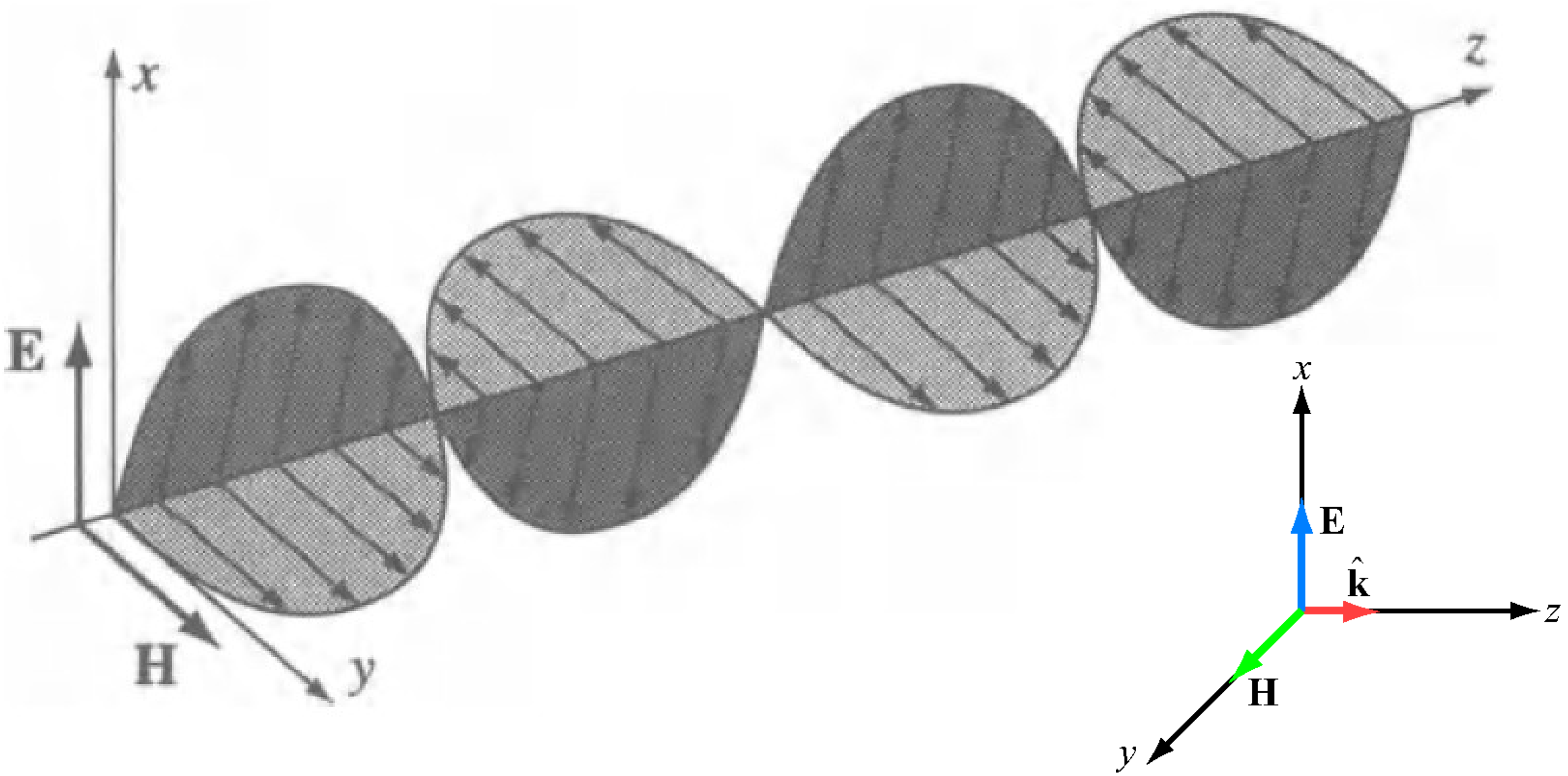
Plane Waves in Free Space

Uniform Plane Wave Travelling in Z direction

- E field → x direction
- H field → y direction
- Wave Propagation → z direction
- E and H are mutually perpendicular
- and they are perpendicular to the direction of propagation
- Hence, there are referred to as "Transverse Electro-Magnetic Waves (TEM Waves)"

Plane Waves in Free Space

Uniform Plane Wave Travelling in Z direction



Plane Waves in Lossless Medium

Plane Waves in Lossless Medium

n a <u>source free region</u>, the homogeneous Helmholtz's quation:

