Introduction

Interpolation is a numerical method used to estimate unknown values between known data points. It constructs new data points within the range of a discrete set of known data. Common methods of interpolation include linear interpolation, where the unknown value is estimated using a straight line between two known points, and polynomial interpolation, which uses higher-degree polynomials for a smoother curve. Interpolation is widely used in fields like engineering, science, and data analysis to approximate values for functions where only discrete measurements are available, allowing for more accurate predictions and analysis.

Lagrange interpolation

Lagrange interpolation is a form of polynomial interpolation that is used to estimate a function passing through a given set of points. The goal is to construct a polynomial that exactly fits the given data points and can be used to approximate the values of the function between those points. The Lagrange form of the interpolation polynomial avoids directly solving systems of equations, making it computationally efficient. This method is particularly useful when the data points are known and fixed, and a single interpolating polynomial is needed.

The motivation behind Lagrange interpolation is to create a simple and flexible method for approximating complex functions or scattered data points, ensuring that the interpolating polynomial passes through all the given points. This makes it valuable in numerical analysis, computer graphics, and data fitting problems.

Problem Statement

Given $n+1$ distinct points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, the goal is to find a polynomial $P(x)$ of degree at most n such that:

$$P(x_i) = y_i, \quad \text{for} \quad i = 0, 1, \ldots, n.$$

The polynomial $P(x)$ interpolates the data points.

1 Lagrange Basis Polynomials

To construct $P(x)$, we define the *Lagrange basis polynomials* $\ell_i(x)$, for $i = 0, 1, \ldots, n$, as follows:

$$\ell_i(x) = \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j}.$$

Each basis polynomial $\ell_i(x)$ has the property:

$$\ell_i(x_j) = \begin{cases} 1, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i \ne j. \end{cases}$$

Thus, $\ell_i(x)$ is 1 at $x = x_i$ and 0 at all other data points.

Lagrange Interpolation Polynomial

Using the Lagrange basis polynomials, the Lagrange interpolation polynomial $P(x)$ is given by:

$$P(x) = \sum_{i=0}^{n} y_i \ell_i(x).$$

This polynomial satisfies $P(x_i) = y_i$ for all i.

Explicit Formula

Expanding the definition of $\ell_i(x)$, the explicit form of the Lagrange interpolation polynomial is:

$$P(x) = \sum_{i=0}^{n} y_i \ell_i(x) = \sum_{i=0}^{n} y_i \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j}.$$

This formula provides a polynomial of degree n that passes through the given $n + 1$ data points.

Theorem 1. *Let $x_0, x_1, \cdots, x_n \in I = [a, b]$ be $n + 1$ distinct nodes and let $f(x)$ be a continuous real-valued function defined on I. Then, there exists a unique polynomial p_n of degree $\le n$ (called Lagrange Formula for Interpolating Polynomial), given by*

$$p_n(x) = \sum_{i=0}^{n} y_i \ell_i(x) = \sum_{i=0}^{n} y_i \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j}.$$

such that

$$p_n(x_i) = f(x_i) \quad i = 0, 1, \cdots, n.$$

Proof. It is evident that the polynomial p_n defined above has a degree of at most n and satisfies the interpolation conditions. It only remains to prove the uniqueness of the polynomial. For this purpose, suppose there exists another interpolating polynomial $q(x)$ of degree $\le n$ that also satisfies the interpolation conditions, and define

$$r(x) = p_n(x) - q(x).$$

Since both p_n and q are polynomials of degree at most n, their difference is also a polynomial of degree no greater than n.. However, we must note that

$$r(x_i) = p_n(x_i) - q(x_) = f(x_i) - f(x_i) = 0 \quad i = 0, 1, \cdots, n.$$

Therefore, we have a polynomial of degree at most n with $n + 1$ distinct roots. The only polynomial that satisfies this condition is the zero polynomial. $\square$

Example

Example 1. *Consider the points* $(1,1),(2,4),(3,9)$. *We want to find the Lagrange interpolation polynomial* $P(x)$ *that passes through these points.*

The Lagrange basis polynomials $\ell_0(x), \ell_1(x), \ell_2(x)$ *are:*

$$\ell_0(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{(x-2)(x-3)}{2},$$

$$\ell_1(x) = \frac{(x-1)(x-3)}{(2-1)(2-3)} = -(x-1)(x-3),$$

$$\ell_2(x) = \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{(x-1)(x-2)}{2}.$$

The interpolation polynomial is:

$$P(x) = 1 \cdot \ell_0(x) + 4 \cdot \ell_1(x) + 9 \cdot \ell_2(x),$$

$$P(x) = \frac{(x-2)(x-3)}{2} - 4(x-1)(x-3) + \frac{9(x-1)(x-2)}{2}.$$

Simplifying, we find that $P(x) = x^2$, *which is the quadratic function that passes through the points* $(1,1),(2,4),(3,9)$.

Error of Lagrange Interpolation

The error of the Lagrange interpolation polynomial $P(x)$ when approximating a smooth function $f(x)$ is given by:

$$E(x) = f(x) - P(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i),$$

where ξ is some point in the interval containing $x_0, x_1, \ldots, x_n$.

Remark 1. *The error depends on the smoothness of* $f(x)$ *and the spacing of the interpolation points. If the interpolation points are too clustered or too sparse, the error can grow significantly.*

Advantages and Limitations of Lagrange Interpolation

Advantages

- The method provides an exact fit to the data points.
- It is conceptually simple and can be applied to any set of distinct points.

