ECC 203 : Electromagnetics and Radiating Systems

Transmission Line Theory

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- Phasors – Recap
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Phasors – Recap

A sinusoidal AC voltage or current or an Electromagnetic Field wave is usually represented as

$$x(t) = A_m \cos(\omega t + \phi)$$

Using Euler's Identity, we write X(t) as:

$$x(t) = \mathrm{Re}\left[A_m e^{j\phi} e^{j\omega t}\right]$$

$$A_m \cos(\omega t + \phi) \longrightarrow \left[A_m e^{j\phi}\right]$$

Time Domain

Complex No. or Phasor Domain

Phasors – Recap

$$x(t) = A_m \cos(\omega t + \phi)$$

$$\text{Phasor} A \Rightarrow \mathrm{Re}[Ae^{j\omega t}] = \mathrm{Re}[A_m e^{j\phi} e^{j\omega t}] = A_m \cos(\omega t + \phi)$$

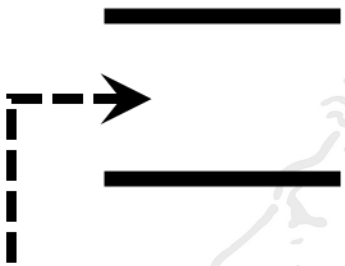
$$\mathrm{Re}[Ae^{j\omega t}] \longleftarrow \left[A_m e^{j\phi}\right]$$

Time Domain

Complex No. or Phasor Domain

Lumped vs. Distributed Analysis

Magic of Distributed Analysis



Looking into the circuit, is it an Open Circuit (i.e. Zin = ∞) ?

Lumped vs. Distributed Analysis

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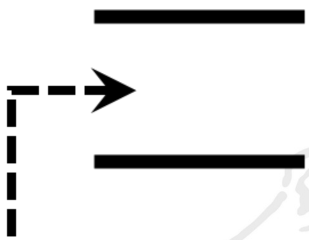

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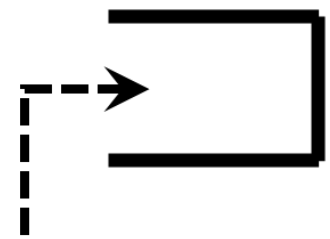
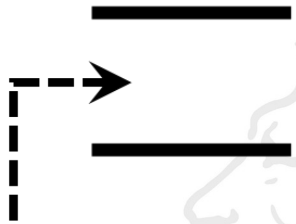


Looking into the circuit, is it a Short Circuit (i.e. Zin = 0) ?

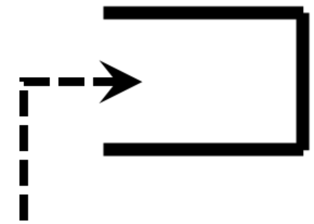
Lumped vs. Distributed Analysis

Magic of Distributed Analysis



Looking into the circuit, is it an Open Circuit (i.e. Zin = ∞) ?

Using Distributed Analysis we can show that this circuit can behave as **Short Circuit** !!!



Looking into the circuit, is it a Short Circuit (i.e. Zin = 0) ?

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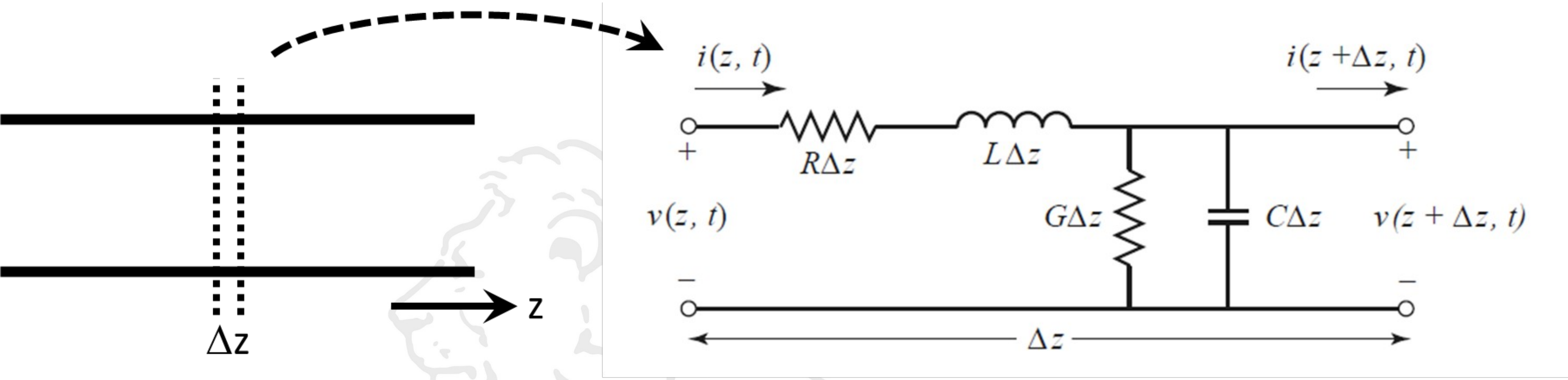
Transmission Line Analysis



