Lecture-04

10/08/2023.

Line integral (path integral) $\Rightarrow$ $\int_a^b A\cdot dl$

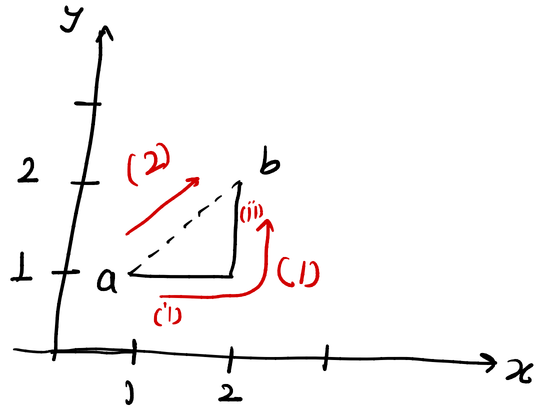
$\oint_L A\cdot dl$ [Circulation of A around L]

Surface integral (flux) $\Rightarrow$ $\int_S A\cdot ds$

$\oint A\cdot ds$

Volume integral $\int_V T\, d\tau$

Ex 1 $\Rightarrow$ Calculate the line integral of the funⁿ $\bar{v} = y^2\hat{x} + 2x(y+1)\hat{y}$ from the point $a = (1,1,0)$ to the point $b = (2,2,0)$, along the path (1) & (2).



Ans $\Rightarrow$ $\bar{v} = y^2\hat{x} + 2x(y+1)\hat{y}$

$d\bar{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

Path(1) Segment (i) $\Rightarrow$ $y = 1$, $dl = dx\,\hat{x}$

$\therefore \int v\cdot dl = \int_1^2 y^2 dx = \int_1^2 1\,dx = 1$

Segment (ii) $\Rightarrow$ $x = 2$, $dl = dy\,\hat{y}$

$\therefore \int v\cdot dl = \int_1^2 2x(y+1)\,dy = \int_1^2 4(y+1)\,dy = 10$

$\therefore$ for Path (1) $\int_a^b \bar{v}\cdot d\bar{l} = 11$ ✓ **Ans**

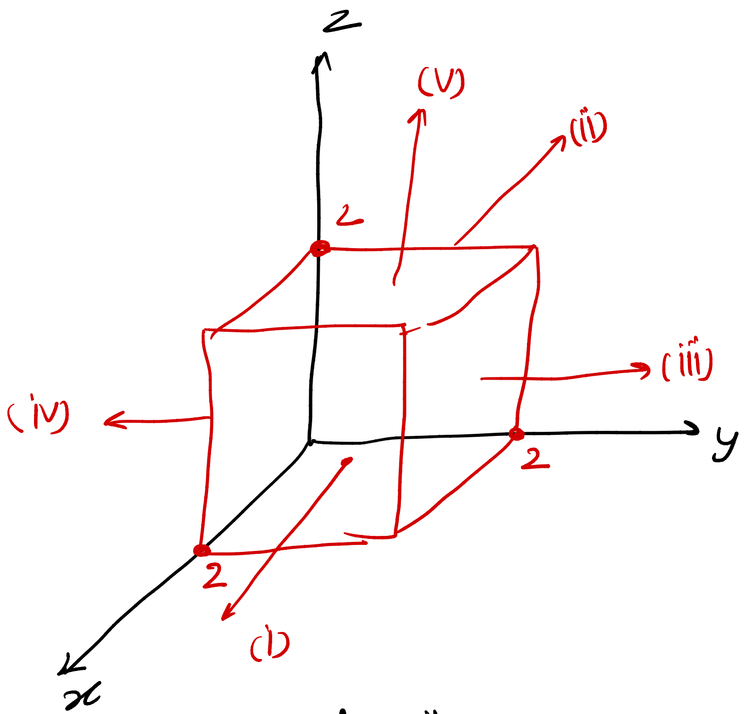
Path (2) $\Rightarrow$ $x = y$

$dx = dy$

$\therefore \int_a^b v\cdot dl = \int_1^2 y^2 dx + 2x(y+1)\,dy \Rightarrow \int_1^2 x^2 dx + 2x(x+1)\,dx$

$= 10$ ✓ **Ans**

Ex2 ⇒ Calculate the surface integral of $\bar{v} = 2xz\,\hat{x} + (x+2)\,\hat{y} + y(z^2-3)\,\hat{z}$ over five sides. Let upward & outward be the +ve direction as indicated by the arrow.