Limitations

- For large n, the Lagrange polynomial can oscillate significantly, especially at the edges (this is known as *Runge's phenomenon*).
- Each time a new point is added, the entire polynomial needs to be recomputed, making it inefficient for large data sets.

Applications of Lagrange Interpolation

Lagrange interpolation is widely used in numerical analysis, computer graphics, and engineering applications, where a smooth curve is required to pass through a set of given data points.

Introduction

Newton's divided difference interpolation is a method for constructing an interpolating polynomial given a set of data points. It is particularly useful because it allows for incremental addition of new points without recalculating the entire polynomial.

Divided Differences

The divided difference of two points (x_0, y_0) and (x_1, y_1) is given by:

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Higher-order divided differences are calculated recursively. For three points:

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

Newton's Divided Difference Polynomial

The Newton form of the interpolating polynomial is:

$$\begin{aligned} P(x) = &f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) \\ &+ \cdots + f[x_0, x_1, \ldots, x_n](x - x_0) \cdots (x - x_{n-1}) \end{aligned}$$

Example

Given the points $(1, 1), (2, 4), (3, 9)$, we calculate the divided differences and construct the polynomial:

Step 1: Calculate Divided Differences

$$\begin{aligned} f[1] &= 1 \\ f[2] &= 4 \\ f[3] &= 9 \\ f[1, 2] &= \frac{f[2] - f[1]}{2 - 1} = \frac{4 - 1}{1} = 3 \\ f[2, 3] &= \frac{f[3] - f[2]}{3 - 2} = \frac{9 - 4}{1} = 5 \\ f[1, 2, 3] &= \frac{f[2, 3] - f[1, 2]}{3 - 1} = \frac{5 - 3}{2} = 1 \end{aligned}$$

Step 2: Construct the Polynomial

Using the divided differences calculated:

$$P_2(x) = f[1] + f[1,2](x-1) + f[1,2,3](x-1)(x-2)$$

Substituting the values:

$$P_2(x) = 1 + 3(x-1) + 1(x-1)(x-2)$$

Expanding this gives:

$$\begin{aligned} P_2(x) &= 1 + 3(x-1) + (x-1)(x-2) \\ &= 1 + 3x - 3 + x^2 - 3x + 2 = x^2 \end{aligned}$$

Thus, the interpolating polynomial is:

$$P_2(x) = x^2$$

Theorem 2. *Let $x_0, x_1, \cdots, x_n \in I = [a,b]$ be $n+1$ distinct nodes and let $f(x)$ be a continuous real-valued function defined on I. Let p_n be the polynomial that interpolates a continuous function $f(x)$ at $n+1$ distinct nodes. The polynomial p_{n+1} that interpolates a continuous function $f(x)$ at $n+2$ distinct nodes x_i for $i = 0, 1, \cdots, n+2$ is given by*

$$p_{n+1}(x) = p_n(x) + f[x_0, x_1, \cdots, x_{n+1}]w_n(x)$$

where

$$f[x_0, x_1, \cdots, x_{n+1}] = \frac{f(x_{n+1}) - p_n(x_{n+1})}{w_n(x_{n+1})}, \quad f[x_0] = f(x_0)$$

is called the $(n+1)$th divided difference of $f(x)$ at points $x_0, x_1, \cdots, x_n$ with

$$w_n(x) = \prod_{i=0}^{n} x - x_i.$$

The above formula is called the Newton Formula for Interpolating Polynomial.

Proof. Given that the interpolation polynomial is unique, it suffices to show that p_{n+1}, as described above, satisfies the interpolation conditions, assuming that p_n already fulfills these conditions.
For $0 \le k \le n$, we have $w_n(x_k) = 0$, which gives us the following:

$$p_{n+1}(x_k) = p_n(x_k) + f[x_0, x_1, \cdots, x_{n+1}]w_n(x_{n+1}) = p_n(x_k) = f(x_k).$$

Thus, p_{n+1} interpolates all points except the last one. To verify for x_{n+1}, we note the following:

$$\begin{aligned} p_{n+1}(x_{n+1}) &= p_n(x_{n+1}) + f[x_0, x_1, \cdots, x_{n+1}]w_n(x_{n+1}) \\ &= p_n(x_{n+1}) + f(x_{n+1}) - p_n(x_{n+1}) = f(x_{n+1}). \end{aligned}$$

Therefore, p_{n+1} interpolates $f(x)$ at all the nodes. Additionally, it is clearly a polynomial of degree at most $n+1$, completing the proof. $\square$

Theorem 3. *Let $x_0, x_1, \cdots, x_n \in I = [a,b]$ be $n+1$ distinct nodes and let $f(x) \in C^{n+1}(I)$. Then for each $\bar{x} \in I$ with $\bar{x} \neq x_i$, $i = 0, 1, \cdots, n$, there is a $x_i \in (a,b)$ such that*

$$e_n(\bar{x}) = \frac{w_n(\bar{x})}{(n+1)!} f^{n+1}(\xi),$$

where

$$w_n(x) = \prod_{i=0}^{n} x - x_i.$$

Proof. Let P_n be the interpolating polynomial of degree $\leq n$ such that $P_n(x_i) = f(x_i)$ for $i = 0, 1, \ldots, n$. Define the error function

$$E(x) = f(x) - P_n(x).$$

Then $E(x_i) = 0$ for each $i = 0, 1, \ldots, n$.