R = series resistance per unit length, for both conductors, in Ω/m.

L = series inductance per unit length, for both conductors, in H/m.

G = shunt conductance per unit length, in S/m.

C = shunt capacitance per unit length, in F/m.

Explain the cause of R, L, G and C in a Transmission Line.

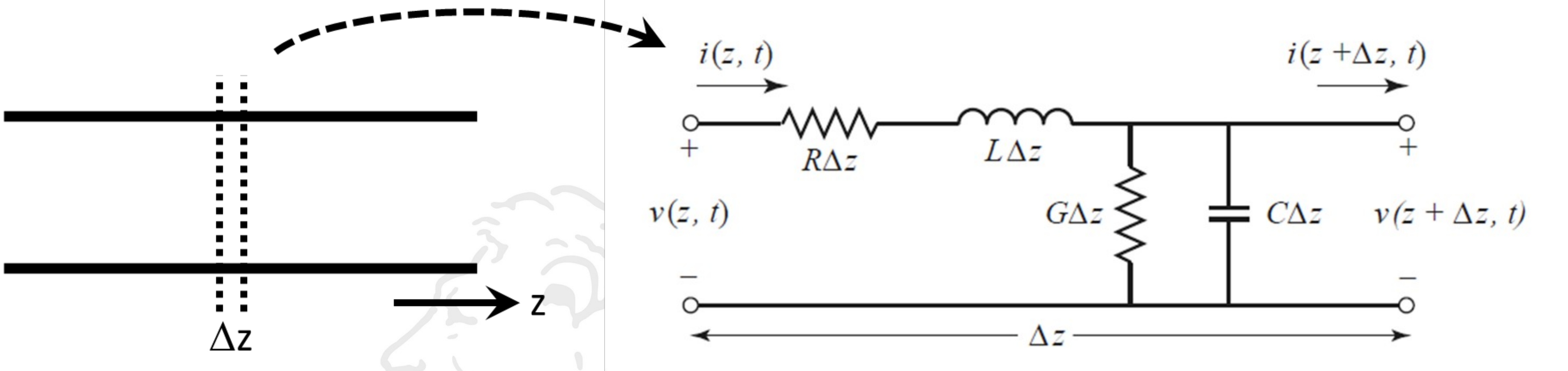
TL Analysis

The series resistance R represents the resistance due to the finite conductivity of the individual conductors, and the shunt conductance G is due to dielectric loss in the material between the conductors. R and G, therefore, represent loss.

The series inductance L represents the total self-inductance of the two conductors, and the shunt capacitance C is due to the close proximity of the two conductors.

A finite length of transmission line can be viewed as a cascade of sections of the equivalent model.

TL Analysis



Applying Kirchhoff's Voltage Law on the Equivalent Model

$$v(z,t) = R\Delta z * i(z,t) + L\Delta z * \frac{\partial i(z,t)}{\partial t} + v(z+\Delta z,t) \qquad (1.1)$$

Applying Kirchhoff's Current Law on the Equivalent Model

$$i(z,t) = G\Delta z * v(z+\Delta z,t) + C\Delta z * \frac{\partial v(z+\Delta z,t)}{\partial t} + i(z+\Delta z,t) \qquad (1.2)$$

TL Analysis

Re-arranging the equations (1.1) and (1.2), we get :

$$v(z,t) = R\Delta z * i(z,t) + L\Delta z * \frac{\partial i(z,t)}{\partial t} + v(z+\Delta z,t) \qquad (1.1)$$

$$\frac{v(z+\Delta z,t) \; - v(z,t)}{\Delta z} = -R * i(z,t) - L * \frac{\partial i(z,t)}{\partial t} \qquad (1.3)$$

$$i(z,t) = G\Delta z * v(z+\Delta z,t) + C\Delta z * \frac{\partial v(z+\Delta z,t)}{\partial t} + i(z+\Delta z,t) \qquad (1.2)$$

$$\frac{i(z+\Delta z,t) \; - \; i(z,t)}{\Delta z} = -G * v(z+\Delta z,t) - C * \frac{\partial v(z+\Delta z,t)}{\partial t} \qquad (1.4)$$

