Assignment 5 solutions

1. Using Maxwell's equations derive the wave equations for electric and magnetic fields and show that in free space electromagnetic waves travel with velocity of light.

Soln 1) Griffith (4th edition) - Introduction to electrodynamics
→ Page 398, section 2.1

2. In free space,

$$\mathbf{E} = \frac{50}{\rho}\cos(10^8 t - kz)\,\mathbf{a}_\rho \text{ V/m}$$

Find k, $\mathbf{J}_d$, and $\mathbf{H}$.

Soln 2) $\vec{E} = \frac{50}{\rho} \cos(10^8 t - kz)\,\hat{\rho}$ V/m

(a) $v = \frac{\omega}{k} \Rightarrow k = \frac{\omega}{v} = \frac{10^8}{3 \times 10^8}\ m^{-1} = 0.33\ m^{-1}$

(b) $J_d = \varepsilon_0 \frac{\partial E}{\partial t} = -\varepsilon_0 \frac{50}{\rho} \sin(10^8 t - kz) \times 10^8\ \hat{\rho}\ Am^{-2}$

(c) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$

$\vec{H} = -\frac{1}{\mu_0} \int (\vec{\nabla} \times \vec{E})\, dt$

$$\vec{\nabla} \times \vec{E} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{k} \\ \partial/\partial\rho & \partial/\partial\phi & \partial/\partial z \\ \frac{50}{\rho}\cos(10^8 t - kz) & 0 & 0 \end{vmatrix}$$

$$= \frac{1}{\rho} \times (-\rho\hat{\phi}) \left[-\frac{\partial}{\partial z} \left(\frac{50}{\rho} \cos(10^8 t - kz) \right) \right]$$

$\vec{\nabla} \times \vec{E} = -\hat{\phi} \times \frac{50}{\rho} \sin(10^8 t - kz) \times (-k)$

$\vec{H} = -\frac{1}{\mu_0} \times \frac{50}{\rho} k\, \hat{\phi} \int \sin(10^8 t - kz)\, dt$

$$\vec{H} = \frac{50\,k}{\rho\,\mu_0} \times \frac{\cos(10^8 t - kz)}{10^8}\,\hat{\phi}$$

Note : For plane em wave you can also use

$$\vec{B} = \frac{\vec{k} \times \vec{E}}{\omega}$$

3. In a certain region,

$$\mathbf{J} = (2y\,\mathbf{a}_x + xz\,\mathbf{a}_y + z^3\,\mathbf{a}_z)\sin 10^4 t \ \mathrm{A/m^2}$$

find ρ_v if $\rho_v(x, y, 0, t) = 0$.

Soln 3) $\quad \vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t}$

$$\vec{\nabla}\cdot\vec{J} = 3z^2 \sin 10^4 t \Rightarrow \rho = -\int(\vec{\nabla}\cdot\vec{J})\,dt$$

$$\Rightarrow \rho_v = -3z^2 \int \sin(10^4 t)\,dt$$

$$= -3z^2 \times \left[-\frac{\cos(10^4 t)}{10^4} + c\right]$$

$$\rho_v = \frac{3z^2}{10^4}\cos(10^4 t) + c'$$

$$\rho_v(x, y, 0, t) = 0$$

$$0 + c' = 0 \Rightarrow c' = 0$$

$$\rho_v = \frac{3z^2}{10^4}\cos(10^4 t)$$

4. Check the following fields are genuine EM fields (i.e., they satisfy Maxwell's equations). Assume that the fields exist in charge-free regions.

(a) $\mathbf{A} = 40\sin(\omega t + 10x)\,\mathbf{a}_z$

(b) $\mathbf{B} = \dfrac{10}{\rho}\cos(\omega t - 2\rho)\,\mathbf{a}_\phi$

(c) $\mathbf{C} = \left(3\rho^2\cot\phi\,\mathbf{a}_\rho + \dfrac{\cos\phi}{\rho}\,\mathbf{a}_\phi\right)\sin\omega t$

(d) $\mathbf{D} = \dfrac{1}{r}\sin\theta\sin(\omega t - 5r)\,\mathbf{a}_\theta$

Soln 4) $\vec{C} = \left[3\rho^2 \cot\phi\,\hat{\rho} + \dfrac{\cos\phi}{\rho}\,\hat{\phi}\right]\sin(\omega t)$

It should satisfy $\vec{\nabla}\cdot\vec{C} = 0$

→ if true then it should also satisfy em wave eq^n to be a valid em field.

