Lecture-30

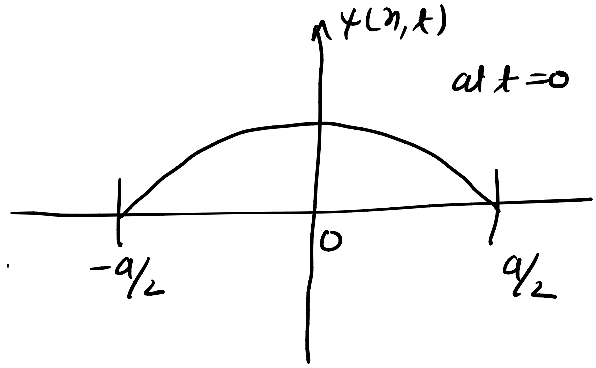
27/10/2023.

Q:⇒ Consider a particle of mass m which is confined in a one dimensional box of length a. Let say it's wave fn for lowest energy state of the particle is

$$\psi(x,t) = A\cos\frac{\pi x}{a}\, e^{-\frac{iEt}{\hbar}} \quad \Big\{ \; -a/2 \; x < a/2$$

$$= 0 \quad \Big\} \; x \leq -a/2 \text{ or } x \geq a/2.$$

Evaluate the expectation value of x, p, x^2 & p^2.



Ans:⇒ $$\int_{-\infty}^{\infty} \psi^*\psi \, dx = 1$$

$A = \sqrt{\frac{2}{a}}$. [Already done]

(i)⇒ $$\bar{x} = \langle x \rangle = \int_{-\infty}^{\infty} \psi^* x \psi \, dx$$

$$= \int_{-a/2}^{a/2} A\cos\frac{\pi x}{a}\, \cancel{e^{\frac{iEt}{\hbar}}}\; x A\cos\frac{\pi x}{a}\, \cancel{e^{-\frac{iEt}{\hbar}}} \, dx$$

$$= A^2 \int_{-a/2}^{a/2} x\cos^2\frac{\pi x}{a} \, dx$$

$$= 0$$

(ii) $$\bar{p} = \int_{-\infty}^{\infty} \psi^* \left(-i\hbar\frac{d}{\partial x}\right)\psi \, dx$$

$$= -i\hbar \int_{-a/2}^{a/2} A\cos\frac{\pi x}{a}\, \cancel{e^{\frac{iEt}{\hbar}}} \left(-\pi/a\right) A\sin\frac{\pi x}{a}\, \cancel{e^{-\frac{iEt}{\hbar}}} \, dx$$

$$= \frac{i\hbar\pi}{a} A^2 \int_{-a/2}^{a/2} \cos\frac{\pi x}{a} \cdot \sin\frac{\pi x}{a} \, dx = 0$$

$$\overline{x^2} = \langle x^2 \rangle = \int_{-\infty}^{\infty} \psi^* x^2 \psi \, dx$$

$$= \int_{-a/2}^{a/2} A \cos\frac{\pi x}{a} \cancel{e^{\frac{iEt}{\hbar}}} x^2 \cdot A \cos\frac{\pi x}{a} \cancel{e^{\frac{-iEt}{\hbar}}} \, dx$$

$$= A^2 \int_{-a/2}^{a/2} x^2 \cos^2\frac{\pi x}{a} \, dx.$$

$$\boxed{\frac{a^3}{4\pi^2}\left(\frac{\pi^2}{6}-1\right).}$$

$$= .033 a^2.$$

$$\overline{p^2} = \langle p^2 \rangle = \int_{-\infty}^{\infty} \psi^* (-i\hbar)^2 \frac{\partial^2 \psi}{\partial x^2} \, dx$$

$$= -\hbar^2 \int_{-\infty}^{\infty} \psi^* \frac{\partial^2 \psi}{\partial x^2} \, dx$$

$$= \frac{\hbar^2 \pi^2}{a^2} \int_{-\infty}^{\infty} \psi^* \psi \, dx$$

$$\psi = A \cos\frac{\pi x}{a} e^{-iEt/\hbar}$$

$$\frac{\partial \psi}{\partial x} = A(-\pi/a) \sin\frac{\pi x}{a} e^{-iEt/\hbar}$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{\pi^2}{a^2} \underbrace{A \cos\frac{\pi x}{a} e^{-iEt/\hbar}}$$

$$= -\frac{\pi^2}{a^2} \psi$$

$$\overline{p^2} = \left(\frac{\hbar\pi}{a}\right)^2.$$

$$\text{Std. } \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$= \sqrt{.33a^2 - 0} = .18 a$$

$$\text{Std } \Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

$$= \sqrt{\left(\frac{\hbar\pi}{a}\right)^2 - 0} = \frac{\hbar\pi}{a}$$

$$\Delta x \cdot \Delta p = .18a \times \frac{\hbar\pi}{a} = .57\hbar < \frac{\hbar}{2}$$

Time - independent Schrodinger eqⁿ ⇒

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\,\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t} \qquad \text{——— (1)}$$

If $\boxed{V(x,t) \Rightarrow V(x)}$ (Almost all system have pot. energy in this form)

then we can use the technique called seperation of Variable.

$$\Psi(x,t) = \psi(x)\,\phi(t).$$

Put in (1)

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)\varphi(t)}{\partial x^2} + V(x)\,\psi(x)\varphi(t) = i\hbar\frac{\partial}{\partial t}\psi(x)\varphi(t).$$

$$\Rightarrow -\frac{\hbar^2}{2m}\varphi(t)\frac{\partial^2\psi(x)}{\partial x^2} + \varphi(t)\,V(x)\psi(x) = i\hbar\,\psi(x)\frac{\partial\varphi(t)}{\partial t}.$$

$\div\ \psi(x)\varphi(t)$

$$\Rightarrow \frac{1}{\psi(x)}\left[-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\,\psi(x)\right] = i\hbar\frac{1}{\varphi(t)}\frac{d\varphi(t)}{dt} = \underset{\text{Const}}{G}$$

$$\frac{1}{\psi(x)}\left[-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\,\psi(x)\right] = G \qquad \text{space eq}^n$$

&

$$i\hbar\frac{1}{\varphi(t)}\frac{d\varphi(t)}{dt} = G \qquad \text{time eq}^n$$

$$\underbrace{i\hbar\frac{d}{dt}}\varphi = G\,\varphi(t)$$

$$E\varphi = G\varphi$$

$$G = E$$

Use the value of G in space eq^n ⇒

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\,\psi(x) = E\,\psi(x)}$$

This is time independent Schrodinger eq^n.

Using G = E in time eq^n -

$$\frac{d\varphi}{dt} = -\frac{iE}{\hbar}\varphi(t)$$

Sol^n

$$\varphi(t) = e^{-\frac{iEt}{\hbar}}$$

$$\boxed{\frac{dy}{dx} = \alpha y \qquad y = e^{\alpha x}}$$

$$\therefore \boxed{\psi(x,t) = \psi(x)\,e^{-\frac{iEt}{\hbar}}}$$

$$* \quad \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi(x) = E\,\psi(x).$$

$\psi(x)$ = eigen function.

E = eigen value.

$\psi(x,t)$ = Wave f^n.