Fix $\bar{x} \in I$ with $\bar{x} \neq x_i$. Define the auxiliary function

$$\Phi(x) = E(x) - \lambda w_n(x),$$

where $w_n(x) = \prod_{i=0}^{n}(x - x_i)$ and λ is chosen such that $\Phi(\bar{x}) = 0$. Since $w_n(\bar{x}) \neq 0$, this λ exists uniquely and is given by

$$\lambda = \frac{E(\bar{x})}{w_n(\bar{x})}.$$

By construction, $\Phi(x)$ vanishes at $x_0, x_1, \ldots, x_n, \bar{x}$, so Φ has $n+2$ distinct zeros in $[a,b]$.

Since $f \in C^{n+1}(I)$ and P_n is a polynomial, $\Phi \in C^{n+1}(I)$. By Rolle's theorem applied repeatedly, there exists a point $\xi \in (a,b)$ such that

$$\Phi^{(n+1)}(\xi) = 0.$$

Now, compute $\Phi^{(n+1)}(x)$:

$$\Phi^{(n+1)}(x) = f^{(n+1)}(x) - \lambda w_n^{(n+1)}(x).$$

Since P_n has degree $\leq n$, its $(n+1)$-th derivative vanishes, and

$$w_n^{(n+1)}(x) = (n+1)!.$$

Thus,

$$\Phi^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \lambda (n+1)! = 0,$$

which implies

$$\lambda = \frac{f^{(n+1)}(\xi)}{(n+1)!}.$$

Substituting back into the definition of λ, we find

$$E(\bar{x}) = \frac{w_n(\bar{x})}{(n+1)!} f^{(n+1)}(\xi).$$

Hence,

$$e_n(\bar{x}) = f(\bar{x}) - P_n(\bar{x}) = \frac{w_n(\bar{x})}{(n+1)!} f^{(n+1)}(\xi),$$

where $\xi \in (a, b)$, as required. $\square$

Conclusion

Newton's divided difference interpolation is an effective technique for constructing polynomials that interpolate a given set of points. Its recursive nature allows for efficient computation, especially when new data points are added.

Introduction

Forward difference interpolation is a method used to estimate the value of a function at a point within the range of a discrete set of known data points. This method is particularly useful when the data points are evenly spaced, as it simplifies the calculation of the interpolation polynomial.

Forward Differences

The forward difference of a function f at a point x_i is defined as:

$$\Delta f[x_i] = f[x_{i+1}] - f[x_i]$$

Higher-order forward differences can be defined recursively as follows:

$$\Delta^k f[x_i] = \Delta(\Delta^{k-1} f[x_i]) = \Delta^{k-1} f[x_{i+1}] - \Delta^{k-1} f[x_i]$$

The k-th forward difference can be computed for a given set of $n+1$ data points $(x_0, f[x_0]), (x_1, f[x_1]), \ldots, (x_n, f[x_n])$.

Newton's Forward Difference Interpolation Formula

The Newton's forward difference interpolation polynomial can be expressed as:

$$\begin{aligned} P_n(x) =& f[x_0] + \frac{(x - x_0)}{h} \Delta f[x_0] + \frac{(x - x_0)(x - x_1)}{2! h^2} \Delta^2 f[x_0] \\ &+ \cdots + \frac{(x - x_0)(x - x_1) \cdots (x - x_{n-1})}{n! h^n} \Delta^n f[x_0] \end{aligned}$$

where $h = x_{i+1} - x_i$ is the uniform spacing between the data points.

Example

Consider the following set of data points:

$$(1, 1), (2, 4), (3, 9), (4, 16)$$

We want to find the interpolating polynomial at $x = 2.5$.

Step 1: Calculate Forward Differences

First, we calculate the forward differences for the function values:

x	$f[x]$	$\Delta f[x]$	$\Delta^2 f[x]$	$\Delta^3 f[x]$
1	1	3	2	0
2	4	5	0	
3	9	7		
4	16			

Calculating the forward differences:

$$\begin{aligned}
\Delta f[x_0] &= f[2] - f[1] = 4 - 1 = 3 \\
\Delta f[x_1] &= f[3] - f[2] = 9 - 4 = 5 \\
\Delta f[x_2] &= f[4] - f[3] = 16 - 9 = 7 \\
\Delta^2 f[x_0] &= \Delta f[x_1] - \Delta f[x_0] = 5 - 3 = 2 \\
\Delta^2 f[x_1] &= \Delta f[x_2] - \Delta f[x_1] = 7 - 5 = 2 \\
\Delta^3 f[x_0] &= \Delta^2 f[x_1] - \Delta^2 f[x_0] = 2 - 2 = 0
\end{aligned}$$

Step 2: Construct the Polynomial

Using the calculated forward differences, we construct the polynomial $P_n(x)$ for $x = 2.5$:

$$h = 1 \quad \text{(spacing between points)}$$

The interpolation polynomial becomes:

$$P_3(x) = f[x_0] + \frac{(x - x_0)}{h} \Delta f[x_0] + \frac{(x - x_0)(x - x_1)}{2! h^2} \Delta^2 f[x_0]$$

Substituting the known values:

$$P_3(2.5) = 1 + \frac{(2.5 - 1)}{1} \cdot 3 + \frac{(2.5 - 1)(2.5 - 2)}{2} \cdot 2$$

Calculating each term:

$$= 1 + 1.5 \cdot 3 + \frac{1.5 \cdot 0.5}{2} \cdot 2$$

Calculating further:

$$= 1 + 4.5 + 0.75 = 6.25$$

Thus, the interpolated value at $x = 2.5$ is $P(2.5) = 6.25$.