$$\nabla^2\vec{C} = \mu_0\varepsilon_0\frac{\partial^2\vec{C}}{\partial t^2}$$

5. Find the state of polarization when the following two plane polarized waves superimpose.

(a) $E_x = E_0\cos(\omega t + kz) \quad E_y = \dfrac{E_0}{\sqrt{2}}\cos(\omega t + kz + \pi)$

(b) $E_x = E_0\cos(kz - \omega t + \dfrac{\pi}{3}) \quad E_y = E_0\cos(kz - \omega t - \dfrac{\pi}{6})$

(c) $E_x = E_0\cos(kz - \omega t + \dfrac{\pi}{4}) \quad E_y = \dfrac{E_0}{\sqrt{2}}\cos(kz - \omega t)$

(d) $E_x = E_0\cos(\omega t + kz) \quad E_y = E_0\cos(\omega t + kz)$

Soln 5) $\underbrace{\Delta\phi}_{\text{phase difference}} = \phi_y - \phi_x$

$E_x = E_{0x}\cos(\omega t - \beta z + \phi_x)\,\hat{i}$

$E_y = E_{0y}\cos(\omega t - \beta z + \phi_y)\,\hat{j}$

(1) Linear polarization

$\Delta\phi = \underbrace{n\pi}$; $n = 0, 1, 2, \ldots$

(2) Circular Polarization

$$E_{0x} = E_{0y}$$

$$\Delta\phi = \pm (2n+1)\frac{\pi}{2} \quad ; \quad n = 0, 1, 2, \ldots$$

(3) Elliptical Polarization

Otherwise

6. A plane harmonic electromagnetic wave propagates in vacuum. The electric field of the wave has amplitude 50 mV/m and frequency 100 MHz. Calculate the root mean square (*rms*) value of the displacement current and the intensity of the wave.

Soln 6) $\vec{E} = 50 \cos(2\pi(100)t - \vec{K}\cdot\vec{r})\ \hat{n}$

(a) $J_d = \varepsilon_0 \dfrac{\partial \vec{E}}{\partial t}$

$$(J_d)_{rms} = \left[\frac{\int_0^\tau J_d^2\, dt}{\int_0^\tau dt}\right]^{1/2}$$

$$(J_d)_{rms} = \frac{(J_d)_0}{\sqrt{2}}$$

where $(J_d)_0$ is the amplitude

Rough Work

$$U = \frac{1}{2}\varepsilon_0 E_0^2$$

$$\text{Energy} = \frac{1}{2}\varepsilon_0 V E_0^2$$

$$I = \frac{\text{Energy}}{tA}$$

$$I = \frac{1}{2}\frac{\varepsilon_0 V E_0^2}{tA}$$

$$I = \frac{1}{2}\varepsilon_0 v E_0^2$$

(b) $\boxed{I = \frac{1}{2}\varepsilon_0 c E_0^2}$ or $I = \langle S \rangle = \dfrac{E_0^2}{2\eta}$

where $\eta = \sqrt{\dfrac{\mu}{\varepsilon}}$

for vacuum, $\eta = \eta_0 = 376.73\ \Omega$

7. Consider an electromagnetic wave polarized in the y-direction propagates in the positive x-direction in vacuum. Using the Maxwell's equation show that

$$\frac{\partial B}{\partial t} = -\frac{\partial E}{\partial x} \quad \text{and} \quad \frac{\partial E}{\partial t} = -c^2 \frac{\partial B}{\partial x}$$

$$\vec{E} = E_0 \cos(\omega t - Kx)\, \hat{j} \quad V/m$$

$$\vec{B} = \frac{\vec{K} \times \vec{E}}{\omega} = \frac{1}{\omega} (K_x \hat{i}) \times (E_0 \cos(\omega t - Kx))\, \hat{j}$$

$$\vec{B} = \frac{1}{\omega} \times K_x E_0 \cos(\omega t - Kx)\, \hat{k}$$

(a) $$\frac{\partial B}{\partial t} = -\frac{\partial E}{\partial x}$$

$$-\frac{K_x E_0}{\omega} \sin(\omega t - Kx)\, \omega = -\left(-E_0 \sin(\omega t - Kx) \times (-K)\right)$$

LHS = RHS

(b) $$\frac{\partial E}{\partial t} = -c^2 \frac{\partial B}{\partial x}$$ (verify yourself)

8. An electromagnetic wave with electric field $\mathbf{E} = 30\cos(\omega t - z)\, \mathbf{a}_x$ V/m propagating in air hits normally a lossless dielectric medium ($\mu = \mu_0$ and $\varepsilon = 4\varepsilon_0$) at $z = 0$. Calculate:

