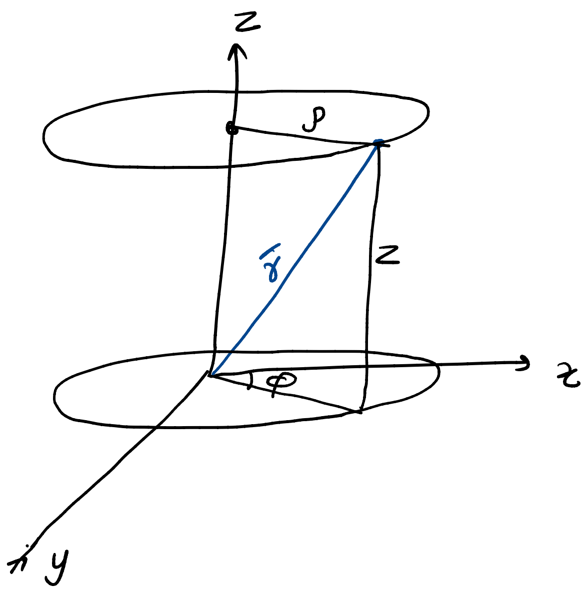
Lecture - 03

8/8/2023.

- The unit vectors change direction from point to point in spherical Polar coordinate system.

Cylindrical coordinate (ρ, φ, z).

u_1, u_2, u_3



Transformation eq^n

$x = \rho \cos\varphi$

$y = \rho \sin\varphi$

$z = z$

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\bar{r} = \rho\cos\varphi\ \hat{i} + \rho\sin\varphi\ \hat{j} + z\hat{k}$

$$\frac{\partial \bar{r}}{\partial \rho} = \cos\varphi\ \hat{i} + \sin\varphi\ \hat{j}$$

$$\left|\frac{\partial r}{\partial \rho}\right| = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1 = h_1$$

$$\therefore\ \hat{\rho} = \cos\varphi\ \hat{i} + \sin\varphi\ \hat{j}$$

$$\frac{\partial \bar{r}}{\partial \varphi} = -\rho\sin\varphi\ \hat{i} + \rho\cos\varphi\ \hat{j}$$
$$= \rho[-\sin\varphi\hat{i} + \cos\varphi\hat{j})$$

$$\left|\frac{\partial r}{\partial \varphi}\right| = \rho = h_2$$

$$\therefore\ \hat{\varphi} = -\sin\varphi\ \hat{i} + \cos\varphi\ \hat{j}$$

$$\frac{\partial r}{\partial z} = \hat{k}$$

$$\left|\frac{\partial r}{\partial z}\right| = 1 = h_3$$

$$\hat{z} = \hat{k}$$

$$d\bar{l} = h_1 du_1 \hat{u}_1 + h_2 du_2 \hat{u}_2 + h_3 du_3 \hat{u}_3$$
$$= d\rho\, \hat{\rho} + r\, d\varphi\, \hat{\varphi} + dz\, \hat{z} \checkmark$$

$$dS = h_1 du_1\, h_2 du_2\, \hat{u}_3$$
$$= 1 \cdot \rho \cdot d\rho\, d\varphi\, \hat{k} \checkmark$$

In Matrix form :-

$$(\hat{i}, \hat{j}, \hat{k}) \longrightarrow (\rho, \varphi, z)$$

$$\begin{bmatrix} \hat{\rho} \\ \hat{\varphi} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

$$\begin{bmatrix} A_\rho \\ A_\varphi \\ A_z \end{bmatrix} = \begin{bmatrix} & \checkmark & \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$(\rho, \varphi, z) \longrightarrow (\hat{i}, \hat{j}, \hat{k})$$

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\rho} \\ \hat{\varphi} \\ \hat{z} \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} & \checkmark & \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\varphi \\ A_z \end{bmatrix}$$

* The unit vectors change direction from point to point except $\hat{z}$

Curvilinear Coordinate :- $\bar{A} = A_1 \hat{u}_1 + A_2 \hat{u}_2 + A_3 \hat{u}_3$

	u_1	u_2	u_3	h_1	h_2	h_3
Cartesian	x	y	z	1	1	1
Spherical	r	θ	φ	1	r	$r\sin\theta$
Cylinderical	ρ	φ	z	1	ρ	1

Q.1 ⇒ Express the vector

$$\bar{A} = y\,\bar{a}_x + (x+z)\,\bar{a}_y$$

in spherical & cylinderical coordinates.