Conclusion

Forward difference interpolation is an efficient method for estimating function values using evenly spaced data points. It provides a straightforward way to construct interpolating polynomials and can be easily adapted for different sets of data.

Introduction

Backward difference interpolation is a numerical method used to estimate the value of a function at a point within the range of a discrete set of known data points. This method is particularly effective when the data points are evenly spaced, allowing for efficient calculation of the interpolation polynomial.

Backward Differences

The backward difference of a function f at a point x_i is defined as:

$$\nabla f[x_i] = f[x_i] - f[x_{i-1}]$$

Higher-order backward differences can be defined recursively as follows:

$$\nabla^k f[x_i] = \nabla(\nabla^{k-1} f[x_i]) = \nabla^{k-1} f[x_i] - \nabla^{k-1} f[x_{i-1}]$$

The k-th backward difference can be computed for a given set of $n+1$ data points $(x_0, f[x_0]), (x_1, f[x_1]), \ldots, (x_n, f[x_n])$.

Newton's Backward Difference Interpolation Formula

The Newton's backward difference interpolation polynomial can be expressed as:

$$\begin{aligned} P_n(x) = & f[x_n] + \frac{(x - x_n)}{h} \nabla f[x_n] + \frac{(x - x_n)(x - x_{n-1})}{2!h^2} \nabla^2 f[x_n] \\ & + \cdots + \frac{(x - x_n)(x - x_{n-1}) \cdots (x - x_{n-k+1})}{k!h^k} \nabla^k f[x_n] \end{aligned}$$

where $h = x_n - x_{n-1}$ is the uniform spacing between the data points.

Example

Consider the following set of data points:

$$(3, 9), (4, 16), (5, 25), (6, 36)$$

We want to find the interpolating polynomial at $x = 4.5$.

Step 1: Calculate Backward Differences

First, we calculate the backward differences for the function values:

x	$f[x]$	$\nabla f[x]$	$\nabla^2 f[x]$	$\nabla^3 f[x]$
3	9	7	1	0
4	16	9		
5	25	11		
6	36			

Calculating the backward differences:

$$\begin{aligned}
\nabla f[x_3] &= f[4] - f[3] = 16 - 9 = 7 \\
\nabla f[x_4] &= f[5] - f[4] = 25 - 16 = 9 \\
\nabla f[x_5] &= f[6] - f[5] = 36 - 25 = 11 \\
\nabla^2 f[x_3] &= \nabla f[x_4] - \nabla f[x_3] = 9 - 7 = 2 \\
\nabla^2 f[x_4] &= \nabla f[x_5] - \nabla f[x_4] = 11 - 9 = 2 \\
\nabla^3 f[x_3] &= \nabla^2 f[x_4] - \nabla^2 f[x_3] = 2 - 2 = 0
\end{aligned}$$

Step 2: Construct the Polynomial

Using the calculated backward differences, we construct the polynomial $P_n(x)$ for $x = 4.5$:

$$h = 1 \quad \text{(spacing between points)}$$

The interpolation polynomial becomes:

$$P_3(x) = f[x_n] + \frac{(x - x_n)}{h} \nabla f[x_n] + \frac{(x - x_n)(x - x_{n-1})}{2!h^2} \nabla^2 f[x_n]$$

Substituting the known values:

$$P_3(4.5) = 36 + \frac{(4.5 - 6)}{1} \cdot 11 + \frac{(4.5 - 6)(4.5 - 5)}{2} \cdot 2$$

Calculating each term:

$$= 36 + (-1.5) \cdot 11 + \frac{(-1.5)(-0.5)}{2} \cdot 2$$

Calculating further:

$$= 36 - 16.5 + 0.75 = 20.25$$

Thus, the interpolated value at $x = 4.5$ is $P(4.5) = 20.25$.

Conclusion

Backward difference interpolation is an efficient method for estimating function values using evenly spaced data points. It provides a straightforward way to construct interpolating polynomials and can be easily adapted for different sets of data.

2 Introduction to Central Difference Interpolation

2.1 Why Central Differences?

- For interpolation near the **middle** of a table, central difference formulas are more accurate
- Gauss formulas use central differences around a central point
- Two main variants: **Forward** and **Backward**

2.2 Key Notation

Let's consider equally spaced data:

- **Step size**: h
- **Central point**: x_0
- **Values**: $x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$
- **Corresponding y-values**: $y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots$

Difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_{-2}	y_{-2}				
		Δy_{-2}			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		
		Δy_{-1}		$\Delta^3 y_{-2}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$
		Δy_0		$\Delta^3 y_{-1}$	
x_1	y_1		$\Delta^2 y_0$		
		Δy_1			
x_2	y_2				