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0$$

$$k = \omega\sqrt{\mu\epsilon} = \frac{2\pi}{\lambda}$$

$$\frac{d^2 E_x}{dz^2} + \omega^2 \mu\varepsilon E_x = 0$$

$$u_p = \frac{dz}{dt} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\ \epsilon}}$$

$$E_x(z) = E_0^+ e^{-jkz} + E_0^- e^{jkz}$$

$$H_y^+ = \frac{1}{-j\omega\mu}\frac{\partial E_x^+(z)}{\partial z}$$

$$H_y^+(z) = \frac{k}{\omega\mu} E_x^+(z)$$

$$H_y^+(z) = \frac{1}{\eta} E_x^+(z)$$

$$\eta = \frac{E_x^+(z)}{H_y^+(z)} = \frac{\omega\mu}{k} = \sqrt{\frac{\mu}{\varepsilon}}$$

Plane Waves in Lossless Medium

Uniform Plane Wave Travelling in Z direction

- E field → x direction
- H field → y direction
- Wave Propagation → z direction
- E and H are mutually perpendicular
- and they are perpendicular to the direction of propagation
- Hence, there are referred to as "Transverse Electro-Magnetic Waves (TEM Waves)"
- Phase velocity is less than "c"

https://em8e.eecs.umich.edu/jsmodules/ulaby_modules.html

Plane Waves in Lossy Medium

Plane Waves in Lossy Medium

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B} = -j\omega\mu\mathbf{H}$$

$$\nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\mathbf{D} = \mathbf{J} + j\omega\varepsilon\mathbf{E}$$

$$\nabla \cdot \varepsilon\mathbf{E} = \rho$$

$$\nabla \times (\nabla \times \mathbf{E}) = -j\omega\mu(\nabla \times \mathbf{H})$$

$$\nabla \times (\nabla \times \mathbf{E}) = -j\omega\mu(\mathbf{J} + j\omega\varepsilon\mathbf{E})$$

$$\nabla \times (\nabla \times \mathbf{E}) = -\nabla^2\mathbf{E} + \nabla(\nabla \cdot \mathbf{E})$$

$$-\nabla^2\mathbf{E} + \nabla(\nabla \cdot \mathbf{E}) = -j\omega\mu(\mathbf{J} + j\omega\varepsilon\mathbf{E})$$

$$\nabla^2\mathbf{E} = \nabla\left(\frac{\rho}{\varepsilon}\right) + j\omega\mu\mathbf{J_s} + j\omega\mu(\sigma\mathbf{E} + j\omega\varepsilon\mathbf{E})$$

In a source free region, the homogeneous Helmholtz's equation:

$$\nabla^2\mathbf{E} = j\omega\mu(\sigma\mathbf{E} + j\omega\varepsilon\mathbf{E})$$

$$\nabla^2\mathbf{E} = j\omega\mu(j\omega\varepsilon)\mathbf{E}$$

Plane Waves in Lossy Medium

n a <u>source free region</u>, the homogeneous Helmholtz's quation:

$$\nabla^2 \mathbf{E} = j\omega\mu(\sigma \mathbf{E} + j\omega\varepsilon \mathbf{E})$$

$$\nabla^2 \mathbf{E} = j\omega\mu(j\omega\varepsilon)\mathbf{E}$$

$$j\omega\varepsilon_c = \sigma + j\omega\varepsilon$$

$$\nabla^2 \mathbf{E} = j\omega\mu(j\omega\varepsilon_c)\mathbf{E} = j\omega\mu\left(j\omega\varepsilon\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]\right)\mathbf{E}$$

$$\nabla^2 \mathbf{E} - j\omega\mu\left(j\omega\varepsilon\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]\right)\mathbf{E} = 0$$

complex permittivity

$$\varepsilon_c = \frac{\sigma + j\omega\varepsilon}{j\omega} = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon\left[1 - j\frac{\sigma}{\omega\varepsilon}\right] = \varepsilon + j\varepsilon''$$

loss tangent

$$\tan\delta_c = \frac{\epsilon''}{\epsilon'} \cong \frac{\sigma}{\omega\epsilon}$$

Plane Waves in Lossy Medium

n a <u>source free region</u>, the homogeneous Helmholtz's quation:

$$\nabla^2 \mathbf{E} - j\omega\mu\left(j\omega\varepsilon\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]\right)\mathbf{E} = 0$$

$$\nabla^2 \mathbf{E} - \gamma^2 \mathbf{E} = 0$$

$$E_x(z) = E_0^+ e^{-\gamma z} + E_0^- e^{+\gamma z}$$

$$E_x(z) = E_0^+ e^{-\alpha z} e^{-j\beta z} + E_0^- e^{+\alpha z} e^{+j\beta z}$$

$$E_x(z,t) = E_0^+ e^{-\alpha z}\cos(\omega t - \beta z) + E_0^- e^{+\alpha z}\cos(\omega t + \beta z)$$

propagation constant

$$\gamma = j\omega\sqrt{\mu\varepsilon}\sqrt{\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]}$$

