Lecture-25

12/10/2023.

Group Velocity $\Rightarrow$ $$\frac{d\omega}{dk} = \frac{d\omega/dv}{dk/dv} \quad \text{———} \quad (1)$$

$$\omega = \frac{mc^2}{\hbar} = \frac{m_0c^2}{\hbar\sqrt{1-\frac{v^2}{c^2}}}$$

$$\frac{d\omega}{dv} = \frac{m_0c^2}{\hbar}\left[-\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left(-\frac{2v}{c^2}\right)\right]$$

$$\frac{d\omega}{dv} = \frac{m_0 v}{\hbar\left(1-\frac{v^2}{c^2}\right)^{3/2}} \quad \text{———} \quad (a)$$

$$k = \frac{p}{\hbar} = \frac{mv}{\hbar} = \frac{m_0 v}{\hbar\sqrt{1-\frac{v^2}{c^2}}}$$

$$\frac{dk}{dv} = \frac{m_0}{\hbar}\left[v\left\{-\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left(-\frac{2v}{c^2}\right)\right\} + 1\cdot\left(1-\frac{v^2}{c^2}\right)^{-1/2}\right]$$

$$= \frac{m_0}{\hbar}\left[\frac{v^2}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-3/2} + \left(1-\frac{v^2}{c^2}\right)^{-1/2}\right]$$

$$= \frac{m_0}{\hbar}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\left[\cancel{\frac{v^2}{c^2}} + \left(1-\cancel{\frac{v^2}{c^2}}\right)\right]$$

$$= \frac{m_0}{\hbar}\frac{1}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} \quad \text{———} \quad (b)$$

$$\therefore\ v_g = \frac{\cancel{m_0} v}{\cancel{\hbar}\cancel{\left(1-\frac{v^2}{c^2}\right)^{3/2}}} \cdot \frac{\cancel{\hbar}\cancel{\left(1-\frac{v^2}{c^2}\right)^{3/2}}}{\cancel{m_0}}$$

$$v_g = v$$

The velocity of the group of matter waves is equal to the velocity of the particle whose motion they govern.

Homework

Q:⇒ An electron has a de-Broglie wavelength of 2×10^{-12} m. Find it's K.E. & the phase & group velocity of it's de-Broglie waves.

Ans:⇒ K.E. = 292 keV.

$v_p = 1.3\,c$

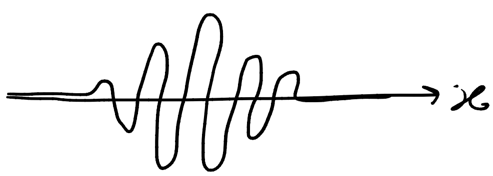
$v_g = .771\,c.$

{c → velocity of light}

Heisenberg Uncertainity Principle ⇒

* In Q.M., a particle is described by a wave packet. Acc. to Born's probability interpretation, the particle can be found anywhere within the wavepacket.

⇒ This implies that the position of the particle is uncertain within the limits of the wavepacket



$$\psi_0(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(k)\,e^{ikx}\,dk$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi_0(x)\,e^{-ikx}\,dx$$

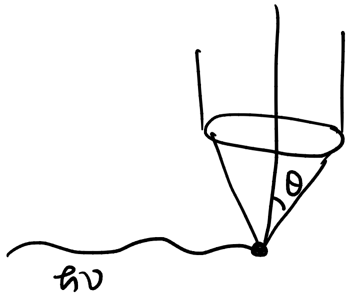
* The width of a wavepacket $\psi_0(x)$ & the width of $\phi(k)$ are not independent, they are correlated by a reciprocal relationship. This reciprocal relationship has a direct connection to the Heisenberg Uncertainty relation.

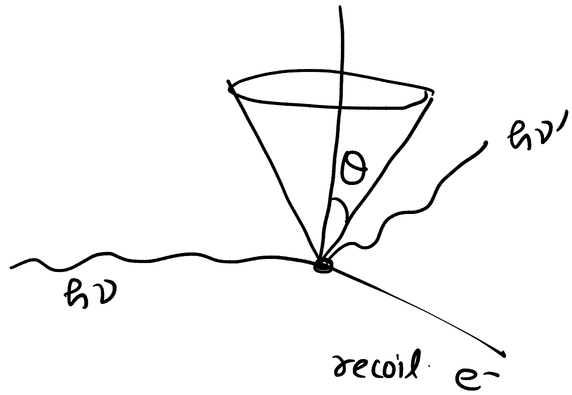
* The product of the uncertanities in determing the position & momentum of particle can never be smaller than $\frac{\hbar}{2}$.

$$\boxed{\begin{aligned}\Delta x\cdot\Delta p_x &\geq \frac{\hbar}{2}\\ \Delta E\cdot\Delta t &\geq \frac{\hbar}{2}\\ \Delta J\cdot\Delta\theta &\geq \frac{\hbar}{2}\end{aligned}}$$

Examples of Uncertainty principle :→

Microscope :→ Let us try to observe the position of an e^- with a very high resolving power microscope.





Limit of resolution, (optical theory) :– $\Delta x = \dfrac{\lambda'}{2\sin\theta}$ ——— (1)

$\Delta x \Rightarrow$ distance b/w two points which can be resolved by the microscope.

⇒ To minimize this uncertainty, Let us use shorter wavelength (say γ ray) compton effect.

Conservation of Momentum :→

$$\frac{h\nu}{c} = \frac{h\nu'}{c}\sin\theta + P_x$$

$$P_x = \frac{h}{c}[\nu - \nu'\sin\theta]$$

$$P_x = \frac{h}{c}[\nu + \nu'\sin\theta)$$

Therefore the uncertainty in Momentum of e^- -

$$\Delta P_x = \frac{h}{c}[\cancel{\nu} + \nu'\sin\theta] - \frac{h}{c}[\cancel{\nu} - \nu'\sin\theta)$$

$$\Delta p_x = \frac{2h}{c}\nu'\sin\theta$$

$$\Delta p_x = \frac{2h}{\lambda'}\sin\theta \quad \text{———} \quad (2)$$

$$\Delta x \cdot \Delta p_x \sim \frac{\lambda'}{2\sin\theta} \cdot \frac{2h}{\lambda'}\sin\theta$$

$$\Delta x \cdot \Delta p_x \sim h$$

$\Delta x \cdot \Delta p_x \geq \hbar/2$ [Validity of Heisenberg's uncertainty].

Q:⇒ A measurement establishes the position of a proton with an accuracy of $1 \times 10^{-11}\,m$. Find the uncertainty in the proton's position 1 second latter. Assum $v \ll c$.

Ans :⇒ $\Delta x \geq 3.15 \times 10^{3}\,m$