2.3 Gauss Forward Interpolation Formula

Recall the defination of Newton forward difference interpolation formula about a base node x_0 is

$$\begin{aligned} y(u) &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 \\ &\quad + \frac{(u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots, \end{aligned}$$

where $u = \frac{x-x_0}{h}$.
Using the fact that

$$\begin{aligned} \Delta y_0 &= \Delta y_{-1} + \Delta^2 y_{-1} \\ \Delta^2 y_0 &= \Delta^2 y_{-1} + \Delta^3 y_{-1} \\ \Delta^3 y_0 &= \Delta^3 y_{-1} + \Delta^4 y_{-1} \\ \Delta^4 y_0 &= \Delta^4 y_{-1} + \Delta^5 y_{-1} \end{aligned}$$

and

$$\begin{aligned} \Delta y_{-1} &= \Delta y_{-2} + \Delta^2 y_{-2} \\ \Delta^2 y_{-1} &= \Delta^2 y_{-2} + \Delta^3 y_{-2} \\ \Delta^3 y_{-1} &= \Delta^3 y_{-2} + \Delta^4 y_{-2} \\ \Delta^4 y_{-1} &= \Delta^4 y_{-2} + \Delta^5 y_{-2} \end{aligned}$$

we obtain

$$\begin{aligned} y(u) &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) \\ &\quad + \frac{u(u-1)(u-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{u(u-1)(u-2)(u-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots \\ &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} \\ &\quad + \frac{u(u-1)(3+u-2)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u-2)(4+u-3)}{4!}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \cdots \\ &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} \\ &\quad + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u-2)(u+1)}{4!}\Delta^4 y_{-2} + \cdots \end{aligned}$$

2.3.1 Formula

For a point $x = x_0 + uh$, where $u = \frac{x-x_0}{h}$:

$$\begin{aligned} y(u) &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} \\ &\quad + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \cdots \end{aligned}$$

2.3.2 Alternate Form (using central differences)

$$
\begin{aligned}
y(u) = y_0 + u\delta y_{1/2} &+ \frac{u(u-1)}{2!}\delta^2 y_0 \\
&+ \frac{(u+1)u(u-1)}{3!}\delta^3 y_{1/2} + \frac{(u+1)u(u-1)(u-2)}{4!}\delta^4 y_0 + \cdots
\end{aligned}
$$

2.4 Gauss Backward Interpolation Formula

Recall the defination of Newton forward difference interpolation formula about a base node x_0 is

$$
\begin{aligned}
y(u) = y_0 + u\Delta y_0 &+ \frac{u(u-1)}{2!}\Delta^2 y_0 \\
&+ \frac{(u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots,
\end{aligned}
$$

where $u = \frac{x-x_0}{h}$.
Using the fact that

$$
\begin{aligned}
\Delta y_0 &= \Delta y_{-1} + \Delta^2 y_{-1} \\
\Delta^2 y_0 &= \Delta^2 y_{-1} + \Delta^3 y_{-1} \\
\Delta^3 y_0 &= \Delta^3 y_{-1} + \Delta^4 y_{-1} \\
\Delta^4 y_0 &= \Delta^4 y_{-1} + \Delta^5 y_{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta y_{-1} &= \Delta y_{-2} + \Delta^2 y_{-2} \\
\Delta^2 y_{-1} &= \Delta^2 y_{-2} + \Delta^3 y_{-2} \\
\Delta^3 y_{-1} &= \Delta^3 y_{-2} + \Delta^4 y_{-2} \\
\Delta^4 y_{-1} &= \Delta^4 y_{-2} + \Delta^5 y_{-2}
\end{aligned}
$$

we obtain

$$
\begin{aligned}
y(u) &= y_0 + u(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{u(u-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) \\
&\quad + \frac{u(u-1)(u-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{u(u-1)(u-2)(u-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots \\
&= y_0 + u\Delta y_{-1} + \frac{u(2+u-1)}{2!}\Delta^2 y_{-1} \\
&\quad + \frac{u(u-1)(3+u-2)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u-2)(4+u-3)}{4!}\Delta^4 y_{-1} + \cdots \\
&= y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} \\
&\quad + \frac{u(u-1)(u+1)}{3!}(\Delta^3 y_{-2} + \Delta^4 y_{-2}) + \frac{u(u-1)(u-2)(u+1)}{4!}(\Delta^4 y_{-2} + \Delta^4 y_{-2}) + \cdots \\
&= y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} \\
&\quad + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-2} + \frac{u(u-1)(4+u-2)(u+1)}{4!}\Delta^4 y_{-2} + \cdots \\
&= y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} \\
&\quad + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-2} + \frac{u(u-1)(u+1)(u+2)}{4!}\Delta^4 y_{-2} + \cdots
\end{aligned}
$$

2.4.1 Formula

For a point $x = x_0 + uh$:

$$
\begin{aligned}
y(u) = y_0 + u\Delta y_{-1} &+ \frac{u(u+1)}{2!}\Delta^2 y_{-1} \\
&+ \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2} + \cdots
\end{aligned}
$$

2.5 Alternate Form (using central differences)

$$
\begin{aligned}
y(u) = y_0 + u\delta y_{-1/2} &+ \frac{u(u+1)}{2!}\delta^2 y_0 \\
&+ \frac{(u+1)u(u-1)}{3!}\delta^3 y_{-1/2} + \frac{(u+1)(u+2)u(u-1)}{4!}\delta^4 y_0 + \cdots
\end{aligned}
$$

Stirling and Bassel interpolations

Stirling interpolation

For n even, we assume that the nodal points are

$$x_{-p}, x_{-(p-1)}, \cdots, x_{-1}, x_0, x_1, \cdots, x_{p-1}, x_p,$$

where $p = n/2$.
We begin by recalling the definition of Gausss forward interpolation formula:

$$\begin{aligned}
y(u) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} \\
+ \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \cdots
\end{aligned}$$

Next we recall the definition of Gauss's Backward intorplation formula:

$$\begin{aligned}
y(u) = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} \\
+ \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2} + \cdots
\end{aligned}$$