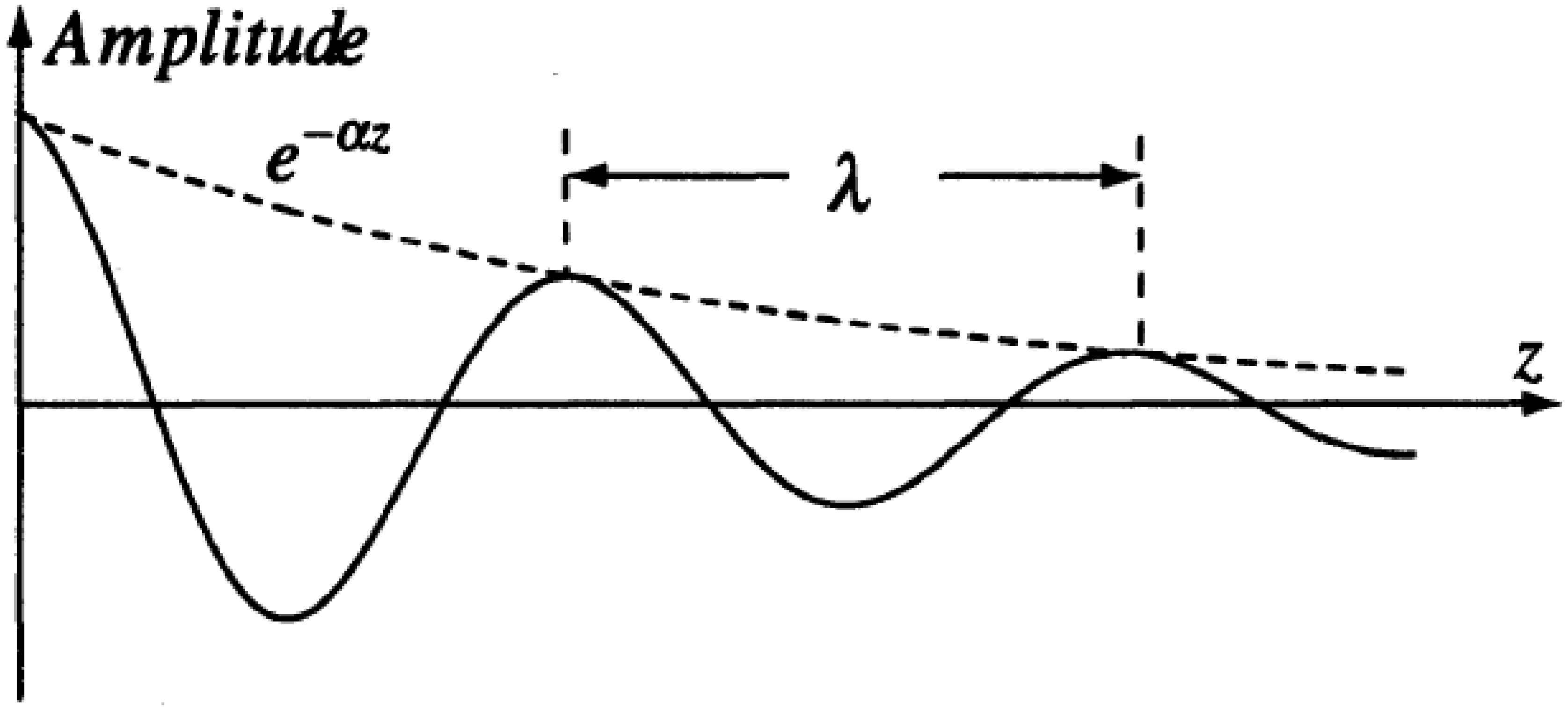
$$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

$$\gamma = \alpha + j\beta$$

attenuation constant

phase constant

Plane Waves in Lossy Medium



Plane Waves in Lossy Medium

n a source free region, the homogeneous Helmholtz's quation:

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]}$$

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]}}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]}}$$

Plane Waves in Lossy Medium

n a <u>source free region</u>, the homogeneous Helmholtz's quation:

$$H_y^+ = \frac{1}{-j\omega\mu}\frac{\partial E_x^+(z)}{\partial z}$$

$$\frac{\partial E_x^+}{\partial z} = \frac{\partial}{\partial z}(E_0^+ e^{-\gamma z}) = -\gamma(E_0^+ e^{-\gamma z}) = -\gamma E_x^+(z)$$

$$-\gamma E_x^+(z) = -j\omega\mu H_y$$

$$\eta = \frac{E_x^+(z)}{H_y^+(z)} = \frac{j\omega\mu}{\gamma} \qquad \eta = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$$

