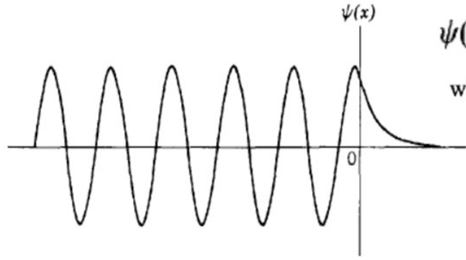
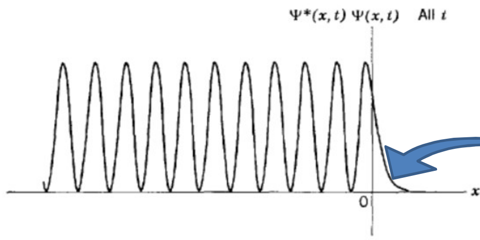


$$\psi(x) = D\cos k_1 x - D\frac{k_2}{k_1}\sin k_1 x \qquad x \le 0$$

which is a real function of x if we take D real.

The wave function can be thought of as oscillating in time according to $e^{-iEt/\hbar}$ with an amplitude whose space dependence is given by $\psi(x)$.



$\Psi(x,t) = \psi(x)\, e^{-iEt/\hbar}$

$$\psi(x) = De^{-k_2 x} \qquad x \ge 0$$

in the region $x > 0$ the probability density

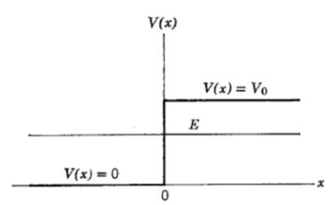
$$\Psi^*\Psi = D^* e^{-k_2 x} e^{+iEt/\hbar} De^{-k_2 x} e^{-iEt/\hbar} = D^* D e^{-2k_2 x}$$

decreases rapidly with increasing x,

Note that the total energy E is a constant.
***Classically,* the particle cannot enter the region *x* > 0**

The reason is $\quad E = \dfrac{p^2}{2m} + V(x) < V(x)$

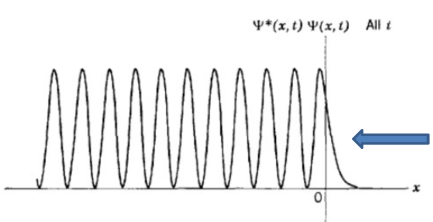
or $\quad \dfrac{p^2}{2m} < 0$



Thus the kinetic energy $p^2/2m$ would be negative in the region $x > 0$.

lead to an imaginary value for the linear momentum p in the region.

This is not allowed!

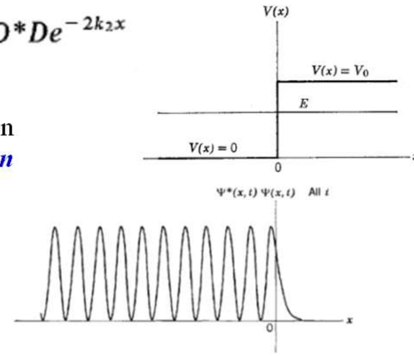


There is a finite probability of finding the particle in the region *x* > 0

This phenomenon, known as *penetration of the classically excluded region*!

$$\Psi^*\Psi = D^* e^{-k_2 x} e^{+iEt/\hbar} D e^{-k_2 x} e^{-iEt/\hbar} = D^* D e^{-2k_2 x}$$

The probability of finding the particle with a coordinate $x > 0$ is only appreciable in a region starting at $x = 0$ and extending in a ***penetration distance*** Δx



$e^{-2k_2 x} \rightarrow 0$ when $x \rightarrow$ much larger than $1/k_2$

Since $k_2 = \sqrt{2m(V_0 - E)}/\hbar$, $\Delta x = \dfrac{\hbar}{\sqrt{2m(V_0 - E)}}$

uncertainty Δp in the momentum, $\Delta p \simeq \dfrac{\hbar}{\Delta x} \simeq \sqrt{2m(V_0 - E)}$

Consequently, the energy of the particle is uncertain by an amount

$$\Delta E \simeq \frac{(\Delta p)^2}{2m} \simeq V_0 - E$$

Example 6-1. Estimate the penetration distance Δx for a very small dust particle, of radius $r = 10^{-6}$ m and density $\rho = 10^4$ kg/m^3, moving at the very low velocity $v = 10^{-2}$ m/sec, if the particle impinges on a potential step of height equal to twice its kinetic energy in the region to the left of the step.