Finally, we take the average of Gausss forward and backward interpolation formulas:

$$\begin{aligned}
y(u) &= y_0 + \frac{u}{2}(\Delta y_0 + \Delta y_{-1}) + \frac{1}{2}\frac{u(u-1+u+1)}{2!}\Delta^2 y_{-1} \\
&\quad + \frac{1}{2}\frac{(u+1)u(u-1)}{3!}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{2}\frac{(u+1)u(u-1)(u-2+u+2)}{4!}\Delta^4 y_{-2} + \cdots \\
&= y_0 + \frac{u}{2}(\Delta y_0 + \Delta y_{-1}) + \frac{u^2}{2!}\Delta^2 y_{-1} \\
&\quad + \frac{1}{2}\frac{(u+1)u(u-1)}{3!}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{(u+1)u^2(u-1)}{4!}\Delta^4 y_{-2} + \cdots
\end{aligned}$$

2.5.1 Formula

For a point $x = x_0 + uh$, where $u = \frac{x-x_0}{h}$:

$$\begin{aligned}
y(u) &= y_0 + \frac{u}{2}(\Delta y_0 + \Delta y_{-1}) + \frac{u^2}{2!}\Delta^2 y_{-1} \\
&\quad + \frac{1}{2}\frac{(u+1)u(u-1)}{3!}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{(u+1)u^2(u-1)}{4!}\Delta^4 y_{-2} + \cdots
\end{aligned}$$

2.6 Alternate Form (using central differences)

The Stirling interpolating polynomial is given by

$$\begin{aligned}
P(x) =& f(x_0) + \frac{u}{2}[\delta f_{1/2} + \delta f_{-1/2}] + \frac{u^2}{2!}\delta^2 f_0 \\
&+ \frac{u(u^2-1^2)}{3!}\frac{1}{2}[\delta^3 f_{1/2} + \delta^3 f_{-1/2}] + \cdots \\
&+ \frac{u(u^2-1^2)\cdots(u^2-(p-1)^2)}{(2p-1)!}\frac{1}{2}[\delta^{2p-1} f_{1/2} + \delta^{2p-1} f_{-1/2}] \\
&+ \frac{u^2(u^2-1^2)\cdots(u^2-(p-1)^2)}{(2p)!}\delta^{2p} f_0
\end{aligned}$$

where $u = (x - x_0)/h$.

Example

Use Stirlings formula to evaluate $f(1.315)$ from the following table:

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$	$\Delta^6 f$
1.0	1.54308						
		0.12544					
1.1	1.66852		0.01670				
		0.14214		0.00141			
1.2	1.81066		0.01811		0.00022		
		0.16025		**0.00163**		**-0.00007**	
1.3	**1.97091**		**0.01974**		**0.00015**		**0.00016**
		0.17999		**0.00178**		**0.00009**	
1.4	2.15090		0.02152		0.00024		
		0.20151		0.00202			
1.5	2.35241		0.02354				
		0.22505					
1.6	2.57746						

Center: $x_0 = 1.3,\ f_0 = 1.97091,\ h = 0.1,\ x = 1.315 \Rightarrow u = \dfrac{1.315 - 1.3}{0.1} = 0.15.$
Stirling formula:

$$\begin{aligned} f(x) \approx f_0 &+ \frac{u}{2}(\Delta f_0 + \Delta f_{-1}) + \frac{u^2}{2!}\Delta^2 f_{-1} \\ &+ \frac{1}{2}\frac{(u+1)u(u-1)}{3!}(\Delta^3 f_{-1} + \Delta^3 f_{-2}) + \frac{(u+1)u^2(u-1)}{4!}\Delta^4 f_{-2} \\ &+ \frac{1}{2}\frac{(u+1)(u+2)u(u-1)(u-2)}{5!}(\Delta^5 f_{-2} + \Delta^5 f_{-3}) + \frac{(u+1)(u+2)u^2(u-1)(u-2)}{6!}\Delta^6 f_{-3} \end{aligned}$$

Compute terms:

$$\begin{aligned} f_0 &= 1.97091, \\ \frac{u}{2}(\Delta f_0 + \Delta f_{-1}) &= \frac{0.15}{2} \times (0.16025 + 0.17999) = 0.025518, \\ \frac{u^2}{2!}\Delta^2 f_{-1} &= \frac{0.15^2}{2} \times 0.01974 = 0.000222075, \\ \frac{1}{2}\frac{(u+1)u(u-1)}{3!}(\Delta^3 f_{-1} + \Delta^3 f_{-2}) &= \frac{0.15(0.15^2 - 1)}{12} \times (0.00163 + 0.00178) \\ &= 0.00004170094. \\ \frac{(u+1)u^2(u-1)}{4!}\Delta^4 f_{-2} &= \frac{0.15^2(0.15^2 - 1)}{24} \times (0.00015) = 0.000000137539 \end{aligned}$$

$$\frac{1}{2}\frac{(u+1)(u+2)u(u-1)(u-2)}{5!}(\Delta^5 f_{-2}+\Delta^5 f_{-3}) = \frac{0.15(0.15^2-1)(0.15^2-4)}{240}\times(-0.00007+0.00009)$$
$$= 0.000000048501$$
$$\frac{(u+1)(u+2)u^2(u-1)(u-2)}{6!}\Delta^6 f_{-3} = \frac{0.15^2(0.15^2-1)(0.15^2-4)}{720}\times(0.00016)$$
$$= 0.0000000194592$$

