Coordinate Systems in Physics

1. Cartesian, Cylindrical, and Spherical Coordinates

Physics problems often require describing positions of points in space. The system you choose depends on the symmetry of the problem.

a. Cartesian (Rectangular) Coordinates

- A point is represented as (x, y, z).
- The axes are mutually perpendicular.
- Simple for problems with cube or brick-like symmetry.

b. Cylindrical Coordinates

- A point is represented as (ρ, ϕ, z).
  - ρ: Radial distance from the z-axis.
  - ϕ: Azimuthal angle from the x-axis (in xy-plane).
  - z: Height above the xy-plane.
- Useful for problems with rotational or axial symmetry (e.g., a long wire, cylinder).

c. Spherical Coordinates

- A point is denoted as (r, θ, ϕ).
  - r: Radial distance from the origin.
  - θ: Polar angle with respect to the z-axis ($0 \leq \theta \leq \pi$).
  - ϕ: Azimuthal angle in the xy-plane ($0 \leq \phi < 2\pi$).
- Suitable for systems with spherical symmetry (e.g., point charges, planet orbits).

2. Interconversion Between Coordinate Systems

The following equations are regularly used to convert a point between systems:

a. Cartesian ↔ Cylindrical

- From Cartesian (x, y, z) to Cylindrical (ρ, ϕ, z):
  - $\rho = \sqrt{x^2 + y^2}$
  - $\phi = \tan^{-1}(y/x)$
  - $z = z$

- From Cylindrical (ρ, ϕ, z) to Cartesian:
  - $x = \rho \cos \phi$
  - $y = \rho \sin \phi$
  - $z = z$

b. Cartesian ↔ Spherical

- From Cartesian (x, y, z) to Spherical (r, θ, ϕ):
  - $r = \sqrt{x^2 + y^2 + z^2}$
  - $\theta = \cos^{-1}\left(\frac{z}{r}\right)$
  - $\phi = \tan^{-1}(y/x)$
- From Spherical (r, θ, ϕ) to Cartesian:
  - $x = r \sin \theta \cos \phi$
  - $y = r \sin \theta \sin \phi$
  - $z = r \cos \theta$

c. Cylindrical ↔ Spherical

- From Cylindrical (ρ, ϕ, z) to Spherical (r, θ, ϕ):
  - $r = \sqrt{\rho^2 + z^2}$
  - $\theta = \tan^{-1}(\rho/z)$
  - $\phi = \phi$
- From Spherical (r, θ, ϕ) to Cylindrical:
  - $\rho = r \sin \theta$
  - $\phi = \phi$
  - $z = r \cos \theta$

3. Angle Between Vectors and Planes

a. Between Two Vectors

For vectors $\vec{A}$ and $\vec{B}$:

- **General formula:**

$$\cos \theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}$$

  where θ is the angle between the vectors.
- **Cartesian example:**

$$\vec{A} = (A_x, A_y, A_z), \quad \vec{B} = (B_x, B_y, B_z)$$
$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

- **Cylindrical** and **spherical**: Express both vectors in Cartesian form using respective conversion formulae, then apply the above.

b. Between a Vector and a Plane

- **Angle between vector $\vec{A}$ and a plane with normal $\vec{n}$:**

$$\sin\theta = \frac{|\vec{A}\cdot\vec{n}|}{|\vec{A}||\vec{n}|}$$

where θ is the angle between the vector and the plane.

4. Dot and Cross Product

a. Dot Product

- **Definition:**

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

- **Components (Cartesian):**

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

- **Properties:**
  - Scalar result.
  - Commutative: $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$
  - Measures the projection of one vector onto another.

b. Cross Product

- **Definition:**

$$\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|\sin\theta\,\hat{n}$$

where $\hat{n}$ is a unit vector perpendicular to the plane containing $\vec{A}$ and $\vec{B}$ (right-hand rule).

- **Components (Cartesian):**

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_yB_z - A_zB_y)\hat{i} - (A_xB_z - A_zB_x)\hat{j} + (A_xB_y - A_yB_x)$$

- **Properties:**
  - Results in a vector.
  - $\vec{A}\times\vec{B} = -(\vec{B}\times\vec{A})$
  - Magnitude is area of parallelogram spanned by $\vec{A}$ and $\vec{B}$.

5. Divergence and Curl

a. Divergence

- Measures how much a vector field spreads out from a point.
- **Mathematical definition:**

$$\text{For } \vec{F} = (F_x, F_y, F_z): \quad \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

  - Zero divergence: field is solenoidal (e.g., magnetic field).

b. Curl

- Measures the tendency of the field to rotate around a point.
- **Mathematical definition:**

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right.$$

  - Zero curl: field is irrotational (e.g., conservative force field).

Key Takeaways

- **Choice of coordinate system** depends on the symmetry of your problem.
- **Interconversion** allows flexible description and easier problem solving.
- **Dot and cross product** are fundamental operations for quantifying angle, projection, and area in vectors.
- **Divergence and curl** are crucial for understanding source/sink and rotational properties of vector fields, especially in electromagnetism and fluid dynamics.

These concepts are foundational and will recur throughout your physics coursework and further study!