Lecture-07

18/08/2023

Three-Dimensional Dirac delta funⁿ :

$$\delta^3(r) = \delta(x)\,\delta(y)\,\delta(z)$$

$$\int_{\text{all space}} \delta^3(r)\, d\tau = \int_{-\infty}^{\infty}\iint \delta(x)\,\delta(y)\,\delta(z)\, dx\, dy\, dz = 1$$

$$\therefore \boxed{\int_{\text{all space}} f(r)\,\delta^3(r-a)\, d\tau = f(a)}$$

$$\therefore \nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta^3(r)$$

$$* \quad \int v\cdot ds = \int \nabla\cdot v\, d\tau$$

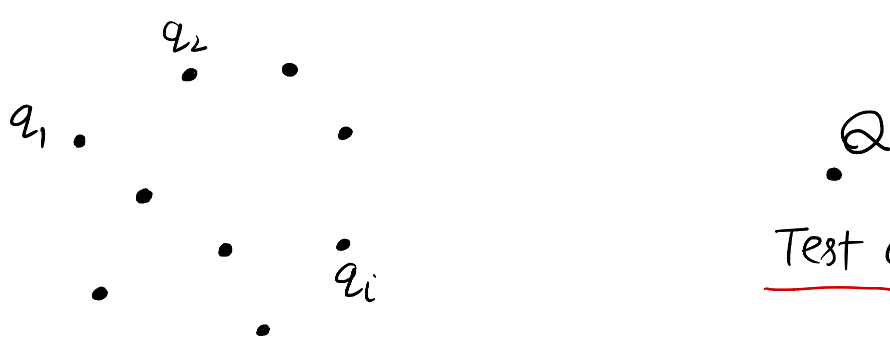
$\int v\cdot ds \rightarrow 4\pi$

$\int \nabla\cdot v\, d\tau \rightarrow \int \nabla\cdot v\, d\tau = \int 4\pi\,\delta^3(r)\, d\tau = 4\pi$ ✓

Electrostatic :

Static (Time independent)

Electric field is produced by a static charge distribution



Test charge (may be moving)

Source charges (Are stationary).

What force do $q_1 \ldots q_i$ exert on Q ?

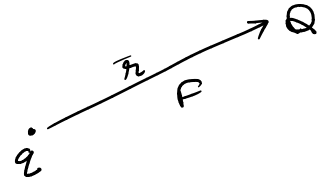
Principle of superposition :

$$F = F_1 + F_2 + \ldots\ldots F_i$$

$F_i \rightarrow$ Force due to q_i alone (ignoring all other charges)

Coulomb's law ⇒

$$\boxed{\bar{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{r}}$$



ε_0 = permittivity of free space
$= 8.85\times10^{-12}\ C^2/N\cdot m^2$

$\frac{1}{4\pi\varepsilon_0} \sim 9\times10^{9}$

* Electric field ⇒

$$F = F_1 + F_2 + \cdots$$
$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 Q}{r_1^2}\hat{r}_1 + \frac{q_2 Q}{r_2^2}\hat{r}_2 + \cdots\right)$$

$$\boxed{F = Q\bar{E}}$$

where,

$$\boxed{\bar{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{r}_i}$$

Continuous charge distribution ⇒

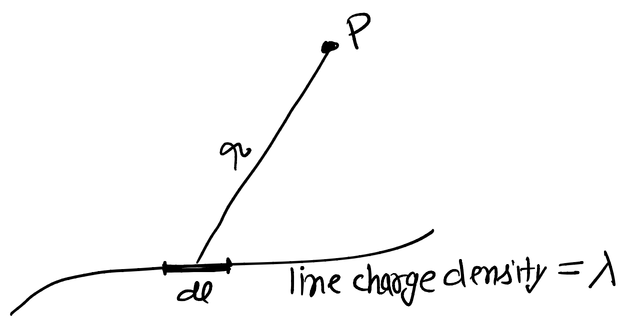
$$\boxed{E(r) = \frac{1}{4\pi\varepsilon_0}\int\frac{1}{r^2}\hat{r}\,dq}$$

(i) ⇒ Charge is spread along a line ⇒

λ = charge per unit length.

