Lecture-24

10/10/2023

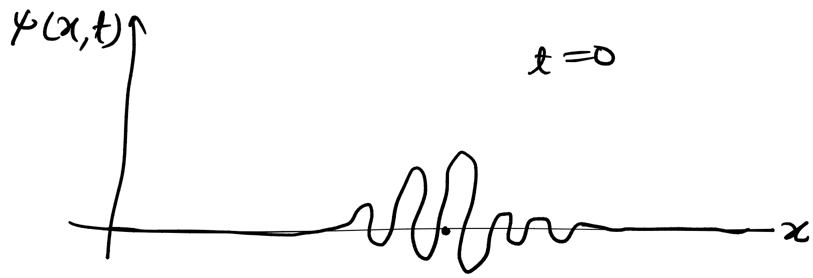
* As quantum particles jointly display particle & wave features, we need to embody them simultaneously.

* A particle is well localized in space. [C.M.]

* In QM. a particle is described by a wave f^n $\Rightarrow$ corresponding to the matter wave associated with particle (de Broglie's conjucture).

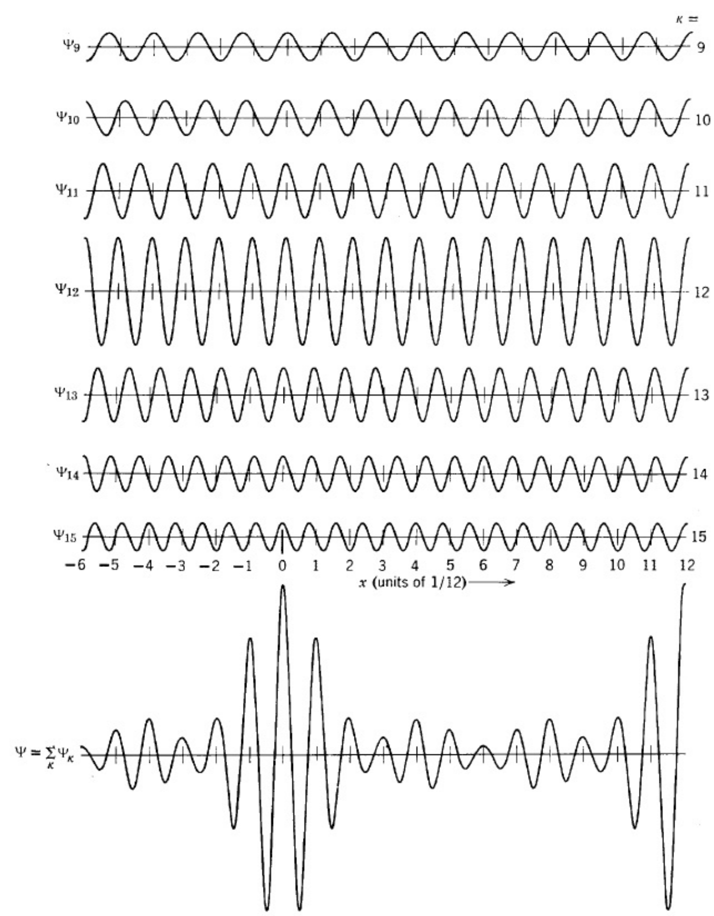
$\Rightarrow$ If the wave f^n is made to vanish everywhere except in the neighborhood of the particle it can then be used to describe the dynamics of the particle.

* A localized wave f^n is called a wave packet.



wavepacket for a particle (de-Broglie wave)

* Localized wave packet can be constructed by superposing waves of slightly different wavelength. The phases & amplitude can be chosen to make the superposition constructive in the desired region & destructive outside it.



$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx-\omega t)} dk$$

$\phi(k)$ is the amplitude of wavepacket.

wavepacket at a given time at $t=0$

$$\psi_0(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{ikx} dk.$$

$$\Rightarrow \phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi_0(x) e^{-ikx} dx$$

* In summary, the particle is represented by a wavepacket that is obtained by adding a large no. of waves of different frequency.

Group & phase velocities ⇒

$$\psi_1 = A \cos(kx-\omega t)$$

$$\psi_2 = A \cos((k+\Delta k)x - (\omega+\Delta\omega)t)$$

$$\psi = \psi_1 + \psi_2$$

$$\boxed{\cos C + \cos d = 2 \cos\frac{C+d}{2} \cdot \cos\frac{C-d}{2}.}$$

$$= 2A \cos\left\{\frac{(k+k+\Delta k)x - (\omega+\omega+\Delta\omega)t}{2}\right\} \cdot \cos\left\{\frac{(k-k-\Delta k)x - (\omega-\omega-\Delta\omega)t}{2}\right\}$$

$$\Rightarrow 2A \cos\left\{\left(\frac{2k+\Delta k}{2}\right)x - \left(\frac{2\omega+\Delta\omega}{2}\right)t\right\} \cdot \cos\left\{\frac{\Delta k x}{2} - \frac{\Delta\omega t}{2}\right\}$$

$$2k+\Delta k \sim 2k$$

$$2\omega+\Delta\omega \sim 2\omega$$

$$\psi \Rightarrow 2A \cos(kx-\omega t) \cdot \cos\left(\frac{\Delta k x}{2} - \frac{\Delta\omega}{2} t\right).$$

⇒ Implies that $\cos(kx-\omega t)$ wave is superimposed by a modulation of wavevector $\Delta k/2$ & angular frequency $\Delta\omega/2$.

* The phase velocity $\boxed{v_p = \frac{\omega}{k}}$

* & the velocity of wave group $v_g = \frac{\Delta\omega}{\Delta k}$

$$\boxed{v_g = \frac{d\omega}{dk}}$$

* **Relationship b/w v_p & v_g ⇒**

$$v_g = \frac{d\omega}{dk} \Rightarrow \frac{d}{dk}(v_p k)$$

$$= v_p \cdot 1 + k\frac{dv_p}{dk}$$

$$\left[k = 2\pi/\lambda \quad dk = -\frac{2\pi}{\lambda^2}d\lambda\right]$$

$$= v_p + \frac{2\pi}{\lambda}\cdot\frac{dv_p}{(-2\pi)d\lambda}\cdot\lambda^2$$

$$\boxed{v_g = v_p - \lambda\frac{dv_p}{d\lambda}}$$

Phase & Group velocity of the de-Broglie waves associated with a particle ⇒

$$v_p = \frac{\omega}{k}$$

$$= \frac{mc^2}{\hbar k}$$

$$\left[E = mc^2 = \hbar\omega \quad \omega = \frac{mc^2}{\hbar}\right]$$

$$\left[\lambda = \frac{h}{p} \quad p = \frac{h}{\lambda}\frac{2\pi}{2\pi} \Rightarrow \hbar k\right]$$

$$= \frac{mc^2}{p}$$

$$= \frac{\cancel{m}c^2}{\cancel{m}v} = \frac{c^2}{v}$$

⇒ This exceeds the velocity of the particle (v) & velocity of light (c).

Thus v_p has no physical significance.