Ans ⇒

$$\left.\begin{aligned} \bar{A} &= A_x\hat{x} + A_y\hat{y} + A_z\hat{z} \\ \bar{A} &= A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi} \\ \bar{A} &= A_\rho\hat{\rho} + A_\phi\hat{\phi} + A_z\hat{z} \end{aligned}\right\} \text{——— (1)}$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x = y \\ A_y = (x+z) \\ A_z = 0 \end{bmatrix}$$

$$\left.\begin{aligned} A_r &= y\sin\theta\cos\phi + (x+z)\sin\theta\sin\phi \\ A_\theta &= y\cos\theta\cos\phi + (x+z)\cos\theta\sin\phi \\ A_\phi &= -y\sin\phi + (x+z)\cos\phi \end{aligned}\right\} \text{——— (2)}$$

get rid of x, y, z ⇒ using the transformation eqⁿ -

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi \qquad \text{put in (2) \& using (1)}$$
$$z = r\cos\theta$$

$$\begin{aligned} \bar{A} = {} & r\left[\sin^2\theta\sin\phi\cos\phi + (\sin\theta\cos\phi + \cos\theta)\sin\theta\sin\phi\right]\hat{a}_r \\ & + r\left[\sin\theta\sin\phi\cos\theta\cos\phi + (\sin\theta\cos\phi + \cos\theta)\cdot\cos\theta\sin\phi\right]\hat{a}_\theta \\ & + r\left[-\sin\theta\sin^2\phi + (\sin\theta\cos\phi + \cos\theta)\cdot\cos\phi\right]\hat{a}_\phi \end{aligned}$$

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x = y \\ A_y = x+z \\ A_z = 0 \end{bmatrix}$$

$$\left.\begin{aligned} A_\rho &= y\cos\varphi + (x+z)\sin\varphi \\ A_\varphi &= -y\sin\varphi + (x+z)\cos\varphi \\ A_z &= 0 \end{aligned}\right\} \longrightarrow (3)$$

transformation eqⁿ -

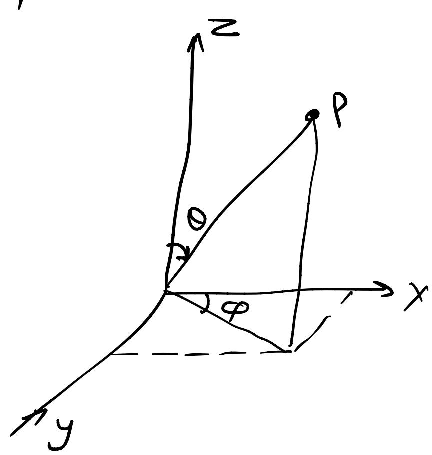
$$x = \rho\cos\varphi$$
$$y = \rho\sin\varphi$$
$$z = z$$

put in (3) & use (1) -

$$A = [\rho\cos\phi\sin\varphi + (\rho\cos\varphi + z)\sin\varphi]\,\hat{\rho} + [-\rho\sin^2\varphi + (\rho\cos\varphi + z)\cos\varphi]\,\hat{\varphi} + 0$$

Q.2 ⇒ Given point $P(-2,6,3)$. Evaluate P in spherical & cylindrical systems.

A ⇒



Spherical -

$$r = \sqrt{x^2+y^2+z^2}$$
$$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$$
$$\varphi = \tan^{-1}\frac{y}{x}$$

cylindrical ⇒

$$\rho = \sqrt{x^2+y^2}$$
$$\varphi = \tan^{-1} y/x$$
$$z = z$$

$$\therefore\ P(-2,6,3) = P(7, 64.62^\circ, 108.43^\circ) = P(6.32, 108.43^\circ, 3)$$

Homework:

Q.3 ⇒ Express the following vectors in Cartesian coordinates -

(a) ⇒ $\bar{A} = \rho z\sin\varphi\,\hat{\rho} + 3\rho\cos\varphi\,\hat{\varphi} + \rho\cos\varphi\sin\varphi\,\hat{z}$