TL Analysis

$$\frac{v(z+\Delta z,t)\ -v(z,t)}{\Delta z} = -R*i(z,t) - L*\frac{\partial i(z,t)}{\partial t} \qquad (1.3)$$

$$\frac{i(z+\Delta z,t)\ -\ i(z,t)}{\Delta z} = -G*v(z+\Delta z,t) - C*\frac{\partial v(z+\Delta z,t)}{\partial t} \qquad (1.4)$$

Taking the limit as Δz → 0, we get following differential equation

$$\frac{\partial \boldsymbol{v}(\boldsymbol{z},\boldsymbol{t})}{\partial \boldsymbol{z}} = -\boldsymbol{R}*\boldsymbol{i}(\boldsymbol{z},\boldsymbol{t}) - \boldsymbol{L}*\frac{\partial \boldsymbol{i}(\boldsymbol{z},\boldsymbol{t})}{\partial \boldsymbol{t}} \qquad (\mathbf{1.5})$$

$$\frac{\partial \boldsymbol{i}(\boldsymbol{z},\boldsymbol{t})}{\partial \boldsymbol{z}} = -\boldsymbol{G}*\boldsymbol{v}(\boldsymbol{z},\boldsymbol{t}) - \boldsymbol{C}*\frac{\partial \boldsymbol{v}(\boldsymbol{z},\boldsymbol{t})}{\partial \boldsymbol{t}} \qquad (\mathbf{1.6})$$

Also known as Telegrapher Equations

TL Analysis

Applying phasors we get the frequency domain equations as below:

$$\frac{dV(z)}{dz} = -(R + j\omega L) * I(z) \qquad (1.7)$$

$$\frac{dI(z)}{dz} = -(G + j\omega C) * V(z) \qquad (1.8)$$

Note that the partial differential equation becomes differential equation and notation of voltage and currents are capital, in phasor domain.

Now, taking the derivative of (1.7) w.r.t z and using (1.8), we get:

$$\frac{d^2V(z)}{dz} = (R + j\omega L) * (G + j\omega C) * V(z) \qquad (1.9)$$

$$\frac{d^2V(z)}{dz} - \gamma^2 * V(z) = 0 \qquad (1.10)$$

where, γ is a complex propagation constant

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L) * (G + j\omega C)} \qquad (1.11)$$

TL Analysis

Homework

Similarly using equations (1.7) and (1.8), show that

$$\frac{d^2 I(z)}{dz} - \gamma^2 * I(z) = 0 \qquad (1.12)$$

TL Analysis

$$\frac{d^2V(z)}{dz} - \gamma^2 * V(z) = 0 \quad (1.10) \qquad \frac{d^2I(z)}{dz} - \gamma^2 * I(z) = 0 \quad (1.12)$$

Equations (1.10) and (1.12) are known as "Wave Equations", whose Travelling Wave solutions are as shown below:

$$\boldsymbol{V(z) = V^{+}e^{-\gamma z} + V^{-}e^{\gamma z}} \qquad \mathbf{(1.13)}$$

$$\boldsymbol{I(z) = I^{+}e^{-\gamma z} + I^{-}e^{\gamma z}} \qquad \mathbf{(1.14)}$$

where, $e^{-\gamma z}$ term represents wave travelling in +z direction, and the $e^{\gamma z}$ term represents wave travelling in -z direction.

From, equations (1.13) and (1.14), we have 4 unknown terms namely $V^{+}, V^{-}, I^{+}$and I^{-}

It is important to note that γ depends only on R, L, G, C and frequency which are all known. R, L, G and C depends only on TL physical structure which is already known.