Sum:

$$\begin{aligned} f(1.315) &\approx 1.97091 + 0.025518 + 0.0002220750.000041700940.000000137539 \\ &\quad + 0.000000048501 + 0.0000000194592 \\ &= \boxed{1.9966083 \approx 1.99661.} \end{aligned}$$

Bassel interpolation

For n odd, we assume that the nodal points are

$$x_{-p}, x_{-(p-1)}, \cdots, x_{-1}, x_0, x_1, \cdots, x_p, x_{p+1},$$

where $p = (n-1)/2$.
Difference Table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_{-2}	y_{-2}					
		Δy_{-2}				
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$			
		Δy_{-1}		$\Delta^3 y_{-2}$		
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$	
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$	
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$			
		Δy_2				
x_3	y_3					

Recall the defination of Newton forward difference interpolation formula about a base node x_0 is

$$\begin{aligned} y(u) &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 \\ &\quad + \frac{(u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots, \end{aligned}$$

where $u = \frac{x-x_0}{h}$.
Using the fact that

$$\begin{aligned}
\Delta y_0 &= \Delta y_{-1} + \Delta^2 y_{-1} \\
\Delta^2 y_0 &= \Delta^2 y_{-1} + \Delta^3 y_{-1} \\
\Delta^3 y_0 &= \Delta^3 y_{-1} + \Delta^4 y_{-1} \\
\Delta^4 y_0 &= \Delta^4 y_{-1} + \Delta^5 y_{-1}
\end{aligned}$$

and

$$\begin{aligned}
\Delta y_{-1} &= \Delta y_{-2} + \Delta^2 y_{-2} \\
\Delta^2 y_{-1} &= \Delta^2 y_{-2} + \Delta^3 y_{-2} \\
\Delta^3 y_{-1} &= \Delta^3 y_{-2} + \Delta^4 y_{-2} \\
\Delta^4 y_{-1} &= \Delta^4 y_{-2} + \Delta^5 y_{-2}
\end{aligned}$$

we obtain

$$\begin{aligned}
y(u) &= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) \\
&\quad + \frac{u(u-1)(u-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{u(u-1)(u-2)(u-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots, \\
&= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}(\Delta^2 y_{-1} \\
&\quad + \frac{u(u-1)(3+u-2)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u-2)(4+u-3)}{4!}\Delta^4 y_{-1} + \cdots, \\
&= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}(\Delta^2 y_{-1} \\
&\quad + \frac{u(u-1)(u+1)}{3!}\Delta^3 y_{-1} + \frac{u(u-1)(u-2)(u+1)}{4!}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \cdots,
\end{aligned}$$

Next we write the defination of Newton forward difference interpolation formula about a base node x_1:

$$\begin{aligned}
y(u+1) &= y_1 + (u-1)\Delta y_1 + \frac{(u-1)(u-2)}{2!}\Delta^2 y_1 \\
&\quad + \frac{((u-1)(u-2)(u-3)}{3!}\Delta^3 y_1 + \frac{(u-1)(u-2)(u-3)(u-4)}{4!}\Delta^4 y_1 + \cdots,
\end{aligned}$$

Using the fact that

$$\begin{aligned}
\Delta y_1 &= \Delta y_0 + \Delta^2 y_0 \\
\Delta^2 y_1 &= \Delta^2 y_0 + \Delta^3 y_0 \\
\Delta^3 y_1 &= \Delta^3 y_0 + \Delta^4 y_0 \\
\Delta^4 y_1 &= \Delta^4 y_0 + \Delta^5 y_0
\end{aligned}$$

and

$$\begin{aligned}
\Delta y_0 &= \Delta y_{-1} + \Delta^2 y_{-1}\\
\Delta^2 y_0 &= \Delta^2 y_{-1} + \Delta^3 y_{-1}\\
\Delta^3 y_0 &= \Delta^3 y_{-1} + \Delta^4 y_{-1}\\
\Delta^4 y_0 &= \Delta^4 y_{-1} + \Delta^5 y_{-1}
\end{aligned}$$

we obtain

$$\begin{aligned}
y(u+1) &= y_1 + (u-1)(\Delta y_0 + \Delta^2 y_0) + \frac{(u-1)(u-2)}{2!}(\Delta^2 y_0 + \Delta^3 y_0)\\
&\quad + \frac{(u-1)(u-2)(u-3)}{3!}(\Delta^3 y_0 + \Delta^4 y_0) + \frac{(u-1)(u-2)(u-3)(u-4)}{4!}(\Delta^4 y_0 + \Delta^5 y_0) + \cdots,\\
&= y_1 + (u-1)\Delta y_0 + \frac{(u-1)(2+u-2)}{2!}\Delta^2 y_0\\
&\quad + \frac{(u-1)(u-2)(3+u-3)}{3!}\Delta^3 y_0 + \frac{(u-1)(u-2)(u-3)(4+u-4)}{4!}\Delta^4 y_0 + \cdots,\\
&= y_1 + (u-1)\Delta y_0 + \frac{(u-1)u}{2!}\Delta^2 y_0\\
&\quad + \frac{(u-1)(u-2)u}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{(u-1)(u-2)(u-3)u}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots,\\
&= y_1 + (u-1)\Delta y_0 + \frac{(u-1)u}{2!}\Delta^2 y_0\\
&\quad + \frac{(u-1)(u-2)u}{3!}\Delta^3 y_{-1} + \frac{(u-1)(u-2)(u+1)u}{4!}\Delta^4 y_{-1} + \cdots,
\end{aligned}$$