(a) the amplitude of electric fields of the reflected ad transmitted waves

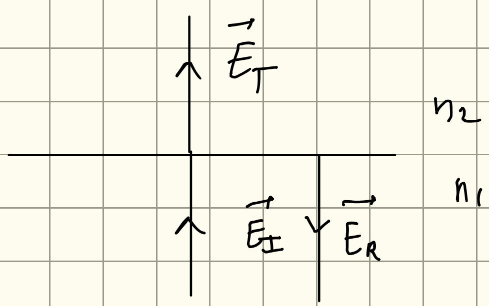
(b) reflection and transmission coefficients

(c) the increment in wavelength when the wave enters into the dielectric medium.

Soln) $\mu = \mu_r \mu_0$; $\varepsilon = \varepsilon_r \varepsilon_0$

$$n = \sqrt{\mu_r \varepsilon_r}$$

$n_1 = 1$; $n_2 = \sqrt{1 \times 4} = 2$



(a)

$$E_{0R} = \left| \frac{n_1 - n_2}{n_1 + n_2} \right| E_{0I}$$

$$E_{0T} = \frac{2n_1}{(n_1 + n_2)} E_{0I}$$

See Griffith

(b) $R = \left(\dfrac{n_1 - n_2}{n_1 + n_2}\right)^2$

$T = \dfrac{4 n_1 n_2}{(n_1 + n_2)^2}$

(c) Frequency doesn't change when wave travels from one medium to another

$\nu_{medium\,1} = \nu_{medium\,2}$ $\qquad v = \nu\lambda$

$\dfrac{v_1}{\lambda_1} = \dfrac{v_2}{\lambda_2}$ $\qquad \nu = \dfrac{v}{\lambda}$

$\dfrac{c}{n_1 \lambda_1} = \dfrac{c}{n_2 \lambda_2}$

$\lambda_2 = \dfrac{n_1}{n_2} \lambda_1$

$\Delta\lambda = \lambda_2 - \lambda_1 = \lambda_1 \left(\dfrac{n_1}{n_2} - 1\right)$

9. A uniform plane wave in a certain medium ($\mu = \mu_0$, $\varepsilon = 4\varepsilon_0$) is given by

$$\mathbf{E} = 12\cos(\omega t - 40\pi x)\mathbf{a}_z \quad \text{V/m}$$

(a) Find ω.

(b) If the wave is normally incident on a dielectric ($\mu = \mu_0$, $\varepsilon = 3.2\varepsilon_0$), determine E_r and E_t.

Soln) (a) $v = \dfrac{\omega}{K} \Rightarrow$

$\omega = vK$

$= \dfrac{1}{\sqrt{\mu\,\varepsilon}} \times 40\pi$

$= \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}\sqrt{\mu_r \varepsilon_r}} \times 40\pi$

$= 40\pi c \times \dfrac{1}{\sqrt{1 \times 4}}$

$\omega = 20\pi c$

(b)

$E_{0R} = \left|\dfrac{n_1 - n_2}{n_1 + n_2}\right| E_{0I}$

$n_1 = 1$

$n_2 = \sqrt{\mu_r \varepsilon_r} = \sqrt{3.2}$

$E_{0T} = \dfrac{2n_1}{(n_1 + n_2)} E_{0I}$

10. An electromagnetic plane wave of wavelength $\lambda = 500\,\text{nm}$ with a circular cross-section of diameter $d = 2\,\text{mm}$ propagates in free space carrying power $P = 5\,\text{mW}$. The wave is linearly polarized along the x-axis and travels in the $+z$ direction. Determine the complete expressions for the electric and magnetic fields of the wave.

Soln) $\lambda = 500\ nm \Rightarrow K = \frac{2\pi}{\lambda} = \frac{2\pi}{500} \times 10^9\ m^{-1}$

$P = 5\ mW$

Intensity $= \frac{1}{2}\varepsilon_0 v E_0^2 = \frac{\text{Power}}{\text{Area}}$

$E_0^2 = \frac{5\times10^{-3}}{\pi\times1\times10^{-6}} \times \frac{2}{\varepsilon_0 c}$

$\omega = v K$

$\omega = c K$

$\vec{E} = E_0 \cos(\omega t - Kz)\ \hat{i}$

$\vec{B} = \frac{\vec{K}\times\vec{E}}{\omega}$ (valid for plane wave)

$\vec{K} = K\hat{z}$