TL Analysis

Let us now express $I^+$and I^- in terms of $V^+$and V^-. From, equations (1.7) we have

$$\frac{dV(z)}{dz} = -(R + j\omega L) * I(z) \qquad (1.7)$$

Using equation (1.13) and (1.14), we get:

$$-\gamma V^+ e^{-\gamma z} + \gamma V^- e^{\gamma z} = -(R + j\omega L) * [I^+ e^{-\gamma z} + I^- e^{\gamma z}] \qquad (1.15)$$

Re-arranging the equation we get:

$$[I^+ e^{-\gamma z} + I^- e^{\gamma z}] = \frac{\gamma}{(R + j\omega L)} [V^+ e^{-\gamma z} - V^- e^{\gamma z}] \qquad (1.16)$$

Let us define the Characteristic Impedance of the TL as Z_0 :

$$\boldsymbol{Z_0 = \frac{(R + j\omega L)}{\gamma} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \qquad (1.17)}$$

Thus, current on the TL is expressed in terms of voltage as:

$$\boldsymbol{[I^+ e^{-\gamma z} + I^- e^{\gamma z}] = \frac{1}{Z_0} [V^+ e^{-\gamma z} - V^- e^{\gamma z}] \qquad (1.18)}$$

It is important to note that Z_0 depends only on R, L, G, C and frequency which are all known. R, L, G and C depends only in TL physical structure which is already known.

TL Analysis

Converting back to the time domain, we can express the voltage waveform as:

$$v(z,t) = [|V^+|e^{-\alpha z}\cos(\omega t - \beta z + \phi^+)] + [|V^-|e^{\alpha z}\, cos(\omega t + \beta z + \phi^-)] \qquad (1.19)$$

where $\phi^{\pm}$is the phase angle of the complex voltage $V^{\pm}$.

Wavelength (λ) is defined as the distance between two successive maxima or minima (or any reference points) on the wave at a fixed instant of time. Hence,

$$(\omega t - \beta z) - [\omega t - \beta(z+\lambda)] = 2\pi \qquad (1.20)$$

$$\lambda = \frac{\mathbf{2}\pi}{\beta} \qquad (\mathbf{1.21})$$

<u>Homework</u>

Phase velocity (v_p) is defined as the velocity at which a fixed phase point on the wave travels. Show that :

$$\boldsymbol{v_p} = \frac{\omega}{\beta} = \boldsymbol{f}\lambda \qquad (\mathbf{1.22})$$

TL Analysis

Homework

A lossless transmission line is one which has no dissipative losses.

Thus, R = G = 0.

Determine the expressions for the following in terms of L, C:

- Propagation Constant (γ)
- Characteristic Impedance (Z_0)
- Expression for Voltage and Current waves on the TL
- Wavelength (λ)
- Phase Velocity (v_p)

TL Analysis

Remember:
$$V(z) = V^+ e^{-\gamma z} + V^- e^{\gamma z} \qquad (1.13)$$

$$I(z) = I^+ e^{-\gamma z} + I^- e^{\gamma z} \qquad (1.14)$$

$$[I^+ e^{-\gamma z} + I^- e^{\gamma z}] = \frac{1}{Z_0} [V^+ e^{-\gamma z} - V^- e^{\gamma z}] \qquad (1.18)$$

So far, we have 2 unknowns, namely, V^+ and V^-. Hence, we need a boundary condition (or a termination) to express V^- in terms of V^+.

For a lossless TL we have:

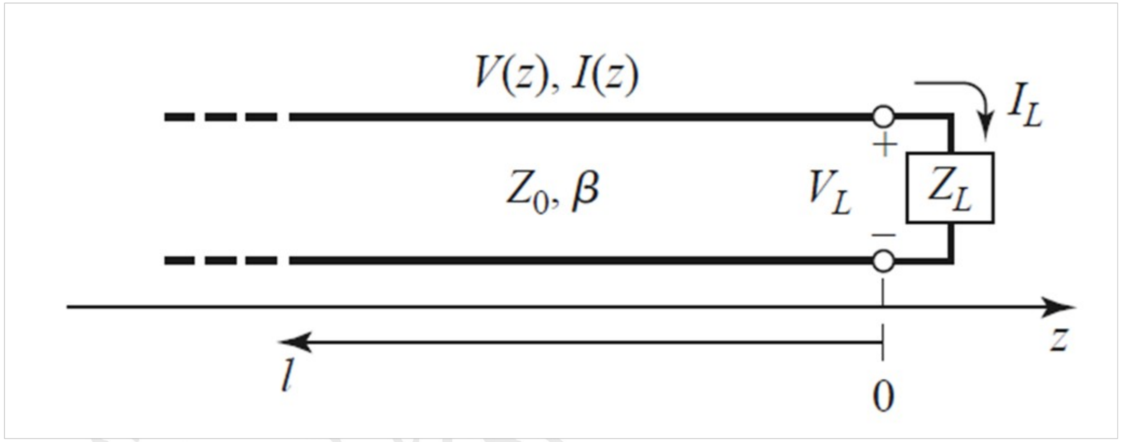
$$V(z) = V^+ e^{-j\beta z} + V^- e^{j\beta z} \qquad (1.19)$$

$$I(z) = I^+ e^{-j\beta z} + I^- e^{j\beta z} \qquad (1.20)$$

$$I(z) = \frac{1}{Z_0} [V^+ e^{-j\beta z} - V^- e^{j\beta z}] \qquad (1.21)$$

TL Analysis

The figure below shows a lossless TL terminated with an arbitrary (known) load impedance Z_L