$dq = \lambda\, dl$

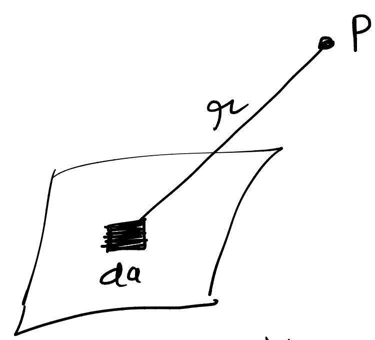
$$E(r) = \frac{1}{4\pi\varepsilon_0}\int\frac{1}{r^2}\hat{r}\,\lambda\, dl \quad ✓$$



(ii) ⇒ Charge is smeared out over a surface ⇒

$dq = \sigma\, da$

$$E(r) = \frac{1}{4\pi\varepsilon_0}\int\frac{1}{r^2}\hat{r}\,\sigma\, da. \quad ✓$$



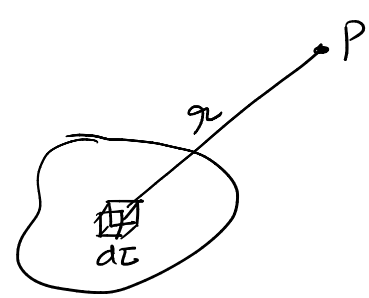
Surface charge density $= \sigma$

(iii) ⇒ Charge fills a volume ⇒

$dq = \rho\, d\tau$

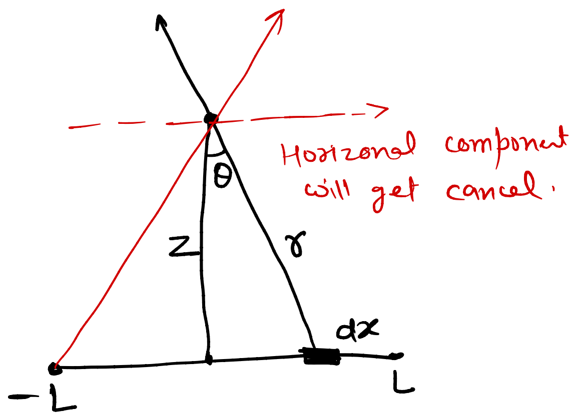
[ρ is charge per unit volume, $d\tau$ is volume element]

$$\bar{E}(r) = \frac{1}{4\pi\varepsilon_0}\int\frac{1}{r^2}\hat{r}\,\rho\, d\tau \quad ✓$$



Ex 1 ⇒ Find the electric field a distance z above the midpoint of a straight line segment of length 2L, which carries a uniform line charge λ.

Ans ⇒



$$\bar{E} = \frac{1}{4\pi\varepsilon_0}\int_{-L}^{L} \frac{1}{r^2}\lambda dx \cos\theta\,\hat{z}$$

$$= \frac{1}{4\pi\varepsilon_0} 2\int_{0}^{L} \frac{1}{r^2}\frac{z}{r} dx$$

$$= \frac{1}{4\pi\varepsilon_0} 2\lambda z\int_{0}^{L} \frac{1}{(z^2+x^2)^{3/2}} dx$$

$$\bar{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}$$

Case I ⇒ $z \gg L$

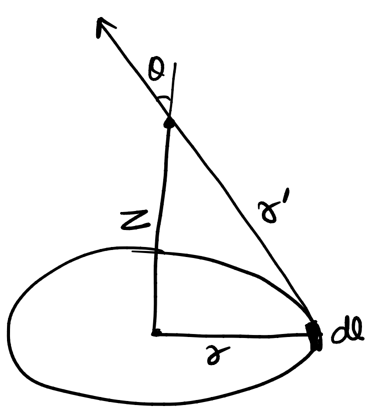
$$E \sim \frac{1}{4\pi\varepsilon_0}\frac{2\lambda L}{z^2}.$$

Case II ⇒ $L \longrightarrow \infty$

$$E \sim \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{z}.$$

Ex 2 ⇒ Find the Electric field, a distance z above the center of a circular loop of radius r, which carries a uniform line charge λ.