Plane Waves in Lossy Medium

Conducting Medium: (σ/ωε) >> 1

$$\gamma \cong j\omega\sqrt{\mu\epsilon}\sqrt{\frac{\sigma}{j\omega\epsilon}} = \sqrt{j}\sqrt{\omega\mu\sigma} = \frac{1+j}{\sqrt{2}}\sqrt{\omega\mu\sigma}$$

$$\sqrt{j} = (e^{j\pi/2})^{1/2} = e^{j\pi/4} = (1+j)/\sqrt{2}$$

$$\gamma = \alpha + j\beta \cong (1+j)\sqrt{\pi f\mu\sigma},$$

$$\alpha = \beta = \sqrt{\pi f \mu\sigma}$$

$$E_x(z) = E_0^+ e^{-z/\delta} e^{-jz/\delta}$$

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{1}{\pi f\mu\sigma}} = \frac{1}{\alpha}$$

is known as the *skin depth* of the wave. It is defined as that distance in which the amplitude of a plane wave is attenuated to 1/ *e* of its original amplitude. The *skin depth* in conductors is very small. In the microwave range, it can be of the order of a few microns (depending on material and frequency)

Plane Waves in Lossy Medium

Conducting Medium: $(\sigma/\omega\varepsilon) \gg 1$

$$v = \frac{\omega}{\beta} = \omega\delta = \sqrt{2\omega\mu\sigma}$$

$$\eta_c = \sqrt{\frac{\mu}{\epsilon_c}} \cong \sqrt{\frac{j\omega\mu}{\sigma}} = (1 + j)\sqrt{\frac{\pi f\mu}{\sigma}}$$

Plane Waves in Lossy Medium

Example : Copper

$$\sigma = 5.80 \times 10^{7} \quad \text{(S/m)},$$
$$\mu = 4\pi \times 10^{-7} \quad \text{(H/m)},$$
$$u_p = 720 \text{ (m/s)} \quad \text{at} \quad 3 \text{ (MHz)},$$

For copper at 3 (MHz), $\lambda = 0.24$ (mm). As a comparison, a 3 (MHz) electromagnetic wave in air has a wavelength of 100 (m).

At 3 MHz

$$\alpha = \sqrt{\pi(3 \times 10^{6})(4\pi \times 10^{-7})(5.80 \times 10^{7})} = 2.62 \times 10^{4} \quad \text{(Np/m)}$$

Since the attenuation factor is $e^{-\alpha z}$, the amplitude of a wave will be attenuated by a factor of $e^{-1} = 0.368$ when it travels a distance $\delta = 1/\alpha$. For copper at 3 (MHz) this distance is $(1/2.62) \times 10^{-4}$ (m), or 0.038 (mm). At 10 (GHz) it is only 0.66 (μm)

Plane Waves in Lossy Medium

Skin Depths, δ in (mm), of Various Materials

Material	σ (S/m)	f = 60 (Hz)	1 (MHz)	1 (GHz)
Silver	6.17×10^7	8.27 (mm)	0.064 (mm)	0.0020 (mm)
Copper	5.80×10^7	8.53	0.066	0.0021
Gold	4.10×10^7	10.14	0.079	0.0025
Aluminum	3.54×10^7	10.92	0.084	0.0027
Iron ($\mu_r \cong 10^3$)	1.00×10^7	0.65	0.005	0.00016
Seawater	4	32 (m)	0.25 (m)	†

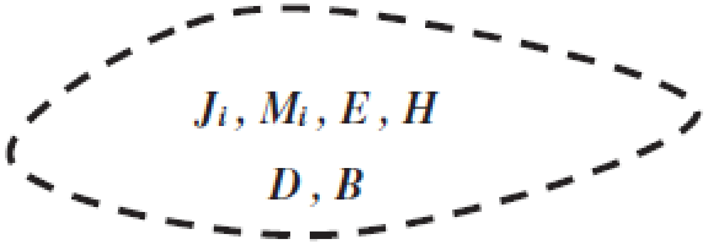
† The ϵ of seawater is approximately $72\epsilon_0$. At $f = 1$ (GHz), $\sigma/\omega\epsilon \cong 1$ (not $\gg 1$). Under these conditions, seawater is not a good conductor

Poynting’s Theorem

Poynting's Theorem

$$\nabla X E = -M_i - \frac{\partial B}{\partial t}$$

$$\nabla X H = J_i + \sigma E + \frac{\partial D}{\partial t}$$



As engineers and physicists, one is often concerned with the power generated by the sources. From circuit theory concepts and its relation with field theory, we know that power generated by sources is given by:

$$-\iiint (H \cdot M_i + E \cdot J_i) dv$$

Now, all that we know are Maxwell Equations :