At z = 0, we know that the impedance is Z_L which is the ratio of total voltage and current. Using equations (1.19) and (1.21) and substituting z = 0, we get:

$$Z_L = \frac{V(0)}{I(0)} \qquad (1.22)$$

$$V(0) = V^+ + V^- \qquad (1.23)$$

$$I(0) = \frac{1}{Z_0}[V^+ - V^-] \qquad (1.24)$$

$$Z_L = Z_0 \frac{V^+ + V^-}{V^+ - V^-} \qquad (1.25)$$

TL Analysis

Re-arranging the equation (1.25) we can express $\boldsymbol{V}^-$ in terms of $\boldsymbol{V}^+$ as below:

$$Z_L = Z_0 \frac{V^+ + V^-}{V^+ - V^-} \qquad (1.25)$$

$$V^- = V^+ \frac{Z_L - Z_0}{Z_L + Z_0} \qquad (1.26)$$

The voltage Reflection Co-efficient (Γ) is defined as the ratio of the amplitude of the reflected voltage wave and the amplitude of the forward voltage wave.

$$\Gamma = \frac{\boldsymbol{V}^-}{\boldsymbol{V}^+} = \frac{\boldsymbol{Z_L - Z_0}}{\boldsymbol{Z_L + Z_0}} \qquad (\mathbf{1.27})$$

The total voltage and current waves on the TL can then be expressed as:

$$\boldsymbol{V(z) = V^+(e^{-j\beta z} + \Gamma e^{j\beta z})} \qquad (\mathbf{1.28})$$

$$\boldsymbol{I(z) = \frac{V^+}{Z_0}(e^{-j\beta z} - \Gamma e^{j\beta z})} \qquad (\mathbf{1.29})$$

TL Analysis

$$V(z) = V^{+}(e^{-j\beta z} + \Gamma e^{j\beta z}) \qquad (1.28)$$

$$I(z) = \frac{V^{+}}{Z_0}(e^{-j\beta z} - \Gamma e^{j\beta z}) \qquad (1.29)$$

From these equations it is seen that the voltage and current on the line consist of a superposition of an incident and a reflected wave; such waves are called standing waves.

Only when Γ = 0 is there no reflected wave. To obtain Γ = 0, the load impedance Z_L must be equal to the characteristic impedance Z_0 of the transmission line, as seen from (1.27).

Such a load is said to be matched to the line since there is no reflection of the incident wave.

It is important to note that now the number of unknown is one i.e. V^{+}, which can be determined from the knowledge of the excitation (or source condition).

TL Analysis

Homework

Now, the time average power flow along the TL at the point z is given as:

$$P_{avg} = \frac{1}{2} Re\{V(z)\, I(z)^*\} \qquad (1.30)$$

Using equations (1.28) and (1.29) show that :

$$\boldsymbol{P_{avg} = \frac{1}{2}\frac{|V^+|^2}{Z_0}(1 - |\Gamma|^2)} \qquad (\mathbf{1.31})$$

Hint : A - A* $= 2\mathrm{j}\ \mathrm{Im}(\mathrm{A})$, hence $\mathrm{Re}\{\mathrm{A} - \mathrm{A}^*\} = 0$

TL Analysis

Voltage Standing Wave Ratio (VSWR or SWR) is defined as the ratio of maximum amplitude of the voltage to the minimum amplitude of the voltage on the TL.