Ans ⇒ <u>For surface (i)</u> :-

$x=2, \quad d\bar{a} = dy\,dz\,\hat{x}$

$$\therefore \int v \cdot d\bar{a} = \int 2xz\,dy\,dz.$$

$$= 4\int_0^2 dy \int_0^2 z\,dz = 16$$

<u>For surface (ii)</u> ⇒ $x = 0, \quad da = -dy\,dz\,\hat{x}$

$$\int v \cdot da = 0$$

<u>For surface (iii)</u> ⇒ $y=2, \quad da = dx\,dz\,\hat{y}$

$$\therefore \int v \cdot da = \int_0^2 (x+2)\,dx \int_0^2 dz = 12$$

<u>For surface (iv)</u> ⇒ $y=0, \quad d\bar{a} = -\,dx\,dz\,\hat{y}$

$$\therefore \int v \cdot da = -\int_0^2 (x+2)\,dx \int_0^2 dz = -12$$

<u>for surface (V)</u> ⇒ $z=2, \quad da = dx\,dy\,\hat{z}$

$$\therefore \int v \cdot da = \int_0^2 \int_0^2 y\,dx\,dy = 4$$

$$\therefore \text{Total flux} = 16 + 0 + 12 - 12 + 4 = 20 \text{ Ans}$$

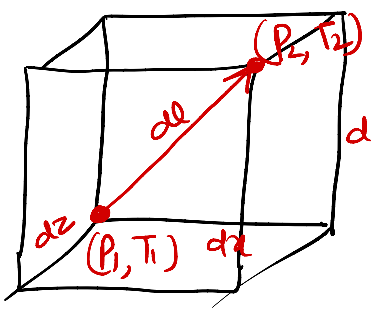
Differencial calculus ⇒ Derivative of fields.

§ Ordinary derivative ⇒ $y = f(x)$ ⟶ funⁿ of one variable.

$\frac{df}{dx}$ ⟶ how rapidly the funⁿ $f(x)$ varies when we change x by a tiny amount dx.

In other words ⇒

$$df = \left(\frac{df}{dx}\right) dx$$



Two points $(P_1$ & $P_2)$
Temp $(T_1$ & $T_2)$

$T(x, y, z)$

$$dT = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz \quad \text{——— (1)}$$

* The left hand side is scalar.
* The R.H.S. has three terms, dx, dy, dz & they are the components of vector.
* This suggest that $\frac{\partial T}{\partial x}$, $\frac{\partial T}{\partial y}$ & $\frac{\partial T}{\partial z}$ should also be the component of a vector.

It can be written as -

$$dT = \left(\frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}\right) \cdot \left(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}\right)$$

$$= \nabla T \cdot dl$$

where,

$$\boxed{\nabla T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}}$$

gradient of T, grad T, del T

∇T ⟶ Vector

⇒ The component of ∇T transform in the same way as the component of $(d\vec{l})$ do.

$\nabla T \Rightarrow$ direction $\Rightarrow$ $\boxed{\nabla T \text{ points in the direction of maximum increase of fun}^n\ T.}$

$|\nabla T| \longrightarrow$ $\boxed{\text{Rate of increase along this max. direction}}$

Q:⇒ Find the gradient of $\bar{r} = \sqrt{x^2+y^2+z^2}$.
(distance from origin)

A ⇒

$$\nabla r = \frac{\partial r}{\partial x}\hat{x} + \frac{\partial r}{\partial y}\hat{y} + \frac{\partial r}{\partial z}\hat{z}$$

$$\Rightarrow \frac{1}{2\sqrt{x^2+y^2+z^2}}\,2x\,\hat{x} + \frac{1}{2\sqrt{x^2+y^2+z^2}}\,2y\,\hat{y} + \frac{1}{\sqrt{x^2+y^2+z^2}}\,2z\,\hat{z}$$

$$= \frac{x\hat{x}+y\hat{y}+z\hat{z}}{\sqrt{x^2+y^2+z^2}} = \frac{\bar{r}}{|r|} = \hat{r}$$

The operator ∇ ⇒

$$\nabla T = \underbrace{\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)}_{\text{del}} T$$

$$\boxed{\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}}$$

∇ is a vector operator. Alone it means nothing.