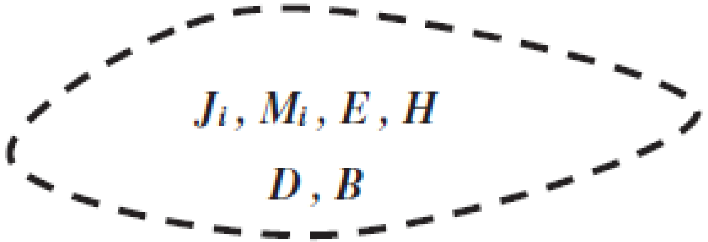
$$-\iiint \left[H \cdot \left(-\nabla X E - \frac{\partial B}{\partial t}\right) + E \cdot \left(\nabla X H - \sigma E - \frac{\partial D}{\partial t}\right)\right] dv$$

$$\iiint \left(H \cdot \frac{\partial B}{\partial t}\right) dv + \iiint \left(E \cdot \frac{\partial D}{\partial t}\right) dv + \iiint (E \cdot \sigma E) dv + \iiint [H \cdot (\nabla X E) - E \cdot (\nabla X H)] dv$$

Poynting's Theorem

$$\iiint \left(H \cdot \frac{\partial B}{\partial t}\right) dv + \iiint \left(E \cdot \frac{\partial D}{\partial t}\right) dv$$

$$+ \iiint (E \cdot \sigma E) dv + \iiint [H \cdot (\nabla X E) - E \cdot (\nabla X H)] dv$$



From familiar circuit theory concepts, we recognize that the first three integrals as: rate of change of stored magnetic energy, rate of change of stored electric energy and power dissipated. However, the term in last integral, i.e. [***H*** • (∇X***E***) − ***E*** • (∇X***H***)] is not that obvious.

$$\nabla \cdot (AXB) = B \cdot (\nabla XA) - A \cdot (\nabla XB)$$

$$\iiint \left(H \cdot \frac{\partial B}{\partial t}\right) dv + \iiint \left(E \cdot \frac{\partial D}{\partial t}\right) dv + \iiint (E \cdot \sigma E) dv + \iiint \nabla \cdot (EXH) dv$$

$$\iiint \left(H \cdot \frac{\partial B}{\partial t}\right) dv + \iiint \left(E \cdot \frac{\partial D}{\partial t}\right) dv + \iiint (E \cdot \sigma E) dv + \oiint (EXH) ds$$

Poynting's Theorem

$$\iiint \left(H \cdot \frac{\partial B}{\partial t}\right) dv + \iiint \left(E \cdot \frac{\partial D}{\partial t}\right) dv$$
$$+ \iiint (E \cdot \sigma E) dv + \oiint (EXH) ds$$

power generated = rate of change of stored energy in magnetic and electric fields + power dissipated + *power radiated*

$$\mathscr{P} = \mathbf{E} \times \mathbf{H}$$

Quantity $\mathscr{P}$ is known as the ***Poynting vector***, which is a power density vector associated with an electromagnetic field.

$$\mathscr{P}_{\mathrm{av}}(z) = \tfrac{1}{2}\mathscr{R}e[\mathbf{E}(z) \times \mathbf{H}^*(z)]$$

$$\mathscr{P}_{\mathrm{av}}(z) = \frac{1}{T}\int_0^T \mathscr{P}(z, t)\, dt$$

Polarization

Polarization

The electric (or magnetic) field intensity of a uniform plane wave has a direction in space. This direction may either be constant or may change as the wave propagates. The polarization of a plane wave is "*the figure traced by the tip of the electric field vector as a function of time, at a fixed point in space*."

Linearly Polarized Wave:

$$\mathbf{E} = \mathbf{a}_x E_x$$

linearly polarized in the x direction.

$$\mathbf{E} = \hat{\mathbf{y}} E_y(z) = \hat{\mathbf{y}} E_y e^{-\gamma z}$$

linearly polarized in the y direction.

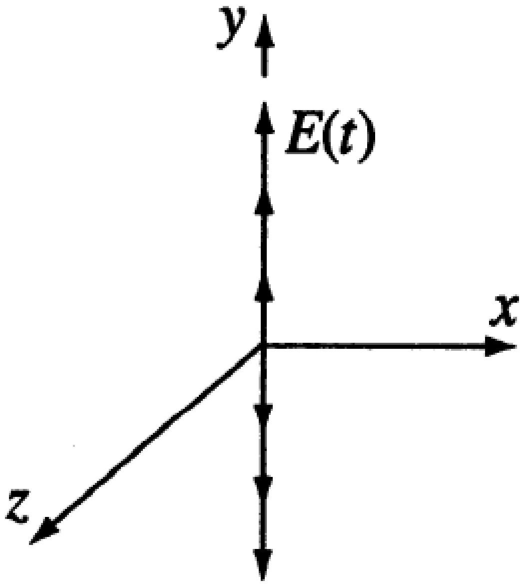
$$\mathbf{E}(z,t) = \hat{\mathbf{x}} E_x e^{-\alpha z}\cos(\omega t - \beta z) + \hat{\mathbf{y}} E_y e^{-\alpha z}\cos(\omega t - \beta z)$$