$$SWR = \frac{V_{max}}{V_{min}} \qquad (1.32)$$

Taking the amplitude of V(z) in equation (1.28) we have [note that Γ is complex number, $\Gamma = |\Gamma|\, e^{j\theta}$]:

$$\boldsymbol{V(z) = V^+(e^{-j\beta z} + \Gamma e^{j\beta z})} \qquad \mathbf{(1.28)}$$

$$|V(z)| = |V^+|(1 + |\Gamma| e^{2j\beta z + j\theta}) \qquad (1.33)$$

At $z = -l$ i.e. at a distance of l from load termination we get:

$$|V(-l)| = |V^+|(1 + |\Gamma| e^{j(\theta - 2\beta l)}) \qquad (1.34)$$

Thus, maximum possible value occurs when $e^{j(\theta-2\beta l)} = 1$, and minimum occurs when $e^{j(\theta-2\beta l)} = -1$. Thus the VSWR is given by expression:

$$\boldsymbol{SWR = \frac{1 + |\Gamma|}{1 - |\Gamma|}} \qquad \mathbf{(1.35)}$$

TL Analysis

Reflection Co-efficient at $z = -l$ i.e. at a distance of l from load termination is given as:

$$\Gamma(-\boldsymbol{l}) = \frac{\boldsymbol{V}^{-}e^{-j\beta l}}{\boldsymbol{V}^{+}e^{j\beta l}} = \Gamma(\mathbf{0})e^{-j2\beta l} \qquad (\mathbf{1.36})$$

Hence, it is important to note that the magnitude of reflection co-efficient does not change along the lossless TL. Only the phase changes as we move along the TL.

As engineers, we are interested in impedance along the TL. Hence, by definition of the impedance at a point z = $-l$, we get:

$$Z_{in} = \frac{V(-l)}{I(-l)} \qquad (1.37)$$

Using expressions (1.28) and (1.29), we get:

$$Z_{in} = Z_0 \frac{V^{+}(e^{j\beta l}+\Gamma e^{-j\beta l})}{V^{+}(e^{j\beta l}-\Gamma e^{-j\beta l})} \qquad (1.38)$$

Using expression (1.27), we get:

$$Z_{in} = Z_0 \frac{(Z_L + Z_0)e^{j\beta l} + (Z_L - Z_0)e^{-j\beta l}}{(Z_L + Z_0)e^{j\beta l} - (Z_L - Z_0)e^{-j\beta l}} \qquad (1.39)$$

TL Analysis

Homework

Reduce the equation (1.39) to the following form:

$$\boldsymbol{Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan(\beta l)}{Z_0 + jZ_L tan(\beta l)}} \qquad \mathbf{(1.40)}$$

TL Analysis

Case Study 1 : $Z_L = Z_0$ (Matched Load)

$$Z_{in} = Z_0 \frac{Z_0 + jZ_0 \tan(\beta l)}{Z_0 + jZ_0 tan(\beta l)}$$

$$Z_{in} = Z_0 \frac{\cancel{Z_0 + jZ_0 \tan(\beta l)}}{\cancel{Z_0 + jZ_0 tan(\beta l)}}$$

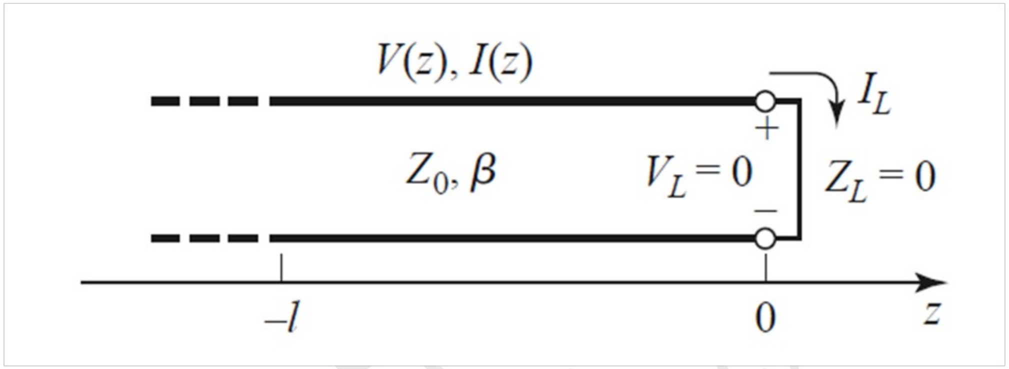
$$Z_{in} = Z_0$$

$$\Gamma_{in} = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} = 0$$

Hence, when terminated with matched load (i.e. Characteristic Impedance Z_0), the reflection co-efficient is zero and thus there is no reflected wave on the TL.

