# Assignment #7

(Jul xx, 2025)

Coordinate Systems and Transformation: Sadiku Chaps 2 & 3, Griffiths Chap 1

**1.** Find the probability for a particle, trapped in a 1-$d$ box defined by $0 \le x \le a$, to be found anywhere between $x = a/4$ and $x = 3a/4$ if it is in the lowest energy state. [Ans. $\frac{1}{2} + \frac{1}{\pi}$]

**2.** Consider $n^{th}$ state of the particle in Problem No.1. Show that as $n \to \infty$ the probability of finding the trapped particle between $x$ and $x + \Delta x$ is $\Delta x/a$.

**3.** Consider a particle of mass m in a 1-$d$ box defined by $0 \le x \le a$. Find the expectation values $\langle x\rangle$, $\langle p\rangle$, $\langle x^2\rangle$ and $\langle p^2\rangle$ for the $n^{\text{th}}$ stationary state. Considering $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$ and $\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$ as root mean square deviations around mean values $\langle x\rangle$ and $\langle p\rangle$ respectively, verify the Heisenberg's uncertainty principle.

[Ans. $a/2, 0, a^2/3 - a^2/2(n\pi)^2, (nh/2a)^2, \Delta x\Delta p = \dfrac{h}{4\pi}\sqrt{\dfrac{(n\pi)^2}{3} - 2}$]

**4.** Consider two wave functions $\psi_1(x) = Ae^{ikx}$ and $\psi_2(x) = Ae^{qx}$, where $k$ and $q$ are positive. Give main characteristics of these two wave functions. How do you define probability density and calculate the probability density for these two wave functions. Whether $\psi_2(x) = Ae^{qx}$ is an admissible wave function?

**5.** Show that $\psi(x) = A\, e^{ikx}$, where $k$ is some constant, is an acceptable eigenfunction. Also normalize it over the region $-a \le x \le a$.

**6.** Solve the time-independent Schrödinger equation with appropriate boundary conditions for an infinite square well centered at the origin,

$$V(x) = \begin{cases} 0 & -\dfrac{a}{2} \le x \le \dfrac{a}{2} \\ \infty & \text{elsewhere} \end{cases}$$

Check that the allowed energies are consistent with those derived for an infinite well of width '$a$' centered at $a/2$. Also check the wave function of this system can be obtained if one uses the substitution $x \to x + a/2$ in the earlier ones.

**7.** Consider a particle in a one dimensional potential

$$V(x) = \begin{cases} \infty & x \le 0 \\ 0 & 0 < x < a/2 \\ V_0 & a/2 \le x < a \\ \infty & x \ge a \end{cases}$$

The energy eigenvalues $E_1$ and $E_2$ of the first two bound states are such that $0 < E_1 < V_0$ and $V_0 < E_2 < \infty$. Schematically sketch the corresponding eigenfunctions.

**8.** Consider the one-dimensional potential step,

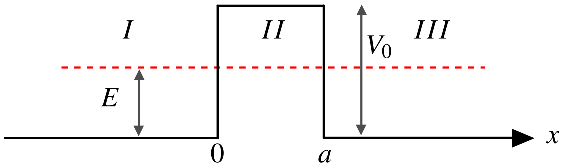
$$V(x) = \begin{cases} 0 & \text{for } x \le 0 \\ V_0 & \text{for } x > 0 \end{cases}$$

A particle with energy $E > V_0$ is incident on the step from the left. Calculate the transmission coefficient. Subsequently the particle is incident on the step from the right. Prove that the transmission coefficients are the same in both cases.

[Ans. $4\sqrt{E(E - V_0)}/(\sqrt{E} + \sqrt{E - V_0})^2$]

**9.** Take $E = 10$ eV and $V_0 = 13.8$ eV in Problem No. 8 and consider an electron incident on the step from the left. Find the distance in which the probability density of finding the electron decreases by a factor of $1/100$ as it penetrates into the classically forbidden region. [Ans. 0.23 nm]

**10.** Consider a rectangular potential barrier of height $V_0$ and width $a$, as shown. A beam of particles, each of mass $m$ and energy $E = V_0$, is incident from the left on this barrier.




(a) Solve the Schrödinger equation for the case $E = V_0$ to find the form for the wave function in the region $0 < x < a$.

(b) Write down the form of the solution in all three regions, and also write down the boundary conditions at $x = 0$ and $x = a$.

(c) Calculate the probability of transmission through the barrier.

**11.** A beam of electrons is incident on a barrier 6 eV high and 0.2 nm wide. Find the energy they should have if 1% of them are to get through the barrier. [Ans. 0.95 eV ]

**12.** Find the expectation value $\langle x \rangle$ for the first two states of a linear harmonic oscillator.

**13.** Plot the ground state and first excited state wave functions for a particle in a one-dimensional harmonic potential. In what ways are these wave functions different from the corresponding wave functions of the particle in a one-dimensional box.

**14.** The first excited state wave function of a linear harmonic oscillator is given by

$$\psi_1 = \left(\frac{8m\nu}{\hbar}\right)^{1/4} y e^{-y^2/2},$$

where $y = \sqrt{\dfrac{2\pi m\nu}{\hbar}}x$. What is the probability of finding the particle in the region $0 \le x \le \infty$. Schematically plot $|\psi_1|^2$ as a function of x showing clearly the classical boundaries.

**15.** The allowed bound state energies for a particle in a harmonic oscillator potential are $\hbar\omega(n + 1/2)$. Using this fact, deduce the allowed bound state energies for a particle of mass m moving in the following half-harmonic oscillator potential:

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2x^2 & \text{for } x > 0 \\ \infty & \text{for } x < 0 \end{cases}$$

This potential represents, for example, a spring that can be stretched but not compressed. [Ans. same expression but with $n = 1, 3, 5, .....$]

**16.** Consider a particle of mass $m$ and energy $E = \frac{1}{2}\hbar\omega$ moving in a one–dimensional potential $V = \frac{1}{2}m\omega^2x^2$. *Write down* the wave function $\psi(x)$. Plot qualitatively quantum mechanical probability distribution and compare it with the classical one. What is the quantum mechanical probability density at the classical amplitude relative to the probability density at the mean position? Find the expectation values of $x$ and the potential energy $V$.

**17.** Suppose that the particle in Problem No.1 has an initial wave function which is a mixture of the first two normalized stationary states: $\Psi(x,0) = A[\psi_1(x) + \psi_2(x)]$. What is the normalization $A$? If the energy of this particle is measured, what are the possible values one might get and what is the probability of getting them? What is the expectation value of the energy for this state? [Ans. $1/\sqrt{2}$, $h^2/8ma^2$, $h^2/2ma^2$, equal probability, $5h^2/16ma^2$]

**18.** Plot the ground state and first excited state wave functions for a particle in a one-dimensional harmonic potential. In what ways are these wave functions different from the corresponding wave functions of the particle in a one-dimensional box.