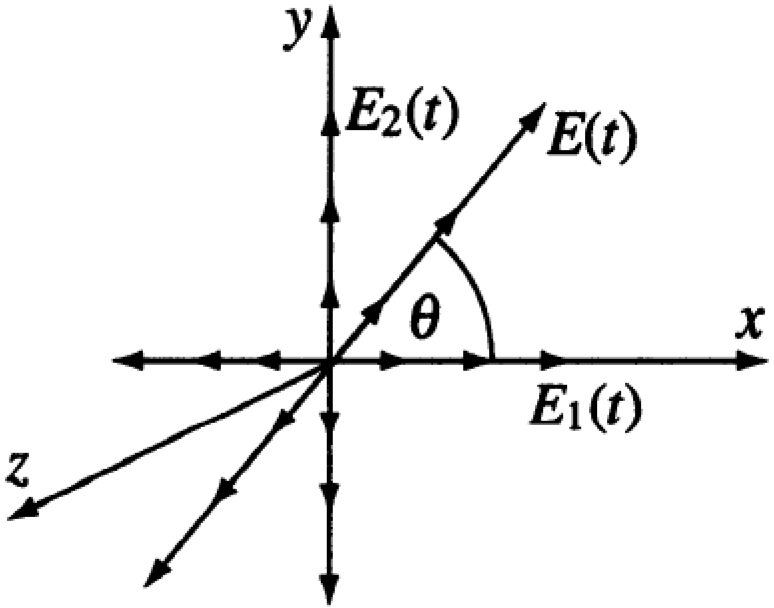
linearly polarized in the θ direction.

linearly polarized wave at an angle arctan(|Ey|/|Ex|)

Polarization

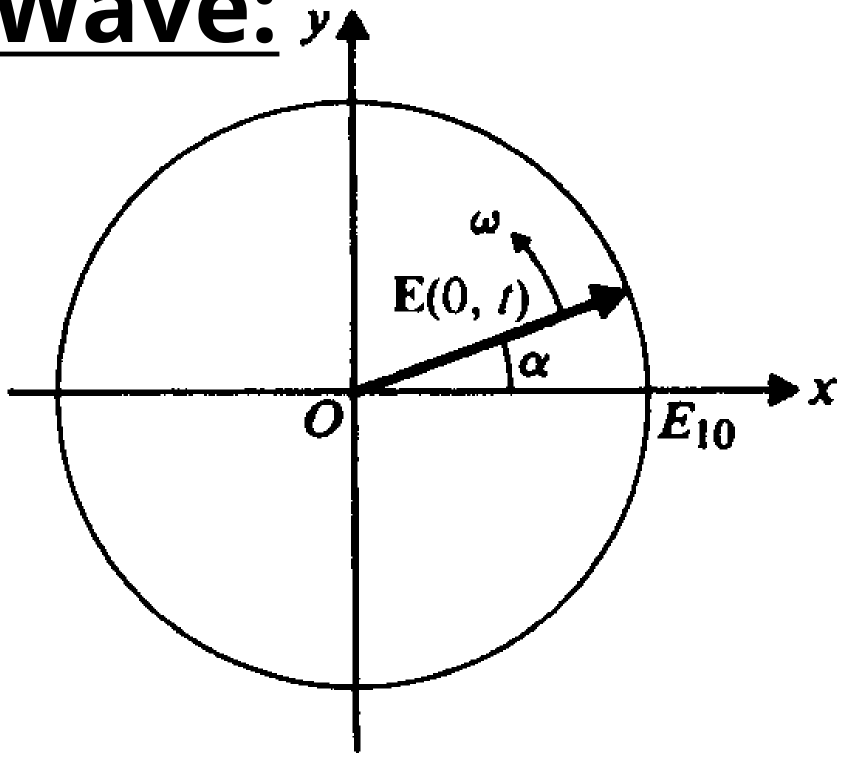
Linearly Polarized Wave:





Polarization

lliptically and Circularly Polarized Wave:



$$\mathbf{E}(z) = \mathbf{a}_x E_1(z) + \mathbf{a}_y E_2(z)$$
$$= \mathbf{a}_x E_{10} e^{-jkz} - \mathbf{a}_y jE_{20} e^{-jkz}$$

$$\mathbf{E}(z, t) = \mathscr{R}e\{[\mathbf{a}_x E_1(z) + \mathbf{a}_y E_2(z)]e^{j\omega t}\}$$
$$= \mathbf{a}_x E_{10} \cos(\omega t - kz) + \mathbf{a}_y E_{20} \cos\left(\omega t - kz - \frac{\pi}{2}\right)$$

$$\mathbf{E}(0, t) = \mathbf{a}_x E_1(0, t) + \mathbf{a}_y E_2(0, t)$$
$$= \mathbf{a}_x E_{10} \cos \omega t + \mathbf{a}_y E_{20} \sin \omega t$$