TL Analysis

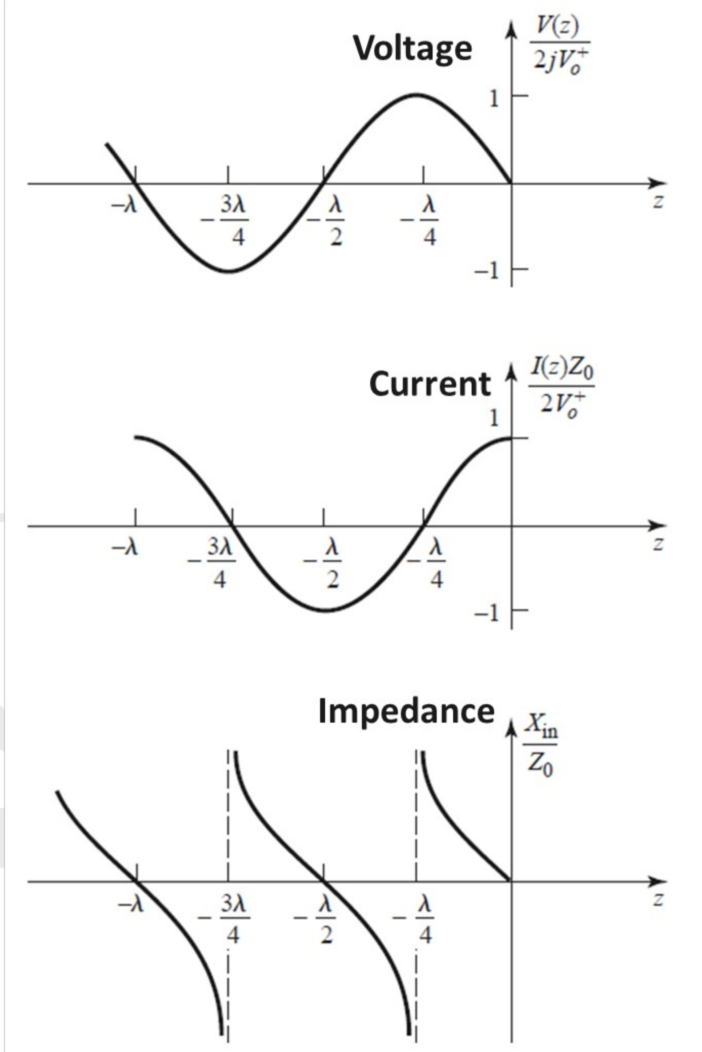
Case Study 2 : $Z_L = 0$ (Short Circuited Load)



$$Z_{in} = Z_0\frac{Z_L + jZ_0\tan(\beta l)}{Z_0 + jZ_L tan(\beta l)} = \cancel{Z_0}\frac{0 + jZ_0\tan(\beta l)}{\cancel{Z_0} + j0tan(\beta l)}$$

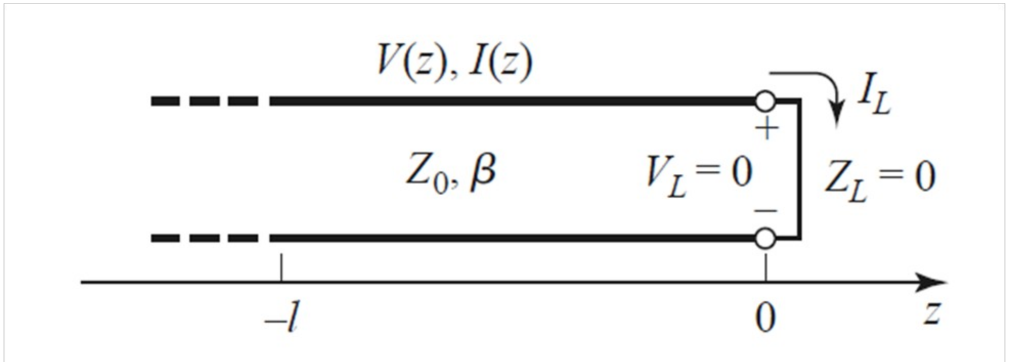
$$\boldsymbol{Z_{in} = jZ_0 tan(\beta l)}$$