Finally, we take the average of both interpolation formulas:

$$\begin{aligned}
y_B(\mu u_{1/2}) &= \frac{y_0 + y_1}{2} + (u-1/2)\Delta y_0 + \frac{1}{2}\frac{u(u-1)}{2!}(\Delta^2 y_{-1} + \Delta^2 y_0)\\
&\quad + \frac{1}{2}\frac{(u+1+u-2)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{1}{2}\frac{u(u-1)(u-2)(u-3)}{4!}(\Delta^4 y_{-2} + \Delta^4 y_{-1}) + \cdots\\
&= \frac{y_0 + y_1}{2} + (u-1/2)\Delta y_0 + \frac{1}{2}\frac{u(u-1)}{2!}(\Delta^2 y_{-1} + \Delta^2 y_0)\\
&\quad + \frac{(u-1/2)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{1}{2}\frac{u(u-1)(u-2)(u+1)}{4!}(\Delta^4 y_{-2} + \Delta^4 y_{-1}) + \cdots
\end{aligned}$$

2.6.1 Formula

For a point $x = x_0 + uh$ and $v = u - 1/2$, where $u = \frac{x - x_0}{h}$:

$$\begin{aligned} y_B(\mu u_{1/2}) &= \frac{y_0 + y_1}{2} + (u - 1/2)\Delta y_0 + \frac{1}{2}\frac{u(u-1)}{2!}(\Delta^2 y_{-1} + \Delta^2 y_0) \\ &\quad + \frac{(u-1/2)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{1}{2}\frac{u(u-1)(u-2)(u+1)}{4!}(\Delta^4 y_{-2} + \Delta^4 y_{-1}) + \cdots \\ &= \frac{y_0 + y_1}{2} + v\Delta y_0 + \frac{1}{2}\frac{v^2 - 1/4}{2!}(\Delta^2 y_{-1} + \Delta^2 y_0) \\ &\quad + \frac{(v^2 - 1/4)v}{3!}\Delta^3 y_{-1} + \frac{1}{2}\frac{(v^2 - 1/4)(v^2 - 9/4)}{4!}(\Delta^4 y_{-2} + \Delta^4 y_{-1}) + \cdots \end{aligned}$$

2.6.2 Alternate Form (using central differences)

$$\begin{aligned} y_B(\mu u_{1/2}) =& \frac{1}{2}(y_0 + y_1) + v\delta y_{1/2} + \frac{v^2 - 1/4}{2!}\frac{1}{2}[\delta^2 y_0 + \delta^2 y_1] \\ &+ \frac{v(v^2 - 1/4)}{3!}\delta^3 y_{1/2} + \cdots \\ &+ \frac{(v^2 - 1/4)\cdots(v^2 - (2p-1)^2/4)}{(2p)!}\frac{1}{2}[\delta^{2p} y_0 + \delta^{2p} y_1] \\ &+ \frac{v(v^2 - 1/4)\cdots(v^2 - (2p-1)^2/4)}{(2p+1)!}\delta^{2p+1} y_{1/2} + \cdots \end{aligned}$$

Given Data

x (deg)	$\cos x$
10	0.176327
12	0.212556
14	0.249328
16	0.286745
18	0.324920
20	0.363970

Step size: $h = 2$, and we want $f(15)$. The point $x = 15$ lies midway between 14 and 16, so we use Bessel's formula centered between these points.

Forward Difference Table

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
10	0.176327					
		0.036229				
12	0.212556		0.000543			
		0.036772		−0.000003		
14	**0.249328**		**0.000540**		**0.000003**	
		0.037417		**0.000000**		**-0.000002**
16	**0.286745**		**0.000540**		**0.000001**	
		0.038175		0.000001		
18	0.324920		0.000541			
		0.039050				
20	0.363970					

Bessel Interpolation Formula

For a point midway between $x_0 = 14$ and $x_1 = 16$:

$$u = \frac{x - x_0}{h} = \frac{15 - 14}{2} = 0.5, \quad v = u - 1/2 = 0.5 - 0.5 = 0.0$$

$$\begin{aligned} f(15^\circ) \approx & \frac{f_0 + f_1}{2} + v\,\Delta f_0 + \frac{(v^2 - 1/4)}{2}\frac{\Delta^2 f_{-1} + \Delta^2 f_0}{2} \\ & + \frac{v(v^2 - 1/4)}{3!}\Delta^3 f_{-1} + \frac{1}{2}\frac{(v^2 - 1/4)(v^2 - 9/4)}{4!}(\Delta^4 f_{-2} + \Delta^4 f_{-1}) + \frac{v(v^2 - 1/4)(v^2 - 9/4)}{5!}\Delta^5 f_{-2}. \end{aligned}$$

Compute Terms

$$\begin{aligned} \frac{f_2 + f_3}{2} &= (0.249328 + 0.286745)/2 = 0.2680365, \\ \frac{(v^2 - 1/4)}{2}\frac{\Delta^2 f_{-1} + \Delta^2 f_0}{2} &= -0.125 \cdot (0.000540 + 0.000540)/2 = 0.0000675, \\ \frac{1}{2}\frac{(v^2 - 1/4)(v^2 - 9/4)}{4!}(\Delta^4 f_{-2} + \Delta^4 f_{-1}) &= (-0.25)(-2.5)/48 \cdot (0.000003 + 0.000001) \approx 0.000000052083 \end{aligned}$$

Sum of Terms

$f(15^\circ) \approx 0.2680365 - 0.0000675 + 0.000000052083 = 0.267969052083 \approx 0.267969$