Hence $\mathbf{E}$, which is the sum of two linearly polarized waves in both space and time quadrature, is ***elliptically polarized*** if $E_{20} \neq E_{10}$, and is ***circularly polarized*** if $E_{20} = E_{10}$. This is a ***right-hand*** or ***positive circularly polarized wave***.

Polarization

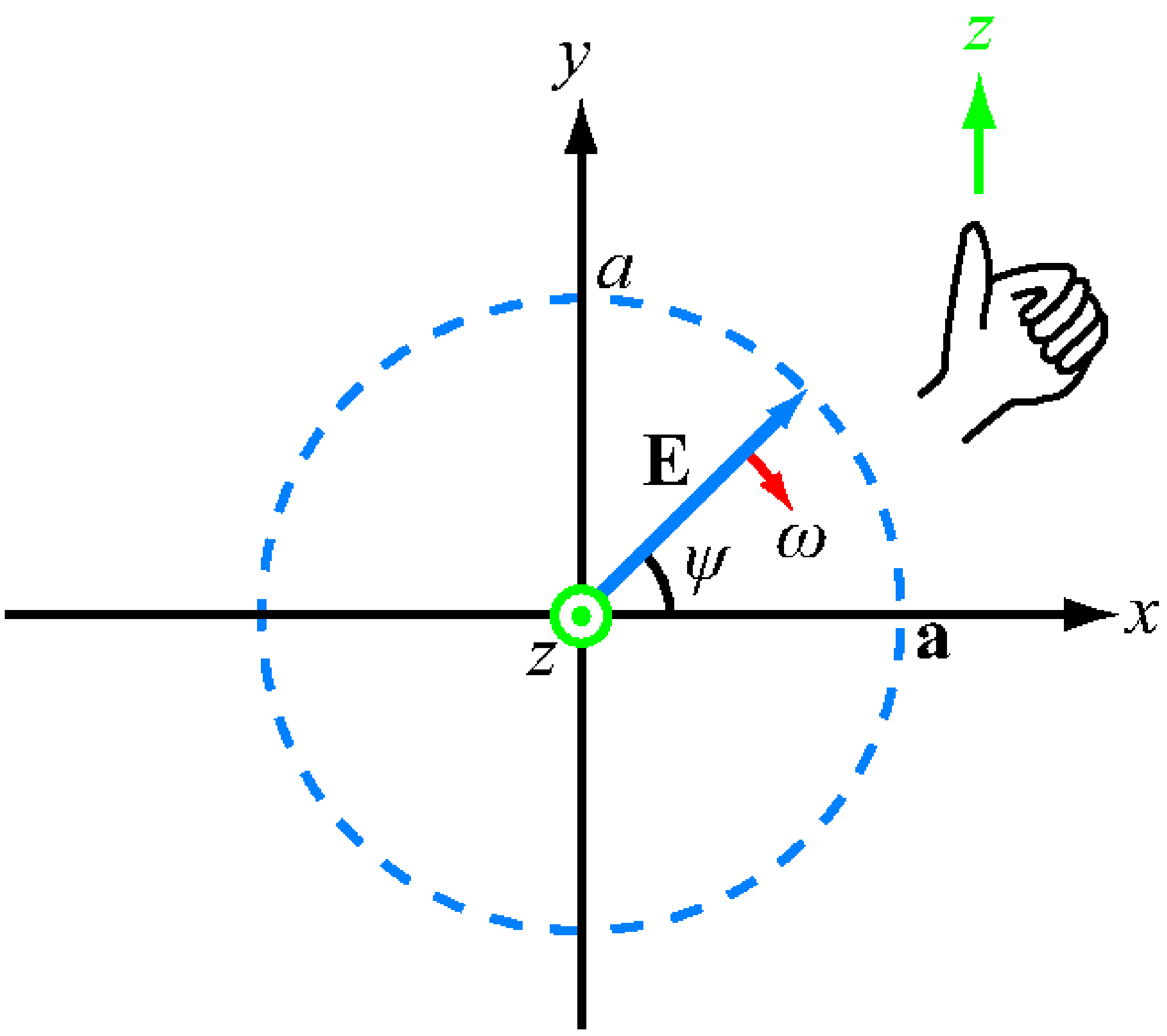
lliptically and Circularly Polarized Wave:

$$\mathbf{E}(z) = \mathbf{a}_x E_{10} e^{-jkz} + \mathbf{a}_y j E_{20} e^{-jkz}$$