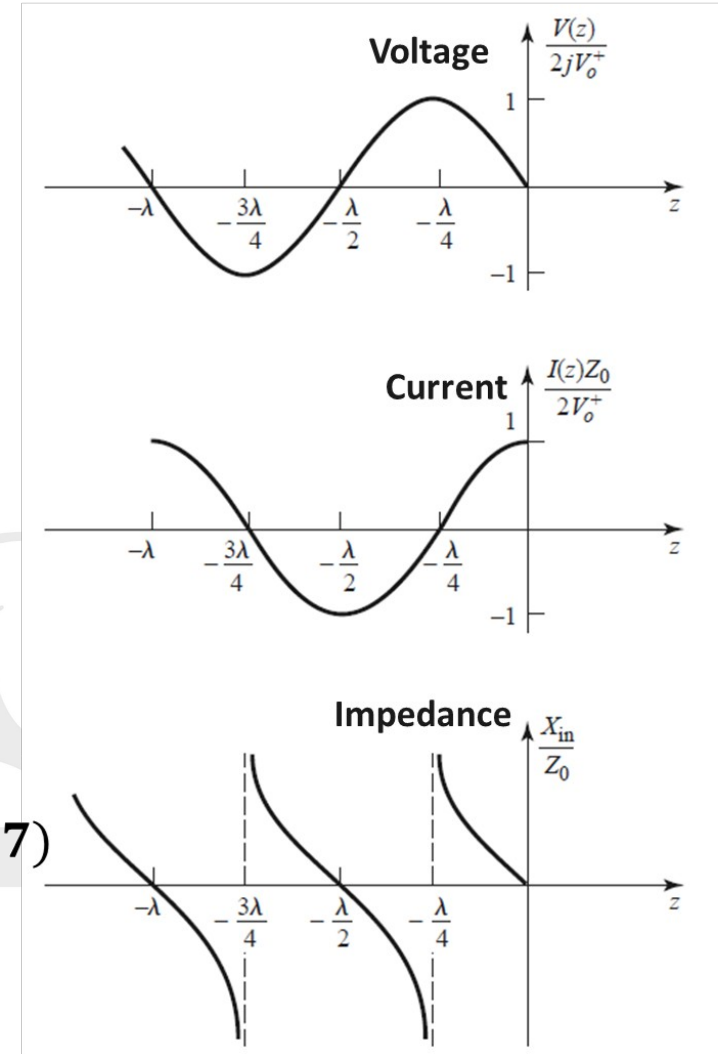
Hence, when terminated with short circuited load (i.e. $Z_L = 0\Omega$), the TL behaves like short circuit, Inductor, Open circuit and Capacitor depending on the length and frequency of operation .



TL Analysis

Case Study 2 : $Z_L = 0$ (Short Circuited Load)





$$\boldsymbol{V(z) = V^+e^{-j\beta z} + V^-e^{j\beta z}} \qquad \mathbf{(1.19)}$$

$$V(z) = V^+e^{-j\beta z} + V^-e^{j\beta z}$$

$$V(z) = V^+(e^{-j\beta z} + \Gamma e^{j\beta z})$$

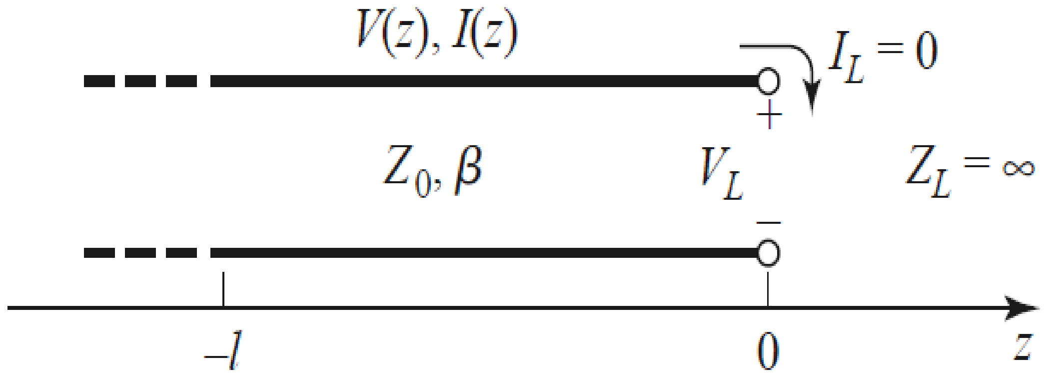
$$\boldsymbol{\Gamma = \frac{V^-}{V^+} = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{0 - Z_0}{0 + Z_0} = -1} \qquad \mathbf{(1.27)}$$

$$V(z) = V^+(e^{-j\beta z} - e^{j\beta z})$$

$$V(z) = -j2V^+sin(\beta z)$$

TL Analysis

Homework



Determine the input impedance as a function of length and frequency. Also determine the voltage and current wave expressions. Using Matlab plot the amplitude of voltage wave and current wave. Using Matlab plot the input impedance.

Thank You

Questions?