Eisberg & Resnick Quantum Mechanics

►The mass of the particle is

$$m = \frac{4}{3}\pi r^3 \rho \simeq 4 \times 10^{-18}\ \text{m}^3 \times 10^4\ \text{kg/m}^3 = 4 \times 10^{-14}\ \text{kg}$$

Its kinetic energy before hitting the step is

$$\frac{1}{2}mv^2 \simeq \frac{1}{2} \times 4 \times 10^{-14}\ \text{kg} \times 10^{-4}\ \text{m}^2/\text{sec}^2 = 2 \times 10^{-18}\ \text{joule}$$

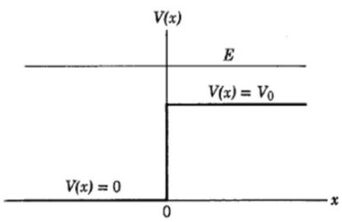
and this is also the value of $(V_0 - E)$. The penetration distance is

$$\Delta x = \frac{\hbar}{\sqrt{2m(V_0 - E)}} \simeq \frac{10^{-34}\ \text{joule-sec}}{\sqrt{2 \times 4 \times 10^{-14}\ \text{kg} \times 2 \times 10^{-18}\ \text{joule}}}$$

$$\simeq 2 \times 10^{-19}\ \text{m}$$

- ➔ **This is many orders of magnitude smaller than what could be detected in any possible measurement.**
- ➔ **For the more massive particles and higher energies, typically considered in classical mechanics, Δx is even smaller.**

THE STEP POTENTIAL (ENERGY GREATER THAN STEP HEIGHT, *E* > *Vo*)



- A particle of total energy ***E*** traveling in the region $x < 0$, in the increasing x direction
- It will suffer an impulsive retarding force

  $F = - dV(x)/dx$ at the point $x = 0$.
- But the impulse will only slow the particle, and it will enter the region $x > 0$, continuing its motion in the direction of increasing x.

Its total energy ***E*** remains constant
Its momentum in $\boldsymbol{x < 0}$ region is $\boldsymbol{p_1}$ where $\boldsymbol{p_1^2/2m = E}$
Its momentum in $\boldsymbol{x > 0}$ region is $\boldsymbol{p_2}$ where $\boldsymbol{p_2^2/2m = E - Vo}$

Predictions of *quantum mechanics* are not so simple!

If E is not too much larger than V_0, the theory predicts that the particle has an appreciable chance of being reflected at the step back into the region $x < 0$, even though it has enough energy to pass over the step into the region $x > 0$.

We learned that the motion of the particle under the influence of the step potential is described by the wave function

$$\Psi(x,t) = \psi(x)e^{-iEt/\hbar}$$

Eigenfunction $\psi(x)$ satisfies the TISE for the potential

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x) \qquad x < 0$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = (E - V_0)\psi(x) \qquad x > 0$$

The eigenfunction $\psi(x)$ also satisfies the conditions requiring finiteness, single valuedness, and continuity, for it and its derivative, particularly at the joining point $x = 0$.

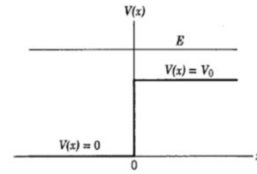
describes the motion of a free particle of momentum p_1.

Its general solution is $\psi(x) = Ae^{ik_1x} + Be^{-ik_1x} \qquad x < 0$

$$k_1 = \frac{\sqrt{2mE}}{\hbar} = \frac{p_1}{\hbar}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = (E - V_0)\psi(x) \qquad x > 0$$

TISE



$$\psi(x) = Ce^{ik_2x} + De^{-ik_2x} \qquad x > 0$$

$$k_2 = \frac{\sqrt{2m(E - V_0)}}{\hbar} = \frac{p_2}{\hbar} \qquad E > V_0$$

The wave function specified by these two forms consists of traveling waves of de Broglie wavelength

$$\lambda_1 = h/p_1 = 2\pi/k_1 \text{ in the region } x < 0,$$

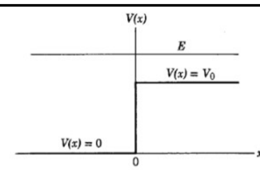
$$\lambda_2 = h/p_2 = 2\pi/k_2 \text{ in the region } x > 0.$$

Classically, a particle initially in the region $x < 0$, and moving towards $x = 0$, would have a probability = 1 of passing the point $x = 0$ and entering the region $x > 0$.

This is not true in quantum mechanics!

$$\psi(x) = Ae^{ik_1x} + Be^{-ik_1x} \qquad x < 0$$

$$\psi(x) = Ce^{ik_2x} + De^{-ik_2x} \qquad x > 0$$



We need to take both terms of the general solution to describe the incident and reflected traveling waves in the region $x < 0$.