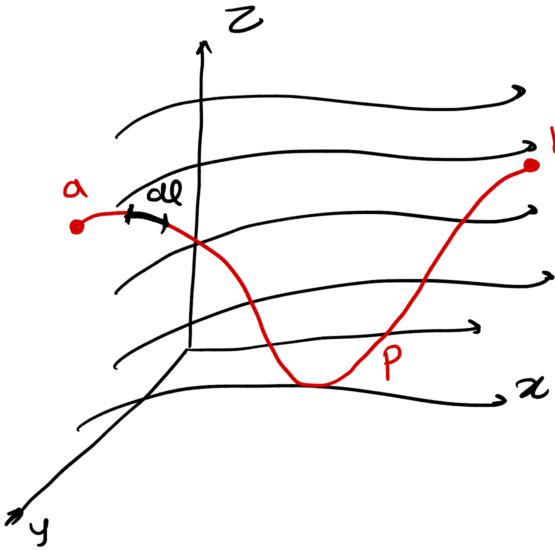
(b) ⇒ $\bar{B} = r^2\hat{r} + \sin\theta\,\hat{\varphi}$

A ⇒ $\bar{A} = \frac{1}{\sqrt{x^2+y^2}}\left[(xyz - 3xy)\,\hat{x} + (zy^2 + 3x^2)\,\hat{y} + xy\,\hat{z}\right]$

$$\bar{B} = \frac{1}{\sqrt{x^2+y^2+z^2}}\left\{[x(x^2+y^2+z^2) - y]\,\hat{x} + [y(x^2+y^2+z^2) + x]\,\hat{y} + z[x^2+y^2+z^2]\,\hat{z}\right\}$$

Integral Calculus :⇒ Line, Surface & Volume Integrals

① Line Integral (path integral) :⇒

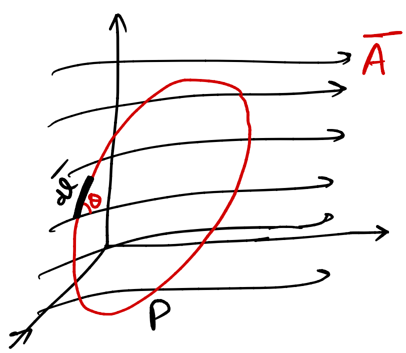


$\bar{A}$ (Vector field)
" function

$P \rightarrow$ prescribed path from a to b.
$dl \rightarrow$ infinitesimal displacement vector.

$$\boxed{\int_{a\,P}^{b} \bar{A}\cdot d\bar{l}} \Rightarrow \int_a^b |A| \cos\theta \, dl$$

* If the path of integration is closed curve :⇒



$$\oint_P \bar{A}\cdot d\bar{l}$$

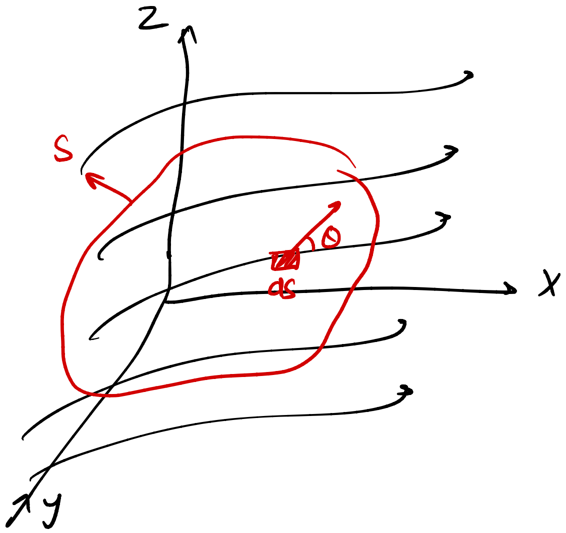
It is also called the circulation of $\bar{A}$ around P.

In general, Line integral depends on the particular path taken from a to b.

* However, there are imp. class of vectors for which the line integral is independent of the path & it is entirely determined by the end points.

⇒ Conservative vector field.
Path independent " ".

(ii) ⇒ Surface integral ⇒ [Flux] ⇒



$$\boxed{\int_S \bar{A}\cdot d\bar{S}}$$

$\bar{A}$ → Vector field " function

$d\bar{S}$ → Infinitesimal area.

If the surface is closed $\oint_S \bar{A}\cdot d\bar{S}$

[outward flux will be +ve]