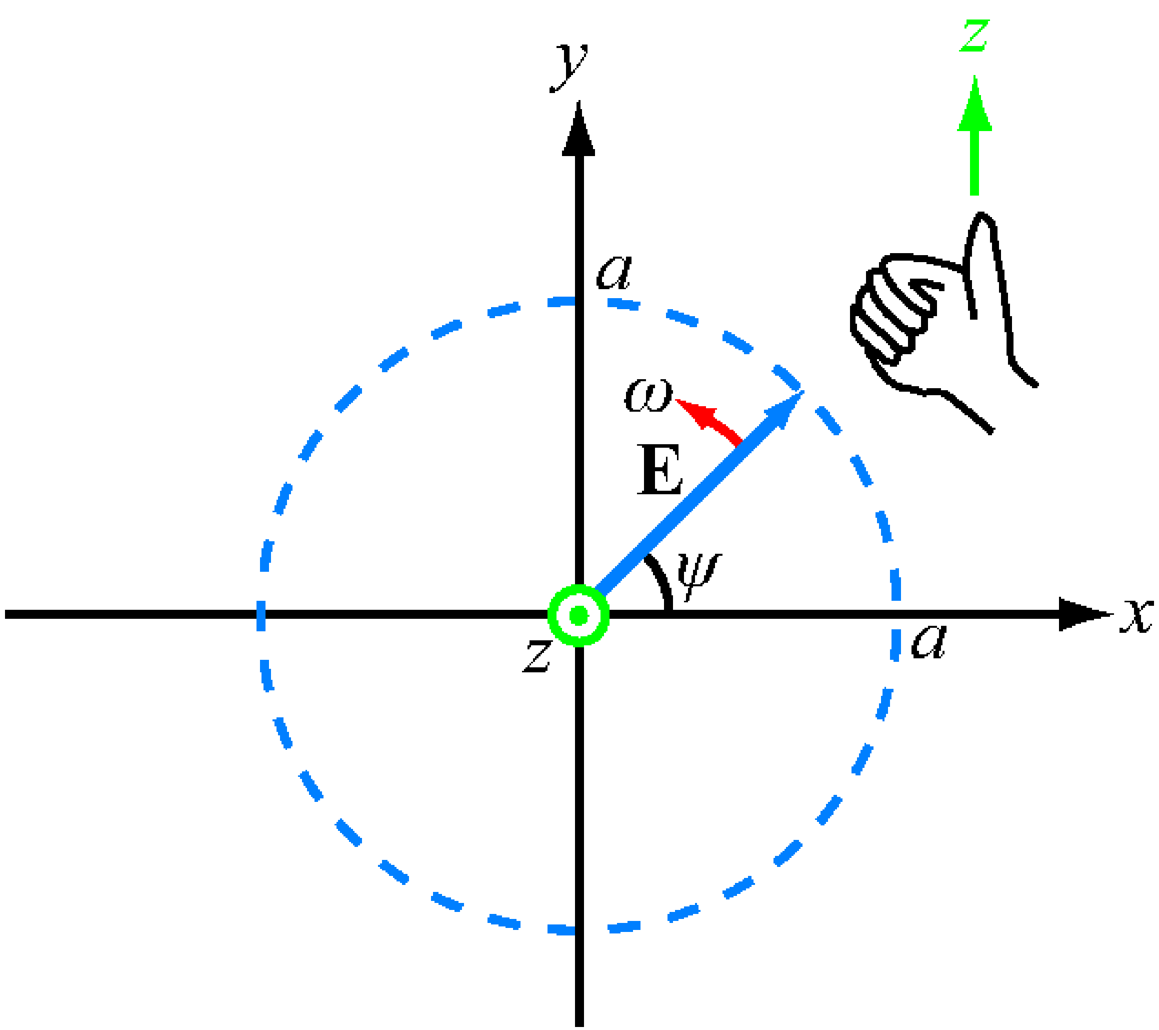
$$\mathbf{E}(z, t) = \mathscr{R}e\{[\mathbf{a}_x E_1(z) + \mathbf{a}_y E_2(z)]e^{j\omega t}\}$$

$$\mathbf{E}(0, t) = \mathbf{a}_x E_{10} \cos \omega t - \mathbf{a}_y E_{20} \sin \omega t$$

If $E_{20} = E_{10}$, $\mathbf{E}$ will be circularly polarized, and its angle measured from the x-axis at $z = 0$ will now be $-\omega t$, indicating that $\mathbf{E}$ will rotate with an angular velocity ω in a *clockwise* direction; this is a ***left-hand*** or ***negative circularly polarized wave***.



(a) LHC polarization



(b) RHC polarization

Thank You