Lecture-36

9/11/2023.

Q ⇒ A particle of mass m, which moves freely inside an infinite pot. well of length a, has the following initial wave fn at $t=0$

$$\psi(x,0) = \frac{A}{\sqrt{a}} \sin\left(\frac{\pi x}{a}\right) + \sqrt{\frac{3}{5a}} \sin\left(\frac{3\pi x}{a}\right) + \frac{1}{\sqrt{5a}} \sin\left(\frac{5\pi x}{a}\right)$$

where A is a real const.

(a) ⇒ Find A so that $\psi(x,0)$ is normalized.

(b) ⇒ Find the wave fn $\psi(x,t)$ at any later time t.

(c) ⇒ If measurement of the energy are carried out, what are the values that will be found and what are the corresponding probabilities?

Calculate the average energy.

A ⇒

$$\varphi_n(x) = \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a}. \qquad E_n = \frac{n^2 h^2}{8ma^2}.$$

$$\psi(x,0) = \frac{A}{\sqrt{a}} \sin\left(\frac{\pi x}{a}\right) + \sqrt{\frac{3}{5a}} \sin\left(\frac{3\pi x}{a}\right) + \frac{1}{\sqrt{5a}} \sin\left(\frac{5\pi x}{a}\right)$$

$$= \frac{A}{\sqrt{2}} \varphi_1(x) + \sqrt{\frac{3}{10}} \varphi_3(x) + \frac{1}{\sqrt{10}} \varphi_5(x)$$

$$1 = \psi^*\psi = \frac{A^2}{2} + \frac{3}{10} + \frac{1}{10}$$

$$A = \sqrt{6/5} \quad \text{A}.$$

$$\psi(x,0) = \sqrt{\frac{3}{5}} \varphi_1(x) + \sqrt{\frac{3}{10}} \varphi_3(x) + \frac{1}{\sqrt{10}} \varphi_5(x).$$

(b) ⇒

$$\psi(x,t) = \sqrt{\frac{3}{5}} \varphi_1(x) e^{-iE_1 t/\hbar} + \sqrt{\frac{3}{10}} \varphi_3(x) e^{-iE_3 t/\hbar} + \frac{1}{\sqrt{10}} \varphi_5(x) e^{-iE_5 t/\hbar}$$

A

(C) $\Rightarrow$ $E_1 = \frac{h^2}{8ma^2}$ $\qquad P_1(E_1) = \frac{3}{5}$

$E_3 = \frac{9h^2}{8ma^2}$ $\qquad P_3(E_3) = \frac{3}{10}$

$E_5 = \frac{25h^2}{8ma^2}$ $\qquad P_5(E_5) = \frac{1}{10}$

Ehrenfest's theorem $\Rightarrow$

* If Q.M. is to be more general than C.M., it must contain C.M. as a limiting case.

* Schroedinger eq^n leads to the Newton's law of motion on average

* The average motion of a wave packet agees with the motion of the corresponding classical particle.

$$\frac{d}{dt}\langle x\rangle = \frac{\langle p\rangle}{m}$$

$$\frac{d}{dt}\langle p\rangle = \langle -\frac{dV}{dx}\rangle$$

Proff $\Rightarrow$ $$\frac{d}{dt}\langle x\rangle = \frac{d}{dt}\int \underset{I}{\psi^*}\, \underset{II}{x\psi}\, dx$$

$$= \int\left(\psi^* x \frac{d\psi}{\partial t} + x\psi\frac{\partial\psi^*}{\partial t}\right)dx$$

$$\left.\begin{aligned} i\hbar\frac{\partial\psi}{\partial t} &= \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V\right)\psi \\ -i\hbar\frac{\partial\psi^*}{\partial t} &= \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V\right)\psi^* \end{aligned}\right\}$$

$$\Rightarrow \int \psi^* x\frac{1}{i\hbar}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + \cancel{V\psi}\right)dx + \int x\psi\frac{1}{i\hbar}\left(\frac{\hbar^2}{2m}\frac{d^2\psi^*}{dx^2} - \cancel{V\psi}\right)dx$$

$$\frac{d}{dt}\langle x\rangle = \frac{i\hbar}{2m}\int \left(x\psi^* \frac{d^2\psi}{dx^2} - x\psi\frac{d^2\psi^*}{dx^2}\right)dx$$

$$= \frac{i\hbar}{2m}\int \underset{\mathrm{I}}{\underline{x}}\, \frac{d}{dx}\underbrace{\left(\psi^*\frac{d\psi}{dx} - \psi\frac{d\psi^*}{dx}\right)}_{\mathrm{II}} dx$$

$$\boxed{\int uv\,dx = u\int v\,dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx}$$

$$\Rightarrow \frac{i\hbar}{2m}\cancel{\left[x\left(\psi^*\frac{d\psi}{dx} - \psi\frac{d\psi^*}{dx}\right)\right]_{-\infty}^{\infty}} - \frac{i\hbar}{2m}\int 1\cdot\left(\psi^*\frac{d\psi}{dx} - \psi\frac{d\psi^*}{dx}\right)dx$$

As $x \to \pm\infty$ ψ & $\frac{d\psi}{dx} \to 0$

$$\Rightarrow -\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\left(\psi^*\frac{d\psi}{dx} - \underline{\psi}\,\underline{\frac{d\psi^*}{dx}}\right)dx$$

$$\Rightarrow -\frac{i\hbar}{2m}\int_{-\infty}^{\infty}\psi^*\frac{d\psi}{dx}dx + \frac{i\hbar}{2m}\left[\cancel{\psi\psi^*}_{-\infty}^{\infty} - \int\frac{d\psi}{dx}\psi^* dx\right]$$

0

$$\Rightarrow -\frac{i\hbar\cancel{2}}{\cancel{2}m}\int_{-\infty}^{\infty}\psi^*\frac{d\psi}{dx}dx$$

$$= \frac{1}{m}\int_{-\infty}^{\infty}\psi^*\left(-i\hbar\frac{d}{dx}\right)\psi\,dx$$

$$\frac{d}{dx}\langle x\rangle = \frac{\langle p\rangle}{m}$$

Similarly,

$$\frac{d}{dt}\langle p\rangle = \left\langle -\frac{dV}{dx}\right\rangle$$