$T\nabla \longrightarrow$ Means nothing, It is an operator.

$\nabla T \longrightarrow$ Vector field.

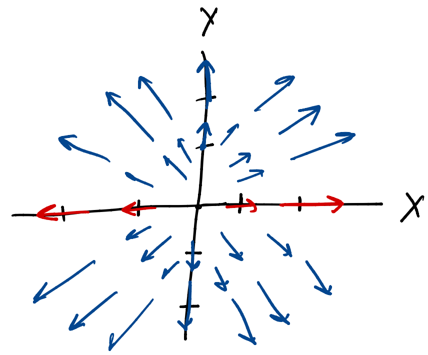
Homework:

Q ⇒ Let $\bar{r}$ be separation vector from a fixed point (x', y', z') to the point (x, y, z) & let r be it's length. Show that -

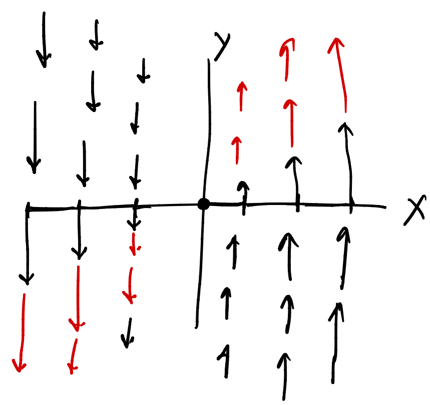
(a) ⇒ $\nabla(r^2) = 2\bar{r}$ (b) ⇒ $\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2}\hat{r}$ (c) ⇒ $\nabla(r^n) = ?$

Sketch of 2-D vector fields ⇒

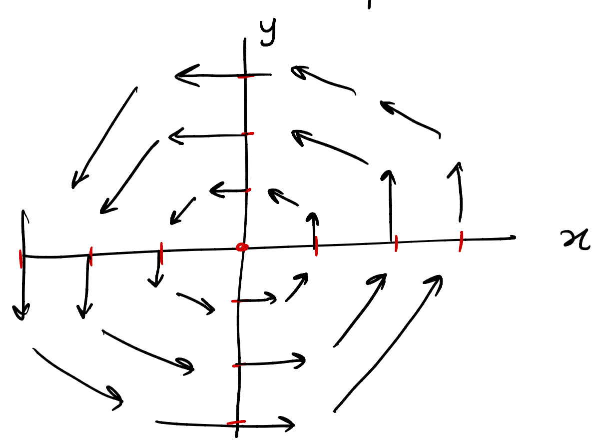
(i) ⇒ $\bar{v} = x\hat{x} + y\hat{y}$



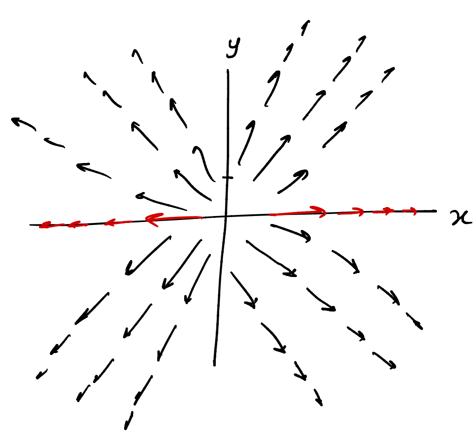
(ii) ⇒ $\bar{v} = x\hat{y}$



(iii) ⇒ $\bar{v} = -y\hat{x} + x\hat{y}$



(iv) ⇒ $\bar{v} = \dfrac{\hat{r}}{r^2}$



$\frac{1}{r^2}$