We do not need the second term of the general solution for region $x > 0$.

$$D = 0$$

The arbitrary constants A, B, and C must be chosen to make $\psi(x)$ and $d\psi(x)/dx$ continuous at $x = 0$.

$$A(e^{ik_1x})_{x=0} + B(e^{-ik_1x})_{x=0} = C(e^{ik_2x})_{x=0} \Longrightarrow A + B = C$$

The second requirement, that the values of the derivatives of the two expressions for $\psi(x)$ be the same at $x = 0$, is satisfied if

$$ik_1A(e^{ik_1x})_{x=0} - ik_1B(e^{-ik_2x})_{x=0} = ik_2C(e^{ik_2x})_{x=0} \Longrightarrow k_1(A - B) = k_2C$$

$$B = \frac{k_1 - k_2}{k_1 + k_2}A \quad \text{and} \quad C = \frac{2k_1}{k_1 + k_2}A$$

Thus the eigenfunction is
$$\psi(x) = \begin{cases} Ae^{ik_1x} + A\dfrac{k_1 - k_2}{k_1 + k_2}e^{-ik_1x} & x \le 0 \\ A\dfrac{2k_1}{k_1 + k_2}e^{ik_2x} & x \ge 0 \end{cases}$$

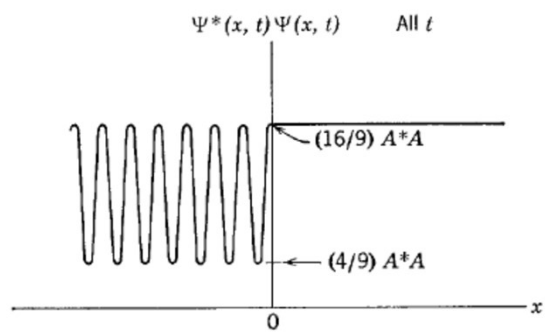
Calculate probability density Ψ*(*x,t*) Ψ(*x,t*) = ψ*(*x*) ψ(*x*) for the wave function Ψ(*x,t*) corresponding to the eigenfunction ψ(*x*) for the case $k_1 = 2k_2$ (a specific case)

We do not plot either the eigenfunction or the wave function, as both are complex.

- In region $x > 0$, the wave function is a pure traveling wave (amplitude 4A/3) traveling to the right, and so the probability density is constant
- In region $x < 0$, the wave function is a combination of the incident traveling wave (amplitude A) moving to the right, and a reflected traveling wave (amplitude A/3) moving to the left.
- As the amplitude of the reflected wave is necessarily smaller than that of the incident wave, the two cannot combine to yield a pure standing wave.

- **Their sum Ψ(*x,t*) in *x* < 0 region is something between a standing wave and a traveling wave.**

It oscillates but has minimum values greater than zero.



The probability density Ψ*Ψ for the eigenfunction when $k_1 = 2k_2$

The ratio of the intensity of the reflected wave to the intensity of the incident wave gives the probability that the particle will be reflected by the potential step back into the region $x < 0$. This probability is the *reflection coefficient* R. That is

$$R = \frac{B^*B}{A^*A} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^* \left(\frac{k_1 - k_2}{k_1 + k_2}\right) = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \quad E > V_0$$

We see from this result that $R < 1$ when $E > V_0$

This is in contrast to the value $R = 1$ when $E < V_0$

What is surprising about the present result is not that $R < 1$, but that $R > 0$

According to the strict definition, the reflection coefficient R *is*

$$R = \frac{v_1 B^*B}{v_1 A^*A} = \frac{B^*B}{A^*A}$$

where v_1 is the velocity of the particle in the region $x < 0$.

$$T = \frac{v_2 C^*C}{v_1 A^*A} = \frac{v_2}{v_1}\left(\frac{2k_1}{k_1 + k_2}\right)^2$$

where v_2 is the velocity of the particle in the region $x > 0$.

$$v_1 = \frac{p_1}{m} = \frac{\hbar k_1}{m} \qquad v_2 = \frac{p_2}{m} = \frac{\hbar k_2}{m}$$

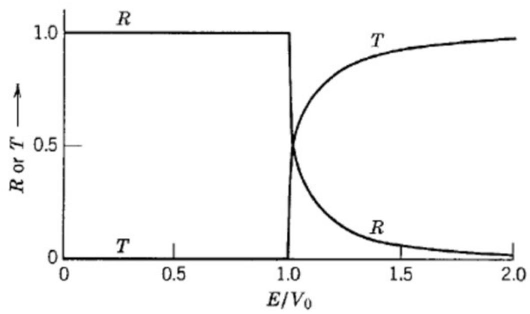
$$T = \frac{k_2}{k_1}\frac{(2k_1)^2}{(k_1 + k_2)^2} = \frac{4k_1 k_2}{(k_1 + k_2)^2} \qquad E > V_0$$

$$R + T = 1$$

This useful relation is the motivation for defining the reflection and transmission coefficients in terms of probability fluxes.

By evaluating k_1 and k_2

$$R = 1 - T = \left(\frac{1 - \sqrt{1 - V_0/E}}{1 + \sqrt{1 - V_0/E}}\right)^2 \qquad \frac{E}{V_0} > 1$$



The figure also plots the results

$$R = 1 - T = 1 \qquad \frac{E}{V_0} < 1$$