Ans ⇒



$$\bar{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{r'^2}\lambda dl \cos\theta\,\hat{z}$$

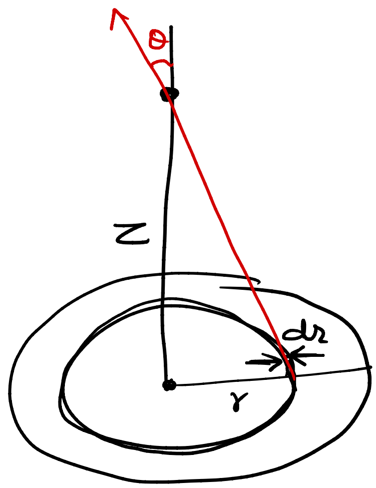
$$= \frac{1}{4\pi\varepsilon_0}\int \frac{1}{r'^2}\lambda\frac{z}{r'} dl$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\lambda z}{(r^2+z^2)^{3/2}}\int dl$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{\lambda z \cdot 2\pi r}{(r^2+z^2)^{3/2}}\hat{z}$$

Ex 3 ⇒ Find the electric field a distance z above the center of a flat circular disk of radius R, which carries a uniform charge σ.

A ⇒



$$E_{ring} = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma z \cdot 2\pi r\, dr}{(r^2+z^2)^{3/2}}$$

$$E = \frac{1}{4\pi\epsilon_0} 2\pi\sigma z \int_0^R \frac{r}{(r^2+z^2)^{3/2}}\, dr$$

$$E = \frac{1}{4\pi\epsilon_0} 2\pi\sigma z \underbrace{\left[\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right]}\hat{z}.$$

Case I ⇒ $z \gg R$

$$\Rightarrow \left[\frac{1}{z} - \frac{1}{z\sqrt{\left(\frac{R}{z}\right)^2+1}}\right]$$

$$\Rightarrow \frac{1}{z}\left[1 - \left(\left(\frac{R}{z}\right)^2+1\right)^{-1/2}\right]$$

$$= \frac{1}{z}\left[\cancel{1} - \left(\cancel{1} - \frac{1}{2}\left(\frac{R}{z}\right)^2\right)\right]$$

$$= \frac{R^2}{2z^3}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{\pi\sigma R^2}{z^2}$$

Case II ⇒ $R \to \infty$

$$E = \frac{1}{4\pi\epsilon_0} 2\pi\sigma \cancel{z} \cdot \frac{1}{\cancel{z}}$$

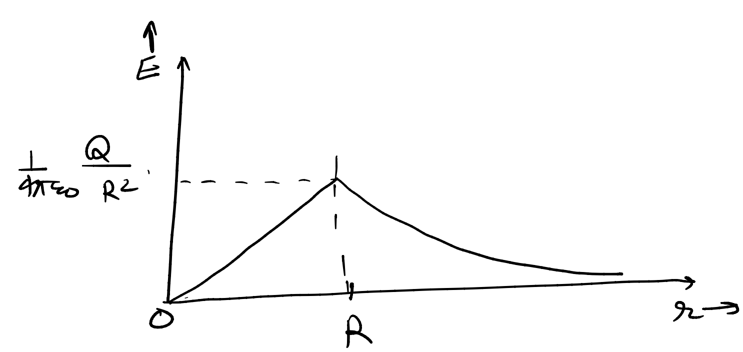
$$E = \frac{\sigma}{2\epsilon_0}\hat{z}$$

Homework.

Ex 4 ⇒ Find out the electric field inside & outside a uniformly charged solid sphere of radius R & total charge q. (charge density ρ)

Ans ⇒ $E_{interior} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3} r \; \hat{r}$

$$E_{out} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$$



Electric displacement (D)
Electric flux density ⇒

* Electric field depends on the medium in which charge is placed.

* Let say we define D such that -

$$\boxed{D = \epsilon E = \epsilon_0 E + P}$$

then $\bar{D}$ is only the fun^n of charge & it is independent of the medium.

* Ex ⇒ For an infinte sheet having uniform surface charge σ ⇒

$$D = \frac{\sigma}{2} \hat{z}$$